# Algorithms for the resolution of stochastic control problems in high dimension by using probabilistic and max-plus methods

Eric Fodjo

université
PARIS-SACLAY

NNT : 2018SACLX034

Thèse de doctorat

# Algorithms for solving stochastic control problems in high dimension by combining probabilistic and max-plus methods

Thèse de doctorat de l'Université Paris-Saclay
préparée à Ecole Polytechnique

Ecole doctorale n°ED 574 Ecole doctorale de mathématiques Hadamard (EDMH)
Spécialité de doctorat : Mathématiques appliquées

Thèse présentée et soutenue à Palaiseau, le 13/07/2018, par

**Mr. Eric Fodjo**

Composition du Jury :

| | |
|---|---|
| Nizar Touzi<br>Professeur, Ecole Polytechnique (CMAP) | Président |
| William McEneaney<br>Professeur, University of California, San Diego | Rapporteur |
| Huyên Pham<br>Professeur, Université de Paris Diderot (LPMA) | Rapporteur |
| Bruno Bouchard<br>Professeur, Université de Paris-Dauphine (Ceremade) | Examinateur |
| Hasnaa Zidani<br>Professeur, ENSTA (Unité des Mathématiques Appliquées) | Examinateur |
| Marianne Akian<br>Docteur, Ecole Polytechnique (CMAP) | Directeur de thèse |

## Remerciements

Je tiens à remercier tout d'abord ma directrice de thèse Marianne Akian qui a accepté de me prendre en thèse dans des conditions loin d'être idéales au vu de mon emploi à temps plein qui m'occupait déjà. Elle a toujours été présente et m'a apporté une aide précieuse tout au long de ce travail de thèse.

Je remercie aussi le dirigeant de mon entreprise Franck Sebban d'avoir été indulgent à mon égard tout au long de cette thèse en faisant preuve de compréhension lorsque mon travail de thèse empiétait sur mon travail en tant que consultant.

Je remercie enfin mes amis, Hippolyte, Mickael, Rija pour ne citer que ceux là et enfin ma famille pour m'avoir épaulé pendant les moments difficiles par lesquels je suis passé au cours de ces années de thèse. Je pense particulièrement à mon grand frère Jean Jacques Kamdem, à mes grandes soeurs Yollande, Carole et Pascaline Fodjo et à ma petite soeur Rachel Fodjo qui ont pu chacun à son tour, assister aux différentes phases par lesquelles je suis passé durant cette thèse. Je remercie également mon père Jean Flaubert Fodjo de m'avoir assisté de loin en m'appelant constamment pour m'encourager.

## Résumé en français de plus de 4000 caractères

Les équations d'Hamilton-Jacobi-Bellman résultent de l'application du principe de la programmation dynamique aux problèmes de contrôle optimal. Les problèmes de contrôle optimal consistent à trouver le paramètre ou contrôle optimisant l'évolution d'un système dynamique contrôlé suivant un certain critère. Le critère peut être de maximiser ou minimiser une certaine fonction dépendante de la trajectoire du système et définie sur un intervalle de temps fini ou infini. On parle alors de problème de contrôle optimal à horizon fini ou infini. Suivant que la nature de la dynamique du système étudié est stochastique ou déterministe, le problème de contrôle optimal est alors stochastique ou déterministe. Dans l'approche de la programmation dynamique, résoudre un problème de controle optimal nécessite de considérer une fonction appelée fonction valeur qui est à tout instant t et pour tout état x, la valeur du critère optimal associé au problème lorsque le système part à l'instant t de l'état x. Le principe de la programmation dynamique qui consiste à observer que toute sous-trajectoire d'une trajectoire optimale du système est également optimale sur l'intervalle de temps réduit considéré, permet d'obtenir que la fonction valeur du problème est solution d'une équation aux dérivées partielles qui est l'équation d'Hamilton-Jacobi-Bellman. Cette solution peut ne pas être une solution classique étant donné que la fonction valeur est en général non dérivable. La notion de solution de viscosité est alors utilisée. Il s'agit d'un type de solutions irrégulières bien adaptées aux problèmes de contrôle optimal. La fonction valeur du problème est ainsi calculée comme solution de viscosité de l'équation d'Hamilton-Jacobi-Bellman en utilisant des méthodes numériques propres aux équations aux dérivées partielles. Cela permet de déduire ensuite le contrôle optimal. Comme méthodes numériques utilisées, on a par exemple la méthode des différences finies, les méthodes semi-Lagrangiennes, les méthodes probabilistes et les méthodes max-plus. La convergence des schémas numériques vers la solution de viscosité d'une équation aux dérivées partielles a été obtenue comme résultat par Barles et Souganidis sous les conditions de consistence, monotonie et stabilité du schéma. Le premier type de méthodes utilisé dans ce cadre est la méthode des différences finies. Elle est assez précise mais a un temps de calcul exponentiel en la dimension de l'espace de l'état du système dynamique considéré. Ainsi, elle est confrontée à ce qu'on appelle la malédiction de la dimension qui fait que pour des espaces de dimension supérieure ou égale à 4, elle est inutilisable avec les puissances de calcul actuelles. Les méthodes semi-Lagrangiennes ont l'avantage de pouvoir être utilisées dans des cas où la méthode des différences finies peut difficilement s'appliquer à cause d'une monotonie difficile à obtenir (Equations aux dérivées partielles(EDP) du second ordre où le coefficient du terme de dérivée seconde est une matrice à diagonale non dominante), mais rencontrent les mêmes limitations que la méthode des différences finies en termes de temps de calcul. Les méthodes probabilistes permettent de dépasser ces limitations en temps de calcul car elles peuvent être appliquées quelquesoit la dimension de l'état du système dynamique. Par contre les schémas probabilistes rencontrés jusqu'ici dans la littérature sont monotones sous des conditions difficilement satisfaisables par l'équation d'Hamilton-Jacobi-Bellman lorsque celle-ci est très fortement non-linéaire. De plus, les méthodes probabilistes sont linéaires alors qu'un problème de contrôle optimal est par essence non linéaire. Cela pose problème quant-à la précision de la solution obtenue par une méthode probabiliste. Les méthodes max-plus quant-à elles, marchent bien pour les problèmes de contrôle optimal déterministes, leur caractère non linéaire permettant cette fois-ci de gagner en précision. Mais pour les problèmes de contrôle optimal stochastiques, leur temps de calcul devient doublement exponentiel en la dimension. Ce qui les rend pires que la méthode des différences finies.

On introduit dans cette thèse une nouvelle méthode alliant méthode probabiliste et méthode max-plus pour profiter de l'adéquation des méthodes probabilistes aux espaces de dimension élevé avec le caractère non linéaire des méthodes max-plus. Cette méthode est appelée méthode probabiliste max-plus. Pour pouvoir être appliquée, elle nécessite un schéma probabiliste monotone. Nous avons donc tout d'abord amélioré les schéma probabilistes existants pour les rendre monotones sous des hypothèses simples. Les problèmes auxquels nous nous intéressons particulièrement sont des problèmes de contrôle stochastique à horizon fini. En utilisant les schémas probabilistes que nous avons introduits, ces problèmes sont résolubles par l'utilisation d'un opérateur probabiliste backward pas à pas en partant de l'expression de la fonction valeur à l'horizon. L'astuce utilisée dans la méthode probabiliste max-plus consiste alors à approximer la fonction valeur à l'horizon par un supremum si le problème de contrôle optimal est un problème de maximization ou un infimum dans le cas contraire, de fonctions appartenant à un espace fonctionnel de dimension faible. Nous utilisons alors un théorème que nous avons introduit dans cette thèse qui donne la max-plus distributivité de tout operateur monotone et sous-homogène agissant sur un supremum de fonctions comme un opérateur intégral. Si les coefficients de l'équation d'Hamilton-Jacobi-Bellman sont tel que l'opérateur probabiliste backward utilisé pour la résoudre conserve pas à pas la forme de la fonction valeur comme supremum ou infimum de fonctions dans l'espace fonctionnel de faible dimension considéré, la méthode probabiliste max-plus est alors encore plus précise en ce sens qu'elle utilise une regression linéaire paramétrique pour déterminer les fonctions entrant dans le supremum ou l'infimum de la fonction valeur en recherchant ces fonctions dans l'espace théorique dans lequel elles sont censées être. Un exemple d'application est présenté dans le cadre du calcul du prix de sur-réplication d'une option dont le cross-gamma peut changer de signe dans un modèle de corrélation incertaine. Le payoff de l'option est alors approximé par un supremum de fonctions quadratiques et la fonction valeur du problème est déduite à chaque pas de temps comme un supremum de fonctions quadratiques dont les coefficients sont déduits par regression. Les calculs sont faits en dimension 2 et 5 en un temps raisonable.

# Contents

# Introduction

## 0.1 Notations

$|\cdot|$ denotes the Euclidean norm in any space $\mathbb{R}^q$, ($q \in \mathbb{N}$, $q \geq 1$). Sometimes, to be more explicit, we denote it $\|\cdot\|_2$. On the space of matrices $\mathbb{R}^{n\times m}$, $|\cdot|$ denotes the matrix norm defined for an element $M$ of $\mathbb{R}^{n\times m}$ by :

$$|M| = \sup_{\substack{r\in\mathbb{R}^m \\ r\neq 0}} \frac{|Mr|}{|r|}.$$

On the space of functions $f$ on a given subspace $A$ of $\mathbb{R}^q$, $|\cdot|$ denotes the sup norm defined by :

$$|f| = \sup_{x\in A} |f(x)|$$

In this definition, $f(x)$ can be an element of $\mathbb{R}^m$ ($m \geq 1$) or an element of $\mathbb{R}^{n\times m}$. We will use sometimes, the following norms for functions defined on a space $A$ when $A \subset \mathbb{R}\times\mathbb{R}^q$.

$$|f|_0 = \sup_{(t,x)\in A} |f(t,x)|,$$

and for

$$[f]_\delta = \sup_{(t,x)\neq(s,y)} \frac{|f(t,x)-f(s,y)|}{(|x-y|+|t-s|^{1/2})^\delta},$$

we define the norm $|\cdot|_\delta$ by :

$$|f|_\delta = |f|_0 + [f]_\delta.$$

For two vectors $u$ and $v$ of $\mathbb{R}^n$, $u\cdot v$ denotes the scalar product of $u$ and $v$.

For a given real number $a$, $a^+ = \max(a,0)$ and $a^- = -\min(a,0)$. These notations extend to real functions where for a real function $w$, $w^+$ is defined by $w^+ : x \mapsto w(x)^+$ and $w^-$ is defined by $w^- : x \mapsto w(x)^-$.

For a given function $v$ having a time variable $t$ and a space variable $x$, $D_x v$ and $D_x^2 v$ denote respectively the gradient and the Hessian of $v$ with respect to the space variable $x$. When there is no ambiguity (the function $v$ has only a space variable), we will denote the gradient and the Hessian respectively by $Dv$ and $D^2 v$.

For a given space $A$, $\mathscr{C}(A)$ denotes the set of continuous functions on $A$. If $A$ is a space of two variables $t$ and $x$ ($t$ for time and $x$ for space), $\mathscr{C}^{1,2}(A)$ denotes the space of functions once continuously differentiable with respect to $t$ and twice continuously differentiable with respect to $x$ and $\mathscr{C}_p(A)$ denotes the set of continuous functions on $A$ with a polynomial growth in the variable $x$. That means that for $\Phi \in \mathscr{C}_p(A)$, there exists $m \in \mathbb{N}^*$ and $K > 0$ such that

$$|\Phi(t,x)| \leq K(1+|x|^m),\ \forall (t,x)\in A.$$

If A is a space of only one variable $x$, this inequality becomes

$$|\Phi(x)| \leq K(1+|x|^m),\ \forall x\in A.$$

$\mathscr{C}^k(A)$ denotes the set of functions $k$ continuously differentiable on $A$ and $\mathscr{C}_p^k(A)$ denotes the subset of $\mathscr{C}^k(A)$ where all the corresponding partial derivatives are of polynomial growth. $\mathscr{C}_\mathrm{b}^k(A)$ denotes the subset of functions of $\mathscr{C}^k(A)$ having all their partial derivatives up to order $k$ bounded.

All these spaces of functions are sometimes denoted respectively as $\mathscr{C}(A,B)$ $\mathscr{C}^{1,2}(A,B)$, $\mathscr{C}_p(A,B)$, $\mathscr{C}^k(A,B)$, $\mathscr{C}_p^k(A,B)$, $\mathscr{C}_{\mathrm{b}}^k(A,B)$ in an equivalent manner where $B$ denotes the space where live the functions values.

$\mathscr{C}^\infty(A)$ denotes the set of function continuously differentiable at any order $k>0$.

$\mathrm{USC}(A)$ denotes the set of real upper semicontinuous functions on $A$ while $\mathrm{LSC}(A)$ denotes the set of real lower semicontinuous functions on $A$. We have the same definition for $\mathrm{USC}(A,\mathbb{R}^q)$ and $\mathrm{LSC}(A,\mathbb{R}^q)$ except that in this case, the functions are $\mathbb{R}^q$-valued functions.

$\chi_{a<b}$ is an indicator function which is equal to 1 if $a<b$ and 0 otherwise.

$\mathbb{S}^+(n)$ and $\mathbb{S}^{++}(n)$ denote respectively the set of symmetric and nonnegative matrices and the set of symmetric and positive definite $n\times n$ matrices. $\mathbb{S}(n)$ denotes the set of symmetric $n\times n$ matrices.

If $A$ is a matrix, $[A]_{ij}$ denotes the coefficient of $A$ at the $i$-th row and the $j$-th column. If $v$ is a vector, $[v]_i$ denotes the $i$-th coefficient of $v$.

Sometimes the Kronecker symbol $\delta_{ij}$ will be used. Its value is 0 if $i\neq j$ and 1 otherwise.

$I$ denotes the identity matrix whose dimensions should be infered from the context.

## 0.2 Context and motivations

Optimal control theory is a vast domain of applied mathematics where mathematicians are interested in solving problems like deterministic optimal control problems with finite horizon, deterministic optimal control problems with infinite horizon, stochastic control problems with finite horizon, stochastic control problems with infinite horizon and stochastic differential games. All these problems consist in finding the value of a variable called a control, such that a given function is optimized in a given sense on trajectories of a given process depending on the control. The term stochastic means that these trajectories are random (the optimization is then done on an expectation) while the term deterministic means that these trajectories are deterministic. Finite horizon means that the process is studied in a bounded time interval while inifinite horizon refers to cases where the process is considered in unbounded time. All these problems are solved in the dynamic programming approach, by computing an optimal function called the value function which satifies a partial differential equation (PDE) called the Hamilon-Jacobi-Bellman equation or in the differential games problems, the Hamilton-Jacobi-Bellman-Isaacs equation.

We are interested particularly here in stochastic optimal control problems with finite horizon despite the fact that we present all the type of optimal control problems aforementioned in Chapter 1. There are a lot of methods that can be used to solve this type of problem. Each of these methods have a positive and a negative aspect. We can divide them in two groups : grid-based methods and probabilistic methods. Grid based methods like finite difference method and Semi-Lagrangian methods applied to the Hamilton-Jacobi-Bellman equation can be very accurate but cannot be used numerically for problems with high space dimension (dimension greater or equal to 4) because the representation of the grid in memory becomes too large. This problem is known as the curse of dimensionality. Probabilistic methods, on the other hand, can at least be used in high dimension; but for that, a good probabilistic interpretation of the stochastic control problem is needed. More recently, a third type of method called the max-plus method has been introduced. However, it was mainly used for deterministic control problems where it was also subject to the curse of dimensionality. There has been a breakthrough in 2007 with the paper [52] of McEneaney where the curse of dimensionality is replaced by a curse of complexity dued to the use of too many functions in the representation of the value function, which can be solved by using a pruning algorithm. After that, the method has also been extended to the stochastic case [43, 54]. As for probabilistic methods, Cheridito, Soner, Touzi and Victoir obtained a probabilistic interpretation of the Hamilton-Jacobi-Bellman equation in [17] which allowed them to build a probabilistic scheme for the resolution of the Hamilton-Jacobi-Bellman equation. This probabilistic scheme have been fully studied in [25] by Fahim, Touzi and Warin.

Nevertheless, the use of the max-plus method in the stochastic case causes an explosion of the curse of complexity related to this method, making any pruning algorithm very hard to implement and the probabilistic scheme resulted from the work of Cheridito, Soner, Touzi and Victoir can only be applied to particular Hamilton-Jacobi-Bellman equation not nonlinear enough with respect to the second order derivatives to represent all the stochastic optimal control problems. The work presented here, tries to solved this two questions.

We introduce two new probabilistic schemes. The first one is monotone under restrictive boundness conditions on the coefficient of the PDE while the second one is monotone under less restrictive conditions allowing it to be used with Hamilton-Jacobi-Bellman equation with unbounded coefficent and unbounded terminal function. We also introduce a new method for solving Hamilton-Jacobi-Bellman equations that is a mix of max-plus and probabilistic methods. We call it a max-plus probabilistic method. With this method, we get rid of the need of a pruning algorithm related to max-plus method as this pruning algorithm is incoporated in the method. On the other hand, we gain the accuracy of max-plus methods as the non linearity of the value function is well represented.

## 0.3 Contributions

In Chapter 1, after having presented the different type of optimal control problems, the related Hamilton-Jacobi-Bellman equations and the results related to viscosity solutions of Hamilton-Jacobi-Bellman equations in bounded and unbounded settings with respect to the coefficients and the terminal function of the PDE, we give an improvement of the result of Da Lio and Ley [23] (Lemma 1.2.2 and Lemma 1.2.3) allowing us to have the existence on any time interval $[0, T]$ of the viscosity solution of an Hamilton-Jacobi-Bellman equation with coefficients and terminal function satisfying growth conditions similar to those of [23].

In Chapter 2, we extend the well known result of convergence of Barles and Souganidis [8] to the case where the viscosity solution lives in the space of unbounded functions (Theorem 2.1.3). This is done after giving some general results on finite difference method.

In Chapter 3, we presents all the results related to the new probabilistic schemes we introduced. After having obtained error estimates of different probabilistic approximations of functions derivatives in the bounded setting, we prove the convergence of each of the two probabilistic schemes that we introduced here and obtain related error estimates. In a second part, we present the result we obtained in the unbounded setting with the second probabilistic scheme that unfortunately did not allow us to obtain the convergence of the scheme. From our point of view, this can be improved and is let to future work.

In Chapter 4, we present the new max-plus probabilistic method that we introduce. We give firstly theoretical related results before describing the method algorithm and the related complexity results.

In Chapter 5, we give the numerical results of the tests we have done which are unfortunately incomplete. Indeed, it took us a lot of time to test the max-plus probabilistic method with the second probabilistic scheme on a different problem than the one presented in this manuscript, but due to numerical issues, which were not dependent on the method or the scheme but rather on the type of problem we chose, we did not get relevant results.

We end this manuscript with a conclusion presenting works that we were not able to complete in the allowed time and that we hope will know a continuation.

# CHAPTER 1

# Optimal control equations and Viscosity solutions

In this section, we define optimal control problems and present the Hamilton-Jacobi-Bellman and Hamilton-Jacobi-Bellman-Isaacs equations. We then give some results encountered in the literature about viscosity solutions which are in general the type of solutions considered for Hamilton-Jacobi-Bellman equations. We also give improvements of results on viscosity solution obtained by Da Lio and Ley in a setting closed to linear quadratic optimal control problems.

## 1.1 Optimal control problems and Hamilton-Jacobi-Bellman equations

We are interested particularly in this thesis in deterministic or stochastic control problems with finite or infinite horizon and differential games problems. We present the general forms of each of these classes of problems in this section. As we are going to see, when using the dynamic programming approach, these problems are solved by finding the value function of the problem which can sometimes be identified as the classical solution of a first-order (for deterministic problems) or second-order (for stochastic problems) partial differential equation called the Hamilton-Jacobi-Bellman equation or the Hamilton-Jacobi-Bellman-Isaacs equation for differential games. We recall here verification theorems allowing to make this identification and which have been extended to the case where the value function is not a classical solution but a viscosity solution. This last notion will be studied in the next section.

### 1.1.1 Deterministic optimal control problems with finite horizon

A deterministic optimal control problem with finite horizon consists in optimizing a functional payoff defined on trajectories of a deterministic control process in bounded time. Let $t$ be the time variable and $x$ the space variable of such a problem. Let $t_0 < t_1$ be two real numbers such that $t \in [t_0, t_1]$. Let $O$ be an open set of $\mathbb{R}^n$ for $n \geq 1$ such that the trajectories of the deterministic control process live in $O$. $O$ may be bounded and in this case, it is supposed that $\partial O$ is a compact manifold of class $C^2$. Let $Q := [t_0, t_1) \times O$. $Q$ is the domain of the problem.

To define the problem, we need to define properly deterministic control processes starting from any point $(t, x) \in \bar{Q}$ and controlled by another process $u$. We will denote these processes by $(x^{t,x,u}(s))_{t \leq s \leq t_1}$ as . We consider the following differential equation.

$$\begin{cases} \dfrac{dx^{t,x,u}(s)}{ds} = f(s, x^{t,x,u}(s), u(s)), \quad t \leq s \leq t_1 \\ x^{t,x,u}(t) = x, \end{cases} \tag{1.1}$$

where for all $s$, $u(s) \in U \subset \mathbb{R}^m$ $(m \geq 1)$, $U$ is a closed set, and $f : [t_0, t_1] \times \mathbb{R}^n \times U \to \mathbb{R}^n$ is a continuous function.

To ensure the existence and unicity of the processes $x^{t,x,u}$, the following conditions on $f$ and the process $u$ are generally imposed.

**A 1.1.1.** *For all $\rho > 0$, there exists $K_\rho > 0$ such that :*

$$|f(t, x, v) - f(t, y, v)| \leq K_\rho |x - y|, \tag{1.2}$$

*for all $t \in [t_0, t_1]$, $x, y \in \mathbb{R}^n$ and $v \in U$ such that $|v| \leq \rho$.*

**A 1.1.2.** *$u(\cdot)$ is a bounded Lebesgue measurable function from $[t, t_1]$ to $U$.*

Each process $x^{t,x,u}$ is the state process starting at time $t$ from $x$ and the allowed processes $u$ are the control processes defined from time $t$. Let $\mathscr{U}^0(t)$ be the set of these control processes.

By definition, the processes $(x^{t,x,u}(s))_{s \geq t}$ stop at $s = t_1$. But, in the case where $O$ is bounded, it is usually required that the processes $(x^{t,x,u}(s))_{s \geq t}$ stay in $\bar{O}$ and stop at any time when they exit $\bar{O}$. So, the processes may stop at a time $\tau < t_1$.

Let us denote by $\tau^{t,x,u}$ the stopping time of a process $x^{t,x,u}$. $\tau^{t,x,u}$ is the exit time of the process $(s, x^{t,x,u}(s))_s$ from the closure $\bar{Q}$ of the domain $Q$.

A running functional payoff that we shall denote by $L$ and a final functional payoff that we shall denote by $\Psi$ are considered such that :

- if $O$ is bounded, $\Psi : ([t_0, t_1) \times \partial O) \cup (\{t_1\} \times \bar{O}) \to \mathbb{R}$ and

$$\Psi(t,x) := \begin{cases} g(t,x) & \text{if } (t,x) \in [t_0,t_1) \times \partial O \\ \psi(x) & \text{if } (t,x) \in \{t_1\} \times \bar{O}, \end{cases} \tag{1.3}$$

  otherwise for $O = \mathbb{R}^n$, $\Psi : \{t_1\} \times \bar{O} \to \mathbb{R}$ and

$$\Psi(t_1, x) = \psi(x),\ x \in \bar{O} \tag{1.4}$$

  with $g \in C([t_0, t_1) \times \partial O)$ and $\psi$ a given function on $\bar{O}$.

- $L \in C([t_0, t_1] \times \bar{O} \times U)$

The overall functional payoff to minimize is a function $J$ given by the following expression :

$$J(t,x;u) := \int_t^{\tau^{t,x,u}} L(s, x^{t,x,u}(s), u(s))ds + \Psi(\tau^{t,x,u}, x^{t,x,u}(\tau^{t,x,u})). \tag{1.5}$$

We will consider here a problem of minimization of the overall functional payoff. The general form of the problem is then to find, for any $(t,x) \in \bar{Q}$, $u^* \in \mathscr{U}^0(t)$ such that :

$$J(t,x;u^*) = \inf_{u \in \mathscr{U}^0(t)} J(t,x;u). \tag{1.6}$$

One may consider for $(t,x) \in \bar{Q}$ a smaller set of controls in the previous minimization. The objective may be to ensure that $\tau^{t,x,u} = t_1$ for all $(t,x) \in \bar{Q}$ by considering only controls whose related state processes do not exit $\bar{O}$ until $t_1$. In such cases, the set of controls considered for the minimization will also depends on $x$. A more general form of the deterministic optimal control problem is then to find, for any $(t,x) \in \bar{Q}$, $u^* \in \mathscr{U}(t,x)$ such that :

$$J(t,x;u^*) = \inf_{u \in \mathscr{U}(t,x)} J(t,x;u), \tag{1.7}$$

where $\mathscr{U}(t,x) \subset \mathscr{U}^0(t)$.

To be able to apply to this problem the dynamic programming principle which will be stated further, the following condition is generally imposed on $\mathscr{U}(t,x)$ :

**A 1.1.3.** *For $u \in \mathscr{U}(t,x)$, let $r \in [t, \tau^{t,x,u}]$ and $u'(\cdot) \in \mathscr{U}(r, x^{t,x,u}(r))$. If we define a control $\tilde{u}$ as :*

$$\tilde{u}(s) := \begin{cases} u(s), & t \le s \le r \\ u'(s), & r < s \le t_1, \end{cases} \tag{1.8}$$

*then $\tilde{u} \in \mathscr{U}(t,x)$.*

For what follows, we will consider the problem formulated through the equation (1.6). In dynamic programming setting, to solve this problem, a function that is called *the value function*, is introduced. It is defined here as follows :

$$V(t,x) := \inf_{u \in \mathscr{U}^0(t)} J(t,x;u). \tag{1.9}$$

The value function respects a principle called the *Dynamic Programming Principle* corresponding to the following identity. For $(t,x) \in \bar{Q}$ and $r \in [t,t_1]$ :

$$\begin{aligned} V(t,x) = \inf_{u(\cdot) \in \mathscr{U}^0(t)} \Bigg[ \int_t^{r \wedge \tau^{t,x,u}} & L(s, x^{t,x,u}(s), u(s)) ds \\ & + g(\tau^{t,x,u}, x^{t,x,u}(\tau^{t,x,u})) \chi_{\tau^{t,x,u} < r} + V(r, x^{t,x,u}(r)) \chi_{r \leq \tau^{t,x,u}} \Bigg] \end{aligned} \tag{1.10}$$

Using this principle, the value function, when smooth enough, is identified as the classical solution of a Partial Differential Equation (PDE). The boundary condition of this PDE depends on whether $O$ is bounded or equal to $\mathbb{R}^n$.

For $O = \mathbb{R}^n$, the PDE is :

$$\begin{cases} \dfrac{\partial}{\partial t} W(t,x) + \inf\limits_{v \in U} \{L(t,x,v) + f(t,x,v) \cdot D_x W(t,x)\} = 0, \quad (t,x) \in Q, \\ W(t_1,x) = \Psi(t_1,x), x \in O. \end{cases} \tag{1.11}$$

For $O$ bounded, the PDE is :

$$\begin{cases} \dfrac{\partial}{\partial t} W(t,x) + \inf\limits_{v \in U} \{L(t,x,v) + f(t,x,v) \cdot D_x W(t,x)\} = 0 \quad (t,x) \in Q \\ W(t_1,x) = \Psi(t_1,x), \quad x \in \bar{O} \\ W(t,x) \leq \Psi(t,x), \quad (t,x) \in [t_0,t_1) \times \partial O. \end{cases} \tag{1.12}$$

The following statement is assumed to be true for $O$ bounded.

**A 1.1.4.** *For every $(s,\xi) \in [t_0,t_1) \times \partial O$, there exists $v(s,\xi) \in U$ such that :*

$$f(s,\xi,v(s,\xi)) \cdot \eta(\xi) > 0,$$

*where $\eta(\xi)$ is the exterior unit normal at $\xi \in \partial O$.*

By convention, the equation of PDE (1.12) and (1.11) is rather written as :

$$-\frac{\partial}{\partial t} W(t,x) + H(t,x,D_x W(t,x)) = 0, \tag{1.13}$$

where for $(t,x,p) \in [t_0,t_1) \times \mathbb{R}^n \times \mathbb{R}^n$

$$H(t,x,p) = \sup_{v \in U} \{-p \cdot f(t,x,v) - L(t,x,v)\}. \tag{1.14}$$

It is the *Hamilton-Jacobi-Bellman equation* for deterministic optimal control problems with finite horizon and the function $H$ is called the *Hamiltonian.*

The verification theorems allowing to solve the optimal control problem using the PDE (1.12) or (1.11) are stated as follows in [28] :

**Theorem 1.1.1** ([28, Theorem I.5.1]). *$O$ is supposed to be equal to $\mathbb{R}^n$. Let $W \in C^1(\bar{Q})$ such that $W$ satisfies* (1.11). *Then :*

$$W(t,x) \leq V(t,x), \forall (t,x) \in \bar{Q}.$$

*Moreover, if there exists $(t,x) \in Q$ and $u^* \in \mathscr{U}^0(t)$ such that :*

$$\begin{aligned} L(s, x^{t,x,u^*}(s), u^*(s)) + &f(s, x^{t,x,u^*}(s), u^*(s)) \cdot D_x W(s, x^{t,x,u^*}(s)) \\ &= - H(s, x^{t,x,u^*}(s), D_x W(s, x^{t,x,u^*}(s))), \end{aligned}$$

*for almost all $s \in [t, t_1]$, then $u^*$ is optimal for initial data $(t,x)$ and $W(t,x) = V(t,x)$.*

**Theorem 1.1.2** ([28, Theorem I.5.2]). *$O$ is supposed to be bounded. Let $W \in C^1(\bar{Q})$ such that $W$ satisfies* (1.12). *Then :*

$$W(t,x) \leq V(t,x), \forall (t,x) \in \bar{Q}.$$

*Moreover, if there exists $(t,x) \in Q$ and $u^* \in \mathscr{U}^0(t)$ such that :*

$$\begin{aligned} L(s, x^{t,x,u^*}(s), u^*(s)) + &f(s, x^{t,x,u^*}(s), u^*(s)) \cdot D_x W(s, x^{t,x,u^*}(s)) \\ &= - H(s, x^{t,x,u^*}(s), D_x W(s, x^{t,x,u^*}(s))), \end{aligned} \tag{1.15}$$

*for almost all $s \in [t, \tau^{t,x,u^*}]$ and $W(\tau^{t,x,u^*}, x^{t,x,u^*}(\tau^{t,x,u^*})) = \Psi(\tau^{t,x,u^*}, x^{t,x,u^*}(\tau^{t,x,u^*}))$ in case $\tau^{t,x,u^*} < t_1$, then $u^*$ is optimal for initial data $(t,x)$ and $W(t,x) = V(t,x)$.*

These theorems allow one to test the optimality of a given control process.

### 1.1.2 Deterministic optimal control problems with infinite horizon

A deterministic optimal control problem with infinite horizon consists in optimizing a functional payoff defined on trajectories of a deterministic control process considered on an unbounded time interval. The unboundness of the time interval may cause definition problems for the deterministic control process and for the overall functional payoff which may be infinite.

To simplify things and by using the same notations as in the previous subsection, let us suppose that the running payoff $L$, the function $g$ involved in the definition of the terminal payoff $\Psi$ and the drift function $f$ do not depend on the time variable as in [28]. In this case, as shown in [28], the dependence of the deterministic optimal problem on the time variable is not relevant anymore. So we may define the state processes and control processes as starting from time 0. The state processes $x^{t,x,u}$ defined in the previous subsection, become the processes $x^{x,u}$ defined by :

$$\begin{cases} \dfrac{dx^{x,u}(s)}{ds} = f(x^{x,u}(s), u(s)), \quad s \geq 0 \\ x^{x,u}(0) = x, \end{cases} \tag{1.16}$$

$f$ verifies a condition similar to A 1.1.1 and $u$ is a Lebesgue measurable function defined on $[0, \infty)$. The set of controls is then denoted by $\mathscr{U}^0$ instead of $\mathscr{U}^0(t)$.

When $O = \mathbb{R}^n$, there is no stopping time anymore. But, for $O$ bounded, a stopping time $\tau^{x,u}$ is considered, which is the exit time of $x^{x,u}$ from $\bar{O}$. In what follows, we use the convention $\tau^{x,u} = \infty$ for $O = \mathbb{R}^n$.

A discount factor with a constant discount rate $\beta$ is introduced in the definition of the overall functional payoff which is then equal to the following :

$$J(x;u) = \int_0^{\tau^{x,u}} e^{-\beta s} L(x^{x,u}(s), u(s))ds + e^{-\beta\tau^{x,u}} g(x^{x,u}(\tau^{x,u}))\chi_{\tau^{x,u}<\infty}, \tag{1.17}$$

where $\beta > 0$, $L \in C(\bar{O} \times U)$, $g \in C(\partial O)$.

The optimal control problem is then defined on controls $u$ such that :

$$\int_0^{\tau^{x,u}} e^{-\beta s}|L(x(s), u(s))|ds < \infty,$$

to ensure that $J(x;u)$ is well defined.

Let $\mathscr{U}_x$ be this set of controls. $\mathscr{U}_x$ is assumed to be non empty for all $x \in \bar{O}$.

As in the previous subsection, we consider here a problem of minimization of the functional payoff. The value function is then defined by :

$$V(x) = \inf_{u\in\mathscr{U}_x} J(x;u),\ x \in \bar{O}.$$

The Dynamic Programming Principle verified by the value function is expressed here by the following identity. For $x \in \bar{O}$ and $r \geq 0$ :

$$\begin{aligned} V(x) = \inf_{u(\cdot)\in\mathscr{U}_x} \Bigg[ &\int_0^{r\wedge\tau^{x,u}} e^{-\beta s} L(x^{x,u}(s), u(s))ds \\ &+ e^{-\beta\tau^{x,u}} g(x^{x,u}(\tau^{x,u}))\chi_{\tau^{x,u}<r} + e^{-\beta r} V(x(r))\chi_{r\leq\tau^{x,u}} \Bigg]. \end{aligned} \tag{1.18}$$

The Hamilton-Jacobi-Bellman equation is in this case, as follows :

$$\beta W(x) + H(x, DW(x)) = 0,\ x \in O, \tag{1.19}$$

where the Hamiltonian $H$ is defined by

$$H(x,p) = \sup_{v\in U} \{-p \cdot f(x,v) - L(x,v)\}$$

for $x, p \in \mathbb{R}^n$.

In [28], the verification theorem is given only for $O$ bounded under the following assumption.

**A 1.1.5.** *For every $\xi \in \partial O$, there exists $v(\xi) \in U$ such that :*

$$f(\xi, v(\xi)) \cdot \eta(\xi) > 0,$$

*where $\eta(\xi)$ denotes the exterior unit normal at $\xi \in \partial O$.*

This assumption is similar to A 1.1.4.

The PDE that should verify the value function is :

$$\begin{cases} \beta W(x) + H(x, DW(x)) = 0,\ x \in O, \\ W(x) \leq g(x), \quad x \in \partial O. \end{cases} \tag{1.20}$$

The verification theorem is then stated as follows.

**Theorem 1.1.3** ([28, Theorem I.7.1]). *Let $W \in C^1(\bar{O})$ such that $W$ satisfies (1.20). If for all $x \in \bar{O}$ and $u \in \mathscr{U}_x$ such that $\tau^{x,u} = \infty$, $\lim_{r \uparrow \infty} e^{-\beta r} W(x^{x,u}(r)) = 0$, then :*

- *$W(x) \leq V(x)$ for all $x \in \bar{O}$,*
- *If there exists $x \in O$ and $u^* \in \mathscr{U}_x$ such that :*

$$\begin{aligned} L(x^{x,u^*}(s), u^*(s)) + & f(x^{x,u^*}(s), u^*(s)) \cdot DW(x^{x,u^*}(s)) \\ & = -H(x^{x,u^*}(s), DW(x^{x,u^*}(s))) \end{aligned} \tag{1.21}$$

  *for almost every $s \in [0, \tau^{x,u^*})$ and $W(x^{x,u^*}(\tau^{x,u^*})) = g(x^{x,u^*}(\tau^{x,u^*}))$ if $\tau^{x,u^*} < \infty$, then $u^*$ is optimal for initial data $x$ and $W(x) = V(x)$.*

### 1.1.3 Stochastic optimal control problems with finite horizon

While a deterministic optimal control problem consists in optimizing a functional payoff on trajectories of deterministic control processes, a stochastic optimal control problem look at the optimization of the expectation of a functional payoff on trajectories of stochastic control processes. As in section 1.1.1, a time variable $t$ in a bounded interval $[t_0, t_1]$ of $\mathbb{R}$ and a space variable $x$ in an open set $O$ of $\mathbb{R}^n$ are considered. $O$ may be bounded. In this case, we suppose that $\partial O$ is a compact of class $C^3$.

Stochastic control processes considered are usually continuous time Markov processes. We will limit ourselves here only to Markov diffusion processes.

Let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t_0 \leq t \leq t_1}, P)$ be a filtered probability space and $(B_t)_{t_0 \leq t \leq t_1}$ be a $\mathbb{R}^d$ Brownian motion defined on this filtered probability space. We denote by $\nu$ the 4-tuple $(\Omega, (\mathscr{F}_t)_{t_0 \leq t \leq t_1}, P, B)$ that is called a reference probability system (see [28]).

We also denote the control processes here by $x^{t,x,u}$ as in Section 1.1.1. They are solutions of the following stochastic differential equation :

$$\begin{cases} dx^{t,x,u}(s) = f(s, x^{t,x,u}(s), u(s))ds + \sigma(s, x^{t,x,u}(s), u(s))dB_s, \ t \leq s \leq t_1, \\ x^{t,x,u}(t) = x, \end{cases} \tag{1.22}$$

where $u(s) \in U$, $U$ closed subset of $\mathbb{R}^m$, $f : [t_0, t_1] \times \mathbb{R}^n \times U \to \mathbb{R}^n$, $\sigma : [t_0, t_1] \times \mathbb{R}^n \times U \to \mathbb{R}^{n \times d}$.

For this stochastic differential equation to have a unique solution $\{\mathscr{F}_s\}$-progressively measurable with continuous sample paths, the following assumptions are usually required.

**A 1.1.6.** *$f$ and $\sigma$ are continuous functions on $[t_0, t_1] \times \mathbb{R}^n \times U$ and for any $v \in U$, $f(\cdot, \cdot, v)$, $\sigma(\cdot, \cdot, v)$ are once continuously differentiable on $[t_0, t_1] \times \mathbb{R}^n$.*

**A 1.1.7.** *There exists a constant $C > 0$ such that :*

$$\begin{aligned} & |f_t| + |f_x| \leq C, \\ & |\sigma_t| + |\sigma_x| \leq C, \\ & |f(t, x, v)| \leq C(1 + |x| + |v|), \\ & |\sigma(t, x, v)| \leq C(1 + |x| + |v|). \end{aligned} \tag{1.23}$$

*where $f_t$, $\sigma_t$, $f_x$, $\sigma_x$ denote respectively the $t$-partial derivatives and the gradients with respect to $x$ of the functions $f$ and $\sigma$. The norm considered for the partial derivatives is the sup norm as defined in section 0.1.*

**A 1.1.8.** *The control processes $u(\cdot)$ are $\{\mathscr{F}_s\}$ progressively measurable and such that*

$$\mathbb{E}^\nu\left[\int_t^{t_1} |u(s)|^2 ds\right] < \infty. \tag{1.24}$$

The control processes $\{\mathscr{F}_s\}$-progressively measurable which verify the inequality (1.24) are called admissible control processes.

As in Section 1.1.1, the process $(x^{t,x,u}(s))_{s\geq t}$ stops by definition at $s = t_1$. However, for $O$ bounded, it will stop at its exit time from $\bar{O}$ if this time is lower than $t_1$. We will denote by $\tau^{t,x,u}$ the stopping time of the process $x^{t,x,u}$. As this process is stochastic and $\mathscr{F}_s$- measurable, $\tau^{t,x,u}$ is random and adapted to the filtration $\mathscr{F}_s$.

The running functional payoff $L$ and the terminal functional payoff $\Psi$ are real functions which verify the following conditions :

**A 1.1.9.** $L \in C([t_0, t_1] \times \bar{O} \times U)$ *and* $\Psi \in C([t_0, t_1] \times \bar{O})$

**A 1.1.10.** *There exists a constant $C$ and an integer $k$ such that :*

$$\begin{aligned} |L(t, x, v)| &\leq C(1 + |x|^k + |v|^k), \\ |\Psi(t, x)| &\leq C(1 + |x|^k). \end{aligned} \tag{1.25}$$

The overall payoff function is then :

$$J^\nu(t, x; u) = \mathbb{E}^\nu\left[\int_t^{\tau^{t,x,u}} L(s, x^{t,x,u}(s), u(s))ds + \Psi(\tau^{t,x,u}, x^{t,x,u}(\tau^{t,x,u}))\right]. \tag{1.26}$$

Let $\mathscr{A}_{t,\nu}$ denote the set of admissible control processes starting from time $t$ in the reference probability system $\nu$. As in the previous sections, we consider here a minimization problem. If the goal is to minimize $J^\nu(t, x, u)$ over the control processes of $\mathscr{A}_{t,\nu}$ for a given reference probability system $\nu$, then the solution of the problem is $\nu$-optimal and the related value function is :

$$V_\nu(t, x) = \inf_{u\in\mathscr{A}_{t,\nu}} J^\nu(t, x; u). \tag{1.27}$$

If the goal is to minimize the overall payoff function $J^\nu(t, x, u)$ over all control processes of the sets $\mathscr{A}_{t,\nu}$ for any reference probability system $\nu$, then a solution of the problem is globally optimal and the related value function is defined by :

$$V_{\mathrm{PM}}(t, x) = \inf_\nu V_\nu(t, x).$$

To study the value function $V_{\mathrm{PM}}$ with the dynamic programming theory, a stronger version of the dynamic programming principle is needed. It is expressed in the following way where $V = V_{\mathrm{PM}}$.

**Definition 1.1.1** (Dynamic programming principle)**.** *For every $\nu, u(\cdot) \in \mathscr{A}_{t\nu}$ and $\{\mathscr{F}_s\}$-stopping time $\theta$,*

$$\begin{aligned} V(t, x) \leq \mathbb{E}^\nu\Bigg[ &\int_t^{\tau^{t,x,u}\wedge\theta} L(s, x^{t,x,u}(s), u(s))ds \\ &+ V(\tau^{t,x,u} \wedge \theta, x^{t,x,u}(\tau^{t,x,u} \wedge \theta))\Bigg]. \end{aligned} \tag{1.28}$$

*For every $\delta > 0$, there exist $\nu$ and $u(\cdot) \in \mathscr{A}_{t\nu}$ such that :*

$$V(t,x) + \delta \geq \mathbb{E}^{\nu}\Bigg[\int_t^{\tau^{t,x,u}\wedge\theta} L(s, x^{t,x,u}(s), u(s))ds \\ + V(\tau^{t,x,u}\wedge\theta, x^{t,x,u}(\tau^{t,x,u}\wedge\theta))\Bigg]. \tag{1.29}$$

The Hamilton-Jacobi-Bellman equation here is :

$$-\frac{\partial}{\partial t}W(t,x) + H(t,x,D_xW(t,x), D_x^2W(t,x)) = 0,\ (t,x) \in [t_0,t_1)\times O, \tag{1.30}$$

where the Hamiltonian $H$ is defined by :

$$H(t,x,p,A) = \sup_{v\in U}\left[-f(t,x,v)\cdot p - \frac{1}{2}\operatorname{Tr}(\sigma\sigma^{\intercal}(t,x,v)A) - L(t,x,v)\right], \tag{1.31}$$

for $(t,x) \in [t_0,t_1)\times O$, $p \in \mathbb{R}^n$, $A \in \mathbb{S}(n)$, set of $n\times n$ symmetric matrices.

The PDE that should verify here the value function $V = V_{\mathrm{PM}} = W$ is then

$$\begin{cases} -\dfrac{\partial}{\partial t}W(t,x) + H(t,x,D_xW(t,x), D_x^2W(t,x)) = 0,\ (t,x) \in Q = [t_0,t_1)\times O \\ W(t,x) = \Psi(t,x),\ (t,x) \in \partial Q. \end{cases} \tag{1.32}$$

This form of the PDE include the case of $O = \mathbb{R}^n$ or $O$ bounded. The verification theorem allowing to identify the value function $V_{\mathrm{PM}}$ as the classical solution of this PDE is then stated as follows.

**Theorem 1.1.4** ([28, Theorem IV.3.1])**.** *Let $W \in C^{1,2}([t_0,t_1)\times O) \cap C_p([t_0,t_1]\times \bar{O})$ be a solution of the PDE* (1.32)*. Then :*

- *$W(t,x) \leq J^{\nu}(t,x;u)$ for any probability reference system $\nu$, control process $u \in \mathscr{A}_{t,\nu}$ and any initial data $(t,x) \in Q$.*
- *If there exist a reference probability system $\nu^* = (\Omega^*, \{\mathscr{F}_s^*\}, P^*, B^*)$ and $u^* \in \mathscr{A}_{t,\nu^*}$ such that :*

$$u^*(s) \in \arg\min\Big[f(s,x^{t,x,u^*}(s),v)\cdot D_xW(s,x^{t,x,u^*}(s)) \\ + \frac{1}{2}\operatorname{Tr}\Big(\sigma\sigma^{\intercal}(s,x^{t,x,u^*}(s),v)D_x^2W(s,x^{t,x,u^*}(s))\Big) + L(s,x^{t,x,u^*}(s),v)\Big]$$

*for Lebesgue$\times P^*$-almost all $(s,\omega) \in [t,\tau^{t,x,u^*}]\times\Omega^*$, then $W(t,x) = V_{\mathrm{PM}}(t,x) = J^{\nu *}(t,x;u^*)$.*

Under sufficient conditions, $V_{\mathrm{PM}} = V_{\nu}$ for all reference probability system $\nu$. We give here a Corollary of Theorem IV.7.1 of [28] stating additional conditions to have this result in the case of $O = \mathbb{R}^n$.

**Corollary 1.1.1** ([28, Theorem IV.7.1])**.** *Assume A 1.1.6-A 1.1.10 and $O = \mathbb{R}^n$. If $U$ is compact, $\sigma$ is bounded and $\Psi(t_1,\cdot) \in C_p^2(\mathbb{R}^n)$, then for all reference probability system $\nu$, $V_{\mathrm{PM}} = V_{\nu}$.*

The optimal control problem presented above, can be generalized with the use of a discount factor with a discount rate function $\beta : [t_0, t_1] \times \mathbb{R}^d \times U \to \mathbb{R}$. The payoff function is then :

$$J^\nu(t,x;u) = \mathbb{E}^\nu \left[ \int_t^{\tau^{t,x,u}} e^{-\int_t^s \beta(r,x^{t,x,u}(r),u(r))dr} L(s, x^{t,x,u}(s), u(s)) ds \right.$$
$$\left. + e^{-\int_t^{\tau^{t,x,u}} \beta(r,x^{t,x,u}(r),u(r))dr} \Psi(\tau^{t,x,u}, x^{t,x,u}(\tau^{t,x,u})) \right].$$

The Hamilton-Jacobi-Bellman equation is

$$-\frac{\partial}{\partial t} W(t,x) + H(t,x,W(t,x),D_x W(t,x), D_x^2 W(t,x)) = 0,\ (t,x) \in [t_0,t_1) \times O,$$

where the Hamiltonian is defined by :

$$H(t,x,r,p,A) = \sup_{v \in U} \left[ -f(t,x,v)\cdot p - \frac{1}{2} \operatorname{Tr}(\sigma\sigma^\intercal(t,x,v)A) + \beta(t,x,v)r - L(t,x,v) \right],$$

for $(t,x) \in [t_0,t_1) \times O$, $r \in \mathbb{R}$, $p \in \mathbb{R}^n$, $A \in \mathbb{S}(n)$.

### 1.1.4 Stochastic optimal control problem with infinite horizon

Stochastic optimal control problems with infinite horizon are stochastic control problems with a time variable in an unbounded interval. In this case, the unboundness of the time variable raises questions of finiteness of the overall payoff.

To fix the ideas, let us consider a stochastic control process with autonomous state dynamics defined by the following stochastic differential equation :

$$dx^{x,u}(s) = f(x^{x,u}(s), u(s))ds + \sigma(x^{x,u}(s), u(s))dB_s, \quad s \geq 0 \tag{1.33}$$
$$x^{x,u}(0) = x, \tag{1.34}$$

with $x \in \mathbb{R}^n$, $u : [0,\infty) \to U \subset \mathbb{R}^m$, $f : \mathbb{R}^n \times U \to \mathbb{R}^n$, $\sigma : \mathbb{R}^n \times U \to \mathbb{R}^{n\times d}$ and $(B_s)_{s\geq 0}$ a $\mathbb{R}^d$-Brownian motion defined on a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_s)_{s\geq 0}, P)$. We denote as in the previous section, by $\nu$ the reference probability system $(\Omega, (\mathscr{F}_s)_{s\geq 0}, P, B)$.

For this stochastic process to be well defined, the following assumptions can be taken :

**A 1.1.11.** *$f$ and $\sigma$ are continuous on $\mathbb{R}^n \times U$.*

**A 1.1.12.** *$f(\cdot, v)$ and $\sigma(\cdot, v)$ are in $C^1(\mathbb{R}^n)$ for any $v \in U$.*

**A 1.1.13.** $|f_x| \leq C$, $|\sigma_x| \leq C$

$$|f(x,v)| \leq C(1 + |x| + |v|), \quad |\sigma(x,v)| \leq C(1 + |x| + |v|)$$

*for some constant $C$.*

**A 1.1.14.** *The control processes $u(\cdot)$ are $\{\mathscr{F}_s\}$ progressively measurable and such that for all $t_1 < \infty$,*

$$\mathbb{E}^\nu \left[ \int_0^{t_1} |u(s)|^2 ds \right] < \infty \tag{1.35}$$

We consider as in the other sections, the exit time of the control process $x^{x,u}$ from $\bar{O}$ that we denote here by $\tau^{x,u}$. If $O = \mathbb{R}^n$, we set $\tau^{x,u} = +\infty$.

The running functional payoff $L$ and the terminal functional payoff $g$ are real functions which verify the following conditions.

**A 1.1.15.** *$L \in C(\mathbb{R}^n \times U)$ and $g \in C(\mathbb{R}^n)$.*

**A 1.1.16.** *There exists a constant $C$ and an integer $k$ such that*

$$|L(x,v)| \leq C(1 + |x|^k + |v|^k)$$

We consider also a discount factor with a constant discount rate $\beta$ in the overall payoff function which is then :

$$J^\nu(x;u) = \mathbb{E}^\nu \left\{ \int_0^{\tau^{x,u}} e^{-\beta s} L(x^{x,u}(s), u(s))ds + \chi_{\tau^{x,u}<\infty} e^{-\beta\tau^{x,u}} g(x^{x,u}(\tau^{x,u})) \right\} \tag{1.36}$$

The set of admissible control processes in the reference probability system $\nu$ is the set $\mathscr{A}_\nu$ of control processes verifying the condition A 1.1.14 and the following condition.

$$\mathbb{E}^\nu \left[ \int_0^{\tau^{x,u}} e^{-\beta s} |L(x^{x,u}(s), u(s))| ds \right] < \infty. \tag{1.37}$$

As in the previous section, we consider here a minimization problem with two cases :

- either a minimization of the overall functional payoff on the controls of $\mathscr{A}_\nu$ in a given reference probability system $\nu$ with a value function defined by

$$V_\nu(x) = \inf_{u \in \mathscr{A}_\nu} J^\nu(x;u),$$

- or a global minimization of the overall functional payoff over all the reference probability systems $\nu$ with a value function defined by

$$V_{\mathrm{PM}}(x) = \inf_\nu V_\nu(x).$$

The Hamilton-Jacobi-Bellman equation here is :

$$\beta W(x) + H(x, DW(x), D^2W(x)) = 0, \quad x \in O, \tag{1.38}$$

where the Hamiltonian $H$ is defined by :

$$H(x,p,A) = \sup_{u \in U} \left\{ -f(x,v) \cdot p - \frac{1}{2} \operatorname{Tr}(\sigma\sigma^\intercal(x,v)A) - L(x,v) \right\}, \tag{1.39}$$

with $x \in O$, $p \in \mathbb{R}^n$, $A \in \mathbb{S}(n)$.
The PDE that should verify the value function $V_{\mathrm{PM}} = W$ is then

$$\begin{cases} \beta W(x) + H(x, DW(x), D^2W(x)) = 0, & x \in O, \\ W(x) = g(x), & x \in \partial O. \end{cases} \tag{1.40}$$

The verification theorem allowing to solve the stochastic optimal control problem using this PDE is then stated as follows :

**Theorem 1.1.5** ( [28, Theorem IV.5.1]). *Let $W \in C^2(O) \cap C_p(\bar{O})$ be a solution of* (1.40) *Then for every $x \in O$ :*

- *$W(x) \leq J^\nu(x; u)$ for any $\nu$-admissible control process $u(\cdot)$ such that :*

$$\liminf_{t_1 \to \infty} e^{-\beta t_1} \mathbb{E}^\nu \left[\chi_{\tau^{x,u} \geq t_1} W(x^{x,u}(t_1))\right] \leq 0. \tag{1.41}$$

- *Suppose that there exist $\nu^* = (\Omega^*, \{\mathscr{F}_s^*\}, P^*, w^*)$ and $u^*(\cdot) \in \mathscr{A}_{\nu^*}$ such that :*

$$\begin{aligned} u^*(s) \in & \arg\min_v \Big[ f(x^{x,u^*}(s), v) \cdot D_x W(x^{x,u^*}(s)) \\ & + \frac{1}{2} \operatorname{Tr} \left( \sigma\sigma^\intercal(x^{x,u^*}(s), v) D_x^2 W(x^{x,u^*}(s)) \right) + L(x^{x,u^*}(s), v) \Big] \end{aligned} \tag{1.42}$$

*for Lebesgue$\times P^*$-almost all $(s, \omega)$ such that $0 \leq s \leq \tau^{x,u^*}(\omega)$ and*

$$\lim_{t_1 \to \infty} e^{-\beta t_1} \mathbb{E}^{\nu*} \left[ \chi_{\tau^{x,u^*} \geq t_1} W(x^{x,u^*}(t_1)) \right] = 0.$$

*Then $W(x) = J(x; u^*)$.*

In the previous theorem, the equality of $W$ to $V_{\text{PM}}$ is not obtained as a result of the theorem. What is rather obtained is the equality of $W$ to $\tilde{V}_{\text{PM}}$ which is the infimum of $J^\nu(x; u)$ among all reference probability system $\nu$ and admissible control $u(\cdot) \in \mathscr{A}_\nu$ such that (1.41) holds. This allows us to state the following corollary.

**Corollary 1.1.2.** *Let $W$ be a function verifying all the conditions of Theorem 1.1.5. If $W$ is bounded, then $W = V_{\text{PM}}$.*

### 1.1.5 Zero-Sum Stochastic differential games

Stochastic differential games is a subproblem of game theory where the players try to optimize their payoffs which depend on a state variable evolving in time according to a stochastic differential equation. In zero-sum stochastic differential games, there are only two players who have adverse goals. One player tries to maximize a given payoff while the other player tries to minimize the same payoff.

We consider here a finite horizon zero-sum stochastic differential game where the corresponding stochastic control process is a Markov diffusion process. Let $x^{t,x,a,b}$ be this stochastic control process with $a$ being the control of player 1 while $b$ denotes the control of player 2. $x^{t,x,a,b}$ is solution of the following stochastic differential equation :

$$\begin{cases} dx^{t,x,a,b}(s) = f(s, x^{t,x,a,b}(s), a(s), b(s))ds + \sigma(s, x^{t,x,a,b}(s), a(s), b(s))dB_s, \ t \leq s \leq t_1 \\ x^{t,x,a,b}(t) = x, \end{cases} \tag{1.43}$$

where $a(s) \in \mathscr{A}$, $b(s) \in \mathscr{B}$, $\mathscr{A}$ closed subset of $\mathbb{R}^m$ and $\mathscr{B}$ closed subset of $\mathbb{R}^p$, $f : [t_0, t_1] \times \mathbb{R}^n \times \mathscr{A} \times \mathscr{B} \to \mathbb{R}^n$, $\sigma : [t_0, t_1] \times \mathbb{R}^n \times \mathscr{A} \times \mathscr{B} \to \mathbb{R}^{n \times d}$ and $(B_s)_{t_0 \leq s \leq t_1}$ a $\mathbb{R}^d$-Brownian motion defined on a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_s)_{t_0 \leq s \leq t_1}, P)$. We let $\nu = (\Omega, (\mathscr{F}_s)_{t_0 \leq s \leq t_1}, P, B)$ denote the related reference probability system.

The conditions for $x^{t,x,a,b}$ to be well defined can be deduced from the conditions of existence of a Markov diffusion process having only one control. We will consider here an adaptation of conditions enumerated in Section 1.1.3.

**A 1.1.17.** *$f$ and $\sigma$ are continuous functions on $[t_0, t_1] \times \mathbb{R}^n \times \mathscr{A} \times \mathscr{B}$ and $f(\cdot, \cdot, a, b)$, $\sigma(\cdot, \cdot, a, b)$ are of once continuously differentiable on $[t_0, t_1] \times \mathbb{R}^n$ for any $(a, b) \in \mathscr{A} \times \mathscr{B}$.*

**A 1.1.18.** *There exists a constant $C > 0$ such that :*

$$\begin{aligned} &|f_t| + |f_x| \leq C, \\ &|\sigma_t| + |\sigma_x| \leq C, \\ &|f(t, x, a, b)| \leq C(1 + |x| + |a| + |b|), \\ &|\sigma(t, x, a, b)| \leq C(1 + |x| + |a| + |b|). \end{aligned} \tag{1.44}$$

**A 1.1.19.** *The control processes $a(\cdot)$ and $b(\cdot)$ are $\{\mathscr{F}_s\}$ progressively measurable and such that,*

$$\begin{aligned} &\mathbb{E}^\nu \left[ \int_t^{t_1} |a(s)|^2 ds \right] < \infty, \\ &\mathbb{E}^\nu \left[ \int_t^{t_1} |b(s)|^2 ds \right] < \infty. \end{aligned} \tag{1.45}$$

These assumptions are weaker than those usually considered in the literature. For the sake of simplicity and just in this section, we will add two more assumptions.

**A 1.1.20.** *$\mathscr{A}$ and $\mathscr{B}$ are compact sets.*

**A 1.1.21.** *$f$ and $\sigma$ are bounded.*

In what follows, we will ignore the dependence of the problem to the reference probability system $\nu$. Indeed, the above conditions of the problem being similar to those of Corollary 1.1.1, we can infer the independence of the problem to the reference probability system even if a formal proof remains to be done.

The state space $O$ is supposed to be equal to $\mathbb{R}^n$, so that the state process only stops at $t_1$.

The running functional payoff $L$ and the terminal functional payoff $\psi$ are real functions which verify the following conditions :

**A 1.1.22.** *$L \in C([t_0, t_1] \times \mathbb{R}^n \times \mathscr{A} \times \mathscr{B})$ and $\psi \in C(\mathbb{R}^n)$.*

**A 1.1.23.** *There exists a constant $C$ and an integer $k$ such that :*

$$\begin{aligned} &|L(t, x, a, b)| \leq C(1 + |x|^k + |a|^k + |b|^k), \\ &|\psi(x)| \leq C(1 + |x|^k). \end{aligned} \tag{1.46}$$

There are then a lower value function and a upper value function to the zero-sum stochastic differential game problem denoted respectively by $V^-$ and $V^+$, and defined by :

$$V^+ = \inf_{a \in \mathscr{A}} \sup_{b \in \mathscr{B}} \mathbb{E} \left[ \int_t^{t_1} L(s, x^{t,x,a,b}(s), a(s), b(s)) ds + \psi(x^{t,x,a,b}(t_1)) \right], \tag{1.47}$$

$$V^- = \sup_{b \in \mathscr{B}} \inf_{a \in \mathscr{A}} \mathbb{E} \left[ \int_t^{t_1} L(s, x^{t,x,a,b}(s), a(s), b(s)) ds + \psi(x^{t,x,a,b}(t_1)) \right]. \tag{1.48}$$

Let us now introduce the notion of **nonanticipative strategies**.

**Definition 1.1.2.** *If $\mathscr{A}(t,t_1)$ and $\mathscr{B}(t,t_1)$ denotes respectively the sets of bounded Lebesgue measurable functions $a : [t,t_1] \to \mathscr{A}$, $b : [t,t_1] \to \mathscr{B}$, a nonanticipative strategy is a map $\alpha : \mathscr{A}(t,t_1) \to \mathscr{B}(t,t_1)$ such that for any time $s$, $(t < s < t_1)$ and any control process $a_1, a_2 \in \mathscr{A}(t,t_1)$, $a_1 = a_2$ almost everywhere in $[t,s]$, implies that $\alpha(a_1) = \alpha(a_2)$ almost everywhere in $[t,s]$.*

It can be shown ([64]) that the following definitions of the lower and upper value functions are equivalent to the ones given in (1.47) and (1.48).

$$V^+ = \inf_{\alpha \in \mathscr{A}_d} \sup_{b \in \mathscr{B}(t,t_1)} \mathbb{E}\left[\int_t^{t_1} L(s, x^{t,x,\alpha(b),b}(s), (\alpha(b))(s), b(s))ds + \psi(x^{t,x,\alpha(b),b}(t_1))\right], \quad (1.49)$$

$$V^- = \sup_{\beta \in \mathscr{B}_d} \inf_{a \in \mathscr{A}(t,t_1)} \mathbb{E}\left[\int_t^{t_1} L(s, x^{t,x,a,\beta(a)}(s), a(s), (\beta(a))(s))ds + \psi(x^{t,x,a,\beta(a)}(t_1))\right], \quad (1.50)$$

where $\mathscr{A}_d$ and $\mathscr{B}_d$ denotes the set of nonanticipative strategies respectively from $\mathscr{B}(t,t_1)$ to $\mathscr{A}(t,t_1)$ and from $\mathscr{A}(t,t_1)$ to $\mathscr{B}(t,t_1)$.

By using these definitions of the upper and the lower value functions, it is possible to obtain a dynamic programming property of these value functions. This allows to obtain then that $V^-$ is at least a viscosity solution of the following PDE :

$$\begin{cases} -\dfrac{\partial}{\partial t}W(t,x) + H^-(t,x,D_xW(t,x),D_x^2W(t,x)) = 0,\ (t,x) \in Q = [t_0,t_1) \times \mathbb{R}^n \\ W(t_1,x) = \psi(x),\ x \in \mathbb{R}^n, \end{cases} \quad (1.51)$$

while $V^+$ is at least the viscosity solution of the following PDE :

$$\begin{cases} -\dfrac{\partial}{\partial t}W(t,x) + H^+(t,x,D_xW(t,x),D_x^2W(t,x)) = 0,\ (t,x) \in Q = [t_0,t_1) \times \mathbb{R}^n \\ W(t_1,x) = \psi(x),\ x \in \mathbb{R}^n, \end{cases} \quad (1.52)$$

where $H^-$ and $H^+$ are respectively the upper and lower Hamiltonians and are defined by :

$$H^-(t,x,p,A) = \inf_{b \in \mathscr{B}} \sup_{a \in \mathscr{A}} \left[-f(t,x,a,b)\cdot p - \frac{1}{2}\operatorname{Tr}(\sigma\sigma^{\intercal}(t,x,a,b)A) - L(t,x,a,b)\right], \quad (1.53)$$

$$H^+(t,x,p,A) = \sup_{a \in \mathscr{A}} \inf_{b \in \mathscr{B}} \left[-f(t,x,a,b)\cdot p - \frac{1}{2}\operatorname{Tr}(\sigma\sigma^{\intercal}(t,x,a,b)A) - L(t,x,a,b)\right] \quad (1.54)$$

for $(t,x) \in [t_0,t_1) \times \mathbb{R}^n$, $p \in \mathbb{R}^n$, $A \in \mathbb{S}(n)$.

The above PDE are called the Hamilton-Jacobi-Isaacs equations. Under the Isaacs condition, which is the equality of the two Hamiltonians, the lower and the upper value functions coincide. One partial differential equation is then obtained. Usually, it is not possible to find a classical solution to this partial differential equation. The value function is then identified to the viscosity solution of the partial differential equation. This type of solution will be the subject of the next section.

## 1.2 Viscosity solutions

We will start this section by a discussion about the verification theorems of the previous section. Each of these theorems supposed the existence of a smooth solution to the Hamilton-Jacobi-Bellman equation and identified the value function of the optimal control problem to the classical solution of this PDE which is at least of the first order. This suppose that the value function of the problem is at least differentiable. However, in most cases, we do not have this differentiability of the value function.

For the deterministic optimal control problems, we can give the following example.

**Example 1.2.1.** *Let us consider a finite horizon deterministic optimal control problem with* $t_0 = 0$, $t_1 = 1$, $O = (-1, 1)$, $U = \mathbb{R}$, $f(t, x, u) = u$, $L(t, x, u) = 1 + \frac{1}{4}u^2$ *and* $\Psi(t, x) = 0$. *This is a problem of calculus of variations. It can be shown that the value function of this problem is :*

$$V(t,x) = \begin{cases} 1 - |x|, & |x| \geq t \\ 1 - t, & |x| < t, \end{cases}$$

*which is not differentiable if* $t = |x|$.

For stochastic control problems, to have a chance to apply the verification theorems stated in Sections 1.1.3 and 1.1.4, an uniform parabolicity is required on the diffusion term which is stated as follows.

**A 1.2.1.** *Let* $a(t, x, u) = \sigma(t, x, u)\sigma^{\intercal}(t, x, u)$. *There exists* $c > 0$ *such that, for all* $(t, x, u) \in [t_0, t_1) \times \mathbb{R}^n \times U$ *(with* $t_1$ *potentially infinite), and* $\xi \in \mathbb{R}^n$,

$$\sum_{i,j=1}^{n} [a(t, x, u)]_{ij} [\xi]_i [\xi]_j \geq c|\xi|^2. \tag{1.55}$$

This condition allows the solution of the Hamilton-Jacobi-Bellman equation to be smooth enough. If this condition is not ensured, there is typically not a classical solution to the PDE. For zero-sum stochastic differential games, it is worse because even the uniform parabolicity is not sufficient to have a smooth solution to the Hamilton-Jacobi-Isaacs equation. However, a generalized solution can be considered. This is also the case for the deterministic optimal control problems when differentiability cannot be obtained everywhere. However, such a generalized solution is not unique in general. This has motivated the introduction of the notion of viscosity solution introduced first in 1984 in a paper of Crandall and Lions [21]. The aim was to characterize the value function. It was first developed in the deterministic case. However an extension to the stochastic case has been introduced in other papers ( [49]). In the following years, many related papers have been published. This ensured the development of the theory making viscosity solutions a standard nowadays in optimal control problems.

We will give here the definition of this notion of viscosity solution and of the related notions of second order subjets and superjets as presented in [20]. We will also recall a theorem of [20] allowing to identify an element of the superjet of a function in particular cases. This theorem is particularly important to prove a comparison principle for a given PDE. It can for example be used in the context of Section 3.2 to obtain Lipschitz continuity of the viscosity solution of the Hamilton-Jacobi-Bellman equation. Existence and unicity results of vicosity solutions for second order Hamilton-Jacobi-Bellman equations will also be given in the case where all the coefficients are bounded and in the linear quadratic case with unbounded coefficients. In this last case, we extend the results of Da Lio and Ley of the paper [23] by obtaining existence results of the viscosity solution on any time interval $[0, T]$ ($T > 0$), rather than only on a small time interval $[0, \tau]$, by using some extra conditions.

Similar verification theorems as the one stated in section 1.1.1-1.1.4 exist in the literature, making the link between the viscosity solution of the Hamilton-Jacobi-Bellman equation and the value function of the related optimal control problem (see [67] and [39] as examples). We will not recall these results here as it will drive us away from the main subject of this work.

### 1.2.1 Definition of Viscosity solutions and of related notions

We want to give here first a definition of viscosity solutions using the notion of subjet and superjet found in [20]. For that, we need to define first what is the subjet and superjet of a function at a given point.

**Definition 1.2.1.** *Let $W$ be a given function defined on $\mathscr{O} \subset \mathbb{R}^N$ and $\hat{x} \in \mathscr{O}$. The second order superjet of $W$ at $\hat{x}$ is the subset denoted by $J^{2,+}_{\mathscr{O}} W(\hat{x})$ of $\mathbb{R}^N \times \mathbb{S}(N)$ such that for any $(p, X) \in J^{2,+}_{\mathscr{O}} W(\hat{x})$ :*

$$W(x) \leq W(\hat{x}) + p \cdot (x - \hat{x}) + \frac{1}{2}(x - \hat{x}) \cdot X(x - \hat{x}) + o(|x - \hat{x}|^2) \quad \text{as } x \to \hat{x},\ x \in \mathscr{O}.$$

*The second order subjet of $W$ at $\hat{x}$ is the subset denoted by $J^{2,-}_{\mathscr{O}} W(\hat{x})$ of $\mathbb{R}^N \times \mathbb{S}(N)$ such that for any $(p, X) \in J^{2,-}_{\mathscr{O}} W(\hat{x})$ :*

$$W(x) \geq W(\hat{x}) + p \cdot (x - \hat{x}) + \frac{1}{2}(x - \hat{x}) \cdot X(x - \hat{x}) + o(|x - \hat{x}|^2) \quad \text{as } x \to \hat{x},\ x \in \mathscr{O}.$$

**Remark 1.2.1.** $-J^{2,+}_{\mathscr{O}}(-W)(\hat{x}) = J^{2,-}_{\mathscr{O}} W(\hat{x})$. *Moreover, the dependence of $J^{2,+}_{\mathscr{O}} W(\hat{x})$ and $J^{2,-}_{\mathscr{O}} W(\hat{x})$ to $\mathscr{O}$ can be removed when $\hat{x}$ is an interior point of $\mathscr{O}$.*

Now we can give the definition of viscosity solutions in the continuous setting.

**Definition 1.2.2** ([20, Definition 2.2])**.** *Let $F$ be a continuous real function on $\mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{S}(N)$ such that for any $r, s \in \mathbb{R}$, $x, p \in \mathbb{R}^N$, $X, Y \in \mathbb{S}(N)$ :*

$$F(x, r, p, X) \leq F(x, s, p, Y)$$

*whenever $r \leq s$ and $Y \leq X$ in the Loewner order on $\mathbb{S}(N)$. Let $\mathscr{O} \subset \mathbb{R}^N$. A viscosity subsolution of $F = 0$ (equivalently a viscosity solution of $F \leq 0$) on $\mathscr{O}$ is a function $W$ upper semicontinuous on $\mathscr{O}$ such that :*

$$F(x, W(x), p, X) \leq 0 \text{ for all } x \in \mathscr{O} \text{ and } (p, X) \in J^{2,+}_{\mathscr{O}} W(x)$$

*Similarly, a viscosity supersolution of $F = 0$ (or viscosity solution of $F \geq 0$) on $\mathscr{O}$ is a function $W$ lower semicontinuous on $\mathscr{O}$ such that :*

$$F(x, W(x), p, X) \geq 0 \text{ for all } x \in \mathscr{O} \text{ and } (p, X) \in J^{2,-}_{\mathscr{O}} W(x)$$

*Finally, $W$ is a viscosity solution of $F = 0$ in $\mathscr{O}$ if it is both a viscosity subsolution and a viscosity supersolution of $F = 0$ in $\mathscr{O}$.*

Another equivalent definition of viscosity solution is the following.

**Definition 1.2.3.** *Let $F$ be a function as in Definition 1.2.2. $W$ is a viscosity solution of*

$$F(x, W(x), DW(x), D^2W(x)) = 0 \tag{1.56}$$

*in $\mathscr{O}$ if and only if $W$ verifies the two following conditions :*

- *$W$ is upper semicontinuous and $\forall \phi \in C^2(\mathscr{O})$, if $\hat{x} \in \mathscr{O}$ is a point where $W - \phi$ reaches a local maximum, then :*

$$F(\hat{x}, W(\hat{x}), D\phi(\hat{x}), D^2\phi(\hat{x})) \leq 0. \tag{1.57}$$

- *$W$ is lower semicontinuous and $\forall \phi \in C^2(\mathscr{O})$, if $\hat{x} \in \mathscr{O}$ is a point where $W - \phi$ reaches a local minimum, then :*

$$F(\hat{x}, W(\hat{x}), D\phi(\hat{x}), D^2\phi(\hat{x})) \geq 0. \tag{1.58}$$

*If $W$ satisfies just the first condition, $W$ is a subsolution of* (1.56)*. If it satisfies only the second condition, it is a supersolution of* (1.56)*.*

In the above definitions, the set $\mathscr{O}$ considered can have a time interval part. So it can be of the form $(t_0, t_1) \times O$, $(t_0, t_1] \times O$ or $[t_0, t_1) \times O$ where $O \subset \mathbb{R}^n = \mathbb{R}^{N-1}$. These definitions can then be applied to parabolic or elliptic PDEs.

In the case of a parabolic PDE, the semijets $J^{2,+}_{\mathscr{O}} W(x)$ and $J^{2,-}_{\mathscr{O}} W(x)$ are denoted respectively by $\mathscr{P}^{2,+}_O W(t, x)$ and $\mathscr{P}^{2,-}_O W(t, x)$ for $\mathscr{O} = (t_0, t_1) \times O$. The superjet of $W$ at $(s, z) \in \mathscr{O}$ becomes the set of $(a, p, X) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}(n)$ such that :

$$\begin{aligned} W(t, x) \leq & W(s, z) + a(t - s) + p \cdot (x - z) + \frac{1}{2}(x - z) \cdot X(x - z) \\ & + o(|t - s| + |x - z|^2) \text{ as } (t, x) \to (s, z),\ (t, x) \in \mathscr{O}. \end{aligned}$$

The subjet of $W$ at $(s, z) \in \mathscr{O}$ is then the set of $(a, p, X) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}(n)$ such that :

$$\begin{aligned} W(t, x) \geq & W(s, z) + a(t - s) + p \cdot (x - z) + \frac{1}{2}(x - z) \cdot X(x - z) \\ & + o(|t - s| + |x - z|^2) \text{ as } (t, x) \to (s, z),\ (t, x) \in \mathscr{O}. \end{aligned}$$

The present work mainly concerns finite horizon stochastic control problems where a parabolic PDE arises as the Hamilton-Jacobi-Bellman equation. We recall now a theorem of [20] in this setting that can be used to obtain Lipschitz results of the viscosity solution of the Hamilton-Jacobi-Bellman equation in the unbounded framework (Similar to Section 3.2). We recall that USC($\mathscr{O}$) is the set of upper semicontinuous functions on $\mathscr{O}$ and LSC($\mathscr{O}$) is the set of lower semicontinuous functions on $\mathscr{O}$.

**Theorem 1.2.1** ([20, Theorem 8.3])**.** *Let $W_i \in$ USC$((0, T) \times O_i)$ for $i = 1, ..., k$ where $O_i$ is a locally compact subset of $\mathbb{R}^{n_i}$. Let $\phi$ be a real-valued function defined on an open neighbourhood of $(0, T) \times O_1 \times ... \times O_k$ and such that $(t, x_1, \ldots, x_k) \mapsto \phi(t, x_1, \ldots, x_k)$ is once continuously* *differentiable in* $t$ *and twice continuously differentiable in* $(x_1, \ldots, x_k) \in O_1 \times \ldots \times O_k$. *Suppose that* $\hat{t} \in (0,T)$, $\hat{x}_i \in O_i$ *for* $i = 1, \ldots, k$ *and*

$$\begin{aligned} w(t, x_1, \ldots, x_k) \equiv & W_1(t,x_1) + \ldots + W_k(t,x_k) - \phi(t, x_1, \ldots, x_k) \\ & \leq w(\hat{t}, \hat{x}_1, \ldots, \hat{x}_k) \end{aligned}$$

*for* $0 < t < T$ *and* $x_i \in O_i$. *Assume, moreover, that there is an* $r > 0$ *such that for every* $M > 0$, *there is a* $C > 0$ *such that for* $i = 1, \ldots, k$

$$\begin{aligned} & b_i \leq C \text{ whenever } (b_i, q_i, X_i) \in \mathscr{P}^{2,+}_{O_i} W_i(t, x_i), \\ & |x_i - \hat{x}_i| + |t - \hat{t}| \leq r \text{ and } |W_i(t,x_i)| + |q_i| + |X_i| \leq M. \end{aligned} \tag{1.59}$$

*Then for each* $\epsilon > 0$ *there are* $X_i \in \mathbb{S}(n_i)$ *such that*

- $(b_i, D_{x_i}\phi(\hat{t}, \hat{x}_1, \ldots, \hat{x}_k), X_i) \in \mathscr{P}^{2,+}_{O_i} W_i(\hat{t}, \hat{x}_i)$ *for* $i = 1, \ldots, k$,
- $$-(\frac{1}{\epsilon} + |A|)I \leq \begin{pmatrix} X_1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & X_k \end{pmatrix} \leq A + \epsilon A^2,$$

  *with* $A = (D^2_x \phi)(\hat{t}, \hat{x}_1, \ldots, \hat{x}_k)$,
- $b_1 + \ldots + b_k = \frac{\partial \phi}{\partial t}(\hat{t}, \hat{x}_1, \ldots, \hat{x}_k)$,

Crandall, Ishii and Lions notice in [20] that the condition (1.59) is guaranteed when $W_i$ is a subsolution of a parabolic equation.

We present this theorem because it can be used to obtain Lipschitz results on the viscosity solution of the Hamilton-Jacobi-Bellman equation with unbounded coefficients similar to the equation of Section 3.2. This Lipschitz result can then be used in the same way as Assellaou, Bokanowski and Zidani in [5] to obtain symmetric error estimates for the convergence of numerical schemes having a solution with a Lipschitz property. A particular probabilistic scheme that we introduce in Chapter 3, fits well in this setting.

### 1.2.2 Existence and Unicity results

We are going to recall now existence and unicity results of viscosity solutions of Hamilton-Jacobi-Bellman and Hamilton-Jacobi-Bellman-Isaacs equations that will be mentioned or used in Chapter 2 and Chapter 3.

The notations in the theorems, lemma and corollary reported here from other papers, have been changed to keep as much as possible a consistency in notations throughout this document.

We start with a result of Barles and Jakobsen on the viscosity solutions of a general switching system which is a generalization of the Hamilton-Jacobi-Bellman equations and the Hamilton-Jacobi-Bellman-Isaacs equations.

**Theorem 1.2.2** ([7, Theorem A.1]). *Let us consider the following system.*

$$F_i(t, x, W, \frac{\partial W_i}{\partial t}, D_x W_i, D^2_x W_i) = 0 \text{ in } Q_T := (0,T] \times \mathbb{R}^n, \quad i \in \mathscr{I} := \{1, \ldots, M\}, \tag{1.60}$$

*with*

$$F_i(t,x,r,p_t,p_x,X) = \max\left\{p_t + \sup_{\alpha\in\mathscr{A}}\inf_{\beta\in\mathscr{B}} \mathscr{L}_i^{\alpha,\beta}(t,x,r_i,p_x,X); r_i - \mathscr{M}_i r\right\},$$

$$\mathscr{L}_i^{\alpha,\beta}(t,x,s,q,X) = -\operatorname{Tr}[a_i^{\alpha,\beta}(t,x)X] - b_i^{\alpha,\beta}(t,x)\cdot q - c_i^{\alpha,\beta}(t,x)s - f_i^{\alpha,\beta}(t,x),$$

$$\mathscr{M}_i r = \min_{j\neq i}\{r_j + k\}$$

*where* $\mathscr{A},\mathscr{B}$ *are compact metric spaces,* $r$ *is a vector* $r=(r_1,\dots,r_M)$*, and* $k>0$ *is a constant (the switching cost).*

*Assume that for any* $\alpha,\beta,i$, $a_i^{\alpha,\beta} = \frac{1}{2}\sigma_i^{\alpha,\beta}\sigma_i^{\alpha,\beta\mathsf{T}}$ *for some* $n\times p$ *matrix* $\sigma_i^{\alpha,\beta}$*. Furthermore, there is a constant* $C$ *independent of* $i,\alpha,\beta,t$ *such that*

$$|\sigma_i^{\alpha,\beta}(t,\cdot)|_1 + |b_i^{\alpha,\beta}(t,\cdot)|_1 + |c_i^{\alpha,\beta}(t,\cdot)|_1 + |f_i^{\alpha,\beta}(t,\cdot)|_1 \le C \tag{1.61}$$

*with* $|\cdot|_1$ *defined in Section 0.1.*

- *If* $W\in \mathrm{USC}(\bar{Q}_T;\mathbb{R}^M)$ *is a subsolution of* (1.60) *bounded above and* $V\in\mathrm{LSC}(\bar{Q}_T;\mathbb{R}^M)$ *is a supersolution of* (1.60) *bounded below, then* $W\le V$ *in* $\bar{Q}_T$*.*
- *There exists a unique bounded continuous viscosity solution* $W$ *of* (1.60)*.*

The above theorem suppose the boundedness of the coefficients of the PDE and the compactness of the control spaces. The new max-plus probabilistic method that this manuscript introduces, can be used in the case of unboundedness of the PDE coefficients and of the control spaces as in linear quadratic problems. The following results due to Da Lio and Ley in [23], give existence and unicity results in this setting.

**Theorem 1.2.3** ([23, Theorem 2.1])**.** *Let us consider the following second order PDE.*

$$\begin{cases} \dfrac{\partial W}{\partial t} + H(t,x,D_xW,D_x^2W) + G(t,x,D_xW,D_x^2W) = 0 \text{ in } (0,T)\times\mathbb{R}^n, \\ W(0,x) = \psi(x) \text{ in } \mathbb{R}^n \end{cases} \tag{1.62}$$

*with* $H$ *and* $G$ *are defined by*

$$H(t,x,p,X) = \inf_{\alpha\in\mathscr{A}}\left\{b(t,x,\alpha)\cdot p + \ell(t,x,\alpha) - \operatorname{Tr}\left[\sigma(t,x,\alpha)\sigma^{\mathsf{T}}(t,x,\alpha)X\right]\right\},$$

$$G(t,x,p,X) = \sup_{\beta\in\mathscr{B}}\left\{-g(t,x,\beta)\cdot p - f(t,x,\beta) - \operatorname{Tr}\left[c(t,x,\beta)c^{\mathsf{T}}(t,x,\beta)X\right]\right\},$$

*for* $(t,x)\in(0,T)\times\mathbb{R}^n$, $p\in\mathbb{R}^n$ *and* $X\in\mathbb{S}(n)$*.*

*Assume that there exist positive constants* $\bar{C}$ *and* $\nu$ *such that :*

- $\mathscr{A}$ *is a subset of a separable complete normed space possibly unbounded.*
- $b\in C([0,T]\times\mathbb{R}^n\times\mathscr{A},\mathbb{R}^n)$ *satisfying for* $x,y\in\mathbb{R}^n$, $t\in[0,T]$, $\alpha\in\mathscr{A}$,

$$|b(t,x,\alpha) - b(t,y,\alpha)| \le \bar{C}(1+|\alpha|)|x-y|$$
$$|b(t,x,\alpha)| \le \bar{C}(1+|x|+|\alpha|);$$

- $\ell \in C([0,T] \times \mathbb{R}^n \times \mathscr{A}, \mathbb{R})$ *satisfying for* $x \in \mathbb{R}^n$, $t \in [0,T]$, $\alpha \in \mathscr{A}$,

$$\bar{C}(1+|x|^2+|\alpha|^2) \geq \ell(t,x,\alpha) \geq \frac{\nu}{2}|\alpha|^2 + \ell_0(t,x,\alpha) \text{ with } \ell_0(t,x,\alpha) \geq -\bar{C}(1+|x|^2)$$

*and for* $R > 0$, *there exists a modulus of continuity* $m_R$ *such that for any* $x, y \in B(0,R)$ *(*$B(0,R)$ *is the open ball of* $\mathbb{R}^n$ *of center* $0$ *and radius* $R$*),* $t \in [0,T], \alpha \in \mathscr{A}$,

$$|\ell(t,x,\alpha) - \ell(t,y,\alpha)| \leq (1+|\alpha|^2) m_R(|x-y|)$$

- $\sigma \in C([0,T] \times \mathbb{R}^n \times \mathscr{A}, \mathbb{R}^{n\times d})$ *is locally Lipschitz with respect to* $x$ *uniformly in* $(t,\alpha) \in [0,T] \times \mathscr{A}$ *and satisfies for every* $t \in [0,T]$, $x \in \mathbb{R}^n$, $\alpha \in \mathscr{A}$,

$$|\sigma(t,x,\alpha)| \leq \bar{C}(1+|x|),$$

- $\mathscr{B}$ *is a bounded subset of a normed space.*

- $g \in C([0,T] \times \mathbb{R}^n \times \mathscr{B}, \mathbb{R}^n)$ *is locally Lipschitz with respect to* $x$ *uniformly in* $(t,\beta) \in [0,T] \times \mathscr{B}$ *and satisfies for every* $t \in [0,T]$, $x \in \mathbb{R}^n$, $\beta \in \mathscr{B}$,

$$|g(t,x,\beta)| \leq \bar{C}(1+|x|),$$

- $f \in C([0,T] \times \mathbb{R}^n \times \mathscr{B}, \mathbb{R})$ *is locally Lipschitz with respect to* $x$ *uniformly in* $(t,\beta) \in [0,T] \times \mathscr{B}$ *and satisfies for every* $t \in [0,T]$, $x \in \mathbb{R}^n$, $\beta \in \mathscr{B}$,

$$|f(t,x,\beta)| \leq \bar{C}(1+|x|^2),$$

- $c \in C([0,T] \times \mathbb{R}^n \times \mathscr{B}, \mathbb{R}^{n\times d})$ *is locally Lipschitz with respect to* $x$ *uniformly in* $(t,\beta) \in [0,T] \times \mathscr{B}$ *and satisfies for every* $t \in [0,T]$, $x \in \mathbb{R}^n$, $\beta \in \mathscr{B}$,

$$|c(t,x,\beta)| \leq \bar{C}(1+|x|),$$

- $\psi \in C(\mathbb{R}^n, \mathbb{R})$ *and*

$$|\psi(x)| \leq \bar{C}(1+|x|^2)$$

*for every* $x \in \mathbb{R}^n$.

*Let* $W \in \mathrm{USC}([0,T] \times \mathbb{R}^n)$ *be a viscosity subsolution of* (1.62) *and* $V \in \mathrm{LSC}([0,T] \times \mathbb{R}^n)$ *be a viscosity supersolution of* (1.62)*. Suppose that* $W$ *and* $V$ *have quadratic growth, i.e there exists* $\hat{C} > 0$ *such that for all* $x \in \mathbb{R}^n$, $t \in [0,T]$,

$$|W(t,x)|, |V(t,x)| \leq \hat{C}(1+|x|^2). \tag{1.63}$$

*Then* $W \leq V$ *in* $[0,T] \times \mathbb{R}^n$.

The above theorem gives the unicity result by stating the comparison principle for the PDE (1.62) and is completed by the following Lemma for the existence result.

**Lemma 1.2.1** ([23, Lemma 2.1])**.** *Consider the same assumptions as in Theorem 1.2.3. If* $K \geq \bar{C}+1$ *and* $\rho$ *are large enough, then* $\underline{W}(t,x) = -Ke^{\rho t}(1+|x|^2)$ *is a viscosity subsolution of* (1.62) *in* $[0,T] \times \mathbb{R}^n$ *and there exists* $0 < \tau \leq T$ *such that* $\bar{W}(t,x) = Ke^{\rho t}(1+|x|^2)$ *is a viscosity supersolution of* (1.62) *in* $[0,\tau] \times \mathbb{R}^n$.

From the two previous statements, the corollary below is deduced.

**Corollary 1.2.1** ([23, Corollary 2.1])**.** *Consider the same assumptions as in Theorem 1.2.3. Then there exists $\tau > 0$ such that there exists a unique continuous viscosity solution of* (1.62) *in $[0,\tau]\times\mathbb{R}^n$ satisfying the growth condition* (1.63)*.*

We notice that the existence of the viscosity solution holds only on a time interval $[0,\tau]$ not necessarily equal to $[0,T]$. Previous work of Ito in [41] has shown the existence of a solution of a similar PDE on any interval $[0,t_1]$ ($t_1 > 0$). But this result needed a uniform parabolicity condition on the PDE and more regularity assumptions were taken on the solution.

We introduce the two following results ensuring the existence of a supersolution of PDE (1.62) on $[0,T]$ by taking some extra assumptions.

**Lemma 1.2.2.** *Consider the assumptions of Theorem 1.2.3. Let $T > 0$. For $K > \bar{C}$, there exists $\rho > 0$ and $C_1 > 0$ such that if $\ell_0(t,x,\alpha) \geq C_1|x|^2 - \bar{C}$, $\bar{W}(t,x) = Ke^{\rho t}(1+|x|^2)$ is a supersolution of PDE* (1.62) *on $[0,T]\times\mathbb{R}^n$. ($\ell_0$ is the function introduced in the assumption of Theorem 1.2.3 on the Lagrangian $\ell$ of PDE* (1.62)*.)*

*Proof.* Let $K > \bar{C}$, $\rho > 0$ and $\bar{W}(t,x) = Ke^{\rho t}(1+|x|^2)$. Let $C_1$ be such that $\ell_0(t,x,\alpha) \geq C_1|x|^2 - \bar{C}$.

$$
\begin{aligned}
&\frac{\partial \bar{W}}{\partial t} + H(t,x,D_x\bar{W},D_x^2\bar{W}) + G(t,x,D_x\bar{W},D_x^2\bar{W}) \\
&\quad = \rho e^{\rho t}K(1+|x|^2) + \inf_{\alpha\in\mathscr{A}}\left\{b(t,x,\alpha)\cdot 2Ke^{\rho t}x + \ell(t,x,\alpha) - \mathrm{Tr}\left(\sigma(t,x,\alpha)\sigma^{\intercal}(t,x,\alpha)2Ke^{\rho t}\right)\right\} \\
&\quad + \sup_{\beta\in\mathscr{B}}\left\{-g(t,x,\beta)\cdot 2Ke^{\rho t}x - f(t,x,\beta) - \mathrm{Tr}\left(c(t,x,\beta)c^{\intercal}(t,x,\beta)2Ke^{\rho t}\right)\right\} \\
&\quad \geq Ke^{\rho t}\Bigg[\rho(1+|x|^2) - 2\bar{C}(1+|x|)|x| + \frac{C_1|x|^2}{Ke^{\rho t}} - 2\bar{C}^2(1+|x|)^2 + \inf_{\alpha\in\mathscr{A}}\left(\frac{\nu|\alpha|^2}{2Ke^{\rho t}} - 2\bar{C}|\alpha||x|\right) \\
&\quad - 2\bar{C}(1+|x|)|x| - \frac{\bar{C}}{Ke^{\rho t}}(1+|x|^2) - 2\bar{C}^2(1+|x|)^2 - \frac{\bar{C}}{K}\Bigg] \\
&\quad \geq Ke^{\rho t}\left[\rho(1+|x|^2) - (6+\frac{2}{K}+8\bar{C})\bar{C}(1+|x|^2) + \left(\frac{C_1}{Ke^{\rho t}} - \frac{2Ke^{\rho t}\bar{C}^2}{\nu}\right)|x|^2\right].
\end{aligned}
$$

For $\rho \geq (6+\frac{2}{K}+8\bar{C})\bar{C}$ and $C_1 \geq \frac{2(Ke^{\rho T}\bar{C})^2}{\nu}$, we then obtain that $\bar{W}$ is a viscosity supersolution of (1.62) on $[0,T]\times\mathbb{R}^n$. $\square$

**Lemma 1.2.3.** *Consider the assumptions of Theorem 1.2.3. Let us suppose that for $(t,x)\in[0,T]\times\mathbb{R}^n$, $\alpha,\alpha_1,\alpha_2\in\mathscr{A}$, $\beta\in\mathscr{B}$ :*

$$
\begin{aligned}
|b(t,x,\alpha_1) - b(t,x,\alpha_2)| &\leq \bar{C}|\alpha_1-\alpha_2|, \\
\ell_0(t,x,\alpha) &\geq -\bar{C}|x|^2, \\
|\sigma(t,x,\alpha)| &\leq \bar{C}|x|, \\
|g(t,x,\beta)| &\leq \bar{C}|x|, \\
|f(t,x,\beta)| &\leq \bar{C}|x|^2, \\
|c(t,x,\beta)| &\leq \bar{C}|x|.
\end{aligned}
$$

*We also suppose that there exists a control* $\alpha_0 \in \mathscr{A}$ *that we will suppose equal to* 0 *such that for a given constant* $C_2$,

$$b(t,x,\alpha_0)\cdot x \geq C_2|x|^2.$$

*Then, for* $K > \bar{C}$ *and* $C_2 > (1+2\bar{C}+\frac{1}{K})\bar{C} + \frac{2K\bar{C}^2}{\nu}$ , $\bar{W}(t,x) = K(1+|x|^2)$ *is a viscosity supersolution of the PDE* (1.62) *on* $[0,T]\times\mathbb{R}^n$.

*Proof.* Let $K > \bar{C}$ and $\bar{W}(t,x) = K(1+|x|^2)$. Consider the assumptions of Lemma 1.2.3.

$$\begin{aligned}
&\frac{\partial \bar{W}}{\partial t} + H(t,x,D_x\bar{W},D_x^2\bar{W}) + G(t,x,D_x\bar{W},D_x^2\bar{W}) \\
&\quad \geq \inf_{\alpha\in\mathscr{A}}\{b(t,x,0)\cdot 2Kx + (b(t,x,\alpha)-b(t,x,0))2Kx + \ell(t,x,\alpha) - \mathrm{Tr}(\sigma(t,x,\alpha)\sigma^{\mathsf{T}}(t,x,\alpha)2K)\} \\
&\quad + \sup_{\beta\in\mathscr{B}}\{-g(t,x,\beta)\cdot 2Kx - f(t,x,\beta) - \mathrm{Tr}\,(c(t,x,\beta)c^{\mathsf{T}}(t,x,\beta)2K)\} \\
&\quad \geq 2K\left[C_2|x|^2 - (\bar{C}+\frac{1}{2K})\bar{C}|x|^2 + \inf_{\alpha\in\mathscr{A}}\left\{\frac{\nu}{4K}|\alpha|^2 - \bar{C}|\alpha||x|\right\} - (1+\bar{C}+\frac{1}{2K})\bar{C}|x|^2\right] \\
&\quad \geq 2K\left[C_2|x|^2 - (1+2\bar{C}+\frac{1}{K})\bar{C}|x|^2 - \frac{2K}{\nu}\bar{C}^2|x|^2\right].
\end{aligned}$$

So for $C_2 \geq (1+2\bar{C}+\frac{1}{K})\bar{C}+\frac{2K\bar{C}^2}{\nu}$, $\bar{W}$ is a viscosity supersolution of (1.62) on $[0,T]\times\mathbb{R}^n$. □

CHAPTER 2

# Numerical methods in Optimal control

In the previous chapter, we presented optimal control problems which are solved by using Hamilton-Jacobi-Bellman (HJB) equations. We have seen that these PDE generally do not have classical solutions and the type of solutions that are looked for in the optimal control theory are viscosity solutions. We do now a brief review of the type of numerical methods in the literature which are used to solve HJB equations and to obtain approximations of their viscosity solutions. We introduce in Section 2.1 an extension of the well known convergence result of monotone numerical schemes due to Barles and Souganidis to be able to apply it to an unbounded setting.

## 2.1 Finite difference methods

The finite difference method is historically the oldest type of numerical method used nowadays to solve partial differential equations. The starting point of the development of this method can be identified with the paper of Courant, Friedrichs and Lewy [18] in 1928 even if some ideas about the method may be found in the literature before this date. The finite difference method knew after that, a bursting period before the introduction in 1960s of the finite element method which became more popular in the subsequent years. However, it remained an active field of research with new finite difference schemes ( [11, 57] for HJB equations and [50, 31, 32, 58, 56] for other types of PDEs) and new results on the generalization of finite difference method results to other types of numerical methods. We can give as examples the theorem on the convergence of approximation schemes for fully non linear second order equations stated by Barles and Souganidis [8] and the order of convergence of approximation schemes for parabolic Hamilton-Jacobi-Bellman equations obtained by Barles and Jakobsen [7] which uses in its proof the method of shaking the coefficients of Krylov [46, 47].

We will give here the definition of the finite difference method and of the related concepts used with it and recall most popular finite difference schemes in optimal control. We will also recall the previously cited generalization results of finite difference methods. We will then introduce as a theorem, an extension of the Barles and Souganidis theorem which will allow us to apply their result in an unbounded setting in Chapter 3. Notations in the subsequent reported theorems have been changed to maintain a consistency in notations throughout this document.

### 2.1.1 Definition and related results in the Literature

When solving a partial differential equation (PDE), a finite difference method consists in :

- discretizing the variables space of the equation by building a grid;
- discretizing the equation which consists in approximating the equation using finite differences instead of partial derivatives such that the resulting equation converges to the initial equation when the grid step goes to zero;
- solving the resulted discretized equation at the grid points by iterative methods. The resulted solution is an approximation of the solution of the initial PDE and its value is computed by interpolation at points which are not grid points.

The discretization of the equation results in the construction of a scheme. It appears in many numerical methods for solving PDE. The particularity of the finite difference method is then the exclusive use of finite differences in the approximation of partial derivatives when building the scheme. The latter has a very important role in the quality of the approximation of the solution of the equation. If the approximated solution obtained with a given scheme converges to the solution of the PDE problem when the discretization step of the grid goes to zero, the scheme is said to be convergent.

It was known for a while that the conditions for a finite difference scheme to be convergent were :

- the consistency of the scheme meaning that the discretized equation must converge to the initial PDE when the grid step goes to zero and the function is regular,

- the stability of the scheme meaning that the solution of the scheme is bounded with respect to the bounds of the equation parameters.

The stability condition was the most challenging one. Depending on the type of scheme, there was a conditional stability (explicit scheme) or an unconditional stability (implicit scheme). The conditional stability for explicit scheme was subject to the CFL condition after Courant, Friedrichs and Lewy who found in their paper [18], that a finite difference method involving a space discretization with step $\Delta x$ and a time discretization with step $\Delta t$ results in a stable explicit finite difference scheme if $\Delta x \geq c\Delta t$ for hyperbolic equations, $c$ being a constant to compute. This has an equivalent for parabolic equations with the inequality $\Delta x^2 \geq c\Delta t$ which is also called the CFL condition.

The conditions of convergence of finite difference schemes have been generalized to PDE problems with viscosity solutions with the following theorem which can be applied to general schemes as we are going to see in Sections 2.2 and 2.3.

**Theorem 2.1.1** ([8, Theorem 2.1]). *Consider a problem of the following form :*

$$F(D^2W, DW, W, x) = 0 \text{ in } \bar{\Omega}, \tag{2.1}$$

*where $\Omega$ is an open subset of $\mathbb{R}^N$ and $F : \mathbb{S}(N) \times \mathbb{R}^N \times \mathbb{R} \times \bar{\Omega}$ has the following form :*

$$F(M,p,r,x) = \begin{cases} H(M,p,r,x) \text{ if } x \in \Omega, \\ r - \Psi(x) \text{ if } x \in \partial\Omega, \end{cases}$$

*for $\Psi$ a given function defined on $\partial\Omega$. $H$ is supposed to be fully non linear. Consider also an approximation scheme of the following form :*

$$S(\rho, x, W^\rho(x), W^\rho) = 0 \text{ in } \bar{\Omega}, \tag{2.2}$$

*where $S : \mathbb{R}^+ \times \bar{\Omega} \times \mathbb{R} \times B(\bar{\Omega}) \to \mathbb{R}$ is locally bounded with $B(\bar{\Omega})$ being the set of bounded functions on $\bar{\Omega}$. Consider the following assumptions.*

1. *$S$ is monotone, which means that :*

$$S(\rho, x, r, W) \leq S(\rho, x, r, V) \tag{2.3}$$

   *if $W \geq V$ for all $\rho \geq 0$, $x \in \bar{\Omega}$, $r \in \mathbb{R}$, $W, V \in B(\bar{\Omega})$.*

2. *$S$ is stable, which means that for all $\rho > 0$, there exists a solution $W^\rho$ of (2.2) such that $W^\rho \in B(\bar{\Omega})$ with a bound independent of $\rho$.*

3. *The scheme $S$ is consistent which means that for all $x \in \bar{\Omega}$ and $\psi \in \mathscr{C}^\infty(\bar{\Omega}) \cap B(\bar{\Omega})$ :*

$$\limsup_{\substack{\rho\to 0\\ y\to x\\ \xi\to 0}} \frac{S(\rho, y, \psi(y)+\xi, \psi+\xi)}{\rho} \leq \limsup_{\substack{y\to x\\ y\in\bar{\Omega}}} F(D^2\psi(y), D\psi(y), \psi(y), y),$$

   *and*

$$\liminf_{\substack{\rho\to 0\\ y\to x\\ \xi\to 0}} \frac{S(\rho, y, \psi(y)+\xi, \psi+\xi)}{\rho} \geq \liminf_{\substack{y\to x\\ y\in\bar{\Omega}}} F(D^2\psi(y), D\psi(y), \psi(y), y),$$

4. (2.1) *has the following strong uniqueness (comparison principle) property : If* $W \in B(\bar{\Omega})$ *is an upper semicontinuous subsolution of* (2.1) *and* $V \in B(\bar{\Omega})$ *is a lower semicontinuous supersolution of* (2.1) *then* $W \leq V$ *on* $\bar{\Omega}$.

*Assume all the assumptions enumerated above. Then, as* $\rho$ *goes to* 0*, the solution* $W^\rho$ *of* (2.2) *converges locally uniformly to the unique continuous viscosity solution of* (2.1).

We can see that the condition of monotonicity on the scheme has been added. There are two very famous examples of monotone finite difference schemes in the literature. The first one (the approximation scheme of Kushner) which is the oldest, is monotone conditionally to the diffusion coefficient matrix and the second one (the approximation scheme of Bonnans Zidani) is unconditionally monotone.

**The approximation scheme of Kushner (see [48])**

This scheme has been introduced by Kushner in 1977. If we consider a partial differential equation containing an operator $L^\alpha$ defined by :

$$L^\alpha \phi(t,x) = \frac{1}{2}\operatorname{Tr}[a^\alpha(t,x) D^2\phi(t,x)] + b^\alpha(t,x) D\phi(t,x), \tag{2.4}$$

the approximation scheme consists in replacing $L^\alpha$ in the PDE by :

$$L_h^\alpha \phi(t,x) = \sum_{i=1}^{n} \left[ \frac{[a^\alpha(t,x)]_{ii}}{2} \Delta_{ii} + \sum_{j\neq i} \left( \frac{[a^{\alpha+}(t,x)]_{ij}}{2} \Delta_{ij}^{+} - \frac{[a^{\alpha-}(t,x)]_{ij}}{2} \Delta_{ij}^{-} \right) + [b^{\alpha+}(t,x)]_i \delta_i^{+} - [b^{\alpha-}(t,x)]_i \delta_i^{-} \right] \phi(t,x) \tag{2.5}$$

where $w^+ = \max(w,0)$, $w^- = -\min(w,0)$ and

$$\begin{aligned}
\delta_i^{\pm} w(t,x) &= \pm\frac{1}{\Delta x}\{w(t,x\pm\Delta x e_i) - w(t,x)\}\\
\Delta_{ii} w(t,x) &= \frac{1}{\Delta x^2}\{w(t,x+\Delta x e_i) - 2w(t,x) + w(t,x-\Delta x e_i)\}\\
\Delta_{ij}^{+} w(t,x) &= \frac{1}{2\Delta x^2}\{2w(t,x) + w(t,x+\Delta x(e_i+e_j)) + w(t,x-\Delta x(e_i+e_j))\}\\
&- \frac{1}{2\Delta x^2}\{w(t,x+\Delta x e_i) + w(t,x+\Delta x e_j) + w(t,x-\Delta x e_i) + w(t,x-\Delta x e_j)\}\\
\Delta_{ij}^{-} w(t,x) &= -\frac{1}{2\Delta x^2}\{2w(t,x) + w(t,x+\Delta x(e_i-e_j)) + w(t,x-\Delta x(e_i-e_j))\}\\
&+ \frac{1}{2\Delta x^2}\{w(t,x+\Delta x e_i) + w(t,x+\Delta x e_j) + w(t,x-\Delta x e_i) + w(t,x-\Delta x e_j)\}
\end{aligned}$$

$\Delta x$ being the space discretization step.

This scheme is then monotone and stable if and only if the matrix $a^\alpha$ is diagonally dominant. The particular discretization of the gradient with respect to the sign of its coefficient in the PDE is also referred as an upwind scheme in the literature. This will be used in a probabilistic form in Section 3.1.

**The approximation scheme of Bonnans and Zidani (see [11])**

It is assumed here that there exists a finite stencil $\bar{\mathscr{S}} \subset \mathbb{Z}^N \backslash \{0\}$ and a set of positive coefficients $\{\bar{a}^\alpha_\beta, \beta \in \bar{\mathscr{S}}\} \subset \mathbb{R}^+$ for each $\alpha \in \mathscr{A}$ such that :

$$a^\alpha(t,x) = \sum_{\beta \in \bar{\mathscr{S}}} \bar{a}^\alpha_\beta(t,x) \beta \beta^\intercal \quad \text{in } Q_T, \quad \alpha \in \mathscr{A}.$$

The operator $L^\alpha$ defined in (2.4) can then be written as :

$$L^\alpha \phi(t,x) = \sum_{\beta \in \bar{\mathscr{S}}} \bar{a}^\alpha_\beta(t,x) D^2_\beta \phi(t,x) + b^\alpha(t,x) D\phi(t,x)$$

where $D^2_\beta = \operatorname{Tr} \beta\beta^\intercal D^2$. The approximation scheme of Bonnans and Zidani consists in replacing $L^\alpha$ by :

$$L^\alpha_h \phi = \sum_{\beta \in \bar{\mathscr{S}}} \bar{a}^\alpha_\beta \Delta_\beta \phi + \sum_{i=1}^N \left[ b^{\alpha+}_i \delta^+_i - b^{\alpha-}_i \delta^-_i \right] \phi \tag{2.6}$$

with the same notations as for the approximation of Kushner for the derivative of order 1, and where

$$\Delta_\beta w(x) = \frac{1}{|\beta|^2 \Delta x^2} \{ w(x + \beta \Delta x) - 2w(x) + w(x - \beta \Delta x) \}.$$

This scheme is unconditionally monotone. In [11], Bonnans and Zidani give explicit conditions of existence of the finite stencil $\bar{\mathscr{S}}$ and techniques to build it systematically.

Once a scheme is known to be convergent, a question remains about how quick is this convergence with the grid step. There comes the notion of order of convergence. It is the power of the grid step such that with an appropriate multiplicative factor, it bounds the error between the approximated solution of the PDE obtained with the scheme and its exact solution. The following theorem is a result on the error bounds for any monotone approximation scheme for a parabolic Hamilton-Jacobi-Bellman equation. It is due to Barles and Jakobsen. For the definition of the norms and notations used in this theorem, see Section 0.1.

**Theorem 2.1.2** ([7, Theorem 3.1])**.** *Let us consider the following partial differential equation :*

$$\frac{\partial W}{\partial t} + F(t,x,W,D_x W, D^2_x W) = 0 \qquad \text{in } Q_T := (0,T] \times \mathbb{R}^n, \tag{2.7}$$

$$W(0,x) = \Psi(x) \quad \text{in } \mathbb{R}^n, \tag{2.8}$$

*where*

$$F(t,x,r,p,X) = \sup_{\alpha \in \mathscr{A}} \{ \mathscr{L}^\alpha(t,x,r,p,X) \},$$

*with*

$$\mathscr{L}^\alpha(t,x,r,p,X) := -\operatorname{Tr}(a^\alpha(t,x)X) - b^\alpha(t,x)p - c^\alpha(t,x)r - f^\alpha(t,x). \tag{2.9}$$

*The coefficients $a^\alpha$, $b^\alpha$, $c^\alpha$, $f^\alpha$ and the terminal data $\Psi$ take values respectively in $\mathbb{S}(n)$, $\mathbb{R}^n$, $\mathbb{R}$, $\mathbb{R}$ and $\mathbb{R}$. Let us also consider the following approximation numerical scheme to* (2.7)-(2.8) *written in the following abstract way :*

$$S(h,t,x,W_h(t,x),[W_h]_{t,x}) = 0 \ \text{in } \mathscr{G}^+_h := \mathscr{G}_h \setminus \{t=0\}, \tag{2.10}$$

$$W_h(0,x) = \Psi_h(x) \ \text{in } \mathscr{G}^0_h := \mathscr{G}_h \cap \{t=0\} \tag{2.11}$$

*where $\mathscr{G}_h$ is the grid related to the scheme, $h = (\Delta x, \Delta t)$ is a multidimensional vector containing the space and the time discretization steps and $[W_h]_{t,x}$ represents typically the values of the function $W_h$ at points of the grid other than $(t, x)$.*
*We consider the following assumptions :*

- ***A1**: For any $\alpha \in \mathscr{A}$, $a^\alpha = \frac{1}{2}\sigma^\alpha \sigma^{\alpha\intercal}$ for some $n \times p$ matrix $\sigma^\alpha$ and there is a constant $K$ independent of $\alpha$ such that :*

$$|\Psi|_1 + |\sigma^\alpha|_1 + |b^\alpha|_1 + |c^\alpha|_1 + |f^\alpha|_1 \leq K$$

- ***A2**: For every $\delta > 0$, there are $M \in \mathbb{N}$ and $\{\alpha_i\}_{i=1}^M \subset \mathscr{A}$ such that for any $\alpha \in \mathscr{A}$ :*

$$\inf_{1 \leq i \leq M} (|\sigma^\alpha - \sigma^{\alpha_i}|_0 + |b^\alpha - b^{\alpha_i}|_0 + |c^\alpha - c^{\alpha_i}|_0 + |f^\alpha - f^{\alpha_i}|_0) \leq \delta$$

- ***S1**: There exists $\lambda, \mu \geq 0$, $h_0 > 0$ such that if $|h| \leq h_0$ (where $|\cdot|$ is the Euclidean norm), $W \leq V$ are functions continuous and bounded on $\mathscr{G}_h$, and $\phi(t) = e^{\mu t}(a + bt) + c$ for $a, b, c \geq 0$, then :*

$$S(h, t, x, r + \phi(t), [W + \phi]_{t,x}) \geq S(h, t, x, r, [V]_{t,x}) + b/2 - \lambda c \text{ in } \mathscr{G}_h^+$$

- ***S2**: For every $h$ and continuous and bounded function $\Phi$ on $\mathscr{G}_h$, the function $(t, x) \mapsto S(h, t, x, \Phi, [\Phi]_{t,x})$ is bounded and continuous in $\mathscr{G}_h^+$ and the function $r \mapsto S(h, t, x, r, [\Phi]_{t,x})$ is uniformly continuous for bounded $r$, uniformly in $(t, x) \in \mathscr{G}_h^+$.*

- ***S3 i)**: There exists a function $E_1(\tilde{K}, h, \epsilon)$ such that for any sequence $\{\psi_\epsilon\}_{\epsilon>0}$ of smooth functions satisfying :*

$$|\partial_t^{\beta_0} D^{\beta'} \psi_\epsilon(t, x)| \leq \tilde{K}\epsilon^{1-2\beta_0-|\beta'|} \text{ in } Q_T \text{ for any } \beta_0 \in \mathbb{N}, \beta' = (\beta'_i)_i \in \mathbb{N}^N,$$

  *($|\beta'| = \sum_{i=1}^N \beta'_i$), the following inequality holds :*

$$S(h, t, x, \psi_\epsilon(t, x), [\psi_\epsilon]_{t,x}) \leq \frac{\partial \psi_\epsilon}{\partial t} + F(t, x, \psi_\epsilon, D\psi_\epsilon, D^2\psi_\epsilon) + E_1(\tilde{K}, h, \epsilon),$$

  *in $\mathscr{G}_h^+$.*

- ***S3 ii)**: There exists a function $E_2(\tilde{K}, h, \epsilon)$ such that for any sequence $\{\psi_\epsilon\}_{\epsilon>0}$ of smooth functions satisfying :*

$$|\partial_t^{\beta_0} D^{\beta'} \psi_\epsilon(t, x)| \leq \tilde{K}\epsilon^{1-2\beta_0-|\beta'|} \text{ in } Q_T \text{ for any } \beta_0 \in \mathbb{N}, \beta' = (\beta'_i)_i \in \mathbb{N}^N,$$

  *($|\cdot|$ being the Euclidean norm and $|\beta'| = \sum_{i=1}^N \beta'_i$), the following inequality holds :*

$$S(h, t, x, \psi_\epsilon(t, x), [\psi_\epsilon]_{t,x}) \geq \frac{\partial \psi_\epsilon}{\partial t} + F(t, x, \psi_\epsilon, D\psi_\epsilon, D^2\psi_\epsilon) - E_2(\tilde{K}, h, \epsilon),$$

  *in $\mathscr{G}_h^+$.*

*Assume (A1), (S1), (S2), and that (2.10) has a unique continuous and bounded solution $W_h$ on $\mathscr{G}_h$. Let $W$ denotes the solution of (2.7)-(2.8) and $h$ a vector of real numbers sufficiently small.*

- *If (S3 i) holds, then there exists a constant $C$ depending only on $\mu$, $K$ in (S1), (A1) such that :*

$$W - W_h \leq e^{\mu t}|(\Psi - \Psi_h)^+|_0 + C\min_{\epsilon>0}\left(\epsilon + E_1(\tilde{K}, h, \epsilon)\right) \text{ in } \mathscr{G}_h,$$

*where $\tilde{K} = |W|_1$.*

- *If (S3 ii) and (A2) holds, then there exists a constant $C$ depending only on $\mu$, $K$ in (S1), (A1) such that :*

$$W - W_h \geq -e^{\mu t}|(\Psi - \Psi_h)^-|_0 - C\min_{\epsilon>0}\left(\epsilon^{1/3} + E_2(\tilde{K}, h, \epsilon)\right) \text{ in } \mathscr{G}_h,$$

*where $\tilde{K} = |W|_1$.*

In the above theorem, the condition **A1** allows one to have the existence and unicity of a viscosity solution to the PDE problem (2.7)-(2.8) according to Theorem 1.2.2.

The above theorem allows one to obtain the following corollary which is a consequence of Theorem 4.1 in [7].

**Corollary 2.1.1.** *Consider the PDE problem* (2.7)-(2.8) *presented in the previous theorem.*

*Let us consider the following approximation scheme for this PDE using the $\theta$-method which is a generalization of the Crank-Nicholson scheme ($\theta = \frac{1}{2}$).*

$$\begin{aligned} W(t,x) =& W(t-\Delta t, x) \\ &- (1-\theta)\Delta t \sup_{\alpha\in\mathscr{A}} \left\{-L_h^\alpha W - c^\alpha W - f^\alpha\right\}(t-\Delta t, x) \\ &- \theta\Delta t \sup_{\alpha\in\mathscr{A}} \left\{-L_h^\alpha W - c^\alpha W - f^\alpha\right\}(t, x) \text{ in } \mathscr{G}_h^+, \end{aligned}$$

*where $L_h^\alpha$ is given by* (2.5) *or* (2.6), *$\Delta t$ being the time discretization step.*

*If a solution $W_h$ to this scheme exists, then under the conditions of stability (CFL conditions), we have the following result :*

$$-e^{\mu t}|(\Psi - \Psi_h)^-|_0 - Ch^{\frac{1}{5}} \leq W - W_h \leq e^{\mu t}|(\Psi - \Psi_h)^+|_0 + Ch^{\frac{1}{2}}$$

*with $h = \sqrt{\Delta x^2 + \Delta t}$.*

### 2.1.2 Barles and Souganidis result in unbounded setting

We want to have a convergence result such as the one of Barles and Souganidis (Theorem 2.1.1) when the PDE solution lives in the space of functions with polynomial growth. This will be particularly useful in Section 3.2 to obtain the convergence of a probabilistic scheme in a linear quadratic style problem. The following theorem gives such a result and is proved using the same tools as for Theorem 2.1.1.

**Theorem 2.1.3.** *Consider the same PDE problem as in Theorem 2.1.1. We replace the set of bounded functions $B(\bar{\Omega})$ in the theorem by the set of functions with a given k-polynomial growth $B^k(\bar{\Omega})$ defined by :*

$$f \in B^k(\bar{\Omega}) \text{ if } \exists C > 0, \forall x \in \bar{\Omega}, \frac{|f(x)|}{1+|x|^k} \leq C.$$

*In this setting, we consider the same assumptions as for Theorem 2.1.1 except for the consistency assumption where for $\mu : x \mapsto 1 + |x|^k$, the following inequalities are considered instead.*

$$\limsup_{\substack{\rho\to 0\\ y\to x\\ \xi\to 0}} \frac{S(\rho, y, \psi(y) + \xi\mu(y), \psi + \xi\mu)}{\rho} \leq \limsup_{\substack{y\to x\\ y\in\bar{\Omega}}} F(D^2\psi(y), D\psi(y), \psi(y), y),$$

*and*

$$\liminf_{\substack{\rho\to 0\\ y\to x\\ \xi\to 0}} \frac{S(\rho, y, \psi(y) + \xi\mu(y), \psi + \xi\mu)}{\rho} \geq \liminf_{\substack{y\to x\\ y\in\bar{\Omega}}} F(D^2\psi(y), D\psi(y), \psi(y), y),$$

*for $x \in \bar{\Omega}$ and $\psi \in \mathscr{C}^\infty(\bar{\Omega}) \cap B^k(\bar{\Omega})$.*

*Then, the result of Theorem 2.1.1 still holds.*

*Proof.* We will redo the proof of Barles and Souganidis with $B^k(\bar{\Omega})$ functions.

Let $\bar{K}(x) = \limsup_{\substack{y\to x\\ \rho\to 0}} \frac{W^\rho(y)}{1+|y|^k}$ and $\underline{\mathrm{K}}(x) = \liminf_{\substack{y\to x\\ \rho\to 0}} \frac{W^\rho(y)}{1+|y|^k}$.

The functions $\bar{W}(x) = \bar{K}(x)(1 + |x|^k)$ and $\underline{\mathrm{W}}(x) = \underline{\mathrm{K}}(x)(1 + |x|^k)$ are respectively upper-semicontinuous and lower-semicontinuous and $\bar{W} \geq \underline{\mathrm{W}}$. If we show that $\bar{W}$ and $\underline{\mathrm{W}}$ are respectively viscosity sub and super solution of equation (2.1), then the comparison principle applied to functions in $B^k(\bar{\Omega})$ will give us the inequality $\bar{W} \leq \underline{\mathrm{W}}$ which will end in an equality so that $W = \bar{W} = \underline{\mathrm{W}} = \lim_{\rho\to 0} W^\rho$ will be the unique viscosity solution of (2.1).

We will show that $\bar{W}$ is a viscosity subsolution of (2.1), the proof of $\underline{\mathrm{W}}$ being a viscosity supersolution of (2.1) having the same pattern.

Let $x \in \Omega$ and $\Phi \in \mathscr{C}^\infty(\bar{\Omega}) \cap B^k(\bar{\Omega})$ such that $0 = (\bar{W} - \Phi)(x)$ is a local maximum of $\bar{W} - \Phi$. $\frac{\bar{W}(x)-\Phi(x)}{1+|x|^k}$ is then also a local maximum of the function $y \mapsto \frac{\bar{W}(y)-\Phi(y)}{1+|y|^k}$.

Let $r_c > 0$ be such that, for all $y$ in the ball $B(x, r_c)$,

$$\frac{\bar{W}(y) - \Phi(y)}{1 + |y|^k} \leq \frac{\bar{W}(x) - \Phi(x)}{1 + |x|^k}.$$

We may suppose without loss of generality, a strict maximum is achieved at $x$ and that $\Phi(y) \geq (1 + |y|^k) \sup_{\substack{z\in\bar{\Omega}\\ \rho}} \frac{W^\rho(z)}{1+|z|^k}$ outside the ball $B(x, r_c)$.

Then, there exists $(\rho_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ such that :

$$\lim_n \rho_n = 0,\ \lim_n y_n = x,\ \lim_n \frac{W^{\rho_n}(y_n) - \Phi(y_n)}{1 + |y_n|^k} = 0$$

and $y_n$ is a global maximum of $y \mapsto \frac{W^{\rho_n}(y)-\Phi(y)}{1+|y|^k}$.

Let $\xi_n := \frac{W^{\rho_n}(y_n)-\Phi(y_n)}{1+|y_n|^k}$. We have $\lim_n \xi_n = 0$ and for all $y \in \bar{\Omega}$, $W^{\rho_n}(y) \leq (1+|y|^k\xi_n+\Phi(y)$. We also have $W^{\rho_n}(y_n) = (1 + |y_n|^k)\xi_n + \Phi(y_n)$. So, by the monotonicity condition,

$$S(\rho_n, y_n, W^{\rho_n}(y_n), W^{\rho_n}) \geq S(\rho_n, y_n, \Phi(y_n) + (1 + |y_n|^k)\xi_n, \Phi + \xi_n\mu),$$

where $\mu : y \mapsto 1 + |y|^k$. We then have :

$$
\begin{aligned}
0 &\geq \liminf_{n} S(\rho_n, y_n, \Phi(y_n) + (1 + |y_n|^k)\xi_n, \Phi + \xi_n \mu) \\
&\geq \liminf_{\substack{\rho \to 0 \\ y \to x \\ \xi \to 0}} S(\rho, y, \Phi(y) + (1 + |y|^k)\xi, \Phi + \xi\mu) \\
&\geq \liminf_{\substack{y \to x \\ y \in \bar{\Omega}}} F(D^2\Phi(y), D\Phi(y), \Phi(y), y),
\end{aligned}
$$

by the consistency assumption.

Hence, we deduce that $\bar{W}$ is a viscosity subsolution of (2.1). We obtain then the result of the theorem. □

## 2.2 Semi-Lagrangian methods

Semi-Lagrangian methods can be found in the literature since 1950s with the famous paper [19] of Courant, Isaacson and Rees. They were related first to the advection equation and were used in the context of atmospheric modelling. However, during the eighties, similar methods appeared in the field of optimal control theory. The first paper refered in the control literature is the paper of Capuzzo Dolcetta ([15]) in 1983 where the author is interested in a deterministic optimal control problem with infinite horizon. Then, followed the paper of Capuzzo Dolcetta and Ishii ([16]) , the papers of Gonzales and Rofman ([37], [38]), the paper of Falcone ([26]) and for the stochastic optimal control problems the paper of Menaldi ([55]) and the paper of Camilli and Falcone ([14]). Recently, Debrabant and Jakobsen have proposed in [24] a unifying framework for the study of these methods.

We will start here by giving a unifying definition of Semi-Lagrangian method and then show some Semi-Lagrangian schemes found in the literature and recall convergence results obtained in [24].

### 2.2.1 Definition of Semi-Lagrangian methods

Semi-Lagrangian methods follow the same steps as finite difference methods as described in the previous section. But, on the contrary of the finite difference schemes where the directions in which the points are taken for the partial derivatives approximations are limited by the grid, the directions taken in Semi-Lagrangian schemes for the partial derivatives approximations are guided by the coefficients of the PDE problem and may not allow to choose a grid point. This may imply the use of an interpolation technique of the value of the solution of the PDE problem at the points chosen for partial derivatives approximations with its value at their closest grid points.

One example of Semi-Lagrangian scheme is the scheme introduced by Capuzzo Dolcetta in [15] and completed with a space discretization by Falcone in [26] where the PDE

$$\max_{\alpha\in\mathscr{A}}\{\lambda W - b^\alpha\cdot DW - f^\alpha\} = 0 \text{ in } \mathbb{R}^n$$

with $\mathscr{A}$ being a finite set, is approximated by

$$\max_{\alpha\in\mathscr{A}}\left\{W^h(x_i) - (1-\lambda h)W^h(x_i + b^\alpha h) - hf^\alpha(x_i)\right\}$$

where $x_i$ is a point of the grid and $h$ is a parameter which can be compared to a time discretization step.

We give below a brief review of Semi-Lagrangian approximation techniques found in the literature to approximate operators of the form (2.9). Assume that $\mathscr{I}_{\Delta x}$ is an interpolation operator using functions values at grid points and $k$ and $h$ are parameters related to the grid discretization steps. The approximation of (2.7)-(2.8) consists in replacing (2.9) by using the following approximations.

- The approximation of Falcone in [26] when $a^\alpha = \frac{1}{2}\sigma^\alpha\sigma^{\alpha\intercal} = 0$ :

$$b^\alpha(t,x)\cdot D_x\phi(t,x) \approx \frac{\mathscr{I}_{\Delta x}\phi(t, x + hb^\alpha(t,x)) - \mathscr{I}_{\Delta x}\phi(t,x)}{h},$$

- the approximation of Crandall and Lions [22] when $b^\alpha = 0$ :

$$\frac{1}{2}\operatorname{Tr}(\sigma^\alpha(t,x)\sigma^{\alpha\intercal}(t,x)D_x^2\phi(t,x)) \approx \sum_{j=1}^{p} \frac{\mathscr{I}_{\Delta x}\phi(t,x+k\sigma_j^\alpha(t,x)) - 2\mathscr{I}_{\Delta x}\phi(t,x) + \mathscr{I}_{\Delta x}\phi(t,x-k\sigma_j^\alpha(t,x))}{2k^2},$$

  where $\sigma^\alpha(t,x) \in \mathbb{R}^{n\times p}$ and $\sigma_j^\alpha(t,x)$ denote the $j$-th column of $\sigma^\alpha(t,x)$. The same notations are considered for the following approximations.

- The corrected version of the approximation of Camilli and Falcone [14]

$$\frac{1}{2}\operatorname{Tr}(\sigma^\alpha(t,x)\sigma^{\alpha\intercal}(t,x)D_x^2\phi(t,x)) + b^\alpha(t,x)\cdot D_x\phi(t,x) \approx \sum_{j=1}^{p} \frac{\mathscr{I}_{\Delta x}\phi(t,x+\sqrt{h}\sigma_j^\alpha(t,x)+\frac{h}{p}b^\alpha(t,x)) - 2\mathscr{I}_{\Delta x}\phi(t,x) + \mathscr{I}_{\Delta x}\phi(t,x-\sqrt{h}\sigma_j^\alpha(t,x)+\frac{h}{p}b^\alpha(t,x))}{2h},$$

- The combination of the approximation of Falcone in [26] and Crandall and Lions in [22]

$$\frac{1}{2}\operatorname{Tr}(\sigma^\alpha(t,x)\sigma^{\alpha\intercal}(t,x)D_x^2\phi(t,x)) + b^\alpha(t,x)\cdot D_x\phi(t,x) \approx \frac{\mathscr{I}_{\Delta x}\phi(t,x+hb^\alpha(t,x)) - \mathscr{I}_{\Delta x}\phi(t,x)}{h} + \sum_{j=1}^{p} \frac{\mathscr{I}_{\Delta x}\phi(t,x+k\sigma_j^\alpha(t,x)) - 2\mathscr{I}_{\Delta x}\phi(t,x) + \mathscr{I}_{\Delta x}\phi(t,x-k\sigma_j^\alpha(t,x))}{2k^2},$$

- The new more efficient version of the approximation of Camilli and Falcone in [14].

$$\frac{1}{2}\operatorname{Tr}(\sigma^\alpha(t,x)\sigma^{\alpha\intercal}(t,x)D_x^2\phi(t,x)) + b^\alpha(t,x)\cdot D\phi(t,x) \approx \sum_{j=1}^{p-1} \frac{\mathscr{I}_{\Delta x}\phi(t,x+k\sigma_j^\alpha(t,x)) - 2\mathscr{I}_{\Delta x}\phi(t,x) + \mathscr{I}_{\Delta x}\phi(t,x-k\sigma_j^\alpha(t,x))}{2k^2} + \frac{\mathscr{I}_{\Delta x}\phi(t,x+k\sigma_p^\alpha(t,x)+k^2b^\alpha(t,x)) - 2\mathscr{I}_{\Delta x}\phi(t,x) + \mathscr{I}_{\Delta x}\phi(t,x-k\sigma_p^\alpha(t,x)+k^2b^\alpha(t,x))}{2k^2}$$

Debrabant and Jakobsen proposed in [24] a unifying and more general form of the above approximations which is the following.

$$\operatorname{Tr}[a^{\alpha,\beta}(t,x)D_x^2\phi(t,x)] + b^{\alpha,\beta}(t,x)\cdot D_x\phi(t,x) \approx L_k^{\alpha,\beta}[\mathscr{I}_{\Delta x}\phi](t,x)$$

with

$$L_k^{\alpha,\beta}[\psi](t,x) := \sum_{i=1}^{M} \frac{\psi(t,x+y_{k,i}^{\alpha,\beta,+}(t,x)) - 2\psi(t,x) + \psi(t,x+y_{k,i}^{\alpha,\beta,-}(t,x))}{2k^2} \tag{2.12}$$

for $k > 0$ and some $M \geq 1$, under some conditions on the functions $y_{k,i}^{\alpha,\beta,+}$ and $y_{k,i}^{\alpha,\beta,-}$ that will be given in the next section.

This unifying form allows Debrabant and Jakobsen to obtain quite general results on Semi-Lagrangian schemes which will be the subject of the next section.

### 2.2.2 Convergence and error bounds for Semi-Lagrangian schemes

We will start with a convergence result of Semi-Lagrangian schemes obtained by Debrabant and Jakobsen in [24] which uses the Theorem of Barles and Souganidis quoted in Theorem 2.1.1. The authors consider the following PDE problem.

$$\frac{\partial W}{\partial t} - \inf_{\alpha\in\mathscr{A}}\sup_{\beta\in\mathscr{B}}\left\{L^{\alpha,\beta}[W](t,x) + c^{\alpha,\beta}(t,x)W + f^{\alpha,\beta}(t,x)\right\} = 0 \text{ in } Q_T := (0,T]\times\mathbb{R}^N \quad (2.13)$$

$$W(0,x) = \Psi(x) \text{ in } \mathbb{R}^N \quad (2.14)$$

where $\mathscr{A}$ and $\mathscr{B}$ are complete metric spaces, and

$$L^{\alpha,\beta}[u](t,x) = \mathrm{Tr}[a^{\alpha,\beta}(t,x)D_x^2u(t,x)] + b^{\alpha,\beta}(t,x)D_xu(t,x).$$

The coefficients $a^{\alpha,\beta}, b^{\alpha,\beta}$, $c^{\alpha,\beta}$, $f^{\alpha,\beta}$ and $\Psi$ take values respectively in $\mathbb{S}(N)$, $\mathbb{R}^N$, $\mathbb{R}$, $\mathbb{R}$ and $\mathbb{R}$. The coefficient $a^{\alpha,\beta}$ is assumed to be positive semi-definite. So the equation may degenerate. The solution is then to be considered as a viscosity solution.

**Theorem 2.2.1** ([24, Theorem 4.2]). *Consider the following scheme for the PDE problem* (2.13)- (2.14).

$$\begin{cases} \delta_{\Delta t_n}W_i^n = \inf_{\alpha\in\mathscr{A}}\sup_{\beta\in\mathscr{B}}\left\{L_k^{\alpha,\beta}[\mathscr{I}_{\Delta x}\bar{W}^{\theta,n}]_i^{n-1+\theta} + c_i^{\alpha,\beta,n-1+\theta}\bar{W}_i^{\theta,n} + f_i^{\alpha,\beta,n-1+\theta}\right\} \text{ in } G \\ W_i^0 = \Psi(x_i) \text{ in } X_{\Delta x}\end{cases} \quad (2.15)$$

*where* $G = (t_n, x_i)_{n,i}$ *is the time-space grid which is a Cartesian product of a time grid* $\mathscr{T}_{\Delta t}$ *and a space grid* $X_{\Delta x}$, $\Delta t_n = t_n - t_{n-1}$, $\max_n \Delta t_n \leq \Delta t$, $L_k^{\alpha,\beta}$ *is defined by* (2.12), $\mathscr{I}_{\Delta x}$ *is a space interpolation operator,* $W_i^n = W(t_n, x_i)$, $f_i^{\alpha,\beta,n-1+\theta} = f^{\alpha,\beta}(t_{n-1}+\theta\Delta t_n, x_i)$, $c_i^{\alpha,\beta,n-1+\theta} = c^{\alpha,\beta}(t_{n-1}+\theta\Delta t_n, x_i)$, $L_k^{\alpha,\beta}[\mathscr{I}_{\Delta x}\bar{W}^{\theta,n}]_i^{n-1+\theta} = L_k^{\alpha,\beta}[\mathscr{I}_{\Delta x}\bar{W}^{\theta,n}](t_{n-1}+\theta\Delta t_n, x_i)$,

$$\delta_{\Delta t_n}W_i^n = \frac{W_i^n - W_i^{n-1}}{\Delta t_n}, \quad \bar{W}_i^{\theta,n} = (1-\theta)W_i^{n-1} + \theta W_i^n,$$

*and*

$$\mathscr{I}_{\Delta x}\bar{W}_\cdot^{\theta,n} = (1-\theta)\mathscr{I}_{\Delta x}W_\cdot^{n-1} + \theta\mathscr{I}_{\Delta x}W_\cdot^n.$$

$\Delta x$ *is considered here as an upper bound of the discretization step of* $X_{\Delta x}$. *Assume the following.*

- ***A1****: For any* $\alpha\in\mathscr{A}$ *and* $\beta\in\mathscr{B}$, $a^{\alpha,\beta} = \frac{1}{2}\sigma^{\alpha,\beta}\sigma^{\alpha,\beta\intercal}$ *for some* $N\times P$ *matrix* $\sigma^{\alpha,\beta}$ *and there is a constant* $K$ *independent of* $\alpha,\beta$ *such that :*

$$|\Psi|_1 + |\sigma^{\alpha,\beta}|_1 + |b^{\alpha,\beta}|_1 + |c^{\alpha,\beta}|_1 + |f^{\alpha,\beta}|_1 \leq K.$$

- ***I1****: There are* $K>0$ *and* $r\in\mathbb{N}$ *such that for all smooth functions* $\Phi$,

$$|(\mathscr{I}_{\Delta x}\Phi) - \Phi|_0 \leq K|D_x^r\Phi|_0\Delta x^r$$

- ***I2***: *There is a set of non-negative functions* $\{w_j(x)\}_j$ *such that*

$$(\mathscr{I}_{\Delta x}\Phi)(x) = \sum_j \Phi(x_j) w_j(x)$$

*and for all* $i, j \in \mathbb{N}$,

$$w_j(x) \geq 0, \quad w_i(x_j) = \delta_{ij} \quad \text{and} \quad \sum_i w_i(x) \equiv 1$$

- ***Y1*** *:*

$$\begin{cases} \sum_{i=1}^{M} [y_{k,i}^{\alpha,\beta,+} + y_{k,i}^{\alpha,\beta,-}] = 2k^2 b^{\alpha,\beta} + O(k^4), \\ \sum_{i=1}^{M} [y_{k,i}^{\alpha,\beta,+} y_{k,i}^{\alpha,\beta,+\intercal} + y_{k,i}^{\alpha,\beta,-} y_{k,i}^{\alpha,\beta,-\intercal}] = 2k^2 \sigma^{\alpha,\beta} \sigma^{\alpha,\beta\intercal} + O(k^4), \\ \sum_{i=1}^{M} [[y_{k,i}^{\alpha,\beta,+}]_{j_1} [y_{k,i}^{\alpha,\beta,+}]_{j_2} [y_{k,i}^{\alpha,\beta,+}]_{j_3} + [y_{k,i}^{\alpha,\beta,-}]_{j_1} [y_{k,i}^{\alpha,\beta,-}]_{j_2} [y_{k,i}^{\alpha,\beta,-}]_{j_3}] = O(k^4), \\ \sum_{i=1}^{M} [[y_{k,i}^{\alpha,\beta,+}]_{j_1} [y_{k,i}^{\alpha,\beta,+}]_{j_2} [y_{k,i}^{\alpha,\beta,+}]_{j_3} [y_{k,i}^{\alpha,\beta,+}]_{j_4} + [y_{k,i}^{\alpha,\beta,-}]_{j_1} [y_{k,i}^{\alpha,\beta,-}]_{j_2} [y_{k,i}^{\alpha,\beta,-}]_{j_3}][y_{k,i}^{\alpha,\beta,-}]_{j_4}] = O(k^4), \end{cases}$$

*for all* $j_1, j_2, j_3, j_4 = 1, 2, \cdots, n$.

- ***CFL condition*** *:*

$$(1-\theta)\Delta t \left[\frac{M}{k^2} - c_i^{\alpha,\beta,n-1+\theta}\right] \leq 1 \quad \text{and} \quad \theta \Delta t c_i^{\alpha,\beta,n-1+\theta} \leq 1 \quad \forall \alpha, \beta, n, i. \tag{2.16}$$

*Then, there exists a unique bounded solution* $U$ *to* (2.15) *and* $U$ *converges uniformly to the solution* $W$ *of* (2.13)- (2.14) *as* $\Delta t, k, \frac{\Delta x^r}{k^2} \to 0$.

In the above Theorem, the condition **A1** allows one to have the existence and unicity of a viscosity solution to the PDE problem (2.13)- (2.14) using Theorem 1.2.2.

The previous Theorem is completed in [24] by a result on the error bound in the convergence of Semi-Lagrangian schemes recalled below.

**Theorem 2.2.2** ([24, Theorem 7.2])**.** *Consider the PDE problem* (2.13)- (2.14) *and its approximation scheme* (2.15) *where the set* $\mathscr{B}$ *is supposed to be a singleton (we can drop then the indexation in* $\beta$*).*

*With the assumptions names of Theorem 2.2.1, assume* ***A1***, ***Y1***, (2.16), $k \in (0,1)$ *and the following condition :*

$$\begin{cases} \sum_{i=1}^{M} [y_{k,i}^{\alpha,+} + y_{k,i}^{\alpha,-}][\tilde{y}_{k,i}^{\alpha,+} + \tilde{y}_{k,i}^{\alpha,-}] \leq 2k^2 (b^\alpha - \tilde{b}^\alpha), \\ \sum_{i=1}^{M} [y_{k,i}^{\alpha,+} y_{k,i}^{\alpha,+\intercal} + y_{k,i}^{\alpha,-} y_{k,i}^{\alpha,-\intercal}] + [\tilde{y}_{k,i}^{\alpha,+} \tilde{y}_{k,i}^{\alpha,+\intercal} + \tilde{y}_{k,i}^{\alpha,-} \tilde{y}_{k,i}^{\alpha,-\intercal}] \\ - [\tilde{y}_{k,i}^{\alpha,+} y_{k,i}^{\alpha,+\intercal} + y_{k,i}^{\alpha,+} \tilde{y}_{k,i}^{\alpha,+\intercal} + \tilde{y}_{k,i}^{\alpha,-} y_{k,i}^{\alpha,-\intercal} + y_{k,i}^{\alpha,-} \tilde{y}_{k,i}^{\alpha,-\intercal}] \\ \leq 2k^2 (\sigma^\alpha - \tilde{\sigma}^\alpha)(\sigma^\alpha - \tilde{\sigma}^\alpha)^\intercal + 2k^4 (b^\alpha - \tilde{b}^\alpha)(b^\alpha - \tilde{b}^\alpha)^\intercal, \end{cases}$$

*where* $\sigma^\alpha, b^\alpha, y_{k,i}^{\alpha,\pm}$ *represent here the values of the functions of the same name at* $(t,x)$ *and* $\tilde{\sigma}^\alpha, \tilde{b}^\alpha, \tilde{y}_{k,i}^{\alpha,\pm}$ *represent the values of these functions at* $(t,y)$. *The above system must then hold for any* $t,x,y$. *If* $W$ *and* $U$ *are bounded solutions respectively of* (2.13)-(2.14) *and* (2.15), *then there exists* $c_0$ *such that for* $\Delta t \in (0, c_0)$,

$$|U - W| \leq C(|1-2\theta|\Delta t^{1/4} + \Delta t^{1/3} + k^{1/2}) \text{ in } G. \tag{2.17}$$

The error bound obtained here seems to be better than the error bound obtained in Corollary 2.1.1 for finite difference schemes. However, in practice, the best error estimate deduced from the above theorem is $\Delta x^{1/5}$ which is achieved when $k = O(\Delta x^{2/5})$. It corresponds then to the negative lower bound of the error obtained in Corollary 2.1.1, the upper bound being of a higher order ($\Delta x^{1/2}$).

There is also a recent result about the error bound of a Semi-Lagrangian scheme in an unbounded setting due to Assellaou, Bokanowski and Zidani ( [5]). It is stated as follows.

**Theorem 2.2.3** ([5, Theorem 4.1]). *Consider the following PDE :*

$$-\frac{\partial W}{\partial t}(t,x) + H(t,x,D_x W(t,x), D_x^2 W(t,x)) = 0, \quad \text{in } (0,T)\times\mathbb{R}^d,$$
$$W(T,x) = \phi(x), \quad \text{in } \mathbb{R}^d,$$

*where*

$$H(t,x,p,X) = \inf_{u\in U}\left\{-\frac{1}{2}\operatorname{Tr}(\sigma(t,x,u)\sigma^\intercal(t,x,u)X) - b(t,x,u)\cdot p\right\}$$

*with* $(t,x,p,X) \in (0,T)\times\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}(d)$. *It is supposed that :*

1. $U$ *is a non empty compact subset of* $\mathbb{R}^q$ $(q>0)$.
2. $b \in \mathscr{C}([0,T]\times\mathbb{R}^d\times U, \mathbb{R}^d)$ *and* $\sigma \in \mathscr{C}([0,T]\times\mathbb{R}^d\times U, \mathbb{R}^{d\times m})$.
3. *There exists* $L_0 > 0$ *such that for any* $(s,t,x,y,u) \in [0,T]\times[0,T]\times\mathbb{R}^d\times\mathbb{R}^d\times U$,

$$|b(t,x,u) - b(s,y,u)| + |\sigma(t,x,u) - \sigma(s,y,u)| \leq L_0(|x-y| + |t-s|^{\frac{1}{2}}),$$

4. $\phi$ *is a continuous real function and there exists a constant* $M_0 > 0$ *such that :*

$$|\phi(x)| \leq M_0(1+|x|), \quad (x\in\mathbb{R}^d).$$

*This PDE has a unique viscosity solution* $W$ *in the space of functions with linear growth. Let* $\sigma_k$ $(1\leq k\leq m)$ *denotes the* $k$*-th column of* $\sigma$ *and* $\bar{\sigma}_k$, $(1\leq k\leq 2m)$ *be defined as follows :*

$$\bar{\sigma}_k(t,x,u) := (-1)^k\sqrt{m}\,\sigma_{\lfloor\frac{k-1}{2}\rfloor}(t,x,u),$$

*where* $\lfloor a\rfloor$ *denotes the integer part of* $a\in\mathbb{R}$.

*Let* $h>0$ *be a time step of a time grid* $(t_n)_{0\leq n\leq N}$, $X_{\Delta x}$ *be a space grid and* $V^n$ $(0\leq n\leq N)$ *be the solution of the scheme :*

$$V^N(x) = \phi(x),$$
$$V^{n-1}(x) = S^h(t_n, x, V^n), \quad \text{for } 1\leq n\leq N,$$

*where for any* $t \in [0,T]$, $x \in X_{\Delta x}$ *and any function* $w : \mathscr{X}_{\Delta x} \to \mathbb{R}$,

$$S^h(t,x,w) := \frac{1}{2m} \max_{u \in U} \left\{ \sum_{k=1}^{2m} [w](x + hb(t,x,u) + \sqrt{h}\bar{\sigma}_k(t,x,u)) \right\},$$

*where* $[w]$ *denotes the linear interpolation of* $w$ *on the grid* $X_{\Delta x}$ *having a grid step* $\Delta x$.
$V^n$ *is completed on points which are not on the time and space grid by bilinear interpolation, which gives the function* $V$.

*Assume that* $\phi$ *is a Lipschitz continuous function with a Lipschitz constant* $L_\phi$. *There exists* $C \geq 0$ *depending only on* $T$ *and* $L_0$, *such that for every* $R > 0$, *we have :*

$$\|V - W\|_{L^\infty(B_R)} \leq CL_\phi \left( R^{7/4} h^{1/4} + \frac{|\Delta x|}{h} \right),$$

*where* $\|w\|_{L^\infty(B_R)}$ *represents the supremum of* $|w(t,x)|$ *for* $(t,x) \in [0,T] \times B_R$, $B_R$ *being the ball of center* $0$ *and radius* $R$.

## 2.3 Probabilistic methods

Finite difference methods and Semi-Lagrangian methods need the discretization of the space variable which leads to a storage and algorithmic complexity exponential in the space dimension and thus cannot be used in high dimension (greater or equal to 4). This problem is known as the curse of dimensionality and was a major stepback in the resolution of many PDEs problems numerically for many years. The introduction of the Feynman-Kac formula in [42] in 1951 by Kac, was a major breakthrough in the numerical methods for solving PDE problems. It related linear PDE problems to stochastic differential equations, requiring just the computation of an expectation to solve them and allowing to use probabilistic methods such as Monte Carlo method to solve these problems. This first article was only about linear PDE problems. However, in the following years, a lot of works have been done to enhance this result and be able to solve non linear partial differential equations in the same way. The notion of Backward Stochastic Differential Equations has been introduced by Bismut [10] in 1973 in the linear case and by Pardoux and Peng [61] in 1990 in the general case. It allowed to enlarge the class of PDE problems that can be solved using relations similar to the Feynman-Kac formula from linear to quasi-linear PDE problems. In 2007, Cheridito, Soner, Touzi and Victoir introduced in [17] the second order Backward Stochastic Differential Equations allowing to have a form of Feynman-Kac formula for fully non linear PDE problems like those which arise in optimal control problems. Their work has been extended by Fahim, Touzi and Warin in [25] who showed that the resulted scheme of [17] can be introduced like a finite difference scheme where partial derivatives are approximated in a probabilistic way instead of using finite differences. These developments introduced the need to compute a conditional expectation. But, in the same time, probabilistic methods evolved to solve this new difficulty. We can cite in this setting the introduction of the Longstaff and Schwartz method in [51], the representation formulae based on Malliavin calculus ([30], [29], [12]) and the quantization-based approach [6, 59]. Further studies followed like the paper of Bouchard and Touzi [13] and the paper of Gobet, Lemor and Warin [36].

We are going firstly to define probabilistic methods and give a description of their different variants introduced in the literature to deal with the computation of the conditional expectation. We will also present related notions such as Backward Stochastic Differential Equation and second-order Backward Stochastic Differential Equation and the results which link them to PDE problems. We will then present some particular probabilistic schemes [25, 40] that we will improve in Section 3.1.

### 2.3.1 Description of probabilistic methods

When applied to a PDE problem, a probabilistic method needs generally a discretization of the time variable if there is any. Then, it consists in simulating one or many stochastic processes according to time-discretized equations on these processes forming the probabilistic scheme, in order to compute in a backward method the solution of the PDE and sometimes its derivatives. It may involve the approximation of an expectation or a conditional expectation. The expectation is generally approximated by the mean of the simulated values which is known to converge to the theoretical expectation when the number of simulated values increase (Law of Large Numbers). The computation of the conditional expectation is more complicated. In the literature, three main approaches have been developed for this calculus. We describe them in the following subsections.

**2.3.1.a Longstaff and Schwartz method** The Longstaff and Schwartz method is based on the fact that the conditional expectation of a random variable $Y$ with respect to a random variable $X$ verify the following condition : There exists a measurable function $f$ from $\mathbb{R}$ to $\mathbb{R}$ which is square integrable on $\sigma(X)$ such that for all measurable functions g from $\mathbb{R}$ to $\mathbb{R}$ which are square integrable on $\sigma(X)$, $\mathbb{E}[(Y - f(X))g(X)] = 0$ ($\sigma(X)$ denotes the $\sigma$-algebra generated by $X$). Then, $\mathbb{E}[Y \mid X] = f(X)$. Solving this condition is equivalent to solve the following problem : Find a measurable function $f$, square integrable on $\sigma(X)$ such that

$$f = \arg\min_{g \in \mathscr{G}} \mathbb{E}\left[(Y - g(X))^2\right],$$

$\mathscr{G}$ being the set of measurable functions square integrable on $\sigma(X)$.

The Longstaff and Schwartz method consists in considering a particular space of finite dimension instead of the set of measurable functions such that this problem can be solved with the least squares method after replacing the expectation by a mean operator on a finite number of simulations. The conditional expectation is then computed using a simple regression. The space considered for $\mathscr{G}$ is generally the space of polynomials with degrees smaller or equal to a given value $k$. But, as shown by Gobet, Lemor and Warin in [36], the space generated by indicator functions on hypercubes or on a Voronoi partition of the state space can also be used.

**2.3.1.b The Malliavin approach** The Malliavin approach for the computation of the conditional expectation of a random variable $Y$ with respect to another random variable $X$ consists in expressing this conditional expectation as a weighted expectation of $Y$ with the expression of the weight obtained by Malliavin calculus. The obtained expectation can then be computed as a mean value over the simulations which converge to the theoretical expectation according to the law of large numbers.

The intuition here consists in using the fact that :

$$\mathbb{E}\left[Y \mid X = x\right] = \frac{\mathbb{E}\left[Y \delta_x(X)\right]}{\mathbb{E}\left[\delta_x(X)\right]}$$

where $\delta_x$ is the Dirac function at the point $x$. Then in the above expectations, the Dirac function is replaced by expectations with an Heaviside function $H$ times an appropriate weight $\pi$, using Malliavin calculus integration by parts. For more details, see [29] and [12].

Bouchard and Touzi in [13] made an analysis of the error due to the use of the Malliavin approach to approximate the conditional expectation in the particular example of the computation of the solution of a decoupled Forward-Backward stochastic differential equation (presented further in this section). They found that this error grows exponentially with the dimension of the state space. This can be considered as another manifestation of the curse of dimensionality in the convergence of probabilistic method. However, on the contrary of grid based methods such as finite difference methods and Semi-Lagrangian methods that cannot even be implemented in high dimension, probabilistic methods have this advantage to be implementable even in high dimension.

**2.3.1.c The quantization approach** This method is described or used in [6, 59, 33]. After a discretization in time common to all the methods presented above, the quantization approach uses also a discretization in space of the simulated stochastic processes. A space grid is built at any time step and a projection of the simulated stochastic processes is done on the grid. The fact that the method uses a space grid can make it comparable to grid based methods such as finite difference and semi-Lagrangian methods except that, in this case, the space grid is stochastic as it is built using processes simulated values.

If we consider then a process $(Y_s)_s$ and a process $(X_s)_s$ for which we want to compute the conditional expectation $\mathbb{E}[Y_{s_{i+1}} \mid X_{s_i}]$ at any time step $s_i, 1 \leq i \leq N-1$ knowing $Y_{s_N}$ as a function of $X_{s_N}$, the quantization method will first apply a discretization of the state space using simulations of $X_{s_i}$, $1 \leq i \leq N$ and build functions $q_i$ known as quantizers which relate any point in the space to a point of the grid corresponding to time $s_i$ (a Voronoi partition is generally used here). Then, transitional probabilities from any point of the grid corresponding to time $s_i$ to any point of the grid corresponding to time $s_{i+1}$ will be computed using statistics on simulations of the process $(X_{s_k})_{1\leq k\leq N}$. Afterwards, values of the conditional expectation $\mathbb{E}[Y_{s_N} \mid X_{s_{N-1}}]$ are then deduced as the sum of the values of $Y_{s_N}$ expressed at the points of the grid corresponding to time $s_N$ weighted by the transitional probabilities. In the same way, from the values of $\mathbb{E}[Y_{s_N} \mid X_{s_{N-1}}]$ at the grid points corresponding to time $s_{N-1}$, the values of $\mathbb{E}[Y_{s_{N-1}} \mid X_{s_{N-2}}]$ can be deduced and so on.

### 2.3.2 Backward Stochastic Differential Equations

The problem we presented above is a simple form of a problem of a Backward stochastic differential equation. Let $\{B_t;\ t \in [0,T]\}$ be a standard $d$-dimensional Brownian process defined on a probability space $(\Omega, \mathscr{F}, P)$, and let $\{\mathscr{F}_t\}$ be its natural filtration. Let $Y$ be a $\mathscr{F}_T$ measurable $k$-dimensional random variable. A Backward stochastic differential equation as generalized by Pardoux and Peng in [61], consists in finding a $\mathscr{F}_t$- adapted pair of stochastic processes $(Y_t, Z_t)_{0\leq t\leq T}$ with values respectively in $\mathbb{R}^k$ and $\mathbb{R}^{k\times d}$ which verify the following equation :

$$Y_t + \int_t^T f(s, Y_s, Z_s)ds + \int_t^T g(s, Y_s, Z_s)dB_s = Y$$

where $f$ is a measurable function from $\Omega \times [0,T] \times \mathbb{R}^k \times \mathbb{R}^{k\times d}$ to $\mathbb{R}^k$ which is supposed to be uniformly Lipschitz with respect to the two last variables, and $g$ is a measurable function from $\Omega \times [0,T] \times \mathbb{R}^k \times \mathbb{R}^{k\times d}$ to $\mathbb{R}^{k\times d}$ such that the mapping $z \mapsto g(s,y,z)$ is a bijection for any $(\omega, s, y)$.

This stochastic differential equation is generally linked to a forward stochastic differential equation in order to obtain an equivalence with PDE. The result is a Forward Backward Stochastic Differential Equation (FBSDE). The solution of the forward stochastic differential equation enters then in the dynamics of the Backward stochastic differential equation. If on the other hand, the solution of the Backward Stochastic Differential equation is included in the terms of the forward stochastic differential equation, the FBSDE is coupled. Otherwise, it is uncoupled.

**Example of uncoupled FBSDE from [60]**

$$\begin{cases} dX_s^{t,x} = b(X_s^{t,x})ds + \sigma(X_s^{t,x})dB_s,\ t \leq s \leq T \\ X_t^{t,x} = x, \\ Y_s^{t,x} = g(X_T^{t,x}) - \int_s^T f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})dr - \int_s^T Z_r^{t,x} dB_r,\ t \leq s \leq T \end{cases}$$

where $b$ is a function from $\mathbb{R}^n$ to $\mathbb{R}^n$ of class $C^3$ with all its derivatives up to order 3 bounded, $\sigma$ is a function from $\mathbb{R}^n$ to $\mathbb{R}^{n\times d}$ of class $C^3$ with all its derivatives up to order 3 bounded, $x \in \mathbb{R}^n$, $g \in C_p^3(\mathbb{R}^n, \mathbb{R}^k)$, $f$ is defined from $[0,T]\times\mathbb{R}^n\times\mathbb{R}^k\times\mathbb{R}^{k\times d}$ to $\mathbb{R}^k$ such that for any $s \in [0,T]$, $(x,y,z)\mapsto f(s,x,y,z)$ is of class $C^3$, $x \mapsto f(s,x,0,0)$ is of class $C^3$ and the first partial derivative of $f$ with respect to $y$ and $z$ are bounded on $[0,T]\times\mathbb{R}^n\times\mathbb{R}^k\times\mathbb{R}^{k\times d}$ as well as their derivatives of order one and two with respect to $x$, $y$, $z$. The unknown processes are then the $\mathscr{F}_s$-adapted processes $(X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x})_{s\in[0,T]}$.

The related PDE is then :

$$\frac{\partial W}{\partial t}(t,x) + \mathscr{L}W(t,x) - f(t,x,W(t,x),(D_xW\sigma)(t,x)) = 0,$$
$$W(T,x) = g(x),$$

where $W : [0,T]\times\mathbb{R}^d \to \mathbb{R}^k,\ (t,x)\mapsto (W_1(t,x),\cdots,W_k(t,x))$, and

$$\mathscr{L}W = \begin{pmatrix} LW_1 \\ \vdots \\ LW_k \end{pmatrix},$$

$$L = \frac{1}{2}\sum_{i,j=1}^{d}(\sigma\sigma^\intercal)_{ij}(t,x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(t,x)\frac{\partial}{\partial x_i}.$$

$W(t,x)$ is then identified to $Y_t^{t,x}$, if $W$ is the viscosity solution of this PDE.

**Example of coupled FBSDE from [9]**

$$\begin{cases} X_t = x + \int_0^t b(s,X_s,Y_s)ds + \int_0^t \sigma(s,X_s,Y_s)dB_s \\ Y_t = g(X_T) - \int_t^T f(s,X_s,Y_s,Z_s)ds - \int_t^T Z_s dB_s,\ t\in[0,T] \end{cases}$$

where $b : [0,T]\times\mathbb{R}^d\times\mathbb{R}\to\mathbb{R}^d$, $\sigma : [0,T]\times\mathbb{R}^d\times\mathbb{R}\to\mathbb{R}^{d\times d}$, $f : [0,T]\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^d\to\mathbb{R}$ and $g : \mathbb{R}^d\to\mathbb{R}$ are deterministic and Lipschitz continuous functions of linear growth which are additionally supposed to satisfy some weak coupling or monotonicity condition.

The related PDE is then :

$$\begin{cases} \frac{\partial W}{\partial t}(t,x) + \frac{1}{2}\operatorname{Tr}\left((\sigma\sigma^\intercal)(t,x,W(t,x))D_x^2W(t,x)\right) \\ \qquad + D_xW(t,x)\cdot b(t,x,W(t,x)) - f(t,x,W(t,x),D_xW(t,x)\sigma(t,x,W(t,x))) = 0, \\ W(T,x) = g(x), \end{cases}$$

and we have $Y_t = W(t,X_t)$ and $Z_t = D_xW(t,X_t)\sigma(t,x,W(t,X_t))$, if $W$ denotes the viscosity solution of this PDE.

Uncoupled FBSDEs can be related to semilinear PDE problems while coupled FBSDEs can be related to quasi-linear PDE problems (see [60], [62], [63]). However, the Hamilton-Jacobi-Bellman equation and the Hamilton-Jacobi-Bellman-Isaacs equation encountered in

Section 1.1 in the stochastic case are either semilinear when the diffusion coefficients do not depend on the controls or fully nonlinear.

To be able to deal with fully nonlinear PDE problems, Cheridito, Soner, Touzi and Victoir introduced the notion of second-order Backward Stochastic Differential Equation. Let $(\mathbb{F}^{t,T}) = (\mathscr{F}^t_s)_{s\in[t,T]}$ be the augmented filtration generated by $(B_s - B_t)_{s\in[t,T]}$. One example of this differential equation consists in finding a quadruple $\mathbb{F}^{t,T}$-adapted process $(Y^{t,x}_s, Z^{t,x}_s, \Gamma^{t,x}_s, A^{t,x}_s)$ taking values in $\mathbb{R}\times\mathbb{R}^d\times\mathbb{S}(d)\times\mathbb{R}^d$ such that given functions $f, g$ and a process $X^{t,x}$ strong solution to the forward stochastic differential equation :

$$\begin{cases} dX^{t,x}_s = b(X^{t,x}_s)ds + \sigma(X^{t,x}_s)dB_s \\ X^{t,x}_t = x \end{cases}$$

with $b:\mathbb{R}^d\to\mathbb{R}^d$ and $\sigma:\mathbb{R}^d\to\mathscr{M}^d_{inv}$ ($\mathscr{M}^d_{inv}$ : set of invertible matrices of $\mathbb{R}^{d\times d}$) Lipschitz of Lipschitz constant smaller or equal to $K$ and such that

$$|b(x)| + |\sigma(x)| \le K(1+|x|^{p_1}),$$

$p_1 \in [0,1]$, we have

$$\begin{cases} dY^{t,x}_s = f(s, X^{t,x}_s, Y^{t,x}_s, Z^{t,x}_s, \Gamma^{t,x}_s)ds + Z^{t,x}_s\cdot dX^{t,x}_s + \frac{1}{2}\operatorname{Tr}\left(\sigma(X^{t,x}_s)\sigma^{\intercal}(X^{t,x}_s)\Gamma^{t,x}_s\right)ds,\ s\in[t,T), \\ dZ^{t,x}_s = A^{t,x}_s ds + \Gamma^{t,x}_s dX^{t,x}_s,\ s\in[t,T), \\ Y_T = g(X_T). \end{cases} \tag{2.18}$$

where $f:[0,T)\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{S}(d)\to\mathbb{R}$ and $g:\mathbb{R}^d\to\mathbb{R}$ are continuous functions.

This stochastic differential equation is related to a fully nonlinear PDE problem according to the following theorem of Cheridito, Soner, Touzi and Victoir in [17].

**Theorem 2.3.1** ([17, Theorem 4.10]). *Let us consider the second order Backward Stochastic Differential Equation (2BSDE)* (2.18) *and the following PDE problem :*

$$-\frac{\partial W}{\partial t}(s,x) + f(s,x,W(s,x),D_xW(s,x),D^2_xW(s,x)) = 0, \quad \text{on } [0,T)\times\mathbb{R}^d, \tag{2.19}$$

$$W(T,x) = g(x), \quad x\in\mathbb{R}^d. \tag{2.20}$$

*Consider the class $\mathscr{A}^{t,x}_m$ ($(t,x)\in[0,T]\times\mathbb{R}^d$ and $m\ge 0$) of processes $(Z_s)_s$ such that :*

$$Z_s = z + \int_t^s A_r dr + \int_t^s \Gamma_r dX^{t,x}_r, \quad s\in[t,T],$$

*where $z\in\mathbb{R}^d$, $(A_s)_{s\in[t,T]}$ is an $\mathbb{R}^d$-valued, $\mathbb{F}^{t,T}$-progressively measurable process, $(\Gamma_s)_{s\in[t,T]}$ is an $\mathbb{S}(d)$- valued, $\mathbb{F}^{t,T}$-progressively measurable process such that :*

$$\max\{|Z_s|,|A_s|,|\Gamma_s|\} \le m(1+|X^{t,x}_s|^{p_4}), \quad \forall s\in[t,T]$$

*and*

$$|\Gamma_r - \Gamma_s| \le m(1+|X^{t,x}_r|^{p_5}+|X^{t,x}_s|^{p_5})(|r-s|+|X^{t,x}_r - X^{t,x}_s|)$$

*for all $r,s\in[t,T]$, where $p_4$ and $p_5$ are fixed positive constants.*

*Set $\mathscr{A}^{t,x} = \cup_{m\ge0}\mathscr{A}^{t,x}_m$ and consider the following assumptions.*

- ***A1*** *: For every $N \geq 1$, there exists a constant $F_N$ such that*

$$|f(t,x,y,z,\gamma) - f(t,x,\tilde{y},z,\gamma)| \leq F_N |y - \tilde{y}|$$

*for all $(t,x,y,\tilde{y},z,\gamma) \in [0,T) \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}(d)$ with*

$$\max\{|x|,|y|,|\tilde{y}|,|z|,|\gamma|\} \leq N.$$

- ***A2*** *: There exists constants $F$ and $p_2 \geq 0$ such that*

$$|f(t,x,y,z,\gamma)| \leq F(1 + |x|^{p_2} + |y| + |z|^{p_2} + |\gamma|^{p_2})$$

*for all $(t,x,y,z,\gamma) \in [0,T) \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}(d)$.*

- ***A3*** *: There exists constants $G$ and $p_3 \geq 0$ such that*

$$|g(x)| \leq G(1 + |x|^{p_3})$$

*for all $x \in \mathbb{R}^d$.*

- ***A4*** *: For all $(t,x,y,z) \in [0,T) \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d$ and $\gamma, \tilde{\gamma} \in \mathbb{S}(d)$,*

$$f(t,x,y,z,\gamma) \geq f(t,x,y,z,\tilde{\gamma}) \quad \text{whenever } \gamma \leq \tilde{\gamma}.$$

- ***A5*** *: For all $t \in [0,T)$, the PDE (2.19) with the terminal condition (2.20) satisfies the comparison principle on $[t,T] \times \mathbb{R}^n$ in the space of functions with a growth exponent $p = \max\{p_2, p_3, p_2 p_4, p_4 + 2p_1\}$, this space of functions being the set of functions $V$ such that :*

$$|V(s,x)| \leq C(1 + |x|^p) \quad \forall (s,x) \in [t,T) \times \mathbb{R}^d.$$

*If there exists $(t,x) \in [0,T) \times \mathbb{R}^d$ such that the 2BSDE (2.18) corresponding to $(X^{t,x}, f, g)$ has a solution $(Y^{t,x}, Z^{t,x}, \Gamma^{t,x}, A^{t,x})$ with $Z^{t,x} \in \mathscr{A}^{t,x}$, then the PDE (2.19) with the terminal condition (2.20) has a unique viscosity solution $W$ on $[t,T] \times \mathbb{R}^d$ in the space of functions with growth exponent $p = \max\{p_2, p_3, p_2 p_4, p_4 + 2p_1\}$, $W$ is continuous on $[t,T] \times \mathbb{R}^d$, and the process $Y^{t,x}$ is almost surely of the form*

$$Y_s^{t,x} = W(s, X_s^{t,x}), \quad s \in [t,T]. \tag{2.21}$$

*In particular, $(Y^{s,x}, Z^{s,x}, \Gamma^{s,x}, A^{s,x})$, is the only solution of the 2BSDE (2.18) corresponding to $(X^{t,x}, f, g)$ with $Z^{s,x} \in \mathscr{A}^{s,x}$.*

The above theorem is mainly used to compute the solution of a PDE problem using equality (2.21). A time discretization approximation of the process $(Y_s^{t,x})_{s \in [t,T]}$ is then computed by using a probabilistic scheme. Cheridito, Soner, Touzi and Victoir, proposed in [17] such a probabilistic scheme that we present below.

Set

$$-F(t,x,y,z,\gamma) = f(t,x,y,z,\gamma) + b(x) \cdot z + \frac{1}{2} \operatorname{Tr}[\sigma(x)\sigma(x)^\intercal \gamma].$$

Let $s_n := t + n(T-t)/N$ and $(Y_{s_n}^N)_n$, $(Z_{s_n}^N)_n$, $(\Gamma_{s_n}^N)_n$ be approximations respectively of $Y^{t,x}$, $Z^{t,x}$, $\Gamma^{t,x}$. The probabilistic scheme in [17] consists in computing $(Y_{s_n}^N)_n$, $(Z_{s_n}^N)_n$, $(\Gamma_{s_n}^N)_n$ using the following (implicit) equations :

$$Y_T^N := g(X_T^{t,x}), \qquad Z_T^N := Dg(X_T^{t,x})$$

and for $n = 1, ..., N$,

$$\begin{aligned}
Y^N_{s_{n-1}} &:= \mathbb{E}[Y^N_{s_n} \mid X^{t,x}_{s_{n-1}}] + F(s_{n-1}, X^{t,x}_{s_{n-1}}, Y^N_{s_{n-1}}, Z^N_{s_{n-1}}, \Gamma^N_{s_{n-1}})(s_n - s_{n-1}) \\
Z^N_{s_{n-1}} &:= \frac{1}{s_n - s_{n-1}}(\sigma(X^{t,x}_{s_{n-1}})^\intercal)^{-1}\mathbb{E}[(B_{s_n} - B_{s_{n-1}})Y^N_{s_n} \mid X^{t,x}_{s_{n-1}}], \\
\Gamma^N_{s_{n-1}} &:= \frac{1}{s_n - s_{n-1}}\mathbb{E}[Z^N_{s_n} \cdot (B_{s_n} - B_{s_{n-1}}) \mid X^{t,x}_{s_{n-1}}]\sigma(X^{t,x}_{s_{n-1}})^{-1}.
\end{aligned}$$

This probabilistic scheme has been the subject of a deeper study in [25]. This study will be presented in the next subsection.

### 2.3.3 Probabilistic scheme for Fully NonLinear PDEs

Fahim, Touzi and Warin in [25] showed that the scheme described at the end of the previous subsection, can be introduced like a finite difference scheme, where partial derivatives are approximated by using probabilistic expressions. The convergence of the scheme can then be proved without using the theory of second order Backward Stochastic Differential Equations, but by using instead theorems related to finite difference schemes such as the Theorem 2.1.1 due to Barles and Souganidis.

Let us consider the PDE of equation (2.19)-(2.20) with the standard $d$-dimensional Brownian motion and the probability space $(\Omega, \mathscr{F}, P)$ introduced in the previous subsection.

As shown in [25], the first hidden step in the scheme of Cheridito and al in [17] is to decompose the function $f$ in two parts, a linear part and a nonlinear part such that the PDE (2.19) can be written as :

$$-\mathscr{L}^X W - F(\cdot, W, D_x W, D^2_x W) = 0, \tag{2.22}$$

with

$$[\mathscr{L}^X \phi](t,x) := \frac{\partial \phi}{\partial t}(t,x) + \underline{b}(t,x) \cdot D_x\phi(t,x) + \frac{1}{2}\,\mathrm{Tr}(\underline{\sigma}(t,x)\underline{\sigma}^\intercal(t,x) D^2_x \phi(t,x)).$$

The scheme can then be introduced by using the fact that $\mathscr{L}^X$ is the infinitesimal generator of a computable diffusion process $X$ and that by the PDE (2.22), $\mathscr{L}^X W = -F(\cdot, W, D_x W, D^2_x W)$. It remains to approximate the function $W$ and its derivatives $D_x W$, $D^2_x W$ used with the function $F$. By using Hermite polynomials and a differentiation in the sense of distributions, this is done in [25] through the result recalled in Lemma 2.3.1. We introduce an Euler discretization $\hat{X}^{t,x}_h := x + \underline{b}(t,x)h + \underline{\sigma}(t,x)(B_{t+h} - B_t)$ of the process $X$ and for $k = 0, 1, 2$, the operators $D^k_x W$ are replaced by the operators

$$\mathscr{D}^k_h W(t,x) := \mathbb{E}[D^k_x W(t+h, \hat{X}^{t,x}_h)].$$

For a given function $\Psi$ on $\mathbb{R}^d$, we let :

$$\mathscr{D}^0_{t,h}[\Psi](x) = \mathbb{E}\left[\Psi(\hat{X}^{t,x}_h)\right], \tag{2.23}$$

$$\mathscr{D}^1_{t,h}[\Psi](x) = \mathbb{E}\left[\Psi(\hat{X}^{t,x}_h)(\underline{\sigma}(t,x)^\intercal)^{-1}\frac{B^t_h}{h}\right], \tag{2.24}$$

$$\mathscr{D}^2_{t,h}[\Psi](x) = \mathbb{E}\left[\Psi(\hat{X}^{t,x}_h)(\underline{\sigma}(t,x)^\intercal)^{-1}\frac{B^t_h B^{t\,\intercal}_h - hI}{h^2}\underline{\sigma}(t,x)^{-1}\right], \tag{2.25}$$

where $B^t_h = B_{t+h} - B_t$, and $I$ is the identity $d \times d$ matrix.

**Lemma 2.3.1** ([25, Lemma 2.1]). *For every function $\phi : (0,T] \times \mathbb{R}^d \to \mathbb{R}$ with exponential growth, we have :*

$$\mathscr{D}_h^0 \phi(t,x) = \mathscr{D}_{t,h}^0[\phi(t+h,\cdot)](x), \tag{2.26}$$
$$\mathscr{D}_h^1 \phi(t,x) = \mathscr{D}_{t,h}^1[\phi(t+h,\cdot)](x), \tag{2.27}$$
$$\mathscr{D}_h^2 \phi(t,x) = \mathscr{D}_{t,h}^2[\phi(t+h,\cdot)](x), \tag{2.28}$$

For a positive integer $n$, $h := T/n$, $t_i = ih$, $i = 0, \cdots, n$, the scheme is then :

$$W^h(t_n, x) = g(x), \quad x \in \mathbb{R}^d,$$
$$W^h(t_i, x) = \mathbb{T}_{t_i,h}[W^h(t_{i+1},\cdot)](t_i, x), \quad x \in \mathbb{R}^d \; i = 0, \cdots, n-1,$$

where $\mathbb{T}_{t,h}$ is defined by

$$\mathbb{T}_{t,h}[\Psi](t,x) := \mathbb{E}[\Psi(\hat{X}_h^{t,x})] + hF(t,x,\mathscr{D}_{t,h}^0[\Psi](x), \mathscr{D}_{t,h}^1[\Psi](x), \mathscr{D}_{t,h}^2[\Psi](x)), \tag{2.29}$$

for $\Psi : \mathbb{R}^d \to \mathbb{R}$ with exponential growth.

Using, Barles and Souganidis theorem, Fahim, Touzi and Warin obtained the following convergence result :

**Theorem 2.3.2** ([25, Theorem 3.6]). *Assume the following on $F$ :*

1. *The nonlinearity $F$ is Lipschitz continuous with respect to $(x,r,p,\gamma)$ uniformly in $t$ and $|F(\cdot,\cdot,0,0,0)|_0 < \infty$;*
2. *$F$ is elliptic and dominated by the diffusion of the linear operator $\mathscr{L}^X$, that is :*

$$D_\gamma F \geq 0 \tag{2.30}$$
$$\mathrm{Tr}[(\underline{\sigma\sigma}^\intercal)^{-1} D_\gamma F] \leq 1 \tag{2.31}$$

   *on $[0,T) \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}(d)$, where $D_\gamma F$ represents the partial gradient of $F$ with respect to its last variable $\gamma$;*

3. *$D_p F \in \mathrm{Image}(D_\gamma F)$ and $|D_p F D_\gamma F^- D_p F|_0 < \infty$ where $D_p F$ is the partial gradient of $F$ with respect to its fourth variable and $D_\gamma F^-$ denotes the pseudo-inverse of $D_\gamma F$.*

*Assume also that $|\underline{b}|_1, |\underline{\sigma}|_1 < \infty$ and $\underline{\sigma}$ is invertible. Also, assume that the fully nonlinear PDE (2.19) has a comparison principle for bounded functions. Then for every bounded Lipschitz function $g$, there exists a bounded function $W$ such that $W^h \to W$ locally uniformly. In addition, $W$ is the unique bounded viscosity solution of problem (2.19)- (2.20).*

One of the most strong hypothesis in the above theorem is the second hypothesis on $F$. It comes from the monotonicity requirement of the scheme.

$$D_\gamma F = -f_\gamma - \frac{1}{2}\underline{\sigma\sigma}^\intercal,$$

where $f_\gamma$ is the partial gradient of $f$ from PDE 2.19 with respect to its last variable. So,

$$\begin{aligned}\mathrm{Tr}[(\underline{\sigma\sigma}^\intercal)^{-1} D_\gamma F] \leq 1 &\Leftrightarrow \mathrm{Tr}\left[(\underline{\sigma\sigma}^\intercal)^{-1}(-f_\gamma - \frac{1}{2}\underline{\sigma\sigma}^\intercal)\right] \leq 1 \\ &\Leftrightarrow \mathrm{Tr}\left[-(\underline{\sigma\sigma}^\intercal)^{-1} f_\gamma - \frac{1}{2} I\right] \leq \mathrm{Tr}\left[\frac{1}{d} I\right].\end{aligned}$$

The last inequality holds if

$$-\underline{\sigma}^{-1} f_\gamma (\underline{\sigma}^\intercal)^{-1} - \frac{1}{2} I \leq \frac{1}{d} I. \tag{2.32}$$

The inequalities here are to be understood in the sense of the Loewner order. As $D_\gamma F$ should also be nonnegative,

$$0 \leq -\underline{\sigma}^{-1} f_\gamma (\underline{\sigma}^\intercal)^{-1} - \frac{1}{2} I \leq \frac{1}{d} I.$$

That give us

$$\underline{\sigma\sigma}^\intercal \leq -2 f_\gamma \leq (1 + \frac{2}{d}) \underline{\sigma\sigma}^\intercal. \tag{2.33}$$

The inequality (2.32) is necessary if all the eigenvalues of $\underline{\sigma}^{-1} f_\gamma (\underline{\sigma}^\intercal)^{-1}$ are equal.

When the dimension $d$ increases, condition (2.33) reduces the non linearity character of $f$ with respect to its last variable $\gamma$ and makes the PDE more close to a quasi-linear PDE than to a fully non linear PDE as encountered in optimal control problems.

Guo, Zhang and Zhuo tried to improve this condition in a subsequent work ([40]) by using instead of a Brownian motion in the above scheme, a trinomial tree with a new kernel for the Hessian approximation. They manage to allow greater variations in the diagonal terms of $f_\gamma$. However, the other terms of $f_\gamma$ must remain almost constant in great dimension.

We note that the probabilistic scheme of Fahim, Touzi and Warin in [25] is equivalent if we replace the Brownian motion $B$ by a binomial or trinomial random walk, to a finite difference scheme. The condition (2.31) is then a stronger than the usual form of the CFL condition which is materialized in dimension greater or equal to 2 by the need to have a diffusion coefficient matrix diagonally dominant. We give more details about this observation in the following discussion.

**Discussion on the scheme of [25] and the condition** (2.31)

Let us first show on examples the behavior of the discretization of [25]. For this, we shall show what happen when the increments of the Brownian motion $B_{t+h} - B_t$ are replaced by any finite valued independent random variables with same law. This will allow us in particular to compare the discretization of [25] with finite difference schemes. Similar comparisons were done in [25] but here we shall discuss in addition the meaning of the constraint (2.31) in this situation that we call here the *critical constraint*.

To simplify the comparison, we drop the dependence of PDE terms in $t$ and consider the case where $f$ is linear and depends only on $\gamma$:

$$f(x, r, p, \gamma) = -\frac{1}{2} \operatorname{Tr}(A\gamma)$$

where $A$ is a $d$-dimensional symmetric positive definite matrix. We assume that $A \geq I$ and choose $\mathscr{L}^X(x, p, \gamma) = \frac{1}{2} \operatorname{Tr}(\gamma)$, that is $\underline{b} \equiv 0$ and $\underline{\sigma} \equiv I$. Hence, $F(x, r, p, \gamma) = \frac{1}{2} \operatorname{Tr}((A - I)\gamma)$.

Then denoting by $N$ any $d$-dimensional normal random variable, we get that the operator $\mathbb{T}_{t,h}$ of (2.29) satisfies:

$$\begin{aligned} \mathbb{T}_{t,h}(\Psi)(x) &= \mathscr{D}_h^0(\Psi)(x) + h \frac{1}{2} \operatorname{Tr}((A - I) \mathscr{D}_h^2(\Psi)(x))) \\ &= \mathbb{E}\left( \Psi(x + \sqrt{h} N)(1 + \frac{1}{2} \operatorname{Tr}((A - I)(N N^\intercal - I))) \right) . \end{aligned} \tag{2.34}$$

This operator is linear, and it is thus order preserving if and only if for almost all values of $N$ the coefficient of $\Psi(x+\sqrt{h}N)$ inside the expectation, that is $(1+\frac{1}{2}\operatorname{Tr}((A-I)(NN^{\mathsf{T}}-I)))$, is nonnegative. The critical constraint $\operatorname{Tr}[(\underline{\sigma\sigma}^{\mathsf{T}})^{-1}D_\gamma F]\leq 1$ is equivalent here to $\frac{1}{2}\operatorname{Tr}(A-I)\leq 1$. This corresponds exactly to the condition that the coefficient of $\Psi(x)$ inside the expectation is nonnegative. Thus, if $N$ is replaced by any random variable taking a finite number of values including 0, the critical constraint is necessary.

Consider the dimension $d=1$ and a simple discretization of $N$ by the random variable taking the values $\pm\nu$ with probability $1/(2\nu^2)$ and the value 0 with probability $1-1/\nu^2$, where $\nu>1$. Then, we obtain

$$\mathbb{T}_{t,h}(\Psi)(x)=\Psi(x)+\frac{b}{2\nu^2}\left(\Psi(x+\sqrt{h}\nu)+\Psi(x-\sqrt{h}\nu)-2\Psi(x)\right)\ , \tag{2.35}$$

with $b=1+\frac{1}{2}(A_{11}-1)(\nu^2-1)$. This scheme is equivalent to an explicit finite difference discretization of (2.19) with a space step $\Delta x=\sqrt{h}\nu$, which is consistent with (2.19) if and only if $b=A_{11}$ and so if and only if $\nu=\sqrt{3}$. In that case, the critical condition $\frac{1}{2}(A_{11}-1)\leq 1$ is necessary for the scheme to be monotone and it is equivalent to the CFL condition $A_{11}h\leq(\Delta x)^2$.

For finite difference schemes, the CFL condition can be satisfied by increasing $\Delta x$. However, here $\Delta x$ is stongly connected to the possible values of $N$ and since the probability of large $N$ is small, one cannot avoid the critical constraint if we keep the discretization (2.28) of $D_x^2W$.

We can note that in this case the expression of $\mathscr{D}^1_{t,h}(\Psi)(x)$ is

$$\mathscr{D}^1_{t,h}(\Psi)(x)=\frac{\Psi(x+\sqrt{h}\nu)-\Psi(x-\sqrt{h}\nu)}{2\sqrt{h}\nu},$$

which corresponds to a centered finite discretization.

Now, consider the dimension $d=2$, and the simple discretization of $N$ where each entry of $N=(N_i)_{i=1,\ldots,d}$ is replaced by a random variable as above, taking the values $\pm\sqrt{3}$ with probability $1/6$ and the value 0 with probability $2/3$. In that case, the critical constraint $\frac{1}{2}(A_{11}+A_{22}-2)\leq 1$ is necessary and sufficient for the discretization to be monotone. We have

$$\begin{aligned}\mathbb{T}_{t,h}(\Psi)(x)=&\mathbb{E}\left(\Psi(x+\sqrt{h}N)(1+\frac{1}{2}\sum_{i,j=1}^{2}(A_{ij}-\delta_{ij})(N_iN_j-\delta_{ij}))\right)\\=&\Psi(x)\frac{2}{9}(2-\operatorname{Tr}(A-I))\\&+\frac{1}{18}\sum_{i=1}^{2}\sum_{\epsilon=\pm1}\Big(\Psi(x+\sqrt{3h}\epsilon e_i)(3(A_{ii}-1)+2-\operatorname{Tr}(A-I))\Big)\\&+\frac{1}{72}\sum_{\epsilon_1=\pm1,\epsilon_2=\pm1}\Big(\Psi(x+\sqrt{3h}(\epsilon_1e_1+\epsilon_2e_2))\\&\qquad\Big(3\Big(\sum_{i,j=1}^{2}(A_{ij}-\delta_{ij})\epsilon_i\epsilon_j\Big)+2-\operatorname{Tr}(A-I)\Big)\Big)\ .\end{aligned}$$

where $(e_1, e_2)$ is the canonical basis of $\mathbb{R}^2$. This discretization can be rewritten as

$$\mathbb{T}_{t,h}(\Psi)(x) = \Psi(x) + \frac{h}{2}\left(\left(\sum_{i,j=1}^{2}(A_{ij} - \delta_{ij}b)D^h_{ij}\Psi(x)\right) + b\Delta^h\Psi(x)\right) \quad ,$$

where $b = (1 + \mathrm{Tr}(A - I))/3$, $D^h_{ij}\Psi$ is the standard 5-point stencil discretization of the partial derivative $\frac{\partial^2\Psi}{\partial x_i \partial x_j}$ on the grid with space step $\Delta x = \sqrt{3h}$ (as above), and $\Delta^h\Psi$ is the discretization of $\Delta\Psi$ using the external vertices of the 9-point stencil (that is the points $x + \Delta x(\pm e_1 + \pm e_2)$). We point out the similarity with the Kushner scheme here. Note also that the critical constraint $\mathrm{Tr}(A - I) \leq 2$ implies $b \leq 1$. Moreover, since $A - I$ is positive semidefinite, then $2|A_{12}| \leq \mathrm{Tr}(A - I) \leq 2$ and $A_{ii} \geq 1$, so $|A_{12}| \leq A_{ii}$ for $i = 1, 2$. The latter condition means that the matrix is diagonally dominant.

In general, we see that the critical condition (2.31) comes from the CFL condition. As the ratio between the equivalent space grid step and the time grid step is fixed, the probabilistic scheme cannot be applied in any case.

We saw that the scheme in [25] is similar to a centered finite difference discretization for the gradient and a Kushner discretization for the Hessian. We improve it in Section 3.1 by introducing a sort of probabilistic upwind scheme to deal with terms of the PDE (2.19) in $D_xW$ and an equivalent in probabilistic world of large stencil schemes such as the scheme of Bonnans and Zidani (see Section 2.1) to deal with terms in $D^2_xW$. This scheme is used in a general new technique combining probabilistic methods to max-plus methods which will be presented in the next sections.

## 2.4 Max-plus methods

Max-plus methods were introduced in the optimal control theory to exploit the max-plus linearity of the semigroups associated to Hamilton-Jacobi-Bellman equations of the first order. Indeed, even if the HJB equation is non linear, optimal control problems can be seen as linear in the max-plus algebra. The first methods proposed in the literature were subject to the curse of dimensionality because they needed the construction of a basis of functions over the state space as in finite element method and thus needed a discretization in space. A basis of functions over the state space was fixed and the value function was projected on this basis at each time step using various techniques.

In 2007, McEneaney proposed in the paper [52] a new max-plus approach where there is no need to project the value function on a fixed basis of functions at each time step. In fact, the value function is still expressed as a linear combination of functions in the max-plus sense, but these functions appear naturally at each time step. The curse of dimensionality of old max-plus approaches disappeared but a new curse appeared called the curse of complexity. This new approach was later extended to the stochastic case with the resolution of HJB equations of the second order ([43, 54, 44]).

We will first define what the max-plus algebra is, then give a brief description of old max-plus approaches before presenting in the deterministic and the stochastic case the curse-of-dimensionality free max-plus approach developed by McEneaney.

### 2.4.1 Definition of Max-plus algebra

The max-plus algebra consists in defining on $\mathbb{R}$ the max operator or a min operator as an addition and the usual addition as a multiplication. The properties of these operators on $\mathbb{R}$ make that they can be used to build a new structure of semi-ring. Indeed, the set $\mathbb{R} \cup \{-\infty\}$ denoted as $\mathbb{R}_{\max}$ equipped with the max operator and the usual addition respectively as "addition" and "multiplication" operators is a semiring called the *max-plus semiring* in the same way as the set $\mathbb{R} \cup \{+\infty\}$ denoted by $\mathbb{R}_{\min}$ equipped with the min operator and the usual addition respectively as "addition" and "multiplication" operators. The last one is called the *min-plus semiring*. The set of functions (resp. upper bounded, continuous, lower semicontinuous, lower semicontinuous and convex) from a subset $\Omega$ of $\mathbb{R}^N$ to $\mathbb{R}_{\max}$ is then a $\mathbb{R}_{\max}$- semimodule while the set of functions (resp. lower bounded, continuous, upper semicontinuous, upper semicontinuous and concave) from $\Omega$ to $\mathbb{R}_{\min}$ is a $\mathbb{R}_{\min}$-semimodule. Using this, the scalar product is redefined on these semimodules in the following way :

$$< f, g >= \sup_{x \in \Omega} f(x) + g(x) \text{ for } f, g \in \mathbb{R}^{\Omega}_{max},$$
$$< f, g >= \inf_{x \in \Omega} f(x) + g(x) \text{ for } f, g \in \mathbb{R}^{\Omega}_{min}.$$

### 2.4.2 Max-plus finite element type approaches

The max-plus approaches for solving first order HJB equations rely on the max-plus linearity of the related Lax-Oleinik semigroup or evolution semigroup. For a deterministic optimal control problem with finite horizon, the Lax-Oleinik semigroup or evolution semigroup is the operator which maps for each horizon $T$, the terminal payoff function $\Psi$ to the value function at time 0. Its expression for a maximization goal is given by

$$S_T[\Psi](x) = \sup_u \left\{ \int_0^T L(x(s), u(s))ds + \Psi(x(T)) \right\}$$
$$\frac{dx(s)}{ds} = f(x(s), u(s)), \quad x(0) = x, \quad x \in \mathbb{R}^d, \quad u(s) \in U.$$

It is a semigroup as for any $t, s \geq 0$, $S_{t+s} = S_t \circ S_s$, using the Dynamic Programming Principle. This property allows to write for a maximization problem, that for a given small time horizon $\delta$, $W(t-\delta, \cdot) = S_\delta(W(t, \cdot))$ where $W$ is the value function.

As we said in the introduction of this subsection, another good property of the Lax-Oleinik semigroup is its max-plus linearity. In fact :

$$S_T[\sup\{f, g\}] = \sup\{S_T[f], S_T[g]\}; \quad S_T[f+\lambda] = S_T[f] + \lambda; \quad \lambda \in \mathbb{R}_{max},$$

where $f, g$ are 2 functions from $\mathbb{R}^d$ to $\mathbb{R}_{\max}$, $\sup\{f, g\}$ is the function obtained by taking the pointwise maximum of $f$ and $g$, and $f + \lambda$ is defined on $\mathbb{R}^d$ such that for $x \in \mathbb{R}^d$, $(f+\lambda)(x) = f(x) + \lambda$. The same observations can be made for a problem with a minimization goal where the max operator and $\mathbb{R}_{\max}$ are replaced by a min operator and $\mathbb{R}_{\min}$.

One approach presented by Fleming and McEneaney in [27] consisted in approximating the terminal payoff function as a linear max-plus combination of some basis functions $(g_i)_i$. Then, the value function was computed inductively by using $S_\delta$ for small time horizon $\delta$, considered as the time step. $S_\delta$ is approximated by a max-plus linear operator over the max-plus semimodule generated by basis functions $(g_i)_i$. The expression of the value function after each time step $\delta$, needed then only the max-plus projection of $(S_\delta(g_i))_i$ on the basis functions $(g_i)_i$ and a max-plus matrix multiplication.

Another approach presented by Akian, Gaubert and Lakhoua in [4] improved the approach presented above by introducing another basis of test functions $(z_j)_j$ such that the following conditions are satisfied :

$$< z_j, \tilde{W}(t-\delta, \cdot) > \leq < z_j, S_\delta(\tilde{W}(t, \cdot)) >, \quad \forall j, \tag{2.36}$$

where $\tilde{W}$ is the approximated value function in this approach.

The highest function $\tilde{W}$ satisfying these conditions is chosen. These conditions introduce a new requirement in the computation of the value function $W$ when the basis functions $(z_j)_j$ are different from the basis functions $(g_i)_i$. It then helps to improve the approximation error of the method. The drawback of this method is that when the functions $(g_i)_i$ and $(z_j)_j$ are not Dirac functions, the condition (2.36) can be interpreted as a zero-sum deterministic game which is not max-plus linear anymore.

All these approaches rely on the choice of basis functions which requires a discretization of the state space as in finite difference methods. It is thus impossible to avoid the curse of dimensionality.

### 2.4.3 Curse of dimensionality free approach of McEneaney in deterministic case

The use of max-plus basis functions in the max-plus methods presented in the above subsection, leads to a particular type of max-plus methods called the max-plus basis methods.

McEneaney introduced a new category of max-plus methods by exploiting the good properties of linear quadratic deterministic optimal control problems which are among the few type of deterministic optimal control problems having a classical solution analytically computable.

Let consider an Hamilton-Jacobi-Bellman equation in the deterministic case with an Hamiltonian $\mathscr{H}$ such that the PDE is written as follows :

$$-\frac{\partial W}{\partial t}(t,x) + \mathscr{H}(t,x,D_xW(t,x)) = 0 \tag{2.37}$$

$$W(T,x) = \Psi(x) \tag{2.38}$$

Let us suppose that the objective of the related optimal control problem is to maximize an overall functional payoff so that $-\mathscr{H}(t,x,D_xW(t,x))$ is a maximum of terms. The first step in the McEneaney approach is to decompose $-\mathscr{H}$ in a max-plus sum as follows :

$$-\mathscr{H}(t,x,D_xW(t,x)) = \max_{m\in\{1,\dots,M\}} \{-\mathscr{H}^m(t,x,D_xW(t,x))\} \tag{2.39}$$

where each $\mathscr{H}^m$ $(m \in \{1,...,M\})$ is the Hamiltonian of a linear quadratic optimal control problem and $M$ is called the complexity of the Hamiltonian $\mathscr{H}$.

Let $(S_{t,t+h})_{t\geq 0}$ that we will denote also by $(S_h^t)_{t\geq 0}$ be the backward evolution operator of the PDE (2.37) from time $t+h$ to time $t$ and in the same way, let $(S_h^{t,m})_{t\geq 0}$ be the backward evolution operator of the PDE (2.37) where the Hamiltonian $\mathscr{H}$ is replaced by the Hamiltonian $\mathscr{H}^m$ for $m \in \{1,...,M\}$. By the dynamic programming principle used to obtain the Hamilton-Jacobi-Bellman equation,

$$\begin{aligned} \frac{S_h^t(W(t+h,\cdot)) - W(t,x)}{h} &\approx \left[\frac{\partial W}{\partial t}(t,x) - \mathscr{H}(t,x,D_xW(t,x))\right] \\ \frac{S_h^{t,m}(W(t+h,\cdot)) - W(t,x)}{h} &\approx \left[\frac{\partial W}{\partial t}(t,x) - \mathscr{H}^m(t,x,D_xW(t,x))\right], \quad \text{when } h\to 0 \end{aligned} \tag{2.40}$$

Using equalities (2.39) and (2.40), one deduces that

$$S_h^t(W(t+h,\cdot)) \approx \max_{m\in\{1,\dots,M\}} S_h^{t,m}[W(t+h,\cdot)] \quad \text{when } h\to 0. \tag{2.41}$$

As for an horizon $T$, $S_{0,T} = S_{0,h}\circ\cdots\circ S_{T-h,T}$ where $h$ is chosen small enough and such that $T/h$ is an integer, the approximation (2.41) allows one to deduce that

$$S_T^0 \approx (\max_{m\in\{1,\dots,M\}} S_h^{0,m})\circ\cdots\circ(\max_{m\in\{1,\dots,M\}} S_h^{ih,m})\cdots\circ(\max_{m\in\{1,\dots,M\}} S_h^{(T/h-1)h,m}).$$

Let $N = T/h$. Each $S_h^{ih,m}$ $m \in \{1,...,M\}$, $i \in \{0,...,N-1\}$ being max-plus linear, the composition operator in the above equality is distributive with respect to the max operator. So

$$S_T^0 \approx \max_{(m_{i_1},\dots,m_{i_N})\in\{1,\dots,M\}^N} S_h^{0,m_{i_1}}\circ\ldots\circ S_h^{(N-1)h,m_{i_N}}.$$

The number of terms in this last max-plus summation is exponential with the complexity $M$. So, even if the curse of dimensionality has disappeared, a curse of complexity appears due to the exponential growth with $M$ of the number of terms to compute in order to obtain the final result $S_T^0$. Pruning algorithms have been proposed to eliminate unnecessary functions in this final max-plus summation (see [53], [34], [35]).

Linear quadratic evolution operators operate on quadratic functions. They transform quadratic functions into quadratic functions. So for the method described above to work, the terminal payoff function must be quadratic or a pointwize maximum of quadratic functions. The value function will then also be a pointwize maximum of quadratic functions. The method is said to be idempotent as it preserves the form of the terminal payoff.

### 2.4.4 Curse of dimensionality free approach of McEneaney in stochastic case

In the stochastic case, the evolution operator of the Hamilton-Jacobi-Bellman equation is no more max-plus linear. This complicates the application of the curse of dimensionality free approach of McEneaney as it is applied to the deterministic case.

To see this, let us consider a stochastic optimal control problem with finite horizon $T$ defined as in section 1.1.3 with trajectories $(x_s^{t,x,u})_s$ in $\mathbb{R}^n$. We drop the dependence on the time variable and we consider a maximization problem instead of a minimization problem in order to use the max-plus algebra $\mathbb{R}_{\max}$. The evolution semigroup is $(S_T)_{T\geq 0}$ such that :

$$S_T[\Phi](x) = \sup\left\{\mathbb{E}\left[\int_0^T L(x^{x,u}(s), u(s))ds + \Phi(x^{x,u}(T))\right]\right\}.$$

Because of the expectation operator, the equality $S_T[\sup\{f,g\}] = \sup\{S_T[f], S_T[g]\}$ may not be true for any function $f$ and $g$. This fact compromises the max-plus linearity of the semigroup $(S_T)_{T\geq 0}$.

In [43, 54], McEneaney, Kaise and Han propose another approach specific to the stochastic case. It uses a sort of max-plus distributivity of the expectation operator over the max operator which is summarized in the equality below :

$$\mathbb{E}\left[\max_{z\in Z}\Phi(w,z)\right] = \max_{\tilde{z}\in\bar{Z}}\mathbb{E}\left[\Phi(w,\tilde{z}(w))\right], \tag{2.42}$$

where $w$ represents here a random variable which lives in a subset $W$ of $\mathbb{R}^d$, $Z$ is a possible continuum set and $\bar{Z}$ denotes the set of measurable functions from $W$ to $Z$.

We can see that like the expectation operator, the evolution semigroup in the stochastic case has this property. It is generalized in Theorem 4.1 to any monotone additively $\alpha$-subhomogeneous operator $\mathscr{T}$ in the following way :

$$\mathscr{T}\left[\max_{z\in Z}\Phi(w,z)\right] = \max_{\tilde{z}\in\bar{Z}}\mathscr{T}\left[\Phi(w,\tilde{z}(w))\right].$$

On the contrary of the max-plus linearity, this distributivity property of the evolution semigroup which we will call the probabilistic distributivity, does not maintain the cardinality of elements present in the initial max operator, but it makes it grows at the power of a functional space. The same algorithm as in the deterministic case presented in the previous subsection, can then be applied here except that the probabilistic distributivity of the evolution operators which replace the max-plus linearity, will make the number of terms in the final maximum or minimum very high. This makes the use of a pruning algorithm critical here, but very hard to implement in practice as the set of elements to which is applied this pruning algorithm is a continuum set. In [54], McEneaney, Kaise and Han propose a pruning algorithm based on a first discretization of this continuum set by a finite set using the properties of compact sets. However they do not explicitly explain how to make this first discretization.

In Chapter 4, we will propose a method which uses the above probabilistic max-plus distributivity property mixed with probabilistic methods. This mix removes the need to do this difficult pruning operation.

# CHAPTER 3

# New probabilistic schemes for stochastic control problems

In this chapter, we describe two new probabilistic monotone schemes for the resolution of an Hamilton-Jacobi-Bellman equation in the stochastic case. The first scheme which is simpler, was introduced in [2] while the second, more complex was introduced in [3]. We make the proof of the convergence of these schemes and obtain related error estimates in a bounded setting first by using the results of Barles and Jakobsen (Theorem 2.1.2). We then present our method and the related results to obtain the convergence of the second scheme in an unbounded quadratic growth setting.

## 3.1 New probabilistic monotone schemes in bounded setting

Let us consider the following Hamilton-Jacobi-Bellman equation :

$$-\frac{\partial W}{\partial t} - \inf_{\alpha\in\mathscr{A}} \{L^\alpha[W](t,x) + c^\alpha(t,x)W + f^\alpha(t,x)\} = 0 \text{ in } Q_T := [0,T)\times\mathbb{R}^d, \tag{3.1}$$

$$W(T,x) = \Psi(x) \text{ in } \mathbb{R}^d, \tag{3.2}$$

where $\mathscr{A}$ is a complete metric space, and

$$L^\alpha[W](t,x) = \frac{1}{2}\operatorname{Tr}[a^\alpha(t,x)D_x^2W(t,x)] + b^\alpha(t,x)\cdot D_xW(t,x).$$

The coefficients $a^\alpha, b^\alpha$, $c^\alpha$, $f^\alpha$ and $\Psi$ take values respectively in $\mathbb{S}(d)$, $\mathbb{R}^d$, $\mathbb{R}$, $\mathbb{R}$ and $\mathbb{R}$.

Let $\{B_t;\ t\in[0,T]\}$ be a standard $d$-dimensional Brownian process defined on a probability space $(\Omega,\mathscr{F},P)$.

As Fahim, Touzi and Warin in [25] (see also Section 2.3.3), we introduce functions $\underline{b}$ : $[0,T]\times\mathbb{R}^d\to\mathbb{R}^d$ and $\underline{\sigma}:[0,T]\times\mathbb{R}^d\to\mathbb{R}^{d\times d}$ such that the PDE (3.1) is written as :

$$-\mathscr{L}^X W - F(\cdot, W, D_xW, D_x^2W) = 0, \tag{3.3}$$

with

$$[\mathscr{L}^X\phi](t,x) := \frac{\partial\phi}{\partial t}(t,x) + \underline{b}(t,x)\cdot D_x\phi(t,x) + \frac{1}{2}\operatorname{Tr}(\underline{\sigma}(t,x)\underline{\sigma}^\intercal(t,x)D_x^2\phi(t,x)).$$

$\mathscr{L}^X$ is the infinitesimal generator of a diffusion process $X$ solution of the stochastic differential equation

$$dX_s = \underline{b}(s,X_s)ds + \underline{\sigma}(s,X_s)dB_s.$$

The Hamiltonian $F$ of (3.3) is then :

$$F(t,x,W,D_xW,D_x^2W) = \inf_{\alpha\in\mathscr{A}} \{G^\alpha[W](t,x) + c^\alpha(t,x)W + f^\alpha(t,x)\}, \tag{3.4}$$

where

$$G^\alpha[W](t,x) = \frac{1}{2}\operatorname{Tr}\left[(a^\alpha - \underline{\sigma\sigma}^\intercal)(t,x)D_x^2W(t,x)\right] + (b^\alpha - \underline{b})(t,x)\cdot D_xW(t,x). \tag{3.5}$$

In the following subsection, we build schemes by replacing $W$, $D_xW$, $D_x^2W$ by conditional expectations in the same spirit as in [25]. The resulted schemes are monotone with simple assumptions on the initial PDE (3.1). The results of the following subsection were presented in [2, 3]. We consider a time discretization $\mathscr{T}_h = \{0,h,2h,...,(n-1)h\}$ with $h=T/n$ and the Euler discretization $\hat{X}$ of the diffusion process $X$ given by :

$$\hat{X}(t+h) = \hat{X}(t) + \underline{b}(t,\hat{X}(t))h + \underline{\sigma}(t,\hat{X}(t))(B_{t+h}-B_t) \tag{3.6}$$

We denote by $\bar{\mathscr{T}}_h$, the set $\{0,h,2h,...,nh\}$.

### 3.1.1 Probabilistic approximation of differential operators and their estimates

We first describe the approximation of the second order derivatives proposed in [2] and estimated in [3]. Consider any matrix $\Sigma \in \mathbb{R}^{d\times\ell}$ with $\ell \in \mathbb{N}$ and let us denote by $\Sigma_{.j},\ j=1,\dots\ell$, its columns. We recall that $\mathscr{C}^k_{\mathrm{b}}([0,T]\times\mathbb{R}^d)$ is the subset of functions of $\mathscr{C}^k([0,T]\times\mathbb{R}^d)$ with bounded derivatives up to order $k$. For any $W \in \mathscr{C}^2([0,T]\times\mathbb{R}^d)$, we have

$$\frac{1}{2}\operatorname{Tr}(\underline{\sigma}(t,x)\Sigma\Sigma^{\mathsf{T}}\,\underline{\sigma}^{\mathsf{T}}(t,x)\,D^2_xW(t,x)) \quad = \quad \frac{1}{2}\sum_{j=1}^{\ell}\Sigma^{\mathsf{T}}_{.j}\,\underline{\sigma}^{\mathsf{T}}(t,x)\,D^2_xW(t,x)\underline{\sigma}(t,x)\Sigma_{.j}\ . \tag{3.7}$$

For any integer $k$, consider the polynomial:

$$\mathscr{P}^2_{\Sigma,k}(w) := \sum_{j=1}^{\ell}\|\Sigma_{.j}\|_2^2\left(c_k\left(\frac{[\Sigma^{\mathsf{T}}\,w]_j}{\|\Sigma_{.j}\|_2}\right)^{4k+2} - d_k\right)\quad , \tag{3.8a}$$

with

$$c_k := \frac{1}{(4k+2)\mathbb{E}\left[N^{4k+2}\right]}\ ,\quad d_k := \frac{1}{4k+2}\ , \tag{3.8b}$$

where $N$ is a one dimensional normal random variable, and where we use the convention that the $j$th term of the sum is zero when $\|\Sigma_{.j}\|_2 = 0$. This is the sum of $\mathscr{P}^2_{\Sigma_{.j},k}(w)$ defined for each column $\Sigma_{.j}$ in the same way as $\mathscr{P}^2_{\Sigma,k}(w)$.

Let $W \in \mathscr{C}^4_{\mathrm{b}}([0,T]\times\mathbb{R}^d)$, and $\hat{X}$ as in (3.6), then, under some conditions, the following expression is an approximation of (3.7) with an error in $O(h)$ uniform in $t$ and $x$ [2, Th. 3.1]:

$$h^{-1}\mathbb{E}\left[W(t+h,\hat{X}(t+h))\mathscr{P}^2_{\Sigma,k}(h^{-1/2}(B_{t+h}-B_t)) \mid \hat{X}(t)=x\right]\quad . \tag{3.9}$$

In order to obtain error estimates for the scheme, we need the more precise following result (Th. 3.1.1).

For $p$ and $q$ two integers and $\phi$ a function from $[0,T]\times\mathbb{R}^d$ to $\mathbb{R}$ with partial derivatives up to order $p$ in $t$ and $q$ in $x$, we introduce the following notation :

$$|\partial^p_t D^q\phi| = \sup_{\substack{(t,x)\in[0,T]\times\mathbb{R}^d\\(\beta_i)_i\in\mathbb{N}^d,\sum_i\beta_i=q}}\left|\frac{\partial^{p+q}\phi}{\partial t^p\partial x_1^{\beta_1}\dots\partial x_d^{\beta_d}}(t,x)\right|.$$

**Theorem 3.1.1.** *Let $\hat{X}$ be as in (3.6), and denote $B^t_h = B_{t+h}-B_t$. Consider any matrix $\Sigma \in \mathbb{R}^{d\times\ell}$ with $\ell \le d$ and any integer $k\in\mathbb{N}$. Assume that $\underline{b}$ and $\underline{\sigma}$ are bounded by some constant $C$ uniformly in $t$ and $x$, and let $M$ be an upper bound of $|\Sigma\Sigma^{\mathsf{T}}|$. Then, there exists $K = K(C,M) > 0$ such that, for all $W \in \mathscr{C}^4_b([0,T]\times\mathbb{R}^d)$, we have, for all $(t,x)\in\mathscr{T}_h\times\mathbb{R}^d$,*

$$\begin{aligned}&\Bigg|h^{-1}\mathbb{E}\left[W(t+h,\hat{X}(t+h))\mathscr{P}^2_{\Sigma,k}(h^{-1/2}B^t_h) \mid \hat{X}(t)=x\right]\\&\quad-\frac{1}{2}\operatorname{Tr}(\underline{\sigma}(t,x)\Sigma\Sigma^{\mathsf{T}}\,\underline{\sigma}^{\mathsf{T}}(t,x)\,D^2_xW(t,x))\Bigg|\\&\le K(1+\sqrt{h})^4\Big[h(|\partial^1_tD^2W|+|\partial^0_tD^3W|+|\partial^0_tD^4W|)+\\&\quad h\sqrt{h}|\partial^1_tD^3W|+h^2|\partial^2_tD^2W|+h^2\sqrt{h}|\partial^3_tD^1W|+h^3|\partial^4_tD^0W|\Big]\ .\end{aligned}$$

To do the proof of this theorem, we will need the following results.

**Lemma 3.1.1.** *Let $A \in \mathbb{S}(d)$. Let $U^\intercal DU$ be the decomposition of $A$ after diagonalization, with $U$ being an orthogonal matrix and $D$ a diagonal matrix. Let $k$ be a positive integer. Let us define $c_k > 0$ as in (3.8b) and $\theta_D \in \mathbb{R}^d$, $K_A \in \mathbb{R}$, and the polynomial $\phi_A$ as follows :*

- $[\theta_D]_i \coloneqq c_k [D]_{ii}$ *for* $i = 1, \dots, d$,
- $K_A = \frac{\mathrm{Tr}(A)}{4k+2}$,
- $\phi_A : \mathbb{R}^d \to \mathbb{R}, Z \mapsto \sum_{i=1}^d ([UZ]_i)^{4k+2}[\theta_D]_i - K_A$.

*The map $\phi_A$ satisfies, for all $p \in \mathbb{R}^d$, $R \in \mathbb{S}(d)$ and $j \in \{1, ..., d\}$,*

$$\mathbb{E}\left[\phi_A(\frac{B_h^t}{\sqrt{h}})\right] = 0 \ , \tag{3.10a}$$

$$\mathbb{E}\left[\phi_A(\frac{B_h^t}{\sqrt{h}}) p^\intercal B_h^t\right] = 0 \ , \tag{3.10b}$$

$$\mathbb{E}\left[{B_h^t}^\intercal R B_h^t \phi_A(\frac{B_h^t}{\sqrt{h}})\right] = h\,\mathrm{Tr}(RA) \ , \tag{3.10c}$$

$$\mathbb{E}\left[[B_h^t]_j {B_h^t}^\intercal R B_h^t \phi_A(\frac{B_h^t}{\sqrt{h}})\right] = 0 \ . \tag{3.10d}$$

*Proof.* For $t \geq 0$, as $\frac{B_h^t}{\sqrt{h}}$ is a Gaussian vector with independent standard normal random coordinates and the matrix $U$ is orthogonal, $U\frac{B_h^t}{\sqrt{h}}$ is still a Gaussian vector with independent standard normal random coordinates. So $[U\frac{B_h^t}{\sqrt{h}}]_i \sim N$, $N$ being a zero-mean, one dimensional normal random variable with identity covariance.

We thus have :

$$\begin{aligned}\mathbb{E}\left[\phi_A(\frac{B_h^t}{\sqrt{h}})\right] =& \sum_{i=1}^d \mathbb{E}\left[N^{4k+2}\right][\theta_D]_i - K_A \\ =& \mathbb{E}\left[N^{4k+2}\right] c_k \sum_{i=1}^d [D]_{ii} - \frac{\mathrm{Tr}(A)}{4k+2}.\end{aligned}$$

From (3.8b), we have $c_k \mathbb{E}\left[N^{4k+2}\right] = \frac{1}{4k+2}$ and since $\mathrm{Tr}(D) = \mathrm{Tr}(A)$, we deduce that

$$\mathbb{E}\left[\phi_A(\frac{B_h^t}{\sqrt{h}})\right] = 0 \ ,$$

which shows (3.10a).
Let $p \in \mathbb{R}^d$, we have

$$\begin{aligned}\mathbb{E}\left[p^\intercal B_h^t \phi_A(\frac{B_h^t}{\sqrt{h}})\right] =& \sqrt{h}\mathbb{E}\left[p^\intercal \frac{B_h^t}{\sqrt{h}} \phi_A(\frac{B_h^t}{\sqrt{h}})\right] \\ =& \sqrt{h}\mathbb{E}\left[(Up)^\intercal (U\frac{B_h^t}{\sqrt{h}}) \left(\sum_{i=1}^d [U\frac{B_h^t}{\sqrt{h}}]_i^{4k+2}[\theta_D]_i\right)\right] \\ & - K_A \mathbb{E}\left[\sqrt{h}(Up)^\intercal (U\frac{B_h^t}{\sqrt{h}})\right] \\ =& 0 \ ,\end{aligned}$$

as each of the terms of this difference can be written as a sum of monomials of odd degree in $[U\frac{B^t_h}{\sqrt{h}}]_i$ $(1 \le i \le d)$ which are independent random variables with standard normal law. This shows (3.10b).
Then, we have to notice that using the formula $\mathbb{E}\left[N^{2p}\right] = \frac{(2p)!}{p!2^p}$, we have $c_k = (\mathbb{E}\left[N^{4k+4} - N^{4k+2}\right])^{-1}$. So, given $R \in \mathbb{S}(d)$, we have

$$\begin{aligned}
\mathbb{E}\left[B^{t\,\intercal}_h R B^t_h \phi_A(\frac{B^t_h}{\sqrt{h}})\right] =& h\mathbb{E}\left[(U\frac{B^t_h}{\sqrt{h}})^\intercal URU^\intercal(U\frac{B^t_h}{\sqrt{h}})\sum_{i=1}^d [U\frac{B^t_h}{\sqrt{h}}]_i^{4k+2}[\theta_D]_i\right] \\
&- K_A\mathbb{E}\left[B^{t\,\intercal}_h R B^t_h\right] \\
=& h\sum_{i\neq j}[URU^\intercal]_{ii}\mathbb{E}[N^{4k+2}][\theta_D]_j + h\sum_{i=1}^d [URU^\intercal]_{ii}\mathbb{E}[N^{4k+4}][\theta_D]_i \\
&- K_A h\,\mathrm{Tr}(R) \\
=& h\sum_{i,j}[URU^\intercal]_{ii}\mathbb{E}[N^{4k+2}][\theta_D]_j + h\sum_{i=1}^d [URU^\intercal]_{ii}\left[\mathbb{E}[N^{4k+4}] - \mathbb{E}[N^{4k+2}]\right][\theta_D]_i \\
&- K_A h\,\mathrm{Tr}(R) \\
=& h\mathbb{E}[N^{4k+2}]c_k\,\mathrm{Tr}(A)\,\mathrm{Tr}(R) + h\,\mathrm{Tr}(URU^\intercal D) - K_A h\,\mathrm{Tr}(R) \\
=& \frac{h}{4k+2}\,\mathrm{Tr}(A)\,\mathrm{Tr}(R) + h\,\mathrm{Tr}(URU^\intercal D) - \frac{\mathrm{Tr}(A)}{4k+2}h\,\mathrm{Tr}(R) \\
=& h\,\mathrm{Tr}(RA)\ ,
\end{aligned}$$

which shows (3.10c).

For $j \in \{1, ..., d\}$, $[B^t_h]_j = e_j^\intercal B^t_h = (Ue_j)^\intercal U B^t_h$ where $e_j$ is the vector of $\mathbb{R}^d$ with 1 at index $j$ and 0 everywhere else.

So :

$$\begin{aligned}
\mathbb{E}\left[[B^t_h]_j B^{t\,\intercal}_h R B^t_h \phi_A(\frac{B^t_h}{\sqrt{h}})\right] =& h\sqrt{h}\mathbb{E}\left[(Ue_j)^\intercal(U\frac{B^t_h}{\sqrt{h}})(U\frac{B^t_h}{\sqrt{h}})^\intercal URU^\intercal(U\frac{B^t_h}{\sqrt{h}})\sum_{i=1}^d[U\frac{B^t_h}{\sqrt{h}}]_i^{4k+2}[\theta_D]_i\right] \\
&- K_A\mathbb{E}\left[[B^t_h]_j B^{t\,\intercal}_h R B^t_h\right] \\
&= 0,
\end{aligned}$$

as this expression can be written as a sum of monomials of odd degree in $[B^t_h]_i$ or $[U\frac{B^t_h}{\sqrt{h}}]_i$ $(1 \le i \le d)$ which are independent random variables with standard normal law. It shows (3.10d). □

**Corollary 3.1.1.** *Let $A \in \mathbb{S}(d)$ be written as $A = \Sigma\Sigma^\intercal$, with $\Sigma \in \mathbb{R}^d$. Let $k$, $c_k$, and $K_A$ be as in Lemma 3.1.1, and define $\phi_A : \mathbb{R}^d \to \mathbb{R}$ by*

$$\phi_A(Z) = c_k(\Sigma^\intercal Z)^{4k+2}/\|\Sigma\|_2^{4k} - K_A\ .$$

*Then (3.10) holds for this definition of $\phi_A$.*

*Proof.* Let $U$ be any orthogonal matrix such that the first row of $U$ is equal to $(\|\Sigma\|_2)^{-1}\Sigma^\intercal$ and let $D$ be the diagonal matrix with the first diagonal term being equal to $\|\Sigma\|_2^2$ and the others beeing 0. Then, we have $A = U^\intercal DU$ and in this case, the map $\phi_A$ of Lemma 3.1.1 coincides with the one of Corollary 3.1.1. So, the corollary follows. □

**Corollary 3.1.2.** *Let* $A \in \mathbb{S}(d)$ *be written as* $A = \sum_{i=1}^{\ell} \lambda_i \Sigma_i \Sigma_i^\intercal$, *with* $\lambda_i \in \mathbb{R}$, $\Sigma_i \in \mathbb{R}^d$ *and* $\ell \geq 1$. *Let* $k$, $c_k$, *and* $K_A$ *be as in Lemma 3.1.1, and define* $\phi_A : \mathbb{R}^d \to \mathbb{R}$ *by*

$$\phi_A(Z) = c_k \sum_{i=1}^{\ell} \lambda_i (\Sigma_i^\intercal Z)^{4k+2} / \|\Sigma_i\|_2^{4k} - K_A \ .$$

*Then* (3.10) *holds for this definition of* $\phi_A$.

*Proof.* From the linearity of $K_A$ with respect to $A$, we get that $\phi_A = \sum_{i=1}^{\ell} \lambda_i \phi_i$ where $\phi_i$ is defined as in Corollary 3.1.1 with $\Sigma_i$ instead of $\Sigma$. Applying Corollary 3.1.1 to each matrix $A_i = \Sigma_i \Sigma_i^\intercal$, and taking the linear combination, $A = \sum_{i=1}^{\ell} \lambda_i A_i$, we deduce (3.10) for $\phi_A$. □

**Corollary 3.1.3.** *The equations* (3.10a)-(3.10d) *hold for* $\phi_A = \mathscr{P}^2_{\Sigma,k}$ *where* $A = \Sigma\Sigma^\intercal$, *with* $\Sigma \in \mathbb{R}^{d \times \ell}$, $\ell \geq 1$.

*Proof.* Apply Corollary 3.1.2 to the expression $A = \sum_{j=1}^{\ell} \Sigma_{.j} \Sigma_{.j}^\intercal$. □

Now, we can prove Theorem 3.1.1.

*Proof of Theorem 3.1.1.* Let $\hat{X}_h^{t,x} := x + \underline{b}(t,x)h + \underline{\sigma}(t,x)B_h^t$.

$$\mathbb{E}\left[W(t+h, \hat{X}(t+h))\mathscr{P}^2_{\Sigma,k}(h^{-1/2}B_h^t) \mid \hat{X}(t) = x\right] = \ \mathbb{E}\left[W(t+h, \hat{X}_h^{t,x})\mathscr{P}^2_{\Sigma,k}(h^{-1/2}B_h^t)\right]$$

By applying the Taylor formula to order 3 to the function $W$ at point $(t,x)$, we obtain :

$$\begin{aligned}
W(t+h, \hat{X}_h^{t,x}) =& W(t,x) + \frac{\partial W}{\partial t}(t,x)h + D_x W(t,x)^\intercal (\underline{b}(t,x)h + \underline{\sigma}(t,x)B_h^t) \\
&+ \frac{1}{2}(\underline{b}(t,x)h + \underline{\sigma}(t,x)B_h^t)^\intercal D_x^2 W(t,x)(\underline{b}(t,x)h + \underline{\sigma}(t,x)B_h^t) \\
&+ \frac{1}{2}h(\underline{b}(t,x)h + \underline{\sigma}(t,x)B_h^t)^\intercal \frac{\partial D_x W}{\partial t}(t,x) + \frac{1}{2}h^2 \frac{\partial^2 W}{\partial t^2}(t,x) \\
&+ \frac{1}{6}\sum_{i,j,k} [\underline{b}(t,x)h + \underline{\sigma}(t,x)B_h^t]_i [\underline{b}(t,x)h + \underline{\sigma}(t,x)B_h^t]_j [\underline{b}(t,x)h + \underline{\sigma}(t,x)B_h^t]_k \\
&\frac{\partial^3 W}{\partial x_i \partial x_j \partial x_k}(t,x) \\
&+ \frac{1}{2}\sum_{i,j} h[\underline{b}(t,x)h + \underline{\sigma}(t,x)B_h^t]_i [\underline{b}(t,x)h + \underline{\sigma}(t,x)B_h^t]_j \frac{\partial^3 W}{\partial t \partial x_i \partial x_j}(t,x) \\
&+ \frac{1}{2}\sum_{i} h^2 [\underline{b}(t,x)h + \underline{\sigma}(t,x)B_h^t]_i \frac{\partial^3 W}{\partial t^2 \partial x_i}(t,x) + \frac{1}{6}h^3 \frac{\partial^3 W}{\partial t^3}(t,x) \\
&+ M^4(W,t,x,h,B_h^t) \ .
\end{aligned} \tag{3.11}$$

where $M^4$ can be interpreted as the remainder in an integral form of the Taylor formula and is a continuous function with respect to its last argument and such that

$$\begin{aligned}|M^4(W,t,x,h,B_h^t)| \leq & \frac{h^4}{24}|\partial_t^4 D^0 W| \\ & + \frac{h^3}{6}\sum_{i=1}^{d} |\partial_t^3 D^1 W||[\underline{b}(t,x)h+\underline{\sigma}(t,x)B_h^t]_i| \\ & + \frac{h^2}{4}\sum_{i,j} |\partial_t^2 D^2 W||[\underline{b}(t,x)h+\underline{\sigma}(t,x)B_h^t]_i||[\underline{b}(t,x)h+\underline{\sigma}(t,x)B_h^t]_j| \\ & + \frac{h}{6}\sum_{i,j,p} |\partial_t^1 D^3 W||[\underline{b}(t,x)h+\underline{\sigma}(t,x)B_h^t]_i||[\underline{b}(t,x)h+\underline{\sigma}(t,x)B_h^t]_j| \\ & \quad |[\underline{b}(t,x)h+\underline{\sigma}(t,x)B_h^t]_p| \\ & + \frac{1}{24}\sum_{i,j,p,q} |\partial_t^0 D^4 W||[\underline{b}(t,x)h+\underline{\sigma}(t,x)B_h^t]_i||[\underline{b}(t,x)h+\underline{\sigma}(t,x)B_h^t]_j| \\ & \quad |[\underline{b}(t,x)h+\underline{\sigma}(t,x)B_h^t]_p||[\underline{b}(t,x)h+\underline{\sigma}(t,x)B_h^t]_q| \ .\end{aligned}$$

By using Corollary 3.1.3, the term

$$\begin{aligned}\frac{1}{2}(\underline{b}(t,x)h+\underline{\sigma}(t,x)B_h^t)^\intercal D_x^2 W(t,x)(\underline{b}(t,x)h+\underline{\sigma}(t,x)B_h^t) = & \frac{h^2}{2}\underline{b}(t,x)^\intercal D_x^2 W(t,x)\underline{b}(t,x) \\ & + h\underline{b}(t,x)^\intercal D_x^2 W(t,x)\underline{\sigma}(t,x)B_h^t \\ & + \frac{1}{2}(B_h^t)^\intercal \underline{\sigma}(t,x)^\intercal D_x^2 W(t,x)\underline{\sigma}(t,x)B_h^t\end{aligned}$$

when multiplied by

$$h^{-1}\mathscr{P}^2_{\Sigma,k}(h^{-1/2}B_h^t),$$

has an expectation equal to

$$\frac{1}{2}\operatorname{Tr}\left(\Sigma\Sigma^\intercal \underline{\sigma}(t,x)^\intercal D_x^2 W(t,x)\underline{\sigma}(t,x)\right).$$

The other terms in

$$h^{-1}\mathbb{E}\left[W(t+h,\hat{X}_h^{t,x})\mathscr{P}^2_{\Sigma,k}(h^{-1/2}B_h^t)\right],$$

are null except terms coming from the product of monomials of degree 2 in $[B_h^t]_i$ $(1 \leq i \leq d)$ by $\mathscr{P}^2_{\Sigma,k}(h^{-1/2}B_h^t)$ and $h^{-1}\mathbb{E}\left[M^4(W,t,x,h,B_h^t)\mathscr{P}^2_{\Sigma,k}(h^{-1/2}B_h^t)\right]$. This last term is bounded by a sum of terms of the form

$$Kh^{p-1}(h+\sqrt{h})^q|\partial_t^p D^q W| \leq Kh^{p-1}(1+\sqrt{h})^4(\sqrt{h})^q|\partial_t^p D^q W|,$$

where $K$ is a constant function of $C$ and $M$ and $p+q=4$. The other terms are bounded by

$$Kh(|\partial_t^1 D^2 W|+|\partial_t^0 D^3 W|) \leq Kh(1+\sqrt{h})^4(|\partial_t^1 D^2 W|+|\partial_t^0 D^3 W|).$$

In this way, we obtain the result of Theorem 3.1.1. □

**Remark 3.1.1.** *When the second order derivatives approximation* (3.9) *will be used in probabilistic schemes in the next subsections, a value of $k$ high enough will be required for these schemes to be monotone. We can see that a high value of $k$ tends to increase the weight of the values of the function $W$ at points far from $(t,x)$ and decrease the weight of the values of the function $W$ at points close to $(t,x)$ in the approximation of the second order derivatives. In this way, approximation* (3.9) *is analogous to a large-stencil finite difference approximation of the second order derivatives. This analogy will be used in the next subsection to name the resulted schemes.*

Let us also introduce the following approximation of the first order derivatives. For any vector $g \in \mathbb{R}^d$, consider the piecewise linear function $\mathscr{P}^1_g$ on $\mathbb{R}^d$ :

$$\mathscr{P}^1_g(w) = 2(g_+ \cdot w_+ + g_- \cdot w_-) \ , \tag{3.12}$$

where for any vector $\mu \in \mathbb{R}^d$, $\mu_+, \mu_- \in \mathbb{R}^d$ are defined such that $[\mu_+]_i = \max([\mu]_i, 0)$, $[\mu_-]_i = -\min([\mu]_i, 0)$. Note that $\mathscr{P}^1_g$ is nonnegative. We shall show that

$$\mathbb{E}\Big[(W(t+h, \hat{X}^{t,x}_h) - W(t,x))\mathscr{P}^1_g(h^{-1}B^t_h)\Big] \tag{3.13}$$

is a monotone approximation of

$$(\underline{\sigma}(x)g) \cdot D_x W(t,x) \ .$$

Before this, let us note that if $\underline{\sigma}(x) = 1$, $\underline{b}(x) = 1$ and $h^{-1/2}B^t_h$ is discretized by a random variable taking the values 1 and $-1$ with probability 1/2, then the discretization (3.13) corresponds to the Kushner (upwind) discretization found in [48]

$$\sum_{i=1}^d \left[ [g_i]_+ \frac{W(t+h, x+h^{1/2}e_i) - W(t,x)}{h^{1/2}} + [g_i]_- \frac{W(t+h, x-h^{1/2}e_i) - W(t,x)}{h^{1/2}} \right] \ .$$

**Theorem 3.1.2.** *Let $\hat{X}^{t,x}_h := x + \underline{b}(t,x)h + \underline{\sigma}(t,x)(B_{t+h} - B_t)$ and denote $B^t_h = B_{t+h} - B_t$. Consider any vector $g \in \mathbb{R}^d$. Assume that $\underline{b}$ and $\underline{\sigma}$ are bounded by some constant $C$ uniformly in $t$ and $x$, and let $M$ be an upper bound of $|g|$. Then, there exists $K = K(C, M) > 0$ such that, for all $W \in \mathscr{C}^2_b([0,T] \times \mathbb{R}^d)$, we have, for all $(t,x) \in \mathscr{T}_h \times \mathbb{R}^d$,*

$$\begin{aligned} &\Big|(\underline{\sigma}(t,x)g) \cdot D_x W(t,x) - \mathbb{E}\Big[(W(t+h, \hat{X}^{t,x}_h) - W(t,x))\mathscr{P}^1_g(h^{-1}B^t_h)\Big]\Big| \\ &\quad \le K(1+\sqrt{h})^2 \Big[\sqrt{h}(|\partial^1_t D^0 W| + |\partial^0_t D^1 W| + |\partial^0_t D^2 W|) \\ &\qquad + h(|\partial^1_t D^1 W|) + h\sqrt{h}|\partial^2_t D^0 W|\Big] \ . \end{aligned}$$

*Proof.* By applying Taylor formula to order 1 to the function $W$ at point $(t,x)$, we obtain :

$$\begin{aligned} W(t+h, \hat{X}^{t,x}) - W(t,x) = &\frac{\partial W}{\partial t}(t,x)h + D_x W(t,x)^\intercal(\underline{b}(t,x)h + \underline{\sigma}(t,x)B^t_h) \\ &+ M^2(W, t, x, h, B^t_h), \end{aligned} \tag{3.14}$$

where $M^2$ can be interpreted as the remainder in an integral form of the Taylor formula and is a continuous function with respect to its last argument and such that :

$$\begin{aligned}|M^2(W,t,x,h,B^t_h)| \leq &\frac{1}{2}\sum_{i,j}|\partial^0_t D^2 W||[\underline{b}(t,x)h+\underline{\sigma}(t,x)B^t_h]_i||[\underline{b}(t,x)h+\underline{\sigma}(t,x)B^t_h]_j| \\ &+h\sum_i |\partial^1_t D^1 W||[\underline{b}(t,x)h+\underline{\sigma}(t,x)B^t_h]_i| + \frac{1}{2}h^2|\partial^2_t D^0 W| \ .\end{aligned}$$

$$\begin{aligned}&\mathbb{E}\left[D_x W(t,x)^\intercal \underline{\sigma}(t,x)B^t_h \mathscr{P}^1_g(h^{-1}B^t_h)\right] \\ &= \frac{2}{h}\mathbb{E}\left[D_x W(t,x)^\intercal \underline{\sigma}(t,x)B^t_h(g_+\cdot(B^t_h)_+ + g_-\cdot(B^t_h)_-)\right] \\ &= \frac{2}{h}\mathbb{E}\left[\left(\sum_{i,j}[D_x W(t,x)]_i[\underline{\sigma}(t,x)]_{ij}[B^t_h]_j\right)\left(\sum_k [g_+]_k[(B^t_h)_+]_k + [g_-]_k[(B^t_h)_-]_k\right)\right] \\ &= \frac{2}{h}\mathbb{E}\left[\sum_{i,j}[D_x W(t,x)]_i[\underline{\sigma}(t,x)]_{ij}[B^t_h]_j\left([g_+]_j[(B^t_h)_+]_j + [g_-]_j[(B^t_h)_-]_j\right)\right] \\ &= \frac{2}{h}\sum_{i,j}[D_x W(t,x)]_i[\underline{\sigma}(t,x)]_{ij}\left([g_+]_j\mathbb{E}\left[[(B^t_h)_+]^2_j\right] - [g_-]_j\mathbb{E}\left[[(B^t_h)_-]^2_j\right]\right).\end{aligned}$$

$$\mathbb{E}\left[[(B^t_h)_+]^2_j\right] = h\mathbb{E}\left[[(\frac{B^t_h}{\sqrt{h}})_+]^2_j\right] = h\mathbb{E}[\max(N,0)^2] = \frac{h}{2},$$

where $N$ is a Gaussian one dimensional variable. In the same way,

$$\mathbb{E}\left[[(B^t_h)_-]^2_j\right] = \frac{h}{2}.$$

So

$$\begin{aligned}\mathbb{E}\left[D_x W(t,x)^\intercal \underline{\sigma}(t,x)B^t_h \mathscr{P}^1_g(h^{-1}B^t_h)\right] &= \sum_{i,j}[D_x W(t,x)]_i[\underline{\sigma}(t,x)]_{ij}([g_+]_j - [g_-]_j) \\ &= D_x W(t,x)\cdot(\underline{\sigma}(t,x)g).\end{aligned} \tag{3.15}$$

The terms of $\mathbb{E}\left[(W(t+h,\hat{X}^{t,x}_h) - W(t,x))\mathscr{P}^1_g(h^{-1}B^t_h)\right]$ involving the partial derivatives of order 2 (through $M^2(W,t,x,h,B^t_h)$) can be bounded by terms of the form

$$Kh^{p-1/2}(h+\sqrt{h})^q|\partial^p_t D^q W| \leq Kh^{p-1/2}(1+\sqrt{h})^2(\sqrt{h})^q|\partial^p_t D^q W|,$$

where $K$ is function of $C$ and $M$, $p+q=2$.
The other terms can be bounded by

$$K\sqrt{h}(|\partial^1_t D^0 W| + |\partial^0_t D^1 W|) \leq K\sqrt{h}(1+\sqrt{h})^2(|\partial^1_t D^0 W| + |\partial^0_t D^1 W|).$$

In this way, we prove Theorem 3.1.2. □

In addition to the approximation of the first and second derivatives, we will need a probabilistic approximation of $\mathscr{L}^X W$.

**Lemma 3.1.2.** *Let* $\hat{X}_h^{t,x} := x+\underline{b}(t,x)h+\underline{\sigma}(t,x)(B_{t+h}-B_t)$. *Assume that* $\underline{b}$ *and* $\underline{\sigma}$ *are bounded by some constant* $C$ *uniformly in* $t$ *and* $x$. *Then, there exists* $K = K(C) > 0$ *such that, for all* $W \in \mathscr{C}_b^4([0,T]\times\mathbb{R}^d)$, *we have, for all* $(t,x) \in \mathscr{T}_h \times \mathbb{R}^d$,

$$\begin{aligned}&\left|h^{-1}\left(\mathbb{E}\left[W(t+h,\hat{X}_h^{t,x})\right]-W(t,x)\right)-\mathscr{L}^X W\right|\\ &\leq K(1+\sqrt{h})^4\Big[h(|\partial_t^0 D^2 W|+|\partial_t^1 D^1 W|+|\partial_t^2 D^0 W|+|\partial_t^0 D^3 W|+|\partial_t^1 D^2 W|+|\partial_t^0 D^4 W|)\\ &\quad +h\sqrt{h}|\partial_t^1 D^3 W|+h^2(|\partial_t^2 D^2 W|+|\partial_t^2 D^1 W|+|\partial_t^3 D^0 W|)+h^2\sqrt{h}|\partial_t^3 D^1 W|\\ &\quad +h^3|\partial_t^4 D^0 W|\Big]\ .\end{aligned}$$

*Proof.* We consider the Taylor development (3.11) of $W(t+h,\hat{X}_h^{t,x})$ that was used to prove Theorem 3.1.1. From this development, we can notice that $\mathscr{L}^X W$ appears naturally in the expression of

$$h^{-1}\left(\mathbb{E}\left[W(t+h,\hat{X}_h^{t,x})\right]-W(t,x)\right).$$

The other terms involving the derivatives of order 4 through $M^4(W,t,x,h,B_h^t)$ are bounded by expressions of the form

$$Kh^{p-1}(h+\sqrt{h})^q|\partial_t^p D^q W| \leq Kh^{p-1}(1+\sqrt{h})^4(\sqrt{h})^q|\partial_t^p D^q W| \tag{3.16}$$

and $K$ function of $C$, and $p+q=4$.

The terms left from the Taylor development of $W(t+h,\hat{X}_h^{t,x})$ are bounded by the following expression :

$$\begin{aligned}&Kh(|\partial_t^0 D^2 W|+|\partial_t^1 D^1 W|+|\partial_t^2 D^0 W|+(1+\sqrt{h})^2(|\partial_t^0 D^3 W|+|\partial_t^1 D^2 W|))\\ &\quad +Kh^2(|\partial_t^2 D^1 W|+|\partial_t^3 D^0 W|)\\ &\leq K(1+\sqrt{h})^4 h(|\partial_t^0 D^2 W|+|\partial_t^1 D^1 W|+|\partial_t^2 D^0 W|+|\partial_t^0 D^3 W|+|\partial_t^1 D^2 W|)\\ &\quad +Kh^2(|\partial_t^2 D^1 W|+|\partial_t^3 D^0 W|)\end{aligned}$$

In this way, we obtain the result of Lemma 3.1.2. □

We add to these results, the estimates of the probabilistic approximation of $D_x W(t,x)$ used by Fahim, Touzi and Warin in [25]. Let for any vector $g \in \mathbb{R}^d$, $\mathscr{P}_g^{1,0}$ be the linear function on $\mathbb{R}^d$ such that :

$$\mathscr{P}_g^{1,0}(w) = g\cdot w. \tag{3.17}$$

**Lemma 3.1.3.** *Let* $\hat{X}_h^{t,x}$ *and* $B_h^t$ *be as in Theorem 3.1.2. Consider any vector* $g \in \mathbb{R}^d$. *Assume that* $\underline{b}$ *and* $\underline{\sigma}$ *are bounded by some constant* $C$ *uniformly in* $t$ *and* $x$, *and let* $M$ *be an upper bound of* $|g|$. *Then, there exists* $K = K(C,M) > 0$ *such that, for all* $W \in \mathscr{C}_b^2([0,T]\times\mathbb{R}^d)$, *we have for all* $(t,x) \in \mathscr{T}_h \times \mathbb{R}^d$,

$$\left|(\underline{\sigma}(t,x)g)\cdot D_x W(t,x)-\mathbb{E}\left[W(t+h,\hat{X}_h^{t,x})\mathscr{P}_g^{1,0}(h^{-1}B_h^t)\right]\right| \tag{3.18}$$

$$\leq K(1+\sqrt{h})^2\left[\sqrt{h}|\partial_t^0 D^2 W|+h|\partial_t^1 D^1 W|+h\sqrt{h}|\partial_t^2 D^0 W|\right] \tag{3.19}$$

*Proof.* We use the same Taylor development (3.14) of $W$ at point $(t,x)$ as in the proof of Theorem 3.1.2. The terms

$$\mathbb{E}\left[W(t,x)\mathscr{P}_g^{1,0}(h^{-1}B_h^t)\right],$$
$$\mathbb{E}\left[\frac{\partial W}{\partial t}(t,x)\mathscr{P}_g^{1,0}(h^{-1}B_h^t)\right],$$
$$\mathbb{E}\left[D_xW(t,x)^\intercal \underline{b}(t,x)h\mathscr{P}_g^{1,0}(h^{-1}B_h^t)\right],$$

are all null as $\mathbb{E}\left[\mathscr{P}_g^{1,0}(h^{-1}B_h^t)\right]=0$.

$$\begin{aligned}
&\mathbb{E}\left[D_xW(t,x)^\intercal \underline{\sigma}(t,x)B_h^t\mathscr{P}_g^{1,0}(h^{-1}B_h^t)\right]\\
&\quad=\frac{1}{h}\mathbb{E}\left[\left(\sum_{i,j}[D_xW(t,x)]_i[\underline{\sigma}(t,x)]_{ij}[B_h^t]_j\right)\left(\sum_k [g]_k[B_h^t]_k\right)\right]\\
&\quad=\frac{1}{h}\mathbb{E}\left[\sum_{i,j}[D_xW(t,x)]_i[\underline{\sigma}(t,x)]_{ij}[B_h^t]_j^2[g]_j\right]\\
&\quad=D_xW(t,x)\cdot(\underline{\sigma}(t,x)g).
\end{aligned}$$

The other terms of $\mathbb{E}\left[W(t+h,\hat{X}_h^{t,x})\mathscr{P}_g^{1,0}(h^{-1}B_h^t)\right]$ coming from $M^2(W,t,x,h,B_h^t)$ can be bounded as in the proof of Theorem 3.1.2. This gives the result of the lemma. □

### 3.1.2 Monotone probabilistic schemes and convergence

We will prove the convergence of two different schemes here under some simple assumptions on the PDE (3.1)-(3.2). The first scheme presented in [2] uses a mix between second order derivatives approximation (3.9) and the scheme of Fahim, touzi and Warin found in [25]. We will call it *large stencil probabilistic scheme.* The second scheme presented in [3], is a better version of the first scheme where the upwind first order derivatives approximation (3.13) is used with the second order derivatives approximation (3.9). We will call it *upwind large stencil probabilistic scheme.*

**3.1.2.a Large stencil probabilistic scheme** The scheme is very similar to the one of Fahim, Touzi and Warin detailed in Section 2.3.3 except that we use approximation (3.9) for the second order derivatives. As this approximation depends on the coefficient matrix by which is multiplied the Hessian $D_x^2W$, it can not be used just as an input of the Hamiltonian $F$ of (3.3). We need the decomposition of $F$ as a infimum of Hamiltonians which are affine in $W$, $D_xW$ and $D_x^2W$. This decomposition is given by (3.4).

For each $\alpha\in\mathscr{A}$, we suppose that there exists a function $\Sigma^\alpha:[0,T]\times\mathbb{R}^d\to\mathbb{R}^{d\times\ell}$ for $\ell\le d$ such that for any $(t,x)\in[0,T]\times\mathbb{R}^d$:

$$(a^\alpha-\underline{\sigma}\underline{\sigma}^\intercal)(t,x)=(\underline{\sigma}\Sigma^\alpha\Sigma^{\alpha\intercal}\underline{\sigma}^\intercal)(t,x). \tag{3.20}$$

One may use for instance the Cholesky factorisation of the matrix $(\underline{\sigma}^{-1}(a^\alpha-\underline{\sigma}\underline{\sigma}^\intercal)(\underline{\sigma}^\intercal)^{-1})(t,x)$. To ensure the existence of this Cholesky factorisation, we suppose that

$$a^\alpha(t,x)\ge(\underline{\sigma}\underline{\sigma}^\intercal)(t,x),$$

and $\underline{\sigma}(t,x)$ is invertible for any $\alpha \in \mathscr{A}$ and $(t,x) \in [0,T] \times \mathbb{R}^d$.

The scheme is then, with the notations introduced previously:

$$\begin{cases} W^h(t,x) := \mathbf{T}^0_{k,h}[W^h(t+h,\cdot)](t,x), \quad (t,x) \in \mathscr{T}_h \times \mathbb{R}^d, \\ W^h(T,x) := \Psi(x), \quad x \in \mathbb{R}^d, \end{cases} \tag{3.21}$$

with

$$\begin{aligned} \mathbf{T}^0_{k,h}[\Phi](t,x) := \mathbb{E}\left[\Phi(\hat{X}^{t,x}_h)\right] + h \inf_{\alpha\in\mathscr{A}} \Bigg\{ & \mathbb{E}\left[\Phi(\hat{X}^{t,x}_h) h^{-1} \mathscr{P}^2_{\Sigma^\alpha(t,x),k}(h^{-1/2}B^t_h)\right] \\ & + \mathbb{E}\left[\Phi(\hat{X}^{t,x}_h) \mathscr{P}^{1,0}_{\underline{\sigma}^{-1}(t,x)(b^\alpha - \underline{b})(t,x)}(h^{-1}B^t_h)\right] \\ & + c^\alpha(t,x)\mathbb{E}\left[\Phi(\hat{X}^{t,x}_h)\right] + f^\alpha(t,x) \Bigg\}. \end{aligned} \tag{3.22}$$

We now establish the conditions under which this scheme is convergent.

**Lemma 3.1.4** (Consistency). *Assume :*

- *$\underline{\sigma}$ and $\underline{b}$ are bounded in $t$ and $x$;*
- *$\underline{\sigma}\underline{\sigma}^\intercal$ is lower bounded by a positive matrix uniformly in $t$ and $x$;*
- *$a^\alpha$, $b^\alpha$ and $c^\alpha$ are bounded in $t$, $x$ uniformly in $\alpha$;*
- *the left hand element of PDE* (3.3) *is continuous in $t$ and $x$.*

*Then, for $\phi \in \mathscr{C}^4_b([0,T] \times \mathbb{R}^d)$,*

$$\begin{aligned} &\lim_{\substack{(s,y)\to(t,x)\\(h,c)\to(0,0)\\s+h\le T}} \frac{(c+\phi)(s,y) - \mathbf{T}^0_{k,h}[(c+\phi)(s+h,\cdot)](s,y)}{h} \\ &= -(\mathscr{L}^X(t,x) + F(t,x,\phi(t,x),D_x\phi(t,x),D^2_x\phi(t,x))). \end{aligned}$$

*Proof.* In the conditions of Lemma 3.1.4, Theorem 3.1.1, Lemma 3.1.2 and Lemma 3.1.3 can be applied. Indeed, the second condition on $\underline{\sigma}\underline{\sigma}^\intercal$ implies that $\underline{\sigma}^{-1}$ is bounded. So with the others conditions, $\Sigma^\alpha$ and $\underline{\sigma}^{-1}(t,x)(b^\alpha - \underline{b})(t,x)$ are bounded.

Let $\phi \in \mathscr{C}^4([0,T] \times \mathbb{R}^d)$ and $(s,y) \in [0,T] \times \mathbb{R}^d$ and $c, h > 0$ small such that $s + h \le T$.

$$\begin{aligned} &\frac{(c+\phi)(s,y) - \mathbf{T}^0_{k,h}[(c+\phi)(s+h,\cdot)](s,y)}{h} = \frac{\phi(s,y) - \mathbb{E}\left[\phi(s+h,\hat{X}^{s,y}_h)\right]}{h} \\ &- \inf_{\alpha\in\mathscr{A}} \Bigg\{ \mathbb{E}\left[\phi(s+h,\hat{X}^{s,y}_h) h^{-1} \mathscr{P}^2_{\Sigma^\alpha(s,y),k}(h^{-1/2}B^s_h)\right] + \mathbb{E}\left[\phi(s+h,\hat{X}^{s,y}_h) \mathscr{P}^{1,0}_{\underline{\sigma}^{-1}(s,y)(b^\alpha-\underline{b})(s,y)}(h^{-1}B^s_h)\right] \\ &+ c^\alpha(s,y)\mathbb{E}\left[\phi(s+h,\hat{X}^{s,y}_h)\right] + f^\alpha(s,y) + c\mathbb{E}\left[h^{-1}\mathscr{P}^2_{\Sigma^\alpha(s,y),k}(h^{-1/2}B^s_h) + \mathscr{P}^{1,0}_{\underline{\sigma}^{-1}(s,y)(b^\alpha-\underline{b})(s,y)}(h^{-1}B^s_h)\right] \\ &+ cc^\alpha(s,y) \Bigg\} \end{aligned}$$

By Theorem 3.1.1 and the definition of $\Sigma^\alpha$,

$$\frac{1}{2}\operatorname{Tr}\left[(a^\alpha - \underline{\sigma\sigma}^\intercal)(s,y)D_x^2\phi(s,y)\right] - O(h) \leq \mathbb{E}\left[\phi(s+h,\hat{X}_h^{s,y})h^{-1}\mathscr{P}^2_{\Sigma^\alpha(s,y),k}(h^{-1/2}B_h^s)\right] \leq \frac{1}{2}\operatorname{Tr}\left[(a^\alpha - \underline{\sigma\sigma}^\intercal)(s,y)D_x^2\phi(s,y)\right] + O(h),$$

where $O(h)$ is uniform in $s$, $y$ and $\alpha$.

In the same way, by Lemma 3.1.3,

$$(b^\alpha - \underline{b})(s,y)\cdot D_x\phi(s,y) - O(\sqrt{h}) \leq \mathbb{E}\left[\phi(s+h,\hat{X}_h^{s,y})\mathscr{P}^{1,0}_{\underline{\sigma}^{-1}(s,y)(b^\alpha-\underline{b})(s,y)}(h^{-1}B_h^s)\right] \leq (b^\alpha - \underline{b})(s,y)\cdot D_x\phi(s,y) + O(\sqrt{h}),$$

where $O(\sqrt{h})$ is uniform in $s$, $y$ and $\alpha$.

By Lemma 3.1.2, we have :

$$\frac{\phi(s,y) - \mathbb{E}\left[\phi(s+h,\hat{X}_h^{s,y})\right]}{h} + O(h) \leq -\mathscr{L}^X\phi(s,y) \leq \frac{\phi(s,y) - \mathbb{E}\left[\phi(s+h,\hat{X}_h^{s,y})\right]}{h} + O(h),$$

where $O(h)$ is uniform in $s$, $y$ and $\alpha$. As $\underline{b}$ and $\underline{\sigma}$ are bounded in $s$ and $y$ and $\phi \in \mathscr{C}_{\mathrm{b}}^4([0,T]\times\mathbb{R}^d)$, $\mathscr{L}^X\phi(s,y) = O(1)$. This allows us to deduce that :

$$\phi(s,y) - O(h) \leq \mathbb{E}\left[\phi(s+h,\hat{X}_h^{s,y})\right] \leq \phi(s,y) + O(h).$$

Then as $c^\alpha$ is bounded in $s$, $y$ uniformly in $\alpha$, we have :

$$c^\alpha(s,y)\phi(s,y) - O(h) \leq c^\alpha(s,y)\mathbb{E}\left[\phi(s+h,\hat{X}_h^{s,y})\right] \leq c^\alpha(s,y)\phi(s,y) + O(h),$$

where in all these inequalities, $O(h)$ is uniform in $s$, $y$ and $\alpha$.

$$c\mathbb{E}\left[\mathscr{P}^2_{\Sigma^\alpha(s,y),k}(h^{-1/2}B_h^s) + \mathscr{P}^{1,0}_{\underline{\sigma}^{-1}(s,y)(b^\alpha-\underline{b})(s,y)}(h^{-1}B_h^s)\right] = 0 \tag{3.23}$$

and $cc^\alpha(s,y) = O(c)$ with $O(c)$ uniform in $s, y, \alpha$ as $c^\alpha$ is bounded. All these observations allow us to deduce that

$$\lim_{\substack{(h,c)\to(0,0)\\ s+h\leq T}} \frac{(c+\phi)(s,y) - \mathbf{T}^0_{k,h}[(c+\phi)(s+h,\cdot)](s,y)}{h} = -(\mathscr{L}^X\phi(s,y) + F(s,y,\phi(s,y),D_x\phi(s,y),D_x^2\phi(s,y))) \underset{(s,y)\to(t,x)}{\longrightarrow} -(\mathscr{L}^X\phi(t,x) + F(t,x,\phi(t,x),D_x\phi(t,x),D_x^2\phi(t,x))).$$

□

Before giving results related to the monotonicity, we are going first to define what we call a monotone operator.

**Definition 3.1.1.** *An operator $T$ is monotone on a space $\mathscr{D}$ of functions if for any couple of functions $\phi$ and $\psi$ in $\mathscr{D}$,*

$$\phi \leq \psi \Rightarrow T[\phi] \leq T[\psi].$$

**Lemma 3.1.5** (Monotonicity)**.** *Assume that :*

- $\underline{\sigma}$ *and* $\underline{b}$ *are bounded in $t$ and $x$;*
- $\underline{\sigma\sigma}^\intercal$ *is lower bounded by a positive matrix uniformly in $t$ and $x$;*
- $a^\alpha$, $b^\alpha$ *and* $c^\alpha$ *are bounded in $t$, $x$ uniformly in $\alpha$;*
- *There exists a bounded map $g^\alpha : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ such that*

$$\underline{\sigma}^{-1}(t,x)(b^\alpha - \underline{b})(t,x) = \Sigma^\alpha(t,x) g^\alpha(t,x).$$

*Then for $k > k_0$ to compute, there exists $h_0$ such that $\mathbf{T}^0_{k,h}$ is monotone for $h \leq h_0$ over the set of bounded functions $\mathbb{R}^d \to \mathbb{R}$ and there exists $C_1$ such that $\mathbf{T}^0_{k,h}$ is $C_1 h$-almost monotone for all $h > 0$ which means that for $\phi$ and $\psi$ bounded functions in $\mathbb{R}^d \to \mathbb{R}$*

$$\phi \leq \psi \Rightarrow \mathbf{T}^0_{k,h}[\phi] \leq \mathbf{T}^0_{k,h}[\psi] + C_1 h \sup(\psi - \phi). \tag{3.24}$$

*Proof.* Let $\phi, \psi : \mathbb{R}^d \to \mathbb{R}$ be bounded such that $\phi \leq \psi$, $h > 0$ and $k \in \mathbb{N}$. Let $g^\alpha$ be a bounded map $[0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ such that

$$\underline{\sigma}^{-1}(t,x)(b^\alpha - \underline{b})(t,x) = \Sigma^\alpha(t,x) g^\alpha(t,x).$$

$$\mathbf{T}^0_{k,h}[\phi](t,x) = \inf_{\alpha \in \mathscr{A}} \left\{ \mathbb{E}\left[ \phi(\hat{X}^{t,x}_h) \mathscr{P}^{t,x,\alpha,k,h}(h^{-1/2} B^t_h) \right] + h f^\alpha(t,x) \right\},$$

where

$$\mathscr{P}^{t,x,\alpha,k,h}(w) = 1 + \mathscr{P}^2_{\Sigma^\alpha(t,x),k}(w) + h \mathscr{P}^{1,0}_{\Sigma^\alpha(t,x) g^\alpha(t,x)}(h^{-1/2} w) + h c^\alpha(t,x).$$

We now use the inequality :

$$\inf_\alpha d^\alpha - \inf_\alpha e^\alpha \geq \inf_\alpha \{ d^\alpha - e^\alpha \}$$

for $(d^\alpha)_\alpha$ and $(e^\alpha)_\alpha$, two families of real numbers indexed by the parameter $\alpha$. This allows us to deduce that :

$$(\mathbf{T}^0_{k,h}[\psi] - \mathbf{T}^0_{k,h}[\phi])(t,x) \geq \inf_{\alpha \in \mathscr{A}} \left\{ \mathbb{E}\left[ (\psi - \phi)(\hat{X}^{t,x}_h) \mathscr{P}^{t,x,\alpha,k,h}(h^{-1/2} B^t_h) \right] \right\}.$$

Let $C$ be a bound of $c^\alpha$ and $g^\alpha$ in $t$, $x$ uniformly in $\alpha$.

$$\mathscr{P}^{t,x,\alpha,k,h}(w) \geq 1 - h^{1/2} C \|\Sigma^\alpha(t,x)^\intercal w\|_2 - hC + \mathscr{P}^2_{\Sigma^\alpha(t,x),k}(w)$$

In the following development, we use the inequality

$$de \leq \frac{d^p}{p} + \frac{e^q}{q}, \tag{3.25}$$

for $d, e > 0$ and $\frac{1}{p} + \frac{1}{q} = 1$. For any matrix $\Sigma \in \mathbb{R}^{d\times\ell}$, $w \in \mathbb{R}^d$, and $\epsilon, \eta > 0$, we have

$$\begin{aligned}
\|\Sigma^\mathsf{T} w\|_2 &\leq \frac{\epsilon}{2}\|\Sigma^\mathsf{T} w\|_2^2 + \frac{1}{2\epsilon} \\
&= \frac{\epsilon}{2}\left(\sum_{j=1}^{\ell}\left(\frac{[\Sigma^\mathsf{T} w]_j}{\|\Sigma_{.j}\|_2}\right)^2 \|\Sigma_{.j}\|_2^2\right) + \frac{1}{2\epsilon} \\
&\leq \frac{\epsilon\eta^{2k}}{4k+2}\left(\sum_{j=1}^{\ell}\left(\frac{[\Sigma^\mathsf{T} w]_j}{\|\Sigma_{.j}\|_2}\right)^{4k+2} \|\Sigma_{.j}\|_2^2\right) + \frac{2k\epsilon}{(4k+2)\eta}\left(\sum_{j=1}^{\ell}\|\Sigma_{.j}\|_2^2\right) + \frac{1}{2\epsilon} \\
&= \frac{\epsilon\eta^{2k}}{4k+2}c_k^{-1}\left(\mathscr{P}_{\Sigma,k}(w) + \frac{\mathrm{Tr}(\Sigma\Sigma^\mathsf{T})}{4k+2}\right) + \frac{2k\epsilon}{(4k+2)\eta}\mathrm{Tr}(\Sigma\Sigma^\mathsf{T}) + \frac{1}{2\epsilon}\ ,
\end{aligned}$$

with $c_k > 0$ as in (3.8). At the first line, we wrote $\|\Sigma^\mathsf{T} w\|_2$ as $\frac{1}{\epsilon}\epsilon\|\Sigma^\mathsf{T} w\|_2$ and used (3.25) with $p = q = 2$ and $d = \epsilon\|\Sigma^\mathsf{T} w\|_2$ and $e = 1$. We used this operation at the third line too by writing $\left(\frac{[\Sigma^\mathsf{T} w]_j}{\|\Sigma_{.j}\|_2}\right)^2$ as $\frac{1}{\eta}\eta\left(\frac{[\Sigma^\mathsf{T} w]_j}{\|\Sigma_{.j}\|_2}\right)^2$, and using inequality (3.25) with $p = 2k+1$, $q = \frac{2k+1}{2k}$, $d = \eta\left(\frac{[\Sigma^\mathsf{T} w]_j}{\|\Sigma_{.j}\|_2}\right)^2$ and $e = 1$.

In the conditions of Lemma 3.1.5, $\Sigma^\alpha$ is bounded, so is $\mathrm{Tr}(\Sigma^\alpha(t,x)\Sigma^\alpha(t,x)^\mathsf{T})$. Let $\bar{a}$ be a bound of $\mathrm{Tr}(\Sigma^\alpha(t,x)\Sigma^\alpha(t,x)^\mathsf{T})$.

Taking $\eta = \epsilon^2$ such that $h^{1/2}C\frac{\epsilon^{4k+1}}{4k+2}c_k^{-1} = 1$ and using that $\mathrm{Tr}(\Sigma^\alpha(t,x)\Sigma^\alpha(t,x)^\mathsf{T}) \leq \bar{a}$, we obtain

$$\mathscr{P}^{t,x,\alpha,k,h}(w) \geq 1 - hC - \frac{\bar{a}}{4k+2} - \frac{h^{1/2}C}{\epsilon}\left(\frac{2k}{4k+2}\bar{a} + \frac{1}{2}\right)\ .$$

We can observe that

$$\begin{aligned}
&h^{1/2}C\frac{\epsilon^{4k+1}}{4k+2}c_k^{-1} = 1 \\
\Rightarrow\ &\epsilon = \left(\frac{(4k+2)c_k}{Ch^{1/2}}\right)^{\frac{1}{(4k+1)}}.
\end{aligned}$$

By letting $C_k^1 = \left(\frac{(4k+2)c_k}{C}\right)^{\frac{1}{(4k+1)}}$, we thus have

$$\begin{aligned}
\frac{h^{1/2}C}{\epsilon} &= \frac{C}{C_k^1}h^{\frac{1}{2}(1+\frac{1}{4k+1})} \\
&= \frac{C}{C_k^1}h^{\frac{2k+1}{4k+1}}.
\end{aligned}$$

Since $\frac{h^{1/2}C}{\epsilon}$ is a multiple of $h^{(2k+1)/(4k+1)}$, there exists a constant $C_k$ depending on $k$ and $\bar{a}$, such that

$$\mathscr{P}^{t,x,\alpha,k,h}(w) \geq L_{k,h} := 1 - hC - \frac{\bar{a}}{4k+2} - C_k h^{(2k+1)/(4k+1)}\ ,$$

for all $w \in \mathbb{R}^d$. Let us choose $k$ such that $\frac{\bar{a}}{4k+2} < 1$. We get that the lower bound $L_{k,h}$ of $\mathscr{P}^{t,x,\alpha,k,h}$ is nonnegative for $h \leq h_0$ for some $h_0 > 0$, which implies that $\mathbf{T}^0_{k,h}$ is monotone. Then, for $h \geq h_0$, $C_k h^{(2k+1)/(4k+1)}/h \leq C'$ for some constant $C' > 0$, which implies that $L_{k,h} \geq -h(C+C')$ for all $h > 0$. This shows that $\mathbf{T}^0_{k,h}$ satisfies (3.24) with $C_1 = (C+C')$. □

**Remark 3.1.2.** *In the condition of Lemma 3.1.5, the existence of the bounded map $g^\alpha$ can be ensured by the fact that $a^\alpha - \underline{\sigma\sigma}^\intercal$ is uniformly lower bounded by a positive matrix.*

We will need the following definition in what follows.

**Definition 3.1.2.** *We say that an operator $T$ between any sets $\mathscr{F}$ and $\mathscr{F}'$ of partially ordered sets of real valued functions, which are stable by the addition of a constant function (identified to a real number), is additively $\alpha$-subhomogeneous if*

$$\lambda \in \mathbb{R}, \lambda \geq 0, \ \phi \in \mathscr{F} \implies T(\phi + \lambda) \leq T(\phi) + \alpha\lambda \ . \tag{3.26}$$

**Lemma 3.1.6.** *Assume $c^\alpha$ is upper bounded in $t$, $x$ uniformly in $\alpha$. Then, $\mathbf{T}^0_{k,h}$ is additively $\alpha_h$ subhomogeneous over the set of bounded continuous functions $\mathbb{R}^d \to \mathbb{R}$, for some constant $\alpha_h = 1 + Ch$ with $C \geq 0$.*

*Proof.* Let $\phi$ be a bounded continuous function from $\mathbb{R}^d$ to $\mathbb{R}$ and $\lambda \geq 0$. Let $C \geq 0$ be a constant such that $c^\alpha \leq C$. Let $(t, x) \in [0, T] \times \mathbb{R}^d$. As :

- $\mathbb{E}\left[\mathscr{P}^2_{\Sigma^\alpha(t,x),k}(h^{-1/2}B^t_h)\right] = 0;$
- $\mathbb{E}\left[\mathscr{P}^{1,0}_{\underline{\sigma}^{-1}(t,x)(b^\alpha - \underline{b})(t,x)}(h^{-1}B^t_h)\right] = 0;$

Definition 3.22, implies that $\mathbf{T}^0_{k,h}[\phi+\lambda] \leq \mathbf{T}^0_{k,h}[\phi]+(1+Ch)\lambda$. Hence, the result of lemma. □

**Lemma 3.1.7** (Stability)**.** *Let assumptions of Lemma 3.1.5 and Lemma 3.1.6 hold. Assume that $f^\alpha$ is bounded in $t$, $x$ uniformly in $\alpha$ and let $W^h$ be the solution of the scheme* (3.21) *where $\Psi$ is a bounded function. Then $W^h$ is bounded, meaning that the scheme is stable.*

*Proof.* $f^\alpha$ bounded uniformly in $t$, $x$ and $\alpha$ implies that there exists $C > 0$ such that $|\mathbf{T}^0_{k,h}[0]| \leq Ch$. The assumptions of Lemma 3.1.5 imply that there exists $C_1$ such that $\mathbf{T}^0_{k,h}$ is $C_1h$-almost monotone. We can take $C_1 = C$. As assumptions of Lemma 3.1.6 also hold, we can also take $C$ such that $\mathbf{T}^0_{k,h}$ is additively $(1 + Ch)$-subhomogeneous. Let us suppose that $|W^h(t + h, \cdot)|$ ($t \in \mathscr{T}_h$) is bounded by a constant $K_{t+h}$. Then, by the $Ch$-almost monotonicity

$$W^h(t, \cdot) \leq \mathbf{T}^0_{k,h}[K_{t+h}] + Ch(2K_{t+h}).$$

The $(1 + Ch)$-subhomogeneity of $\mathbf{T}^0_{k,h}$ implies that

$$\begin{aligned}\mathbf{T}^0_{k,h}[K_{t+h}] &\leq \mathbf{T}^0_{k,h}[0] + (1 + Ch)K_{t+h} \\ &\leq Ch + (1 + Ch)K_{t+h}.\end{aligned}$$

Hence,

$$W^h(t, \cdot) \leq Ch + (1 + 3Ch)K_{t+h}.$$

By symmetry, we obtain that $|W^h(t, \cdot)|$ is bounded by $K_t = Ch + (1 + 3Ch)K_{t+h}$. We suppose also that the bound of $\Psi$ is $C$. We deduce by induction that $|W^h|$ is bounded by $(1 + 3Ch)^{T/h}(1 + 3Ch + C) \leq e^{3CT}(1 + C + 3Ch_0)$. □

**Theorem 3.1.3** (Convergence)**.** *Assume that :*

- *$a^\alpha$, $b^\alpha$, $c^\alpha$, $f^\alpha$, $\Psi$ are bounded in $t$, $x$ uniformly in $\alpha$ and Lipschitz continuous with respect $x$ with a Lipschitz constant independent of $\alpha$ and $t$;*
- *$\underline{\sigma} : [0,T] \times \mathbb{R}^d \to \mathbb{R}^{d\times d}$ and $\underline{b} : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ are continuous functions and such that*
  - *$a^\alpha - \underline{\sigma\sigma}^\intercal$ is lower bounded by a positive matrix uniformly in $t$, $x$ and $\alpha$;*
  - *$\underline{\sigma\sigma}^\intercal$ is lower bounded by a positive matrix uniformly in $t$ and $x$;*
  - *$\underline{b}$ and $\underline{\sigma}$ are bounded;*

*Then, the PDE* (3.1)-(3.2) *has a unique bounded continuous viscosity solution and the scheme* (3.21) *converges to this solution.*

*Proof.* The fact that the PDE (3.1)-(3.2) has a unique bounded continuous viscosity solution comes from the first assumption and Theorem 1.2.2 due to Barles and Jakobsen, which gives us also the fact that the PDE (3.1)-(3.2) satisfies a comparison principle in the space of continuous bounded functions. This first assumption with the other allow the assumptions of Lemma 3.1.4, Lemma 3.1.5, Lemma 3.1.7 to hold. Theorem 2.1.1 due to Barles and Souganidis allow us to conclude that the scheme defined by (3.21) converges to the viscosity solution of (3.1)-(3.2). □

**3.1.2.b Upwind large stencil probabilistic scheme** In this scheme, we use the upwind first order derivatives approximation (3.13) instead of the approximation of Lemma 3.1.3. Moreover, we do not approximate $W$ by $\mathscr{D}^0_{t,h}[W]$ in the Hamiltonian as Fahim, Touzi and Warin in [25].

In the sequel, we will first obtain a raw form of the discretized equation before writing it in the form of a scheme. We will use the notations $\hat{X}^{t,x}_h$ and $B^t_h$ introduced in Section 3.1.1.

Let us define $\Sigma^\alpha$ as in the previous subsection ((3.20)) and $g^\alpha$ such that

$$(b^\alpha - \underline{b})(t,x) = \underline{\sigma}(t,x) g^\alpha(t,x). \tag{3.27}$$

In the PDE (3.3) where the Hamilonan $F$ is given by (3.4), we approximate for $(t,x) \in \mathscr{T}_h \times \mathbb{R}^d$,

$$\mathscr{L}^X W(t,x)$$

by

$$\frac{\mathbb{E}\left[W(t+h,\hat{X}^{t,x}_h)\right] - W(t,x)}{h}.$$

We also approximate for $\alpha \in \mathscr{A}$, $(t,x) \in \mathscr{T}_h \times \mathbb{R}^d$,

$$\frac{1}{2}\operatorname{Tr}\left[(a^\alpha - \underline{\sigma\sigma}^\intercal)(t,x) D^2_x W(t,x)\right]$$

by

$$\frac{1}{h}\mathbb{E}\left[W(t+h,\hat{X}^{t,x}_h)\mathscr{P}^2_{\Sigma^\alpha(t,x),k}(h^{-1/2}B^t_h)\right],$$

and

$$(b^\alpha - \underline{b})(t,x) \cdot D_x W(t,x)$$

by

$$\mathbb{E}\left[(W(t+h,\hat{X}_h^{t,x}) - W(t,x))\mathscr{P}^1_{g^\alpha(t,x)}(h^{-1}B_h^t)\right].$$

This gives us the following equation as an approximation of (3.3).

$$-\frac{\mathbb{E}\left[W(t+h,\hat{X}_h^{t,x})\right] - W(t,x)}{h} - \inf_{\alpha\in\mathscr{A}}\Bigg\{\frac{1}{h}\mathbb{E}\left[W(t+h,\hat{X}_h^{t,x})\mathscr{P}^2_{\Sigma^\alpha(t,x),k}(h^{-1/2}B_h^t)\right] + \mathbb{E}\left[(W(t+h,\hat{X}_h^{t,x}) - W(t,x))\mathscr{P}^1_{g^\alpha(t,x)}(h^{-1}B_h^t)\right] + c^\alpha(t,x)W(t,x) + f^\alpha(t,x)\Bigg\} = 0, \tag{3.28}$$

which is equivalent to :

$$\sup_{\alpha\in\mathscr{A}}\Bigg\{ -\mathbb{E}\left[W(t+h,\hat{X}_h^{t,x})(1+\mathscr{P}^2_{\Sigma^\alpha(t,x),k}(h^{-1/2}B_h^t) + h\mathscr{P}^1_{g^\alpha(t,x)}(h^{-1}B_h^t))\right] + W(t,x)(1+h\mathbb{E}\left[\mathscr{P}^1_{g^\alpha(t,x)}(h^{-1}B_h^t)\right] - hc^\alpha(t,x)) - hf^\alpha(t,x)\Bigg\} = 0. \tag{3.29}$$

Let

$$T^D_{h,\alpha,B}(t,x) := 1 + h\mathbb{E}\left[\mathscr{P}^1_{g^\alpha(t,x)}(h^{-1}B_h^t)\right] - hc^\alpha(t,x),$$

and

$$T^N_{k,h,\alpha,B}(t,x) := 1 + \mathscr{P}^2_{\Sigma^\alpha(t,x),k}(h^{-1/2}B_h^t) + h\mathscr{P}^1_{g^\alpha(t,x)}(h^{-1}B_h^t).$$

**Lemma 3.1.8.** *If $T^D_{h,\alpha,B}(t,x)$ is lower bounded by a positive constant uniformly in $\alpha$ (which happens if $c^\alpha$ is lower bounded uniformly in $\alpha$ and $h$ is small enough), then equation* (3.29) *implies that*

$$W(t,x) = \inf_{\alpha\in\mathscr{A}}\left\{\frac{\mathbb{E}\left[W(t+h,\hat{X}_h^{t,x})T^N_{k,h,\alpha,B}(t,x)\right] + hf^\alpha(t,x)}{T^D_{h,\alpha,B}(t,x)}\right\}. \tag{3.30}$$

*Proof.* Let suppose that equation (3.29) holds. Let $M(t,x) > 0$ be a lower bound of $T^D_{h,\alpha,B}(t,x)$. Then for any $\epsilon > 0$, there exists $\alpha_\epsilon$ such that :

$$\mathbb{E}\left[W(t+h,\hat{X}_h^{t,x})T^N_{k,h,\alpha_\epsilon,B}(t,x)\right] + hf^{\alpha_\epsilon}(t,x) - W(t,x)T^D_{h,\alpha_\epsilon,B}(t,x) \le \epsilon.$$

This gives us :

$$\frac{\mathbb{E}\left[W(t+h,\hat{X}_h^{t,x})T^N_{k,h,\alpha_\epsilon,B}(t,x)\right]+hf^{\alpha_\epsilon}(t,x)}{T^D_{h,\alpha_\epsilon,B}(t,x)} \leq \frac{\epsilon}{T^D_{h,\alpha_\epsilon,B}(t,x)}+W(t,x)$$
$$\Rightarrow \frac{\mathbb{E}\left[W(t+h,\hat{X}_h^{t,x})T^N_{k,h,\alpha_\epsilon,B}(t,x)\right]+hf^{\alpha_\epsilon}(t,x)}{T^D_{h,\alpha_\epsilon,B}(t,x)} \leq \frac{\epsilon}{M(t,x)}+W(t,x).$$

By letting $\epsilon$ go to 0, we obtain that :

$$\begin{aligned}&\inf_{\alpha\in\mathscr{A}}\left\{\frac{\mathbb{E}\left[W(t+h,\hat{X}_h^{t,x})T^N_{k,h,\alpha,B}(t,x)\right]+hf^{\alpha}(t,x)}{T^D_{h,\alpha,B}(t,x)}\right\}\\ &\leq \lim_{\epsilon\to 0}\frac{\mathbb{E}\left[W(t+h,\hat{X}_h^{t,x})T^N_{k,h,\alpha_\epsilon,B}(t,x)\right]+hf^{\alpha_\epsilon}(t,x)}{T^D_{h,\alpha_\epsilon,B}(t,x)}\\ &\leq W(t,x).\end{aligned}$$

On the other hand, for $\alpha\in\mathscr{A}$ :

$$\begin{aligned}&0\leq \mathbb{E}\left[W(t+h,\hat{X}_h^{t,x})T^N_{k,h,\alpha,B}(t,x)\right]+hf^{\alpha}(t,x)-W(t,x)T^D_{h,\alpha,B}(t,x)\\ &\Rightarrow W(t,x)\leq \frac{\mathbb{E}\left[W(t+h,\hat{X}_h^{t,x})T^N_{k,h,\alpha,B}(t,x)\right]+hf^{\alpha}(t,x)}{T^D_{h,\alpha,B}(t,x)}\\ &\Rightarrow W(t,x)\leq \inf_{\alpha\in\mathscr{A}}\left\{\frac{\mathbb{E}\left[W(t+h,\hat{X}_h^{t,x})T^N_{k,h,\alpha,B}(t,x)\right]+hf^{\alpha}(t,x)}{T^D_{h,\alpha,B}(t,x)}\right\}.\end{aligned}$$

Hence the result of the lemma. □

**Lemma 3.1.9.** *If $T^D_{h,\alpha,B}(t,x)$ is lower bounded by a positive constant and upper bounded uniformly in $\alpha$ (which happens if $g^\alpha$ and $c^\alpha$ are bounded), then equation* (3.29) *is equivalent to* (3.30).

*Proof.* The implication (3.29) ⇒ (3.30) has been proved in the previous lemma.

We now suppose that equation (3.30) holds and $T^D_{h,\alpha,B}(t,x)\geq M(t,x)>0$. Let $N(t,x)$ be an upper bound of $T^D_{h,\alpha,B}(t,x)$. Then for any $\epsilon>0$ there exists $\alpha_\epsilon$ such that :

$$\frac{\mathbb{E}\left[W(t+h,\hat{X}_h^{t,x})T^N_{k,h,\alpha_\epsilon,B}(t,x)\right]+hf^{\alpha_\epsilon}(t,x)}{T^D_{h,\alpha_\epsilon,B}(t,x)}\leq W(t,x)+\epsilon.$$

This gives us

$$\begin{aligned}&\mathbb{E}\left[W(t+h,\hat{X}_h^{t,x})T^N_{k,h,\alpha_\epsilon,B}(t,x)\right]+hf^{\alpha_\epsilon}(t,x)\leq W(t,x)T^D_{h,\alpha_\epsilon,B}(t,x)+\epsilon T^D_{h,\alpha_\epsilon,B}(t,x)\\ &\Rightarrow \mathbb{E}\left[W(t+h,\hat{X}_h^{t,x})T^N_{k,h,\alpha_\epsilon,B}(t,x)\right]+hf^{\alpha_\epsilon}(t,x)\leq W(t,x)T^D_{h,\alpha_\epsilon,B}(t,x)+\epsilon N(t,x)\\ &\Rightarrow \mathbb{E}\left[W(t+h,\hat{X}_h^{t,x})T^N_{k,h,\alpha_\epsilon,B}(t,x)\right]+hf^{\alpha_\epsilon}(t,x)-W(t,x)T^D_{h,\alpha_\epsilon,B}(t,x)\leq \epsilon N(t,x)\\ &\Rightarrow \inf_{\alpha\in\mathscr{A}}\left\{\mathbb{E}\left[W(t+h,\hat{X}_h^{t,x})T^N_{k,h,\alpha,B}(t,x)\right]+hf^{\alpha}(t,x)-W(t,x)T^D_{h,\alpha,B}(t,x)\right\}\leq \epsilon N(t,x).\end{aligned}$$

By letting $\epsilon$ goes to 0, we then obtain

$$\inf_{\alpha\in\mathscr{A}}\left\{\mathbb{E}\left[W(t+h,\hat{X}_h^{t,x})T^N_{k,h,\alpha,B}(t,x)\right]+hf^\alpha(t,x)-W(t,x)T^D_{h,\alpha,B}(t,x)\right\}\leq 0.$$

On the other hand, for $\alpha\in\mathscr{A}$ :

$$\begin{aligned}
&\frac{\mathbb{E}\left[W(t+h,\hat{X}_h^{t,x})T^N_{k,h,\alpha,B}(t,x)\right]+hf^\alpha(t,x)}{T^D_{h,\alpha,B}(t,x)}\geq W(t,x)\\
\Rightarrow&\mathbb{E}\left[W(t+h,\hat{X}_h^{t,x})T^N_{k,h,\alpha,B}(t,x)\right]+hf^\alpha(t,x)\geq W(t,x)T^D_{h,\alpha,B}(t,x)\\
\Rightarrow&\mathbb{E}\left[W(t+h,\hat{X}_h^{t,x})T^N_{k,h,\alpha,B}(t,x)\right]+hf^\alpha(t,x)-W(t,x)T^D_{h,\alpha,B}(t,x)\geq 0\\
\Rightarrow&\inf_{\alpha\in\mathscr{A}}\left\{\mathbb{E}\left[W(t+h,\hat{X}_h^{t,x})T^N_{k,h,\alpha,B}(t,x)\right]+hf^\alpha(t,x)-W(t,x)T^D_{h,\alpha,B}(t,x)\right\}\geq 0.
\end{aligned}$$

Hence, in the conditions of the lemma, equation (3.30) implies (3.29). Hence, the result of the lemma. □

Now, let us consider the following scheme :

$$\begin{cases} W^h(t,x):=\mathbf{T}^1_{k,h}[W^h(t+h,\cdot)](t,x), \quad (t,x)\in\mathscr{T}_h\times\mathbb{R}^d\\ W^h(T,x):=\Psi(x), \quad x\in\mathbb{R}^d,\end{cases} \tag{3.31}$$

where

$$\mathbf{T}^1_{k,h}[\Phi](t,x):=\inf_{\alpha\in\mathscr{A}}\left\{\frac{\mathbb{E}\left[\Phi(\hat{X}_h^{t,x})T^N_{k,h,\alpha,B}(t,x)\right]+hf^\alpha(t,x)}{T^D_{h,\alpha,B}(t,x)}\right\}. \tag{3.32}$$

We are going to give the conditions under which this scheme is convergent.

**Lemma 3.1.10** (Consistency)**.** *Assume that :*

- $\underline{\sigma}$ *and* $\underline{b}$ *are uniformly bounded in* $t$ *and* $x$*;*
- $\underline{\sigma\sigma}^\intercal$ *is lower bounded by a positive matrix uniformly in* $t$ *and* $x$*;*
- $a^\alpha$, $b^\alpha$ *and* $c^\alpha$ *are bounded in* $t$, $x$ *uniformly in* $\alpha$*;*
- *the left hand part of PDE* (3.3) *is continuous;*

*Then, the scheme* (3.31) *is consistent with PDE* (3.3)*.*

*Proof.* In the conditions of the lemma, $\underline{\sigma}^{-1}$ is bounded, so is $g^\alpha$, $\Sigma^\alpha$. $c^\alpha$ is also bounded. Then by Lemma 3.1.9, the scheme (3.31) is equivalent to the discretized equation (3.30) which is equivalent to equation (3.28). Moreover, Theorem 3.1.1, Theorem 3.1.2 and Lemma 3.1.2 can be applied.

By using these results and letting $S(h,t,x,W(t,x),W(t+h,\cdot))$ be the left hand side of equation (3.28) for $W \in \mathscr{C}_{\mathrm{b}}^4([0,T]\times\mathbb{R}^d)$, we have :

$$
\begin{aligned}
S(h,s,y,W(s,y)+c,W(s+h,\cdot)+c) = -\mathscr{L}^X W(s,y) + O(h) - \\
\inf_{\alpha\in\mathscr{A}} \Bigg\{ \frac{1}{2}\operatorname{Tr}\left[(a^\alpha - \underline{\sigma\sigma}^\intercal)(s,y)D_x^2 W(s,y)\right] + O(h) \\
+ (b^\alpha - \underline{b})(t,x)\cdot D_x W(t,x) + O(\sqrt{h}) + c^\alpha(t,x)W(t,x) \\
+ f^\alpha(t,x) + c^\alpha(t,x)c \Bigg\},
\end{aligned}
$$

for $(s,y) \in [O,T]\times\mathbb{R}^d$ such that $s+h \le T$. We can notice that $c$ disapear everywhere in the expression of $S(h,s,y,W(s,y)+c,W(s+h,\cdot)+c)$ except as a factor of $c^\alpha(t,x)$ as $\mathbb{E}\left[\mathscr{P}^2_{\Sigma^\alpha(t,x),k}(h^{-1/2}B_h^t)\right] = 0$.

As $c^\alpha$ is bounded uniformly in $t,x,\alpha$, $c^\alpha(t,x)c = O(c)$. So :

$$
\begin{aligned}
\lim_{(h,c)\to(0,0)} & S(h,s,y,W(s,y)+c,W(s+h,\cdot)+c) \\
&= -\mathscr{L}^X W(s,y) - F(s,y,W(s,y),D_x W(s,y),D_x^2 W(s,y)) \\
&\underset{(s,y)\to(t,x)}{\longrightarrow} -\mathscr{L}^X W(t,x) - F(t,x,W(t,x),D_x W(t,x),D_x^2 W(t,x)),
\end{aligned}
$$

because of the continuity of $-\mathscr{L}^X W(\cdot,\cdot) - F(\cdot,\cdot,W(\cdot,\cdot),D_x W(\cdot,\cdot),D_x^2 W(\cdot,\cdot))$ which is an assumption of the lemma. Hence, the consistency of the scheme (3.31) with PDE (3.3). □

**Lemma 3.1.11** (Monotonicity)**.** *Assume that :*

- $\underline{\sigma\sigma}^\intercal$ *is lower bounded by a positive matrix uniformly in* $t$ *and* $x$*;*
- $\underline{\sigma}$ *is bounded in* $t$ *and* $x$*;*
- $a^\alpha$ *is bounded in* $t$ *and* $x$ *uniformly in* $\alpha$*;*

*There exists* $k_0 \in \mathbb{N}$ *such that for* $k > k_0$*, the scheme* (3.31) *is monotone.*

*Proof.* The conditions of the lemma ensure that $\Sigma^\alpha$ is bounded in $t$, $x$ uniformly in $\alpha$. Let $C$ be a bound of $\Sigma^\alpha$.

$$
\begin{aligned}
T^N_{k,h,\alpha,B}(t,x) &\ge 1 - \frac{|\Sigma^\alpha(t,x)|^2}{4k+2} \\
&\ge 1 - \frac{C^2}{4k+2}.
\end{aligned}
$$

Let $k_0$ be such that $\frac{C^2}{4k_0+2} < \frac{1}{2}$. Then for $k > k_0$,

$$T^N_{k,h,\alpha,B}(t,x) > \frac{1}{2} > 0.$$

Hence the monotonicity of the scheme. □

**Lemma 3.1.12.** *If $c^\alpha$ is bounded in $t$ and $x$ uniformly in $\alpha$, then there exists $h_0 > 0$ such that for $h < h_0$, $\mathbf{T}^1_{k,h}$ is additively $\alpha_h$-subhomogeneous over the set of bounded continuous functions $\mathbb{R}^d \to \mathbb{R}$ for some $\alpha_h = 1 + Ch$ with $C \geq 0$.*

*Proof.* As $\mathbb{E}\left[\mathscr{P}^2_{\Sigma^\alpha(t,x),k}(h^{-1/2}B^t_h)\right] = 0$, for $\Phi$ a bounded function from $\mathbb{R}^d$ to $\mathbb{R}$ and $\lambda > 0$, we have

$$\mathbf{T}^1_{k,h}[\Phi + \lambda](t,x) = \inf_{\alpha\in\mathscr{A}} \left\{ \frac{\mathbb{E}\left[\Phi(\hat{X}^{t,x}_h) T^N_{k,h,\alpha,B}(t,x)\right] + hf^\alpha(t,x)}{T^D_{h,\alpha,B}(t,x)} + \lambda \left[1 + \frac{hc^\alpha(t,x)}{T^D_{h,\alpha,B}(t,x)}\right] \right\}.$$

As $c^\alpha$ is bounded in $t$ and $x$ uniformly in $\alpha$, there exists $h_0$ such that for $h < h_0$, $T^D_{h,\alpha,B}(t,x) \geq M > 0$. In this setting, there exists $C > 0$ such that

$$\frac{c^\alpha(t,x)}{T^D_{h,\alpha,B}(t,x)} \leq C,$$

hence the result of the lemma. □

**Lemma 3.1.13** (Stability)**.** *Let the assumptions of Lemma 3.1.11 and Lemma 3.1.12 hold. Assume that $f^\alpha$ is bounded in $t$ and $x$ uniformly in $\alpha$ and let $W^h$ be the solution of the scheme* (3.31) *where $\Psi$ is a bounded function. Then there exists $k_0 \in \mathbb{N}$ and $h_0 > 0$ such that for $k > k_0$ and $h < h_0$, $W^h$ is bounded, meaning that the scheme is stable.*

*Proof.* The assumptions of Lemma 3.1.12 imply that there exists $h_0$ such that for $h < h_0$, the operator $\mathbf{T}^1_{k,h}$ is additively $(1 + Ch)$-subhomogeneous for $C > 0$ and $T^D_{h,\alpha,B}(t,x)$ is lower bounded by a positive constant $M$.

$$\mathbf{T}^1_{k,h}[0](t,x) = h \inf_{\alpha\in\mathscr{A}} \left\{ \frac{f^\alpha(t,x)}{T^D_{h,\alpha,B}(t,x)} \right\}.$$

As $f^\alpha$ is bounded and $T^D_{h,\alpha,B}$ is lower bounded in $t$ and $x$ uniformly in $\alpha$, $\inf_{\alpha\in\mathscr{A}} \left\{ \frac{f^\alpha(t,x)}{T^D_{h,\alpha,B}(t,x)} \right\}$ is bounded. We can take $C$ also as its bound.

Let us suppose that $|W^h(t+h,\cdot)|$ ($t \in \mathscr{T}_h$) is bounded by a constant $K_{t+h}$. By Lemma 3.1.11, there exists $k_0 \in N$ such that for $k > k_0$, the scheme (3.31) is monotone. So for $k > k_0$,

$$\mathbf{T}^1_{k,h}[-K_{t+h}] \leq W^h(t,\cdot) \leq \mathbf{T}^1_{k,h}[K_{t+h}].$$

By the $(1 + Ch)$-subhomogeneity of the operator $\mathbf{T}^1_{k,h}$, we have :

$$\begin{aligned} \mathbf{T}^1_{k,h}[0] &\leq \mathbf{T}^1_{k,h}[-K_{t+h}] + K_{t+h}(1 + Ch) \\ \Rightarrow -Ch - K_{t+h}(1 + Ch) &\leq \mathbf{T}^1_{k,h}[-K_{t+h}], \end{aligned}$$

and

$$\begin{aligned} \mathbf{T}^1_{k,h}[K_{t+h}] &\leq \mathbf{T}^1_{k,h}[0] + K_{t+h}(1 + Ch) \\ \Rightarrow \mathbf{T}^1_{k,h}[K_{t+h}] &\leq Ch + K_{t+h}(1 + Ch). \end{aligned}$$

Hence, $|W^h(t,\cdot)|$ is bounded by $Ch + K_{t+h}(1 + Ch)$. If $C$ is also a bound of $\Psi$, then by induction (we can also use the Gronwall Lemma), we obtain that $|W^h|$ is bounded by $(1 + Ch)^{T/h}(1 + Ch + C) \leq e^{CT}(1 + C + Ch_0)$. □

**Theorem 3.1.4** (Convergence)**.** *Assume that :*

- $a^\alpha$, $b^\alpha$, $c^\alpha$, $f^\alpha$, $\Psi$ *are bounded in* $t$, $x$ *uniformly in* $\alpha$ *and Lipschitz continuous with respect* $x$ *with a Lipschitz constant independent of* $\alpha$ *and* $t$*;*
- $\underline{\sigma} : [0,T] \times \mathbb{R}^d \to \mathbb{R}^{d\times d}$ *and* $\underline{b} : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ *are continuous functions and such that*
  - $a^\alpha \geq \underline{\sigma\sigma}^\intercal$
  - $\underline{\sigma\sigma}^\intercal$ *is lower bounded by a positive matrix uniformly in* $t$ *and* $x$*;*
  - $\underline{b}$ *and* $\underline{\sigma}$ *are bounded;*

*Then, the PDE* (3.1)-(3.2) *has a unique bounded continuous viscosity solution and the scheme* (3.31) *converges to this solution.*

*Proof.* We use Theorem 1.2.2 to obtain the existence and unicity of the viscosity solution of PDE (3.1)-(3.2). The convergence of the scheme (3.31) is then deduced from Theorem 2.1.1 as a consequence of Lemma 3.1.10, Lemma 3.1.11, Lemma 3.1.13 and the comparison principle of the PDE (3.1)-(3.2) in the space of bounded continuous functions obtained from Theorem 1.2.2. □

**Remark 3.1.3.** *We can notice that the convergence of this second probabilistic scheme requires less restrictive conditions than the first one presented in Section 3.1.2.a as* $a^\alpha - \underline{\sigma\sigma}^\intercal$ *should not necessarily be lower bounded by a positive matrix uniformly in* $t$, $x$ *and* $\alpha$*, but is just required to be nonnegative.*

### 3.1.3 Error estimates of probabilistic schemes

We now use Theorem 2.1.2 to obtain error estimates of each scheme presented in the previous subsection. For that, we will need a few lemma to be able to apply Theorem 2.1.2.

**3.1.3.a Large stencil probabilistic scheme** We start by the large stencil probabilistic scheme.

**Lemma 3.1.14.** *Under the conditions of Lemma 3.1.5, there exists* $\lambda, \mu \geq 0$, $h_0 > 0$ *such that if* $h \leq h_0$, $W \leq V$ *are continuous and bounded functions on* $\mathscr{T}_h \times \mathbb{R}^d$*, and* $\phi(t) = e^{\mu(T-t)}(a + b(T-t)) + c$ *for* $a, b, c \geq 0$*, then :*

$$S^1(h,t,x,r+\phi(t),[W+\phi](t+h,\cdot)) \geq S^1(h,t,x,r,V(t+h,\cdot)) + b/2 - \lambda c \text{ in } \mathscr{T}_h \times \mathbb{R}^d,$$

*where*

$$S^1(h,t,x,r,\Phi) := \frac{r - \mathbf{T}^0_{k,h}[\Phi](t,x)}{h}, \tag{3.33}$$

$(t,x) \in \mathscr{T}_h \times \mathbb{R}^d$, $r \in \mathbb{R}$, $\Phi$ *bounded function on* $\mathbb{R}^d$.

*Proof.* Let $\mu, a, b, c \geq 0$, and $\phi$ be defined as in the lemma. Let $W \leq V$ be two continuous and bounded functions on $\mathscr{T}_h \times \mathbb{R}^d$. Under the conditions of Lemma 3.1.5, $c^\alpha$ is bounded and there exists $h_0 > 0$ such that for $h < h_0$, $\mathbf{T}^0_{k,h}$ is monotone. Let $M$ be a bound of $c^\alpha$. By using a development similar to the one of the proof of Lemma 3.1.4, we obtain that :

$$S^1(h,t,x,r+\phi(t),[W+\phi](t+h,\cdot)) \geq S^1(h,t,x,r,W(t+h,\cdot)) + \frac{\phi(t) - (1+Mh)\phi(t+h)}{h}.$$

For $h < h_0$,

$$S^1(h,t,x,r,W(t+h,\cdot)) \geq S^1(h,t,x,r,V(t+h,\cdot)).$$

$$\phi(t) - (1+Mh)\phi(t+h) = e^{\mu(T-t)}(1-(1+Mh)e^{-\mu h})\,(a+b(T-t)) - Mhc \\ + e^{\mu(T-t-h)}(1+Mh)bh$$

For $\mu$ high enough ($\mu > M$), $1-(1+Mh)e^{-\mu h} > 0$ for $h < h_1$. So by taking $\lambda = M$,

$$\frac{\phi(t) - (1+Mh)\phi(t+h)}{h} \geq b - \lambda c \geq b/2 - \lambda c.$$

Hence the result of the lemma. □

**Lemma 3.1.15.** *Under the conditions of Theorem 3.1.3, for every $h > 0$ and continuous and bounded function $\Phi$ on $\bar{\mathscr{T}}_h \times \mathbb{R}^d$, the function $(t,x) \mapsto S^1(h,t,x,\Phi(t,x),\Phi(t+h,\cdot))$ is bounded and continuous on $\mathscr{T}_h \times \mathbb{R}^d$ and the function $r \mapsto S^1(h,t,x,r,\Phi(t+h,\cdot))$ is uniformly continuous for bounded $r$, uniformly in $(t,x) \in \mathscr{T}_h \times \mathbb{R}^d$ with $S^1$ given by equation* (3.33).

*Proof.* Let $h > 0$ and $\Phi$ be a continuous and bounded function on $\bar{\mathscr{T}}_h \times \mathbb{R}^d$. As $\mathscr{T}_h$ is finite, to prove that $(t,x) \mapsto S^1(h,t,x,\Phi(t,x),\Phi(t+h,\cdot))$ is bounded and continuous on $\mathscr{T}_h \times \mathbb{R}^d$, it is sufficient to prove that for any $t \in \mathscr{T}_h$, $x \mapsto S^1(h,t,x,\Phi(t,x),\Phi(t+h,\cdot))$ is continuous and bounded.

$$\begin{aligned} S^1(h,t,x,\Phi(t,x),\Phi(t+h,\cdot)) =& \frac{\Phi(t,x) - \mathbb{E}\left[\Phi(t+h,\hat{X}_h^{t,x})\right]}{h} \\ &- \inf_{\alpha\in\mathscr{A}} \Bigg\{ \mathbb{E}\Big[\Phi(t+h,\hat{X}_h^{t,x})(\mathscr{P}^2_{\Sigma^\alpha(t,x),k}(h^{-1/2}B_h^t) \\ &+ \mathscr{P}^{1,0}_{\sigma^{-1}(t,x)(b^\alpha-\underline{b})(t,x)}(h^{-1}B_h^t) + c^\alpha(t,x))\Big] + f^\alpha(t,x)\Bigg\}. \end{aligned}$$

In the conditions of Theorem 3.1.3, $a^\alpha(t,\cdot)$, $b^\alpha(t,\cdot)$, $c^\alpha(t,\cdot)$ and $f^\alpha(t,\cdot)$ are bounded and Lipschitz continuous with a Lipschitz constant uniform in $t$ and $\alpha$. $\underline{\sigma}$ and $\underline{b}$ are also continuous and bounded and such that $\Sigma^\alpha(t,x)\Sigma^\alpha(t,x)^\intercal$ is lower and upper bounded by a positive matrix uniformly in $t$, $x$ and $\alpha$ and $\underline{\sigma}^{-1}$ is bounded.

$\mathscr{P}^2_{\Sigma^\alpha(t,x),k}(h^{-1/2}B_h^t)$ being the result of polynomial and fractional operations on columns of $\Sigma^\alpha$, it is then continuous in $x$ uniformly in $\alpha$. So is $x \mapsto \mathscr{P}^{1,0}_{\sigma^{-1}(t,x)(b^\alpha-\underline{b})(t,x)}$ for $t \in \mathscr{T}_h$.

We then have that for $t \in \mathscr{T}_h$, the function

$$x \mapsto \mathbb{E}\Big[\Phi(t+h,\hat{X}_h^{t,x})(\mathscr{P}^2_{\Sigma^\alpha(t,x),k}(h^{-1/2}B_h^t) + \mathscr{P}^{1,0}_{\sigma^{-1}(t,x)(b^\alpha-\underline{b})(t,x)}(h^{-1}B_h^t) + c^\alpha(t,x))\Big]$$

is continuous uniformly in $\alpha$ as $\Phi$ is continuous. It is also bounded uniformly in $t$, $x$ and $\alpha$ by an integrable random variable. So by the dominated convergence theorem, the expectation of this function is continuous in $x$ uniformly in $\alpha$. This allows us to conclude that all the part of $S^1$ which is in the infimum on $\alpha \in \mathscr{A}$ is continuous in $x$ uniformly in $\alpha$ and bounded. So this infimum is continuous in $x$ and bounded.

By the dominated convergence theorem, we also obtain that $x \mapsto \mathbb{E}\left[\Phi(t+h, \hat{X}_h^{t,x})\right]$ is continuous in $x$. So, for any $t \in \mathscr{T}_h$, $x \mapsto S^1(h,t,x,\Phi(t,x),\Phi(t+h,\cdot))$ is continuous and bounded.

From the expression of $S^1$, we can see that $r \mapsto S^1(h,t,x,r,\Phi(t+h,\cdot))$ is an affine function with the coefficient of $r$ being $1/h$ and such that for $r=0$, it is finite and bounded uniformly in $t$ and $x$. This function is then uniformly continuous, uniformly in $(t,x) \in \mathscr{T}_h \times \mathbb{R}^d$. □

**Lemma 3.1.16.** *Under the conditions of Theorem 3.1.3, if $W \in C_b^\infty([0,T]\times\mathbb{R}^d)$ is such that :*

$$|\partial_t^p D^q W| \le \tilde{K}\epsilon^{1-2p-q}, \quad \forall p,q \in \mathbb{N}$$

*with $0 < \epsilon \le 1$, $\tilde{K} > 0$, then, when $h$ is small enough :*

$$\begin{aligned}\left|S^1(h,t,x,W(t,x),W(t+h,\cdot)) + \mathscr{L}^X[W](t,x) + F(t,x,W(t,x),D_xW(t,x),D_x^2W(t,x))\right| \\ \le E(\tilde{K},h,\epsilon)\end{aligned}$$

*for all $t \in \mathscr{T}_h$ and $x \in \mathbb{R}^d$ with*

$$E(\tilde{K},h,\epsilon) = K\tilde{K}\left(h\epsilon^{-3}(1+\sqrt{h})^4(1+\sqrt{h}\epsilon^{-1})^4 + \sqrt{h}\epsilon^{-1}(1+\sqrt{h})^2(1+\sqrt{h}\epsilon^{-1})^2\right),$$

*where $K$ depends on bounds of $\underline{\sigma}$, $\underline{b}$, $\underline{\sigma}^{-1}(b^\alpha - \underline{b})$ and $\Sigma^\alpha$ and $S^1$ is given by* (3.33).

*Proof.* We will use the elements of the proof of Theorem 3.1.1, Lemma 3.1.3 and Lemma 3.1.2. We consider $W$ as in the lemma.

From the proof of Theorem 3.1.1, the $S^1(h,t,x,W(t,x),W(t+h,\cdot))$ approximates the second order term in $F$ uniformly with a residual error having two components. The first component is

$$K_1 h(1+\sqrt{h})^4(|\partial_t^1 D^2 W| + |\partial_t^0 D^3 W|)$$

which, considering the conditions of Lemma 3.1.16, gives the following expression

$$K_1 h\tilde{K}(1+\sqrt{h})^4(\epsilon^{-3}+\epsilon^{-2}) \le 2K_1\tilde{K}(1+\sqrt{h})^4 h\epsilon^{-3},$$

$K_1$ being a constant depending on the bounds of $\underline{\sigma}$, $\underline{b}$, $\Sigma^\alpha$.

The second component is a sum of terms of the form

$$\begin{aligned}K_1 h^{p-1}(h+\sqrt{h})^q|\partial_t^p D^q W| &\le K_1 h^{p-1}(1+\sqrt{h})^4(\sqrt{h})^q|\partial_t^p D^q W| \\ &\le K_1(1+\sqrt{h})^4 h^{p-1}(\sqrt{h})^q\tilde{K}\epsilon^{1-2p-q},\end{aligned}$$

for $p+q=4$. This gives us the following expression

$$\begin{aligned}\tilde{K}K_1(1+\sqrt{h})^4\epsilon h^{-1}\left(\frac{h}{\epsilon^2}+\frac{\sqrt{h}}{\epsilon}\right)^4 &= \tilde{K}K_1(1+\sqrt{h})^4\epsilon h^{-1}\left(\frac{\sqrt{h}}{\epsilon}\right)^4(\sqrt{h}\epsilon^{-1}+1)^4 \\ &= \tilde{K}K_1(1+\sqrt{h})^4 h\epsilon^{-3}(1+\sqrt{h}\epsilon^{-1})^4\end{aligned}$$

So the residual error between the second order term of $F$ and its approximation in $S^1$ is bounded by :

$$2K_1\tilde{K}(1+\sqrt{h})^4 h\epsilon^{-3}(1+\sqrt{h}\epsilon^{-1})^4.$$

Using the elements of the proof of Lemma 3.1.3, we also have the first order terms of $F$ that are approximated in $S^1$ uniformly with a residual error being a sum of terms of the form

$$K_2 h^{p-1/2}(1+\sqrt{h})^2(\sqrt{h})^q |\partial_t^p D^q W| \leq K_2 h^{-1/2}(1+\sqrt{h})^2 h^p (\sqrt{h})^q \tilde{K}\epsilon^{1-2p-q},$$

for $p+q=2$, and $K_2$ being a constant depending on the bounds of $\underline{\sigma}^{-1}(b^\alpha - \underline{b})$, $\underline{\sigma}$, $\underline{b}$.

$$\begin{aligned}\sum_{p+q=2} h^p(\sqrt{h})^q \tilde{K}\epsilon^{1-2p-q} &\leq \tilde{K}\epsilon(h\epsilon^{-2}+\sqrt{h}\epsilon^{-1})^2 \\ &\leq \tilde{K}h\epsilon^{-1}(1+\sqrt{h}\epsilon^{-1})^2\end{aligned}$$

So the error estimate in $S^1$ for the first order term of $F$, is bounded by $K_2\tilde{K}\sqrt{h}\epsilon^{-1}(1+\sqrt{h})^2(1+\sqrt{h}\epsilon^{-1})^2$.

Using the elements of the proof of Lemma 3.1.2, we have that $\mathscr{L}^X W$ is approximated in $S^1$ with a residual error having a first component of the form :

$$\begin{aligned}&K_3(1+\sqrt{h})^4 h(|\partial_t^0 D^2 W| + |\partial_t^1 D^1 W| + |\partial_t^2 D^0 W| + |\partial_t^0 D^3 W| + |\partial_t^1 D^2 W|) \\ \leq &K_3(1+\sqrt{h})^4 h\tilde{K}(\epsilon^{-1}+\epsilon^{-2}+\epsilon^{-3}+\epsilon^{-2}+\epsilon^{-3}) \\ \leq &K_3(1+\sqrt{h})^4 5\tilde{K}h\epsilon^{-3},\end{aligned}$$

$K_3$ being a constant dependent on the bounds of $\underline{\sigma}$ and $\underline{b}$.

The second component is exactly as the second error component seen for the approximation of the second order term of $F$ above with $K_3$ instead of $K_1$. This gives us an error bound approximation of $\mathscr{L}^X W$ of

$$5K_3\tilde{K}(1+\sqrt{h})^4 h\epsilon^{-3}(1+\sqrt{h}\epsilon^{-1})^4$$

The third component which is the residual of the form $K_6 h^2(|\partial_t^2 D^1 W| + |\partial_t^3 D^0 W|)$ can be bounded by

$$2K_6 h^2\tilde{K}\epsilon^{-5} \leq 2K_6\tilde{K}h\epsilon^{-3}(1+\sqrt{h}\epsilon^{-1})^4,$$

$K_6$ being a constant dependent on the bounds of $\underline{\sigma}$ and $\underline{b}$.

The approximation of $W$ in $F$ by $\mathbb{E}\left[W(t+h,\hat{X}_h^{t,x})\right]$ in $S^1$ is then bounded by

$$5K_3\tilde{K}(1+\sqrt{h})^4 h^2\epsilon^{-3}(1+\sqrt{h}\epsilon^{-1})^4 + h\|\mathscr{L}^X W\|_\infty,$$

where $\|\mathscr{L}^X W\|_\infty$ is the supremum of $|\mathscr{L}^X W|$ which is bounded by $\tilde{K}(1+2\epsilon^{-1}) = \tilde{K}\epsilon^{-1}(\epsilon+2)$.

As $5K_3\tilde{K}(1+\sqrt{h})^4 h^2\epsilon^{-3}(1+\sqrt{h}\epsilon^{-1})^4 \leq K_4\tilde{K}(1+\sqrt{h})^4 h\epsilon^{-3}(1+\sqrt{h}\epsilon^{-1})^4$ and $h\tilde{K}\epsilon^{-1}(\epsilon+2) \leq K_5\tilde{K}(1+\sqrt{h})^4 h\epsilon^{-3}(1+\sqrt{h}\epsilon^{-1})^4$ for $h$ small, we deduce the result of the lemma. □

We consider the following assumption.

**A 3.1.1.** *For every $\delta > 0$, there are $M \in \mathbb{N}$ and $\{\alpha_i\}_{i=1}^M \subset \mathscr{A}$ such that for any $\alpha \in \mathscr{A}$ :*

$$\inf_{1\leq i\leq M}(|\sigma^\alpha - \sigma^{\alpha_i}|_0 + |b^\alpha - b^{\alpha_i}|_0 + |c^\alpha - c^{\alpha_i}|_0 + |f^\alpha - f^{\alpha_i}|_0) \leq \delta.$$

**Theorem 3.1.5.** *Under the conditions of Theorem 3.1.3 and considering Assumption A 3.1.1, if $W^h$ is the solution of the scheme* (3.21) *and $W$ is the solution of* (3.1)- (3.2)*, then there exists $C_1, C_2$ functions of $|W|_1$ ($|\cdot|_1$ being defined in Section 0.1), such that, for all $(t,x) \in \bar{\mathscr{T}}_h \times \mathbb{R}^d$,*

$$-C_1 h^{1/10} \leq (W^h - W)(t,x) \leq C_2 h^{1/4}.$$

Under the conditions of Theorem 3.1.3, the assumption 3.1.1 can be verified if $\mathscr{A}$ is compact.

*Proof.* Using Lemma 3.1.14, Lemma 3.1.15, Lemma 3.1.16, we can see that under the conditions of Theorem 3.1.3 and with the assumption A 3.1.1, all the assumptions needed to apply Theorem 2.1.2 of Barles and Jakobsen are verified.

Indeed, by taking $\epsilon = h^{1/4}$ in the min expression of the upper bound of the result of Theorem 2.1.2 and $\epsilon = h^{3/10}$ in the lower bound, we obtain the bounds given above.

This gives us the result of the Theorem. □

The bounds we obtain in this case are the same as in Corollary 2.1.1 giving the error bounds estimates for finite difference schemes.

**3.1.3.b Upwind large stencil probabilistic scheme** We are going to obtain in the same way as above, error estimates for the second probabilistic scheme. We will consider $S$ as the operator defined by :

$$S(h,t,x,r,\Phi(\cdot)) := \frac{1}{h} \sup_{\alpha \in \mathscr{A}} \left\{ -\mathbb{E}\left[\Phi(\hat{X}_h^{t,x}) T^N_{k,h,\alpha,B}(t,x)\right] + r T^D_{h,\alpha,B}(t,x) - h f^\alpha(t,x) \right\}.$$

This operator has already been used in the proof of Lemma 3.1.10.

**Lemma 3.1.17.** *Under the conditions of Lemma 3.1.11 and Lemma 3.1.12, there exists $\lambda, \mu \geq 0$, $h_0 > 0$ such that if $h \leq h_0$, $W \leq V$ are functions continuous and bounded on $\mathscr{T}_h \times \mathbb{R}^d$, and $\phi(t) = e^{\mu(T-t)}(a + b(T-t)) + c$ for $a, b, c \geq 0$, then :*

$$S(h,t,x,r+\phi(t),[W+\phi](t+h,\cdot)) \geq S(h,t,x,r,V(t+h,\cdot)) + b/2 - \lambda c \text{ in } \mathscr{T}_h \times \mathbb{R}^d.$$

*Proof.* Let $\mu, a, b, c \geq 0$, and $\phi$ be defined as in the lemma. Let $W \leq V$ be two continuous and bounded functions on $\mathscr{T}_h \times \mathbb{R}^d$ and $(t,x) \in \mathscr{T}_h \times \mathbb{R}^d$.

$$\begin{aligned} S(h,t,x,r+\phi(t),&[W+\phi](t+h,\cdot)) \\ &= \sup_{\alpha \in \mathscr{A}} \Big\{ -\frac{1}{h}\mathbb{E}\left[W(t+h,\hat{X}_h^{t,x}) T^N_{k,h,\alpha,B}(t,x)\right] + \frac{1}{h} r T^D_{h,\alpha,B}(t,x) \\ &\quad - f^\alpha(t,x) + \frac{1}{h}(-\phi(t+h)\mathbb{E}\left[T^N_{k,h,\alpha,B}(t,x)\right] + \phi(t) T^D_{h,\alpha,B}(t,x)) \Big\}. \end{aligned}$$

From the expressions of $T^D_{h,\alpha,B}(t,x)$ and $T^N_{k,h,\alpha,B}(t,x)$ and using the fact that $\phi(t) - \phi(t+h) > 0$,

$\mathbb{E}\left[\mathscr{P}^1_{g^\alpha(t,x)}(h^{-1}B^t_h)\right] \geq 0$ and $|c^\alpha|$ is bounded by $M$,

$$\begin{aligned}-\phi(t+h)\mathbb{E}\left[T^N_{k,h,\alpha,B}(t,x)\right] + \phi(t)T^D_{h,\alpha,B}(t,x) &= (\phi(t)-\phi(t+h))\left(1+\mathbb{E}\left[\mathscr{P}^1_{g^\alpha(t,x)}(h^{-1}B^t_h)\right]\right)\\ &- c^\alpha(t,x)h\phi(t)\\ &\geq (\phi(t)-\phi(t+h)) - c^\alpha(t,x)h\phi(t)\\ &\geq e^{\mu(T-t)}(a+b(T-t))(1-e^{-\mu h}-hM)\\ &+ e^{\mu(T-t-h)}bh - Mhc.\end{aligned}$$

By taking $\mu > M$, there exists $h_0$ such that for $h < h_0$, $1-e^{-\mu h}-hM > 0$. We then have:

$$\begin{aligned}S(h,t,x,r+\phi(t),[W+\phi](t+h,\cdot)) \geq& S(h,t,x,r,W(t+h,\cdot))\\ &+ e^{\mu(T-t-h)}b - Mc\\ &\geq S(h,t,x,r,V(t+h,\cdot))\\ &+ b/2 - Mc,\end{aligned}$$

by Lemma 3.1.11. Hence we obtain the result of the lemma for $\lambda = M$. □

**Lemma 3.1.18.** *Under the conditions of Theorem 3.1.3 which are stronger than the conditions of Theorem 3.1.4, for every $h$ and continuous and bounded function $\Phi$ on $\bar{\mathscr{T}}_h \times \mathbb{R}^d$, the function $(t,x) \mapsto S(h,t,x,\Phi(t,x),\Phi(t+h,\cdot))$ is bounded and continuous on $\mathscr{T}_h \times \mathbb{R}^d$ and the function $r \mapsto S(h,t,x,r,\Phi(t+h,\cdot))$ is uniformly continuous for bounded $r$, uniformly in $(t,x) \in \mathscr{T}_h \times \mathbb{R}^d$.*

*Proof.* Let $\Phi$ be a bounded continuous function on $\bar{\mathscr{T}}_h \times \mathbb{R}^d$. Using the same kind of arguments as in Lemma 3.1.15, we obtain that under conditions of Theorem 3.1.3, $T^N_{k,h,\alpha,B}(t,x)$ and $T^D_{h,\alpha,B}(t,x)$ are continuous in $x$ uniformly in $\alpha$ and bounded for any $t \in \mathscr{T}_h$. Then, by using the properties of $\Phi$ and $f^\alpha$ and the dominated convergence theorem, we conclude that the part of $S(h,t,x,\Phi(t,x),\Phi(t+h,\cdot))$ which is in the supremum on $\alpha$ is continuous in $x$ for any $t \in \mathscr{T}_h$ uniformly in $\alpha$ and bounded. So, the supremum over $\alpha$ is continuous in $x$ for any $t \in \mathscr{T}_h$ and bounded. This allows us to conclude the continuity and boundness of $(t,x) \mapsto S(h,t,x,\Phi(t,x),\Phi(t+h,\cdot))$ on $\mathscr{T}_h \times \mathbb{R}^d$, $\mathscr{T}_h$ being discrete.

$S(h,t,x,r,\Phi(t+h,\cdot))$ is the supremum of affine functions of $r$ which are bounded for bounded $r$ and such that the coefficient ($T^D_{h,\alpha,B}(t,x)$ ) of $r$ is bounded. So it is Lipschitz continuous with respect to $r$ bounded, so uniformly continuous for bounded $r$, uniformly in $(t,x) \in \mathscr{T}_h \times \mathbb{R}^d$. □

**Lemma 3.1.19.** *Under the conditions of Theorem 3.1.4, if $W \in C^\infty_b([0,T]\times\mathbb{R}^d)$ is such that :*

$$|\partial^p_t D^q W| \leq \tilde{K}\epsilon^{1-2p-q}, \quad \forall p,q \in \mathbb{N}$$

*with $0 < \epsilon \leq 1$, $\tilde{K} > 0$, then, when $h$ is small enough :*

$$\begin{aligned}&\left|S(h,t,x,W(t,x),W(t+h,\cdot)) + \mathscr{L}^X[W](t,x) + F(t,x,W(t,x),D_xW(t,x),D^2_xW(t,x))\right|\\ &\leq E(\tilde{K},h,\epsilon)\end{aligned}$$

*for all* $t \in \mathscr{T}_h$ *and* $x \in \mathbb{R}^d$ *with*

$$E(\tilde{K}, h, \epsilon) = K\tilde{K}\left(h\epsilon^{-3}(1+\sqrt{h})^4(1+\sqrt{h}\epsilon^{-1})^4 + \sqrt{h}\epsilon^{-1}(1+\sqrt{h})^2(1+\sqrt{h}\epsilon^{-1})^2\right),$$

*where* $K$ *depends on the bounds of* $\underline{\sigma}$, $\underline{b}$, $g^\alpha$ *and* $\Sigma^\alpha$.

*Proof.* We use exactly the same arguments as in the proof of Lemma 3.1.16. The only difference here is that $W(t,x)$ in $F(t,x,W(t,x),D_xW(t,x),D_x^2W(t,x))$ is not approximated in $S(h,t,x,W(t,x),W(t+h,\cdot))$ and the approximation of the first order derivative term of $F(t,x,W(t,x),D_xW(t,x),D_x^2W(t,x))$ in $S(h,t,x,W(t,x),W(t+h,\cdot))$ results in an additional error (see the proof Theorem 3.1.2) which is :

$$\begin{aligned} K_6\sqrt{h}(|\partial_t^1 D^0| + |\partial_t^0 D^1|) &\le K_6\sqrt{h}\tilde{K}(1+\epsilon^{-1}) \\ &\le K_6\tilde{K}\sqrt{h}\epsilon^{-1}(\epsilon+1) \\ &\le K_7\tilde{K}\sqrt{h}\epsilon^{-1}(1+\sqrt{h})^2(1+\sqrt{h}\epsilon^{-1})^2, \end{aligned}$$

where $K_6$ and $K_7$ depend on the bounds of $\underline{\sigma}$, $\underline{b}$ and $g^\alpha$. Hence we obtain the result of the lemma. □

**Theorem 3.1.6.** *Under the conditions of Theorem 3.1.3 (which are stronger than the conditions of Theorem 3.1.4) and considering the assumption A 3.1.1, if* $W^h$ *is the solution of the scheme* (3.31) *and* $W$ *is the solution of* (3.1)- (3.2), *then there exists* $C_1, C_2$ *functions of* $|W|_1$ *(*$|\cdot|_1$ *defined in Section 0.1), such that, for all* $(t,x) \in \bar{\mathscr{T}}_h \times \mathbb{R}^d$,

$$-C_1 h^{1/10} \le (W^h - W)(t,x) \le C_2 h^{1/4}.$$

*Proof.* Using Lemma 3.1.17, Lemma 3.1.18, Lemma 3.1.19, we can see that under the conditions of Theorem 3.1.3 and with the assumption A 3.1.1, all the assumptions needed to apply Theorem 2.1.2 of Barles and Jakobsen are verified. By the same arguments as for Theorem 3.1.5, we obtain the result of the theorem. □

## 3.2 Extension to PDE with unbounded coefficients

We tried to show that the upwind large stencil probabilistic scheme still converges when the coefficients of the PDE and the terminal function have growth similar to those of a linear quadratic optimal control PDE, but we did not succed. However, we want to present the method and results we obtained in this attempt. We also outline a method used by Assellaou, Bokanowksi and Zidani in [5] to obtain error estimates of a Semi-Lagrangian scheme when the PDE has Lipschitz coefficients with linear growth and the terminal function is also Lipschitz with a linear growth. Indeed, it allows them to obtain the same lower and upper bound of the error, unlike Barles and Jakobsen. Their method relies then on the Lipschitz character not only of the viscosity solution of the PDE but also of the approximated solution obtained with their scheme.

In the following, we present the method and the results that we obtained when trying to show the convergence of the upwind large stencil probabilistic scheme when the PDE coefficients and terminal functions are unbounded and satisfy some conditions (alike linear quadratic optimal control problems conditions).

Let us consider the PDE (3.1)-(3.2) with the following conditions on the coefficients :

**A 3.2.1.** *There exists positive constants $\bar{C}, \nu, C_1, C_2, C_{\text{lip}}$ such that :*

- $\mathscr{A}$ *is unbounded;*
- $c^\alpha = 0$
- *The function $(t, x, \alpha) \mapsto b^\alpha(t, x)$ is continuous and for $x, y \in \mathbb{R}^d$, $t \in [0, T]$, $\alpha \in \mathscr{A}$,*

$$|b^\alpha(t,x) - b^\alpha(t,y)| \le \bar{C}|x-y| \tag{3.34a}$$
$$|b^\alpha(t,x)| \le \bar{C}(1 + |x| + |\alpha|); \tag{3.34b}$$

- *The function $(t, x, \alpha) \mapsto f^\alpha(t, x)$ is continuous and for $x \in \mathbb{R}^d$, $t \in [0, T]$, $\alpha \in \mathscr{A}$,*

$$\frac{\nu}{2}|\alpha|^2 + \ell_0(t,x,\alpha) \le f^\alpha(t,x) \le \bar{C}(1 + |x|^2 + |\alpha|^2) \text{ with } \ell_0(t,x,\alpha) \ge C_1|x|^2 - \bar{C}$$
$$|f^\alpha(t,x) - f^\alpha(t,y)| \le \bar{C}(1 + |x| + |y| + |\alpha|)|x-y|.$$

*where* $C_1 > \frac{2((C_2+1)e^{(6+\frac{2}{C_2}+8\bar{C})\bar{C}T}\bar{C})^2}{\nu}$

- *There exists $\sigma$ which does not depend on $\alpha$ such that $a = \sigma\sigma^\intercal$ and $\sigma \in C([0,T] \times \mathbb{R}^d, \mathbb{R}^{d\times d})$ and for every $t \in [0, T]$, $x, y \in \mathbb{R}^d$,*

$$|\sigma(t,x)| \le \bar{C} \tag{3.35a}$$
$$|\sigma(t,x) - \sigma(t,y)| \le \bar{C}|x-y|. \tag{3.35b}$$

- $\Psi \in C(\mathbb{R}^d, \mathbb{R})$ *and for $x, y \in \mathbb{R}^d$,*

$$|\Psi(x)| \le C_2(1 + |x|^2),$$
$$|\Psi(x) - \Psi(y)| \le C_{\text{lip}}(1 + |x|^2 + |y|^2)|x-y|$$

We can notice that from Corollary 1.2.1 and Lemma 1.2.2, the PDE (3.1)-(3.2) has a unique continuous viscosity solution in the space of functions with quadratic growth in this setting.

We first give error estimates for the approximations (3.9) and (3.13) in this setting.

We consider the space $\mathscr{C}_{\text{qua}}([0,T]\times\mathbb{R}^d)$ of $\mathscr{C}^\infty$ functions on $[0,T]\times\mathbb{R}^d$ with quadratic growth in $x$ such that for any $p\in\mathbb{N}$, $q\geq 2$ and $\phi\in\mathscr{C}_{\text{qua}}$, we have with the notations of Section 3.1 :

$$|\partial_t^p D^0\phi(t,x)| \leq C_{p,0,\phi}(1+|x|^2) \tag{3.36}$$
$$|\partial_t^p D^1\phi(t,x)| \leq C_{p,1,\phi}(1+|x|) \tag{3.37}$$
$$|\partial_t^p D^q\phi(t,x)| \leq C_{p,q,\phi}. \tag{3.38}$$

We hereby, define the constants $C_{r,s,\phi}$ for $r,s\in\mathbb{N}$ and $\phi\in\mathscr{C}_{\text{qua}}([0,T]\times\mathbb{R}^d)$.

**Theorem 3.2.1.** *Let $\hat{X}$ as in (3.6), and denote $B_h^t = B_{t+h}-B_t$. Consider any matrix $\Sigma\in\mathbb{R}^{d\times\ell}$ with $\ell\leq d$ and any integer $k\in\mathbb{N}$. Assume that $\underline{b}$ and $\underline{\sigma}$ are such that there exists a constant $C$ uniform in $t$ and $x$ such that $|\underline{b}(t,x)|\leq C(1+|x|)$ and $|\underline{\sigma}|\leq C$, and let $M$ be an upper bound of $|\Sigma\Sigma^\mathsf{T}|$. Then, there exists $K=K(C,M)>0$ such that, for all $W\in\mathscr{C}_{\text{qua}}([0,T]\times\mathbb{R}^d)$, we have, for all $(t,x)\in\mathscr{T}_h\times\mathbb{R}^d$,*

$$\begin{aligned}\Bigg| & h^{-1}\mathbb{E}\left[W(t+h,\hat{X}(t+h))\mathscr{P}^2_{\Sigma,k}(h^{-1/2}B_h^t)\mid \hat{X}(t)=x\right] \\ & -\frac{1}{2}\operatorname{Tr}(\underline{\sigma}(t,x)\Sigma\Sigma^\mathsf{T}\,\underline{\sigma}^\mathsf{T}(t,x)\,D_x^2W(t,x))\Bigg| \\ & \leq K(1+\sqrt{h})^4(1+|x|^4)\Big[h(C_{1,2,W}+C_{0,3,W}+C_{0,4,W})+ \\ & \quad h\sqrt{h}C_{1,3,W}+h^2C_{2,2,W}+h^2\sqrt{h}C_{3,1,W}+h^3C_{4,0,W}\Big]\,.\end{aligned}$$

*Proof.* The proof is very similar to the proof of Theorem 3.1.1. We use a Taylor formula to order 3 of $W(t+h,\hat{X}_h^{t,x})$ at point $(t,x)$. As seen in the proof of Theorem 3.1.1, by using this Taylor formula, the terms left in $\mathbb{E}\left[W(t+h,\hat{X}_h^{t,x})\mathscr{P}^2_{\Sigma,k}(h^{-1/2}B_h^t)\right]$ are

$$\frac{1}{2}\operatorname{Tr}(\underline{\sigma}(t,x)\Sigma\Sigma^\mathsf{T}\,\underline{\sigma}^\mathsf{T}(t,x)\,D_x^2W(t,x))$$
$$h^{-1}\mathbb{E}\left[M^4(W,t,x,h,B_h^t)\mathscr{P}^2_{\Sigma,k}(h^{-1/2}B_h^t)\right]$$

and the terms coming from the product of monomials of degree 2 in $[B_h^t]_i$ $(1\leq i\leq d)$ by $\mathscr{P}^2_{\Sigma,k}(h^{1/2}B_h^t)$. The difference is in the way of bounding terms in this formula, coming from the bound of $b(t,x)$ which is now $C(1+|x|)$ and the bound of the derivatives. The latter terms are bounded by :

$$Kh(C_{1,2,W}+(1+|x|)C_{0,3,W})\leq Kh(1+\sqrt{h})^4(1+|x|^4)(C_{1,2,W}+C_{0,3,W}).$$

By replacing the derivatives which appear in $M^4(W,t,x,h,B_h^t)$ by their bounds on the interval between $(t,x)$ and $(t+h,\hat{X}_h^{t,x})$, we obtain that $M^4(W,t,x,h,B_h^t)$ is now bounded by :

$$\begin{aligned}|M^4(W,t,x,h,B_h^t)| \leq & \frac{h^4}{24}C_{4,0,W}3(1+|x|^2+|\underline{b}(t,x)h|^2+|\underline{\sigma}(t,x)B_h^t|^2)\\ &+\frac{h^3}{6}\sum_{i=1}^d C_{3,1,W}(1+|x|+|\underline{b}(t,x)h|+|\underline{\sigma}(t,x)B_h^t|)|[\underline{b}(t,x)h+\underline{\sigma}(t,x)B_h^t]_i|\\ &+\frac{h^2}{4}\sum_{i,j}C_{2,2,W}|[\underline{b}(t,x)h+\underline{\sigma}(t,x)B_h^t]_i||[\underline{b}(t,x)h+\underline{\sigma}(t,x)B_h^t]_j|\\ &+\frac{h}{6}\sum_{i,j,p}C_{1,3,W}|[\underline{b}(t,x)h+\underline{\sigma}(t,x)B_h^t]_i||[\underline{b}(t,x)h+\underline{\sigma}(t,x)B_h^t]_j|\\ &\quad |[\underline{b}(t,x)h+\underline{\sigma}(t,x)B_h^t]_p|\\ &+\frac{1}{24}\sum_{i,j,p,q}C_{0,4,W}|[\underline{b}(t,x)h+\underline{\sigma}(t,x)B_h^t]_i||[\underline{b}(t,x)h+\underline{\sigma}(t,x)B_h^t]_j|\\ &\quad |[\underline{b}(t,x)h+\underline{\sigma}(t,x)B_h^t]_p||[\underline{b}(t,x)h+\underline{\sigma}(t,x)B_h^t]_q| \ .\end{aligned}$$

So

$$\begin{aligned}h^{-1}\mathbb{E}\left[M^4(W,t,x,h,B_h^t)\mathscr{P}^2_{\Sigma,k}(h^{-1/2}B_h^t)\right] \leq & \sum_{p+q=4}K_1(1+|x|^{\max\{2-q,0\}})(1+h)^{\max\{2-q,0\}}h^{p-1}\\ &(h(1+|x|)+\sqrt{h})^q C_{p,q,W}\\ \leq & \sum_{p+q=4}K_1[(1+|x|)(1+h)]^{\max\{2-q,0\}}(1+|x|)^q h^{p-1}\\ &(1+\sqrt{h})^q(\sqrt{h})^q C_{p,q,W}\\ \leq & \sum_{p+q=4}K(1+|x|^4)h^{p-1}(1+\sqrt{h})^4(\sqrt{h})^q C_{p,q,W},\end{aligned}$$

with $K_1, K$ depending only on $C$ and $M$. Hence the result of the theorem. □

**Theorem 3.2.2.** *Let $\hat{X}_h^{t,x} := x+\underline{b}(t,x)h+\underline{\sigma}(t,x)(B_{t+h}-B_t)$ and denote $B_h^t = B_{t+h}-B_t$. Consider any map $g^\alpha : [0,T]\times\mathbb{R}^d \to \mathbb{R}^d$ such that $|g^\alpha(t,x)| \leq C(1+|x|+|\alpha|)$, with $C$ uniform in $t$ and $x$. Assume that $\underline{b}$ and $\underline{\sigma}$ are such that $|\underline{b}(t,x)| \leq C(1+|x|)$ and $|\underline{\sigma}(t,x)| \leq C$. Then, there exists $K = K(C) > 0$ such that, for all $W \in \mathscr{C}_{\text{qua}}([0,T]\times\mathbb{R}^d)$, we have, for all $(t,x)\in\mathscr{T}_h\times\mathbb{R}^d$,*

$$\begin{aligned}&\left|(\underline{\sigma}(t,x)g^\alpha(t,x))\cdot D_xW(t,x) - \mathbb{E}\Big[(W(t+h,\hat{X}_h^{t,x})-W(t,x))\mathscr{P}^1_{g^\alpha}(h^{-1}B_h^t)\Big]\right|\\ &\leq K(1+\sqrt{h})^4(1+|x|^2)(1+|x|+|\alpha|)\Big[\sqrt{h}(C_{1,0,W}+C_{0,1,W}+C_{0,2,W})\\ &\qquad +hC_{1,1,W}+h\sqrt{h}C_{2,0,W}\Big] \ .\end{aligned}$$

*Proof.* A Taylor formula applied to $W(t+h,\hat{X}_h^{t,x})$ to order 1 at $(t,x)$ and a development similar to the one of the proof of Theorem 3.1.2 give that :

$$\begin{aligned}\mathbb{E}\Big[(W(t+h,\hat{X}_h^{t,x})-W(t,x))\mathscr{P}^1_{g^\alpha}(h^{-1}B_h^t)\Big] = & h(\frac{\partial W}{\partial t}(t,x)+D_xW(t,x)^\intercal\underline{b}(t,x))\mathbb{E}\left[\mathscr{P}^1_{g^\alpha}(h^{-1}B_h^t)\right]\\ &+(\underline{\sigma}(t,x)g^\alpha(t,x))\cdot D_xW(t,x)\\ &+\mathbb{E}\left[M^2(W,t,x,h,B_h^t)\mathscr{P}^1_{g^\alpha}(h^{-1}B_h^t)\right]\end{aligned}$$

We have :

$$h(\frac{\partial W}{\partial t}(t,x) + D_x W(t,x)^\intercal \underline{b}(t,x))\mathbb{E}\left[\mathscr{P}^1_{g^\alpha}(h^{-1}B^t_h)\right]$$

that is smaller than

$$\bar{C}(1+|x|^2)\sqrt{h}(C_{1,0,W}+C_{0,1,W})|g^\alpha(t,x)|$$

which is smaller than

$$\bar{C}(1+|x|^2)(1+\sqrt{h})^2(1+|x|+|\alpha|)\sqrt{h}(C_{1,0,W}+C_{0,1,W}).$$

By replacing the derivatives involved in $M^2(W,t,x,h,B^t_h)$ by their bounds over the interval between $(t,x)$ and $(t+h,\hat{X}^{t,x}_h)$, we also have

$$\begin{aligned}|M^2(W,t,x,h,B^t_h)| \leq & \frac{1}{2}\sum_{i,j} C_{0,2,W}|[\underline{b}(t,x)h+\underline{\sigma}(t,x)B^t_h]_i||[\underline{b}(t,x)h+\underline{\sigma}(t,x)B^t_h]_j| \\ & + h\sum_i C_{1,1,W}(1+|x|+|\underline{b}(t,x)h|+|\underline{\sigma}(t,x)B^t_h|)|[\underline{b}(t,x)h+\underline{\sigma}(t,x)B^t_h]_i| \\ & + \frac{1}{2}h^2 3C_{2,0,W}(1+|x|^2+|\underline{b}(t,x)h|^2+|\underline{\sigma}(t,x)B^t_h|^2)\ .\end{aligned}$$

So

$$\begin{aligned}\mathbb{E}\left[M^2(W,t,x,h,B^t_h)\mathscr{P}^1_{g^\alpha}(h^{-1}B^t_h)\right] & \leq \textstyle\sum_{p+q=2} \ K_1(1+|x|^{\max\{2-q,0\}})(1+h)^{\max\{2-q,0\}}h^{p-1/2} \\ & \qquad (h(1+|x|)+\sqrt{h})^q C_{p,q,W}(1+|x|+|\alpha|) \\ & \leq \textstyle\sum_{p+q=2} \ K_1((1+|x|)(1+h))^{\max\{2-q,0\}}(1+|x|)^q h^{p-1/2} \\ & \qquad (1+\sqrt{h})^q(\sqrt{h})^q C_{p,q,W}(1+|x|+|\alpha|) \\ & \leq \textstyle\sum_{p+q=2} \ K(1+|x|^2)(1+\sqrt{h})^4 h^{p-1/2}(\sqrt{h})^q \\ & \qquad C_{p,q,W}(1+|x|+|\alpha|),\end{aligned}$$

where $K_1, K$ are constants depending only on $C$. Hence the result of the theorem. □

**Lemma 3.2.1.** *Let $\hat{X}^{t,x}_h := x+\underline{b}(t,x)h+\underline{\sigma}(t,x)(B_{t+h}-B_t)$. Assume that $\underline{b}$ and $\underline{\sigma}$ are such that $|\underline{b}(t,x)| \leq C(1+|x|)$ and $|\underline{\sigma}(t,x)| \leq C$ for some constant $C$ uniform in $t$ and $x$. Then, there exists $K=K(C)>0$ such that, for all $W \in \mathscr{C}_{\text{qua}}([0,T]\times\mathbb{R}^d)$, we have, for all $(t,x) \in \mathscr{T}_h \times \mathbb{R}^d$,*

$$\begin{aligned}&\left|h^{-1}\left(\mathbb{E}\left[W(t+h,\hat{X}^{t,x}_h)\right]-W(t,x)\right)-\mathscr{L}^X W\right| \\ &\leq K(1+\sqrt{h})^4(1+|x|^4)\Big[h(C_{0,2,W}+C_{1,1,W}+C_{2,0,W}+C_{0,3,W}+C_{1,2,W}+C_{0,4,W}) \\ &\quad + h\sqrt{h}C_{1,3,W}+h^2(C_{2,2,W}+C_{2,1,W}+C_{3,0,W})+h^2\sqrt{h}C_{3,1,W}+h^3C_{4,0,W}\Big]\ .\end{aligned}$$

*Proof.* We use also here for the proof a Taylor development of $W(t+h,\hat{X}^{t,x}_h)$ to order 3 at $(t,x)$ which gives us the same result as in the proof of Theorem 3.1.1. The terms which are monomials of degree 1 in $h$, give us $h\mathscr{L}^X W$ in the expression of $\mathbb{E}\left[W(t+h,\hat{X}^{t,x}_h)-W(t,x)\right]$. Using the expression of $M^4(W,t,x,h,B^t_h)$ of the proof of Theorem 3.2.1, $h^{-1}\mathbb{E}\left[M^4(W,t,x,h,B^t_h)\right]$ can be bounded by

$$\sum_{p+q=4} K(1+|x|^4)h^{p-1}(1+\sqrt{h})^4(\sqrt{h})^q C_{p,q,W}.$$

The other terms of $h^{-1}\mathbb{E}\left[W(t+h,\hat{X}^{t,x}_h)-W(t,x)\right]$ which are not null, are bounded by the following terms

$$(K_0+\frac{1}{6}h|\underline{b}(t,x)|^2)C_{0,2,W} \le (1+h)K_1(1+|x|^2)C_{0,2,W} \le hK(1+\sqrt{h})^4(1+|x|^4)C_{0,2,W}$$
$$K_0h|\underline{b}(t,x)||1+|x||C_{1,1,W} \le hK_1(1+|x|^2)C_{1,1,W} \le hK(1+|x|^4)C_{1,1,W}$$
$$K_0h|1+|x|^2|C_{2,0,W} \le hK(1+|x|^4)C_{2,0,W}$$
$$(K_0+\frac{1}{6}h|b(t,x)|^2)h|\underline{b}(t,x)|C_{0,3,W} \le hK_1(1+h)(1+|x|^2)(1+|x|)C_{0,3,W} \le h(1+\sqrt{h})^4K(1+|x|^4)C_{0,3,W}$$
$$(K_0+\frac{1}{2}h|b(t,x)|^2)hC_{1,2,W} \le hK(1+\sqrt{h})^4(1+|x|^4)C_{1,2,W},$$
$$\frac{1}{2}(1+|x|)h^2|b(t,x)|C_{2,1,W} \le K_1(1+|x|^2)h^2C_{2,1,W} \le Kh^2(1+|x|^4)C_{2,1,W},$$
$$\frac{1}{6}(1+|x|^2)h^2C_{3,0,W} \le Kh^2(1+|x|^4)C_{3,0,W}.$$

where $K_0, K_1, K$ are constants depending only at most on $C$. Hence the result of the Lemma. □

We tried to show the convergence of the upwind large stencil probabilistic scheme in the unbounded setting given by Assumption A 3.2.1 by using the improvement of the Barles and Souganidis Theorem that we presented and proved in Section 2.1 (Theorem 2.1.3). However, we did not get all the elements needed to prove the stability of the scheme in this setting. We give below the different results that we obtained.

We consider the assumption :

**A 3.2.2.** *$\underline{\sigma}$ is bounded and $\underline{b}$ has a linear growth with respect to $x$.*

$g^\alpha$ and $\Sigma^\alpha$ are defined as in Section 3.1.2.b meaning that for $\underline{\sigma\sigma}^\mathsf{T}$ uniformly greater than a postive matrix $M_0 \in \mathbb{S}(d)$ and $a(t,x) \ge \underline{\sigma\sigma}^\mathsf{T}$, $\Sigma^\alpha$ is well defined and bounded and $g^\alpha$ is of linear growth in $x$ and $\alpha$. We consider as functionnal space, $B^2([0,T]\times\mathbb{R}^d)$ which is the space of functions with quadratic growth in $x$.

The result of monotonicity of Lemma 3.1.11 still holds in this setting. However, we need to obtain a consistency and a stability result. We will start by the consistency result. We saw in Section 3.1.2.b that the use of the upwind large stencil probabilistic scheme results in the following discretized equation.

$$S(h,t,x,W(t,x),W(t+h,\cdot)) = 0, \quad (t,x) \in \mathscr{T}_h \times \mathbb{R}^d$$

where

$$S(h,t,x,W(t,x),W(t+h,\cdot)) = -\frac{\mathbb{E}\left[W(t+h,\hat{X}_h^{t,x})\right] - W(t,x)}{h} \tag{3.39}$$

$$- \inf_{\alpha\in\mathscr{A}} \left\{ \frac{1}{h}\mathbb{E}\left[W(t+h,\hat{X}_h^{t,x})\mathscr{P}^2_{\Sigma^\alpha(t,x),k}(h^{-1/2}B_h^t)\right] \right. \tag{3.40}$$

$$+ \mathbb{E}\left[(W(t+h,\hat{X}_h^{t,x}) - W(t,x))\mathscr{P}^1_{g^\alpha(t,x)}(h^{-1}B_h^t)\right] \tag{3.41}$$

$$\left. + c^\alpha(t,x)W(t,x) + f^\alpha(t,x) \right\}. \tag{3.42}$$

**Theorem 3.2.3** (Consistency result in the unbounded setting)**.** *Let $\mu : x \mapsto (1+|x|^2)$ and $W \in \mathscr{C}_{\text{qua}}([0,T]\times\mathbb{R}^d)$. If $\underline{\sigma\sigma}^\top$ is uniformly greater than a postive matrix $M_0 \in \mathbb{S}(d)$ and $a(t,x) \geq \underline{\sigma\sigma}^\top$ with $\underline{\sigma}$ and a bounded, then*

$$\lim_{\substack{h\to 0\\(s,y)\to(t,x)\\c\to 0}} S(h,s,y,W(s,y)+c\mu(y),W(s+h,\cdot)+c\mu)$$
$$= -\mathscr{L}^X W(t,x) - F(t,x,W(t,x),D_xW(t,x),D_x^2W(t,x)),$$

*$F$ being the Hamiltonian given in Equation* (3.3)*.*

*Proof.* Let $h,c > 0$ small enough and $y \in \mathbb{R}^d$. Using Theorem 3.2.1, Theorem 3.2.2 and

Lemma 3.2.1, we have

$$
\begin{aligned}
S(h,s,y,W(s,y)+c\mu(y),W(s+h,\cdot)+c\mu) = &-\frac{\mathbb{E}\left[W(s+h,\hat{X}_h^{s,y})\right]-W(s,y)}{h}\\
&-\inf_{\alpha\in\mathscr{A}}\Bigg\{\frac{1}{h}\mathbb{E}\left[W(s+h,\hat{X}_h^{s,y})\mathscr{P}^2_{\Sigma^\alpha(s,y),k}(h^{-1/2}B_h^t)\right]\\
&+\mathbb{E}\left[(W(s+h,\hat{X}_h^{s,y})-W(s,y))\mathscr{P}^1_{g^\alpha(s,y)}(h^{-1}B_h^t)\right]\\
&+f^\alpha(s,y)\\
&+\frac{c}{h}\mathbb{E}\left[(1+|\hat{X}_h^{s,y}|^2)(\mathscr{P}^2_{\Sigma^\alpha(s,y),k}(h^{-1/2}B_h^t))\right]\\
&+c\mathbb{E}\left[(|\hat{X}_h^{s,y}|^2-|y|^2)\mathscr{P}^1_{g^\alpha(s,y)}(h^{-1}B_h^t)\right]\Bigg\}\\
&-\frac{c}{h}\mathbb{E}\left[|\hat{X}_h^{s,y}|^2-|y|^2\right]\\
=&-\mathscr{L}^X W(t,y)+O(Ch(1+|y|^4))-\inf_{\alpha\in\mathscr{A}}\Bigg\{\\
&\frac{1}{2}\operatorname{Tr}\left[(\underline{\sigma}\Sigma^\alpha(\Sigma^\alpha)^\intercal\,\underline{\sigma}^\intercal)(s,y)D_x^2W(s,y)\right]+Kh(1+|y|^4)\\
&+\underline{\sigma}(s,y)g^\alpha(s,y)\cdot D_xW(s,y)+K(1+|y|^2)(1+|y|+|\alpha|)\sqrt{h}\\
&+f^\alpha(s,y)+\frac{c}{h}\mathbb{E}\left[(1+|\hat{X}_h^{s,y}|^2)\mathscr{P}^2_{\Sigma^\alpha(s,y),k}(h^{-1/2}B_h^t)\right]\\
&+c\mathbb{E}\left[(|\hat{X}_h^{s,y}|^2-|y|^2)\mathscr{P}^1_{g^\alpha(s,y)}(h^{-1}B_h^t)\right]\Bigg\}\\
&+\frac{c}{h}\mathbb{E}\left[|\hat{X}_h^{s,y}|^2-|y|^2\right],
\end{aligned}
$$

$K$ being a constant proportional to the constant $K$ of Theorem 3.2.1, Theorem 3.2.2 and Lemma 3.2.1.

$$
\begin{aligned}
|\hat{X}_h^{s,y}|^2 =&\hat{X}_h^{s,y}(\hat{X}_h^{s,y})^\intercal\\
=&|y|^2+h^2|\underline{b}(s,y)|^2+(B_h^t)^\intercal\,(\underline{\sigma}^\intercal\,\underline{\sigma})(s,y)B_h^t\\
&+2(hy^\intercal\,\underline{b}(s,y)+y^\intercal\,\underline{\sigma}(s,y)B_h^t+h\underline{b}(s,y)^\intercal\,\underline{\sigma}(s,y)B_h^t).
\end{aligned}
$$

From the properties of $\mathscr{P}^2_{\Sigma^\alpha(s,y),k}(h^{-1/2}B_h^t)$ deduced in Corollary 3.1.3, we then have

$$
\begin{aligned}
&\mathbb{E}\left[(1+|\hat{X}_h^{s,y}|^2)\mathscr{P}^2_{\Sigma^\alpha(s,y),k}(h^{-1/2}B_h^t)\right]\\
=&h\operatorname{Tr}\left[(\underline{\sigma}\Sigma^\alpha(\Sigma^\alpha)^\intercal\,\underline{\sigma}^\intercal)(s,y)\right],
\end{aligned}
$$

and we can show using same reasoning as in the proof of Theorem 3.1.2 that :

$$
\begin{aligned}
&\mathbb{E}\left[(|\hat{X}_h^{s,y}|^2-|y|^2)\mathscr{P}^1_{g^\alpha(s,y)}(h^{-1}B_h^t)\right]\\
=&(h^2|\underline{b}(s,y)|^2+2hy^\intercal\,\underline{b}(s,y))\mathbb{E}\left[\mathscr{P}^1_{g^\alpha(s,y)}(h^{-1}B_h^t)\right]\\
&+2(y+h\underline{b}(s,y))\cdot(\underline{\sigma}g^\alpha)(s,y)+\mathbb{E}\left[(B_h^t)^\intercal\,(\underline{\sigma}^\intercal\,\underline{\sigma})(s,y)B_h^t\mathscr{P}^1_{g^\alpha(s,y)}(h^{-1}B_h^t)\right],
\end{aligned}
$$

so that

$$c|\mathbb{E}\left[(|\hat{X}_h^{s,y}|^2 - |y|^2)\mathscr{P}^1_{g^\alpha(s,y)}(h^{-1}B_h^t)\right]| \leq K_1(1+|y|^3)\sqrt{h}c + K_1(1+|y|^2)c + K_1((1+|y|^2)\sqrt{h} + 1 + |y|)c|\alpha|,$$

$K_1$ being proportional to the constant $K$ introduced above. We also have

$$\left|\mathbb{E}\left[|\hat{X}_h^{s,y}|^2 - |y|^2\right]\right| \leq h(1+|y|^2)K_2,$$

$K_2$ proportional to $K$. So,

$$\frac{c}{h}\left|\mathbb{E}\left[|\hat{X}_h^{s,y}|^2 - |y|^2\right]\right| \leq c(1+|y|^2)K_2.$$

We thus have :

$$\begin{aligned} S(h,s,y,W(s,y)+c\mu(y),W(s+h,\cdot)+c\mu) = &- \mathscr{L}^X W(s,y) - \inf_{\alpha\in\mathscr{A}} \Bigg\{ \\ &\frac{1}{2}\operatorname{Tr}\left[(\underline{\sigma}\Sigma^\alpha(\Sigma^\alpha)^\mathsf{T}\underline{\sigma}^\mathsf{T})(s,y)D_x^2 W(s,y)\right] \\ &+ \underline{\sigma}(s,y)g^\alpha(s,y)\cdot D_x W(s,y) + f^\alpha(s,y) \\ &+ \mathscr{K}^1(c,h,|y|,\alpha)\Bigg\} + \mathscr{K}^2(c,h,|y|), \end{aligned}$$

with $\mathscr{K}^1$ having a linear growth in $\alpha$ such that it converges to 0 when $c$ and $h$ go to 0 for $\alpha = \alpha_0$ and its growth coefficient with respect to $\alpha$ goes to 0 with $c$ and $h$. We also have $\mathscr{K}^2(c,h,|y|)$ that converges to 0 when $c$ and $h$ go to zero.

These observations with the fact that $f^\alpha$ is strictly convex in $\alpha$ according to Assumption A 3.2.1, allows us to conclude that :

$$\begin{aligned} &\lim_{\substack{h\to 0\\ c\to 0}} S(h,s,y,W(s,y)+c\mu(y),W(s+h,\cdot)+c\mu) \\ &= -\mathscr{L}^X W(s,y) - F(s,y,W(s,y),D_x W(s,y),D_x^2 W(s,y)), \end{aligned}$$

with a convergence uniform in $s$ and $y$ for $s,y$ bounded. Hence,

$$\begin{aligned} &\lim_{\substack{h\to 0\\ c\to 0\\ (s,y)\to(t,x)}} S(h,s,y,W(s,y)+c\mu(y),W(s+h,\cdot)+c\mu) \\ &= -\mathscr{L}^X W(t,x) - F(t,x,W(t,x),D_x W(t,x),D_x^2 W(t,x)). \end{aligned}$$

Hence the result of the theorem. □

For the stability, we tried to use a modified version of the additive $\alpha$- subhomogeneity given in Definition 3.1.2.

**Definition 3.2.1.** *We say that an operator $T$ defined on functions with quadratic growth is additively $\alpha$-subhomogeneous if for $\mu : x \mapsto 1+|x|^2$, for any function with quadratic growth $\phi$ and for any $\lambda > 0$*

$$T(\phi + \lambda\mu) \leq T(\phi) + \alpha\lambda\mu.$$

We obtained the following result.

**Proposition 3.2.1.** *Considering assumptions A 3.2.1 and A 3.2.2, the operator* $\mathbf{T}^1_{k,h}$ *defined by* (3.32) *is additively* $\alpha_h$ *subhomogeneous over the set of functions with quadratic growth, for some constant* $\alpha_h = 1 + C\sqrt{h}$ *with* $C \geq 0$.

*Proof.* Let $\phi$ be a function with quadratic growth and $\mu$ be defined as in Definition 3.2.1. Let $\lambda > 0$.

$$\begin{aligned}\mathbf{T}^1_{k,h}[\phi+\lambda\mu](t,x) &= \inf_{\alpha\in\mathscr{A}}\left\{\frac{\mathbb{E}\left[T^N_{k,h,\alpha,B}(t,x)(\phi+\lambda\mu)(\hat{X}^{t,x}_h)\right]+hf^\alpha(t,x)}{T^D_{h,\alpha,B}(t,x)}\right\}\\ &= \inf_{\alpha\in\mathscr{A}}\left\{\frac{\mathbb{E}\left[T^N_{k,h,\alpha,B}(t,x)\phi(\hat{X}^{t,x}_h)\right]+hf^\alpha(t,x)}{T^D_{h,\alpha,B}(t,x)}+\lambda\frac{\mathbb{E}\left[T^N_{k,h,\alpha,B}(t,x)(1+|\hat{X}^{t,x}_h|^2)\right]}{T^D_{h,\alpha,B}(t,x)}\right\}.\end{aligned}$$

Using the development made in the proof of Theorem 3.2.3, we have :

$$\begin{aligned}\mathbb{E}\left[T^N_{k,h,\alpha,B}(t,x)(1+|\hat{X}^{t,x}_h|^2)\right] =& \mathbb{E}\left[1+|\hat{X}^{t,x}_h|^2\right]+\mathbb{E}\left[(1+|\hat{X}^{t,x}_h|^2)\mathscr{P}^2_{\Sigma^\alpha,k}(h^{-1/2}B^t_h)\right]\\ &+h\mathbb{E}\left[(1+|\hat{X}^{t,x}_h|^2)\mathscr{P}^1_{g^\alpha(t,x)}(h^{-1}B^t_h)\right]\\ =&1+|x|^2+h^2|\underline{b}(t,x)|^2+h\operatorname{Tr}\left[(\underline{\sigma\sigma}^\top)(t,x)\right]+2hx^\top\underline{b}(t,x)\\ &+h\operatorname{Tr}\left[(\underline{\sigma}\Sigma^\alpha(\Sigma^\alpha)^\top\underline{\sigma}^\top)(t,x)\right]\\ &+(1+|x|^2)h\mathbb{E}\left[\mathscr{P}^1_{g^\alpha(t,x)}(h^{-1}B^t_h)\right]\\ &+(h^2|\underline{b}(t,x)|^2+2hx^\top\underline{b}(t,x))h\mathbb{E}\left[\mathscr{P}^1_{g^\alpha(t,x)}(h^{-1}B^t_h)\right]\\ &+2(x+h\underline{b}(t,x))h\cdot(\underline{\sigma}g^\alpha)(t,x)+\mathbb{E}\left[(B^t_h)^\top(\underline{\sigma}^\top\underline{\sigma})(t,x)B^t_h\mathscr{P}^1_{g^\alpha(t,x)}(h^{-1}B^t_h)\right]\\ =&(1+|x|^2+h^2|\underline{b}(t,x)|^2+2hx^\top\underline{b}(t,x))(1+h\mathbb{E}\left[\mathscr{P}^1_{g^\alpha(t,x)}(h^{-1}B^t_h)\right])\\ &+2(x+h\underline{b}(t,x))h\cdot(\underline{\sigma}g^\alpha)(t,x)+h\operatorname{Tr}\left[(\underline{\sigma}\Sigma^\alpha(\Sigma^\alpha)^\top\underline{\sigma}^\top)(t,x)\right]\\ &+h\operatorname{Tr}\left[(\underline{\sigma\sigma}^\top)(t,x)\right].\end{aligned}$$

$$|h^2|\underline{b}(t,x)|^2+2hx^\top\underline{b}(t,x)|\leq Kh(1+|x|^2)$$

$K$ depending only on the linear growth coefficient of $\underline{b}$ in $x$.

$$\begin{aligned}&\Bigg|2(x+h\underline{b}(t,x))h\cdot(\underline{\sigma}g^\alpha)(t,x)+h\operatorname{Tr}\left[(\underline{\sigma}\Sigma^\alpha(\Sigma^\alpha)^\top\underline{\sigma}^\top)(t,x)\right]\\ &+h\operatorname{Tr}\left[(\underline{\sigma\sigma}^\top)(t,x)\right]\Bigg|\\ &\leq hK(1+h)(1+|x|)|g^\alpha(t,x)|+\kappa h,\end{aligned}$$

$\kappa$ depending on the bounds of $\Sigma^\alpha(\Sigma^\alpha)^\intercal$ and $\underline{\sigma\sigma}^\intercal$. We thus have :

$$\begin{aligned}\left|\frac{\mathbb{E}\left[T^N_{k,h,\alpha,B}(t,x)(1+|\hat{X}^{t,x}_h|^2)\right]}{T^D_{h,\alpha,B}(t,x)}\right| &\leq (1+|x|^2)(1+Kh) + \frac{hK(1+h)(1+|x|)|g^\alpha(t,x)|+\kappa h}{1+\tilde{K}\sqrt{h}|g^\alpha(t,x)|} \\ &\leq (1+|x|^2)(1+(K+\kappa)h) + \frac{K}{\tilde{K}}(1+|x|)(1+h)\sqrt{h} \\ &\leq (1+|x|^2)(1+C\sqrt{h}).\end{aligned}$$

Thus the result of the proposition. □

To obtain the stability of the scheme, it would have been preferable to obtain the $\alpha$-subhomogeneity of the scheme operator with a constant $\alpha = (1+Ch)$ for some $C > 0$. As

$$\mathbf{T}^1_{k,h}[0] = \inf_{\alpha\in\mathscr{A}} \frac{f^\alpha(t,x)}{T^D_{h,\alpha,B}(t,x)},$$

with $f^\alpha$ bounded from above and below by convex functions in $\alpha$ plus functions in $x$ with quadratic growth, $\mathbf{T}^1_{k,h}[0]$ is of quadratic growth and a proof similar to the one done in Section 3.1.2.b would have allowed us to obtain the stability of the scheme.

# CHAPTER 4

# New probabilistic max-plus method

We present in this chapter a new method for solving stochastic control problem that we called a probabilistic max-plus method. It has been introduced in [1] where it was used with the probabilistic scheme of Fahim, Touzi and Warin presented in Section 2.3.3. It has then been applied to the schemes presented in Section 3.1.2.a and Section 3.1.2.b respectively in [2] and [3]. The originality of the method is the use of a low number of basis functions in the regression needed to compute the conditional expectation in a probabilistic scheme, while keeping a good approximation of the function solution we try to approximate. In particular, the non linearity of this function is well taken into account. The idea is that, if a probabilistic scheme used with the method keep stable a given space of functions with low dimension, then the method can be used with this scheme with terminal functions expressed as supremum or infimum of these functions depending on the goal of the underlying optimal control problem. We consider here as functionnal space with low dimension, the space of quadratic functions.

We give the theoretical results in the setting of each of the probabilistic schemes described in Section 3.1 and then present the different versions of the algorithm that we have implemented.

## 4.1 Theoretical results

The originality of the method in [1] is that instead of applying a regression estimation to compute $\mathscr{D}^i_{t,h}(W^h(t+h,\cdot))$ as proposed in [25] by Fahim, Touzi and Warin, we approximate $W^h$ by a max-plus linear combination of basic functions (namely quadratic forms) and use the distributivity property stated in Theorem 4.1 which generalizes Theorem 3.1 of McEneaney, Kaise and Han [54, 44]. It also generalizes a more old result stated in Theorem 14.60 of [65] by Rockafellar and Wets along with another result stated in Proposition 6.1 of [66] by Shapiro and Ruszczyński. This allows us to keep a non linearity in the expression of $W^h$ in adequation with the non linearity of the related stochastic control problem.

In the sequel, we denote $\mathscr{W} = \mathbb{R}^d$ and $\mathscr{D}$ the set of measurable functions from $\mathscr{W}$ to $\mathbb{R}$ with at most some given growth or growth rate (for instance with at most exponential growth rate), assuming that it contains the constant functions.

**Theorem 4.1.** *Let $\mathscr{W} = \mathbb{R}^d$ and $\mathscr{D}$ be the set of measurable functions from $\mathscr{W}$ to $\mathbb{R}$ with at most some given growth or growth rate (for instance with at most exponential growth rate), containing the constant functions, and let $G$ be a monotone additively $\alpha$-subhomogeneous operator from $\mathscr{D}$ to $\mathbb{R}$, for some constant $\alpha > 0$. Let $(Z, \mathscr{A})$ be a measurable space, and let $\mathscr{W}$ be endowed with its Borel $\sigma$-algebra. Let $\phi : \mathscr{W} \times Z \to \mathbb{R}$ be a measurable map such that for all $z \in Z$, $\phi(\cdot, z)$ is continuous and belongs to $\mathscr{D}$. Let $v \in \mathscr{D}$ be such that $v(w) = \sup_{z\in Z} \phi(w, z)$. Assume that $v$ is continuous and bounded. Then,*

$$G(v) = \sup_{\bar{z}\in\overline{Z}} G(\bar{\phi}^{\bar{z}})$$

*where $\bar{\phi}^{\bar{z}} : \mathscr{W} \to \mathbb{R},\ w \mapsto \phi(w, \bar{z}(w))$, and*

$$\overline{Z} = \{\bar{z} : \mathscr{W} \to Z,\ \textit{measurable and such that } \bar{\phi}^{\bar{z}} \in \mathscr{D}\}.$$

*Proof.* Since $v$ belongs to $\mathscr{D}$, $G(v)$ is well defined. Similarly, by definition, for all $\bar{z} \in \overline{Z}$, $\bar{\phi}^{\bar{z}}$ belongs to $\mathscr{D}$, so that $G(\bar{\phi}^{\bar{z}})$ is well defined.

Let $\epsilon > 0$. By definition of $v$, for all $w \in \mathscr{W}$, there exists $z^w \in Z$ such that $\phi(w, z^w) \geq v(w) - \epsilon$. Then, since $w' \mapsto \phi(w', z^w)$ and $w' \mapsto v(w')$ are continuous maps $\mathscr{W} \to \mathbb{R}$, there exists $\delta^w > 0$ such that for all $w' \in B(w, \delta^w)$ (the open ball centered at $w$ with radius $\delta^w$), $|\phi(w', z^w) - \phi(w, z^w)| \leq \epsilon$ and $|v(w') - v(w)| \leq \epsilon$. Then, for $w' \in B(w, \delta^w)$, we have

$$\phi(w', z^w) \geq \phi(w, z^w) - \epsilon \geq v(w) - 2\epsilon \geq v(w') - 3\epsilon\ .$$

As $\mathscr{W}$ is the countable union of compact metric spaces, there exists a sequence $(w_i)_{i\geq 0}$ of $\mathscr{W}$ such that $\mathscr{W} = \cup_{i\geq 0} B(w_i, \delta^{w_i})$. Let us denote, for all $i \geq 0$, $\mathscr{W}_i = B(w_i, \delta^{w_i})$ and $\mathscr{W}'_i = \mathscr{W}_i \setminus (\cup_{j<i}\mathscr{W}_j)$. Define the function $z_1$ such that, for all $i \geq 0$, $z_1(w') = z^{w_i}$, for $w' \in \mathscr{W}'_i$. Since $(\mathscr{W}'_i)_{i\geq 0}$ is a countable partition of $\mathscr{W}$ composed of Borel sets, the map $z_1$ is well defined on $\mathscr{W}$ and measurable. Since $\phi$ is measurable, this implies that $\bar{\phi}^{z_1}$ is also measurable. Moreover, by the above properties and the definition of $v$, we have

$$v(w) \geq \bar{\phi}^{z_1}(w) = \phi(w, z_1(w)) \geq v(w) - 3\epsilon, \quad \forall w \in \mathscr{W}\ .$$

Since $v \in \mathscr{D}$, and $\mathscr{D}$ is the set of measurable functions from $\mathscr{W}$ to $\mathbb{R}$ with at most some given growth or growth rate and containing the constant functions, we get that $\bar{\phi}^{z_1}$ has also this growth or growth rate, which implies that $\bar{\phi}^{z_1} \in \mathscr{D}$, so $z_1$ belongs to $\overline{Z}$.

Since $G$ is monotone and additively $\alpha$-subhomogeneous from $\mathscr{D}$ to $\mathbb{R}$, and $\epsilon > 0$, we get that

$$G(v) \geq G(\bar{\phi}^{z_1}) \geq G(v - 3\epsilon) \geq G(v) - 3\alpha\epsilon \ .$$

Then

$$\sup_{\bar{z} \in \overline{Z}} G(\bar{\phi}^{\bar{z}}) \geq G(v) - 3\alpha\epsilon \ .$$

On the other hand, for any $\bar{z} \in \overline{Z}$, $\bar{\phi}^{\bar{z}} \leq v$. So $G(\bar{\phi}^{\bar{z}}) \leq G(v)$. We then have

$$G(v) \geq \sup_{\bar{z} \in \overline{Z}} G(\bar{\phi}^{\bar{z}}) \geq G(v) - 3\alpha\epsilon \ .$$

and since this property holds for all $\epsilon > 0$, we obtain the equality, which shows the assertion of the theorem. □

We will consider in the following a Hamilton-Jacobi-Bellman PDE more general than the one seen previously (PDE (3.1)), with a control $\alpha$ represented by a couple $(m, u)$ where $m$ is a discrete control and $u$ is a control living in a continuum set. We will also suppose without loss of generality, that the PDE coefficients do not depend on the time variable and we will be interested in a maximization problem. The PDE we consider is the following :

$$-\frac{\partial W}{\partial t} - \mathscr{H}(x, W(t,x), DW(t,x), D^2W(t,x)) = 0, \quad x \in \mathbb{R}^d, \ t \in [0, T), \tag{4.1a}$$

$$W(T, x) = \Psi(x), \quad x \in \mathbb{R}^d, \tag{4.1b}$$

where the Hamiltonian $\mathscr{H} : \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}(d) \to \mathbb{R}$ of the above control problem is defined as:

$$\mathscr{H}(x, r, p, \Gamma) := \max_{m \in \mathscr{M}} \mathscr{H}^m(x, r, p, \Gamma) \ , \tag{4.2a}$$

with

$$\mathscr{H}^m(x, r, p, \Gamma) := \sup_{u \in \mathscr{U}} \mathscr{H}^{m,u}(x, r, p, \Gamma) \ , \tag{4.2b}$$

$$\begin{aligned}\mathscr{H}^{m,u}(x, r, p, \Gamma) :=& \frac{1}{2} \operatorname{Tr}(a_u^m(x)\Gamma) + b_u^m(x) \cdot p \\ &+ c_u^m(x) r + f_u^m(x) \ ,\end{aligned} \tag{4.2c}$$

$a_u^m = \sigma_u^m(\sigma_u^m)^\intercal$, $b_u^m$, $c_u^m$, $f_u^m$ taking the place of $a^\alpha$, $b^\alpha$, $c^\alpha$, $f^\alpha$ of Section 3.1. It can be seen that the modifications made here to the initial problem (3.1)-(3.2) of Section 3.1 do not change the validity of the results obtained in Section 3.1.

We rewrite each $\mathscr{H}^m$ ($m \in \mathscr{M}$) in the form (3.3) where the diffusion $X$ and the Hamiltonian $F$ depends now on the discrete control $m$. $\underline{b}$ becomes $\underline{b}^m$ and $\underline{\sigma}$ becomes $\underline{\sigma}^m$.

We consider then the backward operator $T$ related to any probabilistic scheme encountered previously in this document as :

$$T_{t,h}(\phi)(x) = \max_{m \in \mathscr{M}} T^m_{t,h}[\phi](x) \ , \tag{4.3a}$$

with

$$T_{t,h}^{m}[\phi](x) = G_{t,h,x}^{m}(\tilde{\phi}_{t,h,x}^{m}) \tag{4.3b}$$

where

$$S_{t,h}^{m} : \mathbb{R}^d \times \mathscr{W} \to \mathbb{R}^d,\ (x,w) \mapsto S_{t,h}^{m}(x,w) = x + \underline{f}^{m}(x)h + \underline{\sigma}^{m}(x)w \ , \tag{4.3c}$$

$$\tilde{\phi}_{t,h,x}^{m} = \phi(S_{t,h}^{m}(x,\cdot)) \in \mathscr{D} \quad \text{if } \phi \in \mathscr{D} \ . \tag{4.3d}$$

For the probabilistic scheme of Fahim, Touzi and Warin ([25]), $G_{t,h,x}^{m}$ is the operator from $\mathscr{D}$ to $\mathbb{R}$ given by

$$\begin{aligned} G_{t,h,x}^{m}(\tilde{\phi}) &= D_{t,h,m,x}^{0}(\tilde{\phi}) \\ &+ h \max_{u \in \mathscr{U}} \left( \mathscr{G}_1^{m,u}(x, D_{t,h,m,x}^{0}(\tilde{\phi}), D_{t,h,m,x}^{1}(\tilde{\phi}), D_{t,h,m,x}^{2}(\tilde{\phi})) \right) \ , \end{aligned} \tag{4.3e}$$

with

$$\begin{aligned} &D_{t,h,m,x}^{0}(\tilde{\phi}) = \mathbb{E}(\tilde{\phi}(B_{t+h} - B_t)) \ , \\ &D_{t,h,m,x}^{1}(\tilde{\phi}) = \mathbb{E}(\tilde{\phi}(B_{t+h} - B_t)(\underline{\sigma}^{m}(x)^{\mathsf{T}})^{-1} h^{-1}(B_{t+h} - B_t)) \ , \\ &D_{t,h,m,x}^{2}(\tilde{\phi})(x) := h^{-1}\mathbb{E}\left[\tilde{\phi}(B_{t+h} - B_t)(\underline{\sigma}^{m}(x)^{\mathsf{T}})^{-1} \frac{(B_{t+h} - B_t)(B_{t+h} - B_t)^{\mathsf{T}} - hI}{h} \underline{\sigma}^{m}(x)^{-1}\right] \ , \end{aligned}$$

where

$$\mathscr{G}_1^{m,u}(x,r,p,\Gamma) = \frac{1}{2}\operatorname{Tr}\left[(a_u^m(x) - \underline{\sigma}^m(x)\underline{\sigma}^m(x)^{\mathsf{T}})\Gamma\right] + (b_u^m(x) - \underline{b}^m(x)) \cdot p + c_u^m(x) r + f_u^m(x).$$

Indeed, the Euler discretization $\hat{X}^m$ of the diffusion with generator $\mathscr{L}^{X^m}$ satisfies

$$\hat{X}^m(t+h) = S_{t,h}^{m}(\hat{X}^m(t), W_{t+h} - W_t) \ . \tag{4.4}$$

We will redefine these operators in the following subsections to adapt them to the probabilistic scheme considered.

### 4.1.1 Method in the large stencil probabilistic scheme setting

To explain the algorithm, assume that the final reward $\Psi$ of the control problem can be written as the supremum of a finite number of quadratic forms. Denote $\mathscr{Q}_d = \mathbb{S}(d) \times \mathbb{R}^d \times \mathbb{R}$ and let

$$q(x,z) := \frac{1}{2} x^{\mathsf{T}} Q x + b \cdot x + c, \quad \text{with } z = (Q,b,c) \in \mathscr{Q}_d \ , \tag{4.5}$$

be the quadratic form with parameter $z$ applied to the vector $x \in \mathbb{R}^d$. Then for $g_T = q$, we have

$$W^h(T,x) = \Psi(x) = \sup_{z \in Z_T} g_T(x,z)$$

where $Z_T$ is a finite subset of $\mathscr{Q}_d$ and $W^h$ is the approximated solution of the PDE (4.1) computed using a scheme.

We rewrite each $\mathscr{H}^m$ ($m \in \mathscr{M}$) in the form (3.3) where the diffusion $X$ and the Hamiltonian $F$ depends now on the discrete control $m$. $\underline{b}$ becomes $\underline{b}^m$ and $\underline{\sigma}$ becomes $\underline{\sigma}^m$. The function $W^h$ can then be computed by a scheme operator defined as the maximum over $m$ of the large stencil probabilistic scheme operators for Hamilton-Jacobi-Bellman equation with $\mathscr{H}^m$ as Hamiltonian. So the operator $\mathbf{T}^0_{k,h}$ defined by (3.22) in Section 3.1.2.a, can be replaced in this setting by :

$$\mathbf{T}^0_{k,h}[\Phi](t,x) := \max_{m\in\mathscr{M}} \mathbf{T}^{0,m}_{k,h}[\Phi](t,x) \tag{4.6}$$

where each $\mathbf{T}^{0,m}_{k,h}$ is defined by :

$$\begin{aligned}\mathbf{T}^{0,m}_{k,h}[\Phi](t,x) := \mathbb{E}\left[\Phi(\hat{X}^{t,x,m}_h)\right] + h \sup_{u\in\mathscr{U}} \Bigg\{ & \mathbb{E}\left[\Phi(\hat{X}^{t,x,m}_h) h^{-1} \mathscr{P}^2_{\Sigma^m_u(x),k}(h^{-1/2}B^t_h)\right] \\ & + \mathbb{E}\left[\Phi(\hat{X}^{t,x,m}_h)\mathscr{P}^{1,0}_{(\underline{\sigma}^m)^{-1}(x)(b^m_u-\underline{b}^m)(x)}(h^{-1}B^t_h)\right] \\ & + c^m_u(x)\mathbb{E}\left[\Phi(\hat{X}^{t,x,m}_h)\right] + f^m_u(x)\Bigg\},\end{aligned}$$

where $\hat{X}^{t,x,m}_h := x + \underline{b}^m(x)h + \underline{\sigma}^m(x)B^t_h$ and $\Sigma^m_u$ is the equivalent of $\Sigma^\alpha$ defined in Section 3.1.2.a.

In the following, we introduce some notations. Let us consider a continuous function $\phi : \mathbb{R}^d \to \mathbb{R}, x \mapsto \phi(x)$ and define

$$T_{t,h}[\phi](x) := \mathbf{T}^0_{k,h}[\phi](t,x), \tag{4.7a}$$

$$T^m_{t,h}[\phi](x) := \mathbf{T}^{0,m}_{k,h}[\phi](t,x). \tag{4.7b}$$

Let $G^m_{t,h,x}$ be the operator from $\mathscr{D}$ to $\mathbb{R}$ given by

$$\begin{aligned} G^m_{t,h,x}(\tilde{\phi}) = {} & D^0_{t,h,m,x}(\tilde{\phi}) \\ & + h\sup_{u\in\mathscr{U}} \left(\mathscr{G}^{m,u}_1(x, D^0_{t,h,m,x}(\tilde{\phi}), D^1_{t,h,m,x}(\tilde{\phi})) + D^2_{t,h,\Sigma^m_u(x),k}(\tilde{\phi})\right) \ , \end{aligned} \tag{4.7c}$$

with

$$\begin{aligned} D^0_{t,h,m,x}(\tilde{\phi}) &= \mathbb{E}(\tilde{\phi}(B_{t+h} - B_t)) \ , \\ D^1_{t,h,m,x}(\tilde{\phi}) &= \mathbb{E}(\tilde{\phi}(B_{t+h} - B_t)(\underline{\sigma}^m(x)^{\mathsf{T}})^{-1}h^{-1}(B_{t+h} - B_t)) \ , \\ D^2_{t,h,\Sigma,k}(\tilde{\phi})(x) &:= h^{-1}\mathbb{E}\left[\tilde{\phi}(B_{t+h} - B_t)\mathscr{P}_{\Sigma,k}(h^{-1/2}(B_{t+h} - B_t))\right] \ , \end{aligned}$$

and :

$$\mathscr{G}^{m,u}_1(x,r,p) = (b^m_u(x) - \underline{b}^m(x)) \cdot p + c^m_u(x)r + f^m_u(x).$$

Then we have

$$T^m_{t,h}[\phi](x) = G^m_{t,h,x}(\tilde{\phi}^m_{t,h,x}) \ , \tag{4.8}$$

with $\tilde{\phi}^m_{t,h,x}$ and $S^m_{t,h}$ as in (4.3d) and (4.3c).

Using the same arguments as for Lemma 3.1.5 and Lemma 3.1.6, one can obtain the stronger property that for $h \le h_0$, all the operators $G^m_{t,h,x}$ belong to the class of monotone additively $\alpha_h$-subhomogeneous operators from $\mathscr{D}$ to $\mathbb{R}$ if the conditions of these lemma are satisfied. This allows us to apply Theorem 4.1 and thus have the following result.

**Theorem 4.2** ([1, Theorem 2], compare with [54, Theorem 5.1]). *Consider the control problem equivalent to the Hamilton-Jacobi-Bellman equation* (4.1). *Assume that* $\mathscr{U} = \mathbb{R}^p$ *and that for each* $m \in \mathscr{M}$, $c_u^m$ *and* $a_u^m$ *are constant,* $a_u^m$ *is nonsingular,* $b_u^m$ *is affine with respect to* $(x, u)$, $f_u^m$ *is quadratic with respect to* $(x, u)$ *and strictly concave with respect to* $u$, *and that* $\Psi$ *is the supremum of a finite number of quadratic forms. Consider the scheme* (3.21), *with* $\mathbf{T}_{k,h}^0$ *as in* (4.6) *and let* $G_{t,h,x}^m$ *be as in* (4.7). *Assume that* $\underline{\sigma}^m$ *constant and nonsingular,* $\Sigma_u^m$ *constant and nonsingular and* $\underline{b}^m$ *affine. Assume that* $\mathscr{D}$ *is the set of functions with at most quadratic growth. Assume that the operators* $G_{t,h,x}^m$ *belong to the class of monotone additively* $\alpha_h$*-subhomogeneous operators from* $\mathscr{D}$ *to* $\mathbb{R}$, *for some constant* $\alpha_h = 1 + Ch$ *with* $C \geq 0$. *Assume also that the value function* $W^h$ *of* (3.21) *belongs to* $\mathscr{D}$ *and is locally Lipschitz continuous with respect to* $x$. *Then, for all* $t \in \mathscr{T}_h$, *there exists a set* $Z_t$ *and a map* $g_t : \mathbb{R}^d \times Z_t \to \mathbb{R}$ *such that for all* $z \in Z_t$, $g_t(\cdot, z)$ *is a quadratic form and*

$$W^h(t, x) = \sup_{z \in Z_t} g_t(x, z) \ . \tag{4.9}$$

*Moreover, the sets* $Z_t$ *satisfy* $Z_t = \mathscr{M} \times \{\bar{z}_{t+h} : \mathscr{W} \to Z_{t+h} \mid \text{Borel measurable}\}$.

Theorem 4.2 uses the following property which was stated in [1, Lemma 3] without proof, and without the upper bound assumption mentioned in it. We give here the proof of the lemma.

**Lemma 4.3** (Compare with [1, Lemma 3]). *Let us consider the notations and assumptions of Theorem 4.2. Let* $\tilde{z}$ *be a measurable function from* $\mathscr{W}$ *to* $\mathscr{Q}_d$ *and let* $\tilde{q}_x$ *denotes the measurable map* $\mathscr{W} \to \mathbb{R}$, $w \mapsto q(S_{t,h}^m(x, w), \tilde{z}(w))$, *with* $q$ *as in* (4.5). *Assume that there exists* $\bar{z} \in \mathscr{Q}_d$ *such that* $q(x, \tilde{z}(w)) \leq q(x, \bar{z})$ *for all* $x \in \mathbb{R}^d$, *and almost all* $w \in \mathscr{W}$, *and that* $\tilde{q}_x$ *belongs to* $\mathscr{D}$, *for all* $x \in \mathbb{R}^d$. *Then, the function* $x \mapsto G_{t,h,x}^m(\tilde{q}_x)$ *is a quadratic function that is, it can be written as* $q(x, Z)$ *for some* $Z \in \mathscr{Q}_d$.

*Proof.* Since $S_{t,h}^m$ is linear with respect to $x$, $\tilde{q}_x(w)$ is a quadratic function of $x$ the coefficients of which depend on $w$. Then, due to the assumptions that $\underline{\sigma}^m$ and $\Sigma_u^m$ are constant and nonsingular, we get that $D_{t,h,m,x}^i(\tilde{q}_x)$ with $i = 0, 1$, and $D_{t,h,\Sigma_u^m(x),k}^2(\tilde{q}_x)$ are quadratic functions of $x$. Let $G_{t,h,x}^{m,u}(\tilde{\phi})$ denotes the expression in (4.7c) without the maximization in $u$. We get that $G_{t,h,x}^{m,u}(\tilde{q}_x)$ is of the form $K(x, u) + (Ax + Bu) \cdot D_{t,h,m,x}^1(\tilde{q}_x)$, where $K$ is a quadratic function of $(x, u)$, strictly concave with respect to $u$ and $A$ and $B$ are matrices. This also holds if we replace $\tilde{z}(w)$ by $\bar{z}$, that is if we replace $\tilde{q}_x$ by $\tilde{Q}_{t,h,x}^m : w \mapsto q(S_{t,h}^m(x, w), \bar{z})$ with $Q(x) = q(x, \bar{z})$. However in that case, since $Q$ is deterministic, $D_{t,h,m,x}^1(\tilde{Q}_{t,h,x}^m) = \mathscr{D}_{t,h,m}^1(Q)(x) = \mathbb{E}(DQ(S_{t,h}^m(x, B_{t+h} - B_t)))$ which is an affine function of $x$, since $DQ$ is affine. Therefore $G_{t,h,x}^{m,u}(\tilde{Q}_{t,h,x}^m)$ is a quadratic function of $(x, u)$, strictly concave with respect to $u$, so its maximum over $u \in \mathscr{U}$ is a quadratic function of $x$, that we shall denote by $P(x)$.

Since $G_{t,h,x}^m$ is assumed to be monotone from $\mathscr{D}$ to $\mathbb{R}$, we get that $G_{t,h,x}^m(\tilde{q}_x) \leq G_{t,h,x}^m(\tilde{Q}_{t,h,x}^m) = P(x)$. Therefore for all $x \in \mathbb{R}^d$ and $u \in \mathscr{U} = \mathbb{R}^p$, we obtain that $K(x, u) + (Ax + Bu) \cdot D_{t,h,m,x}^1(\tilde{q}_x) = G_{t,h,x}^{m,u}(\tilde{q}_x) \leq P(x)$. So $(Ax + Bu) \cdot D_{t,h,m,x}^1(\tilde{q}_x)$ is a polynomial of degree at most 3 in the variables $x_1, \ldots, x_d, u_1, \ldots, u_p$ upper bounded by a polynomial of degree at most 2. Taking the limit when the $x_i$ and $u_j$ go to $\pm\infty$, we deduce that all the monomials of degree 3 have zero as coefficients, so that $(Ax + Bu) \cdot D_{t,h,m,x}^1(\tilde{q}_x)$ is a quadratic function of $(x, u)$. $D_{t,h,m,x}^1(\tilde{q}_x)$ does not depend on $u$. So $(Ax + Bu) \cdot D_{t,h,m,x}^1(\tilde{q}_x)$ is linear in $u$. Hence, $G_{t,h,x}^{m,u}(\tilde{q}_x)$ is a quadratic function of $(x, u)$, strictly concave with respect to $u$, which implies that its maximum over $u \in \mathscr{U}$, $G_{t,h,x}^m(\tilde{q}_x)$, is a quadratic function of $x$. □

*Proof of Theorem 4.2.* Lemma 4.3 shows in particular the property that each operator $T^m_{t,h}$ such that $T^m_{t,h}[\phi](x) = G^m_{t,h,x}(\tilde{\phi}^m_{t,h,x})$ with $G^m_{t,h,x}$ as in (4.7c), sends a deterministic quadratic form into a quadratic form. Since for any finite number of quadratic forms, there exists a quadratic form which dominate them, the assumptions of Theorem 4.2 imply that $\Psi$ and then all the functions $W^h(t,\cdot)$ are upper bounded by a quadratic form (recall that $\mathscr{M}$ is a finite set). Then, applying Theorem 4.1 to the maps $W^h(t,\cdot)$ and using Lemma 4.3, we get the representation formula (4.9). □

In Theorem 4.2, as in [54, 44, Theorem 5.1], the sets $Z_t$ are infinite for $t < T$. If the Brownian process is discretized in space, the set $\mathscr{W}$ can be replaced by a finite subset, and the sets $Z_t$ become finite. Nevertheless, their cardinality increases at each time step as $\#Z_t = \#\mathscr{M} \times (\#Z_{t+h})^p$ where $p$ is the cardinality of the discretization of $\mathscr{W}$. Then, if all the quadratic functions generated in this way were different, we would obtain that $\#Z_0 = \#\mathscr{M}^{-1/(p-1)} \times (\#\mathscr{M}^{1/(p-1)}\#Z_T)^{p^{T/h}}$ is doubly exponential with respect to the number of time discretization points and more than exponential with respect to $p$. Since the Brownian process is $d$-dimensional, one may need to discretize it with a number $p$ of values which is exponential in the dimension $d$. Hence, the computational time of the resulting method would be worst than the one of a usual grid discretization. In [54], McEneaney, Kaise and Han proposed to apply a pruning method at each time step $t \in \mathscr{T}_h$ to reduce the cardinality of $Z_t$. For this, they assume already that the function $W^h$ is represented as the supremum of the quadratic functions parameterized by a finite set $Z_t$ of $\mathscr{Q}_d$. They show that pruning (that is eliminating elements of $Z_t$) is optimal if one looks for a subset of $\mathscr{Q}_d$ with given size representing $W^h$ as the supremum of the corresponding quadratic functions with a minimal measure of the error. There, the measure of the error is the maximum of the integral of the difference of functions with respect to any probabilistic measure on $\mathbb{R}^d$. Then, restricting the set of probabilistic measures to the set of normal distributions, they propose to use LMI techniques to find the elements of $Z_t$ that can be eliminated. However, whatever the number $N$ of quadratic functions used at the end to represent $W^h$ at each time step is, the computational time of the pruning method is at least in the order of the cardinal of the initial set $Z_t$. Hence, if $Z_t$ is computed as above using a discretization of the Brownian process and the representation of $W^h$ at time $t+h$ already uses $N$ quadratic forms, then $\#Z_t = \#\mathscr{M} \times N^p$, so that it is exponential with respect to $p$ and can then be doubly exponential with respect to the dimension $d$.

In [1], we proposed to compute the expression of the maps $W^h(t,\cdot)$ as a maximum of quadratic forms by using simulations of the processes $\hat{X}^m$. These simulations are not only used for regression estimations of conditional expectations, which are computed there only in the case of random quadratic forms, leading to quadratic forms, but they are also used to fix the "discretization points" $x$ at which the optimal quadratic forms in the expression (4.9) are computed. We will explicitly give the algorithm in Section 4.2 but first let us make the same analysis as above in the upwind large stencil probabilistic scheme setting.

### 4.1.2 Method in the upwind large stencil probabilistic scheme setting

By the same reasoning as in the previous subsection, the following version of the upwind large stencil probabilistic scheme operator $\mathbf{T}^1_{k,h}$ can be used to solve PDE (4.1).

$$\mathbf{T}^1_{k,h}[\Phi](t,x) = \max_{m\in\mathscr{M}} \mathbf{T}^{1,m}_{k,h}[\Phi](t,x),$$

with :

$$\mathbf{T}^{1,m}_{k,h}[\Phi](t,x) := \sup_{u\in\mathscr{U}} \left\{ \frac{\mathbb{E}\left[\Phi(\hat{X}^{t,x,m}_h) T^N_{k,h,m,u,B}(t,x)\right] + h f^m_u(x)}{T^D_{h,m,u,B}(t,x)} \right\},$$

where :

$$T^D_{h,m,u,B}(t,x) := 1 + h\mathbb{E}\left[\mathscr{P}^1_{g^m_u(x)}(h^{-1}B^t_h)\right] - hc^m_u(x),$$

and

$$T^N_{k,h,m,u,B}(t,x) := 1 + \mathscr{P}^2_{\Sigma^m_u(x),k}(h^{-1/2}B^t_h) + h\mathscr{P}^1_{g^m_u(t,x)}(h^{-1}B^t_h),$$

$\Sigma^m_u$ and $g^m_u$ being such that :

$$(a^m_u - \underline{\sigma}^m(\underline{\sigma}^m)^\intercal)(t,x) = (\underline{\sigma}^m\Sigma^m_u(\Sigma^m_u)^\intercal(\underline{\sigma}^m)^\intercal)(t,x), \tag{4.10}$$
$$(b^m_u - \underline{b}^m)(t,x) = \underline{\sigma}(t,x)g^m_u(t,x), \tag{4.11}$$

$\mathscr{P}^1_g$ and $\mathscr{P}^2_{\Sigma,k}$ being defined as in (3.12) and (3.8) and $\hat{X}^{t,x,m}_h := x + \underline{b}^m(x)h + \underline{\sigma}^m(x)B^t_h$.

As in the previous subsection, we define some operators by considering for a continuous function $\phi : \mathbb{R}^d \to \mathbb{R}, x \mapsto \phi(x)$ :

$$T_{t,h}[\phi](x) = \mathbf{T}^1_{k,h}[\phi](t,x), \tag{4.12a}$$

$$T^m_{t,h}[\phi](x) = \mathbf{T}^{1,m}_{k,h}[\phi](t,x). \tag{4.12b}$$

Let $G^m_{t,h,x}$ be given by

$$G^m_{t,h,x}(\tilde{\phi}) \quad = \quad \sup_{u\in\mathscr{U}} \frac{G^N_{t,h,x,m,u}(\tilde{\phi})}{T^D_{h,m,u,B}(x)} \quad , \tag{4.13}$$

with

$$G^N_{t,h,x,m,u}(\tilde{\phi}) \quad = \quad D^0_{t,h}(\tilde{\phi}) + h\big\{f^m_u(x) + D^1_{t,h,g^m_u(x)}(\tilde{\phi}) + D^2_{t,h,\Sigma^m_u(x),k}(\tilde{\phi})\big\} \quad , \tag{4.14}$$

where

$$D^0_{t,h}(\tilde{\phi}) = \mathbb{E}(\tilde{\phi}(B_{t+h} - B_t)) \ ,$$
$$D^1_{t,h,g}(\tilde{\phi}) = \mathbb{E}(\tilde{\phi}(B_{t+h} - B_t)\mathscr{P}^1_g(h^{-1}(B_{t+h} - B_t)) \ ,$$
$$D^2_{t,h,\Sigma,k}(\tilde{\phi})(x) := h^{-1}\mathbb{E}\left[\tilde{\phi}(B_{t+h} - B_t)\mathscr{P}^2_{\Sigma,k}(h^{-1/2}(B_{t+h} - B_t))\right] \ .$$

Then, we have :

$$T^m_{t,h}[\phi](x) = G^m_{t,h,x}(\tilde{\phi}^m_{t,h,x}) \ , \tag{4.15}$$

with $\tilde{\phi}^m_{t,h,x}$ and $S^m_{t,h}$ as in (4.3d) and (4.3c).

Using the same arguments as for Lemma 3.1.11 and Lemma 3.1.12, one can obtain the property that for $h \le h_0$, all the operators $G^m_{t,h,x}$ belong to the class of monotone additively

$\alpha_h$-subhomogeneous operators from $\mathscr{D}$ to $\mathbb{R}$ if the conditions of these lemma are satisfied. This allows us to apply Theorem 4.1. However, we do not have a result similar to Theorem 4.2 here as in this case, the expressions $g^+$ and $g^-$ in the value of $D^1_{t,h,g}(\tilde{\phi})$ prevent the operator $G^m_{t,h,x}$ from sending a convex random quadratic form that is upper bounded by a deterministic quadratic form into a quadratic form.

Despite this fact, one can still obtain the following result.

**Theorem 4.1.1.** *Let us consider the notations and assumptions of Theorem 4.2, except that $\mathbf{T}^0_{k,h}$ is replaced by the operator $\mathbf{T}^1_{k,h}$. Let us use the notations and properties of* (4.12) *and* (4.13)*. Let $\bar{z}$ be a measurable function from $\mathscr{W}$ to $\mathscr{Q}_d$ and $Z \in \mathscr{Q}_d$ be such that $q(x, \bar{z}(W)) \leq q(x, Z)$ for all $x \in \mathbb{R}^d$ and $w \in \mathscr{W}$, where $q$ is as in* (4.5)*. Let $\tilde{q}_x$ be the map $\mathscr{W} \to \mathbb{R}$, $w \mapsto q(S^m_{t,h}(x, w), \bar{z}(w))$, Then, the function $\bar{q} : x \mapsto G^m_{t,x,h}(\tilde{q}_x)$ is upper bounded by a quadratic map. The same property holds for lower bounds.*

*Moreover, there exists $C > 0$, independent of $h$ such that if the map $\bar{z}$ is constant, that is deterministic, and $\|\bar{z}\| \leq K$ for some norm on $\mathscr{Q}_d$. Then, there exists $z \in \mathscr{Q}_d$ such that $\|z - \bar{z}\| \leq C(K + 1)^2 h$ and*

$$|\bar{q}(x) - q(x, z)| \leq C(K + 1)^3 h^{3/2}(|x|^2 + 1)^{3/2}, \quad \textit{for all } x \in \mathbb{R}^d \ .$$

*Proof.* Let $\widetilde{T}_{t,h}$ and $\widetilde{T}^m_{t,h}$, $\widetilde{G}^m_{t,h,x}$ denote respectively the operators $T_{t,h}$, $T^m_{t,h}$ and $G^m_{t,h,x}$ defined in (4.7). The notations $T_{t,h}$, $T^m_{t,h}$ and $G^m_{t,h,x}$ will then refer to Definition (4.12). We introduce $\widetilde{G}_{t,h,x,m,u}$ and $\widetilde{D}^1_{t,h,g}$ such that :

$$\begin{aligned} \widetilde{G}^m_{t,h,x}(\tilde{\phi}) &= \max_{u \in \mathscr{U}} \widetilde{G}_{t,h,x,m,u}(\tilde{\phi}) \ , && (4.16) \\ \widetilde{G}_{t,h,x,m,u}(\tilde{\phi}) &= D^0_{t,h}(\tilde{\phi})(1 + c^m_u(x)h) \\ &\quad + h\big\{ f^m_u(x) + \widetilde{D}^1_{t,h,g^m_u(x)}(\tilde{\phi}) + D^2_{t,h,\Sigma^m_u(x),k}(\tilde{\phi}) \big\} \ , && (4.17) \\ \widetilde{D}^1_{t,h,g}(\tilde{\phi}) &= \mathbb{E}(\tilde{\phi}(B_{t+h} - B_t) g \cdot (h^{-1}(B_{t+h} - B_t)) \ . \end{aligned}$$

Let $\bar{z}$ be a measurable function from $\mathscr{W}$ to $\mathscr{Q}_d$ and $Z \in \mathscr{Q}_d$ be such that $q(x, \bar{z}(w)) \leq q(x, Z)$ for all $x \in \mathbb{R}^d$ and $w \in \mathscr{W}$, where $q$ is as in (4.5). Consider the map $\phi(x) = q(x, Z)$. It satisfies $-CK(1 + |x|^2) \leq \phi(x) \leq CK(1 + |x|^2)$ as soon as $\|Z\| \leq K$. Here and below $\|\cdot\|$ denotes a norm on $\mathscr{Q}_d$ and $C$ is any positive constant independent of $h \leq 1$. Since $G^m_{t,h,x}$ is monotone, we get that $\bar{q}(x) \leq G^m_{t,h,x}(\tilde{\phi}^m_{t,h,x}) = T^m_{t,h}(\phi)(x)$ and a similar result holds for a lower bound. Due to the assumptions on the parameters of the problem, it is easy to show that for any (deterministic) quadratic form, $\widetilde{T}^m_{t,h}(\phi)$ is a quadratic form. Hence, to obtain the two assertions of the theorem, it is sufficient to show that, for any quadratic form $\phi$ with norm $K$, $T^m_{t,h}(\phi)$ is bounded above and below by quadratic forms, the norm of which depend on $K$, $\widetilde{T}^m_{t,h}(\phi)$ is a quadratic form such that the norm of its difference with $\phi$ is bounded by $C(K + 1)^2 h$, and that we have

$$|T^m_{t,h}(\phi)(x) - \widetilde{T}^m_{t,h}(\phi)(x)| \leq C(K + 1)^3 h^{3/2}(|x|^2 + 1)^{3/2}, \quad \text{for all } x \in \mathbb{R}^d \ .$$

Using that $\mathscr{P}^1_g(w) = g \cdot w + |g| \cdot (|w| - \mathbb{E}(|w|) + |g| \cdot \mathbb{E}(|w|)$, we deduce

$$\frac{G^N_{t,h,x,m,u}(\tilde{\phi})}{T^D_{h,m,u,B}(x)} - D^0_{t,h}(\tilde{\phi}) = \frac{\widetilde{G}_{t,h,x,m,u}(\tilde{\phi}) - D^0_{t,h}(\tilde{\phi}) + R_{t,h,g^m_u(x)}(\tilde{\phi})}{T^D_{h,m,u,B}(x)} \ , \qquad (4.18)$$

where

$$R_{t,h,g}(\tilde{\phi}) = \mathbb{E}\left[\tilde{\phi}(B_h^t)|g| \cdot (|B_h^t| - \mathbb{E}(|B_h^t|))\right] \quad \text{with } B_h^t = B_{t+h} - B_t \ ,$$

and $|B_h^t|$ representing the absolute value term by term of $B_h^t$.

Due to the assumptions on the coefficients and on the scheme, $S_{t,h}^m(x,w)$ is affine with respect to $(x,w)$, $(x,u) \mapsto \Sigma_u^m(x)$ is constant and nonsingular, $(x,u) \mapsto g_u^m(x)$ is affine in $(x,u)$, and $(x,u) \mapsto c_u^m(x)$ is constant. Hence the map $\tilde{\phi}_{t,h,x}^m(w)$ is a quadratic function of $(x,w)$. Applying expectations with appropriate factors, we obtain that $D_{t,h}^0(\tilde{\phi}_{t,h,x}^m)$ is a quadratic form, such that the norm of $D_{t,h}^0(\tilde{\phi}_{t,h,x}^m) - \phi(x)$ is bounded by $CKh$, and that $D_{t,h,\Sigma_u^m(x),k}^2(\tilde{\phi}_{t,h,x}^m)$ is a constant (in $x$ and $u$) which can be bounded by $CK$.

Since the coordinates of $B_h^t$ are independent and with zero expectation, we also get that the first order term $\widetilde{D}_{t,h,g^m(x,u)}^1(\tilde{\phi}_{t,h,x}^m)$ in (4.17) is equal to the scalar product of $g_u^m(x)$, which is affine in $(x,u)$, with an affine function of $x$, the norm of which is bounded by $CK$. We deduce that

$$\widetilde{G}_{t,h,x,m,u}(\tilde{\phi}_{t,h,x}^m) - D_{t,h}^0(\tilde{\phi}_{t,h,x}^m) = h(f_u^m(x) + \Psi(x,u)) \ , \tag{4.19}$$

where $\Psi$ is quadratic in $x$ and $u$ with second order derivatives in $u$ equal to 0, with a norm bounded by $CK$ and that $D_{t,h}^0(\tilde{\phi}_{t,h,x}^m)$ is quadratic in $x$, and that their norms are bounded by $CK$. Taking the supremum with respect to $u$ in the previous expression, we deduce that $\widetilde{T}_{t,h}^m(\phi) = \widetilde{G}_{t,h,x}^m(\tilde{\phi}_{t,h,x}^m)$ is quadratic in $x$, and that the norm of its difference with $\phi$ is bounded by $C(K+1)^2h$.

Since $g^m : (x,u) \mapsto g_u^m(x)$ has linear growth,

$$|R_{t,h,g_u^m(x)}(\tilde{\phi}_{t,h,x}^m)| \leq C(1+|u|+|x|)\|\mathbb{E}\left[\tilde{\phi}_{t,h,x}^m(B_h^t)(|B_h^t| - \mathbb{E}(|B_h^t|))\right]\|.$$

Again due to the properties of $\tilde{\phi}_{t,h,x}^m$ and $B_h^t$, we get that the second factor in the former inequality is constant and is bounded by $CKh^{3/2}$.

All together, we deduce that

$$\frac{G_{t,h,x,m,u}^N(\tilde{\phi}_{t,h,x}^m)}{T_{h,m,u,B}^D(x)} - D_{t,h}^0(\tilde{\phi}_{t,h,x}^m) \leq \frac{h(f_u^m(x) + \Psi(x,u)) + CKh^{3/2}(1+|u|+|x|)}{T_{h,m,u,B}^D(x)} \ .$$

Then, using $CKh^{1/2}|u| \leq |u|^2\epsilon/2 + C^2K^2h/(2\epsilon)$, for $\epsilon > 0$ small enough, and similarly for $|x|$, and using that $T_{h,m,u,B}^D(x) \geq 1 - hc_u^m(x) \geq 1/2$ for $h$ small enough, we deduce that the right hand side of the above inequality is bounded above by a quadratic form in $x$, so does the supremum with respect to $u$ of the left hand side. Since $D_{t,h}^0(\tilde{\phi}_{t,h,x}^m)$ is a quadratic form, we deduce that $G_{t,h,x}^m(\tilde{\phi}_{t,h,x}^m) = T_{t,h}^m(\phi)(x)$ is bounded above by a quadratic form. Moreover the norm of this quadratic form is bounded by $K + C(K+1)^2h$. A similar lower bound can be obtained.

To obtain the second assertion of the lemma, we shall now use the following equation

$$\frac{G_{t,h,x,m,u}^N(\tilde{\phi})}{T_{h,m,u,B}^D(x)} - \widetilde{G}_{t,h,x}^m(\tilde{\phi}) = \frac{\widetilde{G}_{t,h,x,m,u}(\tilde{\phi}) - \widetilde{G}_{t,h,x}^m(\tilde{\phi}) + \widetilde{R}_{t,h,g_u^m(x)}(\tilde{\phi})}{T_{h,m,u,B}^D(x)} \ , \tag{4.20}$$

where

$$\widetilde{R}_{t,h,g_u^m(x)}(\tilde{\phi}) = (D_{t,h}^0(\tilde{\phi}) - \widetilde{G}_{t,h,x}^m(\tilde{\phi}))(T_{h,m,u,B}^D(x) - 1) + R_{t,h,g_u^m(x)}(\tilde{\phi}) \ .$$

Using (4.19), we get that for $\tilde{\phi} = \tilde{\phi}^m_{t,h,x}$, $\widetilde{G}_{t,h,x,m,u}(\tilde{\phi}) - \widetilde{G}^m_{t,h,x}(\tilde{\phi})$ can be written in the form $-h(u-L(x))^T Q(u-L(x))$ where $L$ is affine with a norm bounded by $CK$ and $Q$ is a positive definite matrix, independent of $K$, so there exists $\beta > 0$ such that this expression is bounded above by $-\beta h|u-L(x)|^2$. Using (4.19) again, we obtain that $D^0_{t,h}(\tilde{\phi}) - \widetilde{G}^m_{t,h,x}(\tilde{\phi})$ is a quadratic form, the norm of which is bounded by $C(K+1)^2 h$. Moreover, $T^D_{h,m,u,B}(x) - 1 = -c^m_u(x)h + C\sqrt{h}|g^m_u(x)|$ for some norm (the 1-norm) on $\mathbb{R}^d$ and $|R_{t,h,g^m_u(x)}(\tilde{\phi})| \leq CKh^{3/2}|g^m_u(x)|$. We deduce that $|\widetilde{R}_{t,h,g^m_u(x)}(\tilde{\phi})| \leq Ch^2(K+1)^2(1+|x|^2) + C(K+1)^2 h^{3/2}(1+|x|^2)|g^m_u(x)|$. Then, using that $T^D_{t,h,m,u}(x) \geq 1+\delta^m h \geq 1/2$ for $h$ small enough, and that $y \mapsto y/(a+y)$ is increasing with respect to $y > 0$, for any $a > 0$, we obtain

$$\begin{aligned}\frac{|\widetilde{R}_{t,h,g^m_u(x)}(\tilde{\phi})|}{T^D_{h,m,u,B}(x)} &\leq Ch^2(K+1)^2(1+|x|^2) + \frac{C(K+1)^2 h^{3/2}(1+|x|^2)|g^m_u(x)|}{1+C\sqrt{h}|g^m_u(x)|} \\ &\leq Ch^2(K+1)^2(1+|x|^2) + \frac{C(K+1)^2 h(1+|x|^2)A(x,u)}{1+A(x,u)}\end{aligned}$$

for any bound $A(x,u)$ of $h^{1/2}|g^m_u(x)|$. Since $|g^m_u(x)| \leq C(K+1)(1+|x|)+|u-L(x)|$, we can take $A(x,u) = C(K+1)h^{1/2}(1+|x|^2)^{1/2} + \frac{\epsilon}{2C(K+1)^2 h(1+|x|^2)}|u-L(x)|^2 + \frac{C(K+1)^2 h^2(1+|x|^2)}{2\epsilon}$ for any $\epsilon > 0$. Then, bounding above separatly the three terms of the sum in $A(x,u)/(1+A(x,u))$ by lower bounding $1+A(x,u)$, and using the same upper bound $A(x,u)$ of $h^{1/2}\|g^m_u(x)\|$ in the expression of the first summand in (4.20), and that $\widetilde{G}_{t,h,x,m,u}(\tilde{\phi}) - \widetilde{G}^m_{t,h,x}(\tilde{\phi}) \leq -\beta h|u-L(x)|^2 \leq 0$, we deduce for $\epsilon = 2\beta\frac{h}{C}$:

$$\frac{G^N_{t,h,x,m,u}(\tilde{\phi})}{T^D_{h,m,u,B}(x)} - \widetilde{G}^m_{t,h,x}(\tilde{\phi}) \leq C(K+1)^3[h(1+|x|^2)]^{3/2} \ .$$

Then, taking the supremum over $u$, we obtain

$$G^m_{t,h,x}(\tilde{\phi}) - \widetilde{G}^m_{t,h,x}(\tilde{\phi}) \leq C(K+1)^3[h(1+|x|^2)]^{3/2} \ .$$

For the reverse inequality, using that $\widetilde{G}_{t,h,x,m,u}(\tilde{\phi}) - \widetilde{G}^m_{t,h,x}(\tilde{\phi}) = 0$ for $u = L(x)$, and applying the above bound of $\widetilde{R}_{t,h,g^m_u(x)}(\tilde{\phi})$ to $u = L(x)$, we get directly that

$$G^m_{t,h,x}(\tilde{\phi}) - \widetilde{G}^m_{t,h,x}(\tilde{\phi}) \geq -C(K+1)^3[h(1+|x|^2)]^{3/2} \ .$$

□

If the last inequality of Lemma 4.1.1 were true for random maps $\bar{z}$, then one may expect to obtain Equation (4.9) of Theorem 4.2 up to an error in $O(\sqrt{h}(1+\|x\|^2)^{3/2})$. Note that, for this bound to be true, one would also need to show the following Lipschitz property for $T_{t,h}$: if $\phi(x) - \phi'(x) \leq K(1+\|x\|^2)^{3/2}$ for all $x \in \mathbb{R}^d$, and $\phi$ and $\phi'$ have a given quadratic growth, then $T_{t,h}(\phi)(x) - T_{t,h}(\phi')(x) \leq (1+Ch)K(1+\|x\|^2)^{3/2}$ for all $x \in \mathbb{R}^d$.

## 4.2 Description of the algorithm and complexity

### 4.2.1 Description of the algorithm presented in [1]

Using the notations $T^m_{t,h}$ and $G^m_{t,x,h}$ introduced in the previous section whose meaning should be understood depending on the scheme that is used, the first description of the max-plus probabilistic algorithm introduced in [1] was the following. (Some notations were introduced in Section 3.1).

**Algorithm 4.2.1.**
*Input:* *A constant $\epsilon$ giving the precision, and a 5-uple $N=(N_{\rm in},N_{\rm rg},N_x,N_w,N_{\rm m})$ of integers giving the sizes of the samples and the "method of sampling" $N_{\rm m}\in\{1,\dots,5\}$ described below. A finite subset $Z_T$ of $\mathcal{Q}_d$ such that $|\Psi(x)-\max_{z\in Z_T} q(x,z)|\leq\epsilon$, for all $x\in\mathbb{R}^d$, and $\#Z_T\leq M\times N_{\rm in}$, and the operators $T^m_{t,h}$ and $G^m_{t,x,h}$.*
*Output:* *The subsets $Z_t$ of $\mathcal{Q}_d$, for $t\in\mathcal{T}_h\cup\{T\}$, and the approximate value function $W^{h,N}:(\mathcal{T}_h\cup\{T\})\times\mathbb{R}^d\to\mathbb{R}$.*
• *Initialization:* *Let $\hat{X}^m(0)=\hat{X}(0)$, for all $m\in\mathcal{M}$, where $\hat{X}(0)$ is random and independent of the Brownian process. Consider a sample of $(\hat{X}(0),(B_{t+h}-B_t)_{t\in\mathcal{T}_h})$ of size $N_{\rm in}$ indexed by $\omega\in\Omega_{N_{\rm in}}:=\{1,\dots,N_{\rm in}\}$, and denote, for each $t\in\mathcal{T}_h\cup\{T\}$ and $\omega\in\Omega_{N_{\rm in}}$, $\hat{X}^m(t,\omega)$ the value of $\hat{X}^m(t)$ induced by this sample and satisfying $\hat{X}^m(t+h):=\hat{X}^m(t)+\underline{b}^m(\hat{X}^m(t))h+\underline{\sigma}^m(\hat{X}^m(t))(B_{t+h}-B_t)$. Define the function $W^{h,N}(T,\cdot)$ by $W^{h,N}(T,x)=\max_{z\in Z_T} q(x,z)$, for $x\in\mathbb{R}^d$, with $q$ as in (4.5).*
• *For $t=T-h,T-2h,\dots,0$ apply the following 3 steps:*

*(1) For each $\omega\in\Omega_{N_{\rm in}}$ and $m\in\mathcal{M}$, construct a sample $(\omega_1,\omega'_1),\dots,(\omega_{N_{\rm rg}},\omega'_{N_{\rm rg}})$ of elements of $\Omega_{N_{\rm in}}\times\Omega_{N_{\rm in}}$, using the method described below in item named "Method $N_{\rm m}$" and using possibly the constants $N_x$ and $N_w$. Induce the sample $\hat{X}^m(t,\omega_i)$ (resp. $(B_{t+h}-B_t)(\omega'_i)$) for $i\in\Omega_{N_{\rm rg}}$ of $\hat{X}^m(t)$ (resp. $B_{t+h}-B_t$). Denote by $\mathcal{W}^N_t\subset\mathcal{W}$ the set of $(B_{t+h}-B_t)(\omega'_i)$ for $i\in\Omega_{N_{\rm rg}}$.*

*(2) For each $\omega\in\Omega_{N_{\rm in}}$ and $m\in\mathcal{M}$, construct $z_t\in\mathcal{Q}_d$ depending on $\omega$ and $m$ as follows: Let $\bar{z}_{t+h}:\mathcal{W}^N_t\to Z_{t+h}\subset\mathcal{Q}_d$ be such that, for all $i\in\Omega_{N_{\rm rg}}$ we have*

$$\begin{aligned}&W^{h,N}(t+h,S^m_{t,h}(\hat{X}^m(t,\omega),(B_{t+h}-B_t)(\omega'_i)))\\&=q\big(S^m_{t,h}(\hat{X}^m(t,\omega),(B_{t+h}-B_t)(\omega'_i)),\bar{z}_{t+h}((B_{t+h}-B_t)(\omega'_i))\big)\ .\end{aligned}$$

*Extend $\bar{z}_{t+h}$ as a measurable map on $\mathcal{W}$. Let $\tilde{q}^{m,\bar{z}_{t+h}}_{t,x,h}$ be the map $\mathcal{W}\to\mathbb{R}$, $w\mapsto q(x+\underline{b}^m(x)h+\underline{\sigma}^m(x)w,\bar{z}_{t+h}(w))$. Compute an approximation of $x\mapsto G^m_{t,x,h}(\tilde{q}^{m,\bar{z}_{t+h}}_{t,x,h})$ by a linear regression estimation on the set of quadratic forms using the sample $(\hat{X}^m(t,\omega_i),(B_{t+h}-B_t)(\omega'_i))$, with $i\in\Omega_{N_{\rm rg}}$. We obtain $z_t\in\mathcal{Q}_d$ such that $q(x,z_t)\simeq G^m_{t,x,h}(\tilde{q}^{m,\bar{z}_{t+h}}_{t,x,h})$.*

*(3) Denote by $Z_t$ the set of all the $z_t\in\mathcal{Q}_d$ obtained in this way, and define the function $W^{h,N}(t,\cdot)$ by :*

$$W^{h,N}(t,x)=\max_{z\in Z_t} q(x,z)\quad\forall x\in\mathbb{R}^d\ .$$

Let us precise now the different choices $N_{\rm m}$ of the "method of sampling" used in the algorithm:

- **Method 1 :** Assume $N_{\rm rg}=N_{\rm in}$ and take $\omega_i=\omega'_i=i$ for $i\in\Omega_{N_{\rm rg}}$, which means that we take the initial sampling.
- **Method 2 :** Assume $N_{\rm rg}=N_x\times N_w$, with $N_x\leq N_{\rm in}$, and choose once for all $\omega\in\Omega_{N_{\rm in}}$ and $m\in\mathcal{M}$ in the algorithm: a random sampling $\omega_{i,1}$, $i=1,\dots,N_x$ among the elements of $\Omega_{N_{\rm in}}$ and independently a random sampling $\omega'_{1,j}$ $j=1,\dots,N_w$ among the elements of $\Omega_{N_{\rm in}}$, then take the product of samplings, leading to $(\omega_{i,1},\omega'_{1,j})$ for $i=1,\dots,N_x$ and $j=1,\dots,N_w$. Reindexing the sampling, we obtain $(\omega_i,\omega'_i)$ for $i=1,\dots,N_{\rm rg}$.

- **Method 3 :** Do as in Method 2, but choose different samplings for each $\omega \in \Omega_{N_{\mathrm{in}}}$ and $m \in \mathscr{M}$ in the algorithm, independently.
- **Method 4 :** Assume $N_{\mathrm{rg}} = N_x \times N_w$ and $N_w = N_{\mathrm{in}}$ and do as in Method 2, but take the fixed sampling $\omega'_{1,j} = j$ instead of random sampling.
- **Method 5 :** Assume $N_{\mathrm{rg}} = N_{\mathrm{in}}^2$ and do as in Method 2, but take the fixed samplings $\omega_{i,1} = i$ and $\omega'_{1,j} = j$ instead of random samplings.

### 4.2.2 Complexity of the algorithm presented in [1]

Note that no computation is done at Step (3), which gives only a formula (or procedure) to be able to compute the value function at each time step $t$ and point $x \in \mathbb{R}^d$ as a function of the sets $Z_t$. This is what is done for instance to obtain plots. In particular, the algorithm only stores the elements of $Z_t$ which are elements of $\mathscr{Q}_d$. It is easy to see that the sets $Z_t$ of the above algorithm satisfy $\#Z_t \leq M \times N_{\mathrm{in}}$ for all $t \in \mathscr{T}_h$ and $\mathscr{Q}_d$ has dimension $(d+1)(d+2)/2$. The memory space to store the value function at a time step is in the order $M \times N_{\mathrm{in}} \times d^2$, so the maximum space complexity of the algorithm is $O(M \times N_{\mathrm{in}} \times d^2 \times T/h)$. The number of computations at each time step for the optimization (computation of the $\bar{z}_{t+h}$) will be at most in the order of $(M \times N_{\mathrm{in}})^2 \times N_{\mathrm{in}}$ in methods 1 and 5 and at most in the order of $(M \times N_{\mathrm{in}})^2 \times N_w$ when using methods 2,3,4. Moreover, the number of computations at each time step for the regression estimation will be at most in the order of $M \times N_{\mathrm{in}} \times N_{\mathrm{rg}}$ so will be dominated by the number of computations of the optimization step.

### 4.2.3 Description of the algorithm presented in [2]

A particular case of the algorithm described previously was presented in [2], where we added the possibility of having the same operator $\mathscr{L}^{X^m}$ for different $m$, in which case we choose to simulate the process $\hat{X}^m$ only one time for each possible $\mathscr{L}^{X^m}$. Then the number of simulations and quadratic forms decreases. To formalize this, we considered in the algorithm the projection map $\pi$ which sends an element $m$ of $\mathscr{M}$ to a particular element of its equivalence class for the equivalence relation "$m \sim m'$ if $\mathscr{L}^{X^m} = \mathscr{L}^{X^{m'}}$". The algorithm is described in the following.

**Algorithm 4.2.2.** *Input:* *A constant $\epsilon$ giving the precision, a time step $h$ and a horizon time $T$ such that $T/h$ is an integer, a 3-uple $N = (N_{\mathrm{in}}, N_x, N_w)$ of integers giving the sizes of the samples, such that $N_x \leq N_{\mathrm{in}}$, a subset $\overline{\mathscr{M}} \subset \mathscr{M}$ and a projection map $\pi : \mathscr{M} \to \overline{\mathscr{M}}$. A finite subset $Z_T$ of $\mathscr{Q}_d$ such that $|\psi(x) - \max_{z \in Z_T} q(x,z)| \leq \epsilon$, for all $x \in \mathbb{R}^d$, and $\#Z_T \leq \#\overline{\mathscr{M}} \times N_{\mathrm{in}}$. The operators $T_{t,h}$, $S^m_{t,h}$ and $G^m_{t,x,h}$ for $t \in \mathscr{T}_h$ and $m \in \mathscr{M}$, with $\mathscr{L}^{X^m}$ (and thus $S^m_{t,h}$) depending only on $\pi(m)$.*
*Output:* *The subsets $Z_t$ of $\mathscr{Q}_d$, for $t \in \mathscr{T}_h \cup \{T\}$, and the approximate value function $W^{h,N} : (\mathscr{T}_h \cup \{T\}) \times \mathbb{R}^d \to \mathbb{R}$.*
• *Initialization:* *Let $\hat{X}^m(0) = \hat{X}(0)$, for all $m \in \overline{\mathscr{M}}$, where $\hat{X}(0)$ is random and independent of the Brownian process. Consider a sample of $(\hat{X}(0), (B_{t+h} - B_t)_{t \in \mathscr{T}_h})$ of size $N_{\mathrm{in}}$ indexed by $\omega \in \Omega_{N_{\mathrm{in}}} := \{1, \ldots, N_{\mathrm{in}}\}$, and denote, for each $t \in \mathscr{T}_h \cup \{T\}$, $\omega \in \Omega_{N_{\mathrm{in}}}$, and $m \in \overline{\mathscr{M}}$, $\hat{X}^m(t,\omega)$ the value of $\hat{X}^m(t)$ induced by this sample satisfying (4.4). Define the function $W^{h,N}(T,\cdot)$ by $W^{h,N}(T,x) = \max_{z \in Z_T} q(x,z)$, for $x \in \mathbb{R}^d$, with $q$ as in (4.5).*
• *For $t = T-h, T-2h, \ldots, 0$ apply the following 3 steps:*

*(1) Choose a random sampling* $\omega_{i,1},\ i=1,\dots,N_x$ *among the elements of* $\Omega_{N_{\rm in}}$ *and independently a random sampling* $\omega'_{1,j}\ j=1,\dots,N_w$ *among the elements of* $\Omega_{N_{\rm in}}$*, then take the product of samplings, that is consider* $\omega_{(i,j)}=\omega_{i,1}$ *and* $\omega'_{(i,j)}=\omega'_{1,j}$ *for all* $i$ *and* $j$*, leading to* $(\omega_\ell,\omega'_\ell)$ *for* $\ell\in\Omega_{N_{\rm rg}}:=\{1,\dots,N_x\}\times\{1,\dots,N_w\}$.

*Induce the sample* $\hat{X}^m(t,\omega_\ell)$ *(resp.* $(B_{t+h}-B_t)(\omega'_\ell)$*) for* $\ell\in\Omega_{N_{\rm rg}}$ *of* $\hat{X}^m(t)$ *with* $m\in\overline{\mathscr{M}}$ *(resp.* $B_{t+h}-B_t$*). Denote by* $\mathscr{W}_t^N\subset\mathscr{W}$ *the set of* $(B_{t+h}-B_t)(\omega'_\ell)$ *for* $\ell\in\Omega_{N_{\rm rg}}$.

*(2) For each* $\omega\in\Omega_{N_{\rm in}}$ *and* $m\in\overline{\mathscr{M}}$*, denote* $x_t=\hat{X}^m(t,\omega)$ *and construct* $z_t\in\mathscr{Q}_d$ *depending on* $\omega$ *and* $m$ *as follows:*

*(a) Choose* $\bar{z}_{t+h}:\mathscr{W}_t^N\to Z_{t+h}\subset\mathscr{Q}_d$ *such that, for all* $\ell\in\Omega_{N_{\rm rg}}$*, we have*

$$\begin{aligned}&W^{h,N}(t+h,S^m_{t,h}(x_t,(B_{t+h}-B_t)(\omega'_\ell)))\\&\quad=q\big(S^m_{t,h}(x_t,(B_{t+h}-B_t)(\omega'_\ell)),\bar{z}_{t+h}((B_{t+h}-B_t)(\omega'_\ell))\big)\ .\end{aligned}$$

*Extend* $\bar{z}_{t+h}$ *as a measurable map from* $\mathscr{W}$ *to* $\mathscr{Q}_d$*. Let* $\tilde{q}_{t,h,x}$ *be the element of* $\mathscr{D}$ *given by* $w\in\mathscr{W}\mapsto q(S^m_{t,h}(x,w),\bar{z}_{t+h}(w))$.

*(b) For each* $\bar{m}\in\mathscr{M}$ *such that* $\pi(\bar{m})=m$*, compute an approximation of* $x\mapsto G^{\bar{m}}_{t,h,x}(\tilde{q}_{t,h,x})$ *by a linear regression estimation on the set of quadratic forms using the sample* $(\hat{X}^m(t,\omega_\ell),(B_{t+h}-B_t)(\omega'_\ell))$*, with* $\ell\in\Omega_{N_{\rm rg}}$*, and denote by* $z_t^{\bar{m}}\in\mathscr{Q}_d$ *the parameter of the resulting quadratic form.*

*(c) Choose* $z_t\in\mathscr{Q}_d$ *optimal among the* $z_t^{\bar{m}}\in\mathscr{Q}_d$ *at the point* $x_t$*, that is such that* $q(x_t,z_t)=\max_{\pi(\bar{m})=m}q(x_t,z_t^{\bar{m}})$.

*(3) Denote by* $Z_t$ *the set of all the* $z_t\in\mathscr{Q}_d$ *obtained in this way, and define the function* $W^{h,N}(t,\cdot)$ *by :*

$$W^{h,N}(t,x)=\max_{z\in Z_t}q(x,z)\quad\forall x\in\mathbb{R}^d\ .$$

The same algorithm has been given in [3].

### 4.2.4 Complexity of the algorithm presented in [2]

The same remark about the representation of $W^{h,N}$ that has been done in Section 4.2.2, still holds here. Since $Z_t$ satisfy $\#Z_t\leq\#\overline{\mathscr{M}}\times N_{\rm in}$ for all $t\in\mathscr{T}_h$, and $\mathscr{Q}_d$ has dimension $(d+1)(d+2)/2$, the memory space to store the value function at a time step is in the order of $\#\overline{\mathscr{M}}\times N_{\rm in}\times d^2$, so the maximum space complexity of the algorithm is $O(\#\overline{\mathscr{M}}\times N_{\rm in}\times d^2\times T/h)$. Before computing the value function, one need to store the values of all the processes, with a memory space in $O(\#\overline{\mathscr{M}}\times N_{\rm in}\times d\times T/h)$. Moreover, the total number of computations at each time step is in the order of $(\#\overline{\mathscr{M}}\times N_{\rm in})^2\times N_w\times d^2+\#\mathscr{M}\times N_{\rm in}\times(N_x\times N_w\times d^2+N_x\times d^5+d^6)$, where the first term corresponds to step (a) and the second one to step (b). Note also that $N_x$ can be chosen to be in the order of a polynomial in $d$ since the regression is done on the set of quadratic forms, so in general the second term is negligible, and it is also worth to take $\#\overline{\mathscr{M}}$ small.

# CHAPTER 5

# Numerical Results

We present in this chapter our numerical results that we group in two parts. The first section present numerical results of the test of the different options of the new probabilistic max-plus method described in Algorithm 4.2.1 when applied with the probabilistic scheme of Fahim, Touzi and Warin. The second section presents numerical results of the test of Algorithm 4.2.2 with the large stencil probabilistic scheme described in Section 3.1.2.a. In all these tests, we are interested in finding the superhedging price of a basket option in an uncertain correlation model. To test the new probabilistic max-plus method with the upwind large stencil probabilistic scheme presented in Section 3.1.2.b, we looked at a problem of dynamic optimization of a portfolio with transaction costs. However, the characteristics of the problem (utility function) caused us important numerical problems which did not allow us to obtain relevant results.

## 5.1 Test of Algorithm 4.2.1 with the probabilistic scheme of [25]

To test our algorithm, we consider the problem of pricing and hedging an option with uncertain volatility and two underlying processes, studied as an example in Section 3.2 of [45]. There the method proposed is based on a regression on a process involving not only the state but also the (discrete) control.

With the notations of Chapter 4, we consider the case where $d = 2$, $\mathscr{M} = \{\rho_{\min}, \rho_{\max}\}$ with $-1 \leq \rho_{min}, \rho_{max} \leq 1$ , and there is no continuum control, so $u$ is omitted. The dynamics of the controlled processes are given, for all $m \in \mathscr{M}$, by $b^m = 0$, and for $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$,

$$\sigma^m(\xi) = \begin{bmatrix} \sigma_1\xi_1 & 0 \\ \sigma_2 m\xi_2 & \sigma_2\sqrt{1-m^2}\xi_2 \end{bmatrix}$$

with $\sigma_1, \sigma_2 > 0$. The parameters of the overall functional payoff satisfy $c^m = 0$, $f^m = 0$, and, for $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$,

$$\Psi(\xi) = (\xi_1 - \xi_2 - K_1)^+ - (\xi_1 - \xi_2 - K_2)^+$$

with $x^+ = \max(x, 0)$, $K_1 < K_2$.

The two coordinates of the controlled process stay in $\mathbb{R}_+$, the set of positive real numbers. To be in the conditions of Theorem 4.2, we approximate the function $\Psi$ with a supremum of a finite number of concave quadratic forms on a large subset of $\mathbb{R}^2_+$, typically on the subset of $\xi$ such that $\xi_1 - \xi_2 \in [-100, 100]$. Note that since the second derivative of $\Psi$ is $-\infty$ in some points, it is not $c$-semiconvex for any $c > 0$ and bounded domain, so the approximation need to use some quadratic forms with a large negative curvature, and so the algorithm proposed in [54] may not work. The maps $\sigma^m$ for $m \in \mathscr{M}$ are not constant but they are linear. In this setting, the conditions of Theorem 4.2 are not fully satisfied. However, as we are going to see, we still obtain satisfactory results.

We take the same constants as in [45]: $\sigma_1 = 0.4,\ \sigma_2 = 0.3,\ K_1 = -5,\ K_2 = 5,\ T = 0.25$, $\rho_{\min} = -0.8, \rho_{\max} = 0.8$. We fix the time discretization step to $h = 0.01$.

We first tested our algorithm in the case where $\mathscr{M}$ is the singleton $\{\rho_{\min}\}$ or $\{\rho_{\max}\}$, which means that there is no action on the process, so that the true value function can be computed analytically, and compared with the solution obtained by our algorithm. The method $N_{\mathrm{m}} = 1$ gives very bad results even at time $T - h$. The method $N_{\mathrm{m}} = 5$ need too much space and time even for $N_{\mathrm{in}} = 1000$. In Table 5.1, we present for different values of $N = (N_{\mathrm{in}}, N_{\mathrm{rg}}, N_x, N_w, N_{\mathrm{m}})$, with $N_{\mathrm{m}} = 2, 3, 4$, the norm of the error on the value function at time $t = 0$ and states $\xi_2 = 50$ and $\xi_1 \in [20, 80]$. We see that the best method is the second one, and that Method 3 gives very bad results. This may be explained by the introduction of a bias due to the maximization of independent random variables. Note also that the errors for Method 2 are comparable to the standard deviations obtained in [36] by Gobet, Lemor and Warin in the case of similar option problems with a usual regression estimation of the value function.

In view of these results, we present in Figure 5.1 the result obtained for the control problem tested in [45], that is with $\mathscr{M} = \{\rho_{\min}, \rho_{\max}\}$, and $N_{\mathrm{in}} = 1000$, $N_{\mathrm{rg}} = N_x \times N_w$, $N_x = 10$, $N_w = 1000$ and $N_{\mathrm{m}} = 2$. The result is very similar to the one presented in [45].

| $\rho$ | $N_{\rm in}$ | $N_{\rm rg}$ | $N_x$ | $N_w$ | $N_{\rm m}$ | $e_\infty$ | $e_1$ |
|---|---|---|---|---|---|---|---|
| -0.8 | 1000 | 10000 | 10 | 1000 | 2 | 0.521 | 0.173 |
| 0.8 | 1000 | 10000 | 10 | 1000 | 2 | 0.157 | 0.074 |
| -0.8 | 1000 | 1000 | 10 | 100 | 2 | 0.75 | 0.41 |
| 0.8 | 1000 | 1000 | 10 | 100 | 2 | 0.36 | 0.11 |
| -0.8 | 1000 | 1000 | 10 | 100 | 3 | 3.48 | 1.92 |
| 0.8 | 1000 | 1000 | 10 | 100 | 3 | 3.05 | 0.81 |
| -0.8 | 100 | 1000 | 10 | 100 | 2 | 1.95 | 0.46 |
| 0.8 | 100 | 1000 | 10 | 100 | 2 | 1.81 | 0.33 |
| -0.8 | 100 | 10000 | 10 | 1000 | 2 | 2.09 | 0.53 |
| 0.8 | 100 | 10000 | 10 | 1000 | 2 | 1.79 | 0.36 |
| -0.8 | 100 | 1000 | 10 | 100 | 4 | 2.15 | 0.55 |
| 0.8 | 100 | 1000 | 10 | 100 | 4 | 1.80 | 0.39 |

Table 5.1: Sup-norm and normalized $\ell^1$ norm of the error, on the value function with constant $\rho$, at time $t = 0$, and states $\xi_2 = 50$ and $\xi_1 \in [20, 80]$, denoted $e_\infty$ and $e_1$ resp.

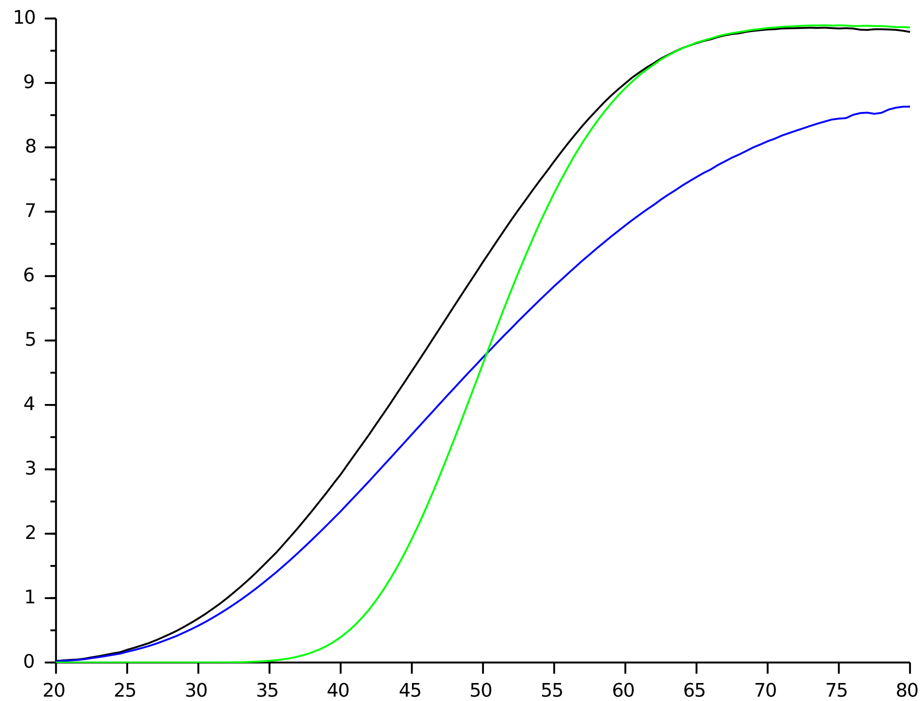



Figure 5.1: Value function obtained at $t = 0$, and $\xi_2 = 50$ as a function of $\xi_1 \in [20, 80]$. Here $N_{\rm in} = 1000$, $N_{\rm rg} = N_x \times N_w$, $N_x = 10$, $N_w = 1000$ and $N_{\rm m} = 2$. In blue, $\rho$ is constant equal to $-0.8$, in green $\rho$ is constant equal to 0.8, and in black $\rho \in \{-0.8, 0.8\}$.

## 5.2 Test of Algorithm 4.2.2 with the probabilistic scheme of Section 3.1.2.a

To illustrate our algorithm, we consider the problem of evaluating the superhedging price of an option under uncertain correlation model with several underlying stocks (the number of which determines the dimension of the problem), with changing sign cross gamma. As we saw in the previous section, the case with two underlying stocks was studied first as an example in Section 3.2 of [45], where the method proposed is based on a regression on a process involving not only the state but also the (discrete) control. In the previous section, we presented our algorithm tests with $\overline{\mathscr{M}} = \mathscr{M}$ on the same 2-dimensional example as [45]. Here we shall consider the same example with $\overline{\mathscr{M}}$ reduced to one element and then consider a similar one with 5 stocks (so in dimension 5). Illustrations are obtained from a C++ implementation of Algorithm 4.2.2, which can easily be adapted to any model.

With the notations of the introduction, the problem has no continuum control, so $u$ is omitted, and for all $m \in \mathscr{M}$, $b^m = 0$ and $c^m = 0 = f^m$. So it reduces to maximize

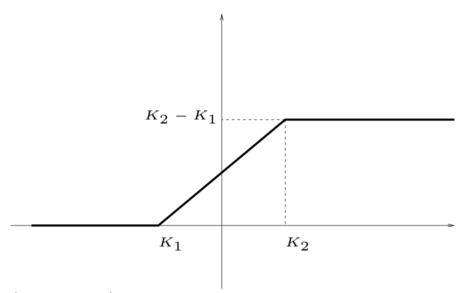
$$J(t, x, \mu) := \mathbb{E}\left[\Psi(\xi_T) \mid \xi_t = x\right] \quad .$$

The dynamics is given by $d\xi_{i,s} = \sigma_i \xi_{i,s} dB_{i,s}$ where the $B_i$ are Brownians with uncertain correlations: $< dB_{i,s}, dB_{j,s} >= [\mu_s]_{ij} ds$ with $\mu_s \in \mathsf{Cor}$, a subset of the set of positive symmetric matrices with 1 on the diagonal. This is equivalent to the condition that

$$[\sigma^m(x)\sigma^m(x)^{\mathsf{T}}]_{ij} = \sigma_i x_i \sigma_j x_j m_{ij}, \quad \text{for } m \in \mathsf{Cor} \quad .$$

Here we assume that $\mathsf{Cor}$ is the convex hull of a finite set $\mathscr{M}$. Since the Hamiltonian of the problem is linear with respect to $m$, the maximum over $\mathsf{Cor}$ is the same as the maximum over $\mathscr{M}$, so we can assume that the correlations satisfy $\mu_s \in \mathscr{M}$. We consider the following final payoff:

$$\begin{aligned}
&\Psi(x) = \Psi_1(\max_{i \text{ odd}} x_i - \min_{j \text{ even}} x_j),\ x \in \mathbb{R}^d,\\
&\Psi_1(x) = (x - K_1)^+ - (x - K_2)^+,\ x \in \mathbb{R},\\
&x^+ = \max(x, 0),\\
&K_1 < K_2.
\end{aligned}$$




Since $\Psi_1$ is nondecreasing, we have $\Psi(x) \geq \Psi_1(x_i - x_j)$, for all $i$ odd and $j$ even. Then, we can lower bound the value function in dimension $d$ by the application of the value function of dimension 2 with volatilities $(\sigma_i, \sigma_j)$ and set $\mathscr{M} = \{-\rho, \rho\}$ with $\rho = \max\{m_{ij}, m \in \mathsf{Cor}\}$. Indeed if we denote by $v^2_{\sigma_i,\sigma_j}$ the value function of the 2-dimensional problem with volatilities $\sigma_i, \sigma_j$ with $\mathscr{M} = \{-\rho, \rho\}$ and by $v^5_{\sigma,\mathsf{Cor}}$ the value function of the 5-dimensional problem, then $v^5_{\sigma,\mathsf{Cor}}(t, x) \geq \sup \mathbb{E}\left[\Psi_1((\xi_T)_i, (\xi_T)_j) \mid \xi_t = x\right]$. Since this depends on $(\xi_T)_i, (\xi_T)_j$ only and that these processes do not depend on the other coordinates $(\xi_T)_k$ with $k \neq i, j$, we get that $v^5_{\sigma,\mathsf{Cor}}(t, x) \geq v^2_{\sigma_i,\sigma_j,\rho}(t, x_i, x_j)$.

Note that all the coordinates of the controlled process stay in $\mathbb{R}_+$, the set of positive real numbers. To be in the conditions of Theorem 4.2, we approximate the function $\Psi_1$ with a supremum of a finite number of quadratic forms on a large subset of $\mathbb{R}$, typically on

$[-1000, 1000]$, so that $\Psi$ is approximated with a supremum of a finite number of quadratic forms on the $x \in \mathbb{R}_+^d$ such that $x_i - x_j \in [-1000, 1000]$. Note that since the second derivative of $\Psi_1$ is $-\infty$ in some points, it is not $c$-semiconvex for any $c > 0$ and bounded domain, so the approximation need to use some quadratic forms with a large negative curvature, and so we are not under the conditions of [54]. Moreover, since the state space is unbounded, one cannot approximate $\Psi$ as a supremum of a finite number of quadratic forms on all the state space as assumed in Algorithm 4.2.2. However, due to stability considerations, the simulated process stays with almost probability one in a ball around the initial point, so that one may expect the value function to be well approximated in a bounded subset of $\mathbb{R}^d$. The maps $\sigma^m$ for $m \in \mathscr{M}$ are not constant but they are linear, and one can choose $\underline{\sigma}$ such that $\underline{\sigma}(x)^{-1}\sigma^m(x)$ is constant and $\underline{b} = 0$, and get that the result of Theorem 4.2 still holds.

In the illustration below, we choose $K_1 = -5,\ K_2 = 5,\ T = 0.25$, the time step $h = 0.01$, the volatilities $\sigma_1 = 0.4,\ \sigma_2 = 0.3,\ \sigma_3 = 0.2, \sigma_4 = 0.3,\ \sigma_5 = 0.4$ and the following correlations sets:

$$\mathscr{M} = \{m = \left[\begin{smallmatrix} 1 & m_{12} \\ m_{12} & 1 \end{smallmatrix}\right] \mid m_{12} = \pm\rho\} \qquad \text{for 2 stocks,}$$

and

$$\mathscr{M} = \{m = \begin{bmatrix} 1 & m_{12} & 0 & 0 & 0 \\ m_{12} & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & m_{45} \\ 0 & 0 & 0 & m_{45} & 1 \end{bmatrix} \mid m_{12} = \pm\rho,\ m_{45} = \pm\rho\} \qquad \text{for 5 stocks.}$$

In dimension 2, we choose $N_x = 10$, $N_w = 1000$ and test several values of simulation size $N_{\text{in}}$, and compare our results with the true solutions that can be computed analytically when $\mathscr{M}$ is a singleton, see Figures 5.2 and 5.3. For $\rho = 0$ or $\rho = 0.4$, $k = 0$ is sufficient in Lemma 3.1.5 (indeed the nonlinearity of the PDE $F = 0$ for $\rho = 0$, so there is no second derivative to discretize), whereas for $\rho = 0.8$, one need to take $k = 2$ to obtain the monotonicity of the scheme. This may explain why a greater sampling size $N_{\text{in}}$ is needed to obtain the convergence for $\rho = 0.8$.

In dimension 5, we choose $N_x = 50$, $N_w = 1000$ and $N_{\text{in}} = 3000$, and compare our results with a lower bound obtained from the results in dimension 2, as explained above, see Figure 5.4. Although, the lower bound appears to be above the value function computed from the Hamilton-Jacobi-Bellman equation in dimension 5, the difference between the value function and the lower bound is small and of the same amount as the difference observed in Figure 5.3 between the value functions computed in dimension 2 with the simulation sizes $N_{\text{in}} = 2000$ and $N_{\text{in}} = 3000$. This indicates that the size of the simulations $N_{\text{in}} = 3000$ is not enough to attain the convergence of the approximation, although the results give already the correct shape of the value function. Such a result would be difficult to obtain with finite difference schemes, and at least will take much more memory space. For instance, the computing time for one time step of a finite difference scheme on a regular grid over $[0, 100]^5$ with 100 steps by coordinate is in $10^{10}$ and is thus comparable with the computing time of Algorithm 4.2.2, $N_{\text{in}}^2 \times N_w \times d^2$, with the above parameters, whereas the memory space needed for the finite difference scheme at each time step is similar to the computing time and is thus much larger than the one needed in Algorithm 4.2.2 (in $N_{\text{in}} \times d^2 = 7.5\,10^5$).

The computation of the value function in dimension 5 took $\simeq$ 19h with the `C++` program compiled with "OpenMP" on a 12 core Intel(R) Xeon(R) CPU $E5 - 2667$ - 2.90GHz with 192Go of RAM (each time iteration taking $\simeq$ 2500s). The main part of the computation time is taken by the optimization part (a) of Algorithm 4.2.2, with a time in $O(N_{\text{in}}^2 \times N_w \times d^2)$. The bottleneck here is in the computation, for each given state $x$ at time $t + h$, of the quadratic form which is maximal in the expression of $W^{h,N}(t+h,x)$. Therefore, a better understanding of this maximization problem is necessary in order to decrease the total computing time. This would allow us to obtain better approximations in dimension 5 in particular, and increase the dimension with a small cost. Such an improvement is left for further work.

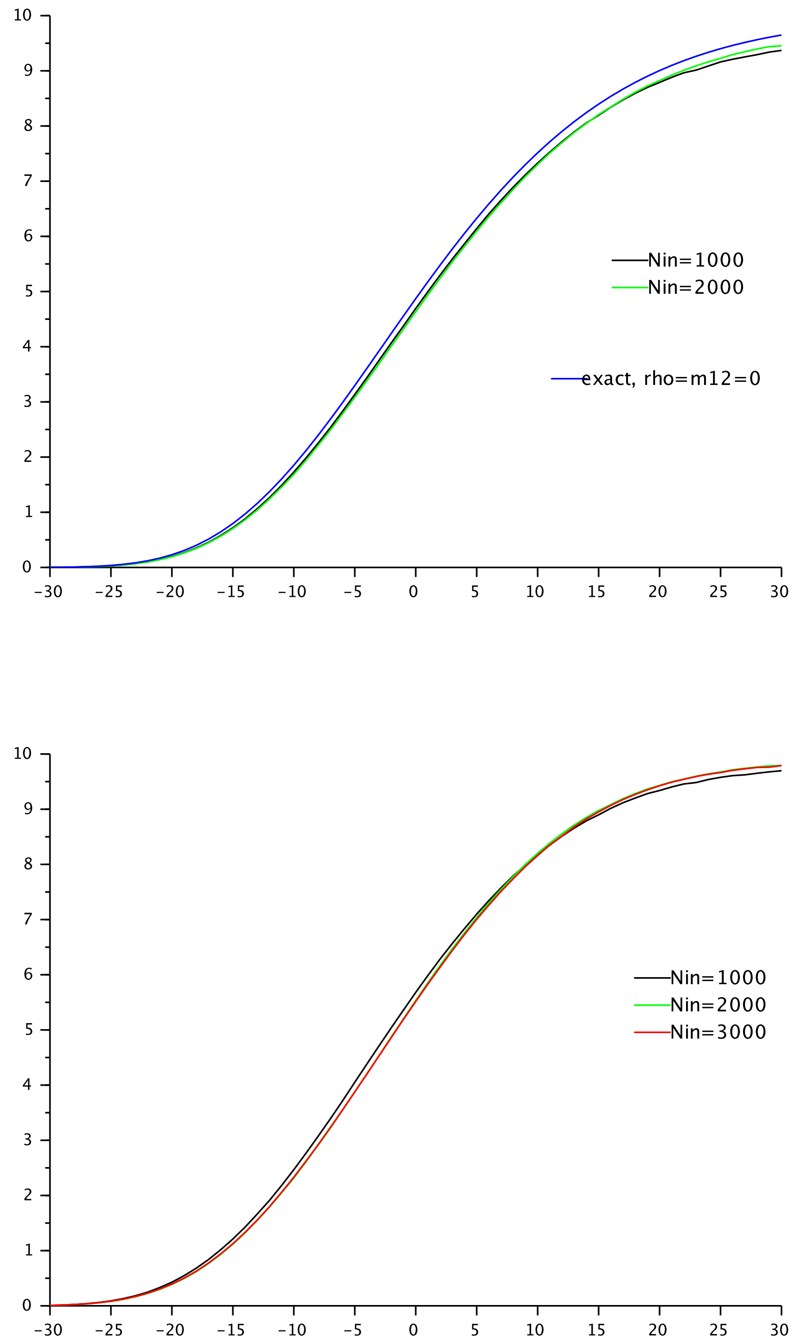



Figure 5.2: Value function in dimension 2, for $\rho = 0$ on top, and $\rho = 0.4$ on bottom, at $t = 0$, and $x_2 = 50$ as a function of $x_1 - x_2$. Here $N_{\text{in}} = 1000, 2000$, or 3000, $N_x = 10$, $N_w = 1000$.

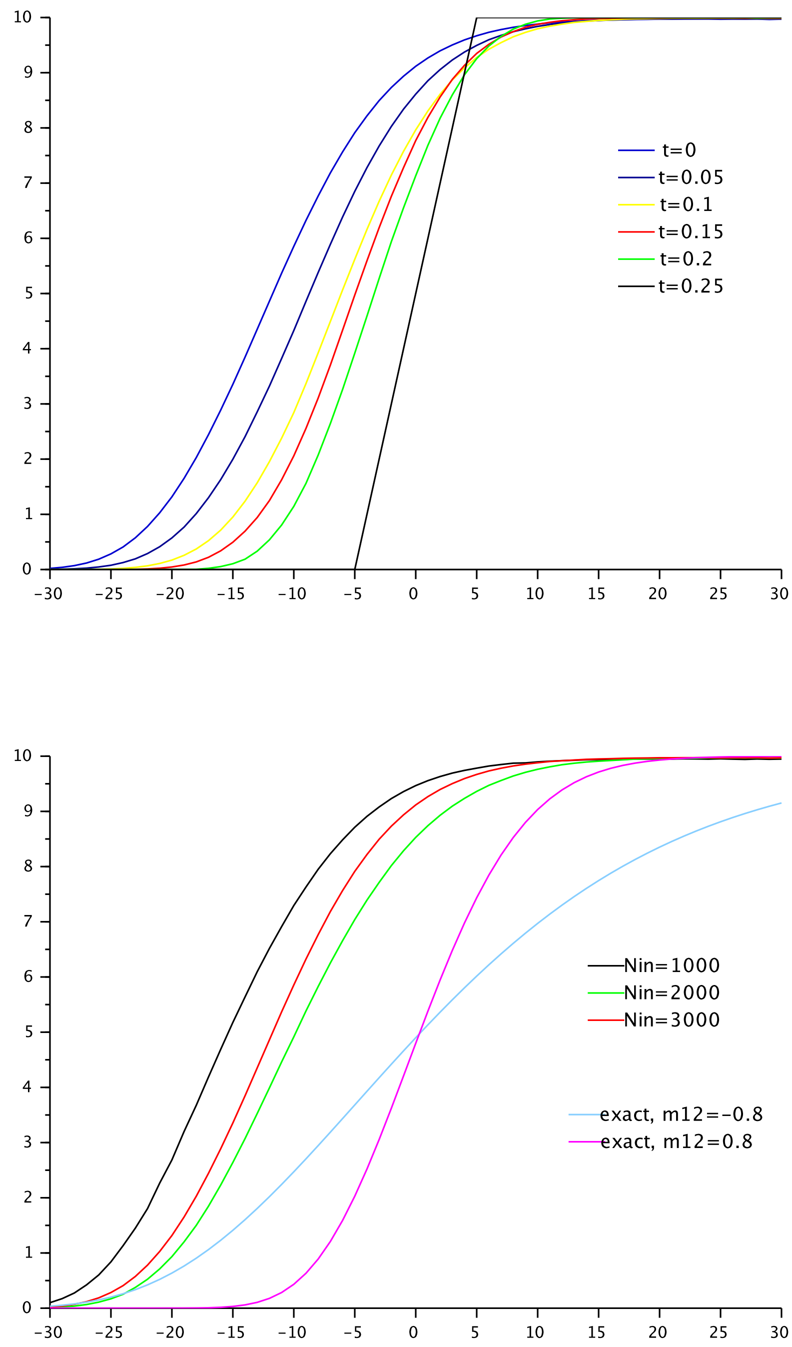



Figure 5.3: Value function in dimension 2, for $\rho = 0.8$, at $x_2 = 50$ as a function of $x_1 - x_2$ obtained with $N_x = 10$, $N_w = 1000$. On top, the value is shown at each time step multiple of 0.05 and is obtained for $N_{\text{in}} = 3000$. On bottom, the value at time $t = 0$ is compared for $N_{\text{in}} = 1000$, 2000 and 3000 and with the exact solution when $\mathscr{M}$ is a singleton.

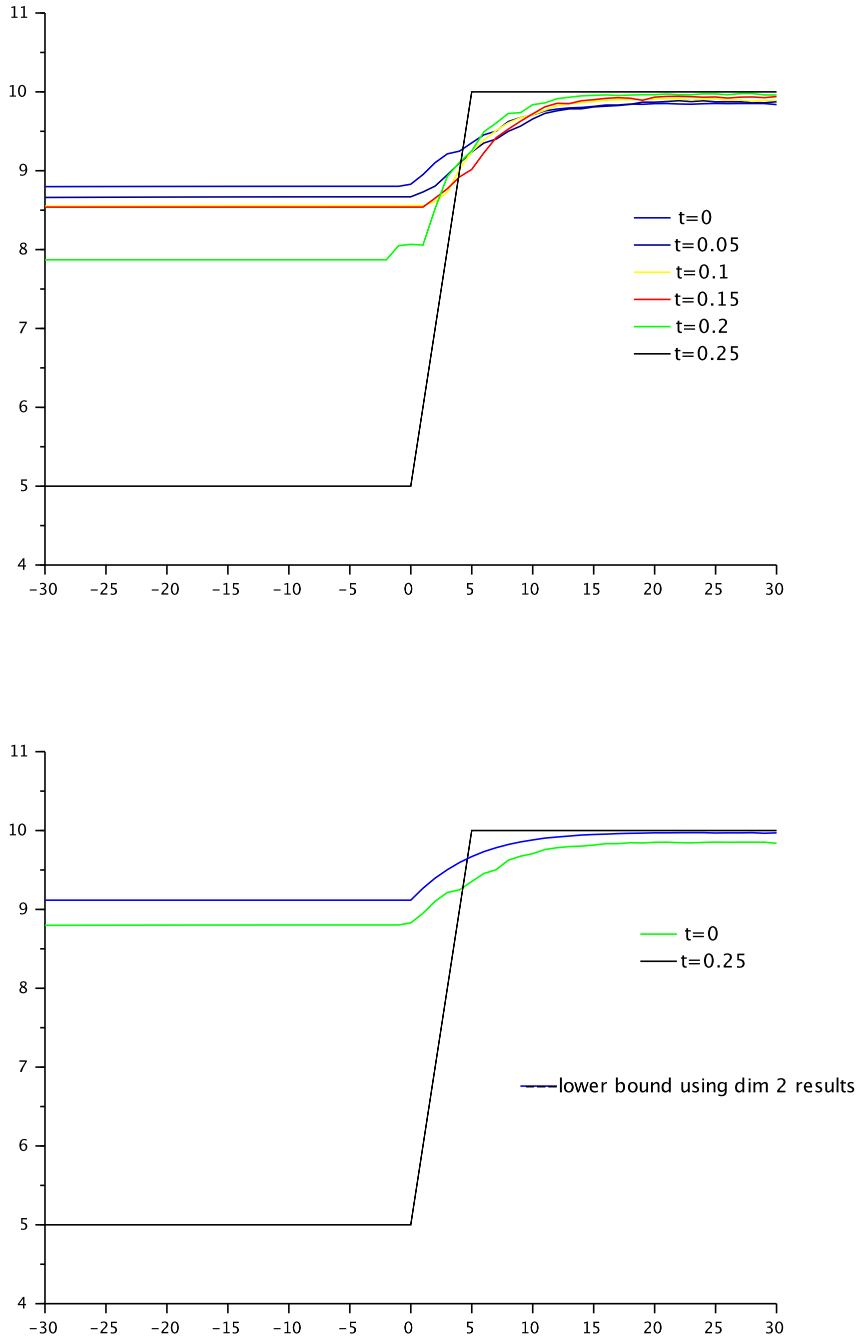



Figure 5.4: Value function for $\rho = 0.8$ in dimension 5, at $x_2 = x_3 = x_4 = x_5 = 50$ as a function of $x_1 - x_2$. Here $N_{\text{in}} = 3000$, $N_x = 50$, $N_w = 1000$. On top, the value is shown at each time step multiple of 0.05. On bottom, the value at time $t = 0$ is compared with a lower bound obtained by using the results in dimension 2.

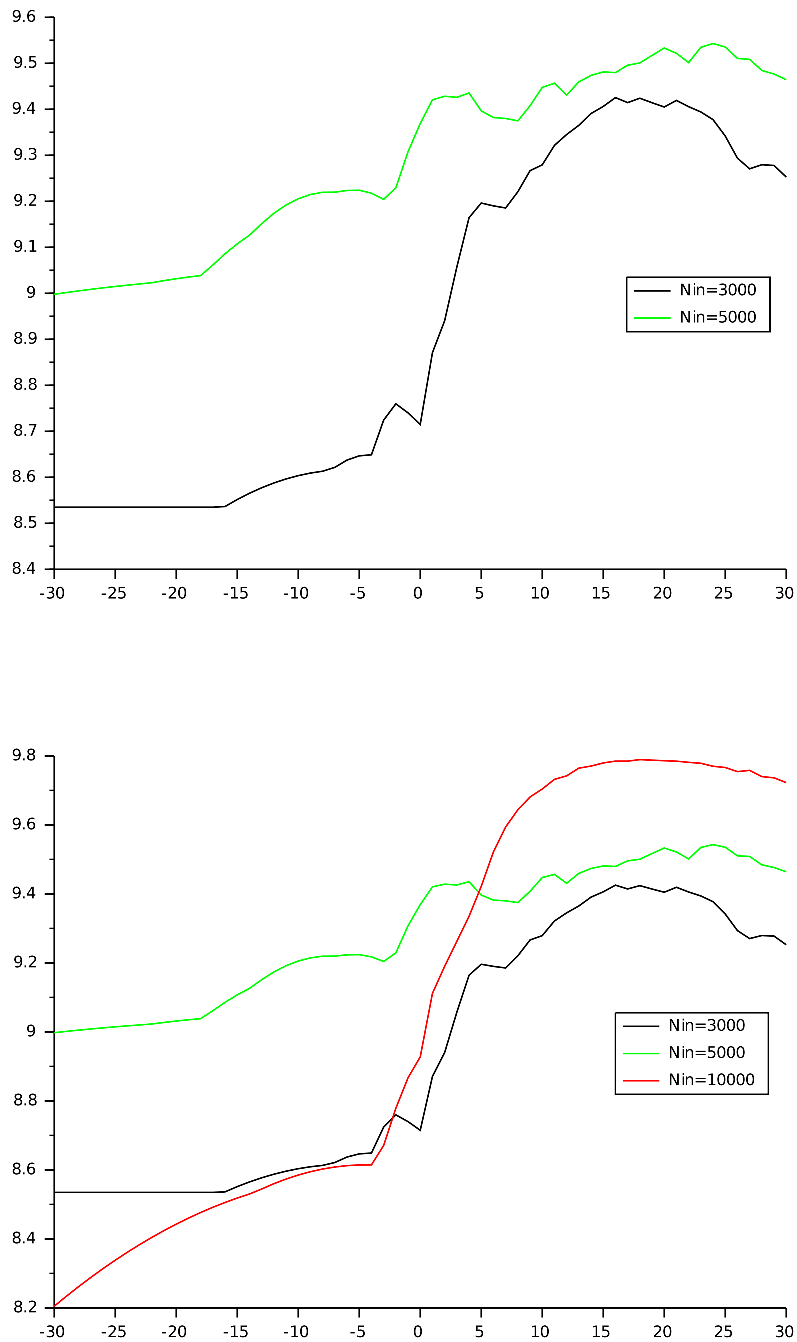



Figure 5.5: Value function for $\rho = 0.8$ in dimension 5, at $x_2 = x_3 = x_4 = x_5 = 50$ as a function of $x_1 - x_2$. On top, $N_{\text{in}} = 3000$ and 5000. On bottom, $N_{\text{in}} = 3000$, 5000 and 10000. $N_x = 50$, $N_w = 1000$. The value is computed using an optimization algorithm for computing $\bar{z}_{t+h}$.

# Conclusion

We introduced new probabilistic schemes to solve Hamilton-Jacobi-Bellman equations in the stochastic case which are monotone and can converge with less restrictions than those of Fahim, Touzi and Warin in [25] and Guo, Zang and Zhuo in [40]. Indeed, the non linearity of the PDE with respect to the second order derivative can be high and the only condition is that the PDE is uniformly elliptic. We proved convergence and obtained error estimates for PDE with bounded coefficients. What remains is to obtain same convergence results in the case of PDE with unbounded coefficients. We think that error estimates can be obtained using the same method as Assellaou, Bokanowski and Zidani in [5] as the solution of the probabilistic scheme we considered for the unbounded setting is Lipschitz continuous. By obtaining Lipschitz result on the viscosity solution of the PDE in the unbounded setting given in Section 3.2 and then taking the same steps as in [5], convergence results and error estimates can then be obtained. Before going this far, we have to notice that the method of [5] allows already to improve the lower bound error estimate that we obtained with the method of Barles and Jakobsen [7] in this manuscript for PDE with bounded coefficients. It can also be generalized to a lot of other schemes where the solution of the scheme is Lipschitz continuous.

The second new result of this work is the max-plus probabilistic method. We showed that we have theoretical results which shows that the convergence of the method is justified at least for one of the probabilistic scheme we presented here in the case of a linear quadratic problem. However, a more precise result giving error estimates of the method with respect to the number of simulations and the other algorithm variables is needed. Some numerical tests have been performed but need to be extended to more difficult problems.

All these remarks are left for further work.

# Bibliography


[1] M. Akian and E. Fodjo. A probabilistic max-plus numerical method for solving stochastic control problems. In 55th Conference on Decision and Control (CDC 2016), Las Vegas, United States, Dec. 2016. Also arXiv:1605.02816.

[2] M. Akian and E. Fodjo. From a monotone probabilistic scheme to a probabilistic max-plus algorithm for solving Hamilton-Jacobi-Bellman equations. In D. Kalise, K. Kunisch, and Z. Rao, editors, Hamilton–Jacobi–Bellman Equations, volume 21 of Radon Series on Computational and Applied Mathematics. De Gruyter, Berlin, 2018. To appear, see also ArXiv:1709.09049.

[3] M. Akian and E. Fodjo. Probabilistic max-plus schemes for solving Hamilton-Jacobi-Bellman equations. ArXiv:1801.01780, 2018.

[4] M. Akian, S. Gaubert, and A. Lakhoua. The max-plus finite element method for solving deterministic optimal control problems: basic properties and convergence analysis. SIAM J. Control Optim., 47(2):817–848, 2008.

[5] M. Assellaou, O. Bokanowski, and H. Zidani. Error Estimates for Second Order Hamilton-Jacobi-Bellman Equations. Approximation of Probabilistic Reachable Sets. Discrete and Continuous Dynamical Systems - Series A (DCDS-A), 35(9):3933 – 3964, Sept. 2015.

[6] V. Bally and G. Pagès. A quantization algorithm for solving multi-dimensional discrete-time optimal stopping problems. Bernoulli, 9(6):1003–1049, 2003.

[7] G. Barles and E. R. Jakobsen. Error bounds for monotone approximation schemes for parabolic Hamilton-Jacobi-Bellman equations. Math. Comp., 76(260):1861–1893, 2007.

[8] G. Barles and P. E. Souganidis. Convergence of approximation schemes for fully nonlinear second order equations. Asymptotic Anal., 4(3):271–283, 1991.

[9] C. Bender and J. Zhang. Time discretization and Markovian iteration for coupled FBS-DEs. Ann. Appl. Probab., 18(1):143–177, 2008.

[10] J.-M. Bismut. Conjugate convex functions in optimal stochastic control. J. Math. Anal. Appl., 44:384–404, 1973.

[11] J. F. Bonnans and H. Zidani. Consistency of generalized finite difference schemes for the stochastic HJB equation. SIAM J. Numer. Anal., 41(3):1008–1021, 2003.

[12] B. Bouchard, I. Ekeland, and N. Touzi. On the Malliavin approach to Monte Carlo approximation of conditional expectations. Finance Stoch., 8(1):45–71, 2004.

[13] B. Bouchard and N. Touzi. Discrete-time approximation and Monte-Carlo simulation of backward stochastic differential equations. Stochastic Process. Appl., 111(2):175–206, 2004.

[14] F. Camilli and M. Falcone. An approximation scheme for the optimal control of diffusion processes. RAIRO Modél. Math. Anal. Numér., 29(1):97–122, 1995.

[15] I. Capuzzo Dolcetta. On a discrete approximation of the Hamilton-Jacobi equation of dynamic programming. Appl. Math. Optim., 10(4):367–377, 1983.

[16] I. Capuzzo-Dolcetta and H. Ishii. Approximate solutions of the Bellman equation of deterministic control theory. Appl. Math. Optim., 11(2):161–181, 1984.

[17] P. Cheridito, H. M. Soner, N. Touzi, and N. Victoir. Second-order backward stochastic differential equations and fully nonlinear parabolic PDEs. Comm. Pure Appl. Math., 60(7):1081–1110, 2007.

[18] R. Courant, K. Friedrichs, and H. Lewy. über die partiellen Differenzengleichungen der mathematischen Physik. Math. Ann., 100(1):32–74, 1928.

[19] R. Courant, E. Isaacson, and M. Rees. On the solution of nonlinear hyperbolic differential equations by finite differences. Comm. Pure. Appl. Math., 5:243–255, 1952.

[20] M. G. Crandall, H. Ishii, and P.-L. Lions. User's guide to viscosity solutions of second order partial differential equations. Bull. Amer. Math. Soc. (N.S.), 27(1):1–67, 1992.

[21] M. G. Crandall and P.-L. Lions. Viscosity solutions of Hamilton-Jacobi equations. Trans. Amer. Math. Soc., 277(1):1–42, 1983.

[22] M. G. Crandall and P.-L. Lions. Convergent difference schemes for nonlinear parabolic equations and mean curvature motion. Numer. Math., 75(1):17–41, 1996.

[23] F. Da Lio and O. Ley. Uniqueness results for second-order Bellman-Isaacs equations under quadratic growth assumptions and applications. SIAM J. Control Optim., 45(1):74–106, 2006.

[24] K. Debrabant and E. R. Jakobsen. Semi-Lagrangian schemes for linear and fully nonlinear diffusion equations. Math. Comp., 82(283):1433–1462, 2013.

[25] A. Fahim, N. Touzi, and X. Warin. A probabilistic numerical method for fully nonlinear parabolic PDEs. Ann. Appl. Probab., 21(4):1322–1364, 2011.

[26] M. Falcone. A numerical approach to the infinite horizon problem of deterministic control theory. Appl. Math. Optim., 15(1):1–13, 1987.

[27] W. H. Fleming and W. M. McEneaney. A max-plus-based algorithm for a Hamilton-Jacobi-Bellman equation of nonlinear filtering. SIAM J. Control Optim., 38(3):683–710, 2000.

[28] W. H. Fleming and H. M. Soner. Controlled Markov Processes and Viscosity Solutions. Springer-Verlag, 1993.

[29] E. Fournié, J.-M. Lasry, J. Lebuchoux, and P.-L. Lions. Applications of Malliavin calculus to Monte-Carlo methods in finance. II. Finance Stoch., 5(2):201–236, 2001.

[30] E. Fournié, J.-M. Lasry, J. Lebuchoux, P.-L. Lions, and N. Touzi. Applications of Malliavin calculus to Monte Carlo methods in finance. Finance Stoch., 3(4):391–412, 1999.

[31] B. D. Froese and A. M. Oberman. Fast finite difference solvers for singular solutions of the elliptic Monge-Ampère equation. J. Comput. Phys., 230(3):818–834, 2011.

[32] B. D. Froese and A. M. Oberman. Convergent filtered schemes for the Monge-Ampère partial differential equation. SIAM J. Numer. Anal., 51(1):423–444, 2013.

[33] P. Gassiat, I. Kharroubi, and H. Pham. Time discretization and quantization methods for optimal multiple switching problem. Stochastic Process. Appl., 122(5):2019–2052, 2012.

[34] S. Gaubert, W. Mceneaney, and Z. Qu. Curse of dimensionality reduction in max-plus based approximation methods: theoretical estimates and improved pruning algorithms. 8pages 5 figures, Sept. 2011.

[35] S. Gaubert, Z. Qu, and S. Sridharan. Bundle-based pruning in the max-plus curse of dimensionality free method. In Proceedings of the 21st International Symposium on Mathematical Theory of Networks and Systems, Groningen, Netherlands, July 2014. See also arXiv:1402.1436.

[36] E. Gobet, J.-P. Lemor, and X. Warin. A regression-based monte carlo method to solve backward stochastic differential equations. The Annals of Applied Probability, 15(3):2172–2202, 2005.

[37] R. Gonzalez and E. Rofman. On deterministic control problems: an approximation procedure for the optimal cost. I. The stationary problem. SIAM J. Control Optim., 23(2):242–266, 1985.

[38] R. Gonzalez and E. Rofman. On deterministic control problems: an approximation procedure for the optimal cost. II. The nonstationary case. SIAM J. Control Optim., 23(2):267–285, 1985.

[39] F. Gozzi, A. Święch, and X. Y. Zhou. Erratum: "A corrected proof of the stochastic verification theorem within the framework of viscosity solutions" [mr2179474]. SIAM J. Control Optim., 48(6):4177–4179, 2010.

[40] W. Guo, J. Zhang, and J. Zhuo. A monotone scheme for high-dimensional fully nonlinear pdes. Ann. Appl. Probab., 25(3):1540–1580, 06 2015.

[41] K. Ito. Existence of solutions to the Hamilton-Jacobi-Bellman equation under quadratic growth conditions. J. Differential Equations, 176(1):1–28, 2001.

[42] M. Kac. On some connections between probability theory and differential and integral equations. In Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability, 1950, pages 189–215. University of California Press, Berkeley and Los Angeles, 1951.

[43] H. Kaise and W. M. McEneaney. Idempotent expansions for continuous-time stochastic control: compact control space. In Proceedings of the 49th IEEE Conference on Decision and Control, Atlanta, Dec. 2010.

[44] H. Kaise and W. M. McEneaney. Idempotent expansions for continuous-time stochastic control. SIAM J. Control Optim., 54(1):73–98, 2016.

[45] I. Kharroubi, N. Langrené, and H. Pham. A numerical algorithm for fully nonlinear HJB equations: an approach by control randomization. Monte Carlo Methods Appl., 20(2):145–165, 2014.

[46] N. V. Krylov. On the rate of convergence of finite-difference approximations for Bellman's equations. Algebra i Analiz, 9(3):245–256, 1997.

[47] N. V. Krylov. On the rate of convergence of finite-difference approximations for Bellman's equations with variable coefficients. Probab. Theory Related Fields, 117(1):1–16, 2000.

[48] H. J. Kushner. Probability methods for approximations in stochastic control and for elliptic equations. Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1977. Mathematics in Science and Engineering, Vol. 129.

[49] P.-L. Lions. Generalized solutions of Hamilton-Jacobi equations, volume 69 of Research Notes in Mathematics. Pitman (Advanced Publishing Program), Boston, Mass.-London, 1982.

[50] G. Loeper and F. Rapetti. Numerical solution of the Monge-Ampère equation by a Newton's algorithm. C. R. Math. Acad. Sci. Paris, 340(4):319–324, 2005.

[51] F. A. Longstaff and E. S. Schwartz. Valuing american options by simulation: A simple least-squares approach. Review of Financial Studies, pages 113–147, 2001.

[52] W. M. McEneaney. A curse-of-dimensionality-free numerical method for solution of certain HJB PDEs. SIAM J. Control Optim., 46(4):1239–1276, 2007.

[53] W. M. McEneaney, A. Deshpande, and S. Gaubert. Curse-of-complexity attenuation in the curse-of-dimensionality-free method for hjb pdes, 2008.

[54] W. M. McEneaney, H. Kaise, and S. H. Han. Idempotent method for continuous-time stochastic control and complexity attenuation. In Proceedings of the 18th IFAC World Congress, 2011, pages 3216–3221, Milano, Italie, 2011.

[55] J.-L. Menaldi. Some estimates for finite difference approximations. SIAM J. Control Optim., 27(3):579–607, 1989.

[56] J.-M. Mirebeau. Minimal stencils for discretizations of anisotropic PDEs preserving causality or the maximum principle. SIAM J. Numer. Anal., 54(3):1582–1611, 2016.

[57] A. M. Oberman. Convergent difference schemes for degenerate elliptic and parabolic equations: Hamilton-Jacobi equations and free boundary problems. SIAM J. Numer. Anal., 44(2):879–895, 2006.

[58] A. M. Oberman. Finite difference methods for the infinity Laplace and $p$-Laplace equations. J. Comput. Appl. Math., 254:65–80, 2013.

[59] G. Pagès, H. Pham, and J. Printems. Optimal quantization methods and applications to numerical problems in finance. In Handbook of computational and numerical methods in finance, pages 253–297. Birkhäuser Boston, Boston, MA, 2004.

[60] É. Pardoux and S. Peng. Backward stochastic differential equations and quasilinear parabolic partial differential equations. In Stochastic partial differential equations and their applications (Charlotte, NC, 1991), volume 176 of Lecture Notes in Control and Inform. Sci., pages 200–217. Springer, Berlin, 1992.

[61] E. Pardoux and S. G. Peng. Adapted solution of a backward stochastic differential equation. Systems Control Lett., 14(1):55–61, 1990.

[62] E. Pardoux and S. Tang. Forward-backward stochastic differential equations and quasilinear parabolic PDEs. Probab. Theory Related Fields, 114(2):123–150, 1999.

[63] S. G. Peng. Probabilistic interpretation for systems of quasilinear parabolic partial differential equations. Stochastics Stochastics Rep., 37(1-2):61–74, 1991.

[64] C. Pierre. Introduction to differential games. 2010.

[65] R. Rockafellar and R. J.-B. Wets. Variational Analysis. Springer Verlag, Heidelberg, Berlin, New York, 1998.

[66] A. Ruszczyński and A. Shapiro. Optimization of convex risk functions. Mathematics of Operations Research, 31:433–452, 2006.

[67] X. Y. Zhou. Verification theorems within the framework of viscosity solutions. J. Math. Anal. Appl., 177(1):208–225, 1993.

universite PARIS-SACLAY
École doctorale de mathématiques Hadamard (EDMH)

**Titre :** Algorithmes de résolution de problèmes de contrôle optimal en grande dimension en combinant des méthodes probabilistes et max-plus

**Mots clés :** Contrôle optimal, programmation dynamique, Equations aux dérivées partielles non linéaires du second ordre

**Résumé :** Les problèmes de contrôle stochastique optimal à horizon fini forment une classe de problèmes de contrôle optimal où interviennent des processus stochastiques considérés sur un intervalle de temps borné. Tout comme beaucoup de problème de contrôle optimal, ces problèmes sont résolus en utilisant le principe de la programmation dynamique qui induit une équation aux dérivées partielles (EDP) appelée équation d'Hamilton-Jacobi-Bellman. Les méthodes basées sur la discrétisation de l'espace sous forme de grille, les méthodes probabilistes ou plus récemment les méthodes max-plus peuvent alors être utilisées pour résoudre cette équation. Cependant, le premier type de méthode est mis en défaut quand un espace à dimension grande est considéré à cause de la malédiction de la dimension tandis que le deuxième type de méthode ne permettait jusqu'ici que de résoudre des problèmes où la non linéarité de l'équation aux dérivées partielles par rapport à la Hessienne n'est pas trop forte. Quant au troisième type de méthode, il entraine une explosion de la complexité de la fonction valeur. Nous introduisons dans cette thèse deux nouveaux schémas probabilistes permettant d'agrandir la classe des problèmes pouvant être résolus par les méthodes probabilistes. L'une est adaptée aux EDP à coefficients bornés tandis que l'autre peut être appliqué aux EDP à coefficients bornés ou non bornés. Nous prouvons la convergence des deux schémas probabilistes et obtenons des estimées de l'erreur de convergence dans le cas d'EDP à coefficients bornés. Nous donnons également quelques résultats sur le comportement du deuxième schéma dans le cas d'EDP à coefficients non bornés. Ensuite, nous introduisons une méthode complètement nouvelle pour résoudre les problèmes de contrôle stochastique optimal à horizon fini que nous appelons la méthode max-plus probabiliste. Elle permet d'utiliser le caractère non linéaire des méthodes max-plus dans un contexte probabiliste tout en contrôlant la complexité de la fonction valeur. Une application au calcul du prix de sur-réplication d'une option dans un modèle de corrélation incertaine est donnée dans le cas d'un espace à dimension 2 et 5.

**Title :** Algorithms for solving stochastic control problems in high dimension by combining probabilistic and max-plus methods

**Keywords :** Optimal control problems, Dynamic programming, Non linear partial differential equations of second order

**Abstract :** Stochastic optimal control problems with finite horizon are a class of optimal control problems where intervene stochastic processes in a bounded time. As many optimal control problems, they are often solved using a dynamic programming approach which results in a second order Partial Differential Equation (PDE) called the Hamilton-Jacobi-Bellman equation. Grid-based methods, probabilistic methods or more recently max-plus methods can be used then to solve this PDE. However, the first type of methods default in a space of high dimension because of the curse of dimensionality while the second type of methods allowed till now to solve only problems where the nonlinearity of the PDE with respect to the second order derivatives is not very high. As for the third type of method, it results in an explosion of the complexity of the value function. We introduce two new probabilistic schemes in order to enlarge the class of problems that can be solved with probabilistic methods. One is adapted to PDE with bounded coefficients while the other can be applied to PDE with bounded or unbounded coefficients. We prove the convergence of the two probabilistic scheme and obtain error estimates in the case of a PDE with bounded coefficients. We also give some results about the behavior of the second probabilistic scheme in the case of a PDE with unbounded coefficients. After that, we introduce a completely new type of method to solve stochastic optimal control problems with finite horizon that we call the max-plus probabilistic method. It allows to add the non linearity feature of max-plus methods to a probabilistic method while controlling the complexity of the value function. An application to the computation of the optimal super replication price of an option in an uncertain correlation model is given in a 5 dimensional space.