Inverse Optimal Control : theoretical study

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Contents

Introduction

Inverse optimal control problem has received a special attention for the last few decades. The renewal interest is due to the growing number of its application. In particular, in modeling the human movements in physiology, which led to a new approach in the domain of humanoid robotics. Actually, a human, as a mechanical system, can be modeled as a control system. Its architecture is such that there are a lot of possibilities to realize each particular task. For any task the chosen movement is stable with respect to the changes of the environment non relevant to the task, and adaptable to the changes if it is needed for the task realization. These characteristics of the movements make them very plausible to be optimal movements and in physiology this optimality paradigm is one of the dominant hypothesis (for more rigorous explanations in context of physiology see [1, 2]). Therefore, the good mathematical framework for the movements is the optimal control framework, that is, the realized movements of the mechanical system minimize some cost function. However, even if it is known that a movement is optimal, the criteria being optimized is hidden. Thus, to model the human movements we should first solve an inverse optimal control problem: given the data of the realized movements and the dynamics of the mechanical system, find the cost function with respect to which the movements are optimal, i.e. the solutions of the corresponding optimal control problem. The inverse optimal control has already proved to be useful in the study of the human locomotion [3] and the arm movements [4].

In humanoid robotics, inverse optimal control is the tool to get the most adapted cost function to then implement the induced command laws in robots. In this view, different movements have been implemented, for example human locomotion [5]. The same scheme is applied to robots which are supposed to act like biological systems other than humans, for example a quadrotor moving as a flying insect, see [6]. Another application in robotics is related to the autonomous robots, autonomous cars in particular (see [7]), which interact with humans and should predict the human actions [8, 9]. In economics when we consider the decision making process of the customers, inverse optimal control is helpful to find a so-called utility function

[10]. In a lot of different contexts an expert performs by intuition very impressive and effective actions which may be better understood via inverse optimal control and also implemented, as robots mimicking the pilot's strategy in [11]. Inverse optimal control theory also gave rise to new methods for stabilization. Indeed, an optimal stabilizing control enjoys particular properties and it is of interest to construct such control. However, to solve the direct optimal control may be difficult both analytically and numerically. Thus it is sometimes better to use inverse optimal control to obtain optimal control policies without an a priori given cost which should be minimized. This method was proposed in [12] to produce a stabilizing feedback control (see also [13]). In case of Linear-Quadratic regulator, the inverse optimal control provides a method for optimal pole placement (see [14]).

From a mathematical point of view, inverse optimal control problems belong to the class of inverse problems where the first question is whether the problem is well posed. Formally, given a dynamics and a class of candidate cost functions, for the corresponding class of direct optimal control problems we can define an operator which maps a cost function to the optimal synthesis, i.e. to the set of all optimal trajectories of the corresponding optimal control problem, for all realizable initial and final points. In the inverse problem we are looking for the inverse operator. For such an inverse problem to be well posed, it should be surjective, injective and stable. By surjectivity we mean that the given set of optimal trajectories contains only minimizing trajectories of the same cost. In general, the surjectivity is difficult to check and in applications it is usually assumed to be satisfied. Injectivity means that there is one to one correspondence between the costs in the class and the optimal synthesis. The injectivity is a serious issue in inverse optimal control. It is easy to see that multiplication of any cost by a constant does not change the minimizers, thus to have injectivity we should normalize the costs in the considered class of cost functions. In general, the proportional costs are not the only costs having the same optimal solutions. Nevertheless, the injectivity may be reached by restricting the class of costs to a smaller one. The stability, that is, the continuity of the inverse operator, means that small perturbations of trajectories imply small perturbations in the cost. This property is important for applications where we never work with exact data.

In control theory, inverse optimal control was first introduced by Kalman in 1964 [15] to relate the mathematical optimal control theory with the real control systems used by engineers. He defined the inverse problem in case of the autonomous linear-quadratic regulator, where the control system is linear and the cost to minimize is quadratic with respect to both control and state. In his formulation the inverse problem was stated as follows. Given a completely controllable constant linear dynamics and constant linear control law, determine all cost functions such that the control law minimizes the performance index (cost function in our terminology). In the same article Kalman considered a single-input case and obtained a necessary and sufficient condition for a stabilizing feedback control to be optimal. Even in the case of such specific class of optimal control problems, one should be careful with the choice of the class of the cost functions. Kalman drew attention to this issue in [15], he noted that every stabilizing or unstable control law is optimal in some sense and to avoid such a trivial conclusion the class of performance indices must be restricted. Kreindler and Jameson showed in [16] that adding the cross index in the cost changes completely the situation and any constant feedback control is optimal. If we extend the class of control problems to non-autonomous linear-quadratic regulators then any linear constant or not feedback is optimal. Another issue met by Kalman is the non-uniqueness of the costs corresponding to a given stabilizing control, even when considering the normalized cost functions. In the paper [17] the authors considered this issue. It was shown that in the single-input case the cost functions corresponding to a stabilizing control law can be parametrized, the number of parameters being equal to the dimension of the state space. In this case one can look for a diagonal matrix in the cost to avoid the non-uniqueness. In the same paper [17], the authors also showed that in the multi-input case the problem is much more complicated. After Kalman the inverse problem for the linear-quadratic regulator was extended to the multi-input case in [18] and a necessary condition and a sufficient condition were obtained. Afterwards, it took a long time to obtain a both necessary and sufficient condition for a constant feedback control to be optimal in the case of a multi-input linear-quadratic regulator. It was first obtained in 1984 in the form of a geometrical condition by Fujii and Narazaki [19] and then in the form of a polynomial matrix equation by Sugimoto and Yamamoto [20] in 1987 (see [21] for a detailed review on the inverse linear-quadratic problem). Each time the reconstruction follows from the condition on the existence of the cost. These methods used for the inverse linear-quadratic regulator rely on the algebraic Riccati equation and thus are impossible to generalize to the other classes of optimal control problems.

Before the definition of Kalman, the inverse problem was already extensively studied in the related domain of the calculus of variation. The problem of calculus of variation can be seen as a special optimal control problem with a trivial dynamics. In the direct problem, given a Lagrangian (usually depending on time, state and state derivative) one is looking for its extremal trajectories. The solutions of the direct problem are trajectories of the Euler-Lagrange equation, which is a second order differential equation. The classical inverse problem of calculus of variations is however stated on the differential equations, that is, is a given differential equation (or system of differential equations) variational? This formulation concerns the question of existence of a Lagrangian such that the given equations are the corresponding Euler-Lagrangian equation. Existence in this case is exactly the surjectivity in our formulation of the well-posedness of the inverse problem. There is an alternative formulation of the inverse problem of calculus of variation which is more relevant to the applications. Namely, does there exist a Lagrangian such that the given differential equations are *equivalent* to the Euler-Lagrangian equation? By equivalence we actually mean that the solutions of the given equations are the solutions of some Euler-Lagrange equation. Mathematically, the equivalence in this case can be represented by a transformation of the given differential equation. This transformation is called variational multiplier or variational integrating factor and the inverse problem with the variational multiplier is called the *multiplier problem*. In contrast to the case of the first formulation of the inverse problem, for which the question of existence was solved, for the second formulation it is still open. Both problems were generalized to higher order differential equations, i.e. the case where Euler-Lagrange equations consist of ordinary differential equations (ODEs) of order higher than 2 which corresponds to Lagrangian depending on derivatives of order higher than one. The problems were also generalized to field theory, that is, the case when the Euler-Lagrange equations consist of partial differential equations (PDEs).

The inverse problem of the calculus of variations was first formally stated in 1887 by Helmholtz. In his paper [22], Helmholtz obtained the necessary condition for a system to be variational, called now the Helmholtz condition. The condition takes the form of a system of differential equations on the coefficients of the given second order differential system. Next, Mayer in 1896 [23] showed that the Helmholtz conditions are locally sufficient. Thus the local existence problem was solved but only in the case of the first formulation. The Helmholtz condition was then generalized to the higher order ODEs and to the PDEs (field theory), see [24]. Notice that the global existence problem is more difficult. It requires special techniques for gluing together the locally defined Lagrangians to obtain a global one or to characterize the topological obstructions for doing that (see [24] for discussions on the global inverse problem). The multiplier problem in the case of dimension one, i.e., with one differential equation of the second order, was solved by Sonin in 1886 [25]. He showed that for any second order differential equation there exists a Lagrangian and exhibited all the solutions. The next significant result of the multiplier problem was obtained in 1941 by Douglas. In his paper [26], Douglas completely classified all 2-dimensional cases using a condition equivalent to the Helmholtz condition and found all the Lagrangians. He also showed that in some cases there is no multiplier such that the transformed equation is the Euler-Lagrange one. Up to now there is no general solution, only special cases are treated. As for the methodology, from the Helmholtz condition we can obtain the system of partial differential equations on the coefficients of the variational multiplier (see the introduction in [27] for the equations on the variational multiplier). Unfortunately, it is not always clear how to solve this system. So far, the multiplier problem is open even in the classical formulation for the second order ODEs, see the review on the recent advances in [28]. As a conclusion, the existence problem of the inverse problem of calculus of variation (in the form adapted for the applications) is still unsolved except in some particular cases since its first consideration in 1886.

The next question after the verification of the variational nature of the vector field is the reconstruction of the Lagrangian itself and the question of the uniqueness of this Lagrangian. This corresponds to the injectivity issue in the inverse optimal control context. Analytically, a Lagrangian can be found from the Euler-Lagrange equation considering the Lagrangian as an unknown function. The local reconstruction of the Lagrangian was proposed by Volterra [29] for the classical second-order inverse problem. The construction was generalized by Vainberg [30] and Tonti [31], the so-obtained Lagrangian being called Tonti-Vainberg Lagrangian. It is a Lagrangian of a very particular form. In the classical case it depends on the second order derivative of the state, but can be turned into a Lagrangian depending only on the first derivative as the highest by adding the total derivative of a well chosen function. In general, the Tonti-Vainberg Lagrangian depends on higher-order derivatives and there may exist other lower order Lagrangians which are more interesting in view of applications (see the introduction of [32, 33]). Another problem is the characterization of all Lagrangians corresponding to a given variational differential equations. There is clearly no uniqueness because of the presence of so-called trivial Lagrangians. Trivial Lagrangians are those Lagrangians for which the Euler Lagrangian equation is identically zero. Thus adding such a Lagrangian to a given one does no change the solutions of the variational problem. There may also exist Lagrangians which are not related via trivial Lagrangians. Such Lagrangians are called alternative Lagrangians and they exist only in the multiplier problem. Their characterization normally uses the form of the variational multiplier and the only treated cases are those where the existence of the multiplier is solved. Therefore, in the case of the multiplier problem it is interesting to obtain uniqueness results up to trivial Lagrangians. Notice that the existence of alternative Lagrangians was shown to be related with the existence of first integrals of the corresponding trajectories. See [33] for advances on the alternative Lagrangians.

Note that the results of the inverse problem of calculus of variations, e.g. the Helmholtz condition, may be applied to some cases of the inverse optimal control problem. Recently, some methods were developed for nonlinear control systems in order to find stabilizing feedback controls. For example, one of these methods is the method of controlled Lagrangians. In this method, instead of a system of differential equations, a system of controlled differential equations is considered (for details on this method see the first chapter of [32]).

After the paper of Kalman, the inverse problem was stated for a vast number of different classes of optimal control problems: for nonlinear [34, 35], discrete [36, 37], stochastic (Markov decision processes [38, 39]) cases and others. Nevertheless, even in the continuous deterministic case, there is no general methodology to treat the problem as well as to overcome the problem related to the ill-posedness of the inverse problem. Indeed, some general nonlinear cases were treated in [34, 35, 40, 41] using the Hamilton-Jacobi-Bellman equation. Yet, the equation is too difficult to analyze as it is a partial differential equation on the unknown value function. There exist two ways to overcome the difficulty, namely, to restrict the class of candidate cost functions to some very specific class (it was done in [34, 35] for a class of infinite horizon integral costs and in [40] for the linear-quadratic regulator) or consider a relaxation to sub-optimal costs as in [41]. As noted in [41], inverse problem with dynamics and costs of general form is highly ill-posed. The set of the cost-functions corresponding to the same optimal solutions is too large, as a consequence it is difficult to analyze its structure. In applications it leads to the problem of picking up an adequate cost. In addition, as the set of costs is large the set of trivial costs is large as well.

In applications one usually seeks an approximate solution. The most common method in robotics is to assume the candidate-cost function to be a linear combination of some basis functions, also sometimes called features. The basis functions in this case are determined via analyses of the physical case under consideration. These functions are usually related with some characteristics of the mechanical system. Such inverse problem is reduced to identification of parameters. The normalization is required and can be done for instance by taking one of the coefficients equal to 1. The corresponding basis function is then assured to be in the cost we are looking for. There exist different methods to determine such coefficients. Some of the methods are: machine learning algorithms [39]; the so-called two level approach, where the coefficients are found by iteratively solving the direct optimal control problem until the solution fits the given data [42]; Lagrange multiplier methods [43] relying on the dynamical programming. Notice that, even if the method via feature functions provides good approximations for well understood models [44], we have no guarantees that the obtained function reflects the structure and special properties of the exact solution. This method is of little interest for the study of the exact inverse optimal control problem. Another existing approach in case of human motions is to concentrate on the given model and to study in details all its characteristics [3, 45, 46]. This method is efficient for concrete cases but do not give an insight into the general inverse problem.

The given data in the inverse optimal control problem is a control system and minimizing trajectories (or/and minimizing control laws). But what if in place of a set of trajectories (parametrized curves) we consider a set of non parametrized curves? This statement of the inverse problem is related to the old geometric problem on the metrics having the same geodesics. In the problem of calculus of variation we can consider minimization of the Energy functional, the functional being defined by a Riemannian metric on the manifold of state space. The extremal solutions of such problem are geodesics. The inverse problem in this case can be formulated as follows. Given a set of geodesics, what is the corresponding Riemannian metric and is it unique? When the metric is not unique, the next question is to study the set of the metrics having the same geodesics. Such metrics are called geodesically equivalent metrics. The first result was obtained by Beltrami in [47] and in 1896 Levi Civita [48] gave a complete classification of locally equivalent metrics. In the end of the last century the problem was revisited in the context of the integrable systems. It appears that the existence of a metric equivalent to a given one implies the existence of a nontrivial first integral of the corresponding Hamiltonian flow [49]. The new approach made use of a so-called orbital diffeomorphism (sometimes called trajectorial diffeomorphism) between the Hamiltonian flows of the equivalent metrics. In particular, this new approach permitted to authors to rediscover the Levi-Civita classification. Next, Zelenko [50] extended the classification to the contact and quasi-contact cases in sub-Riemannian geometry using the same approach. See the complete discussion on the geodesic equivalence in the introduction of Chapter 3 (geodesic equivalence is also called projective equivalence, the latter term is used in Chapter 3).

This dissertation is dedicated to the problem of the well-posedness and more particularly, to the injectivity of the inverse optimal control problem. The analysis is restricted to particular classes of optimal control problem, as it is suggested by the results of previous studies on the inverse optimal control problems. The given data in the inverse problem are a set of trajectories and a control system and the surjectivity is assumed to be satisfied (it is a difficult problem which should be considered on its own). To find the cases when the injectivity of the inverse problem holds one should understand the structure of non-injective cases. This requires introducing a notion of equivalence of cost functions. Equivalent costs are the ones having the same optimal trajectories. The normal Hamiltonian flows corresponding to the equivalent costs are different in the cotangent bundle but they are projected via canonical projection to the same trajectories in the state space. The orbital diffeomorphism in this context is the diffeomorphism which maps the normal Hamiltonian flow of one cost to the normal Hamiltonian flow of the equivalent cost. As will be shown, orbital diffeomorphisms can be a tool to recover the structure of the cases admitting non trivially equivalent costs.

The manuscript is organized as follows. In Chapter 1 we introduce the main notions and the main idea of the approach via the orbital diffeomorphism. We then apply the methodology to several classical classes of optimal control problems. In Chapter 2 we consider the finite horizon Linear Quadratic problem. Notice that this problem is different from the one considered by Kalman. In our case the optimal control law is not constant and not stabilizing. Then we consider two nonlinear cases. First, the sub-Riemannian case, where the control system is linear with respect to the control and the cost is quadratic with respect to the control. This problem is very particular by its geometric structure, which permits to get some special results presented and explained in Chapter 3. In Chapter 4 we consider the control-affine case. It is the generalization of the sub-Riemannian case where the dynamics contains a non-controlled drift. This case is much more complicated than the sub-Riemannian one but we can still carry out the analysis.

Chapter 1

Inverse optimal control

The main topic discussed in this thesis is the injectivity of the inverse optimal problems. The study of injectivity may be reduced to the study of a special equivalence relation on the cost functions. In this chapter we will set the necessary notations and propose a general framework to treat this problem.

1.1 Inverse optimal control problems

1.1.1 Statement of the problem

Let us fix a smooth connected n-dimensional manifold M which represents the state space. We will consider autonomous control systems given by ordinary differential equation

$$\dot{q} = f(q, u), \tag{1.1}$$

the control parameter u takes values in a set $U \subset \mathbb{R}^m$ and for any $u \in U$, the map $q \mapsto f(q, u)$ is a smooth vector field on M.

Definition 1.1. A *trajectory* q_u is a Lipschitz curve $q_u(t) \in M,\ t \in [0, T]$, which satisfies (1.1) for a control $u(\cdot) \in L^\infty([0, T], U)$.

Remark 1.2. Trajectory q_u is Lipschitz with respect to any Riemannian metric on M.

Consider a cost function

$$J(q_u) = \int_0^T L(q_u(t), u(t))dt, \tag{1.2}$$

where $L : M \times U \to \mathbb{R}$ is smooth in both variables. The function L is called a Lagrangian.

A dynamics f and a Lagrangian L define a *class of Optimal Control problems*, as follows: given a pair of points q_0, q_1 and a final time $T > 0$, determine

$$\inf\{J(q_u) \;:\; q_u \text{ trajectory of } \dot{q} = f(q,u) \text{ s.t. } q_u(0) = q_0, q_u(T) = q_1\}. \tag{1.3}$$

We will consider only classes of f, L for which the so-defined optimal control problem admits minimizing solutions q_u whenever $\inf J(q_u) < +\infty$.

Definition 1.3. *The optimal synthesis* $\mathcal{O}$ is the set of trajectories minimizing (1.3) for all $q_0, q_1 \in M$ and $T > 0$ such that $\inf J(q_u) < +\infty$.

The Inverse Optimal Control problem arises as follows: given a dynamics (1.1) and a set Γ of trajectories, find a cost J such that every $\gamma \in \Gamma$, $\gamma : [0,T] \to M$, is a solution of the optimal control problem (1.3) associated with $q_0 = \gamma(0)$ and $q_1 = \gamma(T)$. Let us formalize the inverse problem.

Assume the manifold M and the dynamics f to be fixed. Solving the entire class of optimal control problems (1.3) associated with J for all realizable q_0, q_1, T, we obtain an optimal synthesis $\mathcal{O}(J)$ corresponding to J. We can formally define an operator $\mathcal{F}$ which maps each cost J of the form (1.2) to the corresponding optimal synthesis $\mathcal{O}(J)$.

$$\mathcal{F} : \mathcal{J} \to \Omega \quad \text{such that} \quad \mathcal{F} : J \mapsto \mathcal{O}(J).$$

Note that together with the operator $\mathcal{F}$ we should define the set of cost functions $\mathcal{J}$ and the set Ω where $\mathcal{F}$ takes its values. Thus, solving an inverse optimal control problem amounts to find the image by the inverse operator $\mathcal{F}^{-1}$ of the given set Γ. An inverse problem is well-posed if the operator $\mathcal{F}$ has the following properties:

- surjectivity;
- injectivity;
- continuity of the inverse $\mathcal{F}^{-1}$.

Let us explain each of the properties. The surjectivity of the map $\mathcal{F}$ means that each considered set of trajectories Γ is an optimal synthesis associated with some cost function in the class $\mathcal{J}$, or a subset of such an optimal syntheses in a more general formulation. The injectivity of the map ensures the existence of a unique cost in the considered class of costs $\mathcal{J}$. The continuity of the inverse map asserts that small perturbations of the trajectories imply small perturbations of the costs. This property is important in applications.

In this manuscript the surjectivity is assumed to hold, which is a natural hypothesis in applications. We will concentrate on the question of injectivity. The reconstruction will be considered only in a particular case (linear-quadratic case, see Chapter 2).

Remark 1.4. The inverse optimal control problem can be define for a very general class of costs $\mathcal{J}$. However, we will propose in this thesis to consider the inverse problems of specific classes of costs as it seems to be more efficient. Notice that when we restrict our attention to the class $\mathcal{J}$ of integral costs, i.e. the costs of the form (1.2), a cost $J \in \mathcal{J}$ is completely characterized by its Lagrangian L. Therefore, in this case the map $\mathcal{F}$ can be defined on the set $\mathcal{L}$ of Lagrangians.

1.1.2 Examples

Let us introduce some important examples of optimal control problems.

The Sub-Riemannian case. In this case the control system is defined as follows

$$\dot{q} = \sum_{i=1}^{m} u_i f_i(q), \qquad u = (u_1, \dots, u_m) \in \mathbb{R}^m, \tag{1.4}$$

where $f_1, \dots, f_m$ are smooth vector fields on M. The cost is given by a quadratic functional

$$E(q_u) = \int_0^T u^T Q(q_u) u dt,$$

where $Q(q)$ is a positive definite $(m \times m)$ matrix for any $q \in M$ which depends smoothly on q. The sub-Riemannian problem has a geometric meaning. At each $q \in M$ the vector fields $f_1, \dots, f_m$ span a vector space $D_q = \text{span}\{f_1(q), \dots, f_m(q)\} \subset T_qM$. The union $D = \cup_{q \in M} D_q$ defines a dynamical constraint $\dot{q} \in D_q$ on M equivalent to (1.4). The quadratic form $Q(q)$ defines a scalar product on each D_q and thus it induces a metric g on M such that $g_q(f_i, f_j) = Q(q)_{i,j}$. This implies that the sub-Riemannian problem is the problem of minimization of the energy

$$E = \int_0^T g_{q(t)}(\dot{q}(t), \dot{q}(t)) dt,$$

under the constraint

$$\dot{q} \in D_q.$$

In the inverse sub-Riemannian problem we are looking for a metric corresponding to the given set of minimizers of the energy.

The solutions of this class of direct optimal control problems are closely related to the solutions of the problem on shortest paths connecting the points of M. In this case the dynamical constraint is defined again by $\dot{q} \in D_q$ and the cost is given by the length functional

$$L = \int_0^T \sqrt{g_{q(t)}(\dot{q}(t), \dot{q}(t))}dt.$$

For any trajectory defining a shortest path, its reparameterization is again a shortest path associated with the same metric. As a consequence, the inverse problem in this case can be stated as follows: to determine a metric corresponding to the set of shortest paths given as geometric curves (parametrization is not taken into account). *The geometric inverse problem*, i.e., the inverse problem on the set of geometric curves, is different from the inverse optimal control problem and more difficult. Nevertheless, it can be handled by the same tools, as will be shown later.

The Riemannian case. This case is a special case of the sub-Riemannian problem, where $m \geq n$ and $D = TM$. In the Riemannian problem the dynamical system does not define a constraint and thus the problem belongs to the class of variational problems. Geometric inverse Riemannian problem is classical in Riemannian geometry.

The Linear-Quadratic case. In this case the optimal control problem is defined on $M = \mathbb{R}^n$ with $U = \mathbb{R}^m$ and is associated with a linear control system represented in matrix form by

$$\dot{q} = Aq + Bu,$$

where A is a $(n \times n)$ real-valued matrix and $B = \begin{pmatrix} b_1 & \cdots & b_m \end{pmatrix}$ is a $(n \times m)$ real-valued matrix. The cost is given by a quadratic functional

$$J(q_u) = \int_0^T \left(q_u^\top Q q_u + 2q_u^\top S u + u^\top R u\right) dt,$$

where Q, R, S are real-valued matrices of appropriate dimension.

The control-affine case. The optimal control problem in this case is given by an affine control system

$$\dot{q} = f_0(q) + \sum_{i=1}^{m} u_i f_i(q), \qquad u = (u_1, \dots, u_m) \in \mathbb{R}^m,$$

where $f_0, f_1, \dots, f_m$ are smooth vector fields on M. The non-controlled vector field f_0 is called the *drift*. The cost is given by a quadratic functional

$$J(q_u) = \int_0^T u^T Q(q_u) u dt,$$

where $Q(q)$ is positive definite for any $q \in M$ and depends smoothly on $q \in M$.

Notice that when $f_0 = 0$ this is exactly the sub-Riemannian case and when $M = \mathbb{R}^n$, f_0 is linear in q and $f_1, \ldots, f_m$ are constant, then this is the Linear-Quadratic case with $f_0(q) = Aq$ and $f_i = b_i$.

1.1.3 Injectivity and cost equivalence

Let us fix the dynamics f. The injectivity of the mapping $\mathcal{F}$ depends on the class of costs $\mathcal{J}$ that we choose. The obvious example when the injectivity fails is the case of constantly proportional costs cJ and J. The minimizers of such costs are the same for any triple of time T, initial and final points q_0, q_1, and thus $\mathcal{F}(cJ(u)) = \mathcal{F}(J(u))$. This difficulty can be solved by normalization of the costs in such a way that there is only one representative of the set $\{cJ(u) : c \text{ real positive constant}\}$ in $\mathcal{J}$. In other words we quotient the class $\mathcal{J}$ by the relation of multiplication by a constant. It inspires the notion of equivalence on the set of cost functions.

Remark 1.5. For the integral costs defined by (1.2), the injectivity depends on the class $\mathcal{L}$ of Lagrangians. We can speak about injectivity on $\mathcal{L}$ and the equivalence of the Lagrangians.

Definition 1.6. We say that two costs J and $\tilde{J}$ (or the corresponding Lagrangians) are *equivalent via minimizers* if they define the same optimal synthesis.

This notion defines an equivalence relation on a class of costs $\mathcal{J}$. For a given class $\mathcal{J}$, the existence of two non-proportional costs which are equivalent via minimizers implies that the inverse optimal control problem may not have a unique solution in this class, i.e. $\mathcal{F}$ is not injective.

Definition 1.7. We say that a cost $J \in \mathcal{J}$ is *rigid* if there is no cost in $\mathcal{J}$ equivalent via minimizers to J and non constantly proportional to J.

The inverse problem is injective on $\mathcal{J}$ if all costs in $\mathcal{J}$ are rigid.

In this work we consider the local inverse optimal control problem. In the local problem we work on a neighborhood of some fixed point $q_0 \in M$. Previous studies on the geometric inverse problem on geodesics in Riemannian geometry (see [49,51]) have shown that the topological properties of the manifold may prevent the existence of non-proportional globally equivalent costs. In the inverse optimal control problem one should also take into account the global properties of the minimizers. This suggests that the global problem should be considered separately. Notice that in some cases to consider the local problem is the same as to consider the global problem. It is the case in the linear-quadratic inverse problem, in this case the global equivalence will be considered in the Chapter 2.

Remark 1.8. When we speak about local cost equivalence, we either precise the point $q_0 \in M$ in the neighborhood of which we work or otherwise the point is assumed to be arbitrary.

1.1.4 Example of non-trivially equivalent costs

In general, there may exit costs which are equivalent via minimizers and are not constantly proportional. Let us consider an example of such costs which are non trivially equivalent via minimizers. Let $M = \mathbb{R}^n$ and $U = \mathbb{R}^m$. Assume that there exists coordinates $(x_1, \ldots, x_n)$ of $\mathbb{R}^n$ such that the state q decomposes in $q = (y_1, y_2)$ with $y_1 = (x_1, \ldots, x_k)$, $y_2 = (x_{k+1}, \ldots, x_n)$, and the control u decomposes in $u = (u_1, u_2)$ where $u_1 \in \mathbb{R}^l$, $u_2 \in \mathbb{R}^{m-l}$ in such a way that the dynamics takes the form

$$\begin{cases} \dot{y}_1 = f_1(y_1, u_1), \\ \dot{y}_2 = f_2(y_2, u_2). \end{cases}$$

This means that the dynamics decomposes into two independent parts, its projections on $\mathbb{R}^k$ and on $\mathbb{R}^{n-k}$ in $M = \mathbb{R}^k \times \mathbb{R}^{n-k}$. Let us consider two costs

$$L(q, u) = u_1^T u_1 + u_2^T u_2 \quad \text{and} \quad \tilde{L}(q, u) = \alpha u_1^T u_1 + (1 - \alpha) u_2^T u_2,$$

where α is arbitrary real number such that $0 \leq \alpha \leq 1$. The minimization of the costs can be written as

$$\begin{aligned} \min_u \int_0^T L(q, u) dt &= \min_{u_1} \int_0^T u_1^T u_1 dt + \min_{u_2} \int_0^T u_2^T u_2 dt, \\ \min_u \int_0^T \tilde{L}(q, u) dt &= \alpha \min_{u_1} \int_0^T u_1^T u_1 dt + (1 - \alpha) \min_{u_2} \int_0^T u_2^T u_2 dt. \end{aligned}$$

It is easy to see that the same couple (u_1^*, u_2^*) minimizes both costs independently of α. Therefore, the same minimizing trajectory $(y_1(u_1^*), y_2(u_2^*))$ associated with some initial and final conditions is a solution of the both optimal control problems associated with L and with $\tilde{L}$. If $\alpha \neq \frac{1}{2}$ then the Lagrangians $L, \tilde{L}$ are non-proportional and they are non-trivially equivalent via minimizers. We will see that this kind of examples plays a crucial role in the study of equivalence.

1.2 Geodesic equivalence

1.2.1 Hamiltonian formalism and Pontryagin Maximum Principle

Now let us recall the Hamiltonian formalism. The cotangent bundle T^*M is equipped with the standard symplectic structure defined in the following way. Let

$\pi : T^*M \to M$ be the canonical projection. The *Liouville 1-form* $\eta \in \Lambda^1(T^*M)$ is defined by $\eta_\lambda = \lambda \circ d\pi$ for any $\lambda \in T^*M$. The *canonical symplectic form* on T^*M is a non-degenerate closed 2-form defined by $\sigma = d\eta$. In local canonical coordinates $\lambda = (x,p)$ we have

$$\eta_{(x,p)} = \sum_{i=1}^{n} p_i dx_i, \qquad \sigma_{(x,p)} = \sum_{i=1}^{n} dp_i \wedge dx_i.$$

For any function $h \in C^\infty(T^*M)$ we can associate a vector field $\vec{h}$ on T^*M in the following way.

$$d_\lambda h = \sigma_\lambda(\cdot, \vec{h}(\lambda)) \qquad \text{for every } \lambda \in T^*M.$$

In local canonical coordinates the expression of $\vec{h}$ is

$$\vec{h} = \sum_{i=1}^{n} \frac{\partial h}{\partial p_i}\frac{\partial}{\partial x_i} - \frac{\partial h}{\partial x_i}\frac{\partial}{\partial p_i}.$$

We call h a *Hamiltonian function* (or simply Hamiltonian) and $\vec{h}$ the *Hamiltonian vector field* associated with h. For any $\lambda_0 \in T^*M$ the Hamiltonian vector field defines an integral curve solution of $\dot{\lambda} = \vec{h}(\lambda)$ denoted by $e^{t\vec{h}}\lambda_0$. The operator $e^{t\vec{h}}$ defines the Hamiltonian flow.

Definition 1.9. The *Poisson bracket* of two functions $f, g \in C^\infty(T^*M, \mathbb{R})$ is defined by

$$\{f, g\} = \vec{g}(f),$$

An optimal control problem (1.3) defines a *pseudo-Hamiltonian*, that is, a Hamiltonian parametrized by $(p_0, u) \in \mathbb{R} \times U$. We denote by $\langle p, v\rangle$ for $v \in T_qM,\ p \in T_q^*M$ the action of the covector p on the vector v. For each $(q, p, p_0, u) \in T^*M \times \mathbb{R} \times U$ the pseudo-Hamiltonian is defined as follows

$$\mathcal{H}(q, p, p_0, u) = \langle p, f(q,u)\rangle + p_0 L(q,u).$$

Definition 1.10. An *extremal* associated with (1.3) is a 4-tuple $(q, p, p_0, \tilde{u})$, where $(q(\cdot), p(\cdot))$ is a Lipschitz curve in T^*M, $p_0 \in \mathbb{R}_-$ and $\tilde{u}(\cdot) \in L^\infty([0,T], U)$, such that $(p(t), p_0) \neq 0$ for any $t \in [0,T]$ and

$$\begin{cases} \dot{q}(t) = \frac{\partial}{\partial p}\mathcal{H}(q, p, p_0, \tilde{u}), \\ \dot{p}(t) = -\frac{\partial}{\partial q}\mathcal{H}(q, p, p_0, \tilde{u}), \\ \mathcal{H}(q, p, p_0, \tilde{u}) = \max_{u \in U} \mathcal{H}(q, p, p_0, u). \end{cases} \tag{1.5}$$

An extremal is called *normal* if $p_0 < 0$ and *abnormal* if $p_0 = 0$. The projection $q(\cdot)$ of a normal (resp. abnormal) extremal onto M is called a normal (resp. abnormal) *geodesic*.

Notice that a geodesic can be both normal and abnormal. It is the case when it is a projection of two different extremals, a normal extremal and an abnormal one. The geodesic which is a projection only of abnormal extremals is called strictly abnormal. In the same spirit the strictly normal geodesic is a normal geodesic which is not a projection of abnormal extremals.

Pontryagin Maximum Principle gives a necessary conditions for a curve to be minimizing for the optimal control problem (1.3) in terms of the extremals.

Theorem 1.11 (Pontryagin Maximum Principle)**.** *If a trajectory $q(\cdot)$ is minimizing for the optimal control problem* (1.3) *then $q(\cdot)$ is a geodesic.*

The system (1.5) is invariant under the rescaling of $(p(t), p_0)$ by any positive constant and in case of normal extremals it is usual to fix $p_0 = -\frac{1}{2}$. From now on we will write $\mathcal{H}(q, p, u)$ for the normal pseudo-Hamiltonian associated with $p_0 = -\frac{1}{2}$.

The abnormal extremals do not always exist. For instance, in the Riemannian and the Linear-Quadratic cases there is no non trivial abnormal trajectories. However, in general the abnormal geodesics are present and in some cases they are optimal.

Remark 1.12. By construction, abnormal geodesics depend only on the control system (1.1) and are independent of the cost. So, they are the same for all costs in $\mathcal{J}$ for any choice of the class $\mathcal{J}$.

Let us consider the normal pseudo-Hamiltonian $\mathcal{H}(q, p, u)$. By the smoothness assumptions on dynamics f and Lagrangian L, the pseudo-Hamiltonian is smooth in all variables. Assume in addition that the control set U is open. Then, the maximality condition on $\mathcal{H}$ can be expressed as

$$\frac{\partial}{\partial u}\mathcal{H}(q, p, \tilde{u}) = 0.$$

When the Hamiltonian is strictly convex in u we have moreover the uniqueness of the solution $\tilde{u}$. Assume $\mathcal{H}(q, p, u)$ to have invertible Hessian with respect to u at a maximizing point $\tilde{u}$, we can apply the implicit function theorem to express the maximizing control as $\tilde{u} = \tilde{u}(q, p)$. In this case the maximized Hamiltonian does not depend on the control and defines a usual Hamiltonian $h(q, p) = \mathcal{H}(q, p, \tilde{u}) \in C^\infty(T^*M)$ called the *normal Hamiltonian.* As a consequence, normal extremals are the integral curves of the Hamiltonian system $\dot{\lambda} = \vec{h}(\lambda)$. For any $\lambda_0 \in T^*M$ the trajectory $\lambda(t) = e^{t\vec{h}}\lambda_0$ is a normal extremal and the curve $\gamma(t) = \pi \circ e^{t\vec{h}}\lambda_0$ is a normal geodesic.

Now let us assume in addition that the Hamiltonian's Hessian with respect to u is everywhere negative-definite. Then, by the strong Legendre condition, the sufficient optimality condition holds for small times for any normal geodesic.

Theorem 1.13 ([52, Theorem 1.6])**.** *If the strong Legendre condition holds along the normal extremal $(q(t), p(t), u(t))$, that is, there exists $\alpha > 0$ such that*

$$\frac{\partial^2 \mathcal{H}}{\partial u^2}(q(t), p(t), u(t))(v, v) \leq -\alpha \|v\|^2 \qquad \forall v \in \mathbb{R}^m,$$

then there exists $\epsilon > 0$ so that the geodesic $q(\cdot)$ is locally optimal on the interval $[0, \epsilon]$.

Definition 1.14. We say that a class of costs $\mathcal{J}$ satisfies the strong Legendre condition if any $J \in \mathcal{J}$ satisfies the condition.

Notice that it is not very restrictive to make this assumption on the Hessian, it is satisfied in a lot of useful cases of optimal control problem. In particular, all the examples introduced in the subsection 1.1.2 satisfy this condition. Since, in this work we are interested in the local cost equivalence, the characterization of local minimality of geodesics is sufficient for our purposes.

1.2.2 Geodesic equivalence

The Pontryagin Maximum Principle asserts that the minimizers belong to the set of geodesics and in some cases they even coincide, at least the normal geodesics for small times, by Theorem 1.13. In Riemannian geometry for instance all geodesics are normal and locally minimizing. The geodesics, at least the normal ones, are easier to work with than the minimizers because they are integral curves of the Hamiltonian flow. Notice that the inverse problem of calculus of variations and the geometric inverse problem in Riemannian geometry are stated on geodesics. In view of inverse optimal control problem this inspires the following definition.

Definition 1.15. We say that two costs J and $\tilde{J}$ are equivalent via geodesics if the corresponding classes of optimal control problems have the same geodesics.

Remark 1.16. Two constantly proportional costs are always equivalent via minimizers and via geodesics, so we will call such costs trivially equivalent.

Definition 1.17. A cost $J \in \mathcal{J}$ is said to be *geodesically rigid* if there is no cost in $\mathcal{J}$ which is nontrivially equivalent via geodesics to J.

Remark 1.18. Notice that for two costs to be equivalent via geodesics we only need the costs to have the same strictly normal geodesics. The abnormal geodesics coincide automatically because they depend only on the dynamics.

The following lemma permits to relate the notions of equivalence via geodesics and via minimizers.

Lemma 1.19. *Assume that $\mathcal{J}$ satisfies the strong Legendre condition. If two costs in $\mathcal{J}$ are equivalent via minimizers, then they are equivalent via geodesics.*

Proof. Assume that J and $\tilde{J}$ have the same minimizers. Let γ be a geodesic of J. Either it is an abnormal geodesics, and then it is also an abnormal geodesic of $\tilde{J}$. Or it is a normal geodesic of J, in this case every sufficiently short piece of γ is minimizing for J, therefore γ is a minimizer and thus a geodesic for $\tilde{J}$. In both cases γ is a geodesic of $\tilde{J}$. □

Remark 1.20. Notice that if the costs in $\mathcal{J}$ have the form (1.2) then all the notions can be stated for the Lagrangians in the corresponding set $\mathcal{L}$.

Let us return to the problem of injectivity. In terms of cost equivalence, $\mathcal{F}$ is injective on $\mathcal{J}$ if there is no pair of non-proportional costs in $\mathcal{J}$ which are equivalent via minimizers. This is the case when all costs are rigid. Lemma 1.19 permits to reduce the study of equivalence via minimizers to the study of equivalence via geodesics.

Corollary 1.21. *If every cost in $\mathcal{J}$ is geodesically rigid, then $\mathcal{F}$ is injective.*

Finally, the study of the injectivity of $\mathcal{F}$ can be reduced to the study of the two following questions:

1. For a given $\mathcal{J}$, are all costs in $\mathcal{J}$ rigid?
2. If not, which costs in $\mathcal{J}$ are geodesically equivalent?

Notice that we study the injectivity of $\mathcal{F}$ for a fixed dynamics f. Actually, for different dynamics f and the same class of costs $\mathcal{J}$, $\mathcal{F}$ may be injective or not. Therefore, it is important to understand in which cases the problem is injective and find the structural characteristics which distinguish the injective cases from non-injective ones.

Even in the case of non-injective problem it is important to describe the classes of equivalent costs. In this case the inverse problem is not well-posed but it can be reformulated by reduction of the set of costs $\mathcal{J}$ to another set $\mathcal{C}$ on which $\mathcal{F}$ is injective. The new set can be constructed by representatives from equivalence classes of $\mathcal{J}$, one from each class, to ensure the injectivity.

1.3 Orbital diffeomorphism

We propose a geometrical approach via the orbital diffeomorphism to study the geodesic equivalence of a large class of optimal control problems. The approach will be introduce in the this section and then applied in the next chapters to the examples of Subsection 1.1.2. Let us fix the class of optimal control problems we will work with.

Let the control system on the manifold M be control-affine

$$\dot{q} = f_0(q) + \sum_{i=1}^{m} u_i f_i(q), \qquad u = (u_1, \ldots, u_m) \in \mathbb{R}^m, \tag{1.6}$$

and the cost be a Lagrangian of the form (1.2). In addition, we make the following assumptions.

1. The control system satisfies the weak Hörmander's condition, i.e. ;

$$\mathrm{Lie}_q \left(\{ (\mathrm{ad} f_0)^s f_i : s \geq 0,\ i = 1, \ldots, m \} \right) = T_q M. \tag{A1}$$

2. The function $L(q,u)$ is a Tonelli Lagrangian, i.e.

   - the Hessian of $L(q,u)$ with respect to u is positive definite for all $q \in M$, in particular $L(q,u)$ is strictly convex in u,
   - $L(q,u)$ has superlinear growth, i.e. $L(q,u)/|u| \to +\infty$ when $|u| \to +\infty$.

These assumptions are necessary to have a non-trivial set of strictly normal geodesics and a well-defined normal Hamiltonian. The assumptions permit in particular to apply the theory developed in the previous section and reduce the problem of injectivity to the problem on the geodesic equivalence. This class is however too general and other conditions will be needed. To present the conditions let us first recall some notions and results from geometric control theory. We denote by $\mathcal{L}$ the set of all Lagrangians satisfying the assumptions listed above and by $\mathcal{J}$ the set of the corresponding costs.

Definition 1.22. The Lie bracket of two vector fields f_1, f_2, denoted by $[f_1, f_2]$, is the vector field defined as follows: for any $\phi \in C^1(M, \mathbb{R})$

$$[f_1, f_2]\, \phi = f_1(f_2(\phi)) - f_2(f_1(\phi)),$$

where $f_1(\phi)$ and $f_2(\phi)$ are the Lie derivatives of ϕ in the direction of f_1 and f_2 respectively. For a positive integer l, the notation $(\mathrm{ad} f_1)^l f_2$ stands for $\underbrace{[f_1, \ldots, [f_1}_{l \text{ times}}, f_2]]$.

Definition 1.23. Let F be a family of vector fields on M. The *Lie algebra* spanned by F is the vector space $\mathrm{Lie}(F)$ defined as

$$\mathrm{Lie}(F) = \mathrm{span}\{ \left[f_{i_1}, \left[f_{i_2}, \ldots, \left[f_{i_{k-1}}, f_{i_k} \right] \right] \right] \ :\ k \in \mathbb{N},\ f_{i_1}, \ldots, f_{i_k} \in F \}.$$

The value of Lie(F) at any $q \in M$ is a subspace of the tangent space

$$\mathrm{Lie}_q(F) = \mathrm{span}\{\big[f_{i_1}, \big[f_{i_2}, \ldots, \big[f_{i_{k-1}}, f_{i_k}\big]\big]\big]\,(q) \;:\; k \in \mathbb{N},\; f_{i_1}, \ldots, f_{i_k} \in F\} \subseteq T_qM.$$

Consider now the family of vector fields defined by the control system (1.6),

$$F = \{f_0 + \sum_{i=1}^{m} u_i f_i \;:\; u_i \in \mathbb{R} \text{ for } i = 1, \ldots m\}.$$

We have the following relation

$$\mathrm{Lie}(F) \supseteq \mathrm{Lie}\left(\{(\mathrm{ad} f_0)^s f_i : s \geq 0,\; i = 1, \ldots, m\}\right),$$

which implies in particular that for any $q \in M$ using the weak Hörmander's condition we have

$$\mathrm{Lie}_q(F) = T_qM.$$

Definition 1.24. For any time $T > 0$, the *end-point* map $E^T_{q_0}$ of a control system (1.1) with the initial point $q_0 \in M$ is a map such that

$$E^T_{q_0} : \mathcal{U} \to M, \quad E^T_{q_0}(u) = q_u(T),$$

where $\mathcal{U} \in L^\infty([0,T]\,, U)$ is the set of controls u defining the admissible trajectories q_u on $[0, T]$.

Definition 1.25. For any time $T > 0$ and any point $q_0 \in M$, the *set of reachable points* at time T, denoted $\mathcal{A}_{q_0}(T)$, is the image of the end-point map $E^T_{q_0}$.

Using the results in [53, Chapter 3, Theorem 3] on the reachable sets applied for control-affine systems we have the following standard topological property of the corresponding reachable set.

Theorem 1.26 ([53])**.** *Suppose that $f_0, f_1, \ldots, f_m$ are vector fields on M that define a control-affine system* (1.6)*. If the weak Hörmander's condition* (A1) *is satisfied and the image of $E^T_{q_0}$ is nonempty for $q_0 \in M$ and $T > 0$ then the reachable set $\mathcal{A}_{q_0}(T)$ has a nonempty interior in M.*

To ensure the existence of a minimizer q_u of a cost $J \in \mathcal{J}$ under the conditions $q_u(0) = q_0$ and $q_u(T) \in \mathcal{A}_{q_0}(T)$ we need to introduce some additional assumptions. The classical Filippov theorem can not be applied in our case (see [54]). Hence, other arguments should be found. It follows from the assumptions on the costs in $\mathcal{L}$ (super-linear growth and convexity) that each cost $J \in \mathcal{J}$ is bounded from below. It remains to ensure that the set of trajectories joining two fixed points in time $T < \infty$ is compact, in this case the cost reaches its minimum. To ensure that this condition is satisfied we can adapt the following assumption on the control system from [55]:

$\mathcal{H}1$ For every bounded family $\mathcal{U}$ of admissible controls, there exists a compact subset $K_T \subset M$ such that $q_u(t) \in K_T$ for each $u \in \mathcal{U}$ and $t \in [0, T]$.

Yet, this is not an explicit condition on the control system. Another possibility is to ask the sublinear growth of $f_0, f_1, \ldots, f_m$, in this case $\mathcal{H}1$ holds, see [55]. Putting all together, we conclude that $J \in \mathcal{J}$ admits a minimum on the set of admissible trajectories joining two points.

Theorem 1.27 ([55, Proposition 6]). *Under the assumption $\mathcal{H}1$, for any $q_0 \in M$, $T > 0$ and $q_1 \in \mathcal{A}_{q_0}(T)$, there exists a trajectory of (1.6) joining q_0 to q_1 in time T which minimizes a cost $J \in \mathcal{J}$.*

We refer to [55, Sections 2.1, 2.2] for a discussion on conditions needed for existence of minimizers when the control system is given by a control-affine system.

The strict convexity of Lagrangians in $\mathcal{L}$ permits to define a normal Hamiltonian h for any $L \in \mathcal{L}$. We need to verify the strong Legendre condition to be able to use the framework of geodesically equivalent costs. A Lagrangian $L \in \mathcal{L}$ is strictly convex, therefore, along any normal geodesic $q(\cdot)$ we have

$$\frac{\partial^2 L}{\partial u^2}(q(t), u(t))(v, v) \geq \lambda_{min}(q(t), u(t))\|v\|^2 \qquad t \in [0, T]\,,$$

where $\lambda_{min}(q(t), u(t)) > 0$ is the minimal eigenvalue of $\frac{\partial^2 L}{\partial u^2}(q(t), u(t))$ and on the compact set $[0, T]$ it reaches the minimum $\lambda_{min}(q(t_*), u(t_*)) = \alpha > 0$. As a consequence,

$$\frac{\partial^2 \mathcal{H}}{\partial u^2}(q(t), p(t), u(t))(v, v) = -\frac{\partial^2 L}{\partial u^2}(q(t), u(t))(v, v) \leq -\alpha\|v\|^2,$$

and the strong Legendre condition is satisfied.

Next, we need the existence of strictly normal geodesics. Remind that all costs in $\mathcal{J}$ (and in any other class of costs) have the same abnormal geodesics. Thus, in absence of strictly normal geodesics the geodesic equivalence of the costs is trivial.

Definition 1.28. Let a control $u \in L^\infty([0, T]\,, U)$ be such that the associated trajectory q_u steers q_0 to some point $q_u(T) \in M$. The control u is called *regular* on $[0, T]$ if $dE^T_{q_0}(u)$ is surjective.

A geodesic q_u joining q_0 to $q_u(T)$ is strictly normal if it is associated with a regular control u [56]. The surjectivity of the end-point map is related to the regularity of the value function.

Definition 1.29. Fix a point q_0 and time $T > 0$. For any $q \in M$ the *value function* $V_T(q_0, q)$ associated with $J \in \mathcal{J}$ is defined as follows

$$V_T(q_0, q) = \inf\{J(q_u) : q_u \text{ trajectory of } \dot{q} = f(q, u) \text{ s.t. } q_u(0) = q_0, q_u(T) = q\}.$$

One of the possible conditions to ensure the existence of a strictly normal geodesic is the following.

Theorem 1.30 ([57, Lemma 2.2]). *Let $q_0 \in M$ and $T > 0$ be fixed. If $V_T(q_0, q)$ is smooth at $q \in \mathcal{A}_{q_0}(T)$ and there exists a minimizing trajectory q_u joining q_0 to q in time T, then q_u is the unique minimizer joining the two points in time T and it is strictly normal.*

In some classical cases of optimal control problem it was shown that for any point q_0 and any $T > 0$ the set of smooth points is open and dense in the reachable set $\mathcal{A}_{q_0}(T)$. In particular, it holds in all the cases presented in the next chapters. In the sub-Riemannian case the set of smooth points is open and dense in M, see [58]. In affine case, it was proved in [55] that the set of smooth points is open and dense in the reachable set.

1.3.1 Jacobi curves

We introduce Jacobi curves and all necessary related objects here, for more details we refer to [57, 59, 60]. Consider an optimal control problem (1.3) and a normal geodesic $\gamma(t) \in M$, $t \in [0, T]$. It is the projection on M of an extremal $\lambda(t) = e^{t\vec{h}}\lambda$ for some $\lambda \in T^*M$ i.e. $\pi \circ \lambda(t) = \gamma(t)$. The $2n$-dimensional space $T_\lambda(T^*M)$ endowed with the symplectic form $\sigma_\lambda(\cdot, \cdot)$ is a symplectic vector space. A Lagrangian subspace of this symplectic space is a vector space of dimension n which annihilates the symplectic form. We denote by $\mathcal{V}_{\lambda(t)}$ the vertical subspace $T_{\lambda(t)}(T^*_{\gamma(t)}M)$ of $T_{\lambda(t)}(T^*M)$, it is vertical in the sense that $d\pi(\mathcal{V}_{\lambda(t)}) = 0$. Now we can define the Jacobi curve associated with the normal geodesic $\gamma(t)$.

Definition 1.31. For $\lambda \in T^*M$, we define the *Jacobi curve* $J_\lambda(\cdot)$ as the curve of Lagrangian subspaces of $T_\lambda(T^*M)$ given by

$$J_\lambda(t) = e_*^{-t\vec{h}}\mathcal{V}_{\lambda(t)}, \qquad t \in [0, T].$$

Let us introduce the extensions of a Jacobi curve.

Definition 1.32. For an integer $i \geq 0$, the *ith extension* of the Jacobi curve $J_\lambda(\cdot)$ is defined as

$$J_\lambda^{(i)} = \operatorname{span}\left\{ \frac{d^j}{dt^j} l(0) \ : \ l(s) \in J_\lambda(s)\ \forall s \in [0, T],\ l(\cdot) \text{ smooth },\ 0 \leq j \leq i \right\}.$$

By definition, $J_\lambda^{(i)} \subset J_\lambda^{(i+1)} \subset T_\lambda(T^*M)$, so it is possible to define a flag of these spaces.

Definition 1.33. The *flag of the Jacobi curve* $J_\lambda(\cdot)$ is defined as

$$J_\lambda = J_\lambda^{(0)} \subset J_\lambda^{(1)} \subset \cdots \subset T_\lambda(T^*M).$$

The extensions of a Jacobi curve may be expressed in terms of the Lie brackets.

Lemma 1.34. *Let $q = \pi(\lambda)$. The extensions of the Jacobi curve take the following form:*

$$J_\lambda^{(i)} = \operatorname{span}\left\{(\operatorname{ad}\vec{h})^j Y(\lambda) \ : \ d\pi \circ Y = 0 \text{ near } \lambda,\ 0 \leq j \leq i\right\}.$$

Proof. Let $v \in J_\lambda^{(k)}$, for some integer $k \geq 0$. By definition, $v = \frac{d^s}{dt^s} l(0)$ where $l(\cdot)$ is a curve with $l(t) \in J_\lambda(t)$ for any $t \in [0,T]$, and $s \leq k$ is an integer. Then there exists a vertical vector field Y on T^*M (i.e. $d\pi \circ Y = 0$) such that, for any $t \in [0,T]$,

$$l(t) = e_*^{-t\vec{h}} Y(\lambda(t)),$$

and v writes as

$$v = \frac{d^s}{dt^s} e_*^{-t\vec{h}} Y(e^{t\vec{h}}\lambda)\Big|_{t=0} = (\operatorname{ad}\vec{h})^s Y(\lambda).$$

which proves the result. □

1.3.2 Ample geodesics

Note that the dimension of the spaces $J_\lambda^{(k)}$ for $|k| > 1$ may depend on λ in general. Following [57], we distinguish the geodesics corresponding to the extensions of maximal dimension.

Definition 1.35. The normal geodesic $\gamma(t) = \pi(e^{t\vec{h}}\lambda)$ is said to be *ample at* $t = 0$ if there exists an integer k_0 such that

$$\dim(J_\lambda^{(k_0)}) = 2n, \quad \text{or equivalently,} \quad J_\lambda^{(k_0)} = T_\lambda(T^*M).$$

In that case we say that λ *is ample with respect to the Lagrangian* L.

Notice that if a geodesic is ample at $t = 0$, then it is not abnormal on any small enough interval $[0, \varepsilon]$ (see [57, Prop. 3.6]).

Ample geodesics play a crucial role in the study of equivalence of costs because they are the geodesics characterized by their jets. Let us precise this fact. Fix a nonnegative integer k. For a given curve $\gamma : I \to M$, $I \subset \mathbb{R}$, denote by $j_{t_0}^k\gamma$ the k-jet of γ at the point t_0. Given $q \in M$, we denote by $J_q^k(L)$ the space of k-jets at $t = 0$ of the normal geodesics of L issued from q and parameterized by arclength. We set $J^k(L) = \bigsqcup_{q \in U} J_q^k(L)$.

Define the maps $P^k : T^*M \mapsto J^k(L)$, by

$$P^k(\lambda) = j_0^k \pi(e^{t\vec{h}}\lambda).$$

The properties of the map P^k near a point λ can be described in terms of the kth extension $J_\lambda^{(k)}$ of the Jacobi curve. Let us denote by $\left(J_\lambda^{(k)}\right)^\angle$ the skew-symmetric complement of $J_\lambda^{(k)}$ with respect to the symplectic form σ_λ on $T_\lambda(T^*M)$, i.e.,

$$\left(J_\lambda^{(k)}\right)^\angle = \left\{v \in T_\lambda(T^*M) \ : \ \sigma_\lambda(v, w) = 0 \quad \forall w \in J_\lambda^{(k)}\right\}.$$

Lemma 1.36. *For any integer $k \geq 0$, the kernel of the differential of the map P^k at a point λ satisfies*

$$\ker dP^k(\lambda) \subset \left(J_\lambda^{(k)}\right)^\angle.$$

Proof. Let $\lambda \in T^*M$ and fix a canonical system of coordinates on T^*M near λ. In particular, in such coordinates π is a linear projection.

Let v be a vector in $\ker dP^k(\lambda)$. Then there exists a curve $s \mapsto \lambda_s$ in T^*M such that $\lambda_0 = \lambda$, $\frac{d\lambda_s}{ds}\big|_{s=0} = v$, and the following equalities holds in the fixed coordinate system:

$$\frac{\partial^{l+1}}{\partial t^l \partial s}\left(\pi \circ e^{t\vec{h}}\lambda_s\right)\Big|_{(t,s)=(0,0)} = d\pi \circ \frac{d^l}{dt^l}\left(e_*^{t\vec{h}} v\right)\Big|_{t=0} = 0 \qquad \forall\ 0 \leq l \leq k. \tag{1.7}$$

Consider now $w \in J_\lambda^{(k)}$. Then there exists an integer j, $0 \leq j \leq k$, and a vertical vector field Y (i.e., $d\pi \circ Y = 0$) on T^*M such that w writes as

$$w = \frac{d^j}{dt^j}\left(e_*^{-t\vec{h}} Y(e^{t\vec{h}}\lambda)\right)\Big|_{t=0}.$$

We have

$$\begin{aligned}\sigma_\lambda(v, w) &= \sigma_\lambda\left(v, \frac{d^j}{dt^j}\left(e_*^{-t\vec{h}} Y(e^{t\vec{h}}\lambda)\right)\Big|_{t=0}\right), \\ &= \frac{d^j}{dt^j}\left(\sigma_\lambda\left(v, e_*^{-t\vec{h}} Y(e^{t\vec{h}}\lambda)\right)\right)\Big|_{t=0}.\end{aligned} \tag{1.8}$$

The last equality holds, because we work with the fixed bilinear form σ_λ on the given vector space $T_\lambda T^*M$.

Using now the fact that $e^{t\vec{h}}$ is a symplectomorphism, we obtain

$$\sigma_\lambda(v, w) = \frac{d^j}{dt^j}\left(\sigma_{e^{t\vec{h}}\lambda}\left(e_*^{t\vec{h}} v, Y(e^{t\vec{h}}\lambda)\right)\right)\Big|_{t=0}$$

So far, all equalities starting from (1.8) were obtained in a coordinate-free manner. Now use again the fixed canonical coordinate system on T^*M near λ. In these coordinates, the form σ is in the Darboux form. In particular, the coefficients of this form are constants. Therefore,

$$\sigma_\lambda(v,w) = \sum_{l=1}^{j} \binom{j}{l} \sigma_\lambda(v_l, w_l),$$
$$\text{where} \quad v_l = \frac{d^l}{dt^l}\left(e_*^{t\vec{h}} v\right)\Big|_{t=0} \quad \text{and} \quad w_l = \frac{d^{j-l}}{dt^{j-l}}\left(Y(e^{t\vec{h}}\lambda)\right)\Big|_{t=0}.$$

By (1.7), every vector v_l is vertical. The vectors w_l in the chosen coordinate system are vertical as well since the vector field Y is vertical. As a consequence, $\sigma_\lambda(v_l, w_l) = 0$, which implies $\sigma_\lambda(v,w) = 0$. This completes the proof. □

Remark 1.37. When the Jacobi curve is equiregular (i.e., the dimensions $\dim J^{(k)}_{\lambda(t)}$, $k \in \mathbb{N}$, are constant for t close to 0), the skew-symmetric complement of the kth extension is equal to the kth contractions $J_\lambda^{(-k)}$ of the Jacobi curve (see [60, Lemma 1]). In that case we can show the equality $\ker dP^k(\lambda) = J_\lambda^{(-k)}$.

Since $\dim \left(J_\lambda^{(k)}\right)^{\angle} = 2n - \dim J_\lambda^{(k)}$, we get as a corollary of Lemma 1.36 that ample geodesics are characterized locally by their k-jets for k large enough.

Corollary 1.38. *Let $\lambda \in T^*M$ be ample. Then there exists an integer k_0 such that the map P^{k_0} is an immersion at λ.*

1.3.3 Orbital diffeomorphism on ample geodesics

Consider two Lagrangians L_1 and L_2 from $\mathcal{L}$. We denote by h_1 and h_2 the respective Hamiltonians of L_1 and L_2.

Definition 1.39. We say that $\vec{h}_1$ and $\vec{h}_2$ are *orbitally diffeomorphic* on an open subset V_1 of T^*M if there exists an open subset V_2 of T^*M and a diffeomorphism $\Phi : V_1 \to V_2$ such that Φ is fiber-preserving, i.e. $\pi(\Phi(\lambda)) = \pi(\lambda)$, and Φ sends the integral curves of $\vec{h}_1$ to the integral curves of $\vec{h}_2$, i.e. $\Phi\big(e^{t\vec{h}_1}\lambda\big) = e^{t\vec{h}_2}\big(\Phi(\lambda)\big)$ for all $\lambda \in V$ and $t \in \mathbb{R}$ for which $e^{t\vec{h}_1}\lambda$ is well defined, or, equivalently

$$d\Phi \circ \vec{h}_1(\lambda) = \vec{h}_2(\Phi(\lambda)). \tag{1.9}$$

The map Φ is called an *orbital diffeomorphism* between the extremal flows of L_1 and L_2.

Proposition 1.40. *If $\vec{h}_1$ and $\vec{h}_2$ are orbitally diffeomorphic on a neighborhood of $\pi^{-1}(q_0)$, then L_1, L_2 are equivalent via geodesics at q_0.*

Proof. If $\vec{h}_1$ and $\vec{h}_2$ are orbitally diffeomorphic, then the relation $\Phi\big(e^{t\vec{h}_1}\lambda\big) = e^{t\vec{h}_2}\big(\Phi(\lambda)\big)$ implies that any normal geodesics of L_2 near q_0 satisfies

$$\pi(e^{t\vec{h}_2}\lambda) = \pi \circ \Phi\big(e^{t\vec{h}_1}\big(\Phi^{-1}(\lambda)\big)\big) = \pi \circ e^{t\vec{h}_1}\big(\Phi^{-1}(\lambda)\big),$$

and thus coincides with a normal geodesic of L_1. Since on the other hand abnormal geodesics always coincide, the Lagrangians L_1, L_2 have the same geodesics near q_0.

□

We have actually a kind of converse statement near ample geodesics.

Proposition 1.41. *Assume that the Lagrangians L_1 and L_2 are equivalent via geodesics at q_0. Then, for any $\lambda_1 \in \pi^{-1}(q_0)$ ample with respect to L_1, $\vec{h}_1$ and $\vec{h}_2$ are orbitally diffeomorphic on a neighborhood V_1 of λ_1 in T^*M.*

Proof. Assume that U is a neighborhood of q_0 such that L_1 and L_2 have the same geodesics in U. Then L_1 and L_2 have the same ample geodesics in U. Indeed, a geodesic $\gamma(t) = \pi(e^{t\vec{h}_1}\lambda)$ of L_1 which is ample at $t = 0$ is a geodesics of L_2 as well by assumption, and moreover a normal one since ample geodesics are not abnormal. The conclusion follows then from the fact that being ample at $t = 0$ with respect to L_1 is a property of $\gamma(t) = \pi(e^{t\vec{h}_1}\lambda)$ as an admissible curve (see [57, Proposition 6.15]), and does not depend on the Hamiltonian vector field.

Fix a nonnegative integer k. As in Subsection 1.3.2, for $q \in U$ and $i = 1, 2$, we denote by $J^k_q(L_i)$ the space of k-jets at $t = 0$ of the normal geodesics of L_i issued from q. We set $J^k(L_i) = \bigsqcup_{q\in U} J^k_q(L_i)$ and we define $P^k_i : T^*M \mapsto J^k(L_i)$ by

$$P^k_i(\lambda) = j^k_0 \pi(e^{t\vec{h}_i}\lambda).$$

Let $\lambda_1 \in T^*M \cap \pi^{-1}(q_0)$ be an ample covector with respect to L_1. Then by Corollary 1.38 for a large enough integer k there exists a neighborhood V_1 of λ_1 in T^*M such that the map $P^k_1|_{V_1}$ is a diffeomorphism on its image. Up to reducing V_1 we assume that $\pi(V_1) \subset U$ and that every $\lambda \in V_1$ is ample. As a consequence, every geodesic $\pi(e^{t\vec{h}_1}\lambda)$ with $\lambda \in V_1$ is an ample geodesic with respect to L_2.

Let $\lambda_2 \in \pi^{-1}(q_0)$ be the covector such that the curves $\pi(e^{t\vec{h}_1}\lambda_1)$ and $\pi(e^{t\vec{h}_2}\lambda_2)$ coincide (λ_2 is unique since an ample geodesic is not abnormal). Since λ_2 is ample with respect to L_2, the same argument as above shows that there exists a neighborhood V_2 of λ_2 such that $P^k_2|_{V_2}$ is a diffeomorphism on its image. Up to reducing V_1 and V_2 if necessary, we have a diffeomorphism $\Psi_k : P^k_1(V_1) \subset J^k(L_1) \to P^k_2(V_2) \subset J^k(L_2)$. Thus the map Φ which completes the following diagram into a commutative one,

$$\begin{tikzcd}
V_1 \arrow[r, "\Phi"] \arrow[d, "P^k_1"'] & V_2 \arrow[d, "P^k_2"] \\
P^k_1(V_1) \subset J^k(L_1) \arrow[r, "\Psi_k"] & P^k_2(V_2) \subset J^k(L_2)
\end{tikzcd}$$

defines an orbital diffeomorphism between V_1 and V_2. This completes the proof.

□

Remark 1.42. We have seen in the proof just above that two equivalent costs have the same set of ample geodesics. In the same way, one can prove that they have the same set of strictly normal geodesics. However we can not affirm that they have the same normal geodesics: a geodesic could be both normal and abnormal for L_1 and only abnormal for L_2.

The existence of the orbital diffeomorphism implies an additional condition on the Hamiltonian flows of h_1, h_2.

Proposition 1.43. *If h_1, h_2 are orbitally diffeomorphic and Φ is an orbital diffeomorphism between their extremal flows then the following identity holds*

$$\vec{h}_1(h_2 \circ \Phi(\lambda)) = 0. \tag{1.10}$$

Proof. Let $\lambda_1(t)$ for $t \in [0,T]$ be a trajectory of $\dot{\lambda} = \vec{h}_1(\lambda)$ and lifting of $q(t)$ to $T^*_{q(t)}M$ and $\lambda_2(t)$ be a trajectory of $\dot{\lambda} = \vec{h}_2(\lambda)$ and lifting of $q(t)$ for $t \in [0,T]$. As h_1, h_2 are orbitally diffeomorphic, the orbital diffeomorphism Φ sends λ_1 to λ_2, i.e. for any $t \in [0,T]$

$$\Phi(\lambda_1(t)) = \lambda_2(t).$$

By the property of a Hamiltonian to be constant along its extremals, we have

$$\vec{h}_1(h_2 \circ \Phi(\lambda)) = \frac{d}{dt} h_2(\Phi(\lambda_1(t))) = \frac{d}{dt} h_2(\lambda_2(t)) = \vec{h}_2(h_2(\lambda)) = 0, \qquad t \in [0,T]\,.$$

This ends the proof. □

The roles of the Hamiltonians h_1, h_2 can be interchanged and thus the following identity holds as well

$$\vec{h}_2(h_1 \circ \Phi^{-1}(\lambda)) = 0.$$

This implies the existence of nontrivial fist integrals of the Hamiltonian systems associated with h_1, h_2.

1.3.4 Product structure and symmetries of Jacobi curves

Let us come back to the example of Subsection 1.1.4. The Lagrangians $L, \tilde{L}$ are equivalent via minimizers and by Lemma 1.19 they are equivalent via geodesics. Let us denote the Hamiltonians corresponding to $L, \tilde{L}$ by $h, \tilde{h}$ respectively. Both Hamiltonians $h, \tilde{h}$ can be written in the local coordinates (y_1, y_2, p_1, p_2) as a sum

$$h(y_1, y_2, p_1, p_2) = h_1(y_1, p_1) + h_2(y_2, p_2) \text{ and } \tilde{h}(y_1, y_2, p_1, p_2) = \tilde{h}_1(y_1, p_1) + \tilde{h}_2(y_2, p_2).$$

As a consequence, each extremal $\lambda(t) = e^{t\vec{h}}\lambda$ is a product of two extremals $\lambda(t) = (\lambda_1(t), \lambda_2(t))$, where $\lambda_i(t) = e^{t\vec{h}_i}\lambda_i$ for $i = 1, 2$ and $\lambda = (\lambda_1, \lambda_2)$. The same holds for the extremal $\tilde{\lambda}(t) = e^{t\vec{\tilde{h}}}\tilde{\lambda}$ with $\tilde{\lambda}_1, \tilde{\lambda}_2$ lying in the same spaces as λ_1, λ_2 respectively. Notice also that the Jacobi curve $J_\lambda(t)$ associated with λ decomposes in the product of two Jacobi curves as well

$$J_\lambda(t) = J^1_{\lambda_1}(t) \times J^2_{\lambda_2}(t),$$

for t such that all objects are well defined. By Theorem 1.41, there exists an orbital diffeomorphism Φ which sends the extremals of h to the extremals of $\tilde{h}$. By construction, Φ admit a product structure as well $\Phi = (\Phi_1, \Phi_2)$. Each Φ_i sends the extremals of h_i to the extremals of $\tilde{h}_i$ for $i = 1, 2$.

In this example we start from the optimal control problem which is a product of two optimal control sub-problems. In this case we say that the optimal control problem admits a product structure. It is clear that not all optimal control problems admit a product structure. In this regard, the following question needs to be answered. Does the optimal control problem always admit a product structure if the corresponding cost function admit non trivially equivalent costs? We are also interested in the form of the orbital diffeomorphism. Does a non trivial orbital diffeomorphism always admit a product structure? Yet, we obtained the positive answers to the both questions in the Linear-Quadratic case and the sub-Riemannian case of Carnot groups. The positive answers were already known in Riemannian case and contact and quasi-contact sub-Riemannian cases. We conjecture that the product structure is necessary to have non-trivially equivalent costs. So, the next step should be to generalize the obtained results to other cases of optimal control problems. Notice that if there is a product structure in the optimal control problem, then the Jacobi curve admits the product structure as in the example above.

To conduct the analysis of the stated questions, we may change the framework. In the Subsection 1.3.3 the existence of an orbital diffeomorphism is proved using the notion of Jacobi curves. Actually, the relation between the two objects can be seen from another point of view. The orbital diffeomorphism preserves the Jacobi curves in the following sense. Let $J_\lambda(t), \widetilde{J}_\lambda(t)$ be two Jacobi curves corresponding to the geodesically equivalent Lagrangians $L, \widetilde{L}$ respectively. Assume that Φ is an orbital diffeomorphism which sends the normal extremals of L to the normal extremals of $\widetilde{L}$. The Jacobi curves $J_\lambda(t), \widetilde{J}_\lambda(t)$ are related as follows

$$d\Phi \circ J_\lambda(t) = \widetilde{J}_{\Phi(\lambda)}(t).$$

This can be obtained by differentiation the following equality

$$\Phi(\lambda) = e^{t\vec{\tilde{h}}} \circ \Phi \circ e^{-t\vec{h}}(\lambda)$$

and taking into account

$$d\Phi \circ \mathcal{V}(e^{t\vec{\tilde{h}}}\lambda) = \mathcal{V}(e^{t\vec{h}} \circ \Phi(\lambda)).$$

Thus, in place of studying the orbital diffeomorphism as it is defined now we can study the transformations which send a Jacobi curve of a given cost to the Jacobi curve of an equivalent costs.

Assume for the moment that the Jacobi curves corresponding to the same geodesic are the same, that is $J_\lambda(t) = \widetilde{J}_{\widetilde{\lambda}}(t)$, which is the case for small times due to [57, Proposition 6.15]. In this case the differential of an orbital diffeomorphism represent a symmetry of the Jacobi curves. So we are reduced to study the set of symmetries of Jacobi curves. In this case one can expect that the set of symmetries for a given Jacobi curve is non-trivial if and only if the Jacobi curve admits a product structure. This conjecture is in accordance with the example above. This may be another approach to study the cases of optimal control problems which admit the geodesically equivalent costs. Study of this problem will permit to in particular to understand the structure of the injective cases.

1.4 Geometric inverse problem

The inverse optimal control problem can be stated in more general way if we consider as the data a set Γ of geometric curves γ without taking into account their parametrization. This gives rise to a different inverse problem but the injectivity can be studied in the same way. In this case the inverse problem is stated as follows. Given a dynamics (1.1) and a set Γ of geometric curves we should find a cost J such that for every $\gamma \in \Gamma$ there exists $T > 0$ and a parametrization making the trajectory $\gamma(\cdot) : [0, T] \to M$ a solution of the optimal control problem (1.3) associated with $q_0 = \gamma(0)$ and $q_1 = \gamma(T)$, where $\gamma(0), \gamma(T)$ are the endpoints of γ. We will call the inverse problem stated on unparametrized curves a *geometric inverse problem.*

Notice that the two inverse problems, the inverse optimal control problems and the geometric inverse problem, are different in general as there exist different parametrizations of a curve such that the obtained trajectories are optimal. Nevertheless, the two inverse problems coincide in some cases. They coincide in particular when there is a unique parametrization of a given geometric curve that makes this curve an admissible trajectory. The sufficient condition for the two inverse problems to coincide is the following. If given any realizable curve γ and some its parametrization $\gamma(t),\ t \in [0, T]$ there exist a unique real-valued $\alpha(\cdot)$ such that

$$\dot{\gamma} \in \{\alpha(\gamma) f(\gamma, u) : u \in U\},$$

then the two inverse problem on parametrized curves and the inverse problem on unparameterized curves have the same solutions. This property depends on the control system (1.1) and can be checked by verifying the non-existence of $\alpha \neq 1$

such that at some points $q \in M$ the following equality holds

$$\alpha f(q,u) = f(q,\tilde{u}), \quad u, \tilde{u} \in U.$$

Even if there exists a nontrivial $\alpha(\cdot)$ the further analysis should be conducted. The possibility of reparameterization of minimizers of a cost in $\mathcal{J}$ need to be verified. To admit a reparameterization of trajectories is a strong condition on a control system. For example, in control-affine case $f(q,u) = f_0 + \sum_{i=1}^m u_i f_i$ when the drift f_0 is linearly independent from $f_1, \dots, f_m$ the inverse problems on parametrized and unparametrized curves coincide. Yet this is not always the case, in particular, in the sub-Riemannian case reparameterizations are admissible.

The question of injectivity of the geometric inverse problem induces the notion of equivalent costs just as in the case of the inverse optimal control problem.

Definition 1.44. We say that two costs J and $\tilde{J}$ are *equivalent via unparameterized minimizers* if the optimal synthesis associated with J and $\tilde{J}$ coincide as the set of geometric curves.

And in the same way as before we can consider the equivalence on geodesics.

Definition 1.45. We say that two costs J and $\tilde{J}$ are *equivalent via unparameterized geodesics* if the set of geodesics associated with J and $\tilde{J}$ coincide as the set of geometric curves.

In the Riemannian case both notions of equivalence via parametrized geodesics and via unparameterized geodesics have been studied since the end of the 19th century. The analogue of equivalence via parametrized geodesics is called affine equivalence and the analog of equivalence via unparameterized geodesics is called projective equivalence. The structure of the equivalent costs was completely understood in this case. Levi-Civita's theorem [48] shows that in case of affine equivalence we observe the product structure and in the case of projective equivalence the situation is more complicated but very similar. Let us consider an example which shows that the affine and projective equivalences do not coincide.

Let us consider the Riemannian case of the optimal control problem with $M = \mathbb{R}$. Any metric on $\mathbb{R}$ is of the form $g_x = \alpha(x)dx^2$ and therefore all metrics are conformal. The geodesics are trajectories $x(\cdot)$ such that $\alpha(x(\cdot))\left(\dot{x}(\cdot)\right)^2$ is constant on $[0,T]$. Two different metrics g_1, g_2 define the same geodesics as parametrized curves if and only if the coefficients $\alpha_1(\cdot), \alpha_2(\cdot)$ corresponding to g_1, g_2 respectively are constantly proportional. Thus, there is no non-trivially affinely equivalent metric in this case. On the other hand, for any pair of initial and final points (x_0, x_T) there is only one curve connecting them, so curves are the same for all metrics and thus all metrics on

$\mathbb{R}$ are projectively equivalent. In particular, there exit infinitely many non-trivially projectively equivalent metrics.

To treat the injectivity problem on cost equivalence via unparameterized geodesics we can use the same method via orbital diffeomorphism introduced before but with some modifications. To take into account the possible reparameterization of the curves, the orbital diffeomorphism should be defined as follows.

Consider two costs J_1, J_2 and the corresponding respective Hamiltonians h_1, h_2.

Definition 1.46. We call a map Φ a *projective orbital diffeomorphism* between the extremal flows of J_1 and J_2 and say that $\vec{h}_1$ and $\vec{h}_2$ are *projectively orbitally diffeomorphic* on an open subset V_1 of T^*M if there exists an open subset V_2 of T^*M such that $\Phi : V_1 \to V_2$ is a fiber-preserving diffeomorphism, and Φ sends the integral curves of $\vec{h}_1$ to the integral curves of $\vec{h}_2$, i.e. $\Phi\big(e^{t\vec{h}_1}\lambda\big) = e^{s\vec{h}_2}\big(\Phi(\lambda)\big)$ for all $\lambda \in V$ and $t \in \mathbb{R}$ for which $e^{t\vec{h}_1}\lambda$ is well defined, with $s = s(\lambda, t)$ the reparameterization of the integral curve of h_2 with respect to the integral curve of h_1. Equivalently, it writes as

$$d\Phi \circ \vec{h}_1(\lambda) = \alpha(\lambda)\vec{h}_2(\Phi(\lambda)). \tag{1.11}$$

In the condition (1.11) on the differential of Φ we take into account that it is allowed to send the vector field $\vec{h}_1$ to some vector field proportional to $\vec{h}_2$ and α is the proportionality coefficient. Note that the relation between s and α is $\alpha(\lambda) = \frac{ds}{dt}(\lambda, 0)$.

Chapter 2

Linear-Quadratic case

Inverse optimal control problem was first stated in the infinite horizon linear-quadratic case [15]. Since then the problem was generalized to other cases. In case of linear dynamics and quadratic cost, the non-autonomous problem was considered in [16]. In the finite horizon autonomous case it was first considered in [61]. In their work, authors propose a parameter identification method for reconstructing cost-functions in some canonical class. Adapting their idea we propose a canonical class of linear-quadratic problems for which the orbital diffeomorphism permits to recover the structure of the injective and non-injective cases. This structure and some algebraic equations give rise to an algorithm of cost reconstruction in both injective and non-injective cases.

2.1 Direct problem

Let us consider a linear control system

$$\dot{x} = Ax + Bu, \qquad x \in \mathbb{R}^n,\ u \in \mathbb{R}^m, \tag{2.1}$$

where A is a $(n \times n)$ constant real-valued matrix and B is a $(n \times m)$ constant real-valued matrix and we assume $m < n$. We make the following assumptions on the matrices A, B.

Assumptions 1. *The matrices* (A, B) *defining the dynamics* (2.1) *satisfy:*

A1. B *is of rank* m*;*

A2. (A, B) satisfy the Kalman controllability condition

$$\operatorname{span}\{B, AB, \dots, A^{n-1}B\} = \mathbb{R}^n. \tag{A2}$$

By these assumptions, for any final time $T > 0$, initial point x_0 and final point x_1, there exists an admissible trajectory x_u satisfying (2.1) such that $x_u(0) = x_0$ and $x_u(T) = x_1$.

Every real-valued function

$$L(x, u) = x^\top Qx + 2x^\top Su + u^\top Ru, \tag{2.2}$$

where Q is a $(n \times n)$ matrix, S is a $(n \times m)$ matrix and R is a $(m \times m)$ matrix, defines a class of linear-quadratic optimal control problems: given a time $T > 0$, an initial point x_0 and a final point x_F, minimize the quadratic cost

$$J(x_u) = \int_0^T L(x_u(t), u(t))dt$$

among all trajectories $x_u(\cdot)$ of (2.1) satisfying $x(0) = x_0$ and $x(T) = x_F$.

We need supplementary assumptions on L to ensure that it belongs to the class $\mathcal{L}$ from the Section 1.3 and the necessary properties for applying the framework of the Chapter 1 are satisfied (existence of minimizers, strictly normal geodesics, etc.). In addition we will impose an assumption which permits to express the solutions of optimal control problem in a nice form as will be shown later.

We make the following assumptions on the quadratic cost.

Assumptions 2. *The matrices (Q, S, R) defining the Lagrangian L satisfy:*

B1. $Q = Q^\top \geq 0, \quad R = R^\top > 0, \quad Q - SR^{-1}S^\top \geq 0$;

B2. the matrix
$$\begin{pmatrix} A - BR^{-1}S^\top & BR^{-1}B^\top \\ Q - SR^{-1}S^\top & -A^\top + SR^{-1}B^\top \end{pmatrix}$$
has no eigenvalues on the imaginary axis.

We call an element of the associated class of optimal control problem a linear-quadratic problem or just a *LQ problem.*

Remark 2.1. Under these assumptions, $L(x, u)$ is strictly convex and of super-linear growth with respect to u. Therefore, a Lagrangian (2.2) satisfying the Assumptions 2 belongs to the class $\mathcal{L}$.

Remark 2.2. Notice that in the case of linear control system a reparametrization of the trajectories is not admissible. Suppose, there exists a curve γ which admits two different parametrization. Let $x_u, x_{\tilde{u}}$ be two trajectories realizing different parametrizations of γ on time interval $[0,T]$. Then there exists $\alpha(t) \not\equiv 1$ for any $t \in [0,T]$ such that for any $x \in \gamma$

$$\dot{x} = Ax + B\tilde{u} = \alpha\left(Ax + Bu\right).$$

This implies that $Ax \in \mathrm{Im}B$ and thus $\dot{x} \in \mathrm{Im}B$ on $[0,T]$. The space $\mathrm{Im}B$ is of dimension $m < n$ and the curve γ belongs to $x(0) + \mathrm{Im}B$. On the other hand, the system is assumed to be controllable and therefore for any $x(0)$ and $x(T) \notin \mathrm{Im}B$ there exists a trajectory joining the two points and these trajectories do not admit reparametrization. The reparametrizable trajectories belong to an affine space of codimansion more than 1 and therefore are negligible for the inverse problem. We conclude that the geometric inverse problem in this case coincides with the inverse optimal control problem on almost all trajectories and the others can be excluded from the consideration.

2.1.1 Properties of the optimal control class

The optimal control problem with control system (2.1) satisfying the Assumptions 1 and the cost with quadratic Lagrangian (2.2) satisfying the Assumptions 2 admits a minimizing solution for any final time and pair of initial and final points, as the condition $\mathcal{H}1$ is satisfied in this case and the Theorem 1.27 can be applied. Under the Assumption (A2) and the Assumption B1, the normal Hamiltonian is well defined and moreover all geodesics are strictly normal [54, Chapter 16]. Each geodesic joining two fixed points in a fixed time is globally minimizing and unique (see [62, Theorem A] or [63, Lemma 4]). The question of existence of ample geodesics is trivial in this case, it results from [57, Proposition 3.12] that any geodesic is ample. Let us put together all the results mentioned above in the following statement.

Proposition 2.3. *Under the Assumptions 1 on the control system* (2.1) *and the Assumptions 2 on the Lagrangian* (2.2)*, for any final time $T > 0$, initial point x_0 and final point x_F, there exists a unique minimizing geodesic $x(\cdot)$ such that $x(0) = x_0$ and $x(T) = x_F$, and this geodesic is ample.*

2.1.2 Characterization of the optimal synthesis

Linear-quadratic optimal control problems admit analytic optimal solutions. We will use the characterization of the solutions given in [63]. Consider the algebraic Riccati equation

$$PA + A^\top P - (S + PB)R^{-1}(S^\top + B^\top P) + Q = 0, \tag{2.3}$$

with unknown $(n \times n)$ symmetric real-valued matrix P. This equation admits a unique solution $P_+ \geq 0$ and a unique solution $P_- \leq 0$ such that the matrices

$$A_+ = A - BK_+ \quad \text{and} \quad A_- = A - BK_-, \tag{2.4}$$

where $K_+ = R^{-1}(S^\top + B^\top P_+)$ and $K_- = R^{-1}(S^\top + B^\top P_-)$, are asymptotically stable and asymptotically anti-stable respectively. Then the minimizing solution $x(\cdot)$ of the optimal LQ control problem defined by (2.1), (2.2) is given by

$$x(t) = e^{tA_+}y_+ + e^{tA_-}y_-, \quad t \in [0, T], \tag{2.5}$$

where the vectors y_-, y_+ are the unique solutions of the system

$$\begin{cases} x_F = e^{TA_+}y_+ + e^{TA_-}y_-, \\ x_0 = y_+ + y_-. \end{cases} \tag{2.6}$$

2.2 Inverse problem

The inverse problem for the defined class of linear-quadratic problems has nice properties. The minimizing solutions are defined globally on $\mathbb{R}^n$ and the local optimality conditions are valid globally, therefore, to consider global or local inverse problem is the same. Moreover, in this case any geodesic is minimizing and thus, the equivalence via minimizers and via geodesics define the same equivalence relation on the costs. This permits to define a unique equivalence relation.

In the context of the LQ problem we will consider both problems of injectivity and of cost reconstruction which is the main objective in the inverse problem.

2.2.1 Injectivity and cost equivalence

Let us fix matrices (A, B) which define a control system (2.1) and satisfy Assumptions 1. In this chapter we call $\mathcal{J}_{LQ}$ the set of costs of the form (2.2) satisfying Assumptions 2.

Definition 2.4. We say that two costs $J, \tilde{J}$ in $\mathcal{J}_{LQ}$ are equivalent, and we write $J \sim \tilde{J}$, if they define the same optimal synthesis. We say that two Lagrangians $L, \tilde{L}$ are equivalent $L \sim \tilde{L}$ if the corresponding costs are equivalent.

The algebraic form of the minimizing trajectories implies directly that an optimal synthesis is completely characterized by the pair of $(n \times m)$ matrices K_+, K_-, or equivalently by the pair (A_+, A_-) since the matrix B is injective. We prove now that this characterization is univocal.

Lemma 2.5. *Two equivalent costs define the same pair of matrices* (A_+, A_-). *In other terms, given an optimal synthesis, there exists a unique pair of matrices* (A_+, A_-) *such that any trajectory in the synthesis satisfies* (2.5).

Proof. Consider two equivalent costs J defined by the matrices (Q, S, R) and $\tilde{J}$ by $(\tilde{Q}, \tilde{S}, \tilde{R})$. The two corresponding pairs (A_+, A_-) and $(\tilde{A}_+, \tilde{A}_-)$ define the same minimizing solutions.

Fix $T > 0$. For $i = 1, \dots, n$, let $x_i(\cdot)$ be the minimizing solution between e_i and $e^{TA_+}e_i$, where e_i denotes the ith vector of the canonical basis of $\mathbb{R}^n$. By uniqueness of the solutions of system (2.6), $x_i(t) = e^{tA_+}e_i$. In matrix form $X(t) = (x_1(t) \cdots x_n(t)) = e^{tA_+}$, $t \in [0, T]$.

Now, since $J \sim \tilde{J}$, there exist $(n \times n)$ matrices Y_+, Y_- such that $X(t) = e^{tA_+} = e^{t\tilde{A}_+}Y_+ + e^{t\tilde{A}_-}Y_-$ for $t \in [0, T]$. By analyticity, there holds

$$\|e^{tA_+}\| = \|e^{t\tilde{A}_+}Y_+ + e^{t\tilde{A}_-}Y_-\| \quad \text{for any } t \in [0, +\infty)\,.$$

As $t \to \infty$, $\|e^{A_+t}\| \to 0$ since A_+ is stable, and hence $Y_- = 0$. As a consequence, $e^{tA_+} = e^{t\tilde{A}_+}Y_+$. Now it is sufficient to notice that $X(0) = Y_+ = I$, hence

$$A_+ = \tilde{A}_+.$$

By exchanging the role of A_+ and A_- and taking $t \to -\infty$, we obtain in the same way that $A_- = \tilde{A}_-$. □

2.2.2 Canonical classes

To address the problem of injectivity, we will reduce the inverse problem to a special class of canonical costs containing a representative of each class of equivalence. The idea of restriction to some smaller classes was proposed first in [61], we will further develop the idea and construct a new class of LQ problems.

Let us define a class of costs containing in $\mathcal{J}_{LQ}$ such that each cost $J \in \mathcal{J}_{LQ}$ is equivalent to some cost in the new class.

Lemma 2.6. *The Lagrangian* L *of the form* (2.2) *is equivalent to*

$$\tilde{L} = (u + K_+x)^\top R(u + K_+x).$$

Proof. Given T, x_0, x_F, let $x^*(\cdot)$ be the solution of $\min \int_0^T L(x, u)$ between x_0 and x_F. Clearly, $x^*(\cdot)$ minimizes as well the cost

$$\int_0^T L(x(t), u(t))dt + x_F^\top P_+ x_F - x_0^\top P_+ x_0.$$

Since the constant term in the cost above can be written in integral form as

$$x_F^\top P_+ x_F - x_0^\top P_+ x_0 = \int_0^T 2x^T P_+(Ax + Bu)dt,$$

$x^*(\cdot)$ minimizes $\int_0^T \tilde{L}(x,u)$, where

$$\tilde{L} = x^\top(P_+A + A^T P_+ + Q)x + 2x^\top(S + P_+B)u + u^T Ru.$$

Using the fact that P_+ is a solution of the Riccati equation we get $S + P_+B = K_+^\top R$ and

$$\begin{aligned} P_+A + A^T P_+ &= (S + P_+B)R^{-1}(S^\top + B^T P_+) - Q \\ &= K_+^\top R K_+ - Q. \end{aligned}$$

Putting all together we obtain

$$\tilde{L} = (u + K_+x)^\top R(u + K_+x).$$

We conclude that any minimizer of L is also a minimizer of $\tilde{L}$, which ends the proof. □

This result leads us to introduce the following class of quadratic costs.

Definition 2.7. A *canonical cost* is a cost defined by a Lagrangian of the form

$$L = (u + Kx)^\top R(u + Kx),$$

where R is a symmetric positive definite matrix with determinant equal to 1 and K is a stabilizing matrix, i.e., $A - BK$ is asymptotically stable.

Proposition 2.8. *Any cost J satisfying Assumption 2 is equivalent to a canonical cost $\tilde{J}$. Moreover the matrix K_+ associated with $\tilde{J}$ is $K_+ = K$ (equivalently, $A_+ = A - BK$).*

Proof. From Lemma 2.6, any cost J satisfying Assumption 2 is equivalent to a cost with Lagrangian $\tilde{L} = (u+Kx)^\top R(u+Kx)$, where R is a symmetric positive definite matrix and K is a stabilizing matrix. Since two proportional cost are equivalent and $\det R > 0$, we can assume moreover that $\det R = 1$, which proves the first statement of the lemma.

We are left to prove that the matrix A_+ associated with $\tilde{J}$ is equal to $A - BK$. Fix $T > 0$. For $i = 1, \dots, n$, the minimizing solution between e_i and $e^{T(A-BK)}e_i$ is equal to $x_i(t) = e^{t(A-BK)}e_i$, since the corresponding control $u_i = -Kx_i$ satisfies $\tilde{L}(x_i(t), u_i(t)) \equiv 0$ and the minimizing solution is unique. Let us write in matrix form $X(t) = (x_1(t) \cdots x_n(t)) = e^{t(A-BK)}$, $t \in [0,T]$. Now, from (2.5) there exists $(n \times n)$ matrices Y_+, Y_- such that $X(t) = e^{tA_+}Y_+ + e^{tA_-}Y_-$ for $t \in [0,T]$. Arguing as in the proof of Lemma 2.5 we conclude that $A_+ = A - BK$, which ends the proof. □

2.2.3 Reduced inverse problem

We formulate a reduced inverse optimal control problem as follows: *given a linear-quadratic optimal synthesis Γ, find a canonical cost J such that Γ is the optimal synthesis of J.*

Proposition 2.8 ensures that this problem always has a solution, hence we concentrate now on this reduced problem. What about the injectivity of the reduced problem?

Lemma 2.9. *Let J and $\tilde{J}$ be two canonical costs associated with (R, K) and $(\tilde{R}, \tilde{K})$ respectively. If J and $\tilde{J}$ are equivalent, then $K = \tilde{K}$.*

Proof. If $J \sim \tilde{J}$, then they define the same optimal synthesis Γ. From Lemma 2.5, Γ determines in a unique way the pair of matrices (K_+, K_-) corresponding to J and $\tilde{J}$. And Proposition 2.8 implies that $K_+ = K = \tilde{K}$. □

Corollary 2.10. *Let Γ be an optimal synthesis and (K_+, K_-) the associated pair of matrices. The corresponding reduced inverse optimal control problem is injective if and only if there exists a unique matrix R such that Γ is the optimal synthesis of the canonical cost defined by (R, K_+).*

Note that, by a simple feedback change of the control $v = u + K_+x$, we obtain that Γ is also the optimal synthesis of the optimal control problem with Lagrangian defined by $L = u^\top Ru$ and control system $\dot{x} = A_+x + Bu$. Thus reconstruction of a cost can be decomposed in two steps:

- identify the matrices (A_+, A_-) associated with the given synthesis Γ,
- find R such that Γ is the optimal synthesis of the optimal control problem with cost $L = u^\top Ru$ and control system $\dot{x} = A_+x + Bu$,

the injectivity of the problem depending on the uniqueness of the solution to the second step.

2.3 Characterization of the injective cases

As it was noted in the previous section we can reduce the analysis of injectivity to optimal LQ problems of the form

$$\min_u \int_0^T u^\top Ru \quad \text{s.t.} \quad \begin{cases} \dot{x} = Ax + Bu, \\ x(0) = x_0, \ x(T) = x_F, \end{cases} \tag{2.7}$$

where A is an asymptotically stable matrix and R is a symmetric positive definite matrix with $\det R = 1$ (as usual, the pair (A, B) is assumed to satisfy the Kalman controllability condition (A2) and rank$B = m$). In this context, we write $R \sim \tilde{R}$ if the two canonical Lagrangians $L = u^\top Ru$ and $\tilde{L} = u^\top \tilde{R}u$ are equivalent. The inverse optimal control problem associated with (2.7) is injective if $R \sim \tilde{R}$ implies $R = \tilde{R}$.

2.3.1 Product structure

It appears that a cost associated with $L = u^\top Ru$ may admit non trivial equivalent costs. Let us construct such an example.

Choose a positive integer N and N pairs of positive integers $m_i \leq n_i$, $i = 1, \dots, N$. Set $m = \sum_{i=1}^N m_i$ and $n = \sum_{i=1}^N n_i$. For $i = 1, \dots, N$, choose a controllable linear system

$$\dot{x}_i = A_i x_i + B_i u_i, \qquad x_i \in \mathbb{R}^{n_i}, \quad u_i \in \mathbb{R}^{m_i},$$

with A_i asymptotically stable and B_i of rank m_i, and a canonical Lagrangian $L_i = u_i^\top R_i u_i$. We define a linear-quadratic problem on $\mathbb{R}^n$ with control in $\mathbb{R}^m$ of the form (2.7) by setting

$$A = \begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_N \end{pmatrix}, \quad B = \begin{pmatrix} B_1 & & \\ & \ddots & \\ & & B_N \end{pmatrix},$$

$$\text{and} \quad L = \sum_{i=1}^N u_i^\top R_i u_i, \quad \text{i.e.,} \quad R = \begin{pmatrix} R_1 & & \\ & \ddots & \\ & & R_N \end{pmatrix}. \tag{2.8}$$

Obviously, a trajectory $x(\cdot)$ minimizes the cost associated with L if and only if $x(\cdot) = (x_1(\cdot), \dots, x_N(\cdot))$, where each $x_i(\cdot)$ is a minimizing solution of the problem associated with A_i, B_i, R_i. As a consequence, the Lagrangian L is equivalent to any Lagrangian

$$L_\lambda = \sum_{i=1}^N \lambda_i u_i^\top R_i u_i, \quad \text{i.e.,} \quad R_\lambda = \begin{pmatrix} \lambda_1 R_1 & & \\ & \ddots & \\ & & \lambda_N R_N \end{pmatrix},$$

where $\lambda_1, \dots, \lambda_N$ are positive real numbers satisfying $\det R_\lambda = \prod_i (\lambda_i)^{m_i} = 1$.

We can extend this construction through changes of variables.

Definition 2.11. We say that a LQ optimal control problem defined by $\dot{x} = Ax+Bu$ and $L = (u + Kx)^\top R(u + Kx)$ admits a *product structure* if there exists an integer $N > 1$ and a linear change of coordinates $\tilde{x} = Px$, $\tilde{u} = Mu + Kx$, such that in the new coordinates the problem has the form (2.8) (note that the matrix $\tilde{A}$ is conjugate to $A - BK$ in the new coordinates).

We have seen that, if a problem admits a product structure, then the corresponding inverse problem has many solutions. We will see in Section 2.3.3 that the product structure is actually a necessary and sufficient condition for non injectivity.

2.3.2 Orbital diffeomorphism

Let us consider the Hamiltonian system which drives the extremals in case of LQ problem (2.7). For $(x,p) \in \mathbb{R}^{2n}$ and $u \in \mathbb{R}^m$ the pseudo-Hamiltonian is given by

$$\mathcal{H}(x,p,u) = p^T Ax + p^T Bu - \frac{1}{2}u^T Ru,$$

and the maximizing condition gives $u = R^{-1}Bp$. The normal Hamiltonian $h(x,p)$ associated with (2.7) is, for any $(x,p) \in \mathbb{R}^{2n}$,

$$h(x,p) = p^\top Ax + \frac{1}{2}p^\top BR^{-1}B^\top p.$$

By the Pontryagin Maximum Principle (Theorem 1.11), for every minimizing solution $x(\cdot)$ of (2.7), there exists a curve $p(\cdot)$ in $\mathbb{R}^n$ such that, for any $t \in [0,T]$,

$$\begin{cases} \dot{x}(t) = Ax(t) + BR^{-1}B^\top p(t), \\ \dot{p}(t) = -A^\top p(t). \end{cases} \tag{2.9}$$

Equivalently, an extremal of the LQ problem $(x(\cdot),p(\cdot))$ is a trajectory in $\mathbb{R}^{2n}$ of the Hamiltonian vector field

$$\vec{h}(x,p) = \begin{pmatrix} A & BR^{-1}B^\top \\ 0 & -A^\top \end{pmatrix} \begin{pmatrix} x \\ p \end{pmatrix}.$$

We will show that the equivalence of costs implies a relation on extremals of the corresponding Hamiltonian systems. The relation will be expressed in terms of the orbital diffeomorphisms.

Remark 2.12. Let $L = u^\top Ru$ and $\tilde{L} = u^\top \tilde{R}u$ be two canonical Lagrangians. An *orbital diffeomorphism* associated with the LQ problem (2.7) between the extremals of R and $\tilde{R}$ is a diffeomorphism Φ defined on $\mathbb{R}^n \times \mathbb{R}^n$ which preserves the first component, i.e., $\Phi : (x,p) \mapsto (x, \Phi_2(x,p))$, and which sends the extremals $(x(\cdot),p(\cdot))$ of the optimal control problem (2.7) defined by R to the extremals $(\tilde{x}(\cdot),\tilde{p}(\cdot))$ of the optimal control problem defined by $\tilde{R}$, i.e.

$$\Phi(x(t),p(t)) = (\tilde{x}(t),\tilde{p}(t)).$$

By definition, (1.9) takes the form

$$d\Phi \circ \vec{h}(x(t),p(t)) = \vec{\tilde{h}}(\tilde{x}(t),\tilde{p}(t)). \tag{2.10}$$

Proposition 2.13. *If $L = u^\top Ru$ and $\tilde{L} = u^\top \tilde{R}u$ are equivalent, then there exists an isomorphism $D : \mathbb{R}^n \to \mathbb{R}^n$ such that $\Phi : (x,p) \mapsto (x, Dp)$ is an orbital diffeomorphism between the extremals of R and $\tilde{R}$.*

Proof. Since $R \sim \tilde{R}$, the respective minimizers $x(\cdot)$ and $\tilde{x}(\cdot)$ are equal, and so $\dot{x}(t) \equiv \dot{\tilde{x}}(t)$. Using (2.9), the respective extremal lifts $p(\cdot)$ and $\tilde{p}(\cdot)$ satisfy

$$Ax + BR^{-1}B^\top p = Ax + B\tilde{R}^{-1}B^\top \tilde{p},$$

which implies

$$BR^{-1}B^\top p \equiv B\tilde{R}^{-1}B^\top \tilde{p}.$$

Taking derivatives and using the second equation of (2.9), we obtain for $k \in \mathbb{N}$

$$BR^{-1}B^\top (A^\top)^k p = B\tilde{R}^{-1}B^\top (A^\top)^k \tilde{p}.$$

Then a multiplication by $\tilde{R}(B^\top B)^{-1}B^\top$ on the left gives

$$\tilde{R}R^{-1}B^\top (A^\top)^k p = B^\top (A^\top)^k \tilde{p}.$$

Hence, from the first n derivatives we obtain a system of linear equations

$$\left\{\begin{array}{rcl} \tilde{R}R^{-1}B^\top p &=& B^\top \tilde{p}, \\ \tilde{R}R^{-1}B^\top A^\top p &=& B^\top A^\top \tilde{p}, \\ &\vdots& \\ \tilde{R}R^{-1}B^\top (A^\top)^{n-1} p &=& B^\top (A^\top)^{n-1} \tilde{p}. \end{array}\right. \tag{2.11}$$

Let $C = \begin{pmatrix} B & AB & \cdots & A^{n-1}B \end{pmatrix}$ be the controllability matrix. By controllability assumption, C is of rank n. Denote by M the block-diagonal $(nm \times nm)$ matrix that has n copies of $\tilde{R}R^{-1}$ on the diagonal. System (2.11) can be written as $C^\top \tilde{p} = MC^\top p$, and thus $\tilde{p} = Dp$ with $D = (CC^\top)^{-1}CMC^\top$. This matrix D is invertible and the map $(x,p) \mapsto (x, Dp)$ sends the extremals of the optimal control problem defined by R to the extremals of the one defined by $\tilde{R}$. Therefore, $\Phi(x,p) = (x, Dp)$ is an orbital diffeomorphism between the extremals of R and $\tilde{R}$. □

2.3.3 Injectivity condition

Proposition 2.14. *The Lagrangian L associated with* (2.7) *admits a nonequal equivalent Lagrangian if and only if the optimal control problem* (2.7) *admits a product structure.*

Proof. Let $L = u^\top Ru$ and $\tilde{L} = u^\top \tilde{R}u$ be two nonequal equivalent Lagrangians. Since R and $\tilde{R}$ are symmetric positive definite, there exists a change of coordinates

$u \mapsto v = Pu$ on $\mathbb{R}^m$ such that in the new coordinates $L(v) = v^\top v$ corresponds to the identity matrix I and $\tilde{L}(v) = \sum_i \lambda_i v_i^2$ corresponds to the diagonal matrix Λ with positive diagonal coefficients λ_i. Hence, up to replacing B by BP, we can assume that $\Lambda \sim I$.

By Proposition 2.13, there exists a linear orbital diffeomorphism $(x, p) \mapsto (x, Dp)$ between the extremals of I and Λ. This diffeomorphism satisfies (2.10), which writes as

$$\begin{pmatrix} I & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} A & BB^\top \\ 0 & -A^\top \end{pmatrix} \begin{pmatrix} x \\ p \end{pmatrix} = \begin{pmatrix} A & B\Lambda^{-1}B^\top \\ 0 & -A^\top \end{pmatrix} \begin{pmatrix} x \\ Dp \end{pmatrix}.$$

This implies the following equations on D

$$AD^\top = D^\top A \qquad \text{and} \qquad D^\top B = B\Lambda. \tag{2.12}$$

Let $b_1, \dots, b_m$ be the column vectors of B. The second equality in (2.12) writes as $D^\top b_i = \lambda_i b_i$ for $i = 1, \dots, m$. Applying iteratively the first equality in (2.12) we obtain, for any $k \in \mathbb{N}$,

$$D^\top A^k b_i = \lambda_i A^k b_i \qquad i = 1, \dots, m.$$

Thus $A^k b_i$ is an eigenvector of $D^\top$ associated with the eigenvalue λ_i. From the controllability of the pair (A, B), the set $\{b_1, \dots, b_m, \dots, A^n b_1, \dots, A^n b_m\}$ is of dimension n, and so $D^\top$ is diagonalizable.

Let $E_1, \dots, E_N$ be the eigenspaces of $D^\top$. Note that N is the number of different eigenvalues λ_i of Λ, therefore we have $N > 1$ since $\Lambda \neq I$. The first equality in (2.12) implies that the matrix A preserves every E_j. Moreover, every vector b_i belongs to one of the eigenspaces. Thus, in a basis of $\mathbb{R}^n$ adapted to the decomposition $\mathbb{R}^n = E_1 \oplus \cdots \oplus E_N$, the matrices A and B (up to a reordering of the coordinates u) have block form

$$\bar{A} = \begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_N \end{pmatrix} \quad B = \begin{pmatrix} B_1 & & \\ & \ddots & \\ & & B_N \end{pmatrix},$$

while the Lagrangians $L, \tilde{L}$ are

$$L = \sum_{i=1}^N u_i^\top u_i \quad \tilde{L} = \sum_{i=1}^N \lambda_i u_i^\top u_i.$$

Thus, the optimal control problems defined by L and $\tilde{L}$ have a product structure in the chosen basis. $\square$

Since the number N of elements in a product structure satisfies $1 < N \leq m$, we recover in particular the result of [61] for the single input case.

Corollary 2.15. *In the single input case ($m = 1$), the reduced inverse optimal control problem is injective.*

2.4 Reconstruction

Let us consider now the problem of the reconstruction of the cost in a reduced inverse LQ optimal problem. In this setting the controllable pair (A, B) is fixed, B being assumed to be of rank m. The problem is: given an optimal synthesis Γ, recover the matrices (R, K) of a canonical cost such that Γ is the optimal synthesis of the family of LQ optimal control problems,

$$\min_u \int_0^T (u + Kx)^\top R(u + Kx) \quad \text{s.t.} \quad \dot{x} = Ax + Bu, \tag{2.13}$$

with fixed extremities $x(0) = x_0$, $x(T) = x_F$.

From Lemma 2.5, a unique pair (A_+, A_-) is associated with the set Γ, and thus, a unique $K = K_+$. We do not have uniqueness of R but we can still give a description the set of R which solve the inverse problem.

Proposition 2.16. *Matrix R is a solution to the inverse problem associated with the pair (A_+, A_-) if and only if it satisfies*

$$\begin{aligned} X &= \int_0^\infty e^{tA_+} BR^{-1}B^\top e^{t(A_+)^\top} dt, \\ (A_- - A_+)\, X &= BR^{-1}B^\top, \\ A_+X &= -XA_-^\top. \end{aligned} \tag{2.14}$$

Proof. It follows from (2.3), (2.4) and the analytic expression of $\Delta = P_+ - P_-$ solution of the Lyapunov equation associated to the problem. □

These equations may not determine R in a unique way since the problem may have a product structure and thus many equivalent costs. This issue will be addressed thanks to the following proposition.

Proposition 2.17. *The problem* (2.13) *admits a product structure if and only if there exist a decomposition $\mathbb{R}^n = E_1 \oplus \cdots \oplus E_N$ with $1 < N \leq m$ which is invariant by both A_+ and A_-.*

Proof. Note first that, if the problem admits a product structure, then in appropriate coordinates it splits into $N > 1$ sub-problems, and so do the minimizing solutions and the matrices A_+ and A_-. This gives the decomposition and proves the only if part.

Now, assume that the A_+, A_- associated with (2.13) leave invariant a decomposition $\mathbb{R}^n = E_1 \oplus \cdots \oplus E_N$. Up to a linear feedback change of coordinates $\tilde{x} = M_x x$, $\tilde{u} = u + Kx$, we assume on the one hand that $A_+ = A$, and on the other hand that the matrices A_+, A_- admit a block diagonal form: for $i = 1, \dots, N$, the ith diagonal blocks are $(n_i \times n_i)$ matrices A_+^i, A_-^i respectively, where the integers $n_1, \dots, n_N$ satisfy $n_1 + \cdots + n_N = n$.

From the expression (2.4) of A_- and the Riccati equation (2.3), the matrix P_- (the unique anti-stabilizing solutions of the Riccati equation) satisfies

$$A_+ = -P_-^{-1} A_-^\top P_-.$$

As a consequence, P_- preserves the decomposition, thus P_- is itself block diagonal with $(n_i \times n_i)$ blocks P_-^i, $i = 1, \dots, N$. Moreover, since we assume $A = A_+$, a simple computation using (2.4) shows that the matrices A_-, P_- can be expressed in terms of B and R as

$$BR^{-1}B^\top = (A - A_-)\, P_-^{-1}. \tag{2.15}$$

Let us denote the matrix $BR^{-1}B^\top$ by G. Since all matrices in the right-hand side of (2.15) are block diagonal, the matrix G is block diagonal as well. As a consequence, the image of G admits a decomposition in invariant spaces

$$\mathrm{Im}(G) = V_1 \oplus \cdots \oplus V_N, \qquad V_i \subset E_i,\ i = 1, \dots, N.$$

The image of B contains the image of G and, as B and R are of the same rank, we have rank $(G) = \mathrm{rank}(B)$. Therefore, the images of the both matrices are equal and the image of B admits the same decomposition as the image of G. There exists a linear change of control $v = M_u \tilde{u}$ such that $\tilde{B} = BM_u$ is composed by vectors from each of $V_1, \dots, V_N$ as its columns. Therefore $\tilde{B}$ has block-diagonal form respective to the decomposition $\mathbb{R}^n = E_1 \oplus \cdots \oplus E_N$.

Denote by $\tilde{R}$ the matrix $M_u^\top R M_u$ obtained by the linear change of control. Notice that the matrix G is invariant under linear changes of the control. We have thus the following expression of $\tilde{R}$

$$\tilde{R} = \left(\tilde{B}^\top \tilde{B}\right)^{-1} \tilde{B}^\top G \tilde{B} \left(\tilde{B}^\top \tilde{B}\right)^{-1}.$$

It has block-diagonal form by construction. As a result, we found a change of coordinate in which all matrices in the reduced linear-quadratic problem have respective block diagonal forms with N components and therefore it admits the product structure.

□

From (A_+, A_-) we can deduce either the uniqueness of the cost R, or the existence of several costs but with a particular structure in the optimal control problem. Indeed, in the latter case the decomposition $\mathbb{R}^n = E_1 \oplus \cdots \oplus E_N$ allows to split the problem into several sub-problems of the same form with a smaller number of inputs. Iterating eventually the decomposition (Corollary 2.15 ensures that the iteration will stop), we can assume that each sub-problem is injective. We propose a cost reconstruction method which includes the following steps.

1. Reconstruct A_+, A_- from the trajectories in Γ: this can be done by identification of parameters in (2.5)–(2.6), taking K_+, K_- as the unknown parameters.

2. Check whether A_+, A_- leave invariant a decomposition of $\mathbb{R}^n$; if it is the case, determine the smallest such decomposition.

3. Fix a new basis of $\mathbb{R}^n$ adapted to the above decomposition.

4. Find an $(m \times m)$ invertible matrix M_u such that each column vector b_i of $\tilde{B} = BM_u$ for $i = 1, \ldots, m$ belongs to one of the spaces in the decomposition.

5. Separate the optimal control problem into N independent sub-problems.

6. For each sub-problem, find R_i as the unique positive definite matrix with $\det(R_i) = 1$ which satisfies (2.14) with the ith blocks of matrices $A_+, A_-, \tilde{B}$.

7. Cost R is constructed from $R_1, \ldots, R_N$ by
$$R = (M_u^{-1})^\top \begin{pmatrix} R_1 & & \\ & \ddots & \\ & & R_N \end{pmatrix} M_u^{-1}.$$

8. Set
$$K = \left(B^\top B\right)^{-1} B^\top (A - A_+) .$$

The method gives as an output the matrices R and K such that the optimal synthesis of (2.13) is Γ.

In practice, we expect that the matrices A_+, A_- obtained in the first step will be in general position, and thus will not admit an invariant decomposition. This will eliminate Step 3, which can be difficult from a practical point of view. And the matrix R^{-1} can be obtained directly from the linear equation (2.14). The method will then provide a stable solution to the reduced inverse optimal control problem.

Chapter 3

Sub-Riemannian case

The sub-Riemannian case is a special class of optimal control problems. It is defined by a bracket generating vector distribution D on a manifold M and a Riemannian metric g on D. This structure induces a distance on M defined as the length of the shortest admissible trajectory joining two points. From the control theory point of view an admissible trajectory is a solution of the control system

$$\dot{q} = \sum_{i=1}^{m} u_i f_i(q), \qquad u = (u_1, \dots, u_m) \in \mathbb{R}^m,$$

where $(f_1, \dots, f_m)$ generate D. It is easy to see that any reparametrization of an admissible trajectory is admissible. But what about the optimal trajectories?

It appears that if we consider the length functional as the cost to minimize then it holds as well. If some trajectory is a shortest path joining two points, then any of its reparametrization is a path of same length and thus the shortest path as well. This shows in particular that the geometric inverse problem is well adapted to this class of optimal control problems because the optimal synthesis is characterized by the geometric curves in this case, the parametrization does not matter. As a consequence the inverse problems on the trajectories and on the geometric curves coincide. The injectivity problem in this case inspires the notions of projective equivalence which will be one of the main objects of investigation in this chapter.

On the other hand, if the cost is the energy functional then the described property does not hold anymore even though the solutions of both classes of optimal control problems are related. In this case the only admissible reparameterizations of the minimizers (and of the geodesics) are the affine reparameterizations . Therefore, the inverse problems on the geometric curves and on the trajectories do not coincide and the problem on geometric curves is exactly the problem of the shortest paths on the trajectories. In this situation the injectivity of the inverse problem on the trajectories inspires the notion of affine equivalence which is the second main object of investigation in the chapter.

This chapter is self-contained. Sections 3.1-3.8 are based on the submitted article [64], this is why the chapter's content is redundant and some definition and discourses repeat those introduced in other chapters.

3.1 Introduction to the sub-Riemannian case

In Riemannian geometry, projectively (or geodesically) equivalent metrics are Riemannian metrics on the same manifold which have the same geodesics, up to reparameterization. The study of equivalent metrics dates back to the works of Dini and Levi-Civita in the 19th century. The interest in this notion of equivalence is renewed by recent applications of optimal control theory to the study of human motor control. Indeed, finding the optimality criterion followed by a particular human movement amounts to solve what is called an *inverse optimal control problem* (see for instance [4, 46]): given a set Γ of trajectories and a class of optimal control problems – that is, a pair (dynamical constraint, class $\mathcal{L}$ of infinitesimal costs) –, identify a cost function L in $\mathcal{L}$ such that the elements of Γ are minimizing trajectories of the optimal control problem associated with the integral cost $\int L$. Being the solutions of a same inverse problem defines an equivalence between costs in $\mathcal{L}$ similar to projective equivalence for Riemannian metrics. Our purpose here is to extend and study this kind of equivalence in the context of sub-Riemannian geometry. This is a first step in the direction of a more general goal, which is to give a rigorous theoretical framework to the study of inverse optimal control problems.

A sub-Riemannian manifold is a triple (M, D, g), where M is a smooth manifold, D is a distribution on M (i.e. a subbundle of TM) which is assumed to be bracket generating, and g is a Riemannian metric on D. We say that g is a sub-Riemannian metric on (M, D). Riemannian geometry appears as the particular case where $D = TM$. A horizontal curve is an absolutely continuous curve tangent to D, and for such a curve γ the length and the energy are defined as in Riemannian geometry by respectively $\ell(\gamma) = \int \sqrt{g(\dot\gamma, \dot\gamma)}$ and $E(\gamma) = \int g(\dot\gamma, \dot\gamma)$. A length minimizer (resp. an energy minimizer) is a horizontal curve minimizing the length (resp. the energy) among all the horizontal curves with the same extremities.

The length being independent on the parameterization of the curve, any time reparameterization of a length minimizer is still a length minimizer. On the other hand, a classical consequence of the Cauchy–Schwarz inequality is that the energy minimizers are the length minimizers with constant velocity, i.e. such that $g(\dot{\gamma}, \dot{\gamma})$ is constant along γ. It is then sufficient to describe the energy minimizers, the length minimizers being any time reparameterization of the latter.

It results from the Pontryagin Maximum Principle that energy minimizers are projections of Pontryagin extremals, and can be of two types, normal or abnormal geodesics. These geodesics play a role similar to the one of the solutions of the geodesic equation in Riemannian geometry. We thus extend the definition of equivalence of metrics in the following way.

Definition 3.1. Let M be a manifold and D be a bracket generating distribution on M. Two sub-Riemannian metrics g_1 and g_2 on (M, D) are called *projectively equivalent* at $q_0 \in M$ if they have the same geodesics, up to a reparameterization, in a neighborhood of q_0. They are called *affinely equivalent* at q_0 if they have the same geodesics, up to affine reparameterization, in a neighborhood of q_0.

The trivial example of equivalent metrics is the one of two constantly proportional metrics g and cg, where $c > 0$ is a real number. We thus say that these metrics are *trivially* (projectively or affinely) equivalent. Besides, affine equivalence implies projective equivalence but in general the two notions do not coincide. For instance, on $M = \mathbb{R}$, all metrics are projectively equivalent to each other while two metrics are affinely equivalent if and only if they are trivially equivalent.

Note that if two sub-Riemannian metrics on (M, D) have the same set of length minimizers, then they are projectively equivalent. And if they have the same set of energy minimizers, then they are affinely equivalent. This results from the fact that on one hand normal geodesics are locally energy minimizers, and on the other hand abnormal geodesics are characterized only by the distribution D. Thus projective and affine equivalence are appropriate notions to study inverse optimal control problems where the dynamical constraint is $\dot{\gamma} \in D$ and the class $\mathcal{L}$ is the set of sub-Riemannian metrics. In particular, they allow one to answer to the following questions: given M and D, can we recover g in a unique way, up to a multiplicative constant, from the knowledge of all energy minimizers of (M, D, g)? And from the knowledge of all length minimizers of (M, D, g)? The answer to these questions is positive when the metric presents a kind of rigidity.

Definition 3.2. A metric g on (M, D) is said to be *projectively rigid* (resp. *affinely rigid*) if it admits no non-trivially projectively (resp. affinely) equivalent metric.

We also introduce a weaker notion of rigidity associated with the concept of conformal metrics. Remind that a metric g_2 on (M, D) is said to be *conformal* to another metric g_1 if $g_2 = \alpha^2 g_1$ for some nonvanishing smooth function $\alpha : M \to \mathbb{R}$.

The trivial case of constantly proportional metrics is the particular case where α is constant. Note that two conformal metrics are not projectively equivalent in general (we actually conjecture that the latter situation occurs only when either $\dim M = 1$ or the metrics are constantly proportional to each other).

Definition 3.3. A metric g is said to be *conformally projectively rigid* if any metric projectively equivalent to g is conformal to g.

It is easy to construct examples of metrics which are not projectively rigid. For example, an Euclidean metric on a plane provides such an example. Indeed, its geodesics consist of straight lines. Take the Riemannian metrics on the same plane obtained by the pull-back from the round metric on a sphere placed on this plane via the (inverse of) gnomonic projection, i.e. the stereographic projection with the center in the center of the sphere. Obviously the geodesic of this metric are straight lines as unparameterized curves geodesics but this metric is not constantly proportional to the original metric, because it has nonzero constant Gaussian curvature (see [65, Sect. 3.1]). Note also that by a classical theorem by Beltrami [47], the metrics with constant sectional curvature are the only ones projectively equivalent to the flat ones.

If one extends the notion of equivalence to Lagrangians, then one arrives to the variational version of Hilbert's fourth problem in dimension 2, which was solved by Hamel [66] and provides a very rich class of Lagrangians having straight lines as extremals.

Affine and projective equivalence of Riemannian metrics are actually both classical. From the results of Dini [67], it follows that under natural regularity assumptions a two-dimensional Riemannian metric is non projectively rigid if and only if it is a Liouville surface, i.e., its geodesic flow admits a non-trivial integral which is quadratic with respect to the velocities. This implies that generic Riemannian metrics on surfaces are projectively rigid. In [48], again under natural regularity assumptions, Levi-Civita proved that the same result holds for Riemannian metrics on manifolds of arbitrary dimensions and provided a classification of locally projectively equivalent Riemannian metrics. The affinely equivalent Riemannian metrics are exactly the metrics with the same Levi-Civita connection and the description of the pairs of Riemannian metrics with this property can be attributed to Eisenhart [68]. This description is also closely related to the de Rham decomposition of a Riemannian manifold and the properties of its holonomy group [69].

These classical results in Riemannian case implies in particular that a Riemannian metric that is not rigid with respect to one of the above equivalences satisfies the following two special properties.

1. **Integrability property.** Its geodesic flow possesses a collection of nontrivial integrals quadratic on the fiber and in involution.

2. **Product structure (or separation of variables) property.** Locally the ambient manifold M is a product of at least two manifolds such that the metric is a product of metrics on the factors in the affine case and a sort of twisted product of Riemannian metrics on the factors in the projective case (for a precise meaning of twisting here see formula (3.29) below).

Note that similar relations between separability of the Hamilton-Jacobi equation on a Riemannian manifold and integrability (existence of Killing tensors) were extensively studied by Benenti [70, 71], while a more conceptual explanation of the integrability property, based on the modern language of symplectic geometry was given by Matveev and Topalov [49].

In a proper sub-Riemannian case, the only complete classification of projectively equivalent metrics was done far more recently by Zelenko [50] for contact and quasi-contact sub-Riemannian metrics. The general goal is to extend the above classification results to an arbitrary sub-Riemannian case. By analogy with the Levi-Civita classification we define a wide class of pairs of sub-Riemannian metrics that are projective equivalent, see Section 3.5.1. We call them the *(generalized) Levi-Civita pairs.* Note that the generalized Levi-Civita pairs satisfy both integrability and product structure properties. It turns out that the result of [50] about the contact and quasi-contact case can be actually reformulated in the following way: under a natural regularity assumption the generalized Levi-Civita pairs are the only pairs of projectively equivalent metrics. The natural question is whether this is the case for arbitrary sub-Riemannian case, i.e. *whether under some natural regularity assumption the generalized Levi-Civita pairs are the only pairs of projectively equivalent metrics.*

In the present paper we make several steps toward answering this question by proving the following two general results, which are weaker than the integrability and product structure properties formulated above, but support them. The first result is the existence of at least one integral, which supports the integrability property.

Theorem 3.4. *If a sub-Riemannian metric g is not conformally projectively rigid, then its flow of normal extremals has at least one nontrivial (i.e. not equal to the sub-Riemannian Hamiltonian) integral quadratic on the fibers.*

The second result states that the product structure properties hold at the level of the nilpotent approximations.

Theorem 3.5. *Let M be a smooth manifold, D be a distribution on M, and g_1, g_2 be two sub-Riemannian metrics on (M, D). If g_1, g_2 are projectively equivalent and non conformal to each other, then for q in an open and dense subset of M, the nilpotent approximation $\hat{D}$ of D at q admits a product structure, and the nilpotent approximations $\hat{g}_1, \hat{g}_2$ of the metrics form a Levi-Civita pair with constant coefficients.*

Since no bracket generating rank-2 distribution admits a product structure, we have the following consequence of the previous theorem.

Corollary 3.6. *Any bracket generating sub-Riemannian metric on a rank-2 distribution is affinely rigid and conformally projectively rigid.*

Theorem 3.4 allows us to get the following rigidity property of generic sub-Riemannian metrics on a given distribution.

Theorem 3.7. *Let M be a smooth manifold and D be a distribution on M. A generic sub-Riemannian metric on (M, D) is affinely rigid and conformally projectively rigid.*

Theorem 3.5 allows us to get the following rigidity results for all sub-Riemannian metric of a generic distributions.

Theorem 3.8. *Let m and n be two integers such that $2 \leq m < n$, and assume $(m, n) \neq (4, 6)$ and $m \neq n - 1$ if n is even. Then, given an n-dimensional manifold M and a generic rank-m distribution D on M, any sub-Riemannian metric on (M, D) is affinely rigid and conformally projectively rigid.*

Few words now about the main ideas behind the proofs with references to the corresponding sections of the paper. The problem of the projective equivalence of sub-Riemannian metrics can be reduced to the problem of existence of a fiber preserving orbital diffeomorphism between the flows of normal extremals in the cotangent bundle (orbital diffeomorphism means that it sends normal extremals of one metric to the normal extremals of another one considered as unparameterized curves). In the Riemannian case, if such diffeomorphism exists then it can be easily expressed in terms of the metrics. It is not the case anymore in the proper sub-Riemannian case, which is the main difficulty here. The reason is that, in contrast to the Riemannian case, a sub-Riemannian geodesic is not uniquely determined by its initial point and the velocity at this point (i.e. by its first jet at one point). The order of jet which is needed to determine a geodesic uniquely is controlled by the flag of the Jacobi curves along the corresponding extremal, which were introduced in [60, 72]. In subsections 3.2.2 and 3.2.3 we collect all necessary information about Jacobi curves in order to justify the reduction of the equivalence problem to the existence of a fiberwise diffeomorphism in subsection 3.3.1.

In what follows, for shortness a function which is a polynomial or rational function on each fiber of T^*M will be simply called a polynomial or rational function respectively on T^*M. The equations on a fiber preserving orbital diffeomorphism form a highly overdetermined system of differential equations. In subsection 3.3.2 after certain prolongation process, we reduce this system to a system of infinitely many linear algebraic equations with coefficients which are polynomial functions so that if a solution of this system exists, then it is unique. We refer to this system as the *fundamental algebraic system* for orbital diffeomorphism. Its solution must be a rational function involving quadratic radicals on T^*M.

The analysis of compatibility conditions for this system leads to a set of algebraic conditions. In particular, we show that one specific polynomial function on T^*M is divisible by another specific polynomial function on T^*M. This divisibility condition is equivalent to the existence of an integral for the normal extremal flow, which proves Theorem 3.4. We prove that the non-existence of a non-trivial integral for the geodesic flow of a sub-Riemannian metric is a generic property, adapting the proof of the analogous result for the Riemannian case from [73]. This implies Theorem 3.7.

The idea of the proof of Theorem 3.5 comes from the fact that the filtration of the tangent bundle, generated by the iterative brackets of vector fields tangent to the underlying distribution, induces weighted degrees for polynomial function on T^*M. If we replace all coefficients of the fundamental algebraic system at a point by the components of the highest weighted degree, we will get exactly the fundamental algebraic system for the orbital diffeomorphism related to the flow of normal extremals of the nilpotent approximation of the first metric, see the proof of Theorem 3.61. This and the analysis of conditions for projective equivalence for left invariant sub-Riemannian metrics given in Theorem 3.59 are the main steps of the proof of Theorem 3.5.

Finally, to prove Theorem 3.8 we analyse in Section 3.8 for which pairs (m, n) the generic n-dimensional graded nilpotent Lie algebras generated by the homogeneous component of weight 1 can not be represented as a direct sum of two nonzero graded nilpotent Lie algebras.

3.2 Preliminaries

3.2.1 Sub-Riemannian manifolds

Let us recall some standard notions from sub-Riemannian geometry. Let M be a n-dimensional smooth manifold and D be a rank-m distribution on M, i.e., D is a subbundle of TM of rank m. We define by induction a sequence of modules of vector fields by setting

$$D^1 = \{X : X \text{ is a section of } D\},$$

and, for any integer $k > 1$,

$$D^k = D^{k-1} + \operatorname{span}\{[X, Y] : X \text{ is a section of } D,\ Y \text{ belongs to } D^{k-1}\},$$

where span is taken over the smooth functions on M. The *Lie algebra* $\operatorname{Lie}(D)$ generated by the distribution D is defined as $\operatorname{Lie}(D) = \bigcup_{k\geq 1} D^k$.

For $q \in M$, we denote by $D^k(q)$ and $\operatorname{Lie}(D)(q)$ the subspaces of T_qM

$$D^k(q) = \operatorname{span}\{X(q) : X \in D^k\} \quad \text{and} \quad \operatorname{Lie}(D)(q) = \operatorname{span}\{X(q) : X \in \operatorname{Lie}(D)\}.$$

Definition 3.9. The distribution D is said to be *bracket generating* if at any point $q \in M$ we have $\operatorname{Lie}(D)(q) = T_qM$.

In the rest of the paper, all distributions are supposed to be bracket generating. If D is bracket generating then for any $q \in M$ there exists an integer k such that $D^k(q) = T_qM$. The smallest integer with this property is called the *nonholonomic order* (or simply *the step*) of D at q and it is denoted by $r = r(q)$.

Definition 3.10. A point $q_0 \in M$ is called *regular* if, for every integer $k \geq 1$, the dimension $\dim D^k(q)$ is constant in a neighborhood of q_0.

The *weak derived flag of the distribution* D at q is the following filtration of vector spaces

$$D(q) = D^1(q) \subset D^2(q) \subset \cdots \subset D^r(q) = T_qM. \tag{3.1}$$

For any positive integer k, we set $\dim D^k(q) = m_k(q)$. In particular, $m_1 = m$ and $m_r = n$. We call *weights* at q the integers $w_1(q), \ldots, w_n(q)$ defined by $w_i(q) = s$, if $m_{s-1} < i \leq m_s$, where we set $m_0 = 0$.

Definition 3.11. A set of vector fields $\{X_1, \ldots, X_n\}$ is called a *frame of TM adapted to D* at $q \in M$ if for any integer $i \in \{1, \ldots, n\}$, the vector field X_i belongs to D^{w_i}, and for any integer $k \in \{1, \ldots, r\}$, the vectors $X_1(q), \ldots, X_{m_k}(q)$ form a basis of $D^k(q)$. The *structure coefficients* of the frame are the real-valued functions c_{ij}^k, $i, j, k \in \{1, \ldots, n\}$, defined near q by

$$[X_i, X_j] = \sum_{k=1}^{n} c_{ij}^k X_k.$$

Such a frame can be constructed in the following way. We start by choosing vector fields $X_1, \ldots, X_m \in D^1$ whose values at q form a basis of $D(q)$. Then we choose $m_2 - m$ vector fields $X_{m+1}, \dots, X_{m_2}$ among $\{[X_i, X_j],\ 1 \le i, j \le m\}$ whose values at q form a basis of $D^2(q)$. Continuing in this way we get a set of vector fields $X_1, \dots, X_n$ such that $\operatorname{span}\{X_1(q), \dots, X_{m_k}(q)\} = D^k(q)$ for every integer $k \le r$. In particular, $X_1(q), \dots, X_n(q)$ form a basis of T_qM. Note that if q is a regular point, then a frame adapted at q is also adapted at any point near q.

Choosing now a Riemannian metric g on D, we obtain a sub-Riemannian manifold (M, D, g). By abuse of notations we also say that g is a *sub-Riemannian metric* on (M, D). As mentioned in the introduction, the geodesics of (M, D, g) are the projections on M of the Pontryagin extremals associated with the minimization of energy. There exist two types of geodesics, the normal and abnormal ones. Abnormal geodesics depend only on the distribution D, not on g, hence they are of no use for the study of equivalence of metrics. Normal geodesics admit the following description.

For $q \in M$, we define a norm on T_q^*M by

$$\|p\|_q = \max\left\{\langle p, v\rangle : v \in D(q),\ g(q)(v, v) = 1\right\}, \qquad p \in T_q^*M.$$

The *Hamiltonian* of the sub-Riemannian metric g is the function $h : T^*M \to \mathbb{R}$ defined by

$$h(q, p) = \frac{1}{2}\|p\|_q^2, \qquad q \in M,\ p \in T_q^*M.$$

Definition 3.12. A *normal extremal* is a trajectory $\lambda(\cdot)$ of the Hamiltonian vector field, i.e. $\lambda(t) = e^{t\vec{h}}\lambda_0$ for some $\lambda_0 \in T^*M$. A *normal geodesic* is the projection $\gamma(t) = \pi(\lambda(t))$ of a normal extremal, where $\pi : T^*M \to M$ is the canonical projection.

It is useful to give the expression of $\vec{h}$ in local coordinates. Fix a point $q_0 \in M$ and choose a frame $\{X_1, \dots, X_n\}$ of TM adapted to D at q_0 such that $X_1, \dots, X_m$ is a g-orthonormal frame of D. At any point q in a neighbourhood U of q_0, the basis $X_1(q), \dots, X_n(q)$ of T_qM induces coordinates $(u_1, \dots, u_n)$ on T_q^*M defined as $u_i(q, p) = \langle p, X_i(q)\rangle$. These coordinates in turn induce a basis $\partial_{u_1}, \dots, \partial_{u_n}$ of $T_\lambda(T_q^*M)$ for any $\lambda \in \pi^{-1}(q)$. For $i = 1, \dots, n$, we define the lift Y_i of X_i as the (local) vector field on T^*M such that $\pi_*Y_i = X_i$ and $du_j(Y_i) = 0 \ \ \forall 1 \le j \le n$. The family of vector fields $\{Y_1, \dots, Y_n, \partial_{u_1}, \dots, \partial_{u_n}\}$ obtained in this way is called a *frame of* $T(T^*M)$ *adapted at* q_0. By a standard calculation, we obtain

$$h = \frac{1}{2}\sum_{i=1}^m u_i^2 \qquad \text{and} \qquad \vec{h} = \sum_{i=1}^m u_iY_i + \sum_{i=1}^m \sum_{j,k=1}^n c_{ij}^k u_i u_k \partial_{u_j}. \tag{3.2}$$

Note that a normal geodesic $\gamma(t) = \pi(e^{t\vec{h}}\lambda_0)$ satisfies $g(\dot\gamma(t), \dot\gamma(t)) = 2h(\lambda(t)) = 2h(\lambda_0)$, so the geodesic is arclength parameterized if $h(\lambda_0) = 1/2$.

3.2.2 Jacobi curves

As mentioned in the introduction the notion of Jacobi curve of a normal extremal is important for the considered equivalence problem. This notion, introduced in [74], comes from the notion of Jacobi fields in Riemannian geometry. A Jacobi field is a vector field along a geodesic which carries information about minimizing properties of the geodesic. The Jacobi curve is a generalization of the space of Jacobi fields which can be defined in sub-Riemannian geometry.

For completeness we introduce Jacobi curves and all necessary related objects here, for more details we refer to [57, 59, 60]. Consider a sub-Riemannian manifold (M, D, g) and a normal geodesic $\gamma(t) \in M,\ t \in [0, T]$. It is the projection on M of an extremal $\lambda(t) = e^{t\vec{h}}\lambda$ for some $\lambda \in T^*M$ such that $\pi(\lambda) = \gamma(0)$. The $2n$-dimensional space $T_\lambda(T^*M)$ endowed with the natural symplectic form $\sigma_\lambda(\cdot, \cdot)$ is a symplectic vector space. A Lagrangian subspace of this symplectic space is a vector space of dimension n which annihilates the symplectic form. We denote by $\mathcal{V}_{\lambda(t)}$ the vertical subspace $T_{\lambda(t)}(T^*_{\gamma(t)}M)$ of $T_{\lambda(t)}(T^*M)$, it is vertical in the sense that $\pi_*(\mathcal{V}_{\lambda(t)}) = 0$. Now we can define the Jacobi curve associated with the normal geodesic $\gamma(t)$.

Definition 3.13. For $\lambda \in T^*M$, we define the *Jacobi curve* $J_\lambda(\cdot)$ as the curve of Lagrangian subspaces of $T_\lambda(T^*M)$ given by

$$J_\lambda(t) = e_*^{-t\vec{h}}\mathcal{V}_{\lambda(t)}, \qquad t \in [0, T].$$

We introduce the extensions of a Jacobi curve as an analogue to the Taylor expansions at different orders of a smooth curve.

Definition 3.14. For an integer $i \geq 0$, the *ith extension* of the Jacobi curve $J_\lambda(\cdot)$ is defined as

$$J_\lambda^{(i)} = \operatorname{span}\left\{ \frac{d^j}{dt^j} l(0) \ : \ l(s) \in J_\lambda(s)\ \forall s \in [0, T],\ l(\cdot) \text{ smooth },\ 0 \leq j \leq i \right\}.$$

In other words, these spaces are spanned by all the directions generated by derivatives at $t = 0$ of the standard curves lying in the Jacobi curve. By definition, $J_\lambda^{(i)} \subset J_\lambda^{(i+1)} \subset T_\lambda(T^*M)$, so it is possible to define a flag of these spaces.

Definition 3.15. The *flag of the Jacobi curve* $J_\lambda(\cdot)$ is defined as

$$J_\lambda = J_\lambda^{(0)} \subset J_\lambda^{(1)} \subset \cdots \subset T_\lambda(T^*M).$$

In an adapted frame of $T(T^*M)$, the Jacobi curves can be obtained from iterations of Lie brackets by $\vec{h}$. Let us remind first that, for a positive integer l and a pair of vector fields X, Y, the notation $(\mathrm{ad}X)^l Y$ stands for $\underbrace{[X, \ldots, [X}_{l \text{ times}}, Y]]$.

Lemma 3.16. *Let $q = \pi(\lambda)$. In an adapted frame $\{Y_1, \dots, Y_n, \partial_{u_1}, \dots, \partial_{u_n}\}$ of $T(T^*M)$ at q, the extensions of the Jacobi curve take the following form:*

$$\begin{aligned}
J_\lambda^{(0)} &= \Big\{ v \in T_\lambda(T^*M) \ : \ \pi_* v = 0 \Big\}, \\
J_\lambda^{(1)} &= \Big\{ v \in T_\lambda(T^*M) \ : \ \pi_* v \in D \Big\} = J_\lambda^{(0)} + \mathrm{span}\Big\{ Y_1(\lambda), Y_2(\lambda), \dots, Y_m(\lambda) \Big\}, \\
J_\lambda^{(2)} &= J_\lambda^{(1)} + \mathrm{span}\Big\{ [\vec{h}, Y_1](\lambda), \dots, [\vec{h}, Y_m](\lambda) \Big\}, \\
&\vdots \\
J_\lambda^{(k)} &= J_\lambda^{(k-1)} + \mathrm{span}\Big\{ (\mathrm{ad}\vec{h})^{k-1} Y_1(\lambda), \dots, (\mathrm{ad}\vec{h})^{k-1} Y_m(\lambda) \Big\}.
\end{aligned}$$

Proof. Let $v \in J_\lambda^{(k)}$, for some integer $k \geq 0$. By definition, $v = \frac{d^s}{dt^s} l(0)$ where $l(\cdot)$ is a curve with $l(t) \in J_\lambda(t)$ for any $t \in [0, T]$, and $s \leq k$ is an integer. Then there exists a vertical vector field Y on T^*M (i.e. $\pi_* Y = 0$) such that, for any $t \in [0, T]$,

$$l(t) = e_*^{-t\vec{h}} Y(\lambda(t)),$$

and v writes as

$$v = \frac{d^s}{dt^s} e_*^{-t\vec{h}} Y(e^{t\vec{h}} \lambda) \bigg|_{t=0} = (\mathrm{ad}\vec{h})^s Y(\lambda).$$

As Y is a vertical vector field, in the adapted frame $\{Y_1, \dots, Y_n, \partial_{u_1}, \dots, \partial_{u_n}\}$ it can be written as $Y = \sum_{i=1}^n a_i \partial_{u_i}$. Using the expression (3.2) of $\vec{h}$ in this frame, we obtain

$$\begin{aligned}
[\vec{h}, Y] &= \left[\sum_{i=1}^m u_i Y_i + \sum_{i=1}^m \sum_{j,k=1}^n c_{ij}^k u_i u_k \partial_{u_j}, \sum_{i=1}^n a_i \partial_{u_i} \right] \\
&= \sum_{i=1}^m a_i Y_i \quad \mathrm{mod}\ \mathrm{span}\{\partial_{u_1}, \dots, \partial_{u_n}\}.
\end{aligned}$$

By iteration, we get

$$\begin{aligned}
(\mathrm{ad}\vec{h})^2 Y &= \sum_{i=1}^m a_i\, [\vec{h}, Y_i] \quad \mathrm{mod}\ \mathrm{span}\{\partial_{u_1} \dots, \partial_{u_n}, Y_1, \dots, Y_m\} \\
&\vdots \\
(\mathrm{ad}\vec{h})^s Y &= \sum_{i=1}^m a_i\, (\mathrm{ad}\vec{h})^{s-1} Y_i \\
&\qquad \mathrm{mod}\ \mathrm{span}\{\partial_{u_1} \dots, \partial_{u_n}, Y_i, \dots, (\mathrm{ad}\vec{h})^{s-2} Y_i,\ i = 1, \dots, m\},
\end{aligned}$$

which proves the result. $\square$

3.2.3 Ample geodesics

Note that the dimension of the spaces $J_\lambda^{(k)}$ for $|k| > 1$ may depend on λ in general. Following [57], we distinguish the geodesics corresponding to the extensions of maximal dimension.

Definition 3.17. The normal geodesic $\gamma(t) = \pi(e^{t\vec{h}}\lambda)$ is said to be *ample at* $t = 0$ if there exists an integer k_0 such that

$$\dim(J_\lambda^{(k_0)}) = 2n.$$

In that case we say that λ *is ample with respect to the metric* g.

Notice that if a geodesic is ample at $t = 0$, then it is not abnormal on any small enough interval $[0, \varepsilon]$ (see [57, Prop. 3.6]). It appears that normal geodesic are generically ample in the following sense.

Theorem 3.18 ([57], Proposition 5.23)**.** *For any $q \in M$, the set of ample covectors $\lambda \in \pi^{-1}(q)$ is an open and dense (and hence non empty) subset of T_q^*M.*

Remark 3.19. Two proportional covectors λ and $c\lambda$, $c > 0$, define the same geodesic, up to time reparameterization, and the corresponding extensions of Jacobi curves $J_\lambda^{(k)}$ and $J_{c\lambda}^{(k)}$ have the same dimension. As a consequence, the statement of Theorem 3.18 also holds in $h^{-1}(1/2)$. Namely, the set of ample covectors is open and dense in $\pi^{-1}(q) \cap h^{-1}(1/2)$.

Ample geodesics play a crucial role in the study of equivalence of metrics because they are the geodesics characterized by their jets. Let us precise this fact. Fix a nonnegative integer k. For a given curve $\gamma : I \to M$, $I \subset \mathbb{R}$, denote by $j_{t_0}^k\gamma$ the k-jet of γ at the point t_0. Given $q \in M$, we denote by $J_q^k(g)$ the space of k-jets at $t = 0$ of the normal geodesics of g issued from q and parameterized by arclength. We set $J^k(g) = \bigsqcup_{q \in U} J_q^k(g)$.

Define the maps $P^k : H \mapsto J^k(g)$, where $H = h^{-1}(1/2)$, by

$$P^k(\lambda) = j_0^k \pi(e^{t\vec{h}}\lambda).$$

The properties of the map P^k near a point λ can be described in terms of the kth extension $J_\lambda^{(k)}$ of the Jacobi curve. Let us denote by $\left(J_\lambda^{(k)}\right)^\angle$ the skew-symmetric complement of $J_\lambda^{(k)}$ with respect to the symplectic form σ_λ on $T_\lambda(T^*M)$, i.e.,

$$\left(J_\lambda^{(k)}\right)^\angle = \left\{ v \in T_\lambda(T^*M) \;:\; \sigma_\lambda(v, w) = 0 \quad \forall w \in J_\lambda^{(k)} \right\}.$$

Lemma 3.20. *For any integer $k \geq 0$, the kernel of the differential of the map P^k at a point λ satisfies*

$$\ker dP^k(\lambda) \subset \left(J_\lambda^{(k)}\right)^{\angle}.$$

Proof. Let $\lambda \in H$ and fix a canonical system of coordinates on T^*M near λ. In particular, in such coordinates π is a linear projection.

Let v be a vector in $\ker dP^k(\lambda)$. Then there exists a curve $s \mapsto \lambda_s$ in H such that $\lambda_0 = \lambda$, $\frac{d\lambda_s}{ds}\big|_{s=0} = v$, and the following equalities holds in the fixed coordinate system:

$$\frac{\partial^{l+1}}{\partial t^l \partial s}\left(\pi \circ e^{t\vec{h}}\lambda_s\right)\bigg|_{(t,s)=(0,0)} = d\pi \circ \frac{d^l}{dt^l}\left(e_*^{t\vec{h}}v\right)\bigg|_{t=0} = 0 \qquad \forall\, 0 \leq l \leq k. \tag{3.3}$$

Consider now $w \in J_\lambda^{(k)}$. Then there exists an integer j, $0 \leq j \leq k$, and a vertical vector field Y (i.e., $d\pi \circ Y = 0$) on T^*M such that w writes as

$$w = \frac{d^j}{dt^j}\left(e_*^{-t\vec{h}}Y(e^{t\vec{h}}\lambda)\right)\bigg|_{t=0}.$$

We have

$$\begin{aligned}\sigma_\lambda(v,w) &= \sigma_\lambda\left(v, \frac{d^j}{dt^j}\left(e_*^{-t\vec{h}}Y(e^{t\vec{h}}\lambda)\right)\bigg|_{t=0}\right), \\ &= \frac{d^j}{dt^j}\left(\sigma_\lambda\left(v, e_*^{-t\vec{h}}Y(e^{t\vec{h}}\lambda)\right)\right)\bigg|_{t=0}.\end{aligned} \tag{3.4}$$

The last equality holds, because we work with the fixed bilinear form σ_λ on the given vector space $T_\lambda T^*M$.

Using now that $e^{t\vec{h}}$ is a symplectomorphism, we obtain

$$\sigma_\lambda(v,w) = \frac{d^j}{dt^j}\left(\sigma_{e^{t\vec{h}}\lambda}\left(e_*^{t\vec{h}}v, Y(e^{t\vec{h}}\lambda)\right)\right)\bigg|_{t=0}$$

So far, all equalities starting from (3.4) were obtained in a coordinate-free manner. Now use again the fixed canonical coordinate system on T^*M near λ. In these coordinates, the form σ is in the Darboux form. In particular, the coefficients of this form are constants. Therefore,

$$\sigma_\lambda(v,w) = \sum_{l=1}^{j}\binom{j}{l}\sigma_\lambda(v_l, w_l),$$

$$\text{where} \quad v_l = \frac{d^l}{dt^l}\left(e_*^{t\vec{h}}v\right)\bigg|_{t=0} \quad \text{and} \quad w_l = \frac{d^{j-l}}{dt^{j-l}}\left(Y(e^{t\vec{h}}\lambda)\right)\bigg|_{t=0}.$$

By (3.3), every vector v_l is vertical. The vectors w_l in the chosen coordinate system are vertical as well since the vector field Y is vertical. As a consequence, $\sigma_\lambda(v_l, w_l) = 0$, which implies $\sigma_\lambda(v,w) = 0$. This completes the proof. $\square$

Remark 3.21. When the Jacobi curve is equiregular (i.e., the dimensions $\dim J_{\lambda(t)}^{(-k)}$, $k \in \mathbb{N}$, are constant for t close to 0), the skew-symmetric complement of the kth extension is equal to the kth contractions $J_\lambda^{(-k)}$ of the Jacobi curve (see [60, Lemma 1]). In that case we can show the equality $\ker dP^k(\lambda) = J_\lambda^{(-k)}$.

Since $\dim \left(J_\lambda^{(k)}\right)^\angle = 2n - \dim J_\lambda^{(k)}$, we get as a corollary of Lemma 3.20 that ample geodesics are characterized locally by their k-jets for k large enough.

Corollary 3.22. *Let $\lambda \in T^*M$ be ample. Then there exists an integer k_0 such that the map P^{k_0} is an immersion at λ.*

3.3 Orbital diffeomorphism

Projectively or affinely equivalent metrics have the same geodesics, up to the appropriate reparameterization. But do they have the same (normal) Hamiltonian vector field, up to an appropriate transformation? In particular, is it possible to recover the Hamiltonian vector field of a metric from the knowledge of the geodesics? We will see that both questions have a positive answer near ample geodesics, for which the covector can be obtained from the jets of the geodesics (see Corollary 3.22).

3.3.1 Orbital diffeomorphism on ample geodesics

Fix a manifold M and a bracket generating distribution D on M, and consider two sub-Riemannian metrics g_1 and g_2 on D. We denote by h_1 and h_2 the respective sub-Riemannian Hamiltonians of g_1 and g_2, and by $H_1 = h_1^{-1}(1/2)$ and $H_2 = h_2^{-1}(1/2)$ the respective $\frac{1}{2}$-level sets of these Hamiltonians.

Definition 3.23. We say that $\vec{h}_1$ and $\vec{h}_2$ are *orbitally diffeomorphic* on an open subset V_1 of H_1 if there exists an open subset V_2 of H_2 and a diffeomorphism $\Phi : V_1 \to V_2$ such that Φ is fiber-preserving, i.e. $\pi(\Phi(\lambda)) = \pi(\lambda)$, and Φ sends the integral curves of $\vec{h}_1$ to the integral curves of $\vec{h}_2$, i.e. $\Phi\big(e^{t\vec{h}_1}\lambda\big) = e^{s\vec{h}_2}\big(\Phi(\lambda)\big)$ for all $\lambda \in V_1$ and $t \in \mathbb{R}$ for which $e^{t\vec{h}_1}\lambda$ is well defined, or, equivalently

$$d\Phi \circ \vec{h}_1(\lambda) = \alpha(\lambda)\vec{h}_2(\Phi(\lambda)). \tag{3.5}$$

The map Φ is called an *orbital diffeomorphism* between the extremal flows of g_1 and g_2.

Remark 3.24. In the definition above, the orbital diffeomorphism Φ is defined as a mapping from H_1 to H_2. However it can be easily extended as a mapping $\bar{\Phi}$ from $T^*M \setminus h_1^{-1}(0)$ to itself by rescaling, i.e.,

$$\bar{\Phi}(\lambda) = \sqrt{2h_1(\lambda)}\Phi\left(\frac{\lambda}{\sqrt{2h_1(\lambda)}}\right).$$

This mapping sends the level sets $h_1^{-1}(C^2/2)$ of h_1 to the level sets $h_2^{-1}(C^2/2)$ of h_2, and the integral curves of $\vec{h}_1$ to the ones of $\vec{h}_2$. In particular (3.5) holds with a function $\bar{\alpha}(\lambda) = \alpha(\lambda/\sqrt{2h_1(\lambda)})$.

Proposition 3.25. *If $\vec{h}_1$ and $\vec{h}_2$ are orbitally diffeomorphic on a neighborhood of $H_1 \cap \pi^{-1}(q_0)$, then g_1, g_2 are projectively equivalent at q_0. If in addition the function $\alpha(\lambda)$ in* (3.5) *satisfies $\vec{h}_1(\alpha) = 0$, then g_1, g_2 are affinely equivalent.*

Proof. The first property is obvious. Indeed, if $\vec{h}_1$ and $\vec{h}_2$ are orbitally diffeomorphic, then the relation $\Phi\big(e^{t\vec{h}_1}\lambda\big) = e^{s\vec{h}_2}\big(\Phi(\lambda)\big)$ implies that any normal geodesics of g_2 near q_0 satisfies

$$\pi(e^{s\vec{h}_2}\lambda) = \pi \circ \Phi\big(e^{t\vec{h}_1}\big(\Phi^{-1}(\lambda)\big)\big) = \pi \circ e^{t\vec{h}_1}\big(\Phi^{-1}(\lambda)\big),$$

and thus coincides with a normal geodesic of g_1. Since on the other hand abnormal geodesics always coincide, the metrics g_1, g_2 have the same geodesics near q_0, and thus are projectively equivalent at q_0.

Note that $s = s(\lambda, t)$ is the reparameterization of time and that $\alpha(\lambda) = \frac{ds}{dt}(\lambda, 0)$. If $\vec{h}_1(\alpha) = 0$, then the function α is constant along the geodesics and the time-reparameterization is affine, which implies that the metrics are affinely equivalent. □

We have actually a kind of converse statement near ample geodesics.

Proposition 3.26. *Assume that the sub-Riemannian metrics g_1 and g_2 are projectively equivalent at q_0. Then, for any covector $\lambda_1 \in H_1 \cap \pi^{-1}(q_0)$ ample with respect to g_1, $\vec{h}_1$ and $\vec{h}_2$ are orbitally diffeomorphic on a neighborhood V_1 of λ_1 in H_1.*

If moreover g_1 and g_2 are affinely equivalent at q_0, then the function $\alpha(\lambda)$ in (3.5) *satisfies $\vec{h}_1(\alpha) = 0$.*

Proof. Assume that U is a neighborhood of q_0 such that g_1 and g_2 have the same geodesics in U, up to a reparameterization. Then g_1 and g_2 have the same ample geodesics in U, up to a reparameterization. Indeed, a geodesic $\gamma(t) = \pi(e^{t\vec{h}_1}\lambda)$ of g_1 which is ample at $t = 0$ is a geodesics of g_2 as well by assumption, and moreover a normal one since ample geodesics are not abnormal. The conclusion follows then from the fact that being ample at $t = 0$ with respect to g_1 is a property of $\gamma(t) = \pi(e^{t\vec{h}_1}\lambda)$ as an admissible curve (see [57, Proposition 6.15]), and does not depend neither on the time parameterization nor on the Hamiltonian vector field.

Fix a nonnegative integer k. As in Subsection 3.2.3, for $q \in U$ and $i = 1, 2$, we denote by $J_q^k(g_i)$ the space of k-jets at $t = 0$ of the normal geodesics of g_i issued from q and parameterized by arclength parameter with respect to the sub-Riemannian metric g_i. We set $J^k(g_i) = \bigsqcup_{q \in U} J_q^k(g_i)$ and we define $P_i^k : H_i \mapsto J^k(g_i)$ by

$$P_i^k(\lambda) = j_0^k \pi(e^{t\vec{h}_i}\lambda).$$

Let $\lambda_1 \in H_1 \cap \pi^{-1}(q_0)$ be an ample covector with respect to g_1. Then by Corollary 3.22 for a large enough integer k there exists a neighborhood V_1 of λ_1 in H_1 such that the map $P_1^k|_{V_1}$ is a diffeomorphism on its image. Up to reducing V_1 we assume that $\pi(V_1) \subset U$ and that every $\lambda \in V_1$ is ample. As a consequence, every geodesic $\pi(e^{t\vec{h}_1}\lambda)$ with $\lambda \in V_1$ is an ample geodesic with respect to g_2.

Let $\lambda_2 \in \pi^{-1}(q_0) \cap H_2$ be the covector such that the curves $\pi(e^{t\vec{h}_1}\lambda_1)$ and $\pi(e^{t\vec{h}_2}\lambda_2)$ coincide up to time reparameterization (λ_2 is unique since an ample geodesic is not abnormal). Since λ_2 is ample with respect to g_2, the same argument as above shows that there exists a neighborhood V_2 of λ_2 in H_2 such that $P_2^k|_{V_2}$ is a diffeomorphism on its image. Up to reducing V_1 and V_2 if necessary, the reparameterization of the geodesics from the arclength parameter with respect to g_1 to the arclength parameter with respect to g_2 induces naturally a diffeomorphism $\Psi_k : P_1^k(V_1) \subset J^k(g_1) \to P_2^k(V_2) \subset J^k(g_2)$. Thus the map Φ which completes the following diagram into a commutative one,

$$\begin{tikzcd}
V_1 \subset H_1 \arrow[r, "\Phi"] \arrow[d, "P_1^k"'] & V_2 \subset H_2 \arrow[d, "P_2^k"] \\
P_1^k(V_1) \subset J^k(g_1) \arrow[r, "\Psi_k"] & P_2^k(V_2) \subset J^k(g_2)
\end{tikzcd}$$

defines an orbital diffeomorphism between V_1 and V_2. This completes the proof of the first part of the proposition.

Assume now that g_1 and g_2 are affinely equivalent at q_0. Then the map Φ satisfies $\Phi\big(e^{t\vec{h}_1}\lambda\big) = e^{s\vec{h}_2}\big(\Phi(\lambda)\big)$, where $s = s(\lambda, t)$ is the reparameterization of time and $\frac{ds}{dt}(\lambda, 0) = \alpha(\lambda)$. Since g_1 and g_2 are affinely equivalent, $s(\lambda, t)$ must be affine with respect to t, which implies that $\alpha(e^{t\vec{h}_1}\lambda)$ is constant, and thus $\vec{h}_1(\alpha) = 0$. $\square$

Remark 3.27. We have seen in the proof just above that two projectively equivalent metrics have the same set of ample geodesics. In the same way, one can prove that they have the same set of strictly normal geodesics. However we can not affirm that they have the same normal geodesics: a geodesic could be both normal and abnormal for g_1 and only abnormal for g_2.

3.3.2 Fundamental algebraic system

Let M be a smooth manifold and D be a bracket generating distribution on M. Let us fix two sub-Riemannian metrics g_1, g_2 on (M, D).

Definition 3.28. The *transition operator* at a point $q \in M$ of the pair of metrics (g_1, g_2) is the linear operator $S_q : D_q \to D_q$ such that $g_1(q)(S_q v_1, v_2) = g_2(q)(v_1, v_2)$ for any $v_1, v_2 \in D_q$.

Obviously S_q is a positive g_1-self-adjoint operator and its eigenvalues $\alpha_1^2(q), \dots, \alpha_m^2(q)$ are positive real numbers (we choose $\alpha_1(q), \dots, \alpha_m(q)$ as positive numbers as well). Denote by $N(q)$ the number of distinct eigenvalues of S_q.

Definition 3.29. A point $q \in M$ is said to be *stable with respect to* g_1, g_2 if q is a regular point and $N(\cdot)$ is constant in some neighborhood of q.

The set of regular points and the set of points where $N(q)$ is locally constant are both open and dense in M, and so is the set of stable points.

Let us fix a stable point q_0. In a neighborhood U of q_0 we can choose a g_1-orthonormal frame $X_1, \dots, X_m$ of D whose values at any $q \in U$ diagonalizes S_q, i.e. $X_1(q), \dots, X_m(q)$ are eigenvectors of S_q associated with the eigenvalues $\alpha_1^2(q), \dots, \alpha_m^2(q)$ respectively. Note that $\frac{1}{\alpha_1}X_1, \dots, \frac{1}{\alpha_m}X_m$ form a g_2-orthonormal frame of D. We then complete $X_1, \dots, X_m$ into a frame $\{X_1, \dots, X_n\}$ of TM adapted to D at q_0. We call such a set of vector fields $\{X_1, \dots, X_n\}$ a *(local) frame adapted to the (ordered) pair of metrics* (g_1, g_2).

Let $u = (u_1, \dots, u_n)$ be the coordinates on the fibers T_q^*M induced by this frame, i.e. $u_i(q,p) = \langle p, X_i(q) \rangle$. The Hamiltonian functions h_1 and h_2 associated respectively with g_1 and g_2 write as

$$h_1 = \frac{1}{2}\sum_{i=1}^m u_i^2, \quad h_2 = \frac{1}{2}\sum_{i=1}^m \frac{u_i^2}{\alpha_i^2}.$$

In the corresponding frame $\{Y_1, \dots, Y_n, \partial_{u_1}, \dots, \partial_{u_n}\}$ of $T(T^*M)$, $\vec{h}_1$ has the form (3.2), i.e.,

$$\vec{h}_1 = \sum_{i=1}^m u_i Y_i + \sum_{i=1}^m \sum_{j,k=1}^n c_{ij}^k u_i u_k \partial_{u_j}, \tag{3.6}$$

and a simple computation gives

$$\vec{h}_2 = \sum_{i=1}^m \frac{u_i}{\alpha_i^2} Y_i + \sum_{i=1}^m \sum_{j,k=1}^n \frac{c_{ij}^k}{\alpha_i^2} u_i u_k \partial_{u_j} - \sum_{i=1}^m \sum_{j=1}^n \frac{1}{\alpha_i} X_j(\frac{1}{\alpha_i}) u_i^2 \partial_{u_j}. \tag{3.7}$$

Assume now that $\vec{h}_1$ and $\vec{h}_2$ are orbitally diffeomorphic near $\lambda_0 \in H_1 \cap \pi^{-1}(q_0)$ and let Φ be the corresponding orbital diffeomorphism. Following Remark 3.24, we assume that Φ is defined on a neighborhood V of λ_0 in the whole T^*M. Let us denote by Φ_i, $i = 1, \dots, n$, the coordinates u_i of Φ on the fiber, i.e. $u \circ \Phi(\lambda) = (\Phi_1(\lambda), \Phi_2(\lambda), \dots, \Phi_n(\lambda))$.

Using (3.6) and (3.7), we can write in coordinates the identity (3.5), i.e. $d\Phi \circ \vec{h}_1(\lambda) = \alpha(\lambda)\vec{h}_2(\Phi(\lambda))$, and deduce from there some conditions on the coordinates Φ_i. This computation has been made in [50], we just give the result here (our equations look a bit different than the ones of [50] because we use the structure coefficients c_{ij}^k here instead of the $\bar{c}_{ji}^k$).

Lemma 3.30 ([50], Lemmas 1 and 2). *A smooth fiber-preserving map Φ from an open subset V_1 of H_1 to an open subset V_2 of H_2 satisfies* (3.5) *if and only if the following conditions are satisfied:*

- *the function $\alpha(\lambda)$ is given by*
$$\alpha = \sqrt{\frac{\alpha_1^2 u_1^2 + \cdots + \alpha_m^2 u_m^2}{u_1^2 + \cdots + u_m^2}}, \tag{3.8}$$

- *for $k = 1, \dots, m$,*
$$\Phi_k = \frac{\alpha_k^2 u_k}{\alpha}, \tag{3.9}$$

- *for $j = 1, \dots, m$,*
$$\sum_{k=m+1}^{n} q_{jk} \Phi_k = \frac{R_j}{\alpha}, \tag{3.10}$$
where $q_{jk} = \sum_{i=1}^m c_{ij}^k u_i$ and
$$R_j = \vec{h}_1(\alpha_j^2) u_j + \alpha_j^2 \vec{h}_1(u_j) - \frac{1}{2}\alpha_j^2 u_j \frac{\vec{h}_1(\alpha^2)}{\alpha^2} - \frac{1}{2}\sum_{i=1}^{m} X_j(\alpha_i^2) u_i^2 - \sum_{1 \le i,k \le m} c_{ij}^k \alpha_k^2 u_i u_k,$$

- *for $k = m+1, \dots, n$,*
$$\vec{h}_1(\Phi_k) = \sum_{l=m+1}^{n} q_{kl} \Phi_l + \frac{1}{\alpha} \sum_{i=1}^{m} u_i \left(\alpha_i^2 q_{ki} + \frac{X_k(\alpha_i^2)}{2} u_i \right). \tag{3.11}$$

Remark 3.31. The spectral size N is equal to 1 if and only if g_2 is conformal to g_1 near q_0. In that case $g_2 = \alpha^2 g_1$ and $\alpha_1 = \cdots = \alpha_m = \alpha$. In particular the function α does not depend on u, i.e. $\alpha(\lambda)$ depends only on $\pi(\lambda)$.

This lemma gives directly the values of the first m components of Φ. The difficulty now is to find the other components from (3.10) and (3.11). It is more convenient to replace the differential equations (3.11) by infinitely many linear algebraic equations, forming the *fundamental algebraic system* as described by the following proposition.

Proposition 3.32. *Let Φ be an orbital diffeomorphism between the extremal flows of g_1 and g_2 with coordinates $(\Phi_1, \dots, \Phi_n)$. Set $\widetilde{\Phi} = (\Phi_{m+1}, \dots, \Phi_n)$. Then $\widetilde{\Phi}$ satisfies a linear system of equations,*
$$A\widetilde{\Phi} = b, \tag{3.12}$$

where A is a matrix with $(n-m)$ columns and an infinite number of rows, and b is a column vector with an infinite number of rows. These infinite matrices can be decomposed in layers of m rows as

$$A = \begin{pmatrix} A^1 \\ A^2 \\ \vdots \\ A^s \\ \vdots \end{pmatrix} \qquad \textit{and} \qquad b = \begin{pmatrix} b^1 \\ b^2 \\ \vdots \\ b^s \\ \vdots \end{pmatrix}, \tag{3.13}$$

where the coefficients a^s_{jk} of the $(m \times (n-m))$ matrix A^s, $s \in \mathbb{N}$, are defined by induction as

$$\begin{cases} a^1_{j,k} = q_{jk}, & 1 \le j \le m,\ m < k \le n, \\ a^{s+1}_{j,k} = \vec{h}_1(a^s_{j,k}) + \displaystyle\sum_{l=m+1}^{n} a^s_{j,l} q_{lk}, & 1 \le j \le m,\ m < k \le n, \end{cases} \tag{3.14}$$

(note that the columns of A are numbered from $m+1$ to n according to the indices of $\widetilde{\Phi}$) and the coefficients b^s_j, $1 \le j \le m$, of the vector $b^s \in \mathbb{R}^m$ are defined by

$$\begin{cases} b^1_j = \dfrac{R_j}{\alpha}, \\ b^{s+1}_j = \vec{h}_1(b^s_j) - \dfrac{1}{\alpha} \displaystyle\sum_{k=m+1}^{n} a^s_{j,k} \sum_{i=1}^{m} u_i \left(\alpha_i^2 q_{ki} + \frac{X_k(\alpha_i^2)}{2} u_i \right). \end{cases} \tag{3.15}$$

Note that A is a function of u and this function only depends on the choice of the local frame $\{X_1, \dots, X_n\}$. On the other hand the vector-valued function b depends on $\{X_1, \dots, X_n\}$ and on $\{\alpha_1, \dots, \alpha_m\}$.

Proof. We have to prove that, for every $s \in \mathbb{N}$, the coordinates $\widetilde{\Phi}$ satisfy

$$A^s \widetilde{\Phi} = b^s. \tag{3.16}$$

Observe first that (3.10) is exactly $A^1 \widetilde{\Phi} = b^1$, so (3.16) holds for $s = 1$. Assume by induction that it holds for a given s. Thus we have, for $j = 1, \dots, m$,

$$\sum_{k=m+1}^{n} a^s_{j,k} \Phi_k = b^s_j.$$

Taking the Lie derivative of these expressions by $\vec{h}_1$, we get

$$\sum_{k=m+1}^{n} \vec{h}_1(a^s_{j,k}) \Phi_k + \sum_{k=m+1}^{n} a^s_{j,k} \vec{h}_1(\Phi_k) = \vec{h}_1(b^s_j).$$

Replacing every term $\vec{h}_1(\Phi_k)$ by its expression in (3.11) and reorganizing, we obtain a new linear equation,

$$\sum_{k=m+1}^{n}\left(\vec{h}_1(a^s_{j,k}) + \sum_{l=m+1}^{n} a^s_{j,l} q_{lk}\right)\Phi_k = \\ \vec{h}_1(b^s_j) - \frac{1}{\alpha}\sum_{k=m+1}^{n} a^s_{j,k}\sum_{i=1}^{m} u_i\left(\alpha_i^2 q_{ki} + \frac{X_k(\alpha_i^2)}{2}u_i\right),$$

which is exactly the jth row of $A^{s+1}\widetilde{\Phi} = b^{s+1}$. This ends the induction and then the proof of the proposition. □

3.3.3 Injectivity of the fundamental algebraic system

The matrix A appears to be strongly related to the Jacobi curves, and we will use the properties of the latter to deduce the non degeneracy of A. Let us denote by $u(\lambda)$ the coordinates of $\lambda \in T^*M$.

Proposition 3.33. *If $\lambda \in T^*M \setminus h_1^{-1}(0)$ is ample with respect to g_1, then $A(u(\lambda))$ is injective. As a consequence, there exists at least one $(n-m)\times(n-m)$ minor of the matrix $A(u)$ which is a non identically zero function of u.*

This proposition results directly from the following lemma combined with Theorem 3.18.

Lemma 3.34. *Let s be a positive integer. Denote by A_s the $sm \times (n-m)$ matrix formed by the first s layers of A. Then*

$$\text{rank} A_s(u) = \dim J_\lambda^{(s+1)} - n - m.$$

Proof. We begin by proving that, for any positive integer s,

$$(\text{ad}\vec{h})^s Y_j = \sum_{k=m+1}^{n} a^s_{j,k} Y_k \mod J_\lambda^{(s)}, \quad 1 \le j \le m. \tag{3.17}$$

Remark first that, for $k = 1, \dots, n$,

$$\begin{aligned}[\vec{h}_1, Y_k] = \left[\sum_{i=1}^{m} u_i Y_i + \sum_{i=1}^{m}\sum_{j,l=1}^{n} c^l_{ij} u_i u_l \partial_{u_j}, Y_k\right] &= \sum_{i=1}^{m} u_i [Y_i, Y_k] \mod J_\lambda, \\ &= \sum_{i=1}^{m} u_i \sum_{l=1}^{n} c^l_{ik} Y_l \mod J_\lambda,\end{aligned}$$

which writes as

$$[\vec{h}_1, Y_k] = \sum_{l=1}^{n} q_{kl} Y_l \mod J_\lambda. \tag{3.18}$$

Let us prove (3.17) by induction on s. The case $s = 1$ is a direct consequence of (3.18) since the latter implies that, for $j = 1, \dots, m$,

$$[\vec{h}_1, Y_j] = \sum_{k=m+1}^{n} q_{jk} Y_k + \sum_{k=1}^{m} q_{jk} Y_k \mod J_\lambda = \sum_{k=m+1}^{n} a^1_{j,k} Y_k \mod J^{(1)}_\lambda.$$

Assume now that (3.17) is satisfied for a given s. Using the induction hypothesis, we write

$$(\mathrm{ad}\vec{h}_1)^{s+1} Y_j = \left[\vec{h}_1, (\mathrm{ad}\vec{h}_1)^s Y_j\right] = \left[\vec{h}_1, \sum_{k=m+1}^{n} a^s_{j,k} Y_k\right] \mod J^{(s+1)}_\lambda,$$

since $[\vec{h}_1, J^{(s)}_\lambda] \subset J^{(s+1)}_\lambda$. The last bracket above expands as

$$\begin{aligned}
\left[\vec{h}_1, \sum_{k=m+1}^{n} a^s_{j,k} Y_k\right] &= \sum_{k=m+1}^{n} \vec{h}_1(a^s_{j,k}) Y_k + \sum_{k=m+1}^{n} a^s_{j,k} \left[\vec{h}_1, Y_k\right], \\
&= \sum_{k=m+1}^{n} \vec{h}_1(a^s_{j,k}) Y_k + \sum_{k=m+1}^{n} a^s_{j,k} \sum_{l=1}^{n} q_{kl} Y_l \mod J_\lambda,
\end{aligned}$$

thanks to (3.18). Splitting and renumbering the second sum above, we obtain

$$\begin{aligned}
(\mathrm{ad}\vec{h}_1)^{s+1} Y_j &= \sum_{k=m+1}^{n} \left(\vec{h}_1(a^s_{j,k}) + \sum_{l=m+1}^{n} a^s_{j,l} q_{lk} \right) Y_k + \sum_{l=1}^{m} \sum_{k=m+1}^{n} a^s_{j,k} q_{kl} Y_l \mod J^{(s+1)}_\lambda, \\
&= \sum_{k=m+1}^{n} a^{s+1}_{j,k} Y_k \mod J^{(s+1)}_\lambda,
\end{aligned}$$

which ends the induction and proves (3.17).

Now, from Lemma 3.16, for any positive integer s there holds $J^{(s+1)}_\lambda = J^{(1)}_\lambda + \mathrm{span}\{(\mathrm{ad}\vec{h}_1)^k Y_j(\lambda) \mid 1 \le k \le s,\ 1 \le j \le m\}$. Thus it results from (3.17) that

$$\dim J^{(s+1)}_\lambda = \dim J^{(1)}_\lambda + \mathrm{rank} A_s(u(\lambda)), \quad \text{where } A_s = \begin{pmatrix} A^1 \\ A^2 \\ \vdots \\ A^s \end{pmatrix}.$$

Since $\dim J^{(1)}_\lambda = n + m$ for any λ, the lemma is proved. □

A first consequence of the injectivity of A is that the system of equations $A\widetilde{\Phi} = b$ is a sufficient condition for Φ to be an orbital diffeomorphism.

Proposition 3.35. *Consider a local frame $\{X_1, \dots, X_n\}$ of D on an open subset $U \subset M$, and smooth positive functions $\alpha_1, \dots, \alpha_m$ on U. Let A and b be the associated matrices defined by* (3.14) *and* (3.15)*, and denote by g_1 and g_2 the sub-Riemannian metrics defined locally by the orthonormal frames $X_1, \dots, X_m$ and $\frac{X_1}{\alpha_1}, \dots, \frac{X_m}{\alpha_m}$ respectively.*

Assume $\lambda \in T^*U$ is ample with respect to g_1. Let $\widetilde{\Phi} = (\Phi_{m+1}, \dots, \Phi_n)$ be the solution of $A\widetilde{\Phi} = b$ near λ, and let $\Phi_1, \dots, \Phi_m$ be defined by (3.9)*. Then the local smooth fiber-preserving map $\Phi : H_1 \to H_2$ defined by $u_i \circ \Phi = \Phi_i$, $i = 1, \dots, n$, satisfies* (3.5)*.*

Proof. Following Lemma 3.30, it is sufficient to prove that $\widetilde{\Phi}$ satisfies (3.10) and (3.11) near λ. The equations of the first layer, i.e. $A^1\widetilde{\Phi} = b^1$, are exactly (3.10), hence we are left with the task of proving that $\widetilde{\Phi}$ satisfies (3.11).

Fix a positive integer s and $j \in \{1, \dots, m\}$. Let us write the jth row of the system $A^s\Phi = b^s$,

$$\sum_{k=m+1}^{n} a^s_{j,k}\Phi_k = b^s_j,$$

and differentiate this expression in the direction $\vec{h}_1$. We thus obtain

$$\sum_{k=m+1}^{n} \vec{h}_1(a^s_{j,k})\Phi_k + \sum_{k=m+1}^{n} a^s_{j,k}\vec{h}_1(\Phi_k) = \vec{h}_1(b^s_j).$$

Write now the jth row of the system $A^{s+1}\Phi = b^{s+1}$, replacing the coefficients by their recurrence formula,

$$\sum_{k=m+1}^{n} \vec{h}_1(a^s_{j,k})\Phi_k + \sum_{k,l=m+1}^{n} a^s_{j,l}q_{l,k}\Phi_k$$
$$= \vec{h}_1(b^s_j) - \frac{1}{\alpha}\sum_{k=m+1}^{n} a^s_{j,k}\sum_{i=1}^{m} u_i\left(\alpha_i^2 q_{ki} + \frac{X_k(\alpha_i^2)}{2}u_i\right),$$

and take the difference between the last two formulas. Rearranging the order of summation we obtain

$$\sum_{k=m+1}^{n} a^s_{j,k}\left(\vec{h}_1(\Phi_k) - \sum_{l=m+1}^{n} q_{k,l}\Phi_l + \frac{1}{\alpha}\sum_{i=1}^{m} u_i\left(\alpha_i^2 q_{ki} + \frac{X_k(\alpha_i^2)}{2}u_i\right)\right) = 0. \qquad (3.19)$$

Denote by Ψ_k the terms inside the bracket above, and set $\Psi = (\Psi_{m+1}, \dots, \Psi_n)$. Formula (3.11) for $\widetilde{\Phi}$ is exactly $\Psi = 0$. From (3.19), the vector Ψ satisfies the system $A\Psi = 0$. Moreover, by Proposition 3.33 the matrix $A(u)$ has full rank at $u = u(\lambda)$, and hence in a neighbourhood of $u(\lambda)$ in T^*M. On this neighbourhood Ψ must be identically zero, which implies that $\widetilde{\Phi}$ satisfies (3.11). The statement is proved. $\square$

3.4 First divisibility and consequences

3.4.1 First divisibility

In [50], Zelenko introduced an algebraic condition called *first divisibility condition*, which implies interesting conditions on the eigenvalues α_i^2 and on the structure coefficients.

Consider two sub-Riemannian metric g_1, g_2 on (M, D), a stable point q_0 with respect to these metrics, and introduce as in Section 3.3.2 a frame $\{X_1, \dots, X_n\}$ adapted to (g_1, g_2) and the associated coordinates $(u_1, \dots, u_n)$ on the fibers of T^*M.

Set $\mathcal{P} = \alpha_1^2 u_1^2 + \cdots + \alpha_m^2 u_m^2$, where $\alpha_1^2, \dots, \alpha_m^2$ are the eigenvalues of the transition operator. Note that $\mathcal{P}$ and its Lie-derivative $\vec{h}_1(\mathcal{P})$ are polynomial functions on the fiber, i.e. polynomial functions of u (see [50, Eq. (2.30)] for an intrinsic definition of $\mathcal{P}$). We say that the ordered pair of sub-Riemannian metrics (g_1, g_2) satisfies *the first divisibility condition* if the polynomial $\vec{h}_1(\mathcal{P})$ is divisible by $\mathcal{P}$.

Proposition 3.36 ([50], Proposition 6)**.** *If (g_1, g_2) and (g_2, g_1) satisfy the first divisibility condition in a neighborhood U of a stable point q_0, then for any $q \in U$ the following properties hold:*

- *for any $1 \leq i, j \leq m$, $\quad [X_i, X_j](q) \notin D(q) \;\Rightarrow\; \alpha_i(q) = \alpha_j(q)$;*
- $X_i\left(\frac{\alpha_j^2}{\alpha_i^2}\right) = 2c_{ij}^j\left(1 - \frac{\alpha_j^2}{\alpha_i^2}\right)$ *for any* $0 \leq i, j \leq m$*;*
- $X_i\left(\frac{\alpha_j^2}{\alpha_i}\right) = 0, \quad \alpha_j \neq \alpha_j$*;*
- $X_i\left(\frac{\alpha_j}{\alpha_k}\right) = 0, \quad \alpha_i \neq \alpha_j,\; \alpha_i \neq \alpha_k$.

It appears actually that this condition is always fulfilled by pairs of metrics whose Hamiltonian vector fields are orbitally diffeomorphic.

Proposition 3.37. *If $\vec{h}_1, \vec{h}_2$ are orbitally diffeomorphic near some $\lambda \in \pi^{-1}(q_0)$, then (g_1, g_2) and (g_2, g_1) satisfy the first divisibility condition near q_0.*

Proof. Let Φ be the orbital diffeomorphism between the extremal flows of g_1, g_2. From Proposition 3.32, the $n - m$ last coordinates of Φ satisfy $A\widetilde{\Phi} = b$. Let us give first some algebraic properties of the components Φ_i.

Notice that
$$\alpha^2 = \frac{\mathcal{P}}{h_1},$$
which implies that
$$\frac{\vec{h}_1(\mathcal{P})}{\mathcal{P}} = \frac{\vec{h}_1(\alpha^2)}{\alpha^2}. \tag{3.20}$$

Using this remark, a simple induction argument shows that, for any positive integer s, there exists a constant $C_s > 0$ and polynomial functions $\mathrm{pol}_{s,j}(u)$ on the fiber such that

$$b_j^s = \frac{C_s \alpha_j^2 u_j}{\alpha} \left(\frac{\vec{h}_1(\mathcal{P})}{\mathcal{P}} \right)^s + \frac{1}{\alpha \mathcal{P}^{s-1}} \mathrm{pol}_{s,j}(u), \qquad j = 1, \ldots, m. \tag{3.21}$$

From Proposition 3.33, the matrix A admits at least one nonzero maximal minor δ. Since all coefficients of A are polynomial functions of u, δ is in turn polynomial in u. Using Cramer's rule, we deduce from (3.21) that there exists an integer S such that, for $i = m+1, \ldots, n$,

$$\Phi_i = \frac{1}{\alpha \delta \mathcal{P}^S} \times \text{polynomial in } u. \tag{3.22}$$

Let us prove now the divisibility of $\vec{h}_1(\mathcal{P})$ by $\mathcal{P}$. Choose an arbitrary large integer s ($s > S$) and $j \in \{1, \ldots, m\}$, and consider the jth equation of the sth layer of the system (3.12),

$$a_{j,m+1}^s \Phi_{m+1} + \cdots + a_{j,n-m}^s \Phi_{n-m} = b_j^s.$$

Recall that all coefficients $a_{i,j}^s$ are polynomial functions of u. Substituting expressions (3.21) and (3.22) for b_j^s and the Φ_i's respectively, we get,

$$\frac{C_s \alpha_j^2 u_j}{\alpha} \left(\frac{\vec{h}_1(\mathcal{P})}{\mathcal{P}} \right)^s + \frac{1}{\alpha \mathcal{P}^{s-1}} \mathrm{pol}_{s,j}(u) = \frac{1}{\alpha \delta \mathcal{P}^S} \times \text{polynomial in } u.$$

Multiplying by $\alpha \mathcal{P}^S$, we obtain finally,

$$C_s \alpha_j^2 \frac{u_j \vec{h}_1(\mathcal{P})^s}{\mathcal{P}^{s-S}} = \frac{1}{\mathcal{P}^{s-S-1}} \mathrm{pol}_{s,j}(u) + \frac{1}{\delta} \times \text{polynomial in } u. \tag{3.23}$$

Assume by contradiction that $\vec{h}_1(\mathcal{P})$ is not divisible by $\mathcal{P}$. Let k be the maximal nonnegative integer such that δ is divisible by $\mathcal{P}^k$ and take $s > k + S$. Taking into account that $\mathcal{P}$ is a positive quadratic form, and thus it is irreducible over $\mathbb{R}$, we have that the exponent of $1/\mathcal{P}$ in the left-hand side of (3.23) is strictly bigger that the one in the right-hand side. We have a contradiction, which completes the proof. $\square$

This proposition has several consequences. The first one is an obvious corollary of Propositions 3.36 and 3.37. Let us introduce first some notations. Let $N = N(q_0)$ be the number of distinct eigenvalues of the transition operator S_q for q near q_0. We assume that the eigenvalues α_i^2, $i = 1, \ldots, m$, are numbered in such a way that $\alpha_1^2, \ldots, \alpha_N^2$ are the N distinct ones. For $\ell = 1, \ldots, N$, we denote by I_ℓ the set of indices $i \in \{1, \ldots, m\}$ such that $\alpha_i = \alpha_\ell$.

Corollary 3.38. *Assume $\vec{h}_1, \vec{h}_2$ are orbitally diffeomorphic near some $\lambda \in \pi^{-1}(q_0)$. Then, for any $\ell, \ell' \in \{1, \dots, N\}$, $\ell \neq \ell'$,*

(i) $[X_\ell, X_{\ell'}] \in D^1$;

(ii) if $X \in \mathrm{Lie}\{X_i \ : \ i \in I_\ell\}$*, then* $X\left(\frac{\alpha_{\ell'}^2}{\alpha_\ell}\right) = 0$;

(iii) if $X \in \mathrm{Lie}\{X_i \ : \ i \in I_\ell\}$ *and* $\ell'' \neq \ell$*, then* $X\left(\frac{\alpha_{\ell'}}{\alpha_{\ell''}}\right) = 0$.

The second consequence results directly from the definition of first-divisibility.

Lemma 3.39. *If $\vec{h}_1, \vec{h}_2$ are locally orbitally diffeomorphic, then*

$$\vec{h}_1(\mathcal{P}) = Q\mathcal{P}, \quad \text{where} \quad Q = \sum_{i=1}^{m} \frac{X_i(\alpha_i^2)}{\alpha_i^2} u_i. \tag{3.24}$$

Proof. From Proposition 3.37, the third degree polynomial $\vec{h}_1(\mathcal{P})$ is divisible by the quadratic polynomial $\mathcal{P}$. Hence there exists a linear function $Q = \sum_{j=1}^{n} p_j u_j$ such that

$$\vec{h}_1(\mathcal{P}) = Q\mathcal{P} = \Big(\sum_{j=1}^{n} p_j u_j\Big)\Big(\sum_{i=1}^{m} \alpha_i^2 u_i^2\Big).$$

On the other hand, using the expression (3.6) of $\vec{h}_1$, we get

$$\vec{h}_1(\mathcal{P}) = \sum_{i,j=1}^{m} X_i(\alpha_j^2) u_i u_j^2 + \sum_{i,j=1}^{m} \sum_{k=1}^{n} 2c_{ij}^k \alpha_j^2 u_i u_j u_k.$$

Identifying the coefficients of the monomials u_i^3 and $u_i^2 u_j$, $1 \leq i \leq m$, $m < j \leq n$, in the two expressions above, we obtain respectively

$$p_i = \frac{X_i(\alpha_i^2)}{\alpha_i^2} \text{ for } 1 \leq i \leq m, \quad p_i = 0 \text{ for } m < i \leq n.$$

□

3.4.2 Existence of first integrals

An important consequence of the first-divisibility property is the existence of quadratic first integrals for the Hamiltonian flow. Let g_1, g_2 be two sub-Riemannian metrics on (M, D), and q_0 be a stable point w.r.t. g_1, g_2. Proceeding as above, we assume that the eigenvalues α_i^2, $i = 1, \dots, m$, are numbered in such a way that $\alpha_1^2, \dots, \alpha_N^2$ are the N distinct ones. We introduce also a frame $\{X_1, \dots, X_n\}$ adapted to (g_1, g_2), the associated coordinates $(u_1, \dots, u_n)$ on the fibers of T^*M, and the polynomial

$$\mathcal{P} = \sum_{i=1}^{m} \alpha_i^2 u_i^2.$$

Proposition 3.40. *If $\vec{h}_1$ and $\vec{h}_2$ are orbitally diffeomorphic near some $\lambda \in \pi^{-1}(q_0)$, then the function*

$$F = \Big(\prod_{\ell=1}^{N} \alpha_\ell^2\Big)^{-\frac{2}{N+1}} \mathcal{P}$$

is a first integral of the normal extremal flow of g_1, i.e.

$$\vec{h}_1(F) = 0.$$

Note that, in the Riemannian case (i.e. $D = TM$), the existence of this quadratic first integral was shown by Levi-Civita in [48] (see also [49], where this integral is attributed to Painlevé).

Proof. Set $f = \Big(\prod_{\ell=1}^{N} \alpha_\ell^2\Big)^{-\frac{2}{N+1}}$. Using Lemma 3.39 we get

$$\vec{h}_1(F) = \vec{h}_1(f\mathcal{P}) = \big(\vec{h}_1(f) + fQ\big)\mathcal{P}. \tag{3.25}$$

Further, using the expression (3.6) of $\vec{h}_1$, we have

$$\begin{aligned} \vec{h}_1(f) &= -\frac{2}{N+1} \sum_{i=1}^{m} \left(\Big(\prod_{\ell=1}^{N} \alpha_\ell^2\Big)^{-\frac{2}{N+1}-1} u_i \sum_{\ell=1}^{N} \big(\prod_{k\neq\ell} \alpha_k^2\big) X_i(\alpha_\ell^2) \right) \\ &= -\frac{2}{N+1} f \sum_{i=1}^{m} u_i \sum_{\ell=1}^{N} \frac{X_i(\alpha_\ell^2)}{\alpha_\ell^2}. \end{aligned} \tag{3.26}$$

Notice now that Corollary 3.38, *(ii)*, implies that,

$$\text{if } \alpha_j \neq \alpha_i, \qquad \text{then} \quad X_i(\alpha_j^2) = \frac{\alpha_j^2 X_i(\alpha_i^2)}{2\alpha_i^2}.$$

Plugging this into (3.26), we get

$$\vec{h}_1(f) = -\frac{2}{N+1}\Big(\frac{N-1}{2} + 1\Big) f \sum_{i=1}^{m} \frac{X_i(\alpha_i^2)}{\alpha_i^2} u_i = -fQ.$$

By (3.25) we obtain $\vec{h}_1(F) = 0$, which completes the proof. □

The normal extremal flow of g_1 already admits h_1 as a quadratic first integral, and F is not proportional to h_1 except when $N = 1$, which corresponds to the case where g_1 and g_2 are conformal to each other. This proves Theorem 3.4. The existence of several quadratic first integrals appears to be a strong condition on the metric.

Proposition 3.41. *Let (M, D) be fixed. The normal extremal flow of a generic sub-Riemannian metric on (M, D) admit no other non-trivial quadratic first integral than its Hamiltonian.*

Proposition 3.41 is the generalization to sub-Riemannian metrics of a result stated for Riemannian metrics in [73], namely Corollary 3 of Theorem 1. It is then sufficient to show the following result, which is the exact generalization to the sub-Riemannian case of that Theorem 1 (in the case of polynomials of degree $d = 2$).

Proposition 3.42. *Let D be a Lie-bracket generating distribution on an open ball $B \subset \mathbb{R}^n$ and g be a smooth metric on D. Then, for any $\varepsilon > 0$ there exists a metric $\widetilde{g}$ on D which is ε-close to g in the C^∞-topology, and $\varepsilon' > 0$ such that for any C^2 metric g' on D which is ε'-close to $\widetilde{g}$ in the C^2-topology, the normal extremal flow of g' does not admit a non-trivial quadratic first-integral (non-trivial means non proportional to the Hamiltonian $h_{g'}$ associated with g').*

Note that we work on an open subset of $\mathbb{R}^n$ and not in a general manifold since, as noticed in [73], it is sufficient to prove the result locally. Thus we identify T^*B to $B \times \mathbb{R}^n$ and we write a covector $\lambda \in T^*B$ as a pair (x, p), where $x = \pi(\lambda)$.

The proof of Theorem 1 in [73] goes as follows. Choose k sets of N points[1] in B, $S_\ell = \{x_{\ell,1}, \dots, x_{\ell,N}\}$, $\ell = 1, \dots, k$, where $N = n(n+1)/2$ and k is an integer larger than 4. Then consider the initial covectors associated with all the geodesics joining the points in different sets. The existence of a quadratic first-integral implies strong constraints on these covectors. If the points are in "general" position, small and localized perturbations of the metric along the geodesics make these constraints incompatible, which prevents the existence of a quadratic first-integral.

This argument is very general, it is not specific to Riemannian geometry. It only requires the following assumption on the kN points:

(H.1) no three of the points $x_{1,1}, \dots, x_{k,N}$ lie on one normal geodesic;

(H.2) for every sets $S_i \neq S_j$ and every point $x \in S_i$, there exists a 2-decisive set (see below) $p_1, \dots, p_N \in T_x^*B \simeq \mathbb{R}^n$ such that

$$S_j = \{\pi \circ e^{\vec{h}_g}(x, p_1), \dots, \pi \circ e^{\vec{h}_g}(x, p_N)\};$$

(H.3) for every pair of sets $S_i \neq S_j$ and every pair of points $x \in S_i$ and $y \in S_j$, let $p \in T_x^*B \simeq \mathbb{R}^n$ be the covector such that $y = \pi \circ e^{\vec{h}_g}(x, p)$; then perturbations $\widetilde{g}$ of the metric g localized near one point of the geodesic $\pi \circ e^{t\vec{h}_g}(x, p)$, $t \in (0, 1)$, generate a neighbourhood of $e^{\vec{h}_g}(x, p)$ in T^*B, i.e. the map

$$\widetilde{g} \mapsto e^{\vec{h}_{\widetilde{g}}}(x, p)$$

is a submersion at $\widetilde{g} = g$.

[1] In [73], the sets of points are labelled $A = \{A_1, \dots, A_N\}$, $B_\ell = \{B_{\ell,1}, \dots, B_{\ell,N}\}$, $\ell = 1, \dots, \kappa$, $C = \{C_1, \dots, C_N\}$, with $\kappa = k - 2$ greater than 2.

As a consequence, if any sub-Riemannian metric g admits kN points satisfying **(H.1)**–**(H.3)**, then Proposition 3.42 can be proved in the same way as [73, Theorem 1]. Thus we are reduced to proving the existence of such sets of points.

Remark 3.43. A set of $N = n(n+1)/2$ vectors of $\mathbb{R}^n$ is called *2-decisive* if the values of any quadratic polynomial on this set determine the polynomial. Clearly, the set of 2-decisive sets is open and dense in the set of N-tuples of vectors of $\mathbb{R}^n$.

Let us first study the perturbation property of **(H.3)**. We denote by $\mathcal{G}$ the set of sub-Riemannian C^2 metrics on D. Locally $\mathcal{G}$ can be identified with an open subset of the Banach space $\mathcal{S}$ of C^2 maps from B to the set of symmetric $(m \times m)$ matrices.

Lemma 3.44. *Let g be a sub-Riemannian metric and $\lambda_0 \in T^*B$ be an ample covector with respect to g. Then the map*

$$\psi : \widetilde{g} \in \mathcal{G} \mapsto e^{\vec{h}_{\widetilde{g}}}(\lambda_0) \in T^*B$$

is a submersion at $\widetilde{g} = g$.

Proof. From standard results on the dependance of differential equations with respect to a parameter, the differential of ψ at g can be written as

$$D_g\psi : \widetilde{g} \in \mathcal{S} \mapsto e_*^{\vec{h}_g} \int_0^1 e_*^{-s\vec{h}_g} \left(\frac{\partial \vec{h}_g(\lambda(s))}{\partial g}(g) \cdot \widetilde{g} \right) ds,$$

where $\lambda(s) = e^{s\vec{h}_g}\lambda_0$, $s \in [0,1]$. Now, we can easily verify that, for a given $\lambda \in T^*B$, the image of the partial differential of $\vec{h}_g$ with respect to g is

$$\mathrm{Im}\left(\frac{\partial \vec{h}_g(\lambda)}{\partial g}(g) \right) = \{ v \in T_\lambda(T^*B) \ : \ \pi_* v \in D \} = J_\lambda^{(1)},$$

where the last equality comes from Lemma 3.16. As a consequence, the image of the linear map $D_g\psi$ satisfies

$$\mathrm{Im} D_g\psi = e_*^{\vec{h}_g} \int_0^1 e_*^{-s\vec{h}_g} J_{\lambda(s)}^{(1)} ds,$$

and ψ is a submersion at g if and only if

$$\mathrm{span}\left\{ e_*^{-s\vec{h}_g} J_{\lambda(s)}^{(1)} \ : \ s \in [0,1] \right\} = T_{\lambda_0}(T^*B). \tag{3.27}$$

Assume by contradiction that (3.27) does not hold. Then there exists $p \in T_{\lambda_0}^*(T^*B)$ such that $\langle p, e_*^{-s\vec{h}_g} J_{\lambda(s)}^{(1)} \rangle = 0$ for all $s \in [0,1]$. Note that $J_{\lambda_0}(s) \subset e_*^{-s\vec{h}_g} J_{\lambda(s)}^{(1)}$ (see Definition 3.13). Hence, for all smooth curve $l(\cdot)$ such that $l(s) \in J_{\lambda_0}(s)$ for all

$s \in [0,1]$, we have $\langle p, l(s)\rangle \equiv 0$. Taking the derivatives with respect to s at 0, we get

$$\langle p, \frac{d^j}{dt^j} l(0)\rangle = 0 \quad \text{for all integer } j.$$

From Definition 3.2.2 this implies $\langle p, J_{\lambda_0}^{(k)}\rangle = 0$ for any integer k, which contradicts the fact that λ_0 is ample. Thus (3.27) holds and ψ is a submersion at g.

As a direct consequence of this lemma, if **(H.2)** is satisfied with ample covectors p_i, then **(H.3)** is satisfied as well.

Let x be a point in B and $\exp_x$ be the exponential mapping at x, $\exp_x : p \in T_x^*B \to \pi \circ e^{\vec{h}_g}(x,p) \in B$. Since conjugate times are isolated from 0 along a geodesic which is ample at $t = 0$ (see for instance [75, Cor. 8.47]), for any ample covector p the map $\exp_x$ is locally open near tp for t small enough. Let us denote by $\mathcal{A}_x$ the set of N-tuples of ample covectors $(p_1,\dots,p_N)$ in $(T_x^*B)^N$ which are 2-decisive, and set

$$S(x) = \{(\exp_x(p_1),\dots,\exp_x(p_N)) \in B^N \ : \ (p_1,\dots,p_N) \in \mathcal{A}_x\}.$$

By Remark 3.43 and Theorem 3.18, the set $\mathcal{A}_x$ is open and dense in $(T_x^*B)^N$. It results then from the local openness of the exponential map that $S(x)$ has a non empty interior with $(x,\dots,x) \in \overline{\text{int}S(x)}$.

We are now in a position to give the construction of sets of N points S_ℓ, $\ell = 1,\dots,k$, satisfying **(H.1)–(H.3)**. The properties above ensure that we can choose $S_1 = \{x_{1,1},\dots,x_{1,N}\} \in B^N$ such that no three points are aligned and such that the intersection

$$\bigcap_{i=1}^{N} \text{int}S(x_{1,i})$$

is non empty. We then choose $S_2 = \{x_{2,1},\dots,x_{2,N}\}$ in this intersection such that no three points in $S_1 \cup S_2$ are aligned and such that the intersection of all sets $\text{int}S(x_{1,i}) \cap \text{int}S(x_{2,i})$ is non empty. Iterating this construction we obtain k sets of N points satisfying **(H.1)–(H.3)**. This together with the argument in [73] shows Proposition 3.42 and then Proposition 3.41. □

Theorem 3.7 is a direct consequence of this proposition and Proposition 3.40.

3.4.3 Consequences on affine equivalence

Proposition 3.45. *If two sub-Riemannian metrics g_1, g_2 on (M, D) are affinely equivalent on an open connected subset $U \subset M$, then all the eigenvalues $\alpha_1^2,\dots,\alpha_m^2$ of the transition operator are constant.*

Proof. Let g_1, g_2 be two affinely equivalent metrics on U, and let $q_0 \in U$ be a stable point with respect to g_1, g_2. From Proposition 3.26, $\vec{h}_1$ and $\vec{h}_2$ are locally orbitally diffeomorphic and $\vec{h}_1(\alpha) = 0$. Using equality (3.20) we get that $\vec{h}_1(\mathcal{P}) = 0$.

From Lemma 3.39, $\vec{h}_1(\mathcal{P}) = Q\mathcal{P}$. Hence $Q = 0$, which implies that $X_i(\alpha_i^2) = 0$ for $i = 1, \dots, m$. Using Corollary 3.38 *(ii)*, we obtain $X_i(\alpha_j^2)$ for any $i, j \in \{1, \dots, m\}$, and since the vector fields $X_1, \dots, X_m$ are bracket-generating, we finally get that $\alpha_1^2, \dots, \alpha_m^2$ are constant near q_0.

Thus the eigenvalues $\alpha_1^2, \dots, \alpha_m^2$ are continuous functions on U which are locally constant near stable points. Since the set of stable points is dense in U, we conclude that all eigenvalues are constant. □

Remark 3.46. It results from (3.24) that, if all α_i's are constants, then $\vec{h}_1(\mathcal{P}) = 0$, which in turn implies $\vec{h}_1(\alpha) = 0$ by (3.20). Combining this remark with Propositions 3.26 and 3.25, we obtain the following result: if two metrics are projectively equivalent and the transition operator has constant eigenvalues, then the metrics are affinely equivalent.

Corollary 3.47. *If a sub-Riemannian metric is conformally projectively rigid, then it is affinely rigid.*

Proof. Let g_1 be a conformally projectively rigid sub-Riemannian metric. If a metric g_2 is affinely equivalent to g_1, then it is also projectively equivalent to g_1, and by hypothesis $g_2 = \alpha^2 g_1$. Hence α^2 is the unique eigenvalue of the transition operator and is constant by Proposition 3.45, which implies that g_2 is trivially equivalent to g_1. □

Remark 3.48. The polynomial equation $\vec{h}_1(\mathcal{P}) = 0$ has other consequences than $X_i(\alpha_i^2) = 0$, namely:

- $c_{ij}^i = 0$ if $\alpha_i \neq \alpha_j$, $i, j \in \{1, \dots, m\}$ (see also the second line of Prop 3.36);
- $c_{ij}^k + c_{kj}^i = 0$ if $\alpha_i = \alpha_k \neq \alpha_j$, $i, j, k \in \{1, \dots, m\}$;
- for every $i, j, k \in \{1, \dots, m\}$,

$$\alpha_i^2(c_{ji}^k + c_{ki}^j) + \alpha_j^2(c_{ij}^k + c_{kj}^i) + \alpha_k^2(c_{ik}^j + c_{jk}^i) = 0.$$

3.5 Levi-Civita pairs

3.5.1 Definition and the main open question

Let us introduce a special case of non-trivially projectively and affinely equivalent metrics. First we define a distribution which admits a product structure.

Fix positive integers N, $n_1, \dots, n_N$, and set $n = n_1 + \cdots + n_N$. We denote the canonical coordinates on $\mathbb{R}^n = \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_N}$ by $x = (\bar{x}_1, \dots, \bar{x}_N)$, where

$\bar{x}_\ell = (x_\ell^1, \ldots, x_\ell^{n_\ell})$. For any $\ell \in \{1, \ldots, N\}$, let D_ℓ be a Lie bracket generating distribution on $\mathbb{R}^{n_\ell}$. We define the *product distribution* $D = D_1 \times \cdots \times D_N$ on $\mathbb{R}^n$ by

$$D(x) = \left\{v \in T_x\mathbb{R}^n : (\pi_\ell)_*(v) \in D_\ell\big(\pi_\ell(x)\big),\ \ell = 1, \ldots, N\right\}, \tag{3.28}$$

where $\pi_\ell : \mathbb{R}^n \to \mathbb{R}^{n_\ell}$, $\ell = 1, \ldots, N$, are the canonical projection.

Definition 3.49. We say that a distribution D on a n-dimensional manifold M admits *a nontrivial product structure* at $q \in M$ if there is a local coordinate system in a neighbourhood of q in which D takes the form of a product distribution with $N \geq 2$ factors.

Note that the case $N = 1$ is trivial since any distribution can be written in local coordinates as a product distribution with one factor.

Given a product distribution $D = D_1 \times \cdots \times D_N$ on $\mathbb{R}^n$, we choose for every $\ell \in \{1, \ldots, N\}$ a sub-Riemannian metric $\bar{g}_\ell$ on $(\mathbb{R}^{n_\ell}, D_\ell)$ and a function β_ℓ depending only on the variables $\bar{x}_\ell$ such that β_ℓ is constant if $n_\ell > 1$ and $\beta_\ell(0) \neq \beta_{\ell'}(0)$ for $\ell \neq \ell'$. We define two sub-Riemannian metrics g_1, g_2 on $(\mathbb{R}^n, D)$ by

$$\begin{cases} g_1(x)(\dot{x}, \dot{x}) = \displaystyle\sum_{\ell=1}^{N} \gamma_\ell(x)\, \bar{g}_\ell(\bar{x}_\ell)(\dot{\bar{x}}_\ell, \dot{\bar{x}}_\ell), \\ g_2(x)(\dot{x}, \dot{x}) = \displaystyle\sum_{\ell=1}^{N} \alpha_\ell^2(x)\gamma_\ell(x)\, \bar{g}_\ell(\bar{x}_\ell)(\dot{\bar{x}}_\ell, \dot{\bar{x}}_\ell), \end{cases} \tag{3.29}$$

where

$$\alpha_\ell^2(x) = \beta_\ell(\bar{x}_\ell) \prod_{\ell'=1}^{N} \beta_{\ell'}(\bar{x}_{\ell'}), \qquad \gamma_\ell(x) = \prod_{\ell' \neq \ell} \left| \frac{1}{\beta_{\ell'}(\bar{x}_{\ell'})} - \frac{1}{\beta_\ell(\bar{x}_\ell)} \right|. \tag{3.30}$$

Definition 3.50. Let D be a distribution on an n-dimensional manifold M. We say that a pair (g_1, g_2) of sub-Riemannian metrics on (M, D) form a *(generalized) Levi-Civita pair* at a point $q \in M$, if there is a local coordinate system in a neighbourhood of q, in which D takes the form of a product distribution and the metrics g_1 and g_2 have the form (3.29). We say that such a pair has *constant coefficients* if the coordinate system can be chosen so that the functions β_ℓ, $\ell = 1, \ldots, N$, are constant (and so all functions α_ℓ^2 and γ_ℓ are constant too).

This definition is inspired by the classification in the Riemannian case appearing in [48]. Note however that, in the Riemannian case, the distribution $D = TM$ takes the form of a product in any system of coordinates, so that Levi-Civita pairs always exist locally.

Remark 3.51. A Levi-Civita pair with $N = 1$ is a pair of conformal metrics, $g_2 = \alpha_1^2 g_1$. If moreover $n > 1$, two such metrics are actually constantly proportional. Thus, when $n > 1$, the metrics of a Levi-Civita pair are constantly proportional if and only if $N = 1$.

Proposition 3.52. *The two metrics of a Levi-Civita pair are projectively equivalent. They are affinely equivalent if the pair has constant coefficients.*

Proof. Let (g_1, g_2) be a Levi-Civita pair on a distribution D of a manifold M, and fix a point $q_0 \in M$. In local coordinates, the metrics g_1, g_2 take the form (3.29) and the distribution D is the product distribution $D = D_1 \times \cdots \times D_N$ on $\mathbb{R}^n = \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_N}$ defined by (3.28).

Let us construct a frame adapted to (g_1, g_2). For any integer $1 \le \ell \le N$, we choose vector fields $Y_1^\ell, \dots, Y_{k_\ell}^\ell$, where $k_\ell = \dim D_\ell$, of the form $Y_i^\ell = \sum_{j=1}^{n_\ell} a_{ij}^\ell(x^\ell)\partial_{x_j^\ell}$ such that $\{Y_1^\ell, \dots, Y_{k_\ell}^\ell\}$ is a frame of D_ℓ and is orthonormal with respect to $\bar{g}_\ell$. We complete $\{Y_1^\ell, \dots, Y_{k_\ell}^\ell\}$ into a frame adapted to the flag $D_\ell \subset D_\ell^2 \subset \cdots \subset T\mathbb{R}^{n_\ell}$ by adding vector fields $X_{k_\ell+1}^\ell, \dots, X_{n_\ell}^\ell$ of the form $[Y_{i_1}^\ell, \dots, [Y_{i_{k-1}}^\ell, Y_{i_k}^\ell]]$. Moreover, setting $X_i^\ell = \frac{1}{\sqrt{\gamma_\ell}} Y_i^\ell$ for $i = 1, \dots, k_\ell$, we obtain a g_1-orthonormal frame $\{X_1^\ell, \dots, X_{k_\ell}^\ell\}$ of D_ℓ.

Grouping all together, we have obtained a frame $\{X_i^\ell,\ 1 \le \ell \le N,\ i = 1, \dots, k_\ell\}$ of D which is g_1-orthonormal and g_2-orthogonal, and a frame $\{X_i^\ell,\ 1 \le \ell \le N,\ i = 1, \dots, n_\ell\}$ of $T\mathbb{R}^n$ which is adapted to the pair (g_1, g_2). To simplify the notations we denote by $\{X_1, \dots, X_m\}$ the frame of D and by $\{X_1, \dots, X_n\}$ the frame of $T\mathbb{R}^n$. For $i = 1, \dots, n$, we denote by $\ell(i)$ the integer in $\{1, \dots, N\}$ such that X_i is of the form $X_j^{\ell(i)}$.

The special form of the constructed adapted frame and the form of (3.30) imply the following properties of the structure coefficients c_{ij}^k:

- if $\ell(i) \ne \ell(j)$, then $c_{ij}^k = 0$ if $k \ne i$ or j; moreover,

$$c_{ij}^j = \begin{cases} \frac{\alpha_{\ell(j)}^2 X_i(\alpha_{\ell(i)}^2)}{4\alpha_{\ell(i)}^2\left(\alpha_{\ell(j)}^2 - \alpha_{\ell(i)}^2\right)}, & \text{if } j \le m; \\ 0, & \text{if } j > m; \end{cases} \tag{3.31}$$

- if $\ell(i) = \ell(j) \le \ell(k)$, then $c_{ij}^k = 0$.

Notice also that we can obtain the following relationship from (3.30),

$$X_i(\alpha_{\ell(j)}^2) = \frac{\alpha_{\ell(j)}^2}{2\alpha_{\ell(i)}^2} X_i(\alpha_{\ell(i)}^2), \qquad \text{if } \ell(i) \ne \ell(j). \tag{3.32}$$

These formulas permit us to simplify the equations (3.10) and (3.11) which characterize an orbital diffeomorphism. To simplify (3.10), we have to compute R_j. For this, we first show that the first divisibility condition holds for our choice of adapted frame (it results directly from the use of (3.31) and (3.32) in the computation of $\vec{h}_1(\mathcal{P})$). Then we use the following formula (see [50, Lemma 3]),

$$\begin{aligned} R_j = &\sum_{i=1}^{m}(1-\delta_{ij})\left((\alpha_j^2-\alpha_i^2)c_{ji}^i - \frac{X_j(\alpha_i^2)}{2}\right)u_i^2 + \sum_{i=1}^{m}(1-\delta_{ij})\frac{\alpha_i^2}{2\alpha_j^2}X_i\left(\frac{\alpha_j^4}{\alpha_i^2}\right)u_iu_j \\ &+\sum_{i=1}^{m}\sum_{k=1}^{m}(1-\delta_{ik})(\alpha_j^2-\alpha_k^2)c_{ji}^k u_iu_k + \alpha_j^2\sum_{i=1}^{m}\sum_{k=m+1}^{n}c_{ji}^k u_iu_k. \end{aligned}$$

We substitute the structure coefficients by the expressions shown above and use the property of functions $\beta_\ell(\bar{x}_\ell)$ to be constant if x_ℓ is of dimension more then one. We get $R_j = \alpha_j^2\sum_{i=1}^{m}\sum_{k=m+1}^{n}c_{ij}^k u_iu_k$. We finally obtain a simplified form of (3.10),

$$\sum_{k=m+1}^{n} q_{jk}\Phi_k = \frac{\alpha_j^2}{\alpha}\sum_{k=m+1}^{n} q_{jk}u_k \qquad 1\le j\le m.$$

To simplify (3.11), it is sufficient to notice that $X_i^s(\alpha_i^2) = 0$ if $|I_s| > 1$. Setting $\Phi_i = \frac{\alpha_i^2 u_i}{\alpha}$ for $i = 1,\dots,m$ as in (3.9), we obtain

$$\vec{h}_1(\Phi_s) = \sum_{k=1}^{n} q_{sk}\Phi_k.$$

To summarize, there exists an orbital diffeomorphism between $\vec{h}_1$ and $\vec{h}_2$ if the following equations have a solution:

$$\sum_{k=m+1}^{n} q_{jk}\Phi_k = \frac{\alpha_j^2}{\alpha}\sum_{k=m+1}^{n} q_{jk}u_k \qquad 1\le j\le m,$$

$$\vec{h}_1(\Phi_s) = \sum_{k=1}^{n} q_{sk}\Phi_k \qquad m+1\le s\le n.$$

It appears that $\Phi_k = \frac{\alpha_k^2 u_k}{\alpha}$, $k = m+1,\dots,n$, obviously satisfy this system. Thus $\vec{h}_1$ and $\vec{h}_2$ are orbitally diffeomorphic and, by Proposition 3.25, g_1, g_2 are projectively equivalent.

In the case of a pair with constant coefficients, all α_i are constant and thus $\vec{h}_1(\alpha^2) = 0$. Applying again Proposition 3.25, we deduce that the metrics of a Levi-Civita pair with constant coefficients are affinely equivalent. This ends the proof of Proposition 3.52.

□

The main open question is *whether under some natural regularity assumption the generalized Levi-Civita pairs are the only pairs of the projectively equivalent metrics.*

3.5.2 Levi-Civita theorem and its generalizations

The preceding question has a positive answer in the Riemannian case, that is when $D = TM$. Indeed, in that case the local classification of projectively equivalent metrics near generic points has been established by [48] in any dimension. The classification of affinely equivalent metrics is a consequence of [68, Th. p. 303]. We summarize all these results in the following theorem.

Theorem 3.53. *Assume* $\dim M > 1$. *Then two Riemannian metrics on M are non-trivially projectively equivalent in a neighbourhood of a stable point q if and only if they form a Levi-Civita pair at q. They are moreover affinely equivalent if the pair has constant coefficients.*

We can actually give a rather short explanation of the classification of affinely equivalent Riemannian metric based on the de Rham decomposition theorem, [69]. Indeed, a simple analysis of the geodesic equation implies that two Riemannian metrics are affinely equivalent if and only if they have the same Levi-Civita connection. Since the Levi-Civita connection is parallel with respect to the metric, a metric with given Levi-Civita connection on a connected manifold is determined by its value at one point q. Besides, it must be invariant with respect to the holonomy group (or the reduced holonomy group for the local version of the problem). If the action of the holonomy group on T_qM is irreducible, the Riemannian metric is uniquely determined by its Levi-Civita connection, i.e. it is affinely rigid. On the other hand, if the action of the holonomy group is reducible, then by the de Rham decomposition theorem the Riemannian metric becomes the direct product of Riemannian metrics and any metric which is affinely equivalent to it is such that the metrics can be represented as in (3.29) with all functions β_ℓ being constant.

Our main open question has a positive answer as well for sub-Riemannian metrics on contact and quasi-contact distributions, which are typical cases of corank 1 distributions (i.e. $m = n - 1$). Recall that a *contact distribution* D on a $(2k+1)$-dimensional manifold M, $k > 0$, is a rank-$2k$ distribution for which there exists a 1-form ω such that at every $q \in M$, $D(q) = \ker \omega(q)$ and $d\omega(q)|_{D(q)}$ is non-degenerate. A *quasi-contact distribution* D on a $2k$-dimensional manifold M, $k > 1$, is a rank-$(2k-1)$ distribution for which there exists a 1-form ω such that at every $q \in M$, $D(q) = \ker \omega(q)$ and $d\omega(q)|_{D(q)}$ has a one-dimensional kernel. The main result of [50] can be formulated in the following way.

Theorem 3.54 ([50]). *Two sub-Riemannian metrics on a contact or a quasi-contact distribution are non-trivially projectively equivalent at a stable point q if and only if they form a Levi-Civita pair at q.*

Remark 3.55. This theorem and Proposition 3.45 imply that, in the contact and quasi-contact cases, two affinely equivalent metrics form a Levi-Civita pair with constant coefficients.

Since contact distributions are never locally equivalent to a non-trivial product distribution, they admit only Levi-Civita pairs with $N = 1$.

Corollary 3.56. *On a contact distribution, every sub-Riemannian metric is projectively rigid.*

For a generic corank one distribution D on an odd dimensional manifold M, there is an open and dense subset of M where D is locally contact. By continuity we obtain the following result.

Corollary 3.57. *Let M be an odd-dimensional manifold. Then, for a generic corank one distribution on M, all metrics are projectively rigid.*

3.6 Left-invariant metrics on Carnot groups

Let us study the particular case of affine and projective equivalence of left-invariant sub-Riemannian metrics on Carnot groups. This case plays an important role in sub-Riemannian geometry since Carnot groups appear as tangent cones to sub-Riemannian manifolds near generic points.

Definition 3.58. A Carnot group $\mathbb{G}$ of step $r \geq 1$ is a connected and simply connected nilpotent Lie group whose Lie algebra $\mathfrak{g}$ admits a step r grading

$$\mathfrak{g} = V^1 \oplus \cdots \oplus V^r,$$

and is generated by its first component, that is, $[V^j, V^1] = V^{j+1}$ for $1 \leq j \leq r-1$. A graded Lie algebra satisfying the last property is called *fundamental.*

A Carnot group is canonically endowed with a bracket generating distribution $D_{\mathbb{G}}$: identifying $\mathfrak{g}$ with the tangent space $T_e\mathbb{G}$ to $\mathbb{G}$ at the identity e, $D_{\mathbb{G}}$ is the distribution spanned by the left-invariant vector fields whose value at the identity belongs to V^1. Hence $D^k_{\mathbb{G}}(e) = V^1 \oplus \cdots \oplus V^k$ for $k \leq r$, and the step r of the Carnot group is exactly the step of the distribution.

Given an inner product on V^1, we can extend it to a Riemannian metric on $D_{\mathbb{G}}$ by left-translations. Such a sub-Riemannian metric on $(\mathbb{G}, D_{\mathbb{G}})$ is called a *left-invariant sub-Riemannian metric on* $\mathbb{G}$.

Theorem 3.59. *Let g_1, g_2 be two left-invariant sub-Riemannian metrics on a Carnot group $\mathbb{G}$. If g_1 and g_2 are non-trivially projectively equivalent, then $D_{\mathbb{G}}$ admits a non-trivial product structure and (g_1, g_2) is a Levi-Civita pair with constant coefficients.*

Proof. Let g_1, g_2 be two left-invariant sub-Riemannian metrics on $\mathbb{G}$ which are non-trivially projectively equivalent. Set $D = D_{\mathbb{G}}$. Since both metrics g_1 and g_2 are obtained by left-invariant extensions of inner products on V^1, it is clear that the eigenvalues $\alpha_1^2, \ldots, \alpha_m^2$ of the transition operator are constant. Thus the number N of distinct eigenvalues is constant and every point of $\mathbb{G}$ is stable. Note that N is necessarily greater than one, otherwise the metrics would be proportional, i.e. trivially equivalent.

We choose the numbering of the eigenvalues α_i^2, $i = 1, \ldots, m$, in such a way that $\alpha_1^2, \ldots, \alpha_N^2$ are the N distinct ones. Let $X_1, \ldots, X_m$ be a g_1-orthonormal basis of V^1 such that $\frac{1}{\alpha_1} X_1, \ldots, \frac{1}{\alpha_m} X_m$ is orthonormal with respect to g_2. For $\ell = 1, \ldots, N$, we denote by I_ℓ the set of indices $i \in \{1, \ldots, m\}$ such that $\alpha_i = \alpha_\ell$, and by V_ℓ^1 the linear subspace of V^1 generated by the vectors X_i, $i \in I_\ell$. We get

$$V^1 = V_1^1 \oplus \cdots \oplus V_N^1. \tag{3.33}$$

Each subspace V_ℓ^1, $\ell = 1, \ldots, N$, generates a graded Lie subalgebra of $\mathfrak{g}$,

$$\mathfrak{g}_\ell = V_\ell^1 \oplus \cdots \oplus V_\ell^r, \quad \text{where } V_\ell^{k+1} = [V_\ell^k, V_\ell^1].$$

Moreover, from Corollary 3.38 *(i)*, we have

$$[V_\ell^1, V_{\ell'}^1] = 0 \quad \text{for all } \ell \neq \ell' \in \{1, \ldots, N\}.$$

Using the Jacobi identity, this relation can be generalized as

$$[V_\ell^k, V_{\ell'}^s] = 0 \quad \text{for } \ell \neq \ell' \in \{1, \ldots, N\},\ k, s \in \{1, \ldots, r\}. \tag{3.34}$$

Hence each homogeneous component V^k, $k = 1, \ldots, r$, admits a decomposition into a sum $V^k = V_1^k + \cdots + V_N^k$, and the Lie algebra $\mathfrak{g}$ writes as

$$\mathfrak{g} = \mathfrak{g}_1 + \cdots + \mathfrak{g}_N. \tag{3.35}$$

Note that, if the sum (3.35) is a direct one, then (3.33) implies that the distribution D admits a product structure $D = D_1 \times \cdots \times D_N$, where D_ℓ, $\ell = 1, \ldots, N$, is the distribution spanned by the left-invariant vector fields whose value at the identity belongs to V_ℓ^1. Thus, in order to prove that D admits a non-trivial product structure, it is sufficient to prove that the sum (3.35) is a direct sum, i.e. $\mathfrak{g}_\ell \cap \mathfrak{g}_{\ell'}$ is reduced to zero when $\ell \neq \ell'$.

The first step is to complete $\{X_1, \ldots, X_m\}$ into a basis of $\mathfrak{g}$ adapted to the grading $\mathfrak{g} = V^1 \oplus \cdots \oplus V^r$. For $k = 2, \ldots, r$, we construct a basis of $V^k = V_1^k + \cdots + V_N^k$ as follows. Fix first a basis of $\cap_{1 \leq \ell \leq N} V_\ell^k$; then complete it into a basis of $\operatorname{span}\{\cup_{1 \leq i \leq N} \left(\cap_{\ell \neq i} V_\ell^k\right)\}$ with vectors from $\cup_{1 \leq i \leq N} \left(\cap_{\ell \neq i} V_\ell^k\right)$; then to a basis of span$\{\cup_{1\leq i\leq N}\left(\cup_{1\leq j\leq N}\left(\cap_{\ell\neq i,\ell\neq j}V_\ell^k\right)\right)\}$ with vectors from $\cup_{1\leq i\leq N}\left(\cup_{1\leq j\leq N}\left(\cap_{\ell\neq i,\ell\neq j}V_\ell^k\right)\right)$ and so on. At the last step, complete the obtained set of vectors into a basis of V^k.

By collecting the basis of $V^1, V^2, \dots, V^r$, we obtain a basis $\{X_1, \dots, X_n\}$ of $\mathfrak{g}$. By abuse of notations, we keep the notation X_i to denote the left-invariant vector field whose value at identity is X_i. We have constructed in this way a frame $\{X_1, \dots, X_n\}$ of $T\mathbb{G}$ with the following properties:

- it is by construction a frame adapted to (g_1, g_2);
- it contains a basis of every D_ℓ^k, $k = 1, \dots, r$, $\ell = 1, \dots, N$; for $\ell = 1, \dots, N$, we denote by $\mathcal{L}(I_\ell)$ the set of indices such that $\{X_i,\ i \in \mathcal{L}(I_\ell)\}$ is a basis of D_ℓ^r;
- from (3.34), $[X_i, X_j] = 0$ if i and j belong to two different sets $\mathcal{L}(I_\ell)$; this implies the following property of the structure coefficients:
$$\text{if } i, j, k \text{ do not belong to the same } \mathcal{L}(I_\ell), \text{ then } c_{ij}^k = 0; \tag{3.36}$$
- all structure coefficients are constant since the vector fields are left-invariant; moreover,
$$\text{if } w_k \neq w_i + w_j, \text{ then } c_{ij}^k = 0, \tag{3.37}$$
where as usual w_i is the smallest integer l such that $X_i \in D^l$.

The property $\mathfrak{g}_\ell \cap \mathfrak{g}_{\ell'} = \{0\}$ is equivalent to $\mathcal{L}(I_\ell) \cap \mathcal{L}(I_{\ell'}) = \emptyset$, so we have to prove that the latter holds for any $\ell \neq \ell'$.

Now, Proposition 3.26 implies that the Hamiltonian vector fields of g_1 and g_2 are orbitally diffeomorphic near any ample covector. And, from Proposition 3.32, in the coordinates $(u_1, \dots, u_n)$ associated with the frame $\{X_1, \dots, X_n\}$, the orbital diffeomorphism satisfies the fundamental algebraic system $A\widetilde{\Phi} = b$.

Let us compute first the matrix b. Fix $\ell \in \{1, \dots, N\}$ and $j \in I_\ell$. Using (3.37) and the fact that the α_i's are constant, there holds

$$b_j^1 = \frac{\alpha_\ell^2}{\alpha}\vec{h}_1(u_j) = \frac{\alpha_\ell^2}{\alpha}\sum_{k=m+1}^{n} a_{j,k}^1 u_k, \qquad \text{and} \qquad b_j^{s+1} = \vec{h}_1(b_j^s) \ \text{ for } s \geq 1.$$

An easy induction argument gives the value

$$b_j^s = \frac{\alpha_\ell^2}{\alpha}\sum_{k=m+1}^{n} a_{j,k}^s u_k, \quad s \in \mathbb{N}.$$

Thus the system of equations $A\widetilde{\Phi} = b$ can be rewritten as

$$\sum_{k=m+1}^{n} a_{j,k}^s \Phi_k = \frac{\alpha_\ell^2}{\alpha}\sum_{k=m+1}^{n} a_{j,k}^s u_k \qquad \text{for every } j \in I_\ell,\ \ell \in \{1, \dots, N\},\ s \in \mathbb{N}.$$

In other terms, $A\widetilde{\Phi} = b$ splits into N systems of equations indexed by $\ell = 1, \dots, N$ of the form

$$\sum_{k=m+1}^{n} a^s_{j,k} \left(\Phi_k - \frac{\alpha_\ell^2}{\alpha} u_k \right) = 0 \qquad \text{for every } j \in I_\ell,\ s \in \mathbb{N}. \tag{3.38}$$

Let us have a closer look to the coefficients $a^s_{j,k}$. Fix as before $\ell \in \{1, \dots, N\}$ and $j \in I_\ell$. First we have

$$a^1_{j,k} = q_{jk} = \sum_{i=1}^{m} c^k_{ij} u_i = \sum_{i \in I_\ell} c^k_{ij} u_i,$$

due to (3.36). Using again the latter relation and the other properties of the structure coefficients, an easy induction argument shows that the recurrence formula for $a^s_{j,k}$, $s \in \mathbb{N}$, is

$$a^{s+1}_{j,k} = \sum_{i \in I_\ell} u_i \vec{u}_i (a^s_{j,k}) + \sum_{l \in \mathcal{L}(I_\ell)} a^s_{j,l} \sum_{i \in I_\ell} c^k_{il} u_i.$$

As a consequence of this formula:

- if $k \notin \mathcal{L}(I_\ell)$, then $a^s_{j,k} = 0$; hence, for a fixed $\ell \in \{1, \dots, N\}$, the system (3.38) writes as
$$\sum_{k \in \mathcal{L}(I_\ell)} a^s_{j,k} \left(\Phi_k - \frac{\alpha_\ell^2}{\alpha} u_k \right) = 0 \qquad \text{for every } j \in I_\ell,\ s \in \mathbb{N}; \tag{3.39}$$
- if $k \in \mathcal{L}(I_\ell)$, then $a^s_{j,k}$ is the corresponding coefficient of the matrix A associated with the family of vector fields $\{X_i, i \in \mathcal{L}(I_\ell)\}$; from Proposition 3.33, the latter matrix has maximal rank for almost every u, thus (3.39) implies
$$\Phi_k = \frac{\alpha_\ell^2}{\alpha} u_k \qquad \text{for every } k \in I_\ell. \tag{3.40}$$

Now, assume that there exists two indices ℓ, ℓ' in $\{1, \dots, N\}$ such that the intersection $\mathcal{L}(I_\ell) \cap \mathcal{L}(I_{\ell'})$ is non empty. For $k \in \mathcal{L}(I_\ell) \cap \mathcal{L}(I_{\ell'})$, we have from (3.40)

$$\Phi_k = \frac{\alpha_\ell^2}{\alpha} u_k = \frac{\alpha_{\ell'}^2}{\alpha} u_k,$$

which implies $\ell = \ell'$. Hence $\mathcal{L}(I_\ell) \cap \mathcal{L}(I_{\ell'}) = \emptyset$ for any $\ell \neq \ell'$, which implies that $\mathfrak{g}$ is decomposed into a direct sum $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_N$. We conclude that the distribution D admits a product structure $D = D_1 \times \cdots \times D_N$ which is non-trivial since $N > 1$. This proves the first part of the theorem.

It remains to prove that (g_1, g_2) form a Levi-Civita pair on D. Set $n_\ell = \dim \mathfrak{g}_\ell$ for $\ell = 1, \dots, N$ and define coordinates $x = (\bar{x}_1, \dots, \bar{x}_N)$ on $\mathbb{G}$, where $\bar{x}_\ell = (x^1_\ell, \dots, x^{n_\ell}_\ell)$, by

$$x \mapsto \exp\left(\sum_{i \in \mathcal{L}(I_1)} x^i_1 X_i \right) \cdots \exp\left(\sum_{i \in \mathcal{L}(I_N)} x^i_N X_i \right).$$

In these coordinates, a vector field X_i with $i \in \mathcal{L}(I_\ell)$, $\ell = 1, \dots, N$, depends only on the coordinates $\bar{x}_\ell$ and can be considered as a vector field on $\mathbb{R}^{n_\ell}$ (with coordinates $\bar{x}_\ell$). Thus D_ℓ can be identified with a distribution on $\mathbb{R}^{n_\ell}$. Let $\bar{g}_\ell$ be the sub-Riemannian metric on $(\mathbb{R}^{n_\ell}, D_\ell)$ for which the vector fields X_i, $i \in I_\ell$, form an orthonormal frame. We have the following expressions in coordinates:

$$\begin{cases} g_1(x)(\dot{x}, \dot{x}) = \sum_{\ell=1}^{N} \bar{g}_\ell(\bar{x}_\ell)(\dot{\bar{x}}_\ell, \dot{\bar{x}}_\ell), \\ g_2(x)(\dot{x}, \dot{x}) = \sum_{\ell=1}^{N} \alpha_\ell^2 \, \bar{g}_\ell(\bar{x}_\ell)(\dot{\bar{x}}_\ell, \dot{\bar{x}}_\ell). \end{cases}$$

Hence g_1, g_2 form a Levi-Civita pair on D with constant coefficients and the theorem is proved. $\square$

Remark 3.60. Note that we use the hypothesis of projective equivalence between g_1 and g_2 only to deduce the existence of a solution to the fundamental algebraic system. So we have actually proved a stronger result than Theorem 3.59, namely: *if g_1 and g_2 are non proportional and if the corresponding fundamental algebraic system $A\widetilde{\Phi} = b$ admits a solution, then $D_{\mathbb{G}}$ admits a nontrivial product structure and (g_1, g_2) is a Levi-Civita pair with constant coefficients.*

3.7 Nilpotent approximation of equivalent metrics

3.7.1 Nilpotent approximation

Let (M, D, g) be a sub-Riemannian manifold and $q_0 \in M$ be a regular point. The nilpotent approximation of (M, D, g) at q_0 is another sub-Riemannian manifold, denoted by $(\hat{M}, \hat{D}, \hat{g})$, which has a particular structure: $\hat{M}$ is a Carnot group, $\hat{D} = D_{\hat{M}}$ is the canonical distribution on $\hat{M}$, and $\hat{g}$ is a left-invariant sub-Riemannian metric on $(\hat{M}, \hat{D})$.

Below we briefly recall the construction of the nilpotent approximation in a form convenient for us here, following the foundational paper [76] in nilpotent differential geometry. For equivalent description using privileged coordinates or metric tangent space approach see [77,78].

Let $V^1 = D(q_0)$ and, for an integer $i > 1$, $V^i = D^i(q_0)/D^{i-1}(q_0)$. The graded space

$$\mathfrak{g} = \bigoplus_{i=1}^{r} V^i \tag{3.41}$$

associated with the filtration (3.1) at q_0 is endowed with the natural structure of a fundamental graded Lie algebra: if $X \in V^i$ and $Y \in V^j$, then for any vector fields

$\widetilde{X}$ and $\widetilde{Y}$ tangent to D^i and D^j respectively in a neighborhood of q_0 and such that $\widetilde{X}(q_0) = X$, $\widetilde{Y}(q_0) = Y$, the vector $[\widetilde{X}, \widetilde{Y}](q_0)$ is well-defined modulo $D^{i+j-1}(q_0)$, i.e. $[X, Y] := [\widetilde{X}, \widetilde{Y}](q_0)$ is a well-defined element of V^{i+j}. The graded Lie algebra $\mathfrak{g}$ is called the *Tanaka symbol of the distribution D at q_0*. Note that since D generates the weak derived flag (3.1), the space V^1 generates the Lie algebra $\mathfrak{g}$. Therefore, $\mathfrak{g}$ is a fundamental graded Lie algebra. As a consequence, the connected simply-connected Lie group $\hat{M}$ with Lie algebra $\mathfrak{g}$ is a Carnot group.

Let us denote by $\hat{D}$ the left-invariant distribution on $\hat{M}$ such that $\hat{D}(e) = V^1$, where e is the identity of $\hat{M}$. The metric g on D induces an inner product $g(q_0)$ on V^1, and so a left-invariant sub-Riemannian metric $\hat{g}$ on $(\hat{M}, \hat{D})$. The constructed sub-Riemannian manifold $(\hat{M}, \hat{D}, \hat{g})$ is called the *nilpotent approximation of (M, D, g) at q_0*.

Consider a frame $\{X_1, \dots, X_n\}$ of TM adapted to D at $q_0 \in M$ and such that $X_1, \dots, X_m$ are g-orthonormal. For every $i \in \{1, \dots, n\}$, $X_i(q_0)$ can be identified by the construction above to an element of $\mathfrak{g}$, which defines a left-invariant vector field $\hat{X}_i$ on $\hat{M}$. Then $\hat{X}_1, \dots, \hat{X}_m$ are $\hat{g}$-orthonormal and $\{\hat{X}_1, \dots, \hat{X}_n\}$ is a frame of $T\hat{M}$ adapted to $\hat{D}$ at any point of $\hat{M}$. The structure coefficients $\hat{c}_{ij}^k$ of this frame satisfy:

$$\begin{cases} \hat{c}_{ij}^k \equiv c_{ij}^k(q_0) & \text{if } \ w_i + w_j = w_k; \\ \hat{c}_{ij}^k \equiv 0 & \text{if } \ w_i + w_j \neq w_k. \end{cases} \tag{3.42}$$

3.7.2 Equivalence for nilpotent approximations

Let (M, D, g_1) and (M, D, g_2) be two sub-Riemannian manifolds. We fix a point q_0 which is stable with respect to g_1, g_2 and we denote by $(\hat{M}, \hat{D}, \hat{g}_i)$, $i = 1, 2$, the nilpotent approximation of (M, D, g_i) at q_0.

Theorem 3.61. *If g_1, g_2 are projectively equivalent and not conformal to each other near q_0, then $\hat{D}$ admits a product structure and $(\hat{g}_1, \hat{g}_2)$ is a Levi-Civita pair with constant coefficients.*

To prove this result we need first some technical results.

Let g_1, g_2 be two non-trivially projectively equivalent metrics. By Proposition 3.26, their Hamiltonian vector fields are orbitally diffeomorphic near any ample covector. We choose a frame $\{X_1, \dots, X_n\}$ adapted to (g_1, g_2) near q_0. It induces (see subsection 3.7.1) a frame $\{\hat{X}_1, \dots, \hat{X}_n\}$ of $T\hat{M}$ adapted to $\hat{D}$ which has by construction the following properties: $\hat{X}_1, \dots, \hat{X}_m$ is $\hat{g}_1$-orthonormal and $\frac{1}{\alpha_1(q_0)}\hat{X}_1, \dots, \frac{1}{\alpha_m(q_0)}\hat{X}_m$ is $\hat{g}_2$-orthonormal, where $\alpha_1^2(q), \dots, \alpha_m^2(q)$ are the eigenvalues of the transition operator at q between g_1 and g_2. Note that the transition operator between $\hat{g}_1$ and $\hat{g}_2$ has the same eigenvalues $\alpha_1^2(q_0), \dots, \alpha_m^2(q_0)$ at any point of $\hat{M}$.

Remark 3.62. The metrics $\hat{g}_1$ and $\hat{g}_2$ are proportional if and only if all $\alpha_i^2(q_0)$'s are equal, i.e. if g_1, g_2 are conformal to one another near q_0 (recall that q_0 is stable). The hypothesis of the theorem rules out this possibility.

Recall that the data of a frame $\{X_1, \dots, X_n\}$ of TM and of eigenvalues $\alpha_1^2, \dots, \alpha_m^2$ allows to construct infinite matrices A and b by the formulas (3.13)–(3.15). Each element of these matrices $A = A(q)(u)$ and $b = b(q)(u)$ is a function of q in a neighbourhood of q_0 and of $u \in \mathbb{R}^n$. Similarly, denote by $\hat{A}$ and $\hat{b}$ the matrices constructed by using $\{\hat{X}_1, \dots, \hat{X}_n\}$ as a frame and $\alpha_1^2(q_0), \dots, \alpha_m^2(q_0)$ as eigenvalues in the formulas (3.13)–(3.15). Each element of $\hat{A} = \hat{A}(\hat{q})(u)$ and $\hat{b} = \hat{b}(\hat{q})(u)$ is a function of $\hat{q}$ in $\hat{M}$ and of $u \in \mathbb{R}^n$. Finally, the elements of the matrices A, b, $\hat{A}$ and $\hat{b}$ are denoted by $a_{j,k}^s$, b_j^s, $\hat{a}_{j,k}^s$ and $\hat{b}_j^s$ respectively. Let us introduce the notion of *weighted degree* $\deg_w$ for a polynomial with n variables. For a monomial $m = u_1^{\beta_1} \cdots u_n^{\beta_n}$, we set $\deg_w(m) = \sum_{i=1}^n \beta_i w_i$. Then the weighted degree $\deg_w(P)$ of a polynomial function $P = P(u_1, \dots, u_n)$ is the largest weighted degree of the monomials of P. A polynomial is said to be *w-homogeneous* if all its monomials are of the same weighted degree.

Lemma 3.63. *For any $s \in \mathbb{N}$, $1 \leq j \leq m$, and $m+1 \leq k \leq n$, there hold:*

- *for every $q \in M$ near q_0, the element $a_{j,k}^s(q)$ is a polynomial in $u_1, \dots, u_n$ of weighted degree*
$$\deg_w(a_{j,k}^s(q)) \leq 2s - w_k + 1;$$
- *the function $\hat{a}_{j,k}^s$ does not depend on $\hat{q} \in \hat{M}$ and is a w-homogeneous polynomial in $u_1, \dots, u_n$ of weighted degree*
$$\deg_w(\hat{a}_{j,k}^s) = 2s - w_k + 1;$$
- *the homogeneous term of highest weighted degree in $a_{j,k}^s(q_0)$ is $\hat{a}_{j,k}^s$, that is,*
$$a_{j,k}^s(q_0)(u) = \hat{a}_{j,k}^s(u) + \operatorname{poly}(u_1, \dots, u_n),$$
with $\deg_w(\operatorname{poly}) < \deg_w(\hat{a}_{j,k}^s)$.

Proof. Notice first that a structure coefficient c_{ij}^l is zero if $w_l > w_i + w_j$; and second that, for any polynomial P,
$$\deg_w\big(\vec{h}_1(P)\big) \leq \deg_w(P) + 2.$$
An easy induction argument based on (3.14) allows then to prove the first item. The second and the third item are proven in the same way by using moreover (3.42). □

Lemma 3.64. *For any $s \in \mathbb{N}$ and $1 \leq j \leq m$, there hold:*

- *for every $q \in M$ near q_0, αb_j^s is a polynomial in $u_1, \dots, u_n$ of weighted degree*

$$\deg_w(\alpha b_j^s) \leq 2s + 1;$$

- *the function $\alpha(q_0)\hat{b}_j^s$ does not depend on $\hat{q} \in \hat{M}$ and is a w-homogeneous polynomial in $u_1, \dots, u_n$ of weighted degree*

$$\deg_w(\alpha(q_0)\hat{b}_j^s) = 2s + 1;$$

- *the homogeneous term of highest weighted degree in $b_j^s(q_0)$ is $\hat{b}_j^s$, that is,*

$$\alpha b_j^s(q_0)(u) = \alpha \hat{b}_j^s(u) + \mathrm{poly}(u_1, \dots, u_n),$$

  with $\deg_w(\mathrm{poly}) < \deg_w(\alpha \hat{b}_j^s)$.

Proof. Note first that, by (3.20) and Lemma 3.39, $\vec{h}_1(\alpha^2)/\alpha^2 = Q$. Thus it is a polynomial function of u of weighted degree 1. As a consequence, the terms R_j, $j = 1, \dots, m$, are polynomials of weighted degree 3. Using then the recurrence formula (3.15) and the fact that

$$\alpha \vec{h}_1(\frac{1}{\alpha}) = -\frac{1}{2}\frac{\vec{h}_1(\alpha^2)}{\alpha^2} = -\frac{Q}{2},$$

an easy induction argument shows the first item.

The second and the third item are proven in the same way by using moreover (3.42). □

The detailed calculations for the proofs in the two lemmas above can be found in Section 3.10.

Lemma 3.65. *Assume that a minor of the matrix $\hat{A}$ (resp. of the matrix $\begin{pmatrix} \hat{A} & \alpha(q_0)\hat{b} \end{pmatrix}$) is nonzero. Then the corresponding minor - same rows and columns - of A (resp. of $\begin{pmatrix} A & \alpha b \end{pmatrix}$) is nonzero as well near q_0.*

Proof. An arbitrary $(l \times l)$ minor $m(A)$ of the matrix A has the form

$$m(A) = \sum_{\sigma \in \mathfrak{S}_l} \mathrm{sgn}(\sigma)\ a^{s_1}_{j_1, k_{\sigma(1)}} \cdots a^{s_l}_{j_l, k_{\sigma(l)}}.$$

As a consequence of Lemma 3.63, each term in this sum is a polynomial function of u of weighted degree $\leq 2\sum_i s_i - \sum_i w_{k_i} + l$. Moreover, the homogeneous part of $m(A(q_0))$ of weighted degree $2\sum_i s_i - \sum_i w_{k_i} + l$ is equal to

$$m(\hat{A}) = \sum_{\sigma \in \mathfrak{S}_l} \mathrm{sgn}(\sigma)\ \hat{a}^{s_1}_{j_1, k_{\sigma(1)}} \cdots \hat{a}^{s_l}_{j_l, k_{\sigma(l)}}.$$

Hence, if $m(\hat{A}) \neq 0$, then $m(A(q_0))$ is nonzero and so is $m(A(q))$ for q near q_0.

The same argument holds for the minors of $\begin{pmatrix} \hat{A} & \alpha(q_0)\hat{b} \end{pmatrix}$ and $\begin{pmatrix} A & \alpha b \end{pmatrix}$ by using Lemma 3.64. □

Lemma 3.66. *The algebraic system $\hat{A}\hat{\Phi} = \hat{b}$ admits a solution $\hat{\Phi}$.*

Proof. Since $\vec{h}_1$ and $\vec{h}_2$ are locally orbitally diffeomorphic, there exists an orbital diffeomorphism Φ between the extremal flows of (g_1, g_2) with coordinates $(\Phi_1, \dots, \Phi_n)$ in the system of coordinates associated with the frame $\{X_1, \dots, X_n\}$. Then from Proposition 3.32 $\widetilde{\Phi} = (\Phi_{m+1}, \dots, \Phi_n)$ satisfies $A\widetilde{\Phi} = b$. Introducing the nonzero function α defined by (3.8), this can be rewritten as $A\alpha\widetilde{\Phi} - \alpha b = 0$, i.e.

$$\begin{pmatrix} A & \alpha b \end{pmatrix} \begin{pmatrix} \alpha\widetilde{\Phi} \\ -1 \end{pmatrix} = 0.$$

Thus $\begin{pmatrix} A & b \end{pmatrix}$ is not of full rank, or equivalently, any maximal minor of the latter matrix is zero. The contraposition of Lemma 3.65 implies that any maximal minor of $\begin{pmatrix} \hat{A} & \alpha(q_0)\hat{b} \end{pmatrix}$ is zero as well, thus this matrix is not of full rank.

Any element of $\ker \begin{pmatrix} \hat{A} & \alpha(q_0)\hat{b} \end{pmatrix}$ is a function of $u \in \mathbb{R}^n$ with values in $\mathbb{R}^{n-m} \times \mathbb{R}$. Since $\hat{A}$ is of full rank by Proposition 3.33, and since $\alpha(q_0)$ is nonzero, there exists $\Psi \in \ker \begin{pmatrix} \hat{A} & \alpha(q_0)\hat{b} \end{pmatrix}$ of the form $\Psi = (\alpha(q_0)\hat{\Phi}, -1)$. In other terms, $\hat{\Phi}$ satisfies $\hat{A}\hat{\Phi} = \hat{b}$. □

Proof of Theorem 3.61. The metrics $\hat{g}_1$ and $\hat{g}_2$ are left-invariant metrics on the Carnot group $\hat{M}$ and by Remark 3.62 they are not proportional. Moreover by the lemma above, the fundamental algebraic system associated with $\hat{g}_1$ and $\hat{g}_2$ admits a solution. Theorem 3.61 follows then from Remark 3.60. □

3.8 Genericity of indecomposable fundamental graded Lie algebras

This section is dedicated to the proof of Theorem 3.8. From Theorem 3.61, the existence of projectively equivalent and non conformal metrics implies that the nilpotent approximation of D at generic points admits a product structure. Thus we have to show that, under the hypothesis of the theorem on (m, n), generic nilpotent approximations do not have a product structure.

Remark first that, when $n \geq \frac{m(m+1)}{2}$, generic distributions are free up to the second step at generic points, i.e. D^2 is a distribution of rank $\frac{m(m+1)}{2}$ near these points. The nilpotent approximation of such a distribution does not admit a product structure, therefore the statement of Theorem 3.8 holds for these values of (m, n).

Consider now a pair (m, n) such that $2 \leq m < n \leq \frac{m(m+1)}{2}$. We denote by GNLA$(m, n)$ the set of all n-dimensional step 2 graded Lie algebras generated by the homogeneous component V_1 of dimension m. Theorem 3.8 results directly from the following proposition.

Proposition 3.67. *Except the following two cases:*

1. $m = n - 1$ *with even* n,

2. $(m, n) = (4, 6)$,

a generic element of GNLA(m, n) *cannot be represented as a direct sum of two graded Lie algebras.*

Proof. Let $\mathfrak{g} = V_1 \oplus V_2$ be a step 2 graded Lie algebra. This algebra can be described by the *Levi operator*

$$\mathcal{L}_q : \wedge^2 V_1 \to V_2,$$

which sends $(X, Y) \in \wedge^2 V_1$ to $[X, Y]$ or, equivalently, by the dual operator $\mathcal{L}_q^* : V_2^* \to \wedge^2 V_1^*$. Denote by $\Omega_{\mathfrak{g}}$ the image of the latter operator.

Since $\mathfrak{g}$ is generated by V_1, the space $\Omega_{\mathfrak{g}}$ is a $(n-m)$-dimensional subspace of the space $\wedge^2 V_1^*$ of all skew-symmetric forms on V_1. The set GNLA(m, n) is in a bijective correspondence with the orbits of $(n-m)$-dimensional subspace of $\wedge^2 V_1^*$ under the natural action of $GL(V_1)$. This reduces our question to an analysis of orbits in Grassmannians of $\wedge^2 V_1^*$ under the natural action of $GL(V_1)$.

Given a subspace W of V_1 denote by A_W the space of all skew-symmetric forms with kernel W. A graded Lie algebra $\mathfrak{g} = V_1 \oplus V_2$ is a direct sum of two graded Lie algebras if and only if there is a splitting

$$V_1 = V_1^1 \oplus V_1^2 \tag{3.43}$$

(with each summand being nonzero) such that the corresponding subspace of $\wedge^2 V_1^*$ can be represented as

$$\Omega_{\mathfrak{g}} = \Omega_{\mathfrak{g}}^1 \oplus \Omega_{\mathfrak{g}}^2, \qquad \Omega_{\mathfrak{g}}^1 \subset A_{V_1^2}, \quad \Omega_{\mathfrak{g}}^2 \subset A_{V_1^1}. \tag{3.44}$$

In this case we will say that the space $\Omega_{\mathfrak{g}}$ is *decomposable* with respect to the splitting (3.43). The condition on $\Omega_{\mathfrak{g}}^1$ and $\Omega_{\mathfrak{g}}^2$ in (3.44) is equivalent to require that, in some basis of V_1, the elements of $\Omega_{\mathfrak{g}}^1$ have the block-diagonal matrix representation $\left(\begin{array}{c|c} A_1 & 0 \\ \hline 0 & 0 \end{array}\right)$ and the elements of $\Omega_{\mathfrak{g}}^2$ have the block-diagonal matrix representation $\left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & A_2 \end{array}\right)$, where the corresponding blocks have the same nonzero size. Note that we do not exclude that one of the subspaces $\Omega_{\mathfrak{g}}^i$ is equal to zero. In this case the space $\Omega_{\mathfrak{g}}$ itself must consist of forms with a common nontrivial kernel. This corresponds to the situation where one of the summands in the decomposition of $\mathfrak{g}$ into a direct sum is commutative.

We will distinguish several cases, depending on the value of the corank $n - m$ and on the parity of m.

1. The case $n - m = 1$**.** In this case the space $\Omega_{\mathfrak{g}}$ is a line in the space $\wedge^2 V_1^*$. It is decomposable if and only if $\Omega_{\mathfrak{g}}$ is generated by a degenerate skew-symmetric form, one of the subspaces $\Omega_{\mathfrak{g}}^i$ being zero. The latter condition is satisfied by a generic line in $\wedge^2 V_1^*$ if and only if $\dim V_1$ is odd, or equivalently, when n is even.

2. The case $n - m = 2$**.** In this case $\Omega_{\mathfrak{g}}$ is a plane in the space $\wedge^2 V_1^*$. The orbits of planes of $\wedge^2 V_1^*$ under the natural action of $GL(V_1)$ are in bijective correspondence with the equivalence classes of *pencils of skew-symmetric 2-forms*, which are linear combinations $\lambda A + \mu B$ of two skew-symmetric 2-forms A, B with real parameters λ, μ. The classification of these pencils is classical, we give here some basic definitions and results and we refer the reader to [79] (based on [80]) for more details.

Let us consider a pencil of skew-symmetric 2-forms $\lambda A + \mu B$, identified to a pencil of skew-symmetric matrices in some basis of the space V_1. The pencil is called *regular* if its determinant is a non-zero polynomial, it is called *singular* otherwise. A regular pencil is characterized by its *elementary divisors*, defined as follows. Consider the greatest common divisor of all rank-k minors of the pencil for the integers k for which it makes sense. The elementary divisors of the pencil are the simple factors (with their multiplicity) of these greatest common divisors for all possible k. In case of skew-symmetric pencils, all elementary divisors come in pairs. A singular pencil is characterized by its elementary divisors and its *minimal indices* (also called Kronecker indices in [79]). The special property of a singular pencil is that there exists a nonzero homogeneous polynomial branch of kernels $\lambda, \mu \mapsto v(\lambda, \mu)$, i.e. for any $\lambda, \mu \in \mathbb{R}$, the vector $v(\lambda, \mu)$ is a nonzero element of $\ker(\lambda A + \mu B)$. The *first minimal index* is the minimal possible degree of a polynomial $v(\lambda, \mu)$. We do not need the other indices here, so we do not define them.

The pencils defined in different basis of V_1 and associated to different elements of the same $GL(V_1)$-orbit of a skew-symmetric form are called equivalent. The following result give the normal forms of skew-symmetric pencils.

Theorem 3.68 ([79]). *A skew-symmetric pencil $\lambda A + \mu B$ with minimal indices $m_1 \leq m_2 \leq \cdots \leq m_p$ and elementary divisors $(\mu + a_1\lambda)^{l_1}, (\mu + a_1\lambda)^{l_1}, \ldots, (\mu + a_q\lambda)^{l_q}, (\mu + a_q\lambda)^{l_q}$ and $(\lambda)^{f_1}, (\lambda)^{f_1}, \ldots, (\lambda)^{f_s}, (\lambda)^{f_s}$ is equivalent to the skew-symmetric pencil Q of the following form,*

$$Q = \left(\begin{array}{c|c} \mathcal{M} & 0 \\ \hline 0 & \mathcal{F} \end{array} \right),$$

where the singular part $\mathcal{M}$ and the regular part $\mathcal{F}$ satisfy

$$\mathcal{M} = \begin{pmatrix} M_1 & & \\ & \ddots & \\ & & M_q \end{pmatrix}, \qquad \mathcal{F} = \begin{pmatrix} E_1(a_1) & & & & & \\ & \ddots & & & & \\ & & E_q(a_q) & & & \\ & & & F_1 & & \\ & & & & \ddots & \\ & & & & & F_s \end{pmatrix},$$

all blocks M_i, E_i, F_i being skew-symmetric, M_i of size $(2m_i+1)\times(2m_i+1)$, E_i of size $2l_i \times 2l_i$ and F_i of size $2f_i \times 2f_i$.

Remark 3.69. Note that the only possible zero blocks are the blocks $E_i(a_i)$ with $\frac{\mu}{\lambda} = -a_i$ and $l_i = 1$, and F_j with $\lambda = 0$ and $f_j = 1$.

Let us return to the plane Ω_g considered as a pencil. The cases of odd-dimensional and even-dimensional V_1 are treated again separately.

2(a) The subcase when $\dim V_1$ **is odd,** $\dim V_1 = 2k+1$**.** In this case all forms in the pencil Ω_g are degenerate, so the pencil is singular. From the dimension of the blocks in the normal form, we see that the first minimal index is not greater than k. Moreover, for generic pencils this first minimal index has its maximal possible value, thus it is equal to k.

On the other hand, if the pencil $\Omega_{\mathfrak{g}}$ is decomposable, then its first minimal index must be equal to zero, i.e. all forms of the pencil have a common nontrivial kernel. Indeed, assume that $\Omega_{\mathfrak{g}}$ is decomposable with respect to the splitting (3.43) with decomposition (3.44). The statement is clear if one of the spaces $\Omega_{\mathfrak{g}}^i$ in (3.44) is zero. The remaining possibility is that the spaces $\Omega_{\mathfrak{g}}^i$ are both one-dimensional. Without loss of generality we assume that V_1^1 is odd-dimensional. Then all forms on the line $\Omega_{\mathfrak{g}}^1$ have a nontrivial kernel in V_1^1 and this kernel is common for all forms in $\Omega_{\mathfrak{g}}$.

Since $k > 0$ ($\dim V_1 \geq 2$), we conclude that generic pencils are not decomposable.

2(b) The subcase when $\dim V_1$ **is even.** In this case generic pencils are regular. Generic regular pencils of skew-symmetric forms have only simple elementary divisors, i.e. linear and not nontrivial powers of linear, such that each divisor appear only twice.

Now consider a decomposable regular pencil $\Omega_{\mathfrak{g}}$ with respect to the splitting (3.43) with decomposition (3.44). Then

$$\Omega_g = \{\lambda\omega_1 + \mu\omega_2 : \lambda, \mu \in \mathbb{R}\}, \tag{3.45}$$

where the form ω_i generates $\Omega_{\mathfrak{g}}^i$, $i = 1, 2$. One can see by the normal form in Theorem 3.68 that the elementary divisors of this pencil can be only of the form λ

or μ of multiplicity one. The pencil satisfy the genericity property of the previous paragraph if and only if the set of elementary divisors is $\{\lambda, \lambda, \mu, \mu\}$, i.e. when $m = 4$. Consequently $n = 6$. So, decomposibility on an open set can occur only if $(m, n) = (4, 6)$. Conversely, if $(m, n) = (4, 6)$ and Ω_g is as in (3.45), then Pfaffian$\big(\lambda\omega_1 + \mu\omega_2\big)$ is a quadratic form in λ and μ and the pencil Ω_g is decomposible if and only if this form is sign-indefinite, which implies that decomposibility occurs on an open set in this case.

Remark 3.70. In the case $m = 4$, if the pencil $\Omega_g = \{\lambda\omega_1 + \mu\omega_2 : \lambda, \mu \in \mathbb{R}\}$ is decomposable, then the subspaces V_1^1 and V_1^2 in the splitting (3.43) are uniquely defined. Indeed, in this case there are exactly two degenerate forms: these are the lines on which Pfaffian$(\lambda\omega_1 + \mu\omega_2) = 0$ and the subspaces V_i^j are kernels of these forms. We will call this splitting $V_1 = V_1^1 \oplus V_1^2$ the canonical splitting corresponding to the decomposable pencil Ω_g.

3. The case $n - m > 2$**.** We will reduce this case to the case $n - m = 2$.

3(a) The subcase when $\dim V_1$ **is odd, i.e.** $\dim V_1 = 2k + 1$. Assume that $\Omega_{\mathfrak{g}}$ is decomposable with respect to the splitting (3.43) and, without loss of generality, that $\dim V_1^1$ is odd and equal to $2l + 1$, $l < k$. Then it is easy to see on the normal form that the first minimal index of any plane in $\Omega_{\mathfrak{g}}$ is not greater than l. On the other hand, a generic plane in a generic $(n - m)$-dimensional subspace of $\wedge^2 V_1^*$ has first minimal index k. This proves the statement of the theorem in this case.

3(b) The subcase when $\dim V_1$ **is even.** First, assume that $m = \dim V_1 > 4$. Then by item 2(b) a generic $(n - m)$-dimensional subspace of $\wedge^2 V_1$ contains an indecomposable plane, therefore the original $(n - m)$-dimensional subspace is indecomposable.

Now assume that $m = \dim V_1 = 4$. Then a generic $(n-m)$-dimensional subspace of $\wedge^2 V_1$ either contains an indecomposable plane or contains two planes such that the canonical decomposition of V_1 corresponding to these planes, as defined in Remark 3.70, do not coincide. This implies that generic $(n - m)$-dimensional subspaces of $\wedge^2 V_1$ are indecomposable. This case ends the proof. □

3.9 Conformal case

The results of the previous sections show that on a distribution whose nilpotent approximation does not admit a product structure, all projectively equivalent metrics are conformal. This is the case in particular for a generic distribution. In the Riemannian case two projectively equivalent metrics which are conformal are necessarily constantly proportional, as follows directly from the Levi-Civita pair form (3.29).

In (pseudo-)Riemannian geometry, the relations between the projective and conformal metric transformations were studied by H. Weyl. In [81] he demonstrated that transformations that are simultaneously projective and conformal are necessarily constant scaling of the metric. That was important for rapidly developing relativity theory. For our problem of metric equivalence the result of Weyl implies the projective rigidity of the conformally projectively rigid metrics in the Riemannian case. His arguments rely however on the Weyl tensors which are defined by the Riemann curvature tensor and thus are not applicable in the context of sub-Riemannian geometry. Therefore we have to use a different approach to address the question.

In view of the geometric inverse problem, the conformal rigidity of metrics is not satisfactory as it does not imply the injectivity of the inverse problem. Thus, it is of special importance to recover the relations between the projectively equivalent conformal metrics with the goal to approach the Weyl result in the sub-Riemannian case. In this section we make some steps toward this goal.

3.9.1 The fundamental algebraic system in the conformal case

Let M be a manifold and D be a bracket generating distribution on M. We consider two sub-Riemannian metrics on (M, D) that are both conformal and projectively equivalent. Let us denote these metrics by g and $\alpha^2 g$, where $\alpha : M \to \mathbb{R}$ is a never vanishing smooth function.

From Proposition 3.26, there exists local orbital diffeomorphisms Φ between the Hamiltonian vector fields associated with g and $\alpha^2 g$ near ample covectors. Choosing appropriate coordinates u on the fiber and using Lemma 3.30 and Proposition 3.32, we have the following properties for the coordinates u_i of Φ.

- The function $\alpha \circ \pi$ on T^*M coincide with the function $\alpha(\lambda)$ associated with an orbital diffeomorphism by (3.5) (this result follows from (3.8) since all eigenvalues $\alpha_1^2, \dots, \alpha_m^2$ are equal to α^2). As a consequence, by (3.9)

$$\Phi_k = \alpha u_k \qquad \text{for } k = 1, \dots, m.$$

- Equation (3.10) writes as: for $j = 1, \dots, m$,

$$\sum_{k=m+1}^{n} q_{jk}(\Phi_k - \alpha u_k) = \sum_{i=1}^{m} (X_i(\alpha) u_j - X_j(\alpha) u_i) u_i,$$

where $q_{jk} = \sum_{i=1}^{m} c_{ij}^k u_i$.

- Equation (3.11) writes as: for $k = m+1, \dots, n$,

$$\vec{h}(\Phi_k - \alpha u_k) = \sum_{l=m+1}^{n} q_{kl}(\Phi_l - \alpha u_l) + \sum_{i=1}^{m} (X_k(\alpha)u_i - X_i(\alpha)u_k)u_i.$$

- Setting $\widetilde{u} = (u_{m+1}, \dots, u_n)$ and $\widetilde{\Phi} = (\Phi_{m+1}, \dots, \Phi_n)$, the fundamental algebraic system (3.12) writes as

$$A(\widetilde{\Phi} - \alpha \widetilde{u}) = d, \tag{3.46}$$

where A is the matrix defined by (3.14) and d is a column vector with an infinite number of rows which can can be decomposed in layers of m rows as

$$d = \begin{pmatrix} d^1 \\ d^2 \\ \vdots \\ d^s \\ \vdots \end{pmatrix},$$

where the coefficients d^s_j, $1 \leq j \leq m$, of the vector $d^s \in \mathbb{R}^m$ are defined by

$$\begin{cases} d^1_j = \displaystyle\sum_{i=1}^{m} (X_i(\alpha)u_j - X_j(\alpha)u_i)u_i, \\ d^{s+1}_j = \vec{h}_1(d^s_j) + \displaystyle\sum_{k=m+1}^{n} a^s_{j,k} \sum_{i=1}^{m} u_i \left(X_i(\alpha)u_k - X_k(\alpha)u_i\right). \end{cases} \tag{3.47}$$

The fundamental algebraic system (3.46) implies that the coordinates of Φ are rational functions on the fibers. Proving that g and $\alpha^2 g$ are proportional actually amounts to prove that these coordinates are polynomial, as stated below.

Proposition 3.71. *If there exists a local orbital diffeomorphism Φ which is polynomial on the fibers, then g and $\alpha^2 g$ are locally constantly proportional, i.e., α is constant.*

Before giving the proof of this result, we need to study the consequence of the fundamental algebraic system on the nilpotent approximation.

Fix a regular point q_0 and denote by $(\hat{M}, \hat{D})$ the nilpotent approximation of (M, D) at q_0. We argue as in the proof of Theorem 3.61, with the same notations. In particular $\{\hat{X}_1, \dots, \hat{X}_n\}$ is a frame of $T\hat{M}$ adapted to $\hat{D}$ such that $\hat{X}_1, \dots, \hat{X}_m$ is $\hat{g}$-orthonormal and $\hat{A}$ is the matrix of Proposition 3.32 constructed by using $\{\hat{X}_1, \dots, \hat{X}_n\}$ as a frame.

Lemma 3.72. *There exists* $\Psi = (\Psi_{m+1}, \ldots, \Psi_n)$ *solution of*

$$\hat{A}\Psi = \hat{d}, \tag{3.48}$$

where, for any $s \in \mathbb{N}$ *and* $1 \le j \le m$, $\hat{d}_j^s$ *is defined by*

$$\begin{cases} \hat{d}_j^1 = \displaystyle\sum_{i=1}^m (X_i(\alpha)(q_0)u_j - X_j(\alpha)(q_0)u_i)u_i, \\ \hat{d}_j^{s+1} = \vec{\hat{h}}_1(\hat{d}_j^s) + \displaystyle\sum_{k=m+1}^n \hat{a}_{j,k}^s \sum_{i=1}^m u_i X_i(\alpha)(q_0)u_k. \end{cases}$$

Proof. An easy induction argument based on equations (3.47) shows the following result, similar to Lemma 3.64: for any $s \in \mathbb{N}$ and $1 \le j \le m$, there hold:

- for every $q \in M$ near q_0, d_j^s is a polynomial in $u_1, \ldots, u_n$ of weighted degree
$$\deg_w(d_j^s) \le 2s;$$
- the homogeneous term of highest weighted degree in $d_j^s(q_0)$ is $\hat{d}_j^s$.

It results from (3.46) that $\begin{pmatrix} A & d \end{pmatrix}$ is not of full rank, thus also the matrix $\begin{pmatrix} \hat{A} & \hat{d} \end{pmatrix}$ is not of full rank. Since $\hat{A}$ is of full rank by Proposition 3.33, there exists an element in ker $\begin{pmatrix} \hat{A} & \hat{d} \end{pmatrix}$ of the form $(\Psi, -1)$, which ends the proof. $\square$

Using all equations above it is easy to show that Ψ has the following properties.

(i) Each Ψ_k, $k = m+1, \ldots, n$, is a rational function which is:

- homogeneous of degree 1 w.r.t. the usual degree;
- w-homogeneous with $\deg_w(\Psi_k) = w_k - 1$.

(ii) For $j = 1, \ldots, m$, we have

$$\sum_{\{k \,:\, w_k = 2\}} \sum_{i=1}^m \hat{c}_{ij}^k u_i \Psi_k = \sum_{i=1}^m (\alpha^i u_j - \alpha^j u_i)u_i, \tag{3.49}$$

where $\alpha^j = X_i(\alpha)(q_0)$.

(iii) For $k = m+1, \ldots, n$, we have

$$\vec{\hat{h}}(\Psi_k) = \sum_{\{l \,:\, w_l = w_k + 1\}} \sum_{i=1}^m \hat{c}_{ik}^l u_i \Psi_l - \sum_{i=1}^m \alpha^i u_k u_i. \tag{3.50}$$

Lemma 3.73. *Assume that the map* Ψ *given in Lemma 3.72 is polynomial. Then*

$$X_1(\alpha)(q_0) = \cdots = X_m(\alpha)(q_0) = 0.$$

Proof. By hypothesis, every Ψ_k, $k = m+1, \dots, n$, is a polynomial. Moreover, by Property i above, Ψ_k is a linear function of u and depends only on the coordinates u_l of weight $w_l = w_k - 1$. To simplify the notations, we use the following convention: given a positive integer s, an index k_s denotes an index of weight $w_{k_s} = s$ and $\sum_{k_s}$ denotes $\sum_{\{k_s\,:\,w_{k_s}=s\}}$. With this notation we have, for every k_s,

$$\Psi_{k_s} = \sum_{k_{s-1}} \varepsilon_{k_s k_{s-1}} u_{k_{s-1}}, \tag{3.51}$$

where the coefficients $\varepsilon_{k_s k_{s-1}}$ are real numbers. Taking the derivative along $\vec{\hat{h}}$ we obtain

$$\vec{\hat{h}}(\Psi_{k_s}) = \sum_{i=1}^{m} \sum_{k_{s-1}, l_s} \varepsilon_{k_s k_{s-1}} \hat{c}_{i k_{s-1}}^{l_s} u_i u_{l_s}. \tag{3.52}$$

On the other hand, plugging (3.51) into (3.50), we get

$$\vec{\hat{h}}(\Psi_{k_s}) = \sum_{i=1}^{m} \sum_{l_s, l_{s+1}} \hat{c}_{i k_s}^{l_{s+1}} \varepsilon_{l_{s+1} l_s} u_i u_{l_s} - \sum_{i=1}^{m} \alpha^i u_k u_i. \tag{3.53}$$

Fix an index $i \in \{1, \dots, m\}$. By identifying the coefficients of the monomial $u_i u_{k_s}$ in (3.52) and (3.53), we obtain the following equality,

$$\sum_{k_{s-1}} \varepsilon_{k_s k_{s-1}} \hat{c}_{i k_{s-1}}^{k_s} = \sum_{k_{s+1}} \hat{c}_{i k_s}^{k_{s+1}} \varepsilon_{k_{s+1} k_s} - \alpha^i,$$

and, after a summation on the $n_s - n_{s-1}$ indices k_s,

$$\sum_{k_{s-1}, k_s} \hat{c}_{i k_{s-1}}^{k_s} \varepsilon_{k_s k_{s-1}} = \sum_{k_s, k_{s+1}} \hat{c}_{i k_s}^{k_{s+1}} \varepsilon_{k_{s+1} k_s} - (n_s - n_{s-1})\alpha^i.$$

Set $K_i(s) = \sum_{k_s, k_{s+1}} \hat{c}_{i k_s}^{k_{s+1}} \varepsilon_{k_{s+1} k_s}$. Then the above equality writes as

$$K_i(s-1) = K_i(s) - (n_s - n_{s-1})\alpha^i \qquad \text{for } s > 1.$$

Note that $K_i(r) = 0$ since r is the nilpotency step. Hence,

$$K_i(1) = -(n_r - n_1)\alpha^i = -(n-m)\alpha^i. \tag{3.54}$$

Now, by plugging (3.51) in (3.49), we have, for $j = 1, \dots, m$:

$$\sum_{k_2, k_1} \sum_{i=1}^{m} \hat{c}_{ij}^{k_2} \varepsilon_{k_2 k_1} u_i u_{k_1} = \sum_{i=1}^{m} (\alpha^i u_j - \alpha^j u_i) u_i.$$

Given an index $i \in \{1, \dots, m\}$, the identification of coefficient of $u_i u_j$ in this equality gives

$$\sum_{k_2} \hat{c}_{ij}^{k_2} \varepsilon_{k_2 j} = (1 - \delta_{ij})\alpha^i,$$

and by summation on the indices $j = k_1$, we obtain

$$K_i(1) = (m-1)\alpha^i.$$

This equation and (3.54) imply $\alpha^i = 0$, which ends the proof. $\square$

Proof of Proposition 3.71. Assume Φ to be defined on an open subset U of T^*M. Fix a regular point q_0 in $\pi(U)$ and let $(\hat{M}, \hat{D})$ be the nilpotent approximation of (M, D) at q_0.

Let $\hat{\delta}$ be a nonzero maximal minor of $\hat{A}$. It is a w-homogeneous polynomial which is the homogeneous part of highest weighted degree of the corresponding minor (same rows and columns) δ of A, which is nonzero as well. It results easily from (3.46) that, for $k = m+1, \dots, n$, we have $\Phi_k - \alpha u_k = p_k/\delta$ where $d_w(p_k) \leq d_w(\delta) + 2$, and $\Psi_k = \hat{p}_k/\hat{\delta}$, where $\hat{p}_k$ is the homogeneous part (eventually zero) of weighted degree $d_w(\delta) + 2$ in p_k. From the hypothesis of the theorem, p_k/δ is polynomial, therefore $\Psi_k = \hat{p}_k/\hat{\delta}$ is polynomial as well and by Lemma 3.73 we get $X_i(\alpha)(q_0) = 0$, $i = 1, \dots, m$.

Since regular points form an open and dense subset of $\pi(U)$, the functions $X_i(\alpha)$, $i = 1, \dots, m$, are identically zero on $\pi(U)$. The family $X_1, \dots, X_m$ being a Lie-bracket generating family, we thus obtain that α is locally constant. $\square$

A remark on Lemma 3.72

Let $\hat{\alpha}$ be the real-valued function on $\hat{M}$ defined by

$$\begin{cases} \hat{\alpha}(0) = \alpha(q_0), \\ \hat{X}_i(\hat{\alpha}) \equiv X_i(\alpha)(q_0) \qquad i = 1, \dots, m, \\ \hat{X}_k(\hat{\alpha}) \equiv 0, \qquad k = m+1, \dots, n. \end{cases}$$

In a system of privileged coordinates z at q_0 such that $X_i(z_j)(q_0) = \delta_{ij}$, $\hat{\alpha}$ writes as

$$\hat{\alpha} = \alpha(q_0) + \sum_{i=1}^{m} z_i X_i(\alpha)(q_0).$$

The existence of the mapping Ψ in Lemma 3.72 may be interpreted as follows.

Lemma 3.74. *There exists a fiber-preserving map $\hat{\Phi} : T^*\hat{M} \to T^*\hat{M}$ such that, on a neighbourhood of every ample covector (w.r.t. $\hat{g}$), $\hat{\Phi}$ is smooth and sends the integral curves of the Hamiltonian vector fields of the metric $\hat{g}$ to the ones of $\hat{\alpha}^2\hat{g}$.*

Proof. Note that $\hat{d}$ is the vector d constructed by using $\{\hat{X}_1, \dots, \hat{X}_n\}$ as a frame and $\hat{\alpha}$ as conformal coefficient in (3.47). Let Ψ be the solution of $\hat{A}\Psi = \hat{d}$ and set

$$\begin{cases} \hat{\Phi}_k = \hat{\alpha} u_k, \qquad k = 1, \dots, m, \\ \hat{\Phi}_k = \Psi_k + \hat{\alpha} u_k, \qquad k = m+1, \dots, n. \end{cases}$$

Define $\hat{\Phi} : T^*\hat{M} \to T^*\hat{M}$ as the fiber-preserving map such that $u \circ \hat{\Phi} = (\hat{\Phi}_1, \dots, \hat{\Phi}_n)$. It results from Proposition 3.35 that $\hat{\Phi}$ sends the extremal flows of $\hat{g}$ to the one of $\hat{\alpha}^2\hat{g}$ near any ample covector. $\square$

3.9.2 A Geometric condition

Consider the case where (M, D) are real analytic (for instance a nilpotent approximation). Locally we can consider a complex manifold $\mathbb{C}M$, a *complexification of* M, by extending the transition functions between chart, which are real analytic by definition, to analytic functions. We can also consider the cotangent bundle of $\mathbb{C}M$ whose fibers are complex vector spaces. We can extend the sub-Riemannian Hamiltonian analytically to this bundle and consider the corresponding Hamiltonian vector field. We also can define complex normal, abnormal, and strictly normal sub-Riemannian geodesics, Jacobi curve and the corresponding osculating flag for every complex normal extremal.

Theorem 3.75 (Sub-Riemannian Weyl theorem). *Let (M, D, g) be a smooth sub-Riemannian manifold. Assume that its nilpotent approximation at every point satisfies the following property: for every positive $\kappa \in \mathbb{N}$, there is no $(n-2-\kappa)$-parametric family of corank-κ non strictly normal complexified geodesics through a point. Then any metric which is simultaneously conformal and projective to g is constantly proportional to g.*

Remark 3.76. The hypothesis of the theorem above holds in particular when the nilpotent approximations do not have non strictly normal complexified geodesics.

Proof. Let $\alpha^2 g$ be a metric conformal to g. Consider a regular point $q_0 \in M$, the nilpotent approximation $(\hat{M}, \hat{D}, \hat{g})$ of (M, D, g) at q_0, and the map Ψ given by Lemma 3.72. We only need to show that Ψ is polynomial, the conclusion will then follow from Lemma 3.73 and the same argument as in the end of the proof of Proposition 3.71.

Let k be a positive integer and $\hat{A}_k$ be the truncation of the fundamental matrix $\hat{A}$ up to the kth layer. We choose k large enough so that at least one maximal minor of $\hat{A}_k$ is nonzero. Note that since we consider the nilpotent approximation, the coefficients of the matrix $\hat{A}_k$ are polynomials in u with constant coefficients.

From now on we work on the complexified manifold $\mathbb{C}\hat{M}$, and, identifying locally $T^*\mathbb{C}\hat{M}$ with $\mathbb{C}\hat{M} \times \mathbb{C}^n$, we consider all polynomials on the fibers as polynomials on $\mathbb{C}^n$. Let $\mathcal{S}_0 \subset \mathbb{C}^n$ be the common zero level set of all maximal minors of $\hat{A}_k$.

It results from (3.48) that for any nonzero minor δ of $\hat{A}_k$ we have

$$\Psi_i = \frac{p_i}{\delta}, \quad m+1 \leq i \leq n, \tag{3.55}$$

where p_i is a polynomial. Cancelling the greatest common factor of the collection of polynomials $\{\delta, p_{m+1}, \dots, p_n\}$, we get the collection of polynomials $\{\widetilde{\delta}, \widetilde{p}_{m+1}, \dots, \widetilde{p}_n\}$ with the greatest common factor equal to constant and such that

$$\Psi_i = \frac{\widetilde{p}_i}{\widetilde{\delta}}. \tag{3.56}$$

Besides, substituting (3.56) into (3.50) we get

$$\vec{\hat{h}}(\widetilde{\delta})\widetilde{p}_i \text{ is divisible by } \widetilde{\delta}. \tag{3.57}$$

Let us show that under the assumption of Theorem 3.75 $\widetilde{\delta}$ is constant. Assuming the converse, there is an irreducible polynomial δ_1 such that $\widetilde{\delta} = \delta_1^s p$, where s is a positive integer and p is a polynomial such that p and δ_1 are coprime. By constructions, there exists $j \in \{m+1, \dots n\}$ such that $\widetilde{p}_j$ is not divisible by δ_1, otherwise δ_1 is a nonconstant common factor of the collection $\{\widetilde{\delta}, \widetilde{p}_{m+1}, \dots, \widetilde{p}_n\}$.

Consider this particular j. Although the polynomials $\widetilde{p}_j$ and $\widetilde{\delta}$ are not coprime in general, if we further reduce the expression (3.56) for Ψ_j to the lowest terms (i.e. such that the numerator and denominator will be coprime), then the denominator will be divisible by δ_1. Note that in (3.55) we can use any nonzero maximal minor δ of $\hat{A}_k$ and the expression for Ψ_j in the lowest terms is unique and does not depend on the initial choice of the nonzero maximal minor δ. Hence, δ_1 is a common divisor of all maximal minors of $\hat{A}_k$ and the zero-level set $\mathcal{S}_1$ of δ_1 belongs to the set $\mathcal{S}_0$.

Since $\widetilde{p}_j$ and p are not divisible by δ_1, by the Hilbert Nullstellensatz there is a nonzero open subset $\mathcal{S}_2$ of $\mathcal{S}_1$, where $\widetilde{p}_j$ and p are nonzero and such that $\mathcal{S}_2$ is submanifold of $\mathbb{C}^n$. From (3.57), there holds

$$\vec{\hat{h}}(\widetilde{\delta})\widetilde{p}_j = \Big(sp\delta_1^{s-1}\vec{\hat{h}}(\delta_1) + \delta_1^s\vec{\hat{h}}(p)\Big)\widetilde{p}_j \text{ is divisible by } \widetilde{\delta} = \delta_1^s p,$$

which implies that

$$\Big(sp\vec{\hat{h}}(\delta_1) + \delta_1\vec{\hat{h}}(p)\Big)\widetilde{p}_j \text{ is divisible by } \delta_1 p.$$

Since $\widetilde{p}_j$ and p are nonzero on $\mathcal{S}_2$, we deduce that $\vec{\hat{h}}(\delta_1)$ is zero on $\mathcal{S}_2$, and so that $\vec{\hat{h}}$ is tangent to $\mathcal{S}_2$. Therefore any complexified normal extremal of $(\hat{M}, \hat{D}, \hat{g})$ starting at a point of $\mathcal{S}_2$ will stay in $\mathcal{S}_2$.

Consider such a normal extremal $\lambda(t)$, $t \in [0, T]$, in $\mathcal{S}_2$. Since $\mathcal{S}_2 \subset \mathcal{S}_0$, due to [64, Lemma 3.12] there holds $\dim J_{\lambda(t)}^{(k+1)} < 2n$ for all $t \in [0, T]$, and so $\dim J_{\lambda(t)}^{(n-m)} < 2n$ for all $t \in [0, T]$ by assuming k large enough. But the Jacobi curve $J_\lambda(t)$ is equiregular for t in an open dense subset of $[0, T]$, thus for these times t we have, for any $s \in \mathbb{N}$, $\dim J_{\lambda(t)}^{(s)} = \dim J_{\lambda(t)}^{(n-m)} < 2n$, and the normal geodesic $\gamma(\cdot) = \pi(\lambda(\cdot))$ is not ample at t. As a consequence, $\gamma(\cdot)$ is a non strictly normal geodesic of $(\hat{M}, \hat{D}, \hat{g})$ (see [57, Prop. 3.12] and remind that the nilpotent approximation is analytic).

Hence, the $(n-1)$-dimensional submanifold $\mathcal{S}_2$ admits locally a foliation by a $(n-2)$-parameters family of normal extremals which project to non strictly geodesics. Anyone of these geodesics may admit several normal extremal lifts, the dimension of the space of these extremals being given by the corank κ of the geodesic. Thus we have proven that there exists, for at least one positive integer κ, a $(n-2-\kappa)$-parametric family of corank-κ non strictly normal complexified geodesics through a point, which contradicts our assumptions. So, $\widetilde{\delta}$ is constant and all Ψ_i are polynomials, which concludes the proof. $\square$

3.10 Appendix

Here we present calculations of weighted degrees of the fundamental system in case of nilpotent approximation.

Fix a regular point q_0. Let $(\hat{M}, \hat{D})$ be the nilpotent approximation of (M, D) at q_0. Consider a frame $\{X_1, \ldots, X_n\}$ of TM adapted to D at $q_0 \in M$ and such that $X_1, \ldots, X_m$ are g-orthonormal. Let $\{\hat{X}_1, \ldots, \hat{X}_n\}$ be the corresponding frame of $T\hat{M}$ adapted to $\hat{D}$ such that $\hat{X}_1, \ldots, \hat{X}_m$ is $\hat{g}$-orthonormal. Let $\hat{A}, \hat{b}$ be the matrices from Proposition 3.32 constructed by using $\{\hat{X}_1, \ldots, \hat{X}_n\}$ as a frame and $\alpha_1^2(q_0), \ldots, \alpha_m^2(q_0)$ as eigenvalues in the formulas (3.13)–(3.15). We present here the calculations of degrees of the coefficients of $\hat{A}$ and $\hat{b}$ appearing in Lemmas 3.63, 3.64. The following convention is used in the proofs: given a positive integer k, an index l_k denotes an index of weight $w_{l_k} = k$ and $\sum_{l_k}$ denotes $\sum_{\{l_k \,:\, w_{l_k}=k\}}$. First, notice that $\vec{h}_1$ is homogeneous of weighted degree 2 and for any polynomial P we have

$$\deg_w\big(\vec{h}_1(P)\big) \leq \deg_w(P) + 2. \tag{3.58}$$

Moreover, if P is a homogeneous weighted polynomial then (3.58) becomes an equality if $\vec{h}_1(P)$ is nonzero. Let r be the step of D, then we have $1 \leq \deg_w(u_i) \leq r$ for $i = 1, \ldots, n$. In case of the nilpotent approximation, (3.15) takes the form

$$\alpha \hat{b}_j^s = \alpha_j(q_0) \sum_{k=m+1}^{n} \hat{a}_{j,k}^s u_k, \qquad 1 \leq j \leq m, \ s \in \mathbb{N},$$

and (3.14) takes the form

$$\begin{cases} \hat{a}_{j,k}^1 = \hat{q}_{jk}, & 1 \leq j \leq m, \ m < k \leq n, \\ \hat{a}_{j,k}^{s+1} = \vec{h}_N(\hat{a}_{j,k}^s) + \displaystyle\sum_{\{l \,:\, w_l = w_k - 1\}} \hat{a}_{j,l}^s \hat{q}_{lk}, & 1 \leq j \leq m, \ m < k \leq n, \end{cases} \tag{3.59}$$

where $\vec{h}_N$ stands for the Hamiltonian vector field associated with the nilpotent approximation, i.e., with $\hat{X}_1, \ldots, \hat{X}_n$ and $\hat{g}$. Notice also that $\hat{q}_{jk} = 0$ if $w_k \neq 2$ and $\hat{q}_{jl_2} = q_{jl_2}(q_0)$.

Lemma 3.77. *An element $\hat{a}^s_{j,k}$, for any $s \in \mathbb{N}$, $1 \leq j \leq m$, $2 \leq k \leq r$, is either zero or a weighted homogeneous polynomial of degree $2s - w_k + 1$.*

Proof. The proof is by induction on the number s of the layer. The statement holds for $s = 1$. Indeed, by definition, $\hat{a}^1_{j,l_2} = \hat{q}_{jl_2} = \sum_{i=1}^m \hat{c}^{l_2}_{ij} u_i$ is a homogeneous polynomial of degree 1. Suppose the statement to be true for any number of layers $< s$, let us prove it for s. By (3.58) and the inductive assumption for $s-1$, we have

$$\deg_w \left(\vec{h}(\hat{a}^{s-1}_{j,l_k})\right) = \deg_w \left(\hat{a}^{s-1}_{j,l_k}\right) + 2 = 2s - k + 1,$$
$$\deg_w \left(\textstyle\sum_{l_{k-1}} \hat{a}^{s-1}_{j,l_{k-1}} \hat{q}_{l_{k-1},l_k}\right) = \deg_w \left(\hat{a}^{s-1}_{j,l_{k-1}}\right) + 1 = 2s - k + 1.$$

Combining this with the recurrence formula (3.59), we complete the proof. $\square$

Corollary 3.78. *An element $\alpha\hat{b}^s_j$, for any $s \in \mathbb{N}$, $1 \leq j \leq m$, is either zero or a weighted homogeneous polynomial of degree $2s + 1$.*

Corollary 3.79. *The degree of nonzero elements in $\hat{A}$ changes along the layer and from layer to layer by the following formulas, for all $s \in \mathbb{N}$, $1 \leq j \leq m$, and $2 \leq k \leq r$,*

$$\deg_w(\hat{a}^s_{j,l_{k+1}}) = \deg_w(\hat{a}^s_{j,l_k}) - 1, \quad \deg_w(\hat{a}^s_{j,l_{k+1}}) = \deg_w(\hat{a}^{s-1}_{j,l_{k+1}}) + 2.$$

Let A, b be from Proposition 3.32 constructed by using $\{X_1, \ldots, X_n\}$ as a frame and $\alpha^2_1(\cdot), \ldots, \alpha^2_m(\cdot)$ as eigenvalues in the formulas (3.13)–(3.15).

Lemma 3.80. *For any $s \in \mathbb{N}$, $j = 1, \ldots, m$, $k = 2, \ldots, r$, let $a^s_{i,j}$ be an element of A and $\hat{a}^s_{i,j}$ be the corresponding element of $\hat{A}$, then*

$$a^s_{j,l_k}(q_0) = \hat{a}^s_{j,l_k} + \mathrm{pol}(u_1, \ldots, u_n), \tag{3.62}$$

where $\deg_w(\mathrm{pol}) < \deg_w(\hat{a}^s_{j,l_k})$.

Proof. The proof is again by induction on the number s of layer. Let $s = 1$ then by (3.42) we have for any $1 \leq j \leq m$,

$$a^1_{j,l_2}(q_0) = \sum_{i=1}^m c^{l_2}_{ij}(q_0) u_i = \hat{a}^1_{j,l_2},$$

and for l_k with $k > 2$,

$$a^1_{j,l_k} = \hat{a}^1_{j,l_k} = 0.$$

This agrees with (3.62). Assume now that the statement holds for any number of layers $\leq s$, then for any $2 \leq k \leq r$ and $1 \leq j \leq m$,

$$a^s_{j,l_k}(q_0) = \hat{a}^s_{j,l_k} + \bar{a}^s_{j,l_k},$$

where $\deg_w\left(\bar{a}^s_{j,l_k}\right) < \deg_w\left(\hat{a}^s_{j,l_k}\right)$. In the rest of the prove we write a^s_{j,l_k} for $a^s_{j,l_k}(q_0)$ to simplify notations. Let us show the statement for $s+1$. Using (3.14) for a^{s+1}_{j,l_k} and (3.59) for $\hat{a}^{s+1}_{j,l_k}$, we have

$$\begin{aligned} a^{s+1}_{j,l_k} &= \vec{h}_1(a^s_{j,l_k}) + \sum_{\{l_s\,:\,s=k-1,\ldots,r\}} a^s_{j,l_s} q_{l_s l_k}, \\ \hat{a}^{s+1}_{j,l_k} &= \vec{h}_N(\hat{a}^s_{j,l_k}) + \sum_{l_{k-1}} \hat{a}^s_{j,l_{k-1}} \hat{q}_{l_{k-1} l_k}, \end{aligned}$$

We will show (3.62) for a^{s+1}_{j,l_k} by considering the two kinds of terms in the sum above separately. The rest of the prove is based on straightforward calculations. First, using the inductive assumption we obtain the degree of $\vec{h}(a^s_{j,l_k})$. Let us denote $\hat{a} = \hat{a}^s_{j,l_k}$ and $\bar{a} = \bar{a}^s_{j,l_k}$. We decompose $\vec{h}(a^s_{j,l_k})$ in elementary terms

$$\vec{h}(a^s_{j,l_k}) = \vec{h}(\hat{a}) + \vec{h}(\bar{a}) =$$

$$\sum_{i=1}^m \sum_{j_1,j_2=1}^n c^{j_2}_{ij_1} u_i u_{j_2} \partial_{u_{j_1}}(\hat{a}) + \sum_{i=i}^m u_i X_i(\bar{a}) + \sum_{i=1}^m \sum_{j_1,j_2=1}^n c^{j_2}_{ij_1} u_i u_{j_2} \partial_{u_{j_1}}(\bar{a}),$$

where

$$\sum_{i=1}^m \sum_{j_1,j_2=1}^n c^{j_2}_{ij_1} u_i u_{j_2} \partial_{u_{j_1}}(\hat{a}) = \sum_{i=1}^m \sum_{w_{j_1}+1=w_{j_2}} \hat{c}^{j_2}_{ij_1} u_i u_{j_2} \partial_{u_{j_2}}(\hat{a}) + \sum_{i=1}^m \sum_{w_{j_2} \leq w_{j_1}} \hat{c}^{j_2}_{ij_1} u_i u_{j_2} \partial_{u_{j_1}}(\hat{a}).$$

The weighted degree of each term above is compared with the weighted degree of $\hat{a}$ and $\bar{a}$ in the following list.

- $\deg_w(\sum_{i=1}^m \sum_{w_{j_1}+1=w_{j_2}} \hat{c}^{j_2}_{ij_1} u_i u_{j_2} \partial_{u_{j_1}}(\hat{a})) = \deg_w(\hat{a}) + 2;$
- $\deg_w(\sum_{i=1}^m \sum_{w_{j_2} \leq w_{j_1}} \hat{c}^{j_2}_{ij_1} u_i u_{j_2} \partial_{u_{j_1}}(\hat{a})) \leq \deg_w(\hat{a}) + 1;$
- $\deg_w(\sum_{i=i}^m u_i X_i(\bar{a}) = \deg_w(\bar{a}) + 1 < \deg_w(\hat{a}) + 1;$
- $\deg_w(\sum_{i=1}^m \sum_{j_1,j_2=1}^n c^{j_2}_{ij_1} u_i u_{j_2} \partial_{u_{j_1}}(\bar{a})) \leq \deg_w(\bar{a}) + 2 < \deg_w(\hat{a}) + 2.$

It is easy to see that the first element from the list is of the highest weighted degree. Moreover for this element there holds

$$\sum_{i=1}^m \sum_{w_j+1=w_k} \hat{c}^k_{ij} u_i u_k \partial_{u_j}(\hat{a}) = \vec{h}_N(\hat{a}).$$

Therefore $\vec{h}(\hat{a}^s_{j,l_k})$ is the homogeneous polynomial of the highest weighted degree in $\vec{h}(a^s_{j,l_k})$.

The same kind of calculation for the term $\sum_{\{l_s\,:\,s=k-1,\dots,r\}} a^s_{j,l_s} q_{l_s l_k}$ leads to the following expansion

$$\sum_{\{l_s\,:\,s=k-1,\dots,r\}} a^s_{j,l_s} q_{l_s l_k} = \sum_{l_{k-1}} \hat{a}^s_{j,l_{k-1}} \hat{q}_{l_{k-1} l_k} + \sum_{\{l_s\,:\,s=k,\dots,r\}} \hat{a}^s_{j,l_s} q_{l_s l_k} + \sum_{\{l_s\,:\,s=k-1,\dots,r\}} \bar{a}^s_{j,l_s} q_{l_s l_k}.$$

There are following relations on weighted degree of the terms from the sum above.

- $\deg_w(\hat{q}_{l_s l_k}) = \deg_w(q_{l_s l_k}) = 1$, for any l_s, l_k for which $q_{l_s l_k}, \hat{q}_{l_s l_k}$ are not zero;
- $\deg_w(\hat{a}^s_{j,k_1}) > \deg_w(\hat{a}^s_{j,k_2})$ if $w_{k_1} < w_{k_2}$;
- $\deg_w(\hat{a}^s_{j,k_1}) > \deg_w(\bar{a}^s_{j,k_2})$ if $w_{k_1} \leq w_{k_2}$.

We conclude that $\sum_{l_{k-1}} \hat{a}^s_{j,l_{k-1}} \hat{q}_{l_{k-1} l_k}$ is the term of the highest weighted degree in $\sum_{\{l_s\,:\,s=k-1,\dots,r\}} a^s_{j,l_s} q_{l_s l_k}$. Thus, we deduce that for $s+1$ we have

$$a^{s+1}_{j,l_k}(q_0) = \hat{a}^{s+1}_{j,l_k} + \bar{a}^{s+1}_{j,l_k} \quad \text{with} \quad \deg_w\left(\bar{a}^{s+1}_{j,l_k}\right) < \deg_w\left(\hat{a}^{s+1}_{j,l_k}\right),$$

this ends the inductive step and proves the statement of the theorem. □

Proposition 3.81. *For any $1 \leq j \leq m$ and $s \in \mathbb{N}$, let $\hat{b}^s_j$ be an element of $\hat{b}$ and $b^s_j(q_0)$ an element of b. Then*

$$\alpha b^s_j(q_0)(u) = \alpha \hat{b}^s_j(u) + \mathrm{poly}(u_1, \dots, u_n),$$

where $\deg_w(\mathrm{poly}) < \deg_w(\alpha \hat{b}^s_j)$.

Proof. We show it by induction on s. For $s = 1$, $\hat{b}^1_j$ and $b^1_j(q_0)$ have the following form for any $1 \leq j \leq m$,

$$\alpha \hat{b}^1_j = \alpha^2_j(q_0) \sum_{i=1}^{m} \sum_{l_2} \hat{c}^{l_2}_{ij} u_i u_{l_2} \quad \text{and} \quad \alpha b^1_j(q_0) = R_j(q_0),$$

where

$$R_j(q_0) = \alpha^2_j(q_0) \sum_{i=1}^{m} \sum_{l_2} \hat{c}^{l_2}_{ij} u_i u_{l_2} + \sum_{i,k=1}^{m} a_{i,j,k}(q_0) u_i u_k + \sum_{i=1}^{m} b_{i,j}(q_0) u_i u_j,$$

and $a_{i,j,k}, b_{i,j}$ are some functions of $q \in M$. Thus, for the first layer the statement is proven. We suppose that the statement is true for all the layers $\leq s$ and therefore $b^s_j(q_0) = \hat{b}^s_j + \bar{b}^s_j$ with $\deg_w(\bar{b}^s_j) < \deg_w(\hat{b}^s_j)$. For $s+1$ we have for $1 \leq j \leq m$,

$$\begin{aligned}
\alpha\hat{b}_j^{s+1} &= \alpha_j^2(q_0) \sum_{\{l_k:k=2,\dots,s+2\}} \hat{a}_{j,l_k}^{s+1} u_{l_k}, \\
\alpha b_j^{s+1}(q_0) &= \ \alpha\vec{h}_1(b_j^s) + \sum_{\{l_k:k=2,\dots,s+1\}} \alpha a_{j,l_k}^s \mathrm{pol}_{l_k}(u),
\end{aligned}$$

where $\mathrm{pol}_{l_k}(u)$ is some polynomial of weighted degree 2, where by inductive assumption, we have in the first term in the sum above $\vec{h}_1(\hat{b}_j^s + \bar{b}_j^s)$. Finally we get

$$\begin{aligned}
\alpha b_j^{s+1}(q_0) = &\ \alpha\vec{h}_1(\hat{b}_j^s) + \alpha\vec{h}_1(\bar{b}_j^s) + \sum_{\{l_k:k=2,\dots,Z+1\}} \alpha(\hat{a}_{j,l_k}^s + \bar{a}_{j,l_k}^s)\mathrm{pol}_{l_k}(u) \\
&= \alpha\hat{b}_j^{s+1} + \hat{\mathrm{pol}} + \alpha\vec{h}_1(\bar{b}_j^s) + \sum_{\{l_k:k=2,\dots,s+1\}} \alpha(\hat{a}_{j,l_k}^s + \bar{a}_{j,l_k}^s)\mathrm{pol}_{l_k}(u),
\end{aligned}$$

where $\vec{h}_1(\hat{b}_j^s) = \vec{h}_N(\hat{b}_j^s) + \hat{\mathrm{pol}}$ and $\deg_w\left(\hat{\mathrm{pol}}\right) < \deg_w\left(\vec{h}_N(\hat{b}_j^s)\right)$. Comparing degrees, we deduce that $\alpha\hat{b}_j^{s+1}$ is of the strictly higher weighted degree. This ends the proof of the inductive step and the proof of the proposition. $\square$

Chapter 4

Control-Affine case

Many ideas developed in the Chapter 3 for the sub-Riemannian case are still valid in the control-affine case. In particular, the orbital diffeomorphism satisfies an algebraic system similar to the fundamental system in the sub-Riemannian case. In this chapter we develop the method in the control-affine case, and we take advantage of it to give details of some steps of reasoning that we had omitted in the sub-Riemannian case (because we were referring to [50]). We will also try to show until which limits the results of the sub-Riemannian case are valid in the control-affine case and at which key-points they fail. We treat the simple case of control-affine problem with $n = m + 1$ where we show that, similarly to Theorem 3.61, the existence of equivalent costs implies a product structure for the nilpotent approximation. There are important difficulties to overcome in order to prove such a result in the general control-affine case and this will be explained later on. Another problem that will be treated is the existence of ample geodesics, which is not known in this setting.

4.1 Problem statement

4.1.1 Direct and inverse problems

Consider a control-affine system on M

$$\dot{q} = f_0(q) + \sum_{i=1}^{m} u_i f_i(q), \qquad u = (u_1, \dots, u_m) \in \mathbb{R}^m, \tag{4.1}$$

where $f_0, f_1, \dots, f_m$ are smooth vector fields on M. We make the following assumptions.

Assumptions 3. *The vector fields $f_0, f_1, \dots, f_m$ defining the dynamics* (4.1) *satisfy:*

A1. The weak Hörmander condition at any $q \in M$,

$$\mathrm{Lie}_q\left(\{(\mathrm{ad} f_0)^s f_i : s \geq 0,\ i = 1, \dots, m\}\right) = T_q M, \tag{4.2}$$

A2. For every bounded family $\mathcal{U}$ of admissible controls, there exists a compact subset $K_T \subset M$ such that $q_u(t) \in K_T$ for each $u \in \mathcal{U}$ and $t \in [0, T]$.

We consider an integral cost given by a quadratic Lagrangian

$$L(q, u) = u^T R(q) u, \tag{4.3}$$

and we make the following assumptions on R.

Assumptions 4. *The mapping $q \mapsto R(q)$ satisfies:*

B1. $q \mapsto R(q)$ is smooth,

B2. $R^T(q) = R(q)$ is positive definite $(m \times m)$ real valued matrix for any $q \in M$.

Remark 4.1. The dynamics and cost defined above satisfy the conditions of Section 1.3 and we can use the approach via orbital diffeomorphism.

Remark 4.2. In the control-affine case the possibility of different parameterizations of a curve depends on the property of f_0 to be lineyarly independent from $f_1, \dots, f_m$. If the vector fields satisfy $f_0 \notin \mathrm{span}\{f_1, \dots, f_m\}$ at all points of a curve, then only one parametrization is admissible. In our framework we will work on sets where it holds at all points. Therefore, the geometric inverse problem coincides with the inverse optimal control problem in our case. In general, this question is more complicated for this kind of control systems.

4.1.2 Equivalent costs

We will study the problem of injectivity of the inverse optimal control problem via the equivalence of costs. We fix a control-affine system (4.1) and consider two Lagrangians

$$L_1(q, u) = u^T R_1(q) u \quad \text{and} \quad L_2(q, u) = u^T R_2(q) u,$$

with R_1, R_2 satisfying Assumptions 4.

For the further study we want to work with the optimal control problems in the simplest possible form. Using feedback equivalence of control systems we can replace an optimal control problem by a simpler one which has the same solutions. Let us explain it in more details.

Definition 4.3. Let us consider two affine control systems on M

$$\dot{q} = f_0(q) + \sum_{i=1}^{m} u_i f_i(q) \qquad \text{and} \qquad \dot{q} = \tilde{f}_0(q) + \sum_{i=1}^{m} v_i \tilde{f}_i(q).$$

We say that the two systems are *feedback equivalent* if there exists a feedback transformation of the control $\psi : M \times \mathbb{R}^m \to \mathbb{R}^m$, such that

$$f_0(q) + \sum_{i=1}^{m} u_i f_i(q) = f_0(q) + \sum_{i=1}^{m} \psi_i(q, v) f_i(q) = \tilde{f}_0(q) + \sum_{i=1}^{m} v_i \tilde{f}_i(q).$$

Remark 4.4. Usually, the feed-back equivalence also contains a pure state transformation but in our case it is just a change of variables on M and is irrelevant to the considered problem. That is why we have defined a pure feedback transformation.

Notice that when the optimal control problem is stated, the feedback transformation changes the cost as well. Thus this induces a feedback equivalence between optimal control problems. Notice that feedback equivalent control systems have the same admissible trajectories and feedback equivalent optimal control problems have the same optimal solutions. Let us apply a feedback transformation to the two optimal control problems defined by the control system (4.1) and the Lagrangians L_1 and L_2. The new Lagrangians are of the form

$$L_1(q, u) = \psi(q, u)^T R_1(q) \psi(q, u) \quad \text{and} \quad L_2(q, u) = \psi(q, u)^T R_2(q) \psi(q, u).$$

Since we want to deal only with quadratic costs, we consider only feedback transformations which are linear in u, that is, of the form $\psi(q, u) = \Psi(q) u$.

At each point $q \in M$ we can define a transition $(m \times m)$ matrix-function

$$S(q) = R_1(q)^{-1} R_2(q).$$

Let $N(q)$ be the number of distinct eigenvalues of $S(q)$.

Definition 4.5. A point $q_0 \in M$ is called *stable* with respect to the pair (R_1, R_2) if $N(q)$ is constant in some neighborhood of q_0.

Notice that the set of stable points is open and dense in M.

Definition 4.6. A point $q_0 \in M$ is called *regular* if $\operatorname{rank}\{f_0, f_1, \ldots, f_m\} = m + 1$ in some neighborhood of q_0.

The set of all regular points is open in M. To have an open and dense intersection of the sets of stable and regular point, we assume that the vector fields $f_0, f_1, \ldots, f_m$ are chosen in such a way that the set of the regular points is dense. Otherwise, the conclusions of the chapter are valid only at the intersection of the open sets of regular and stable points if it is not empty.

Remark 4.7. Notice that near regular points $f_0 \notin \text{span}\{f_1, \dots, f_m\}$. Thus no reparameterization of trajectories is admissible and the inverse problems on the geometric curves and on trajectories coincide.

Fix a point $q_0 \in M$ which is regular and stable with respect to (R_1, R_2) and a neighborhood U of q_0 containing only regular and stable points. There exists a feedback transformation $\Psi(q)u$ such that $\Psi(q)^T R_1(q)\Psi(q) = I$ (identity matrix) and $\Psi(q)^T R_2(q)\Psi(q)$ is a diagonal matrix with positive-valued functions $\alpha_1^2(q), \dots, \alpha_m^2(q)$ on the diagonal. To simplify the notations, we write sometimes $\alpha_1^2, \dots, \alpha_m^2$ for $\alpha_1^2(q), \dots, \alpha_m^2(q)$. The costs L_1, L_2 take the forms

$$L_1 = \sum_{i=1}^m u_i^2 \qquad \text{and} \qquad L_2 = \sum_{i=1}^m \alpha_i^2 u_i^2. \tag{4.4}$$

We denote H_1, H_2 the normal pseudo-Hamiltonians corresponding to the costs L_1, L_2 respectively, in the local coordinates (q, p) of T^*U they are of the form

$$H_1(p, q, u) = \langle p, f_0(q)\rangle + \sum_{i=1}^m u_i \langle p, f_i(q)\rangle - \frac{1}{2}\sum_{i=1}^m u_i^2;$$
$$H_2(p, q, u) = \langle p, f_0(q)\rangle + \sum_{i=1}^m u_i \langle p, f_i(q)\rangle - \frac{1}{2}\sum_{i=1}^m \alpha_i^2(q) u_i^2.$$

The maximizing conditions for each $u_j,\ j = 1, \dots, m$ are the following

$$\frac{\partial H_1}{\partial u_j} = \langle p, f_j\rangle - u_j = 0 \quad \Rightarrow \qquad u_j = \langle p, f_j\rangle;$$
$$\frac{\partial H_2}{\partial u_j} = \langle p, f_j\rangle - \alpha_j^2 u_j = 0 \quad \Rightarrow \qquad u_j = \frac{\langle p, f_j\rangle}{\alpha_j^2}.$$

As a consequence, the normal Hamiltonians h_1, h_2 take the forms

$$h_1(p, q) = \langle p, f_0(q)\rangle + \frac{1}{2}\sum_{i=1}^m \langle p, f_j(q)\rangle^2;$$
$$h_2(p, q) = \langle p, f_0(q)\rangle + \frac{1}{2}\sum_{i=1}^m \frac{\langle p, f_j(q)\rangle^2}{\alpha_j^2(q)}.$$

4.1.3 Adapted coordinates

Following [54], we define coordinates in T_q^*M adapted to the frame of the control system and which simplify the form of the normal Hamiltonians. First let us define a local frame of TM near q_0. We complete the frame $f_1, \dots, f_m, f_{m+1} = f_0$ to linearly independent $f_1, \dots, f_n$ in such a way that $\text{span}\{f_1(q), \dots, f_n(q)\} = T_qM$ for any $q \in U$. Under assumption (4.2), each $f_i,\ i = m+1, \dots, n$ can be obtained as a linear combination of iterated Lie brackets of $f_1, \ldots, f_m, f_{m+1}$. On the other hand, Lie bracket of any pair of vector fields f_i, f_j can be written as

$$[f_i, f_j] = \sum_{k=1}^{n} c_{ij}^k(q) f_k,$$

where the functions c_{ij}^k are called the *structure coefficients*. For any $q \in U$ and any $p \in T_q^*U$ the functions $\langle p, f_1\rangle, \ldots, \langle p, f_n\rangle$ define coordinates of p. Let us denote $u_i = \langle p, f_i\rangle$ for $i = 1, \ldots, n$. The Hamiltonians h_1, h_2 can be expressed in these coordinates as follows

$$\begin{aligned} h_1 &= u_{m+1} + \frac{1}{2}\sum_{i=1}^{m} u_i^2; \\ h_2 &= u_{m+1} + \frac{1}{2}\sum_{i=1}^{m} \frac{u_i^2}{\alpha_i^2}. \end{aligned} \tag{4.5}$$

To write the vector fields $\vec{h}_1, \vec{h}_2$ in coordinates, let us choose a frame of $T_\lambda T^*U$. The coordinates $(u_1, \ldots, u_n)$ induce a basis $\partial_{u_1}, \ldots, \partial_{u_n}$ of $T_\lambda(T_q^*U)$ for any $\lambda \in \pi^{-1}(q)$. For $i = 1, \ldots, n$, we define the lift F_i of f_i as the vector field on T^*U such that $\pi_* F_i = f_i$ and $du_j(F_i) = 0$, $\forall 1 \leq j \leq n$. The family of vector fields $\{F_1, \ldots, F_n; \partial_{u_1}, \ldots, \partial_{u_n}\}$ obtained in this way constitutes a frame of $T_\lambda T^*U$.

$$\begin{aligned} \vec{h}_1 &= F_{m+1} + \sum_{i=1}^{m} u_i F_i + \sum_{j=1}^{n} b_j \partial_{u_i}; \\ \vec{h}_2 &= F_{m+1} + \sum_{i=1}^{m} \frac{u_i}{\alpha_i^2} F_i + \sum_{j=1}^{n} \bar{b}_j \partial_{u_i}. \end{aligned}$$

To calculate the coefficients $b_j, \bar{b}_j$ we use the notations of [82, Appendix 4]. First let us calculate b_j. For any $j = 1, \ldots, m$, the coefficient b_j is given by the following equation

$$b_j = \vec{h}_1(u_j) = \{u_j, h_1\},$$

where $\{\cdot, \cdot\} : C^\infty(T^*M, \mathbb{R}) \times C^\infty(T^*M, \mathbb{R}) \to C^\infty(T^*M, \mathbb{R})$ is the Poisson bracket. In (symplectic) canonical coordinates $(q_1, \ldots, q_n, p_1, \ldots, p_n)$ on T^*M the Poisson bracket of two functions $f, g \in C^\infty(T^*M, \mathbb{R})$ can be calculated as

$$\{f, g\} = \sum_{k=1}^{n} \frac{\partial f}{\partial q_k}\frac{\partial g}{\partial p_k} - \frac{\partial f}{\partial q_k}\frac{\partial g}{\partial p_k}.$$

This formula applied to u_i, u_j for $i, j = 1, \ldots, n$ gives

$$\{u_i, u_j\} = \sum_{k=1}^{n} \frac{\partial u_i}{\partial q_k}\frac{\partial u_j}{\partial p_k} - \frac{\partial u_j}{\partial q_k}\frac{\partial u_i}{\partial p_k} = \langle p, df_i \circ f_j - df_j \circ f_i\rangle = \langle p, [f_j, f_i]\rangle.$$

As a consequence,

$$\{u_i, u_j\} = \sum_{k=1}^{n} c_{ji}^k u_k.$$

By linearity of the Poisson bracket and the Leibniz's rule, we finally obtain

$$b_j = \{u_j, u_0 + \frac{1}{2}\sum_{i=1}^{m} u_i^2\} = \{u_j, u_0\} + \sum_{i=1}^{m} u_i\{u_j, u_i\} = \sum_{k=1}^{n} c_{m+1j}^k u_k + \sum_{k=1}^{n}\sum_{i=1}^{m} c_{ij}^k u_i u_k.$$

Thus the vector field $\vec{h}_1$ takes the form

$$\vec{h}_1 = F_{m+1} + \sum_{i=1}^{m} u_i F_i + \sum_{j=1}^{n}\left(\sum_{k=1}^{n} c_{m+1j}^k u_k + \sum_{k=1}^{n}\sum_{i=1}^{m} c_{ij}^k u_i u_k\right)\partial_{u_j}.$$

In the same way we can calculate the coefficients $\bar{b}_j$.

$$\bar{b}_j = \vec{h}_2(u_j) = \{u_j, u_0\} + \sum_{i=1}^{m}\frac{u_i}{\alpha_i}\{u_j, \frac{u_i}{\alpha_i}\},$$

By the Leibniz's property of the Lie bracket, we obtain

$$\left[\frac{f_i}{\alpha_i}, f_j\right] = \frac{1}{\alpha_i}\left[f_i, f_j\right] - f_j\left(\frac{1}{\alpha_i}\right) f_i,$$

and thus

$$\langle p, \left[\frac{f_i}{\alpha_i}, f_j\right]\rangle = \frac{1}{\alpha_i}\sum_{k=1}^{n} c_{ij}^k u_k - f_j\left(\frac{1}{\alpha_i}\right) u_i.$$

Finally, putting all together we get the coefficient $\bar{b}_j$,

$$\bar{b}_j = \sum_{k=1}^{n} c_{m+1j}^k u_k + \sum_{k=1}^{n}\sum_{i=1}^{m} c_{ij}^k \frac{1}{\alpha_i^2} u_i u_k - \sum_{i=1}^{m}\frac{u_i}{\alpha_i} f_j\left(\frac{1}{\alpha_i}\right) u_i,$$

and $\vec{h}_2$ takes the form

$$\vec{h}_2 = F_{m+1} + \sum_{i=1}^{m}\frac{u_i}{\alpha_i^2} F_i + \sum_{j=1}^{n}\left(\sum_{k=1}^{n} c_{m+1j}^k u_k + \sum_{k=1}^{n}\sum_{i=1}^{m} c_{ij}^k \frac{1}{\alpha_i^2} u_i u_k + \sum_{i=1}^{m}\frac{f_j(\alpha_i)}{\alpha_i^3} u_i^2\right)\partial_{u_i}.$$

4.2 Orbital diffeomorphism

4.2.1 Orbital diffeomorphism in adapted coordinates

Let Φ be an orbital diffeomorphism between the extremals flows of h_1, h_2 on some neighborhood $V \in T^*U$ of λ_0 such that $\pi(\lambda_0) = q_0$. In the local coordinates $(q_1, \ldots, q_n; u_1, \ldots, u_n)$ on T^*U, Φ maps $(q_1, \ldots, q_n; u_1, \ldots, u_n)$ to $(q_1, \ldots, q_n; \Phi_1, \ldots, \Phi_n)$

where $\Phi_i(\lambda) = u_i(\Phi(\lambda))$. By definition, the orbital diffeomorphism satisfies (1.9), we will write this condition in the frame $\{F_1, \ldots, F_n, \partial_{u_1}, \ldots, \partial_{u_n}\}$. First, notice that by definition of the differential $d\Phi$, for any function $f \in C^1(T^*M, \mathbb{R})$ and any $\lambda \in T^*M$

$$d\Phi \circ \vec{h}_1(f)(\lambda) = \vec{h}_1 \left(f \circ \Phi\right)(\lambda).$$

Applying this formula to some coordinate function $f(\lambda) = u_j(\lambda)$ we get

$$d\Phi \circ \vec{h}_1\left(u_j\right) = \vec{h}_1\left(u_j \circ \Phi\right) = \vec{h}_1\left(\Phi_j\right).$$

The orbital diffeomorphism is fiber-preserving, i.e., $\pi \circ \Phi(\lambda) = \pi(\lambda)$. This implies

$$d\pi\left(d\Phi \circ \vec{h}_1\right) = d\pi(\vec{h}_1),$$

and thus $d\Phi \circ \vec{h}_1$ takes the form

$$d\Phi \circ \vec{h}_1 = F_{m+1} + \sum_{i=1}^{m} u_i F_i + \sum_{j=1}^{n} \vec{h}_1\left(\Phi_j\right)\partial_{u_j}.$$

Proposition 4.8. *A smooth fiber-preserving map Φ on an open set V satisfies* (1.9) *if and only if the coordinates $(\Phi_1, \ldots, \Phi_n)$ of Φ satisfy the following conditions:*

- *for $k = 1, \ldots, m$,*
$$\Phi_k = \alpha_k^2 u_k, \tag{4.6}$$

- *for $j = 1, \ldots, m$,*
$$\sum_{k=m+1}^{n} q_{jk}\Phi_k = R_j, \tag{4.7}$$
where $q_{jk} = c_{m+1j}^k + \sum_{i=1}^m c_{ij}^k u_i$ and
$$R_j = \vec{h}_1\left(\alpha_j^2 u_j\right) - \frac{1}{2}\sum_{i=1}^{m} f_j(\alpha_i^2)u_i^2 - \sum_{k=1}^{m} q_{jk}u_k\alpha_k^2,$$

- *for $k = m+1, \ldots, n$,*
$$\vec{h}_1(\Phi_k) = \sum_{l=m+1}^{n} q_{kl}\Phi_l + \sum_{i=1}^{m}\left(q_{ki}u_i\alpha_i^2 + \frac{1}{2}f_k(\alpha_i^2)u_i^2\right). \tag{4.8}$$

Proof. Let us calculate the first m values $\Phi_1, \ldots, \Phi_m$. For that, we can just project (1.9) on M.

$$d\pi\left(d\Phi \circ \vec{h}_1\right) = f_0 + \sum_{i=1}^{m} u_i f_i;$$
$$d\pi\left(\vec{h}_2(\Phi)\right) = f_0 + \sum_{i=1}^{m} \frac{\Phi_i}{\alpha_i^2} f_i.$$

The equality of these two expressions (1.9) implies that for any $i = 1, \ldots, m$ the corresponding coordinates of Φ are

$$\Phi_i = \alpha_i^2 u_i.$$

Now we can calculate the complete equations (1.9) in the chosen frame.

$$d\Phi \circ \vec{h}_1 = F_{m+1} + \sum_{i=1}^{m} u_i F_i + \sum_{j=1}^{m} \vec{h}_1 \left(\alpha_j^2 u_j\right) + \sum_{j=m+1}^{n} \vec{h}_1 \left(\Phi_j\right) \partial_{u_j};$$

$$\vec{h}_2(\Phi) = F_{m+1} + \sum_{i=1}^{m} u_i F_i + \sum_{j=1}^{n} \left(\sum_{k=1}^{n} \left(c_{m+1j}^k + \sum_{i=1}^{m} c_{ij}^k u_i \right) \Phi_k + \sum_{i=1}^{m} f_j(\alpha_i)\alpha_i u_i^2 \right) \partial_{u_i}.$$

The equality of the coefficients of ∂_{u_j} gives for $j = 1, \ldots, m$,

$$\vec{h}_1 \left(\alpha_j^2 u_j\right) = \sum_{k=1}^{m} \left(c_{m+1j}^k + \sum_{i=1}^{m} c_{ij}^k u_i \right) u_k \alpha_k^2 + \sum_{i=1}^{m} f_j(\alpha_i)\alpha_i u_i^2 + \sum_{k=m+1}^{n} \left(c_{m+1j}^k + \sum_{i=1}^{m} c_{ij}^k u_i \right) \Phi_k,$$

which can be seen as a linear algebraic system on the variables Φ_k

$$\sum_{k=m+1}^{n} \left(c_{m+1j}^k + \sum_{i=1}^{m} c_{ij}^k u_i \right) \Phi_k = \vec{h}_1 \left(\alpha_j^2 u_j\right) - \frac{1}{2} \sum_{i=1}^{m} f_j(\alpha_i^2) u_i^2 - \sum_{k=1}^{m} \left(c_{m+1j}^k + \sum_{i=1}^{m} c_{ij}^k u_i \right) u_k \alpha_k^2.$$

Let us denote

$$q_{jk} = c_{m+1j}^k + \sum_{i=1}^{m} c_{ij}^k u_i.$$

The above algebraic system writes, for any $j = 1, \ldots, m$, as

$$\sum_{k=m+1}^{n} q_{jk} \Phi_k = \vec{h}_1 \left(\alpha_j^2 u_j\right) - \frac{1}{2} \sum_{i=1}^{m} f_j(\alpha_i^2) u_i^2 - \sum_{k=1}^{m} q_{jk} u_k \alpha_k^2. \tag{4.9}$$

To simplify the expression, let us denote the right-hand side of (4.9) by R_j, i.e. for $j = 1, \ldots, m$,

$$R_j = \vec{h}_1 \left(\alpha_j^2 u_j\right) - \frac{1}{2} \sum_{i=1}^{m} f_j(\alpha_i^2) u_i^2 - \sum_{k=1}^{m} q_{jk} u_k \alpha_k^2.$$

For the next coefficients of ∂_{u_j}, $j = m+1, \ldots, n$, the equations are

$$\vec{h}_1(\Phi_j) = \sum_{k=m+1}^{n} q_{jk} \Phi_k + \sum_{k=1}^{m} \left(q_{jk} u_k \alpha_k^2 + \frac{1}{2} f_j(\alpha_k^2) u_k^2 \right).$$

As a result, we have equations on all Φ_k, $k = 1, \ldots, n$ and this ends the proof. $\square$

Remark 4.9. Notice that the similar equations appear in the sub-Riemannian case in Lemma 3.30 and were obtained in [50]. In the sub-Riemannian case, they were obtained in the same way as it is presented here but adapted to the sub-Riemannian case. We presented all the arguments in details for the sake of completeness.

4.2.2 Fundamental system in affine case

Proposition 4.8 provides the system of differential equations on Φ. These equations can be rewritten in algebraic form.

Proposition 4.10. *Let Φ be an orbital diffeomorphism between the extremal flows of h_1 and h_2, with coordinates $(\Phi_1, \dots, \Phi_n)$. Set $\widetilde{\Phi} = (\Phi_{m+1}, \dots, \Phi_n)$. Then $\widetilde{\Phi}$ satisfies a linear system of equations,*

$$A\widetilde{\Phi} = b, \tag{4.10}$$

where A is a matrix with $(n-m)$ columns and an infinite number of rows, and b is a column vector with an infinite number of rows. These infinite matrices can be decomposed in layers of m rows as

$$A = \begin{pmatrix} A^1 \\ A^2 \\ \vdots \\ A^s \\ \vdots \end{pmatrix} \qquad \text{and} \qquad b = \begin{pmatrix} b^1 \\ b^2 \\ \vdots \\ b^s \\ \vdots \end{pmatrix}, \tag{4.11}$$

where the coefficients a^s_{jk} of the $(m \times (n-m))$ matrix A^s, $s \in \mathbb{N}$, are defined by induction as

$$\begin{cases} a^1_{j,k} = q_{jk}, & 1 \le j \le m, \ m < k \le n, \\ a^{s+1}_{j,k} = \vec{h}_1(a^s_{j,k}) + \displaystyle\sum_{l=m+1}^{n} a^s_{j,l} q_{lk}, & 1 \le j \le m, \ m < k \le n, \end{cases} \tag{4.12}$$

(note that the columns of A are numbered from $m+1$ to n according to the indices of $\widetilde{\Phi}$) and the coefficients b^s_j, $1 \le j \le m$, of the vector $b^s \in \mathbb{R}^m$ are defined by

$$\begin{cases} b^1_j = R_j, \\ b^{s+1}_j = \vec{h}_1(b^s_j) - \displaystyle\sum_{k=m+1}^{n} a^s_{j,k} \sum_{i=1}^{m} \left(q_{ki} u_i \alpha_i^2 + \frac{1}{2} f_k(\alpha_i^2) u_i^2 \right). \end{cases} \tag{4.13}$$

Proof. The algebraic equations can be obtained from (4.7), (4.8) by the following procedure. First, let us take the derivative of (4.7) in direction $\vec{h}_1$, we obtain

$$\sum_{k=m+1}^{n} \vec{h}_1(q_{jk})\Phi_k + \sum_{k=m+1}^{n} q_{jk}\vec{h}_1(\Phi_k) = \vec{h}_1(R_j).$$

Now let us substitute $\vec{h}_1(\Phi_k)$ by its value from (4.8)

$$\sum_{k=m+1}^{n} \vec{h}_1(q_{jk})\Phi_k + \sum_{l=m+1}^{n} q_{jl}\left(\sum_{k=m+1}^{n} q_{lk}\Phi_k + \sum_{k=1}^{m}\left(q_{lk}u_k\alpha_k^2 + \frac{1}{2}f_l(\alpha_k^2)u_k^2\right)\right) = \vec{h}_1(R_j).$$

As a result, we obtain an algebraic system of m equations on Φ_k, for $j = 1, \ldots, m$,

$$\sum_{k=m+1}^{n}\left(\vec{h}_1(q_{jk}) + \sum_{l=m+1}^{n} q_{jl}q_{lk}\right)\Phi_k = \vec{h}_1(R_j) - \sum_{k=1}^{m}\sum_{l=m+1}^{n} q_{jl}\left(q_{lk}u_k\alpha_k^2 + \frac{1}{2}f_l(\alpha_k^2)u_k^2\right).$$

Repeating N times the procedure of applying to the system of m new equations the derivative in direction $\vec{h}_1$ and substituting $\vec{h}_1(\Phi_k)$ for $k = m+1, \ldots, n$, by values from (4.8), we obtain the linear system (4.11) with N blocks. Repeating the procedure further we obtain the infinite number of blocks and thus the desired result. □

We call (4.10) the *fundamental system* of algebraic equations.

Remark 4.11. Notice that all the coefficients of the system (4.10) are polynomial in $u_1, \ldots, u_n$. Therefore if there exists $\lambda \in T^*M$ such that $\pi(\lambda) = q_0$ with coordinates $u_1^*, \ldots, u_n^*$ for which A has a nonzero minor of rank $n-m$, then this minor is a nonzero polynomial of $u_1, \ldots, u_n$ and the matrix is injective almost everywhere on $T_{q_0}^*M$. The injectivity of A implies the existence of a unique solution $\widetilde{\Phi}$.

4.2.3 Injectivity of the fundamental system

The coefficients $a_{j,k}^s$ from (4.12) are related with the extensions of Jacobi curve and this will permit us to identify the injectivity property of A at particular $\lambda \in T_{q_0}^*M$.

In an adapted frame $\{F_1, \ldots, F_n; \partial_{u_1}, \ldots, \partial_{u_n}\}$ of $T(T^*U)$, the extensions of a Jacobi curve associated with the affine system (4.1) and some Lagrangian $L(q,u) = u^T R(q) u$ can be obtained from iterations of Lie brackets by the correspond Hamiltonian vector field

$$\vec{h} = F_{m+1} + \sum_{i=1}^{m} u_i F_i + \sum_{i=1}^{n} c_i(q,u)\partial_{u_i}. \tag{4.14}$$

Lemma 4.12. *Let $q = \pi(\lambda)$. In an adapted frame $\{F_1, \ldots, F_n, \partial_{u_1}, \ldots, \partial_{u_n}\}$ of*

$T(T^*M)$ at q, the extensions of the Jacobi curve take the following form:

$$
\begin{aligned}
J_\lambda^{(0)} &= \Big\{ v \in T_\lambda(T^*M) \ :\ d\pi \circ v = 0 \Big\}, \\
J_\lambda^{(1)} &= \Big\{ v \in T_\lambda(T^*M) \ :\ d\pi \circ v \in D \Big\} = J_\lambda^{(0)} + \operatorname{span}\Big\{ F_1(\lambda), \dots, F_m(\lambda) \Big\}, \\
J_\lambda^{(2)} &= J_\lambda^{(1)} + \operatorname{span}\Big\{ [\vec{h}, F_1](\lambda), \dots, [\vec{h}, F_m](\lambda) \Big\}, \\
&\vdots \\
J_\lambda^{(k)} &= J_\lambda^{(k-1)} + \operatorname{span}\Big\{ (\operatorname{ad}\vec{h})^{k-1} F_1(\lambda), \dots, (\operatorname{ad}\vec{h})^{k-1} F_m(\lambda) \Big\}.
\end{aligned}
$$

Proof. Let $v \in J_\lambda^{(k)}$, for some integer $k \geq 0$. By the Lemma 1.34, there exists a vertical vector field Y on T^*M (i.e. $d\pi \circ Y = 0$) such that, for any $t \in [0, T]$,

$$
v = (\operatorname{ad}\vec{h})^s Y(\lambda).
$$

As Y is a vertical vector field, in the adapted frame $\{F_1, \dots, F_n, \partial_{u_1}, \dots, \partial_{u_n}\}$ it can be written as $Y = \sum_{i=1}^n a_i \partial_{u_i}$. Using the expression (4.14) of $\vec{h}$ in this frame, we obtain

$$
\begin{aligned}
[\vec{h}, Y] &= \left[F_{m+1} + \sum_{i=1}^m u_i F_i + \sum_{i=1}^n c_i \partial_{u_i}, \sum_{i=1}^n a_i \partial_{u_i} \right] \\
&= \sum_{i=1}^m a_i F_i \quad \operatorname{mod} \operatorname{span}\{\partial_{u_1}, \dots, \partial_{u_n}\}.
\end{aligned}
$$

By iteration, we get

$$
\begin{aligned}
(\operatorname{ad}\vec{h})^2 Y &= \sum_{i=1}^m a_i \, [\vec{h}, F_i] \quad \operatorname{mod} \operatorname{span}\{\partial_{u_1} \dots, \partial_{u_n}, F_1, \dots, F_m\} \\
&\vdots \\
(\operatorname{ad}\vec{h})^s Y &= \sum_{i=1}^m a_i \, (\operatorname{ad}\vec{h})^{s-1} F_i \\
&\qquad \operatorname{mod} \operatorname{span}\{\partial_{u_1} \dots, \partial_{u_n}, F_i, \dots, (\operatorname{ad}\vec{h})^{s-2} F_i, \ i = 1, \dots, m\},
\end{aligned}
$$

which proves the result. □

Proposition 4.13. *If $\lambda_0 \in T^*M$ is ample with respect to L_1, then $A(u(\lambda_0))$ is injective. As a consequence, there exists at least one $(n-m) \times (n-m)$ minor of the matrix $A(u)$ which is a non identically zero function of u.*

This proposition results directly from the following lemma.

Lemma 4.14. *Let s be a positive integer. Denote by A_s the $sm \times (n-m)$ matrix formed by the first s layers of A. Then*

$$
\operatorname{rank} A_s(u) = \dim J_\lambda^{(s+1)} - n - m.
$$

Proof. We begin by proving that, for any positive integer s,

$$(\mathrm{ad}\vec{h})^s F_j = \sum_{k=m+1}^{n} a_{j,k}^s F_k \mod J_\lambda^{(s)}, \quad 1 \le j \le m. \tag{4.15}$$

Remark first that, for $k = 1, \dots, n$,

$$\begin{aligned}
[\vec{h}_1, F_k] &= \left[F_{m+1} + \sum_{i=1}^{m} u_i F_i + \sum_{j=1}^{n} \sum_{k=1}^{n} \left(c_{m+1j}^k + \sum_{i=1}^{m} c_{ij}^k u_i \right) u_k \partial_{u_j}, F_k \right] \\
&= [F_{m+1}, F_k] + \sum_{i=1}^{m} u_i [F_i, F_k] \mod J_\lambda, \\
&= \sum_{l=1}^{n} \left(c_{m+1k}^l + \sum_{i=1}^{m} u_i c_{ik}^l \right) F_l \mod J_\lambda,
\end{aligned}$$

which writes as

$$[\vec{h}_1, F_k] = \sum_{l=1}^{n} q_{kl} F_l \mod J_\lambda. \tag{4.16}$$

Let us prove (4.15) by induction on s. The case $s = 1$ is a direct consequence of (4.16) since the latter implies that, for $j = 1, \dots, m$,

$$[\vec{h}_1, F_j] = \sum_{k=m+1}^{n} q_{jk} F_k + \sum_{k=1}^{m} q_{jk} F_k \mod J_\lambda = \sum_{k=m+1}^{n} a_{j,k}^1 F_k \mod J_\lambda^{(1)}.$$

Assume now that (4.15) is satisfied for a given s. Using the induction hypothesis, we write

$$(\mathrm{ad}\vec{h}_1)^{s+1} F_j = \left[\vec{h}_1, (\mathrm{ad}\vec{h}_1)^s F_j \right] = \left[\vec{h}_1, \sum_{k=m+1}^{n} a_{j,k}^s F_k \right] \mod J_\lambda^{(s+1)},$$

since $[\vec{h}_1, J_\lambda^{(s)}] \subset J_\lambda^{(s+1)}$. The last bracket above expands as

$$\begin{aligned}
\left[\vec{h}_1, \sum_{k=m+1}^{n} a_{j,k}^s F_k \right] &= \sum_{k=m+1}^{n} \vec{h}_1(a_{j,k}^s) F_k + \sum_{k=m+1}^{n} a_{j,k}^s \left[\vec{h}_1, F_k \right], \\
&= \sum_{k=m+1}^{n} \vec{h}_1(a_{j,k}^s) F_k + \sum_{k=m+1}^{n} a_{j,k}^s \sum_{l=1}^{n} q_{kl} F_l \mod J_\lambda,
\end{aligned}$$

thanks to (4.16). Splitting and renumbering the second sum above, we obtain

$$\begin{aligned}
(\mathrm{ad}\vec{h}_1)^{s+1} F_j &= \sum_{k=m+1}^{n} \left(\vec{h}_1(a_{j,k}^s) + \sum_{l=m+1}^{n} a_{j,l}^s q_{lk} \right) F_k + \sum_{l=1}^{m} \sum_{k=m+1}^{n} a_{j,k}^s q_{kl} F_l \mod J_\lambda^{(s+1)}, \\
&= \sum_{k=m+1}^{n} a_{j,k}^{s+1} F_k \mod J_\lambda^{(s+1)},
\end{aligned}$$

which ends the induction and proves (4.15).

Now, from Lemma 4.12, for any positive integer s there holds $J_\lambda^{(s+1)} = J_\lambda^{(1)} + \text{span}\{(\text{ad}\vec{h}_1)^k F_j(\lambda) \mid 1 \leq k \leq s,\ 1 \leq j \leq m\}$. Thus it results from (4.15) that

$$\dim J_\lambda^{(s+1)} = \dim J_\lambda^{(1)} + \text{rank} A_s(u(\lambda)), \quad \text{where } A_s = \begin{pmatrix} A^1 \\ A^2 \\ \vdots \\ A^s \end{pmatrix}.$$

Since $\dim J_\lambda^{(1)} = n + m$ for any λ, the lemma is proved. $\square$

A first consequence of the injectivity of A is that the system of equations $A\widetilde{\Phi} = b$ is a sufficient condition for Φ to be an orbital diffeomorphism.

Proposition 4.15. *Fix a control-affine system* (4.1) *on an open regular subset $U \subset M$, and smooth positive functions $\alpha_1, \dots, \alpha_m$ on U. Let A and b be the associated matrices defined by* (4.12) *and* (4.13)*, and denote by L_1 and L_2 two costs of the form* (4.4)*.*

Assume A to be injective at $\lambda \in T^*U$. If there exists a local smooth fiber-preserving map $\Phi : T^*U \to T^*U$ defined by $u_i \circ \Phi = \Phi_i$, $i = 1, \dots, n$ with $\widetilde{\Phi} = (\Phi_{m+1}, \dots, \Phi_n)$ solution of $A\widetilde{\Phi} = b$ near λ, and $\Phi_1, \dots, \Phi_m$ defined by (4.6)*, then Φ satisfies* (1.9)*.*

Proof. Following Lemma 4.8, it is sufficient to prove that $\widetilde{\Phi}$ satisfies (4.7) and (4.8) near λ. The equations of the first layer, i.e. $A^1\widetilde{\Phi} = b^1$, are exactly (4.7), hence we are left with the task of proving that $\widetilde{\Phi}$ satisfies (4.8).

Fix a positive integer s and $j \in \{1, \dots, m\}$. Let us write the jth row of the system $A^s\Phi = b^s$,

$$\sum_{k=m+1}^{n} a_{j,k}^s \Phi_k = b_j^s,$$

and differentiate this expression in the direction $\vec{h}_1$. We thus obtain

$$\sum_{k=m+1}^{n} \vec{h}_1(a_{j,k}^s)\Phi_k + \sum_{k=m+1}^{n} a_{j,k}^s \vec{h}_1(\Phi_k) = \vec{h}_1(b_j^s).$$

Write now the jth row of the system $A^{s+1}\Phi = b^{s+1}$, replacing the coefficients by their recurrence formula,

$$\sum_{k=m+1}^{n} \vec{h}_1(a_{j,k}^s)\Phi_k + \sum_{k,l=m+1}^{n} a_{j,l}^s q_{l,k}\Phi_k = \vec{h}_1(b_j^s) - \sum_{k=m+1}^{n} a_{j,k}^s \sum_{i=1}^{m} u_i \left(\alpha_i^2 q_{ki} + \frac{f_k(\alpha_i^2)}{2} u_i \right),$$

and take the difference between the last two formulas. Rearranging the order of summation we obtain

$$\sum_{k=m+1}^{n} a_{j,k}^{s} \left(\vec{h}_1(\Phi_k) - \sum_{l=m+1}^{n} q_{k,l}\Phi_l - \sum_{i=1}^{m} u_i \left(\alpha_i^2 q_{ki} + \frac{f_k(\alpha_i^2)}{2} u_i \right) \right) = 0. \tag{4.17}$$

Denote by Ψ_k the terms inside the bracket above, and set $\Psi = (\Psi_{m+1}, \dots, \Psi_n)$. Formula (4.8) for $\widetilde{\Phi}$ is exactly $\Psi = 0$. From (4.17), the vector Ψ satisfies the system $A\Psi = 0$. Moreover, by Proposition 4.13 the matrix $A(u)$ has full rank at $u = u(\lambda)$, and hence in a neighborhood of $u(\lambda)$ in T^*M. On this neighborhood Ψ must be identically zero, which implies that $\widetilde{\Phi}$ satisfies (4.8). The statement is proved. □

Remark 4.16. Notice that by the previous proposition, if we are able to find a fiber-preserving diffeomorphism on the cotangent bundle T^*U of some regular open set $U \subset M$ which satisfies (4.10) where A is injective then the constructed diffeomorphism is orbital and by the Proposition 1.40 the Lagrangians L_1, L_2 of the form (4.4) are equivalent via geodesics on U.

4.2.4 Additional results

The injectivity of A from (4.10) at some ample $\lambda_0 \in T^*M$ such that $\pi(\lambda) = q_0$ permits to deduce the existence of many ample $\lambda \in T^*M$ such that $\pi(\lambda) = q_0$.

Proposition 4.17. *If $\lambda_0 \in T^*M$ is ample at q_0 with respect to L_1, then there exists an open and dense set in $T_{q_0}M$ of ample $\lambda \in T^*M$ with respect L_1.*

Proof. By Proposition 4.13, $A(u(\lambda_0))$ is injective and thus has a non-zero maximal minor $\delta_0\,(u(\lambda_0)) \neq 0$. By construction, each coefficient (4.12) of A is a polynomial in u thus δ_0 is a nonzero polynomial function of u. The set of non-ample $\lambda \in T^*_{q_0}M$ is the zero level-set of δ_0. The level-set is an algebraic variety of positive codimension and thus its complement is open and dense. This ends the proof. □

In addition to equations (4.10), the existence of an orbital diffeomorphism implies the following condition which is an analog in the affine case of the first divisibility condition (see Section 3.4.1).

Proposition 4.18. *If $\vec{h}_1, \vec{h}_2$ are orbitally diffeomorphic and Φ is an orbital diffeomorphism between their extremal flows, then the following identity holds in coordinates $(q, u_1, \dots, u_n)$*

$$\vec{h}_1(\Phi_{m+1} + \frac{1}{2}\sum_{i=1}^{m} \alpha_i^2 u_i^2) = 0. \tag{4.18}$$

Proof. In the coordinates $(q, u_1, \dots, u_n)$ we have

$$h_2 \circ \Phi(\lambda) = \Phi_{m+1} + \sum_{i=1}^{m} \alpha_i^2 u_i^2,$$

Applying (1.10) to this expression, we obtain the desired result. □

Notice that if the system (4.10) is injective then Φ_{m+1} can be expressed as a function of $u_1, \dots, u_n$ by solving (4.10). The linear system (4.10) has all coefficients polynomial in $u_1, \dots, u_n$ and by the Cramer's rule, Φ_k for any $k = 1, \dots, n$ is a rational function of $u_1, \dots, u_n$.

Corollary 4.19. *The Hamiltonian system associated with h_1 admits a rational first integral along an ample geodesic*

$$\Phi_{m+1}(u_1, \dots, u_n) + \sum_{i=1}^{m} \alpha_i^2 u_i^2.$$

4.3 Injectivity of the inverse problem

Let us return to the injectivity problem. Putting together the results of Proposition 4.17, Proposition 4.10, Proposition 1.40, we obtain the following corollary.

Corollary 4.20. *Let $q_0 \in M$ be a regular point.*

1. *If there exists an ample λ_0 such that $\pi(\lambda_0) = q_0$ and if the system $A(u(\lambda_0))\Phi = b(u(\lambda_0))$ has a unique solution, then L_1, L_2 are equivalent via geodesics at q_0.*

2. *If there exists an ample λ_0 such that $\pi(\lambda_0) = q_0$ and the system $A(u(\lambda_0))\Phi = b(u(\lambda_0))$ admits solutions only when $\alpha_1 = \dots = \alpha_m = \mathrm{const}$, then the inverse optimal control problem is injective.*

Thus, the existence of an ample geodesic is crucial to use the orbital diffeomorphism. In the control-affine case the existence of an ample geodesic starting at a given point is still an open question except in some particular cases. Let us overview the cases where the existence can be affirmed.

4.3.1 Analytic case

It appears that we have the existence of an ample geodesic in the analytic control-affine case. To see it let us introduce some auxiliary definitions.

Let M be an analytic manifold and $f_0, \dots, f_m$ analytic vector fields on M. Fix a point $q_0 \in M$.

Any strictly normal geodesic corresponds to a regular control u characterized by the surjectivity of $dE^T_{q_0}(u)$, by Definition 1.28. The differential $dE^T_{q_0}$ is related with the solution of the linearized system near the trajectory q_u. In some local coordinates x near q_u defined on $[0,T]$, for any $t \in [0,T]$ the linearized system is defined by

$$\begin{cases} \dot{x}(t) = A(t)x(t) + B(t)v(t), \\ x(0) = 0; \end{cases} \tag{4.19}$$

where

$$A(t) = \frac{\partial f_0}{\partial x}(q_u(t), u(t)) + \sum_{i=1}^{m} u_i \frac{\partial f_i}{\partial x}(q_u(t), u(t)), \qquad B(t) = \begin{pmatrix} f_1 & \cdots & f_m \end{pmatrix}.$$

The differential of the endpoint map is defined by $d_u E^T_{q_0}(v) = x_v(T)$ for any $v \in L^\infty([0,T], \mathbb{R}^m)$. Notice that if q_u is a normal geodesic associated with an analytic Lagrangian (4.1), then q_u, $A(\cdot)$ and $B(\cdot)$ are analytic functions of t.

Consider the recurrent sequence of matrix-functions $B_i,\ i \in \mathbb{N}$ defined by

$$B_0(t) = B(t), \qquad B_{i+1}(t) = A(t)B_i(t) - \dot{B}_i(t), \qquad \text{for any } t \in [0,T].$$

The analog of the Kalman's controllability condition for autonomous systems holds in analytic case of non-autonomous systems.

Theorem 4.21 ([56, Theorem 2.3.2]). *The system* (4.19) *with analytic* A, B *is controllable in time* $T > 0$ *if and only if for any* $t \in [0,T]$ *there holds*

$$\operatorname{span}\{B_i(t)v : v \in R^m,\ i \in \mathbb{N}\} = \mathbb{R}^n. \tag{4.20}$$

By [57, Section 3.2], the condition (4.20) is related with the ample and strictly normal geodesics in the following way.

Proposition 4.22. *Let* γ *be a normal geodesic on* $[0,T]$*, then*

- γ *is strictly normal if and only if the linearized system is controllable in time* T*,*
- γ *is ample at time* $t \in [0,T]$ *if and only if* $\operatorname{span}\{B_i(t)v : v \in R^m,\ i \in \mathbb{N}\} = \mathbb{R}^n$.

Corollary 4.23 ([57, Proposition 3.12]). *Let* γ *be a normal geodesic on* $[0,T]$*. If* γ *is ample at* 0 *then it is strictly normal on* $[0,t]$ *for any* $t \in [0,T]$*. If* γ *is strictly normal on* $[0,T]$ *then it is ample at any* $t \in [0,T]$*.*

Let $V_T(q_0,\cdot)$ be the value function corresponding to the cost (4.3) with analytic $R(\cdot)$ and the dynamics (4.1) associated to the fixed $f_0, f_1, \ldots, f_m$. From Theorem 1.30, the smooth points of $V_T(q_0,\cdot)$ are the ones which are related to q_0 by strictly normal minimizers. In the control-affine case the following result on the set of smooth points was recently proved in [55].

Theorem 4.24 ([55, Theorem 2]). *The set of smooth points of $V_T(q_0, \cdot)$ related to q_0 by minimizing trajectories is open and dense in* $\mathrm{int}\mathcal{A}_{q_0}(T)$*. In particular it is not empty.*

From Theorem 4.24 and Corollary 4.23 we have the existence of an ample geodesic. As q_0 is an arbitrary point of M, we have the existence at any $q \in M$.

Theorem 4.25. *In the analytic category, for any point $q \in M$ there exists an ample $\lambda \in T^*M$ with respect to a Lagrangian of the form* (4.3) *such that $\pi(\lambda) = q$.*

Taking into account Proposition 4.17, we obtain the following corollary.

Corollary 4.26. *At a regular point q_0, the set of ample $\lambda \in T^*M$, $\pi(\lambda) = q_0$, with respect to a Lagrangian of the form* (4.3) *is open and dense in $T^*_{q_0}M$.*

For smooth dynamics and cost, we will show the existence for a particular class of control-affine systems using an adapted nilpotent approximation.

4.3.2 Nilpotent approximation

Fix a regular point $q_0 \in M$ and smooth vector fields $f_0, f_1, \ldots, f_m$ satisfying Assumptions 3. Just as in sub-Riemannian case we can define a local graded structure at q_0 and the corresponding nilpotent approximation of the affine system (4.1). The construction of the nilpotent approximation for the control-affine case was defined in [83] and another nice exposition can be found in [84]. Let us briefly present the nilpotent approximation in a convenient form. For the rigorous definitions and proofs we refer to [83, 84]. We use the notions related to nilpotent approximation from Section 3.7 and constantly refer to this section to see the similarities and differences with the sub-Riemannian case.

Let us introduce some notations. For any $q \in M$, set

$$\begin{aligned} \mathcal{L}_f &= \mathrm{Lie}\left(\{(\mathrm{ad} f_0)^s f_i : s \geq 0,\ i = 1, \ldots, m\}\right), \\ \mathcal{L}_f(q) &= \mathrm{Lie}_q\left(\{(\mathrm{ad} f_0)^s f_i : s \geq 0,\ i = 1, \ldots, m\}\right). \end{aligned}$$

Let U be a regular neighborhood of q_0. A graded structure on U is defined by the vector fields $f_0, f_1, \ldots, f_m$ and their respective weights $l_0, l_1, \ldots, l_m$. We choose the weights in such a way that the normal Hamiltonian (4.5) is homogeneous. The natural choice in our case is

$$l_0 = 2,\ l_1 = \cdots = l_m = 1, \tag{4.21}$$

as in this case the Hamiltonian is homogeneous of weighted degree 2. The weights induce a filtration of the Lie algebra spanned by $f_0, f_1, \ldots, f_m$ in the following way.

Let us define a sequence of spaces

$$\begin{aligned}
&D_1 = \operatorname{span}\{f_1, \dots, f_m\};\\
&D_2 = \operatorname{span}\{f_0,\ [f_i, f_j]\,,\ f_i\ :\ i, j = 1, \dots, m\};\\
&D_3 = \operatorname{span}\{[f_0, f_i]\,, [[f_0, f_i]\,, f_k]\,, f_0,\ [f_i, f_j]\,,\ f_i\ :\ i, j, k = 1, \dots, m\};\\
&\quad\vdots
\end{aligned} \tag{4.22}$$

and we have

$$D_1 \subseteq D_2 \subseteq \dots \subseteq \operatorname{Lie}(f_0, f_1, \dots, f_m)\,, \qquad \operatorname{Lie}(f_0, f_1, \dots, f_m) = \bigcup_{k \geq 1} D_k.$$

By the weak Hörmander condition (4.2), there exists $r \in \mathbb{N}$ such that at any $q \in U$, $D_r(q) = T_q M$. The smallest such r we call *step of the filtration.* As a consequence, at any $q \in U$ the following filtration of vector spaces is well defined

$$D_1(q) \subset D_2(q) \subset \dots \subset D_r(q) = T_q M. \tag{4.23}$$

Just as in the sub-Riemannian case in (3.41), the filtration defines a grading structure and the weighted degrees, denoted by $\deg_w$, of vector fields and functions (see Section 3.7.1). Moreover, by [83], there exist coordinates on a neighborhood of q_0 adapted to the grading. Based on the coordinates we can construct the vector fields $\hat{f}_0, \hat{f}_1, \dots, \hat{f}_m$, each $\hat{f}_i$ is the homogeneous part of the highest weighted degree of f_i and $\hat{f}_i(q_0) = f_i(q_0)$ for $i = 0, 1, \dots, m$. Therefore, $\hat{f}_0$ is of weighted degree 2 and $\hat{f}_i$ is of weighted degree 1 for any $i = 1, \dots, m$. The vector fields $\hat{f}_0, \hat{f}_1, \dots, \hat{f}_m$ are called *nilpotent approximation* of $f_0, f_1, \dots, f_m$. Based on the procedure described in Definition 3.11, we can construct a frame $f_1, \dots, f_n$ adapted to the filtration (4.23) with the particular choice of $f_{m+1} = f_0$. Remind that we work in the neighborhood of regular point q_0 and thus, f_0 is linearly independent from $f_1, \dots, f_m$. The corresponding frame of the nilpotent approximation $\hat{f}_1, \dots, \hat{f}_n$ consist of homogeneous polynomial vector fields, each $\hat{f}_i$ is of the weighted degree k corresponding to the strata D_k/D_{k-1} to which it belongs.

Remark 4.27. In the control-affine case, the nilpotent approximation satisfies

$$\operatorname{Lie}_{q_0}(f_0, f_1, \dots, f_m) = \operatorname{Lie}_{q_0}\left(\hat{f}_0, \hat{f}_1, \dots, \hat{f}_m\right), \tag{4.24}$$

but the following inclusion is strict in general

$$\mathcal{L}_{\hat{f}}(q_0) \subset \mathcal{L}_f(q_0). \tag{4.25}$$

Notice that the weak Hörmander condition (4.2) does not hold for the nilpotent approximation in the cases where the inclusion in (4.25) is strict. In the sub-Riemannian case we usually ask the strong Hörmander condition (equivalent to the condition in Definition 3.9) therefore this problem does not appear. More precisely, the problem comes from the fact that Lie $\left(\hat{f}_0, \hat{f}_1, \ldots, \hat{f}_m\right)$ is nilpotent and corresponds to the filtration (4.23) but the filtration is not adapted to $\mathcal{L}_{\hat{f}}$. In particular, the step of the filtration is less than the step of $\mathcal{L}_{\hat{f}}$, in general. The step of $\mathcal{L}_{\hat{f}}$ being defined as the least integer $\widetilde{r}$ such that at some neighborhood of q_0

$$\mathcal{L}_{\hat{f}}(q) = \mathrm{Lie}_q\left(\{(\mathrm{ad}\hat{f}_0)^s \hat{f}_i : 0 \le s \le \widetilde{r},\ i = 1, \ldots, m\}\right).$$

We can still get the equality in (4.25) by introducing supplementary assumptions on the vector fields $f_0, f_1, \ldots, f_m$. The most general assumption on $f_0, f_1, \ldots, f_m$ which permits to keep the weak Hörmander's condition and the chosen weights is the following

$$f_0 \in \mathrm{span}\{f_i, [f_i, f_j] \ :\ i, j = 1, \ldots, m\}. \tag{4.26}$$

Less restrictive assumptions on f_0 are possible but for more restrictive classes of $f_1, \ldots, f_m$.

Proposition 4.28. *Assume* $f_0, f_1, \ldots, f_m$ *satisfy* (4.26). *Then the weak Hörmander condition* (4.2) *holds for the nilpotent approximation* $\hat{f}_0, \hat{f}_1, \ldots, \hat{f}_m$.

Proof. In this case the step of the filtration is $r > 2$ or otherwise $m = n$. By the regularity assumption, f_0 is linearly independent from $f_1, \ldots, f_m$ on U, thus $m < n$. The filtration (4.23) is adapted to $\mathcal{L}_f$ in the following sense. There exists a neighborhood of q_0 such that at any q in this neighborhood

$$D_k(q) = \mathrm{Lie}_q\left(\{(\mathrm{ad}f_0)^s f_i : 0 \le s \le k-1,\ i = 1, \ldots, m\}\right), \quad k = 1, \ldots, r.$$

Therefore, we have

$$\mathrm{Lie}_q\{f_0, f_1, \ldots, f_m\} = \mathcal{L}_f(q), \qquad \mathrm{Lie}_q\{\hat{f}_0, \hat{f}_1, \ldots, \hat{f}_m\} = \mathcal{L}_{\hat{f}}(q).$$

Together with (4.24) this gives the desired result and ends the proof. □

Let $f_0, f_1, \ldots, f_m$ satisfy (4.26) and $\hat{f}_0, \hat{f}_1, \ldots, \hat{f}_m$ be the corresponding nilpotent approximation. Let A be the matrix from (4.10) corresponding to the adapted frame $f_1, \ldots, f_n$ and $\hat{A}$ be the matrix corresponding to $\hat{f}_1, \ldots, \hat{f}_n$. The same properties of the structure coefficients as in (3.42) hold in this case.

Proposition 4.29. *For any* $s \in \mathbb{N},\ j = 1, \ldots, m,\ k = m+1, m+2, \ldots, n$, *let* $a^s_{i,j}$ *be an element* (4.12) *of* A *and* $\hat{a}^s_{i,j}$ *be the corresponding element of* $\hat{A}$, *then*

$$a^s_{j,k} = \hat{a}^s_{j,k} + \mathrm{pol}(u_1, \ldots, u_n),$$

where $\deg_w(\mathrm{pol}) < \deg_w(\hat{a}^s_{j,k})$.

Proof. The result is obtained with the same calculation as in Lemma 3.63. □

Corollary 4.30. *A maximal minor of $\hat{A}$ is either zero or the homogeneous polynomial of the highest weighted degree of the corresponding minor of A.*

Remark 4.31. Notice, that to obtain the homogeneity of $\hat{a}^s_{i,j}$ we need the homogeneity in the weighted sense of the normal Hamiltonian. This holds automatically in the sub-Riemannian case, but in the control-affine case, it holds only if the weights for $f_0, f_1, \dots, f_m$ are chosen as in (4.21).

Proposition 4.32. *Let $f_0, f_1, \dots, f_m$ satisfy the Assumptions 3 and* (4.26)*, and $R(\cdot)$ satisfy Assumptions 4. At any regular point $q_0 \in M$ the control-affine system* (4.1) *admits an ample $\lambda_0 \in T^*M$ with respect to the Lagrangian* (4.3) *defined by $R(\cdot)$ such that $\pi(\lambda_0) = q_0$.*

Proof. Let $\hat{f}_0, \hat{f}_1, \dots, \hat{f}_m$ be the nilpotent approximation defined above. The value of $R(\cdot)$ at q_0 defines a constant matrix R. Up to a simultaneous change of $f_1, \dots, f_m$ and $\hat{f}_1, \dots, \hat{f}_m$, we assume $R(\cdot)$ and R to be the identity matrices. Notice that such a change of the control system does not affect the condition (4.26), as the new vector fields belong to the same D_1.

Both the dynamics (4.1) defined by $\hat{f}_0, \hat{f}_1, \dots, \hat{f}_m$ and the cost (4.3) defined by the identity matrix are analytic by construction. By Corollary 4.26, there exists an ample $\lambda_0 \in T^*M$ such that $\pi(\lambda_0) = q_0$. Let us consider the matrices $A(q_0), \hat{A}(q_0)$ corresponding to $f_0, f_1, \dots, f_m$ and $\hat{f}_0, \hat{f}_1, \dots, \hat{f}_m$ respectively.

By Proposition 4.13, the matrix $\hat{A}(q_0)$ is injective at $u(\lambda_0)$. By Corollary 4.30, the matrix $A(q_0)$ is injective at $u(\lambda_0)$ as well. We conclude that λ_0 is ample with respect to the initial dynamics and cost. □

By Proposition 4.17, the set of ample $\lambda \in T^*M$ such that $\pi(\lambda) = q_0$ is moreover open and dense.

4.4 Product structure

As in the linear-quadratic and sub-Riemannian cases we can define the product structure in the control-affine case.

Definition 4.33. We say that an optimal control problem defined by (4.1) and (4.3) *admits a product structure* at $q_0 \in M$ if there exists a system of coordinates $(x_1, \dots, x_n)$ in some neighborhood of q_0 such that the dynamics and the cost admit a simultaneous separation of the variables $(x_1, \dots, x_n) = (y_1, \dots, y_N)$ for some $1 \leq$

$N \leq n$. Namely, the dynamics takes the form

$$\begin{cases} \dot{y}_1 = f_0^1(y_1) + \sum_{i=1}^{m_1} u_i f_i(y_1), \\ \vdots \\ \dot{y}_N = f_0^N(y_N) + \sum_{i=m_{N-1}+1}^{m} u_i f_i(y_N); \end{cases} \tag{4.27}$$

and the Lagrangian takes the form

$$L(y_1, \dots, y_N) = u_1^T R_1(y_1) u_1 + \dots + u_N^T R_N(y_N) u_N,$$

with $u_k = (u_{m_{k-1}+1}, \dots, u_{m_k})$ for $k = 1, \dots, N$, and $0 = m_0 < m_1 < \dots < m_N = m$ are some integers.

It is clear that if a control-affine problem admits a product structure then it admits non-trivially equivalent via minimizers (and geodesics) costs. The next question is: can we obtain the converse implication? In the sub-Riemannian case the product structure can be seen at the level of nilpotent approximation. We will see that in a particular case of the control-affine system for which the nilpotent approximation is well-defined, we can obtain the same result with further consequences on the initial problem.

Let us fix a regular point $q_0 \in M$ and assume that $f_0, f_1, \dots, f_m$ satisfy (4.26) and Assumptions 3. Let $\hat{f}_0, \hat{f}_1, \dots, \hat{f}_m$ be the corresponding nilpotent approximation at q_0. Let us construct the matrices $\hat{A}, \hat{b}$ from (4.11) corresponding to the adapted frame $\hat{f}_0, \hat{f}_1, \dots, \hat{f}_n$ and to positive real values $\alpha_1, \dots, \alpha_m$. For any $s \in \mathbb{N}$ and $j = 1, \dots, m$, the element $\hat{b}_j^s$ from (4.13) takes the form

$$\hat{b}_j^s = \sum_{k=m+1}^{n} \alpha_j \hat{a}_{j,k}^s u_k.$$

For any $s \in \mathbb{N}$ and $j = 1, \dots, m$, the $(s-1+j)$th equation of system (4.10) takes the form

$$\sum_{k=m+1}^{n} \hat{a}_{j,k}^s \Phi_k = \sum_{k=m+1}^{n} \alpha_j \hat{a}_{j,k}^s u_k. \tag{4.28}$$

4.4.1 Special case $n = m + 1$

In the case $n = m + 1$ it is easy to solve the system (4.28).

Lemma 4.34. *Let $n = m + 1$. Assume that the system $\hat{A}\widetilde{\Phi} = \hat{b}$ admits a solution. Then $\Phi_{m+1} = \alpha_j u_{m+1}$ for some j such that $1 \leq j \leq m$ and the control-affine problem with dynamics (4.1) defined by $\hat{f}_0, \hat{f}_1, \dots, \hat{f}_m$ and cost defined by identity matrix admits a product structure.*

Proof. In case $n = m + 1$, the matrix $\hat{A}$ is a vector with infinite number of elements and the system (4.28) contains one unknown variable Φ_{m+1}. Taking into account the properties of the nilpotent approximation $\hat{f}_0, \hat{f}_1, \ldots, \hat{f}_m$, we have

$$\begin{cases} q_{jk} = \sum_{i=1}^{m} \hat{c}_{ij}^{k} u_i, & j = 1, \ldots, m, \ k = m+1; \\ q_{jk} = 0, & \text{otherwise.} \end{cases}$$

From (4.28) with $s = 1$, for any $j = 1, \ldots, m$ we have

$$\hat{a}_{j,m+1}^{1} \Phi_{m+1} = \alpha_j \hat{a}_{j,m+1}^{1} u_{m+1}.$$

By the assumption (4.26), there exist i, j, $1 \leq i, j \leq m$, such that $\hat{c}_{ij}^{m+1} \neq 0$. Therefore, there exists j^* such that $\hat{a}_{j^*,m+1}^{1} \neq 0$ and we have

$$\Phi_{m+1} = \alpha_{j*} u_{m+1},$$

and for any $j = 1, \ldots, m$ such that $\alpha_j \neq \alpha_{j*}$ we have

$$\hat{a}_{j,m+1}^{1} = q_{jm+1} = 0.$$

From the last equality, if $\alpha_j \neq \alpha_{j*}$ then $c_{ij}^{m+1} = 0$ for any $i = 1, \ldots, m$. On the other hand, as we are in the case of the nilpotent approximation, $c_{i,j}^{k} \neq 0$ only if $k = m+1$. As a consequence, if $\left[\hat{f}_i, \hat{f}_j\right] \neq 0$ then $\alpha_i = \alpha_{j^*}$. We define the coordinates $(x_1, \ldots, x_{m+1}) \in \mathbb{R}^{m+1}$ as the inverse of the map

$$(x_1, \ldots, x_{m+1}) \mapsto \exp\left(x_{m+1}\hat{f}_0 + \sum_{\{i\,:\,\alpha_i = \alpha_{j^*}\}} x_i \hat{f}_i\right) \circ \exp\left(\sum_{\{i\,:\,\alpha_i \neq \alpha_{j^*}\}} x_i \hat{f}_i\right).$$

In these coordinates, the control-affine system admits a separation of variables (y_1, y_2), such that $y_1 = (x_{m+1}, \{x_i : \alpha_i = \alpha_{j^*}\})$ and $y_2 = (\{x_i : \alpha_i \neq \alpha_{j^*}\})$, and takes the form

$$\begin{cases} \dot{y}_1 = \hat{f}_0(y_1) + \sum_{\{i\,:\,\alpha_i = \alpha_{j^*}\}} u_i \hat{f}_i(y_1), \\ \dot{y}_2 = \sum_{\{i\,:\,\alpha_i \neq \alpha_{j^*}\}} u_i \hat{f}_i(y_2). \end{cases} \tag{4.29}$$

The dynamics is of the form (4.27) and together with the Lagrangian

$$L = \sum_{i\,:\,\alpha_i = \alpha_{j^*}} u_i^2 + \sum_{i\,:\,\alpha_i \neq \alpha_{j^*}} u_i^2$$

they form a control-affine problem with product structure, this ends the proof.

□

In the following proposition $\widetilde{\Phi}$ and $\widetilde{\Psi}$ are two different vector-variables that we use to emphasize that the systems are independent, their solutions are different in general.

Lemma 4.35. *If the system* $A\widetilde{\Phi} = b$ *corresponding to* $f_0, f_1, \ldots, f_m$ *and positive* $\alpha_1(\cdot), \ldots, \alpha_m(\cdot)$ *admits a solution at* q_0 *then* $\hat{A}\widetilde{\Psi} = \hat{b}$ *corresponding to* $\hat{f}_0, \hat{f}_1, \ldots, \hat{f}_m$ *and constants* $\alpha_1(q_0), \ldots, \alpha_m(q_0)$ *admits a solution as well.*

Proof. The result is obtained with the same of arguments as in Lemma 3.66. □

Proposition 4.36. *Assume* $n = m+1$ *and the costs* L_1, L_2 *of the form* (4.4) *to be equivalent via geodesics at* q_0*. Then the control-affine problem corresponding to the nilpotent approximation* $\hat{f}_0, \hat{f}_1, \ldots, \hat{f}_m$ *and to* L_1 *admits a product structure.*

Proof. By Propositions 1.41 and 4.10, the system $A\widetilde{\Phi} = b$ admits a solution. By Lemma 4.35, the corresponding $\hat{A}\widetilde{\Psi} = \hat{b}$ admits a solutions as well. Then by Lemma 4.34, the control-affine problem corresponding to the nilpotent approximation $\hat{f}_0, \hat{f}_1, \ldots, \hat{f}_m$ with dynamics (4.29) and Lagrangian

$$L_1 = \sum_{i\,:\,\alpha_i = \alpha_{j^*}} u_i^2 + \sum_{i\,:\,\alpha_i \neq \alpha_{j^*}} u_i^2$$

admits a product structure. □

Notice that if $\left[\hat{f}_i, \hat{f}_j\right] = 0$ then $[f_i, f_j] = 0 \mod D_1$, where D_1 is the first strata in (4.22). We thus obtain the following corollary.

Corollary 4.37. *Assume* $n = m+1$*. If the costs* L_1, L_2 *of the form* (4.4) *are equivalent via geodesics at* q_0*, then*

$$[f_i, f_j] \notin D_1 \quad \textit{if and only if} \quad \alpha_i = \alpha_j. \tag{4.30}$$

Remark 4.38. In the sub-Riemannian case, (4.30) follows from the first divisibility condition (see Proposition 3.36). In the control-affine case, the analog of the first divisibility condition is (4.18), but it does not permit to conclude in general because it contains an unknown variable Φ_{m+1}.

The relations (4.30) have a special meaning for the product structure of a control-affine problem. In the sub-Riemannian case the product structure of the nilpotent approximation is based on two essential properties:

1. $\hat{f}_1, \ldots, \hat{f}_m$ decomposes in components $\hat{f}_1, \ldots, \hat{f}_k$ and $\hat{f}_{k+1}, \ldots, \hat{f}_m$ such that the Lie algebra decomposes accordingly

$$\mathrm{Lie}\{\hat{f}_1, \ldots, \hat{f}_m\} = \mathrm{Lie}\{\hat{f}_1, \ldots, \hat{f}_k\} \oplus \mathrm{Lie}\{\hat{f}_{k+1}, \ldots, \hat{f}_m\},$$

2. the two Lie algebras $\mathrm{Lie}\{\hat{f}_1, \ldots, \hat{f}_k\}$ and $\mathrm{Lie}\{\hat{f}_{k+1}, \ldots, \hat{f}_m\}$ commute.

If these two properties are satisfied then in the exponential coordinates the distribution spanned by $\hat{f}_1, \ldots, \hat{f}_m$ has a product structure as in Definition 3.49. The second property is the direct consequence of the condition (4.30) applied to $\hat{f}_1, \ldots, \hat{f}_m$. The first property then follows from the special form of the fundamental system using the second property. Therefore, the second property is needed to obtain the first one.

In the control-affine case the condition on the Lie algebra is more complicated in general. The fundamental system (4.28) in this case has the same form as in the sub-Riemannian case but we do not have the condition (4.30) and can not conclude as in the proof of Theorem 3.59. Therefore, the absence of the condition (4.30) is crucial in the control-affine case.

Chapter 5

Conclusions and perspectives

The main topic of this dissertation is the injectivity of the inverse optimal control problem. We presented a general framework which can be applied to diverse classes of optimal control problems. Then we applied the proposed methodology to specific classes. Let us summarize what have been achieved and what can be done in the future.

5.1 Conclusions on the general approach

In Chapter 1 we formalized the inverse optimal control problem. The first issue that is met in inverse problems is the ill-posedness, and in particular the problem of injectivity. The study of injectivity amounts to describe the costs which have the same minimizing solutions. This inspires the definition of an equivalence relation on the set of the costs, where, for a fixed dynamics, the equivalent costs are those which have the same optimal synthesis. Here we focused on the local problem. This allows us to consider the equivalence on geodesics, which have nicer properties than minimizers and are still meaningful for the injectivity problem.

There are two kinds of geodesics, namely, normal and abnormal ones, these two have different value for the inverse problem. The abnormal geodesics are independent from the cost and thus do not carry any useful information. On the other hand, normal geodesics are projections of the extremals, which are solutions of Hamiltonian systems. Notice that the Hamiltonian is constructed from the cost, and in general, the extremals corresponding to different costs which are projected to the same geodesics do not coincide. Therefore studying the difference between these kinds of extremals permits to conclude on the equivalent costs. The difficulty is that a geodesic may be normal and abnormal at the same time, and thus, a projection of different extremals. It appears that the good choice is to work with the ample geodesics. An ample geodesic is strictly normal at any time, therefore it admits a unique normal extremal lift. Moreover, as we have shown, an ample geodesic is characterized by its finite jet at the initial point (just as in Riemannian geometry, where the geodesics are solutions of the second-order geodesic equations and are defined by the initial point and the initial velocity). The last property permitted in particular to construct an orbital diffeomorphism which is fiber-preserving, therefore preserves the geodesics, and sends the normal extremals of a cost to the normal extremals of an equivalent cost. We have shown that near ample geodesics the existence of such an orbital diffeomorphism is necessary and sufficient for two costs to have the same geodesics. Our new approach is to reduce the study of injectivity to the study of the orbital diffeomorphism. We then applied it to the classical cases of optimal control problems, namely, linear-quadratic problem, sub-Riemannian problem and control-affine problem.

5.2 Linear-quadratic case

5.2.1 Conclusions

In Chapter 2 we considered the linear-quadratic case. In this case the set of costs is parametrized by three matrices (Q, R, S) coming from the definition of a quadratic Lagrangian. Therefore, the study of the cost equivalence reduces to the study of the equivalence on matrix-triplets. The linear-quadratic case enjoys a lot of simplifying properties. One of them is that the equivalence via geodesics and the equivalence via minimizers coincide. Therefore, even in non-injective case the same classes are defined by both kinds of equivalence. Notice that this is not the case in general. Thus, in non-injective case we can achieve injectivity by reducing the class of quadratic costs to the smaller one which contains a unique representative of each equivalence class. We made the same kind of reduction by introducing a canonical class of costs. In this class we reduced the number of parameters from the triplets (Q, R, S) to one matrix R, and there is still a representative of each equivalence class. The inverse problem is not injective on this class but we can apply the approach via the orbital diffeomorphism which works especially well in this case.

Based on the canonical class of costs we constructed a new class of linear quadratic problems for which we applied the approach via the orbital diffeomorphism. In this case the orbital diffeomorphism has a simple form, namely, it is a constant linear transformation. Analysis of the eigenspaces of the orbital diffeomorphism has led us to the conclusion that the existence of non-trivially equivalent costs (parametrized by R) implies that the underlying linear-quadratic problems admit the product structure. A product structure in this case means that a linear-quadratic problem decomposes in independent linear-quadratic problems. Notice that we can construct non-trivially equivalent costs for any linear dynamics thanks to the Brunovsky normal form, in which the control-system decomposes in independent components. But a pair of a linear dynamics and a quadratic cost does not always decompose in independent components simultaneously.

In the case of a product structure, the minimizing solutions are products of the solutions of the smaller-dimensional problems. In the linear-quadratic case the solutions have a nice algebraic form. All solutions in an optimal synthesis are parametrized by a stabilizing and an anti-stabilizing matrices (A_+, A_-). As a consequence, if all solutions admit a product form in some coordinates then (A_+, A_-) do also admit a product form in the same coordinates. Based on this observation we have proposed a cost-reconstruction algorithm. In this algorithm we first test if matrix-parameters (A_+, A_-) obtained from the given set of trajectories admit a simultaneous block-diagonal structure in some coordinates. If there is no such, then the inverse problem is injective and the unique solution R can be found from algebraic equations. If it is not the case, then we decompose the matrices (A_+, A_-) into smaller-dimensional blocks until each pair of sub-matrices corresponds to an injective problem, which is solved again by algebraic equations.

5.2.2 Perspectives

The first step of the proposed reconstruction method is not well adapted for numerical treatment. There are some adapted numerical approaches to treat this issue but there is also another point of view on this problem. In general, when we are dealing with approximative data in numerical analysis, in mathematical terms we are actually dealing with generic data. In our case it means a generic pair (A_+, A_-) in the set of possible parameters for the linear-quadratic problem with the given dynamical constraint. A generic pair of matrices in the set of pairs of stabilizing and anti-stabilizing matrices does not admit a simultaneous product structure. But in our case the pair (A_+, A_-) is obtained from the same Riccati equation and therefore the two matrices are related by some supplementary conditions. As a consequence, the structure of the set of such (A_+, A_-) should be understood to conclude on the properties of a generic pair. If there is no simultaneous product structure of a generic pair then the proposed algorithm reduces to the resolutions of algebraic equations.

Another direction for further investigations is to understand what is the meaning of the product structure of the new defined class of linear-quadratic problems in terms of the initial linear-quadratic problem on the canonical class of costs. In other word, given a linear dynamics, which matrices R correspond to a product structure in the reformulated linear-quadratic problem? Knowing the set of such R we will know that, on its complement in the set of positive definite symmetric matrices, the inverse problem with the initial dynamics is injective.

5.3 Sub-Riemannian case

5.3.1 Conclusions

The inverse optimal control problem is defined on trajectories but it can be defined as well on geometric curves instead of trajectories, in this case we call it the geometric problem. The geometric problem is much more complicated than the problem on the trajectories, but in the sub-Riemannian case the two problems are closely related. The inverse geometric problem inspires the notion of projective equivalence on the costs and the inverse optimal control problem inspires the notion of affine equivalence. We considered both equivalences within the framework of sub-Riemannian geometry in Chapter 3. The injectivity problem in the sub-Riemannian case is the following: given a manifold and a distribution, can we determine a metric from the given set of geodesics in a unique way? In the sub-Riemannian case it has been proven [57] that at any point of the manifold the set of ample covectors is open and dense. Therefore, the approach via orbital diffeomorphisms can be applied and we used it to study the two kinds of metric equivalence.

We have shown that in the sub-Riemannian case the orbital diffeomorphism is a solution of an algebraic system called the fundamental system. This system enjoys specific properties, in particular, its coefficients are homogeneous polynomials on the fiber. Another important property is its relation with the Jacobi curves. This relation implies the injectivity of the fundamental system on ample covectors. An analysis of the system allowed us to obtain a so-called first divisibility condition. This condition plays a key role in the study of projective and affine equivalences as it implies important relations between the Lie algebra of the distribution and the coefficients of the equivalent metrics.

Using the first divisibility condition we showed that if a distribution admits non-proportional equivalent metrics, then its nilpotent approximation admits a product structure. In the Riemannian case, the distribution is the whole tangent bundle and locally it always admits a product structure, but in a general sub-Riemannian setting it is not the case. Indeed, the Lie algebra of a distribution which admits a product structure has to satisfy strong conditions and it is reflected notably at the level of the nilpotent approximation. We have shown that for a generic distribution (except quasi-contact distributions and the case (4,6)) the nilpotent approximation does not admit a product structure. This implies the injectivity of the inverse optimal control problem on generic distributions.

For the geometric inverse problem the above condition is not sufficient and to prove the injectivity we need to show that the projectively equivalent conformal metrics are constantly proportional. In the Riemannian geometry, this is known as the Weyl theorem, but it is not applicable to the sub-Riemannian case. However, we were able to show that the Weyl theorem holds if the orbital diffeomorphism is polynomial on the fiber.

Another consequence of the first divisibility condition is the existence of non-trivial quadratic first-integral for geodesics of non-trivially equivalent metrics. The existence of such integrals is a strong condition on the metric. In the Riemannian case it does not hold generically and we extended this result to the sub-Riemannian case. As a consequence, a generic metric on a given distribution does not admit a non-proportional equivalent metric, and therefore it does not admit a non-trivially affine equivalent metric. Here again, we are missing a Weyl theorem to conclude that a generic metric does not admit a non-trivially projectively equivalent metric.

Another direction in the study of projective and affine equivalences is to derive the structure of the equivalent metrics. The known results in the Riemannian case [48] and contact and quasi-contact sub-Riemannian cases [50] suggest the so-called Levi-Civita pair form of the equivalent metrics. Our conjecture is that Levi-Civita pairs are the only possible pairs of projectively/affinely equivalent metrics in a general sub-Riemannian case. We approached this conjecture by showing that it holds in Carnot groups and for the nilpotent approximation.

5.3.2 Perspectives

A first direction for the future investigations is the Weyl theorem in the sub-Riemannian case with the goal to understand if it holds in general. The next question that should be addressed in this view: is the orbital diffeomorphism always polynomial? Or at least for generic distributions? We proved a first result in this direction but there is still a lot of work to do.

The second direction is to prove the conjecture on the Levi-Civita pair. At the present state it is clear that new arguments are needed to show it. One of the possible approaches could be to linearise the sub-Riemannian problem along an ample geodesic and to study the so-obtained non-autonomous linear-quadratic problem.

The results obtained in the sub-Riemannian case could also be used in applications, for instance to model the visual cortex. Indeed, models of the visual cortex

V1 show that the behavior of this cortex is governed by a sub-Riemannian geometry defined by a contact distribution. It would then be interesting to identify the metric associated to this sub-Riemannian geometry. In the contact case the inverse problem is injective and therefore the underlying metric is unique. So we could try to develop a metric reconstruction algorithm to recover numerically this metric.

More generally, in cases where the injectivity is established, it would be interesting to address the issue of reconstruction of the metric. In the cases of small dimension, one could rely on existing normal forms in the literature and try to identify the associated invariants. In particular in the three-dimensional contact case (as in the application to the cortex above) one could use the normal forms of [85]. The description of the cut-locus given in this paper is also an element on which we could rely, the cut-locus being one of the data of the optimal synthesis.

5.4 Control-affine case

5.4.1 Conclusions

In the control-affine case which was considered in Chapter 4 we followed the methodology from the sub-Riemannian case to obtain a fundamental algebraic system. However, in this case the algebraic system is more complicated, in particular, its coefficients are not homogeneous anymore. The condition similar to the first divisibility condition case depends on an unknown coordinate of the orbital diffeomorphism and thus does not permit to obtain any conclusion directly. Moreover the nilpotent approximation is not always well defined, only in a few restrictive cases of control-affine systems. Even in these cases the fundamental algebraic system is difficult to treat in absence of supplementary conditions on the Lie algebra as the ones obtained in the sub-Riemannian case. Nevertheless, we have treated the simplest case $n = m + 1$, where n is the dimension of the manifold and m is the dimension of the control. In this case we showed that the nilpotent approximation has a product structure, just as in the sub-Riemannian case. Via nilpotent approximation, we also obtained the same relations on the Lie-algebra as we obtained in the sub-Riemannian case via the first divisibility condition. Therefore we deduce that this kind of relations is not the proper sub-Riemannian characteristic.

Another important issue in the control-affine case is the existence of an ample geodesic. In the analytic case we obtained the existence of an ample geodesic from the results on smooth points [55]. In the smooth case the existence is an open question except the case when the drift is generated by the Lie brackets of the controlled vector fields. In this case, the nilpotent approximation was used to show the existence of an ample geodesic in the following way. The nilpotent approximation induces an analytic control-affine system and therefore, we applied to it the result of the analytic case and then used the properties of the fundamental system. Actually, the fundamental system itself is the tool to show the existence, because its injectivity on a covector implies that this covector is ample. Therefore, if we can verify the injectivity using exterior arguments, then we can deduce the existence at an ample covector, and thus, the ample geodesic.

5.4.2 Perspectives

For further investigations in the control-affine case it would be interesting to make all computations in the case of $n = m + 1$. It would allow us to see in a proper control-affine case the structure of the cases admitting non-trivially equivalent costs and to fix the picture of the possible structures for different cases of optimal control problems. We should also understand the meaning of the product structure in this case. However, for further study of the control-affine case in its full generality a new approach should be found.

5.5 New general approach and conjecture

The study of the symmetries of the Jacobi curves (see Section 1.3.4) is a promising idea, it could complete the considered cases and be further applied to very general inverse problems. In this approach, the study of orbital diffeomorphisms for the same geodesics would be reduced to the study of the symmetries of the corresponding Jacobi curves. We conjecture that a nontrivial symmetry exists if and only if the Jacobi curve admits a structure of a product of Jacobi curves of lower dimension. All the known cases agree with this conjecture and it reflects the general structure observed in all the results, namely, the product structure of the optimal control problem admitting equivalent costs and absence of product structure in the injective cases.

Résumé en français

Le problème du contrôle optimal inverse fait l'objet d'une attention particulière au cours des dernières décennies. Le regain d'intérêt est dû au nombre croissant d'applications. En particulier dans la modélisation des mouvements humains en physiologie, ce qui a conduit à une nouvelle approche dans le domaine de la robotique humanoide. Un être humain, en tant que système mécanique, peut être modélisé comme un système de commande. Son architecture est telle qu'il y a beaucoup de possibilités pour réaliser chaque tâche particulière. Pour toute tâche, le mouvement choisi est stable vis-à-vis des changements d'environnement non pertinents pour la tâche et adaptable aux changements s'il est nécessaire pour la réalisation de la tâche. Ces caractéristiques des mouvements les rendent très plausibles pour être des mouvements optimaux et en physiologie, ce paradigme d'optimalité est l'une des hypothèses dominantes (pour des explications plus rigoureuses dans le contexte de la physiologie, voir [1, 2]). Par conséquent, le bon cadre mathématique pour les mouvements est le cadre de contrêle optimal, c'est-à-dire que les mouvements réalisés par le système mécanique minimisent certaines fonctions de coût. Cependant, même si on sait qu'un mouvement est optimal, les critères optimisés sont cachés. Ainsi, pour modéliser les mouvements humains, nous devons d'abord résoudre un problème de contrôle optimal inverse: compte tenu des données des mouvements réalisés et de la dynamique du système mécanique, trouvez la fonction de coût par rapport à laquelle les mouvements sont optimaux, c'est-à-dire, les solutions du problème de contrôle optimal correspondant. Le contrôle optimal inverse s'est déjà révélé utile dans l'étude de la locomotion humaine [3] et des mouvements des bras [4].

En robotique humanoide, le contrôle optimal inverse est l'outil permettant d'obtenir la fonction de coût la plus adaptée pour ensuite mettre en oeuvre les lois de commande induites dans les robots. Dans cette perspective, des mouvements différents ont été mis en oeuvre, par exemple la locomotion humaine [5]. Le même schéma est appliqué aux robots censés agir comme des systèmes biologiques autres que l'homme, par exemple un quadrotor se déplacant comme un insecte volant, voir [6]. Une autre application en robotique concerne les robots autonomes, les voitures autonomes en particulier (voir [7]), qui interagissent avec les humains et devraient prédire les actions humaines [8, 9]. En économie, lorsque nous considérons le processus de prise de décision des clients, le contrôle optimal inverse est utile pour trouver une fonction d'utilité [10]. Dans de nombreux contextes différents, un expert effectue par intuition des actions très impressionnantes et efficaces qui peuvent être mieux comprises via un contrôle optimal inverse et également mises en oeuvre, par exemple comme dans [11] sous la forme de robots imitant la stratégie du pilote. La théorie du contrôle optimal inverse a également donné lieu à de nouvelles méthodes de stabilisation, une telle méthode a été proposée dans [12] pour produire un contrôle de retour stabilisant (voir aussi [13]). Dans le cas d'un régulateur linéaire-quadratique, le contrôle optimal inverse fournit une méthode pour un placement des pôles optimal (voir [14]).

D'un point de vue mathématique, le problème de contrôle optimal inverse appartient à la classe des problèmes inverses où la première question est de savoir si le problème est bien posé. Formellement, étant donné une dynamique et une classe de fonctions de coût candidates, pour la classe de problèmes de contrôle optimal direct correspondante on peut définir un opérateur qui associe une fonction de coût à la synthèse optimale, c'est-à-dire à l'ensemble des trajectoires optimales pour tous les points initiaux et finaux réalisables. Dans le problème inverse, nous recherchons l'opérateur inverse. Pour qu'un tel problème inverse soit bien posé, il doit être surjectif, injectif et stable. Par surjectivité, nous entendons que l'ensemble de trajectoires donné ne contient que des trajectoires minimisantes le même coût. En général, la surjectivité est difficile à vérifier et, dans les applications, on suppose qu'elle est satisfaite. L'injectivité signifie qu'il existe une correspondance univoque entre les coûts de la classe et la synthèse optimale. Il est facile de voir que la multiplication de tout coût par une constante ne modifie pas les minimiseurs; par conséquent, pour obtenir l'injectivité, nous devrions normaliser les coûts dans la classe de fonctions de coût considérée. En général, les coûts proportionnels ne sont pas les seuls coûts à avoir les mêmes solutions optimales. Néanmoins, l'injectivité peut être atteinte en limitant la classe de coûts à une classe inférieure. La stabilité, c'est-à-dire la continuité de l'opérateur inverse, signifie que de petites perturbations de trajectoires impliquent de petites perturbations de coût. Cette propriété est importante pour les applications où nous ne travaillons jamais avec des données exactes.

Cette thèse est consacrée à la caractérisation du problème bien posé et plus particulièrement à l'injectivité du problème de contrôle optimal inverse. L'analyse est limitée à certaines classes de problèmes de contrôle optimal. La surjectivité est supposée être satisfaite. Pour trouver les cas où l'injectivité du problème inverse est vérifiée, on a étudié la structure des cas non injectifs. Cela nécessite l'introduction d'une notion d'équivalence des fonctions de coût. Les coûts équivalents sont ceux ayant les mêmes trajectoires optimales. Les flots hamiltoniens normaux correspondant aux coûts équivalents sont différents dans le fibré cotangent mais ils sont projetés via une projection canonique sur les mêmes trajectoires dans l'espace d'états. Le difféomorphisme orbital dans ce contexte est le difféomorphisme qui associe le flot hamiltonien normal d'un coût au flot hamiltonien normal du coût équivalent. Le difféomorphisme orbital nous a permis de dégager la structure des cas admettant des coûts équivalents.

Le manuscrit est organisé comme suit. Dans le chapitre 1 on a introduit les notions principales et l'idée centrale de l'approche par le difféomorphisme orbital. Nous avons ensuite appliqué la méthodologie à plusieurs classes classiques de problèmes de contrôle optimal. Dans le chapitre 2, nous avons considéré le problème linéaire-quadratique à l'horizon fini où une critère d'injectivité et une description des cas non-injective ont été obtenus. Dans ce cas nous avons réussi a trouver un algorithme de reconstruction du coût unique pour toute synthèse optimal donnée. Nous avons ensuite considéré deux cas non linéaires. Premièrement, le cas sous-riemannien, où le système de contrôle est linéaire par rapport au contrôle et le coût est quadratique par rapport au contrôle. Ce problème est très particulier par sa structure géométrique, ce qui nous a permis d'obtenir des résultats importants présentés au chapitre 3 sur injectivité générique et sur la structure des cas non-injectifs. Dans le chapitre 4, nous avons considéré le cas contrôle-affine. C'est la généralisation du cas sous-riemannien où la dynamique contient une dérive non contrôlée. Ce cas est beaucoup plus compliqué que le cas sous-riemannien mais nous avons arrivé a effectuer l'analyse des propriétés spécifiques de cette classe.

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École doctorale de mathématiques Hadamard (EDMH)

Titre : Contrôle Optimal Inverse : étude théorique

Mots Clefs : contrôle optimal, contrôle géométrique, neurophysiologie

Résumé : Cette thèse s'insère dans un projet plus vaste, dont le but est de s'attaquer aux fondements mathématiques du problème inverse en contrôle optimal afin de dégager une méthodologie générale utilisable en neurophysiologie. Les deux questions essentielles sont : (a) l'unicité d'un coût pour une synthèse optimale donnée (injectivité); (b) la reconstruction du coût à partir de la synthèse. Pour des classes de coût générales, le problème apparaît très difficile même avec une dynamique triviale. On a donc attaqué l'injectivité pour des classes de problèmes spéciales: avec un coût quadratique, la dynamique étant soit non-holonome, soit affine en le contrôle. Les résultats obtenus ont permis de traiter la reconstruction pour le probléme linéaire-quadratique.

Title : Inverse Optimal Control: theoretical study

Keys words : optimal control, geometric control, neurophysiology

Abstract : This PhD thesis is part of a larger project, whose aim is to address the mathematical foundations of the inverse problem in optimal control in order to reach a general methodology usable in neurophysiology. The two key questions are : (a) the uniqueness of a cost for a given optimal synthesis (injectivity) ; (b) the reconstruction of the cost from the synthesis. For general classes of costs, the problem seems very difficult even with a trivial dynamics. We treat the injectivity question for special classes of problems, namely, the problems with quadratic cost and a dynamics, which is either non-holonomic (sub-Riemannian geometry) or control-affine. Based on the obtained results, we propose a reconstruction algorithm for the linear-quadratic problem.