Exponential stabilization of quantum systems subject to non-demolition measurements in continuous time

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En fait, leur fusée n'était pas très, très au point, mais ils avaient calculé qu'elle avait quand même une chance sur un million de marcher. Et ils se dépêchaient de bien rater les 999 999 premiers essais pour être sûrs que le millionième marche.

—*Les Shadoks, Saison 1, Épisode 27.* Jacques Rouxel

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Chapter 1

Résumé

1.1 Contexte

Les progrès des méthodes experimentales pour mesurer et manipuler des systèmes quantiques individuels avec une grande précision, comme dans la première expérience du contrôle quantique en temps réel [61], encouragent le développement de lois de contrôle automatique pour manipuler l'état d'un système quantique. Aujourd'hui, la communauté des physiciens s'oriente vers des conceptions d'ingénierie prometteuses pour le calcul quantique [21, 30, 54, 58, 20]. L'un des éléments de base est la stabilisation d'un état quantique. Cette procédure de stabilisation pourrait être présente lors de la préparation d'un état fragile et fortement intriqué qui pourrait être utilisé sur un ordinateur quantique [53]. La procédure de stabilisation pourrait également être utilisée dans des applications de métrologie quantique [29]. Les applications de contrôle, en tant que partie de la couche sous-jacente correspondant au traitement quantique robuste de l'information, visent à concevoir des méthodes pour protéger l'information quantique en présence de perturbations [36, 28, 1, 60, 2].

L'utilisation des mesures quantiques non-destructives (QND) — en anglais *quantum non-demolition measurement* — est devenue un élément fondamental pour le contrôle quantique basé sur la mesure [38]. Une mesure QND est essentiellement une version en temps continu du postulat de projection [10], où effectuer une mesure en temps continu fait que l'état quantique converge progressivement vers un état propre de l'opérateur de mesure, mais l'état résultant est aléatoire. Chacun de ces états propres reste donc non perturbé sous la rétroaction quantique associée à la dynamique de la mesure. Les états propres QND sont donc des états d'équilibre stables naturels d'un système quantique mesuré, et la stabilisation d'un état propre cible grâce à une loi de contrôle appropriée constitue un élément de base pour des procédures de contrôle plus complexes, tels que la stabilization des varietés d'états quantiques comme réquis pour implementer algorithmes de correction d'erreurs quantiques [1].

Les contrôles basés sur la mesure reposent principalement sur deux architectures : le *contrôle basé sur l'état* et le *contrôle proportionnel.* Dans l'approche basée sur l'état [74], le contrôleur suppose qu'un observateur d'état (aussi appelé «filtre quantique de Bayes») capture de manière efficace l'évolution de l'état quantique. En effet, plusieurs articles ont developpé des techniques pour stabiliser un état QND cible [50, 71, 69, 70, 44]. Tandis que les systèmes quantiques deviennent de plus en plus complexes, plusieurs difficultés dans cette approche doivent encore être surmontées. Parmi celles-ci, les lois de contrôle proposées dans les travaux précédents sont assez complexes; la mise en marche basée sur un observateur d'état quantique deviendra rapidement difficile avec la dimension croissante des systèmes, comme requis pour les technologies quantiques visant le développement de l'ordinateur quantique. En outre, l'étude de convergence est compliquée, et aucun de ces schémas de rétroaction ne permet d'assurer une convergence exponentielle. Il est bien connu que la convergence exponentielle est un indicateur de la robustesse lorsque le système interagirait avec d'autres sous-systèmes ou serait sujet à certaines perturbations. En revanche, on peut montrer que la convergence du système en boucle ouverte sous la mesure QND vers l'ensemble de ses états stationnaires est exponentielle. L'absence d'une propriété similaire démontré pour la sélection d'un état QND cible sous l'effet d'un contrôleur d'état quantique est donc une thématique de recherche ouverte.

Dans le cas de contrôle proportionnel, aussi connu dans la littérature en physique sous le nom de «rétroaction markovienne» —en anglais *Markovian feedback*— la mesure est simplement renvoyée directement au système. Remarquablement, on peut voir l'équation en boucle fermée comme une reformulation des opérateurs de dissipation tel que l'état cible est l'unique état stationnaire de la dynamique [73]. De plus, la convergence exponentielle vers l'état cible peut être montrée avec des preuves algébriques directes [64, 65, 63]. Néanmoins, il peut être démontré que les états QND ne peuvent pas être stabilisés dans cette architecture du contrôle. En effet, il semble qu'un contrôleur plus complexe doit être utilisé afin de biaiser l'évolution stochastique vers un état propre cible.

Les développements historiques sur la conception des systèmes de contrôle classiques, avec des résultats majeurs tels que le régulateur Proportionnel Intégral Derivée (*PID*) [9, 40], encouragent à rechercher des alternatives aux architectures des systèmes de contrôle exposés toute a l'heure, telles que l'utilisation des filtres passe-bas qui pourraient être plus faciles à implementer plutôt qu'un filtre d'état quantique complet.

La lacune existante entre les deux architectures de contrôle quantique, contrôle proportionnel est contrôle basé sur l'état, n'est pas là en raison d'un manque d'intérêt. Cependant, des problèmes tels que vérifier que le système en boucle fermée soit bien posé [15, 18, 5], ou l'analyse de convergence en boucle fermée font qu'il est difficile de trouver des alternatives. Efforts dans cette direction considerent la formulation de filtres réduits ou filtres passe bas qui permettent de faciliter les calculs en ligne [52, 47, 60]. On peut considérer aussi éviter tout filtre, en concevant des contrôles pré-calculés hors-ligne et mis en œuvre sous forme des signaux variant dans le temps [77, 48]. Sinon, il est tout à fait possible de considerer uniquement l'intensité de la mesure comme paramètre de contrôle pour encourager des «sauts» vers un état particulier [67, 68, 13].

1.2 Énoncé du problème et idée principale

Conformément aux éléments précédents, les objectifs principaux de cette thèse sont

- *La stabilisation exponentielle d'un état propre QND prescrit,*
- *Identifier des opportunités vers l'implémentation des lois de contrôle calculables de manière efficace dans un contexte expérimental.*

En termes d'ingénierie du contrôle, l'idée principale poursuivie dans cette thèse pour atteindre les deux objectifs repose sur

Utiliser du bruit pour piloter l'actionneur.

Plus précisément, dans cette thèse il est presenté une nouvelle architecture de contrôle quantique, dans laquelle l'actionneur unitaire est piloté par un processus stochastique. En effet, il consiste en l'application d'un signal de commande piloté par un mouvement brownien, dont la variance est régulée par un gain dépendant de l'état quantique. Dans ce *contrôle assisté par le bruit*, la conception de la loi du contrôle[1] profite de la dynamique du bruit brownien et de la dynamique en boucle ouverte du système QND pour réaliser la stabilisation globale : lorsque le système s'approche d'un état propre aléatoire indesiré de l'opérateur de mesure, il suffit d'injecter du bruit pour «secouer» le système, en decourageant la convergence du système vers ces situations non-voulus, le système finise pour converger a l'état cible.

L'analyse de stabilité est étudiée à travers de la version stochastique de le méthode directe de Lyapunov. Dans cette thèse, on montre que l'utilisation du bruit permet d'obtenir des fonctions de Lyapunov globales avec décroissance exponentielle [41]. Donc la théorie basique sur la convergence des martingales en temps continu suffit pour assurer la convergence exponentielle globale vers l'état propre cible.

De plus, l'analyse de la stabilité de la dynamique en boucle fermée suggère l'utilisation de filtres réduits pour calculer la loi de contrôle. Contrôler en «secouant» l'état ne nécessite que des informations sur les populations autour des états propres QND. Étant donné que cette information est

[1] cf. épigraphe.

disponible avec le processus de mesure, cela signifie que seule l'information classique sur l'état est nécessaire pour piloter l'état quantique vers un état propre QND cible. De plus, on peut montrer que les termes associés au contrôle unitaire stochastique admettent soit une expression sous forme fermée, soit une approximation en lesdites populations de l'état propre. Ainsi, le filtre ne dépend que des probabilités classiques. Formulations de filtres reduits de ce genre, qui croît linearement en function des paramètres comme la dimension de systèmes augmente, pourrait voir un role plus prominent dans des technologies quantiques avancées.

1.3 Plan de la thèse

La structure de cette thèse est la suivante:

- Le chapitre 3 présente d'abord les modèles mathématiques correspondant aux mesures QND en temps continu. Le comportement dynamique des systèmes QND à boucle ouverte est présenté. Il est montré via une fonction de Lyapunov originale que, pour toute condition initiale, l'état converge de manière exponentielle vers l'ensemble des états propres QND, l'ensemble de vecteurs propres de la matrice hermitienne de la mesure. Cette propriété de «multistabilité stochastique» signifie que le système converge vers l'un des états QND possibles, mais en moyenne, le système ne bouge pas vers un état QND en particulier. Nous présentons ensuite le problème de contrôle, qui consiste à stabiliser de manière exponentielle un état propre QND parmi l'ensemble existant en boucle ouverte. Les architectures de contrôle existantes sont rappelées. Enfin, la stratégie de commande constituant le coeur de cette thèse est présentée.

  La présentation de ce chapitre sur l'exposition des travaux précédents, et la preuve de convergence exponentielle vers l'ensemble des états propres QND suit les lignes des articles [22, 24].

- Dans le chapitre 4 nous utilisons le contrôle proportionnel et le contrôle assisté par bruit pour stabiliser de manière exponentielle un qubit sujet aux mesures QND en temps continu. Dans le contrôle proportionnel, il peut être facilement montré que presque tout état pur peut être stabilisé de manière exponentielle, à condition de disposer d'une efficacité de mesure parfaite. Néanmoins, lorsque l'on vise des états proches a un état propre QND du qubit, le taux de convergence décroît vers zéro. Pour aborder la stabilisation des deux états propres QND de l'opérateur de mesure, nous utilisons l'approche assistée par bruit. La stabilisation exponentielle est montrée via des arguments classiques de la théorie stochastique de Lyapunov. En ce qui concerne la loi de contrôle, le contrôleur dépend uniquement de la coordonnée correspondant à l'axe de mesure, ce qui permet de déduire un filtre d'ordre réduit pour obtenir une stabilisation exponentielle.

Ce chapitre adapte les idées trouvées dans [22] pour obtenir la stabilisation exponentielle d'un qubit. Ici la stratégie utilise un bruit éxogene, au lieu d'une version dynamique du feedback Markovien utilisé dans ledit papier.

- Dans le chapitre 5 nous étendons la stratégie de contrôle assistée par bruit à un système QND à plusieurs niveaux, en utilisant un seule contrôle avec un Hamiltonian fixe. L'extension diffère sensiblement du qubit en raison de la présence de multiples états propres orthogonaux à l'état cible, et sur la manière dont l'hamiltonien du contrôle «connecte» les états QND. La stratégie du control est de «secouer» le système lorsqu'il s'approche d'états propres indésirables en y appliquant un bruit fort, tout en s'appuyant sur la dynamique en boucle ouverte pour atteindre progressivement la cible. Pour montrer que le bruit éloigne le système des voisinages des états propres QND non-voulus, des méthodes basiques de la théorie des graphes impliquant l'inversion d'une matrice laplacienne sont utilisées pour montrer la décroissance exponentielle autour desdits états. Ensuite, l'analyse sur la fonction de Lyapunov montre que, en combinaison avec la dynamique en boucle ouverte, on peut établir une convergence exponentielle de la fonction de Lyapunov en boucle fermée. Cette fonction est une supermartingale avec une décroissance exponentielle, impliquant la stabilité exponentielle globale. Le schéma de contrôle et son analyse de stabilité suggèrent l'utilisation d'un filtre approximatif qui ne fait que suivre les populations des états propres de l'opérateur de mesure. Ceci consiste à remplacer les termes correspondant à la dissipation faite par le bruit par la matrice laplacienne associée au graphe de connectivité induit par le hamiltonien d'actionnement.

  Ce chapitre est basé principalement sur [24], ici on ajoute quelques explications additionnelles.

- Le chapitre 6 étudie le problème de correction d'erreurs quantiques en temps continu pour le code de correction d'erreurs à trois qubits. Cela implique de rendre une variété cible d'états quantiques globalement attractive. Nous adaptons la loi du contrôle assisté par bruit aux spécificités du problème de la correction d'erreurs. Il est possible de montrer que la dynamique en boucle fermée résultante stabilise la variété cible de manière exponentielle. Néanmoins, il y a quelques différences par rapport aux problémes de stabilisation précédents qui doivent être prises en compte ici: d'abord, ayant une donnée initiale appartenant à la varieté cible, il est nécessaire de montrer qu'elle est protégée des erreurs detectées ; deuxièmement, les actions de contrôle doivent être minimisées pour que des erreurs additionnelles à cause de ladite action soient évitées. En outre, une formulation de filtre d'ordre réduit est présenté, qui est basé sur des probabilités classiques. L'étude sur la protection induite par la dynamique de correction d'erreur en boucle fermée contre les perturbations est étudié numériquement. Les performances de cette loi de contrôle et du filtre réduit sont étudiées numériquement. La procédure de contrôle considerée dans ce chapitre constitue une première extension à plusieurs entrées et sorties — en anglais *multi-input multi-output* (MIMO) — du contrôle assisté par bruit considéré dans les chapitres précédents.

Les résultats de ce chapitre suivent l'article [23].

Chapter 2

Introduction

2.1 Background

We are now at a stage where measuring and controlling quantum systems is experimentally realizable, as in the first real time quantum feedback experiment [61]. Now, the physics community has been moving towards engineering designs that are truly promising for quantum computation [21, 30, 54, 58, 20]. One of the elementary building blocks is the stabilization of a quantum state. This stabilization procedure could be present while preparing a fragile and highly entangled state as an input to be used on a quantum computer [53]. It could be used as well in applications of quantum metrology [29]. Applications of feedback, as part of the control layer underlying robust quantum information processing devices, aim towards developing continuous-time methods that can be used to protect quantum information in the presence of perturbations [1, 49]. The design of the control layer has motivated the study of different quantum feedback schemes, notably *coherent feedback*, *reservoir engineering* and *measurement-based feedback.*

Coherent feedback is a formulation where the control signals, sensors and controller, are quantum mechanical [46]. This concept has given rise to mathematical formulations that have promising capabilities for modelling and analysis of networks of interconnected quantum systems [35, 39]. In close relation, reservoir engineering techniques consists on engineering a coupling between a system of interest and a dissipative ancilliary quantum system in the aim of mantaining the coherence of some particular quantum states [57, 49]; the coupling is done such that the entropy of the main system is evacuated through the dissipation of the ancilliary one.

At the core of this thesis is the use of continuous-time measurement-based feedback. In this feedback scheme, a continuous-time signal provides weak information, associated to a weak backaction, on the quantum state. Quantum non-demolition (QND) measurements [38] are a standard element for measurement-based feedback control. A QND measurement is essentially a continuous-time version of the projection postulate, characterized by a Hermitian operator. Performing a continuous measurement makes the quantum state progressively converge to a random eigenstate of the measurement operator. Each eigenstate is a steady state of the dynamics, and hence remains unperturbed under the backaction associated to this quantum measurement. Thus, the use of a QND measurement can be seen as a non-deterministic preparation tool, and an additional feedback control is necessary for preparing a particular QND eigenstate. The goal of this thesis is to address this control layer in continuous-time.

Measurement-based feedback controls have been based mainly on two feedback structures: *state feedback* and *static output feedback*. In the state feedback approach [74], the controller assumes that a state observer (often called the "quantum Bayes filter") efficiently captures the evolution of the quantum state.

Several papers have indeed considered ways to stabilize a target QND eigenstate [50, 71, 69, 70, 44] under state feedback. As engineered quantum systems become more complex, there are still several issues that need to be addressed. Among them, the feedback laws proposed in existing work can be quite complex; the implementation based on a quantum state observer would scale poorly with increasing system dimension as will be needed by advanced quantum technologies. In addition, the associated convergence analysis is rather involved and none of these feedback schemes are shown to ensure exponential convergence. It is well-known that exponential convergence is an indication of robustness when the system would interact with other subsystems or would perform subject to some perturbations. Moreover, it can be shown that the convergence of the open-loop system under QND measurement towards the set of its steady states is exponential. The absence of a proven similar property for the selection of one target QND steady state under quantum state feedback thus appears as an avoidable gap.

In the static output feedback, also known in the physics literature as *Markovian feedback* [74], the measurement output is fed-back into the system. In analogy with classical systems, the feedback is dependent on constants that define a set-point for the closed-loop dynamics. Remarkably, we can consider the closed-loop equation as a reformulation of the dissipation operators such that target state is the only steady state of the closed-loop dynamics [73, 74]. Moreover, exponential convergence of the target state does hold with direct algebraic proofs [64, 65, 63]. Unfortunately, this feedback is not without inconvenients: for one, perfect stabilization of a target pure state depends on having a perfect measurement efficiency, that is, there is no loss of information from the measurement device to the environment. Additionally, it can be shown that QND eigenstates cannot be stabilized under this setting. Indeed, a more involved controller needs to be used in order to bias the stochastic evolution towards a target eigenstate.

The developments of classical control systems design, with major results such as PID control [9, 40], encourage to look for alternatives to the previous two quantum feedback architectures, and motivate the search of dynamical feedback approaches, such as the use of reduced quantum filters that are easier to compute rather than a full quantum state filter.

The gap that exists currently between the two measurement-based quantum feedback structures, static output feedback and state feedback, is not due to lack of interest. However, problems like well-posedeness of the closed-loop system [15, 18, 5] and convergence analysis seem to make difficult to find alternatives. Some efforts in this direction include low dimensional formulations of quantum filters or low-pass filters that would be easier to compute online [52, 47, 60]. Another possibility is on avoiding a quantum filter altogether, by proposing pre-computed controls that are computed off-line and implemented as time-varying signals [77, 48]. Else, it is as well possible to consider the mesurement strength as a control parameter to encourage "jumps" towards a particular state [67, 68, 13].

2.2 Problem statement and main idea

In accordance with the previous exposition, there are two main goals that drive this Thesis

- *Achieve exponential stabilization of a prescribed QND eigenstate,*
- *Identify opportunities towards the implementation of efficiently computable controls on an experimental setup,*

In control engineering terms, the main idea pursued in this Thesis to address both goals is

Use an exogenous noise to drive the actuator.

More precisely, in this Thesis we propose a new actuation strategy where the unitary actuator is driven by a stochastic process. It consists on the application of a control signal driven by a Brownian motion, whose variance is regulated by a state-dependent gain.

In this *noise-assisted feedback* architecture, the control design strategy[1] exploits the QND dynamics to achieve global stabilization: since in open-loop the system converges to a random eigenstate of the measurement operator, it suffices to apply noise to "shake" away the system when it is close to an undesired eigenstate; by selectively rejecting undesired situations, the stochastic evolution is biased to the target.

The control synthesis is be setup as a stabilization problem through the stochastic version of Lyapunov's second method. In this Thesis, we show

[1] cf. ephigraph.

that the stochastic actuation strategy, driven by Brownian noise, allows to obtain closed-loop Lyapunov functions with exponential decay, thus standard martingale theory is enough to ensure the global exponential convergence towards the target eigenstate.

Interestingly, the stability analysis of the closed-loop dynamics suggests the use of reduced filters to compute the feedback law. Controlling by "shaking" the state just needs information on the populations on the QND eigenstates. Since this information is available from the measurement operator, this means that only classical information on the state is needed to drive the quantum state towards a target QND eigenstate. Furthermore, it can be shown that the terms associated to the stochastic unitary control admit either a closed form expression or an approximation in terms of the said eigenstate populations, thus the filter only depends classical probabilities. Formulations of reduced-order filters of this type, scaling linearly in terms of its parameters as the dimension of the system increases, could see an increasing role as needed for advanced quantum technologies

2.3 Thesis outline

The structure of this Thesis is as follows:

- Chapter 3 first presents the mathematical models corresponding to continuous QND measurements. The dynamical behaviour of the open-loop QND systems is introduced. We show via an original Lyapunov function that, for any initial condition, the state converges exponentially towards the set of QND eigenstates. This "stochastic multistability" property means that the state converges to one of a few steady state situations, but on average it does not move to any particular one. We then present the control problem, which is to exponentially stabilize a QND eigenstate amongst the open-loop set. The associated existing feedback designs are overviewed. Lastly, I present the control strategy that will be common for the rest of this Thesis.

  The presentation of this Chapter on precedent works and the proof of exponential convergence towards the set of QND eigenstates follows mostly the lines of the articles [22, 24].

- Chapter 4 provides the continuous-time static output feedback and noise-assisted feedback to exponentially stabilize a qubit under QND measurements. In static output feedback, it can be easily shown that any pure state on the qubit can be exponentially stabilized when having a high measurement efficiency. Nevertheless, when aiming towards a QND eigenstate, the convergence rate decreases towards zero. To address the two QND eigenstates of the measurement operator, we introduce the noise-assisted approach. Exponential stabilization is shown via standard Lyapunov arguments. With respect to other feedback laws, here controller only depends on the single coordinate along the measurement axis, this allows to derive a reduced order filter to achieve exponential stabilization.

This Chapter adapts the ideas found on [22] to exponentially stabilize a qubit. The main difference with said article is the use of an external Brownian noise instead of a dynamical modification of Markovian feedback.

- Chapter 5 extends the noise assisted feedback strategy to a multi-level QND system. The extension differs substantially from the qubit due to the presence of multiple eigenstates orthogonal to the target, and how the feedback Hamiltonian connects the QND eigenstates. The control strategy "shakes" the system away from undesired eigenstates by applying strong noise there, while relying on the open-loop dynamics to reach the target. To show that noise drives the state away undesired eigenstate, methods of graph theory involving the inversion of a Laplacian matrix are used to show exponential decay around the undesired QND eigenstates. Next, analysis done on the Lyapunov function exploits the open-loop dynamics to show exponential decay to the target. Convergence analysis is done using standard stochastic Lyapunov methods; with the two properties said above, the control strategy ensures that the closed-loop Lyapunov function is a supermartingale with exponential decay in all the state space, implying a progressive approach to the target state. The feedback scheme and its stability analysis suggest the use of an approximate filter which only tracks the populations of the eigenstates of the measurement operator. This consists on replacing the terms corresponding to the dissipation induced by the noise with the Laplacian matrix associated with the connectivity graph of the feedback Hamiltonian.

  This Chapter follows mainly [23] with additional explanations.

- Chapter 6 studies the standard quantum error correction using the three-qubit bit-flip code in continuous-time. This entails rendering a target manifold of quantum states globally attractive, that states within the manifold should remain unperturbed. We show that the resulting closed-loop dynamics can be shown to stabilize the target manifold exponentially. There are however some differences with respect to the previous stabilization problems: first, given an initial data on the target manifold, it is necessary to show how the said data is protected from errors; second, control actions must be minimized in order to avoid inducing more errors. It is further presented a reduced-order filter formulation with classical probabilities. The performance of the control law to protect information against disturbances is studied numerically, as the that of the reduced filter. The feedback scheme considered in this chapter is a first extension of the noise assisted feedbcack scheme to a *multi-input multi-output* (MIMO) setting.

The results of this Chapter are based on [23].

Chapter 3

Quantum non-demolition measurements and feedback

Dans ce chapitre, nous introduisons les modèles mathématiques correspondant aux mesures QND en temps continu. Nous presentons le comportement dynamique des systèmes QND à boucle ouverte. Il est montré via une fonction de Lyapunov originale que, pour toute condition initiale, l'état converge de manière exponentielle vers l'ensemble des états propres QND, l'ensemble de vecteurs propres de la matrice hermitienne de la mesure. Cette propriété de «multistabilité stochastique» signifie que le système converge vers l'un des états QND possibles, mais en moyenne, le système ne bouge pas vers un état QND en particulier. Nous présentons ensuite le problème de contrôle, qui consiste à stabiliser de manière exponentielle un état propre QND parmi l'ensemble existant en boucle ouverte. Nous rappelerons aussi les architectures de contrôle existantes. Enfin, nous presenterons la stratégie de commande constituant le coeur de cette thèse.

The concept of measurement and feedback, which are so natural in engineering applications, becomes more subtle in the quantum domain. In order to control the system, a measurement must be performed, but due to the *measurement back-action*, this would invariably perturb the system. The formalism of *stochastic master equations* [25, 11] provides a mathematical framework to model this measurement process. It is then possible to use the information obtained from the measurements to influence the behavior of a quantum system.

Next, we need to make sure that the measurement iteslf does not perturb the target state. A special case of stochastic master equations are those that model *quantum non-demolition* (QND) measurements, which are a continuous-time versions of the projection postulate. The eigenstates of the measurement operator, called QND eigenstates, are equilibria of the measurement dynamics and thus naturally protected from the measurement backaction. Then, a QND measurement can be chosen advantageously such that its eigenstates/eigenspaces contain states/subspaces suitable for quantum computation. Thus, an appropiate QND measurement can itself be considered as an open-loop preparation tool for quantum states [33], although the resulting state will be random. Then, added to a fixed QND measurement, feedback control can be used for preparing a target QND eigenstate with probability one.

Throughout this chapter, we overview some properties of stochastic master equations and in paricular of QND systems, which are considered as our open-loop models. Afterwards we will present the feedback stabilization problem, the previous work around this problem, and lastly address the control methodology that will be pursued in this Thesis.

3.1 Open loop-models: continuous QND measurements

The mathematical setup for quantum measurements in continuous time is that of stochastic master equations, for a more complete introduction, the reader is encouraged to follow the book of A. Barchielli & M. Gregoratti [11] for a mathematical introduction, or the lecture notes of H. Carmichael [25]. We do not consider here discrete-time models, for which the reader is invited to read the previous works [12, 4, 3, 6], addressing discrete-time quantum measurements and feedback control.

We work in a finite dimensional complex Hilbert space $\mathcal{H}$ isomorphic to $\mathbb{C}^n$. The *kets* $|x\rangle$ are complex vectors on $\mathbb{C}^n$ and $\langle x|$ its complex conjugate transpose called *bra*, $\mathrm{Tr}\,(\,\cdot\,)$ denotes the trace operator, $[\,\cdot,\ \cdot]$ denotes the commutator and $\|\cdot\|$ is the norm induced by the inner product $\langle\,\cdot||\cdot\,\rangle$.

The state space is set of density matrices as

$$\mathcal{S} = \{\rho \in \mathbb{C}^{n\times n} : \rho \geq 0, \rho = \rho^\dagger, \mathrm{Tr}\,(\rho) = 1\},$$

$\rho^\dagger$ is the Hermitian conjugate of ρ, $\rho \geq 0$ means that ρ is positive semidefinite, i.e. $\langle\psi|\rho|\psi\rangle \geq 0$ for all $\psi \in \mathcal{H}$. The set of density matrices $\mathcal{S}$ is the convex hull of the set of pure states

$$\mathcal{P} = \{\rho \in \mathbb{C}^{n\times n} : \rho = |\psi\rangle\langle\psi|, \psi \in \mathbb{C}^n, \|\psi\|^2 = 1\},$$

and mixed states are the states for which $\mathrm{Tr}\left(\rho^2\right) < 1$.

The *open-loop systems* describing a quantum system subject to several continuous measurements correspond to Itō stochastic differential equations of the form:

$$\begin{aligned} d\rho &= \sum_\mu \mathcal{D}_{L_\mu}(\rho)dt + \sqrt{\eta}\mathcal{M}_{L_\mu}(\rho)dW\mu\ , \qquad (3.1)\\ dY_\mu &= \sqrt{\eta}\,\mathrm{Tr}\left((L_\mu + L_\mu^\dagger)\rho\right)dt + dW_\mu\ , \end{aligned}$$

where we have defined the superoperators

$$\mathcal{D}(L,\rho) = L\rho L^\dagger - \frac{1}{2}(L^\dagger L\rho + \rho L^\dagger L)\ ,$$
$$\mathcal{M}(L,\rho) = L\rho + \rho L^\dagger - Tr(\rho(L+L^\dagger))\rho\ ,$$

Each W_μ is an independent Brownian motion, i.e. $dW_\mu\, dW_{\mu'} = \delta_{\mu,\mu'}$ and each Y_μ corresponds to an independent measurement output channel [11]. The linear operator L_μ is the *measurement operator*, and each $\eta_\mu \in [0,1]$ represents the corresponding measurement efficiencies, i.e. the ratio of the corresponding channel linking the system to the outside world which is effectively captured by the measurement device; channels μ with $\eta_\mu = 0$ represent pure loss channels.

We recall here the result of existence and uniqueness of solutions of (3.1)

Theorem 3.1.1. *The open-loop system* (3.1) *admits a unique solution* (ρ_t). *The set of states* $\mathcal{S}$ *is a positively invariant set under* (3.1)*, i.e., the solution takes values in the set of states* $\mathcal{S}$ *and it is defined for all* $t \geq 0$.

Proof. This result has been shown, e.g., in [56, 11, 50]. Existence and uniqueness of solutions can be shown via standard arguments of SDE's defined on compact sets. The assertion that $\mathcal{S}$ is a positively invariant set of (3.2) is more subtle, but it can be seen in the following way: remark that, with Itō's rule, Equation (3.2) admits the following formulation

$$\rho_{t+dt} = \frac{\mathbf{M}_{dY}\rho_t\mathbf{M}_{dY}^\dagger + \sum_\mu(1-\eta_\mu)L_\mu\rho_t L_\mu^\dagger dt}{\mathrm{Tr}\left(\mathbf{M}_{dY}\rho_t\mathbf{M}_{dY}^\dagger + \sum_\mu(1-\eta_\mu)L_\mu\rho_t L_\mu^\dagger dt\right)},$$

with $\mathbf{M}_{dY} = I - \frac{1}{2}\sum_\mu L_\mu^\dagger L dt + \sum_\mu \sqrt{\eta_\mu}L_\mu dY_\mu$. Then it becomes clear that Eq. (3.2) preserves trace and positive semi-definiteness of ρ. □

Quantum non-demolition measurements are characterized by a Hermitian measurement operator, i.e., $L_\mu = L_\mu^\dagger$. In this general setting, the open-loop system (3.2) allows to weakly measure non-commuting observables in parallel [37]. As such, operators must commute in order that Eq. (3.2) admits steady states —cf. Chapter 6. We consider here the case of a single QND measurement

$$d\rho = \left(L\rho L - \tfrac{1}{2}L^2\rho - \tfrac{1}{2}\rho L^2\right)dt + \sqrt{\eta}(L\rho + \rho L - 2\,\mathrm{Tr}\,(\rho L)\,\rho)dW, \qquad (3.2)$$

We associate to L its spectral decomposition $L = \sum_{\ell=1}^d \lambda_\ell \Pi_\ell$; λ_ℓ are the d-distinct eigenvalues of L with corresponding projection operators Π_ℓ resolving the identity. We define the *population* of the eigenspace associated with Π_ℓ as

$$\boldsymbol{p}_\ell(\rho_t) := \mathrm{Tr}\,(\rho_t\Pi_\ell) \geq 0, \quad \sum_\ell \mathrm{Tr}\,(\rho_t\Pi_\ell) = 1,\ \forall t \geq 0.$$

We summarize in the following Lemma, the asymptotic behaviour of an open-loop QND system with a single measurement. It formalizes the point of view of a QND measurement as a continuous-time analog of the projection postulate [71, 50, 16]; in addition, we provide an estimate of the exponential convergence rate towards the set of steady states via an original Lyapunov function.

Lemma 3.1.1. *Consider the open-loop system* (3.2) *and the subspace populations* $\boldsymbol{p}_\ell(\rho)$, $1 \leq \ell \leq d$. *Then*

(i) For any ℓ, *the subspace population* $\boldsymbol{p}_\ell(\rho_t)$ *is a martingale for all* $t \geq 0$, *i.e.* $\mathbb{E}[\boldsymbol{p}_\ell(\rho_t)] = \boldsymbol{p}_\ell(\rho_0)$.

(ii) Given an initial condition $\rho_0 \in \mathcal{S}$, *if there exists* ℓ *such that* $\boldsymbol{p}_\ell(\rho_0) = 1$ *and* $\boldsymbol{p}_{k\neq\ell}(\rho_0) = 0$, *then* ρ_0 *is a steady state of* (3.2).

(iii) The Lyapunov function

$$V_o(\rho) = \sum_{k=1}^{d} \sum_{k'<k} \sqrt{\boldsymbol{p}_k(\rho)}\sqrt{\boldsymbol{p}_{k'}(\rho)}, \tag{3.3}$$

decreases exponentially as

$$\mathbb{E}[V_o(\rho_t)] \leq \exp(-rt)V_o(\rho_0),$$

for all $t \geq 0$ *and* $\rho_0 \in \mathcal{S}$ *with rate*

$$r = \tfrac{1}{2}\eta \min_{k,k'}(\lambda_k - \lambda_{k'})^2.$$

In this sense, system (3.2) *converges towards the set of invariant states described in point* (ii).

Proof. For each ℓ, the subspace populations $\boldsymbol{p}_\ell(\rho)$ satisfy the stochastic differential equation (SDE)

$$d\boldsymbol{p}_\ell(\rho) = \mathrm{Tr}\left((L\rho L - \tfrac{1}{2}L^2\rho - \tfrac{1}{2}\rho L^2)\Pi_\ell\right)dt - \sqrt{\eta}\,\mathrm{Tr}\left((L\rho + \rho L - 2\,\mathrm{Tr}\left(\rho L\right)\rho)\Pi_\ell\right)dW$$
$$= \mathrm{Tr}\left(\rho(L\Pi_\ell L - \tfrac{1}{2}L^2\Pi_\ell - \tfrac{1}{2}\Pi_\ell L^2)\right)dt - 2\sqrt{\eta}(\,\mathrm{Tr}\left(L\Pi_\ell\rho\right) - \mathrm{Tr}\left(\rho L\right)\mathrm{Tr}\left(\rho\Pi_\ell\right))dW$$
$$= 2\sqrt{\eta}(\lambda_\ell - \sum_{s=1}^{d}\lambda_s\boldsymbol{p}_s(\rho))\boldsymbol{p}_\ell(\rho)dW.$$

Taking the expectation yields $\frac{d}{dt}\mathbb{E}[\boldsymbol{p}_\ell(\rho)] = 0$, so indeed $\boldsymbol{p}_\ell(\rho)$ is a martingale, i.e. $\mathbb{E}[\boldsymbol{p}_\ell(\rho_t)\ ] = \boldsymbol{p}_\ell(\rho_0)$, $\forall t \geq 0$.

Take ρ_0 such that $\boldsymbol{p}_\ell(\rho_0) = 1$. Plugging into Eq. (3.2) we have

$$\mathcal{D}_L(\rho_\ell) = L\rho_0 L - \tfrac{1}{2}L^2\rho_0 - \tfrac{1}{2}\rho_0 L^2 = \lambda_\ell^2\rho_0 - \tfrac{1}{2}(2\lambda_\ell^2\rho_0) = 0,$$

and

$$\mathcal{M}_L = \sqrt{\eta}(L\rho_0 + \rho_0 L - 2\,\mathrm{Tr}\,(\rho_0 L)\,\rho_0) = 2\sqrt{\eta}(\lambda_\ell \rho_0 - \mathrm{Tr}\,(\lambda_\ell \rho_0)\,\rho_0) = 0.$$

Thus ρ_0 is a steady state of (3.2).

As for point (iii), V_o is a positive definite function on $\mathcal{S}$ and 0 only when $\boldsymbol{p}_\ell(\rho) = 1$ for some ℓ. It remains to check that it is a supermartingale with exponential decay. From Itō's formula, the variable $\xi_k := \sqrt{\boldsymbol{p}_k}$ satisfies the SDE

$$d\xi_k = -\tfrac{1}{2}\eta(\lambda_k - \varpi(\xi))^2 \xi_k dt + \sqrt{\eta}(\lambda_k - \varpi(\xi))\xi_k dW, \tag{3.4}$$

with $\xi = (\xi_k)_{1\le \ell \le d}$ and $\varpi(\xi) = \sum_{s=1}^d \lambda_s \xi_s^2$.

Under the ξ coordinates,

$$V_o(\xi) = \sum_{k'=1}^{d} \sum_{k'<k} \xi_k \xi_{k'}.$$

We detail the computations for a single product of $\xi_k\xi_{k'}$, as the other terms in the summation are computed in the same manner. From Itō's formula

$$\begin{aligned}
d(\xi_k\xi_{k'}) &= (d\xi_k)\xi_{k'} + \xi_k(d\xi_{k'}) + (d\xi_k)(d\xi_{k'})\\
&= -\tfrac{1}{2}\eta(\lambda_k - \varpi(\xi))^2 \xi_k\xi_{k'} dt + \sqrt{\eta}(\lambda_k - \varpi(\xi))\xi_k\xi_{k'} dW\\
&\quad - \tfrac{1}{2}\eta(\lambda_{k'} - \varpi(\xi))^2 \xi_k\xi_{k'} dt + \sqrt{\eta}(\lambda_{k'} - \varpi(\xi))\xi_k\xi_{k'} dW\\
&\quad + \eta(\lambda_k - \varpi(\xi))(\lambda_{k'} - \varpi(\xi))\xi_{k'}\xi_k dt\\
&= -\tfrac{1}{2}\eta(\lambda_k - \lambda_{k'})^2 \xi_{k'}\xi_k dt + \sqrt{\eta}(\lambda_k + \lambda_{k'} - 2\varpi(\xi))\xi_k\xi_{k'} dW.
\end{aligned}$$

The expectation satisfies $d\mathbb{E}[\xi_k\xi_{k'}] = -\frac{1}{2}\eta(\lambda_k - \lambda_{k'})^2 \mathbb{E}[\xi_{k'}\xi_k]dt$. Then the Markov generator ,detailed in Appendix (A.3), associated to $\sum_{k'=1}^d \sum_{k'<k} d\mathbb{E}[\xi_k\xi_{k'}]$ reads

$$\mathcal{A}V_o = -\frac{\eta}{2} \sum_{k'=1}^{d} \sum_{k'<k} (\lambda_\ell - \lambda_{\ell'})^2\, \xi_k\xi_{k'}.$$

Since each component of $\xi(t)$ remains non-negative for all t, this readily yields

$$\mathcal{A}V_o \le -\frac{\eta}{2}\left(\min_{k',k\neq k'} (\lambda_k - \lambda_{k'})^2\right) V_o.$$

By Theorem A.0.1 in Appendix, V_o decays exponentially towards zero, implying that solutions of (3.2) converge with exponential speed towards the set of states described in point (ii). □

3.2 Control problem and existing feedback designs

QND measurements are a common element in measurement based feedback. Part of this consideration is the practical interest of such measurements for engineering quantum systems: from property *ii*) of Lemma 3.1.1, steady states remain unperturbed from the measurement process. On the other hand, QND measurements pose interesting challenges: as seen in Lemma 3.1.1, the open-loop system stochastically converges to one of a few steady-state situations, but on the average it does not move closer to any particular one. It is then our goal to bias this average to a target eigenstate.

The control objective is to ensure convergence to a target QND eigenstate, indexed by $\ell \in \{1, \dots, d\}$ for all realizations. More precisely, we will design a real-valued continuous stochastic control process v, depending on the state ρ, such that $\lim_{t\to\infty} \mathbb{E}[\boldsymbol{p}_\ell(\rho)] = 1$ with exponential convergence rate, for any initial condition $\rho_0 \in \mathcal{S}$.

Feedback actions are incorporated as a unitary actuators of the form $U_t = \exp(-iHdv)$, where $H = H^\dagger$ is the actuation Hamiltonian, and dv is the control process that drives the actuator. To the QND dynamics (3.1), we add unitary actuators of form

$$\rho_{t+dt} = \sum_\nu U_\nu(t)(\rho_t + d\rho_t)U_\nu(t)^\dagger \tag{3.5}$$

with m measurements channels $\{Y_\mu\}_{1\le\mu\le m}$.

We aim to showing exponential convergence via global Lyapunov functions $V(\rho)$ such that $V(\rho_t)$ is a supermartingale with exponential decay for all $t \geq 0$ and for all $\rho_0 \in \mathcal{S}$.

3.2.1 Static output feedback

We consider the simplest possible feedback control scheme, a proportional output feedback of the form

$$dv = f dt + \sigma dY,$$

where f, σ are constants. Since the control signal depends on a stochastic process, care is needed in order to derive the closed-loop dynamics.

Since dv depends on a stochastic proces, care is needed to derive the closed-loop dynamics. The derivation of the closed loop dynamics was first done in [73], here we only provide a simpler formulation. We relate Eq. (3.1) with (3.5) via the Baker-Campbell-Hausdorff (BCH) formula

$$e^{xH}Pe^{-xH} = \sum_j T_j \frac{x^j}{j!},$$

the terms T_j of the series expansion on the right hand side of are defined recursively as $T_0 = P$ and $T_{j+1} = [H, T_j]$ for $j \geq 0$. We identify the right hand side up-to second order of the BCH formula with Itō's formula. Take $P = \rho_t + d\rho_t$ and consider Itō rules ($dW^2 = dt$, $dWdt = dt^2 = 0$). The closed-loop equation reads

$$
\begin{aligned}
d\rho = -if[H,\rho]dt - i\sigma\sqrt{\eta}[H, L\rho+\rho L]dt \\
+ \mathcal{D}_L(\rho)dt + \sigma^2\mathcal{D}_H(\rho)dt \\
+ \sqrt{\eta}\mathcal{D}_L(\rho)dW - i\sigma[F,\rho]dW.
\end{aligned}
$$

The simplicity of the feedback scheme makes it attractive for experimental implementations, since there is no control processing overhead in the feedback loop. This feedback equation has been derived as well using the rules of the quantum stochastic calculus [35]. The next Lemma shows that the controller (3.6) cannot be tuned to achieve global asymptotic stability towards a prescribed QND eigenstate Π_ℓ.

Lemma 3.2.1. *Consider the closed-loop system* (3.6) *with* $L = L^\dagger$. *There does not exist* H, f *and* σ *such that the closed-loop system* (3.6) *converges towards a chosen QND closed-loop steady-state, i.e.* Π_ℓ *for some chosen* $\ell \in \{1, \dots, d\}$.

Proof. If Π_ℓ is a steady-state of (3.6) and since f, σ are constants, then

$$-if[H,\Pi_\ell] - i\sigma\sqrt{\eta}[H, L\Pi_\ell + \Pi_\ell L] + \mathcal{D}_L(\Pi_\ell) + \sigma^2\mathcal{D}_H(\Pi_\ell) = 0,$$

and

$$\sqrt{\eta}\mathcal{M}(L,\Pi_\ell) - i\sigma[H,\Pi_\ell] = 0.$$

Since $L\Pi_\ell = \lambda_\ell\Pi_\ell$, the second condition implies that $\sigma[H,\Pi_\ell] = 0$. By plugging this into the other steady-state constraint, we get $f[H,\Pi_\ell] = 0$ and $\sigma^2\mathcal{D}_H(\Pi_\ell) = -\frac{\sigma^2}{2}[H,[H,\Pi_\ell]] = 0$. If $f = \sigma = 0$, we are in open-loop and Π_ℓ is not globally asymptotically stable. When σ or f are not zero, we must have $[H,\Pi_\ell] = 0$. Then we have $d\,\mathrm{Tr}\,(\rho\Pi_\ell) = 0$. Thus $\mathrm{Tr}\,(\rho_t\Pi_\ell) = \mathrm{Tr}\,(\rho_0\Pi_\ell)$ is a time invariant and we cannot have convergence towards ρ_ℓ. □

In this static output feedback [74, 72], we can consider the role of the control as to advantageously changing the dissipation operators of the average dynamics, as it can be rewritten as

$$
\begin{aligned}
d\rho = -if[H,\rho]dt - \tfrac{i\sigma\sqrt{\eta}}{2}[HL+LH,\rho]dt \\
+ \mathcal{D}_{(L-i\sigma\sqrt{\eta}H)}(\rho)dt + \mathcal{D}_{(i\sigma\sqrt{1-\eta}H)}(\rho)dt \\
+ \sqrt{\eta}\mathcal{M}_{(L-i\sqrt{\eta}\sigma H)}(\rho)dW - (1-\eta)i\sigma[H,\rho]dW. \quad (3.6)
\end{aligned}
$$

Markovian feedback is constrained to having measurement efficiency $\eta = 1$ in order to use this feedback for stabilizing a target pure state. When $\eta < 1$, the state is subject to significant noise, namely, purity is limited by the dissipation term $\mathcal{D}_{(i\sigma\sqrt{1-\eta}H)}(\rho)$. With perfect measurement efficiency, exponential stability of the target state follows from algebraic arguments, provided that there is some liberty on adding open loop Hamiltonians, say H_o, $\tilde{H} = H + H_o$. The following theorem, taken from [64, 65] gives general conditions for stabilization under static output feedback.

Theorem 3.2.1. *Consider the closed-loop system* (3.6) *with* $\eta = 1$. *Set the Hilbert space decomposition* $\mathcal{H} = \mathcal{H}_S \oplus \mathcal{H}_R$ *and let* Π_S, Π_R *be the orthogonal projections on* $\mathcal{H}_S, \mathcal{H}_R$ *respectively. Under the Matrix decomposition*

$$A = \begin{bmatrix} A_S & A_P \\ A_Q & A_R \end{bmatrix}, \tag{3.7}$$

let $\tilde{L} = L - iH$ *and* $\tilde{H} = LH + HL^\dagger + H_o$ *fullfil the conditions* $\tilde{L}_Q = 0$ *and* $\tilde{H}_P = -i\frac{1}{2}\tilde{L}_S^\dagger \tilde{L}_P$.

Then for any $\rho_0 \in \mathcal{S}$ *the solution of Eq.* (3.6) *converges to the largest invariant set contained in*

$$\{\rho \in \mathcal{S} : \mathrm{Tr}\,(\Pi_R \rho) = 0\}, \tag{3.8}$$

Moreover, if the invariant set (3.8) *consists of a single pure state* ρ_d, *then* $\mathbb{E}(\rho_t) \to \rho_d$ *with exponential rate given by the minimum eigenvalue of* $\tilde{L}_P^\dagger \tilde{L}_P$.

Given a fixed QND measurement operator L and for a target state ρ_d such that $[\rho_d, L] \neq 0$, there always exist a feedback Hamiltonian H such that ρ_d is globally stable. The explicit construction is not trivial and depends on the specific form of L, explicit algorithms are provided in [64, 63]. The need for a single pure state goes back to [34].

To summarize, static output feedback is a simple feedback that avoids the overhead of needing a quantum state estimator to compute the control law; furthermore, states stabilizable under this feedback are exponentially stable. The main observation that we can take from this approach, is the use of a stochastic process, the measurement output, to drive the control fields. Just keeping f and σ constant does not allow for QND stabilization, so a dynamical approach is necessary. State feedback is one alternative.

3.2.2 State feedback

State feedback is a control approach where a the control signal is dependent on the quantum state ρ, computed through a quantum filter. Previous works [31, 32] have considered a closed-loop model where the control signal is a function of the state implemented through a deterministic actuation, that is

$$dv = f(\rho)dt$$

which corresponds to a closed-loop equation of the form

$$d\rho = -if(\rho)[H, \rho]dt + \mathcal{D}_L(\rho)dt + \sqrt{\eta}\mathcal{M}_L(\rho)dW. \tag{3.9}$$

QND eigenstate stabilization by quantum state feedback

Stabilization by quantum state feedback of QND eigenstates has been explored and solved in several papers [71, 76, 50, 69]. Although they all start with Lyapunov control techniques and the open-loop martingale

$$\tilde{V}(\rho) = 1 - \mathrm{Tr}\,(\rho\Pi_\ell)\,, \tag{3.10}$$

they also involve specific stochastic arguments combining arguments on the support of the closed-loop trajectories with Doob inequalities in order to establish the global asymptotic almost sure convergence for the closed-loop system [50, 69]. Let us explain the basic reason why it is difficult, as already highlighted in [71], to construct quantum state feedback and a strict closed-loop Lyapunov function in this case.

From (3.9), the Lyapunov function candidate (3.10) satisfies $d\mathbb{E}[\tilde{V}(\rho)] = f(\rho)\,\mathrm{Tr}\,(i[H,\rho]\Pi_\ell)\,dt$. The quantum state feedback

$$f(\rho) = -\,\mathrm{Tr}\,(i[H,\rho]\Pi_\ell) \tag{3.11}$$

makes this expectation decreasing. It does not mean, that for all realizations, $\mathrm{Tr}\,(\rho_t\Pi_\ell) \to 1$ as $t \to \infty$. In particular, any QND state $\Pi_{\ell'}$ with $\ell' \neq \ell$ is a closed-loop steady state since $\mathrm{Tr}\,(i[F,\Pi_{\ell'}]\Pi_\ell) = 0$, for any choice of H. In a stochastic convergence setting, such steady states are harder to treat, as there will be realizations that can even converge to maxima of a stochastic Lyapunov function.

It could be tempting to follow the deterministic intuition and try to define a local region of atraction. Unfortunately, due to the stochastic nature of the problem, a local stability result on its own is of no interest. To develop further on this remark, let $0 < \gamma < 1$ and consider level sets like $\mathcal{Q}_{<1-\gamma} = \{\rho = \tilde{V}(\rho) < 1-\gamma\}$, similarly $\mathcal{Q}_{<1-\gamma/2}$, so $\mathcal{Q}_{<1-\gamma} \subset \mathcal{Q}_{<1-\gamma/2}$. Then the set $\mathcal{S} \setminus \mathcal{Q}_{<1-\gamma/2}$ contains all QND eigenstates $\Pi_{\ell'}$, $\ell' \neq \ell$, and $\frac{d}{dt}\mathbb{E}[V(\rho)] \leq 0$ for all $\rho \in \mathcal{Q}_{<1-\gamma/2}$ with the control the control 3.11. In the deterministic theory this is enough to show that any trajectory that starts at $\mathcal{Q}_{<1-\gamma/2}$ remains there, and it converges towards Π_ℓ. Unfortunately for, stochastic systems, this statement is weaker: Doob's supermartingale inequality (see e.g., [42, Chapter 2]), provides a bound in probability Pr for exiting the set $\mathcal{Q}_{<1-\gamma/2}$ with initial condition on $\mathcal{Q}_{<1-\gamma}$. We have for all $\rho_0 \in \mathcal{Q}_{<1-\gamma}$

$$\Pr\big(\sup_{0\leq t<\infty} \tilde{V}(\rho_t) \geq 1/2\big) \leq \frac{V(\rho_0)}{1-\gamma/2} \leq \frac{1-\gamma}{1-\gamma/2} < 1.$$

The above probability estimate holds for any $0 < \gamma < 1$. The question is now whether or not this probability estimate gives any relevant information on how well the control improves convergence to the target. Take any initial state diagonal on the eigenbasis of L, and such that $1 - \mathrm{Tr}\,(\rho\rho_0) < 1 - \gamma$.

With this initial condition, $f(\rho_0) = -\operatorname{Tr}(i[H,\rho_0]\Pi_\ell) = 0$ and Eq. (3.9) remains in open-loop regime for all $t \geq 0$, i.e. Π_ℓ does not attract the state more than any other eigenstate $\Pi_{\ell'}$ for $\ell' \neq \ell$. This illustrates that a closed-loop Lyapunov function having a minimum at $\rho = \Pi_\ell$ does not automatically hint at the fact that the latter is more attractive than other critical points.

To achieve a global stability result, analytical approaches have followed a stabilization strategy that consists on defining a local stability region $\mathcal{Q}$, where $V(\rho)$ is decreasing in expectation, and to show that trajectories that start outside, or that exit such level set will attain back in finite time. The controls are explicitly designed to apply perturbations which remove the undesired steady states [50, 70]. These previous works have only considered the context of quantum spin systems (cf. Section 5.5 of Chapter 5), that is, systems of dimension $n = 2J + 1$ where the measurement operator L is of the form

$$L = \sum_{m=0}^{2J} (J-m)|J\text{-}m\rangle\langle J\text{-}m|$$

and the actuator Hamiltonian H is a tri diagonal matrix

$$H = \sum_{m=0}^{2J-1} \frac{\sqrt{(m+1)(2J-m)}}{2i} \left(|J\text{-}m\rangle\langle J\text{-}m\text{-}1| - |J\text{-}m\text{-}1\rangle\langle J\text{-}m|\right)$$

thus $\Pi_\ell = |J\text{-}\ell\rangle\langle J\text{-}\ell|$, $\ell = 0, \ldots, 2J$. The biggest limitation when constraining to quantum spin systems is that the structure of the actuator Hamiltonian, that is, the way it couples different eigenstates, could change on other types of physical systems.

In [50], global stabilization of quantum spin systems is achieved by introducing a control with hysteresis that gives a constant input whenever $\tilde{V}(\rho)$ is close to one, and then switching to the Lyapunov feedback 3.11 once ρ is close enough to the goal state Π_ℓ. The control law is the following

Control Law 3.2.1 (Mirrahimi & Van Handel [50])**.** *Consider the system* (3.9) *evolving in the set S. Let* Π_ℓ *be a rank-1 projector of* L *and let* $\gamma > 0$. *Consider the following control law:*

1. $f(\rho_t) = -\operatorname{Tr}(i[H,\rho_t]\Pi_\ell)$ *if* $\operatorname{Tr}(\rho_t\Pi_\ell) \geq \gamma$.

2. $f(\rho_t) = 1$ *if* $\operatorname{Tr}(\rho_t\Pi_\ell) \leq \gamma/2$.

3. *If* $\rho_t \in \mathcal{B} = \{\rho : \gamma/2 < \operatorname{Tr}(\rho\Pi_\ell) < \gamma\}$, *then* $f(\rho_t) = -\operatorname{Tr}(i[H,\rho]\Pi\ell)$ *if* ρ_t *las entered* $\mathcal{B}$ *through the boundary* $\operatorname{Tr}(\rho\Pi_\ell) = \gamma$, *and* $f(\rho_t) = 1$ *otherwise.*

Then there exists $\gamma > 0$ *s.t.* $f(\rho_t)$ *globally stabilizes* (3.9) *around* Π_ℓ *as* $t \to \infty$.

The design of the hysteresis could pose problems for real-time implementation. Moreover, the presence of a constant input could excite spurious dynamics that are often neglected when performing the approximations yielding to the closed-loop model. This motivates the search of smooth control laws.

Tsumura [70] proposes a continuous approach, again for quantum spin systems, where the control is perturbed by the addition of a term that only vanishes on the target state. The control law is

Control Law 3.2.2 (Tsumura [70])**.** *The control law*

$$f(\rho) = -\alpha \operatorname{Tr}(i[H,\rho]\Pi_\ell) + \beta(\lambda_\ell - \operatorname{Tr}(\rho L)), \quad \alpha, \beta > 0, \quad \frac{\beta^2}{8\alpha\eta} < 1,$$

globally stabilizes (3.9) *around* Π_ℓ *as* $t \to \infty$.

The convergence proof then involves a combination of Lyapunov function on part of the state space, and probabilistic arguments on the other part. With the feedback (3.2.2), the set $\{\rho \in \mathcal{S} : V(\rho) \leq \frac{\beta^2}{8\alpha\eta}\}$ satisfies $d\mathbb{E}(V) \leq 0$ defining a kind of attraction set for the control Lyapunov function. Thus trajectories that do not exit this region of attraction will converge to the target. Outside the region of attraction, both feedback schemes 3.2.1 and 3.2.2 rely on ensuring that trajectories will reach again this region of attraction. This involves specific arguments of the closed-loop trajectories involving the support associated to the SDE (3.9). We refer to [50, 69, 70] for more details.

While convergence is proven in a proper probabilistic sense, all those methods only establish asymptotic convergence. Having a stronger global convergence result like exponential convergence is important for robustness and estimation of convergence speed. The robustness is practically important in particular towards unmodeled dynamics. This could concern other system elements like actuators, but at very least there is the quantum filter which estimates the state ρ from the measurement inputs with a finite convergence speed. The convergence speed is particularly important in applications where feedback control is applied to protect the fragile quantum systems from perturbing effects, like decoherence. The strongest result with deterministic actuation, as far as our knowledge goes, is given in Liang et. al. [44]; the authors provide an estimate of the Lyapunov exponent for a qubit, valid for the final approach of the target state after an unspecified final initial transient.

As far as we know, a state feedback scheme ensuring global exponential convergence via simple Lyapunov arguments has remained an open issue.

Engineering issues: implementation of the closed-loop filter

The implementation of the quantum filter is not without challlenges: when the dimension n of the system is large, the filter requires to store and update in real-time the $n(n-1)/2$ components of the estimated density matrix. This estimate is computed via the quantum filter:

$$d\hat{\rho} = -if(\hat{\rho})[H,\hat{\rho}]dt + \mathcal{D}_L(\hat{\rho})dt + \sqrt{\eta}\mathcal{M}_L(\hat{\rho})(dY - 2\,\mathrm{Tr}\,(\hat{\rho}L)\,dt). \qquad (3.12)$$

Usually, it is assumed that $\rho_0 = \hat{\rho}_0$. Then, standard results in quantum filtering theory [18, 5, 50, 19] guarantee that, the quantum filter (3.12) is the best estimate of the true state ρ conditioned to the observations Y_τ, $\tau \in [0\,,\,t]$, that is, $\hat{\rho}_t = \mathbb{E}[\rho\,|\,Y_{0\leq\tau\leq t}]$. A separation principle [18] allows us to just consider (3.9) as the closed-loop system. This mathematical property is useful to do control design, since one needs to analyze only the system (3.9) rather than analyzing the extended plant-observer system $(\rho,\hat{\rho})$ described by Equations (3.9) and (3.12).

In practice the filter (3.12) must be computed in real-time. This can pose several challenges in experimental settings; as the dimension of quantum systems grows, the implementation of the filter under experimental constraints— like the lifetime of the real quantum system— becomes unpractical. Measurement-based quantum feedback that relies as less as possible on real-time computations is the way to go for the coming years. A first step towards this goal is on finding opportunities for formulating reduced order filters as to avoid to compute the conditional density matrix in real time.

From our point of view, for QND systems the biggest hint towards a reduction of the filter comes from the structure fixed by the measurement operator L. In open-loop, the dynamics can be reduced exactly to the d-equations of the populations $\boldsymbol{p}_k(\rho)_{1\leq k\leq d}$ (cf. Lemma 3.1.1), that is, it is only necessary to track the diagonal elements of ρ in the eigenbasis fixed by L. In the current setting, the only term that poses problems for a similar reduction in Eq. (3.9) is the actuation term $-i\,f(\rho)[H,\rho]dt$. Indeed, from any $j \in \{1,\ldots,d\}$, $\boldsymbol{p}_j(\rho)$ satisfies

$$d\boldsymbol{p}_j(\rho) = -f(\rho)\,\mathrm{Tr}\,(i[H,\Pi_j]\rho)\,dt + 2\sqrt{\eta}((\lambda_j - \sum_{s=1}^{d}\lambda_s\boldsymbol{p}_s(\rho)))\boldsymbol{p}_j(\rho)dW,$$

The actuation term $\mathrm{Tr}\,(i[H,\Pi_j]\rho)$ cannot be expressed as linear combinations of the populations $\{\boldsymbol{p}_k(\rho)\}_{1\leq k\leq d}$. If that would be the case, then $i[H,\Pi_j]$ would be diagonal in the eigenbasis of L, thus $0 = [i[H,\Pi_j],L] = -[i[L,H],\Pi_j]$, but this cannot be possible as it is assumed that $[H,L]\neq 0$ to enable non-trivial transitions. The term $\mathrm{Tr}\,(i[H,\Pi_j]\rho)$ corresponds to off-diagonal elements of the density matrix ρ in the eigenbasis fixed by the measurement eigenprojectors— the so called *quantum coherences.*

It seems that a deterministic actuation strategy precludes any significant reduction for the quantum filter. In contrast to static output feedback, where there is no control processing overhead involved, this seems dissapointing. There have been efforts two bridge the gap between the two quantum feedback architechtures, in order to keep the control processing overhead at a minimum, as it appears unnecessary for QND eigenstate stabilization. In the present task, the asymptotic behavior is directly visible on the measurement signal. Indeed, the measurement signal corresponding to $\boldsymbol{p}_k(\rho_t) = 1$ for all $t \geq 0$ is $Y_t = Y_0 + 2\lambda_k t + W_t$, with thus an expectation that directly informs on the eigenstate via the drift $\lambda_k\, t$, and a standard deviation in $\sqrt{t}$. This suggests to use the measurement output explicitly in the feedback loop [48, 77, 22]. Recent works have presented schemes that use the measurement signal in the feedback loop with a quantum filter, that is, the controls are of the form

$$dv_t = f_t dt + \sigma_t dY.$$

In [48, 77], local optimal controls are formulated in having f_t and σ_t as state-dependent controls, but do not provide stability statements. Moreover, in those works they have proposed the use of pre-computed controls that avoid the use of a quantum filter. We proposed a similar feedback scheme in [22] restricted for the case of a qubit. In this case we constructed controls $f(\rho),\ \sigma(\rho)$ to exponentially stabilize the eigenstate of a qubit measurement operator. Moreover, we showed that the controller did not needed to track quantum coherences, it only depended on tracking the populations on the eigenstates of the qubit, but we did not went further on showing a simplified setup that avoided the use of a full quantum filter.

We noticed that the main contribution for these results, was the use of the measurement output as a particular choice of a stochastic process driving the control. In this case, the measurement output is *correlated* with the Brownian motion corresponding to the measurement backaction. The main idea of this thesis revolves around the use of *exogenous* and *independent* Brownian motions to drive the control fields.

3.3 Main contribution: noise-assisted feedback stabilization

There are two main issues that we want to address for stabilization of a QND eigenstate:

- *Achieve exponential stabilization of a prescribed QND eigenstate.* Lemma 3.1.1 indicates that the open-loop system (3.2) approaches exponentially the set of QND eigenstates, the resulting state being at random. Then a point of view on the role of feedback is that it has to discourage the system to converge towards any undesired situation. The challenge is to show that this procedure induces exponential convergence towards the target state.

- *Identify opportunities towards the implementation of efficiently computable controls on an experimental setup.* Global stabilization of a QND eigenstate can be achieved by using a quantum filter, but the explicit dependence of known control laws on quantum coherences preclude a simpler implementation. Developing feedback controls that are dependent only on observed quantities, like monitoring the population on a target eigenstate, and the formulation of reduced models that avoid the computation of the full quantum state.

The main idea to address this two problems is

Use an external noise to drive the actuator.

Here the control law dv present on the unitary actuation e^{-iHdv} will have attached a gain computed via a quantum filter, but now we let the state feedback signal dv to be driven by Brownian noise, i.e. $dv = \sigma(\rho_t)dB$, where B_t a standard Brownian motion independent of W_t. We will construct controls continuously differentiable $\sigma(\rho)$. Our approach for m measurements and c controls translates to the closed-loop SDE:

$$d\rho = \sum_{\mu=1}^{m} \mathcal{D}_{L_\mu}(\rho)dt + \sqrt{\eta}\mathcal{M}_{L_\mu}(\rho)dW + \sum_{\nu=1}^{c} \sigma_\nu(\rho)^2 \mathcal{D}_{H_\nu}(\rho)dt - i\sigma_\nu(\rho)[H_\nu, \rho]dB_\nu, \tag{3.13}$$

Where $dB_\nu\, dB_{\nu'} = \delta_{\nu,\nu'}dt$, $dW_\mu\, dW_{\mu'} = \delta_{\mu\mu'}dt$, $dB_\nu dW_\mu = 0$.

Theorem 3.3.1. *Let $\sigma(\rho)$ be a smooth function of ρ. Then the closed-loop system* (3.13) *admits a unique solution on the set $\mathcal{S}$.*

Proof. From Eq. (3.13), the smooth control $\sigma(\rho)$ is bounded since it is defined on a compact set and the terms $\mathcal{D}_H(\rho) = -\frac{1}{2}[H, [H, \rho]]$ and $-i[H, \rho]$ are Lipschitz in ρ. Terms corresponding to the QND measurement fullfills the existence and solution properties by Theorem (3.1.1). Putting all together, existence and uniqueness of solutions on (3.13) follows from standard arguments of SDE's. □

The way we address the two problems at the beginning of this section is

- *Exponential stabilization via noise-assisted feedback.* The main idea relies on the fact that the open loop system (3.2) stochastically converges to one of a few steady-state situations, i.e. to a state supported on one of the eigenspaces of the measurement operator, but on the average does not move closer to any particular state. Thus it suffices that the controller tracks the eigenspace populations $\{\boldsymbol{p}_k(\rho)\}_{1\leq k\leq d}$, that is, the diagonal elements on the eigenbasis fixed by the measurement.

  Control design consists then on activating noise only when the state is close to a bad equilibrium, in order to "shake it" away from it.

As the control selectively rejects the undesired situations, we rely on the open-loop dynamics to progressively reach the target. We will show that this procedure induces global exponential convergence to a target eigenstate Π_ℓ. Convergence analysis is done by defining global Lyapunov functions that are supermartingales with exponential decay for all initial condition.

- *Reduced order filtering.* Moreover, we address the design of reduced filters that only tracks the eigenspace populations. The construction relies first on the fact that B_ν is a Brownian motion independent of each internal Brownian motion W_μ, and thus independent of the history of all the measurements channels. Thus, the filter (3.13) follows the same conditional statistics than

$$d\hat{\rho} = \sum_{\mu=1}^{m} \mathcal{D}_{L_\mu}(\hat{\rho})dt + \sqrt{\eta\mu}\mathcal{M}_{L_\mu}(\hat{\rho})dW + \sum_{\nu=1}^{c} \sigma_\nu(\hat{\rho})^2 \mathcal{D}_{H_\nu}(\hat{\rho})dt, \quad (3.14)$$

  where $\hat{\rho}_t = \mathbb{E}[\rho_t|\{Y_\mu(\tau)\}_{1\leq\mu\leq m}, \tau \in [0,t]]$. With this reduction, we bypass the need of tracking the quantum coherences that were present under the deterministic actuation of the filter (3.9). In the sequel we will address how to provide either direct reductions, or approximations of this conditional filter by a filter that only tracks the populations of the measurement operator. With this we have effectively a filter that computes classical probabilities.

Chapter 4

Exponential stabilization of a qubit

Dans ce nous utilisons le contrôle proportionnel et le contrôle assisté par bruit pour stabiliser de manière exponentielle un qubit sujet aux mesures QND en temps continu. Dans le contrôle proportionnel, il peut être facilement montré que presque tout état pur peut être stabilisé de manière exponentielle, à condition de disposer d'une efficacité de mesure parfaite. Néanmoins, lorsque l'on vise des états proches a un état propre QND du qubit, le taux de convergence décroît vers zéro. Pour aborder la stabilisation des deux états propres QND de l'opérateur de mesure, nous utilisons l'approche assistée par bruit. La stabilisation exponentielle est montrée via des arguments classiques de la théorie stochastique de Lyapunov. En ce qui concerne la loi de contrôle, nous developpons une loi du contrôle qui ne dépend uniquement que de la coordonnée correspondant à l'axe de mesure, ce qui permet de déduire un filtre d'ordre réduit pour obtenir une stabilisation exponentielle.

4.1 Introduction

The low dimension and simple topology of a qubit make the control problem easy to visualize. We will use this setup to illustrate the control methodology and associated implementation questions. We evaluate the use of static output strategy which, with perfect measurement efficiency, stabilizes exponentially any state but the QND eigenstates of the qubit. Then we show how the use of noise-assisted feedback aexponentially stabilizes an eigenstate of the measurement operator.

Lastly, we present a proposal for a reduced order filter that could be efficiently computed in practice. Regarding the feedback structure, static output feedback avoids the overhead of a state estimator. With noise assisted feedback we need a quantum state observer to compute the control law, but the situation is not as bad as it seems: first, the control law only depends

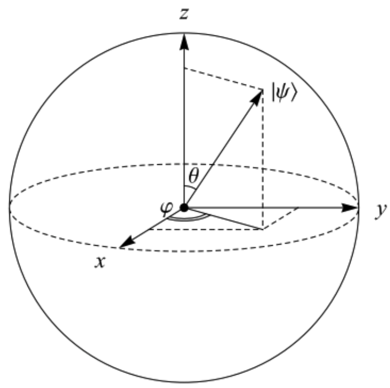


Figure 4.1: Angles θ and ϕ on the Bloch sphere.

on tracking the eigenstates of the measurement operator. Second, the use of noise allows to formulate a reduced order filter that only computes these eigenstates, thus avoiding to compute any other off-diagonal element of the density matrix. This may not seem as a large gain in a qubit system, but it serves as a proof of principle that will be revisited in later chapters.

The results of this chapter are based on the investigations done in [22], published in the Proceedings of the 2018 IEEE Conference on Decision and Control.

4.2 Qubit system

Qubit systems are defined on a two-dimensional Hilbert space $\mathcal{H} = \mathbb{C}^2$. Since the state ρ is Hermitian, positive and of trace 1, it can be decomposed on the Pauli matrix basis

$$\rho = \frac{1}{2}(\boldsymbol{I} + x\boldsymbol{X} + y\boldsymbol{Y} + z\boldsymbol{Z}), \tag{4.1}$$

where $(x, y, z)^\intercal := (\mathrm{Tr}\,(\rho\boldsymbol{X})\,, \mathrm{Tr}\,(\rho\boldsymbol{Y})\,, \mathrm{Tr}\,(\rho\boldsymbol{Z}))^\intercal$ are the coordinates of the Bloch vector and the Pauli matrices

$$\boldsymbol{X} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \boldsymbol{Y} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \boldsymbol{Z} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \tag{4.2}$$

which together with the identity form a basis of the set of 2×2 Hermitian matrices when this are multiplied by real coefficients.

It is useful to use as well the following representation for pure states on the Bloch sphere

$$\rho = \frac{1}{2} \begin{bmatrix} 1 + \cos(\theta) & e^{-i\varphi}\sin(\theta) \\ e^{i\varphi}\sin(\theta) & 1 - \cos(\theta) \end{bmatrix}$$

the Bloch coordinates for pure states read

$$x = \sin\theta\cos\varphi,$$
$$y = \sin\theta\sin\varphi,$$
$$z = \cos\theta.$$

We will consider a qubit being measured with the operator $\boldsymbol{Z}$. Let $\{\Pi_{Z+1}, \Pi_{Z-1}\}$ denote the eigenstates of $\boldsymbol{Z}$ with eigenvalues $\{+1, -1\}$ respectively, thus $\boldsymbol{Z} = \Pi_{Z+1} - \Pi_{Z-1}$. On the Bloch sphere, these states correspond to the north pole and south pole respectively, and measurement of the $\boldsymbol{Z}$ matrix corresponds to measurements along the z-axis of the sphere.

The open-loop system describing a measurement on the qubit obeys the SDE

$$d\rho = \Gamma\Big(\boldsymbol{Z}\rho\boldsymbol{Z} - \rho\Big)dt - \sqrt{\eta\Gamma}\Big(\boldsymbol{Z}\rho + \rho\boldsymbol{Z} - 2\,\mathrm{Tr}\,(\rho\boldsymbol{Z})\,\rho\Big)dW, \tag{4.3}$$

The following proposition follows directly from Lemma 3.1.1

Proposition 4.2.1. *For the open-loop system* (4.3)*, the Lyapunov function* $V_o(\rho) = \sqrt{1 - \mathrm{Tr}\,(\rho\boldsymbol{Z})^2}$ *decays as* $\mathbb{E}[V_o(\rho_t)] \leq e^{-2\eta\Gamma t}V_o(\rho_0), \quad \forall t \geq 0.$

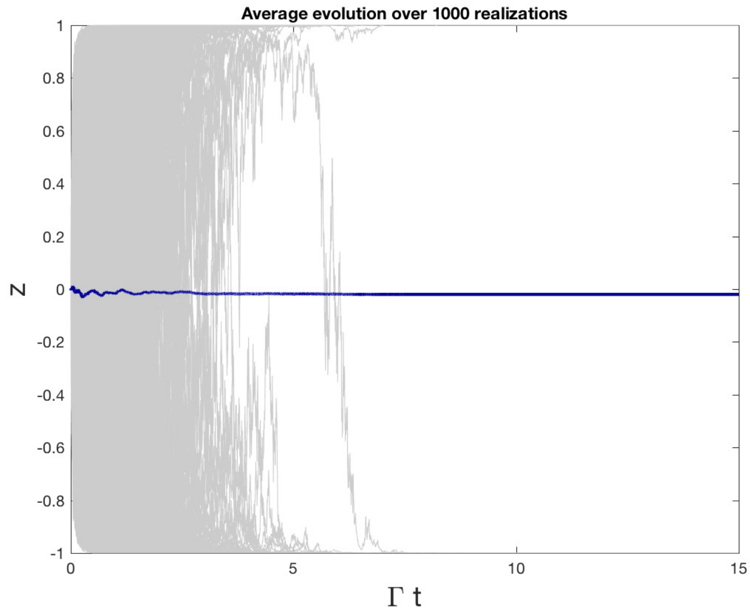
Simulations of the open-loop dynamics are provided in Figure 4.2a.

Our control problem is then to stabilize a target eigenstate $\{\Pi_{Z+1}, \Pi_{Z+1}\}$. We choose Π_{Z+1} as the target eigenstate, the procedure to stabilize Π_{Z+1} follows in the same manner.

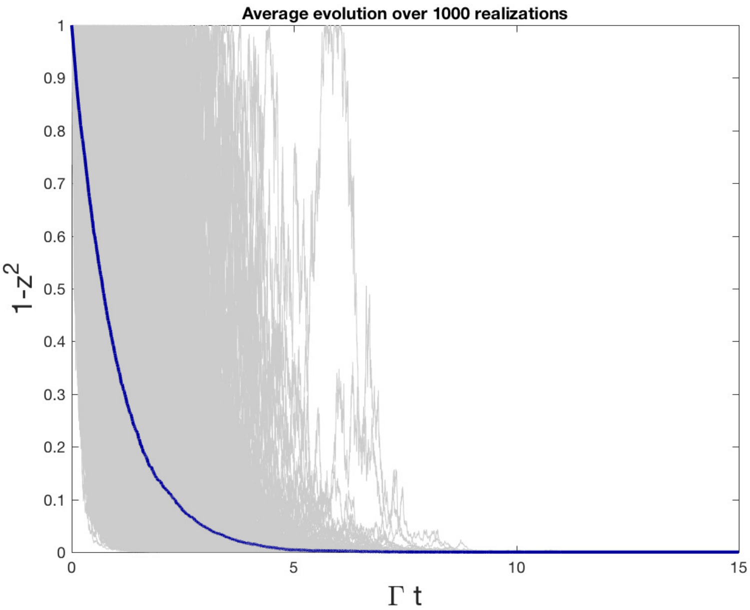
We now compare the static output feedback with our noise assisted control design. The former is attractive because it is easy to show exponential convergence and that the controller only depends on constants. The analysis for the latter is relatively more involved, and it requires a state estimator in order to compute the feedback law. However we will show that this is not such a big issue, the use of noise allows to consider reduced order filters that estimate only the z-axis. Since the Boch sphere depends on three variables, this may seem a small advantage with respect to other known control laws that might require full knowledge of the state, but it serves as a proof of principle that later will be extended to higher dimensional systems.

4.3 Static output feedback on a qubit

As we exposed in the introduction, there is an obstruction for static output feedback to stabilize a QND eigenstate. One could ask about defining the static output feedback gains as to approach arbitrarily a target QND eigenstate. The next Proposition indeed shows that, at least for a two-level system, any other pure state can be exponentially stabilized by a static output feedback with a fixed measurement operator and detection efficiency $\eta = 1$. We use the standard Pauli matrix and Bloch sphere notation for the qubit system.



(a) Simulation of the qubit system (4.3). We take $\Gamma = 1$, $\eta = 0.5$. In gray: 100 individual trajectories with $\rho_0 = \boldsymbol{I}/2$ ($z_0 = 0$). The ensemble average (in blue) remains constant around 0, in agreement with the martingale property.



(b) Exponential decay of the qubit system (4.3) towards the set $\{\Pi_{Z+1}, \Pi_{Z-1}\}$, in terms of the Lyapunov function $V(\rho) = 1 - \mathrm{Tr}\,(\rho\boldsymbol{Z})^2$. The ensemble average depicted in blue, decays with rate 1, in agreement with Proposition 4.2.1.

Figure 4.2: Simulations of the open loop system

Proposition 4.3.1. *Consider* (3.6) *with* $n = 2$, $\eta = 1$, *and* $L = \sqrt{\Gamma}\boldsymbol{Z}$, $H = \boldsymbol{Y}$. *Take* $\bar{\theta} \notin \{k\pi : k \in \mathbb{Z}\}$ *and set* $f = -\Gamma \sin(\bar{\theta})\cos(\bar{\theta})$, $\kappa = \sqrt{\Gamma}\sin(\bar{\theta})$. *Then the closed-loop system exponentially stabilizes the pure state:*

$$\bar{\rho} = \frac{1}{2}\begin{bmatrix} 1+\cos(\bar{\theta}) & \sin(\bar{\theta}) \\ \sin(\bar{\theta}) & 1-\cos(\bar{\theta}) \end{bmatrix},$$

as the Lyapunov function

$$V(\rho) = 1 - \operatorname{Tr}(\rho\bar{\rho})$$

decreases according to $\mathbb{E}(V(\rho_t)) = e^{-rt}V(\rho_0)\ \forall t \geq 0$ *with rate* $r = 2\Gamma(\sin\bar{\theta})^2$.

The proof is based on standard matrix manipulations showing $d\mathbb{E}(V) = -\Gamma(\sin\bar{\theta})^2\mathbb{E}(V)\,dt$. By modifying the actuation Hamiltonian as $F = U_{\bar{\alpha}}\boldsymbol{Y}U_{\bar{\alpha}}^\dagger$, where $U_{\bar{\alpha}} = \exp(-i\bar{\alpha}\boldsymbol{Z}/2)$ is a rotation of angle $\bar{\alpha}$ around the z axes of the Bloch sphere associated to the qubit, the same feedback gains stabilize $U_{\bar{\alpha}}\bar{\rho}U_{\bar{\alpha}}^\dagger$. With $\bar{\alpha} \in [0, 2\pi]$ and $\bar{\theta} \in]0, \pi[$, any pure state different from the two eigenstates of $\boldsymbol{Z}$ are thus obtained via $U_{\bar{\alpha}}\bar{\rho}U_{\bar{\alpha}}^\dagger$.

While we have shown that we could approach the eigenstates of the measurement $\boldsymbol{Z}$, we can directly note two things: First, the associated convergence rate approaches to zero as the target state approaches an eigenstate of $\boldsymbol{Z}$ (cf. Lemma 3.2.1). Second, exact exponential stabilization depends on having a perfect detection device: as soon as $\eta < 1$, the closed-loop system is subject to significant noise, which limits the rate of purity towards the target. Simulations are provided in Figure 4.3.

4.4 Noise assisted stabilization of Qubit eigenstates

It was already highlighted in [71] the difficulties associated to designing a globally stabilizing control law even in the case of a qubit. It was noted there that in principle it suffices that the control drives a continuous field that only vanishes on the target. This is the approach followed in the literature, e.g. [44, 69, 71, 50]. The point of view adopted here is, in a sense, weaker: the control field on (3.13) will be for a large parte of the state space, turned-off. It is only when the state is closed to an undesired situation that the controller will be turned on. It suffices to show that this procedure induces exponential convergence. We can readily illustrate on the qubit how the use of noise allows us to exponentially stabilize a target eigenstate. From (3.13) with $L = \Gamma\boldsymbol{Z}$, $\Gamma > 0$, and $H = \boldsymbol{Y}$, the closed loop dynamics read

$$\begin{aligned} d\rho = \Gamma\Big(\boldsymbol{Z}\rho\boldsymbol{Z} - \rho\Big)dt - \sqrt{\eta}\Gamma\Big(\boldsymbol{Z}\rho + \rho\boldsymbol{Z} - 2\operatorname{Tr}(\rho\boldsymbol{Z})\,\rho\Big)dW \\ + \sigma(\rho)^2(\boldsymbol{Y}\rho\boldsymbol{Y} - \rho)dt - i\sigma(\rho)^2[\boldsymbol{Y}, \rho]dB \quad (4.4) \end{aligned}$$

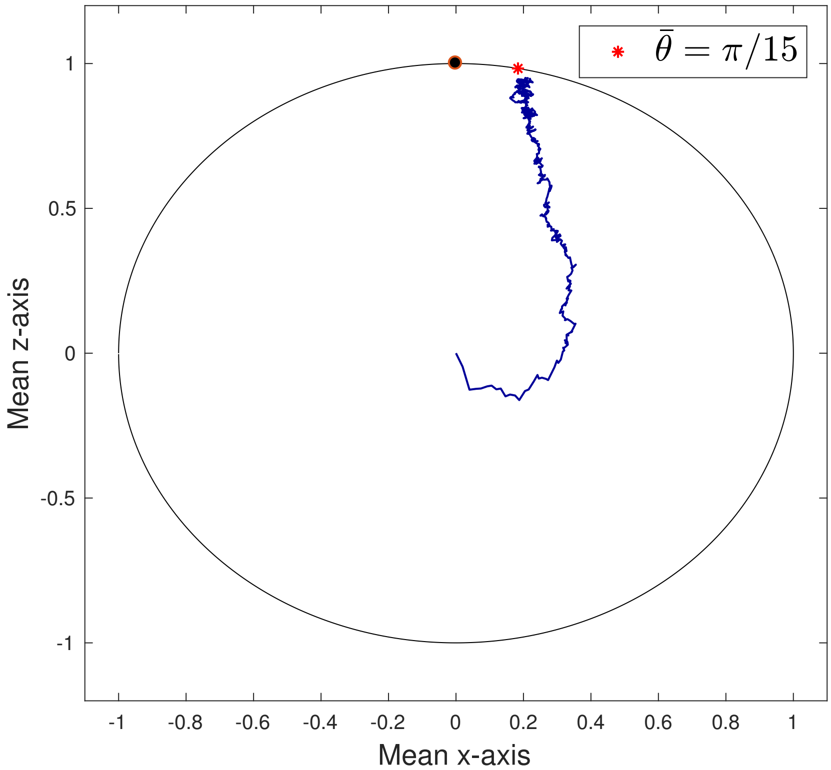


Figure 4.3: Simulation of the qubit closed-loop system under the static output feedback of Proposition 4.3.1. The initial state is $\rho_0 = \boldsymbol{I}/2$, and the target state $\tilde{\rho}$ of the form given in Proposition 4.3.1. On Bloch sphere coordinates, the average state approaches the target state $\tilde{\rho}$, which has been chosen close to the north pole Π_{Z+1}.

We will target the stabilization of the eigenstate Π_{Z+1} of the measurement operator $\boldsymbol{Z}$, stabilization of the other state Π_{Z-1} follows the same lines.

Control Law 4.4.1. *Let $\bar{\sigma} > 0$, and define $\sigma(\rho)$ as*

$$\sigma(\rho) = \bar{\sigma}\ \varphi\left(\frac{\mathrm{Tr}\,(\rho\Pi_{Z-1}) - p_{min}}{p_{max} - p_{min}}\right) \tag{4.5}$$

where $\varphi \geq 0$ is a smooth function on $[0,1]$ defined as $\varphi(]-\infty,0]) = 0$ and $\varphi(]1,\infty) = 1$, and parameters $\bar{\sigma} > 0$, $1 > p_{max} > p_{min} > 1/2$.

Proposition 4.4.1. *Consider the closed-loop system* (4.4) *with control law as given in* (4.5). *Let $p \in]1/2, 1]$ and set the control parameters $\bar{\sigma} = \sqrt{\frac{c\eta\Gamma}{2p_{max}-1}}$, $p_{min} > p$. Then the Lyapunov function*

$$V(\rho) = \sqrt{1 - \mathrm{Tr}\,(\rho\Pi_{Z+1})} \tag{4.6}$$

satisfies $\mathbb{E}[V(\rho_t)] < e^{-\eta\Gamma r t}V(\rho_0)$ for all $t \geq 0$, and the state Π_{Z+1} is globally exponentially stable with rate $r = \min\{c/2\,,\ \frac{1}{2}\}$.

Proof. We compute the formula for the generator $\mathcal{A}V(\rho)$

$$\begin{aligned}\mathcal{A}V(\rho) = &-\frac{\sigma(\rho)^2\big(\,\mathrm{Tr}\,(\rho\Pi_{Z-1}) - \mathrm{Tr}\,(\rho\Pi_{Z+1})\,\big)}{2\sqrt{1 - \mathrm{Tr}\,(\rho\Pi_{Z+1})}} \\ &- \frac{\eta\Gamma}{2(1 - \mathrm{Tr}\,(\rho\Pi_{Z+1}))^{3/2}}(1 - \mathrm{Tr}\,(\rho\boldsymbol{Z}))^2\,\mathrm{Tr}\,(\rho\Pi_{Z+1})^2 \\ &- \frac{\sigma(\rho)^2}{2(1 - \mathrm{Tr}\,(\rho\Pi_{Z+1}))^{3/2}}\,\mathrm{Tr}\,(i[\boldsymbol{Y},\rho]\Pi_{Z+1})^2\end{aligned}$$

The main idea of the control law 4.4.1 is that $\sigma(\rho)$ is close to 1 as soon as $\mathrm{Tr}\,(\rho\Pi_{Z-1}) > p_{\min} > p$. Accordingly, we choose the parameter $p_{\max}$ close to $p_{\min}$ such that the region where $\sigma(\rho)$ is monotonically decreasing is small. When $\mathrm{Tr}\,(\rho\Pi_{Z-1}) \geq p_{\max}$, we have

$$\mathcal{A}V(\rho) \leq -\frac{\sigma(\rho)^2\big(\,\mathrm{Tr}\,(\rho\Pi_{Z-1}) - \mathrm{Tr}\,(\rho\Pi_{Z+1})\,\big)}{2\sqrt{1 - \mathrm{Tr}\,(\rho\Pi_{Z+1})}} \leq -\frac{\sigma(\rho)^2(2p-1)}{2} \leq -\frac{\eta\Gamma c}{2}V(\rho)$$

Otherwise $\sigma(\rho) = 0$ when $\mathrm{Tr}\,(\rho\Pi_{Z+1}) \geq 1 - p > 1/2$ and

$$\mathcal{A}V(\rho) < -2\eta\Gamma\sqrt{(1 - \mathrm{Tr}\,(\rho\Pi_{Z+1})}\,\mathrm{Tr}\,(\rho\Pi_{Z+1})^2 \leq -\tfrac{\eta\Gamma}{2}V(\rho).$$

Since $V(\rho) = 0$ when $\rho = \Pi_{Z+1}$ this implies that Π_{Z+1} is globally exponentially stable with rate r as described above. $\square$

Simulations of the closed loop qubit system are provided in Figure 4.5a. Compared to relying on the pure static output feedback of Proposition 4.3.1 to stabilize a state close to Π_{Z+1}, the average exponential decay is about three times faster.

Some remarks are useful to gain intuition on the role of noise on achieving exponential stabilization. Consider the system (4.4) on the (x, y, z)-coordinates of the Bloch sphere. These read

$$dz = 2\sqrt{\eta\Gamma}(1-z^2)dW - 2\sigma(z)^2 z dt + 2\sigma(z) x dB \tag{4.7}$$

$$dy = -2\Gamma y dt + 2\sqrt{\eta\Gamma} y z dW \tag{4.8}$$

$$dx = -2\Gamma x dt - 2\sqrt{\eta\Gamma} x z dW - 2\sigma(z)^2 x dt - 2\sigma(z) z dB. \tag{4.9}$$

Note that when $y_0 = 0$, the equation of the y-coordinate is decoupled from (z, x), and so we can restrict the dynamics to the ball $\{(x, z) : x^2 + z^2 \leq 1\}$.

We consider first two situations for the variable $\sigma(z)$ that follows the control logic 4.4.1, which in Bloch coordinates reads

$$\sigma(z) = \bar{\sigma}\,\varphi\Big(\frac{(1-z)/2 - p_{\min}}{p_{\max} - p_{\min}}\Big).$$

Take $\sigma(z) = \bar{\sigma} > 0$. Added to the measurement dynamics, the terms depending on σ correspond —up-to a factor 2σ— to a standard Brownian motion on the circle (see, e.g. [55, Chapter 5]). A Brownian motion on the circle does not change the purity of the state on the Bloch sphere, but on average it decays exponentially towards $(0, 0)$ with rate $2\sigma^2$. Intuitively, if the gain σ is strong enough, this average dynamics would dominate the measurement dynamics and the state would be driven towards the center of the Bloch sphere. At $(0, 0)$, $\sigma(z) = 0$ and Eq. (4.7) is in open-loop, and by 3.1.1, the probability that $z_t \to 1$ is less than or equal to $1/2$. The role of the smooth control law 4.4.1 is to re-activate noise from as soon as $z_t < 1 - 2p_{\min} < 0$, to drive again the trajectory towards the centre of the circle, and give another chance for z to converge towards 1. The use of a smooth control law ensures continuity of the closed-loop trajectories.

Estimating the speed of convergence

The Lyapunov function used in Proposition 4.4.1 is a natural measure of a distance for the two-level system (corresponding to the trace fidelity/Bures distance), but there is an under-estimation of the convergence rate. Having a tighter bound is important specially towards providing an estimate of robustness. By design, the convergence rate is limited by the open-loop convergence rate, which from Lemma 3.1.1 is $r = 2\eta\Gamma$. The reason for it is that control actions are unitary, they cannot modify the purity of the state. We propose a Lyapunov function to provide a tighter bound, a description of the parameters is given in figure 4.4a.

$$\tilde{V}(\rho) = \begin{cases} \frac{\sqrt{(1-z)(1+\alpha+(1-\alpha)z)}}{\sqrt{2(1+\alpha)}}, & z := \mathrm{Tr}\,(\rho \boldsymbol{Z}) \geq 0, \\ \frac{1}{\sqrt{2}}(1-\alpha^2+\alpha\sqrt{\alpha^2 - 2\frac{\alpha}{1+\alpha}z}), & z < 0 \end{cases} \tag{4.10}$$

For $z \geq 0$, the function (4.11) corresponds to a re-scaling $(1-z) \mapsto (1-\alpha)(1-\alpha)$ with respect to the open-loop Lyapunov function. When $\alpha = 1$, it corresponds to the Lyapunov function (4.6). The idea is that the three functions have the same tangent around $z = 1$, thus (4.11) correctly measures the distance around the target. For $z < 0$ we have just chosen a monotonically decreasing, concave function with derivative defined at $z = 0$.

Theorem 4.4.1. *Consider the closed-loop system* (4.4) *and the Lyapunov function* (4.10) *for some small parameter* $\alpha > 0$. *Under the control law 4.4.1, set* $p_{max} > p_{max} > p > 1/2$, *and choose the control gain* $\bar{\sigma}$ *as*

$$\bar{\sigma}^2 \geq \sup_{z \in [-1,-2p+1]} \left(\frac{2\eta\gamma/(1+\alpha)^2}{m(z)\left(|z|(\alpha - \frac{2z}{1+\alpha})(1+\alpha)\right)} \right),$$

where

$$m(z) = \frac{2\alpha^3(1-z)^2}{(1+\alpha)^2(\alpha - \frac{2\alpha z}{1+\alpha})^{3/2}(\,1-\alpha^2+\alpha\sqrt{\alpha^2-2\alpha z/(1+\alpha)}\,)}\,.$$

Then $\mathcal{A}\tilde{V}(\rho) \leq -\frac{2\eta\Gamma}{(1+\alpha)^2}\tilde{V}(\rho)$ *for all* $\rho \in \mathcal{S}$.

Proof. We analyze the closed-loop evolution for the two regions $z \geq 0$ and $z \leq 0$. Using Itō's formula we get:

- For $z \geq 0$, $\sigma(z) = 0$ by control law 4.4.1 , thus the behavior of $\tilde{V}$ in open loop is:

$$\mathcal{A}\tilde{V}(\rho) = -2\eta\Gamma\left(\frac{1+z}{(1+\alpha)+(1\text{-}\alpha)z}\right)^2 \tilde{V}\,dt \leq \frac{-2\eta\Gamma}{(1+\alpha)^2}\,\tilde{V}(\rho)\,dt\,.$$

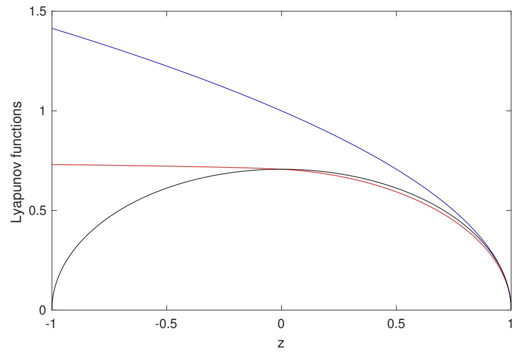
- For $z \leq -2p+1$, and $\sigma \neq 0$, we compute:

$$\begin{aligned}\mathcal{A}\tilde{V} = &\frac{2\alpha^3}{(1+\alpha)^2} \cdot \frac{(1-z)}{(\alpha^2-2\alpha z/(1+\alpha))^{3/2}} \\ &\cdot \frac{\left(-(\,\eta\Gamma(1+z)^2+\sigma^2 x^2+\sigma^2|z|(1+\alpha)(\alpha-\frac{2z}{1+\alpha})\,)\,(1-z)\right)}{1-\alpha^2+\alpha\sqrt{\alpha^2-2\alpha z/(1+\alpha)}} \cdot \tilde{V}\,dt\,.\end{aligned} \tag{4.11}$$

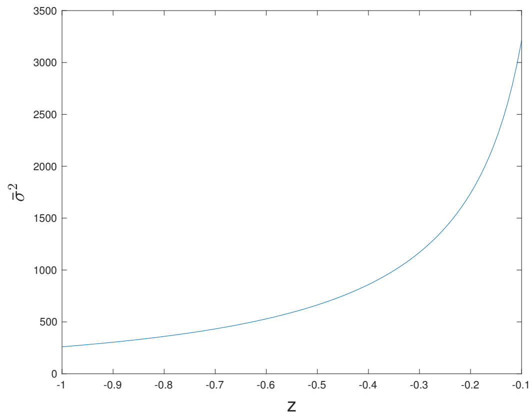
 We can then achieve a convergence rate r if we can choose

$$\bar{\sigma}^2 \geq \left(\frac{r}{m(z)} - \eta\Gamma(1+z)^2\right)\ /\ \left(|z|(\alpha - \frac{2z}{1+\alpha})(1+\alpha)\right),$$

 for all $z < 0$. To get a finite σ with this formula we need $\frac{r}{m(z)} - \eta(1+z)^2$ negative at $z = 0$, or converging to zero at least linearly in $|z|$ when z converges to zero from below. One checks that this is indeed the case for $r \leq \eta\Gamma m(0) = 2\eta\Gamma/(1+\alpha)^2$. This rate matches the one for $z \geq 0$.



(a) In black: open-loop Lyapunov function (3.3). In blue, closed-loop function (4.6). In red: proposed function (4.11) with $\alpha = 0.1$.



(b) Selection of optimal gain $\bar{\sigma}$ with $\eta = .8$, $\gamma = 1$, $\alpha = .1$ and $2p - 1 = .1$.

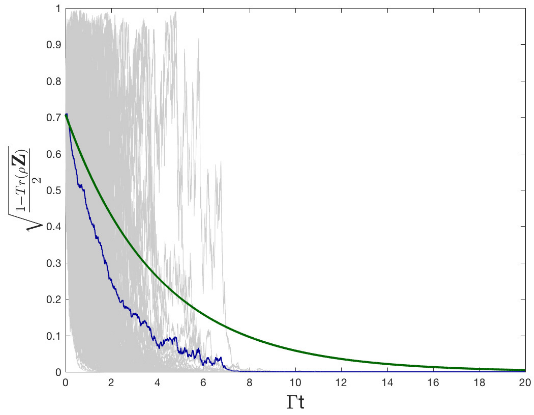
Figure 4.4: Comparison of Lyapunov functions on the qubit and selection of optimal gain

$\square$

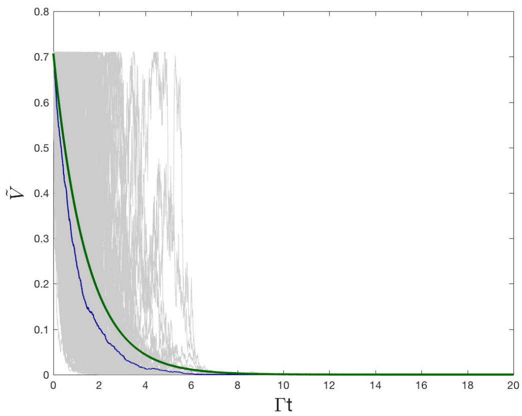
One can find numerically the value for $\bar{\sigma}$. An example is given in figure 4.4b, and simulations comparing with the standard Lyapunov function can be found in figure 4.5b.

4.5 Reduced order filtering on the qubit

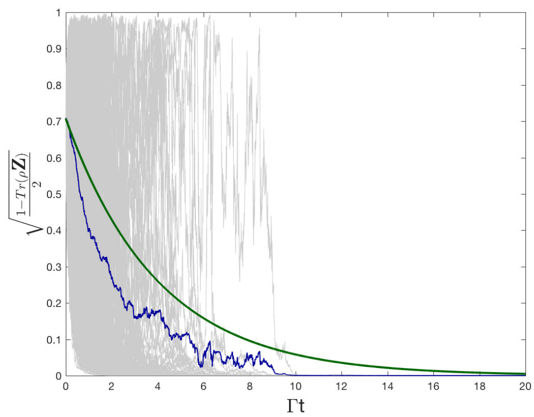
The control that drives the strength of the noise still depends on computing the control law from a quantum filter, which provides the best estimate of the state based on the measurement outcomes. Interestingly, the stability analysis shows that it is sufficient to consider only the measured coordinate to do control. With this in mind, we will introduce a reduced order fil-



(a) Simulation of the controlled qubit system of Proposition 4.4.1, plotting $\sqrt{1 - \mathrm{Tr}\,(\rho\Pi_{Z+1})} = \sqrt{\frac{1-\mathrm{Tr}(\rho\boldsymbol{Z})}{2}}$ Here $\bar{\sigma} = 8$ and $p_{\min} = 0.6$, $p_{\max} = .7$ $\eta = 0.5$, and $\Gamma = 1$. In gray: 1000 individual trajectories with $\rho_0 = \boldsymbol{I}/2$. In blue: ensemble average. In green: Bound from Proposition 4.4.1.



(b) Simulation of the controlled qubit system of Proposition 4.4.1, under the Lyapunov function $\tilde{V}$ of Theorem 4.4.1. From the control law 4.4.1, $p = .55$, $p_{\max} = .65$, $p_{\min} = .6$. The control gain $\bar{\sigma}^2 = 3500$ is chosen from figure 4.4b. In gray: 200 individual trajectories with $\rho_0 = \boldsymbol{I}/2$. In blue: ensemble average. In green: Bound from Theorem 4.4.1.



(c) Simulation of the controlled qubit system with the reduced filter (4.12).

Figure 4.5: Stabilization of Qubit QND eigenstates.

ter which will be later extended in other high dimensional settings. More precisely, consider the quantum filter for this two-level system which reads

$$d\rho = \Gamma\Big(\boldsymbol{Z}\rho\boldsymbol{Z}-\rho\Big)dt-\sqrt{\eta\Gamma}\Big(\boldsymbol{Z}\rho+\rho\boldsymbol{Z}-2\,\mathrm{Tr}\,(\rho\boldsymbol{Z})\,\rho\Big)(dY-2\sqrt{\eta\Gamma}\,\mathrm{Tr}\,(\rho\boldsymbol{Z})\,dt)$$
$$+\,\sigma(\rho)^2(\boldsymbol{Y}\rho\boldsymbol{Y}-\rho)dt - i\sigma(\rho)^2[\boldsymbol{Y},\rho]dB$$

where $dY = 2\sqrt{\eta\Gamma}\,\mathrm{Tr}\,(\rho\boldsymbol{Z})\,dt+dW$ is the measurement outcome. We can consider the conditional expectation with respect to Y_t, then the stochastic term dependent on B_t vanishes because the noises are conditionally independent. We consider then the state $\widehat{\rho}$, the Bayesian estimate of ρ_t knowing ρ_0 and the measurements Y_t, $t \in [0,T]$, but not B_t. This state reads

$$d\widehat{\rho} = \Gamma\Big(\boldsymbol{Z}\widehat{\rho}\boldsymbol{Z}-\widehat{\rho}\Big)dt-\sqrt{\eta\Gamma}\Big(\boldsymbol{Z}\widehat{\rho}+\widehat{\rho}\boldsymbol{Z}-2\,\mathrm{Tr}\,(\widehat{\rho}\boldsymbol{Z})\,\widehat{\rho}\Big)(dY-2\sqrt{\eta\Gamma}\,\mathrm{Tr}\,(\widehat{\rho}\boldsymbol{Z})\,dt)$$
$$+\,\sigma(\widehat{\rho})^2(\boldsymbol{Y}\widehat{\rho}\boldsymbol{Y}-\widehat{\rho})dt.$$

Since $\boldsymbol{Y}\boldsymbol{Z}\boldsymbol{Y} = -\boldsymbol{Z}$, the dissipation $\sigma(\widehat{\rho})^2(\boldsymbol{Y}\widehat{\rho}\boldsymbol{Y}-\widehat{\rho})$ induced by the noise can be expressed in terms of the state $\widehat{z} := \mathrm{Tr}\,(\widehat{\rho}\boldsymbol{Z})$. This obeys the SDE

$$d\widehat{z} = \sqrt{\eta\Gamma}(1-\widehat{z}^2)(dY-2\sqrt{\eta\Gamma}\widehat{z}dt) - \sigma^2(\widehat{z})\widehat{z}dt. \tag{4.12}$$

We have a fiter formulation based only on knowledge of the z-measurement axis. In the sequel we will propose higher dimensional extensions of this reduced-order formulation, where only the eigenstate populations are needed to compute the feedback law. This corresponds to constructing filters based on classical probabilities. This is a point of depart from the usual measurement based feedback, which always assumes an underlying quantum filter, and control laws usually considered tracking the coherences between populations. Numerical simulations, provided in Figure 4.5c suggest that this filter is robust against wrongly estimated parameters where the closed-loop convergence rate is slightly slower.

4.6 Moving beyond a qubit: generation of GHZ states

We can readily extend the ideas developed on a qubit for other type of systems.

A Greenberger-Horne-Zeilinger (GHZ) state for d-qubits ($d > 2$) is an entangled state of the form

$$|GHZ\rangle_d = \frac{|0\rangle^{\otimes d} + |1\rangle^{\otimes d}}{\sqrt{2}}.$$

To prepare this highly entangled state, the measurement operator

$$L = \sum_{s,s',\ s<s'} \boldsymbol{Z}_s\boldsymbol{Z}_{s'}$$

is a natural choice, as the state $|GHZ\rangle$ belongs to the eigenspace of L with corresponding eigenvalue $d(d-1)/2$. This measurement operator possesses full permutation symmetry and simultaneous bit-flip symmetry which induces a symmetric subspace common to the measurements of dimension $\frac{d+1}{2}$ if d is odd and $\frac{d}{2}+1$ if d is even. It is instructive to consider the case $d=3$ with control goal to prepare the target state $|GHZ\rangle_3$, as it will serve as a point of departure towards the next section on quantum error correction.

Denote as Π_{GHZ} the projector associated with the eigenspace of $L = \boldsymbol{Z}_1\boldsymbol{Z}_2 + \boldsymbol{Z}_2\boldsymbol{Z}_3 + \boldsymbol{Z}_1\boldsymbol{Z}_3$ with eigenvalue 3. Its complement $I - \Pi_{GHZ}$ corresponds to the eigenspace with eigenvalue -1. To preserve the bit-flip symmetry and thus remain on the symmetric subspace of $\mathcal{H}$ generated by the measurement, we choose as feedback Hamiltonian $H = (\boldsymbol{X}_1 + \boldsymbol{X}_2 + \boldsymbol{X}_3)$ driven by the single Brownian motion B_t.

The projector Π_{GHZ} is not of rank 1, but we have a perfect symmetry in the system. As such, with the two projectors this system can be seen as an effective qubit system. From the techniques that we developed on the qubit, we consider the control law

Control Law 4.6.1. *Let $\bar{\sigma} > 0$, and define $\sigma(\rho)$ as*

$$\sigma(\rho) = \bar{\sigma}\ \varphi\left(\frac{(1-\mathrm{Tr}\,(\rho\Pi_{GHZ})) - p_{min}}{p_{max} - p_{min}}\right) \tag{4.13}$$

where $\varphi \geq 0$ is a smooth function on $[0,1]$ defined as $\varphi(]-\infty,0]) = 0$ and $\varphi([1,\infty) = 1$, and parameters $\bar{\sigma} > 0$, $1 > p_{max} > p_{min} > 1/2$.

The controller does not indicate at all, if the asymptotic eigenstate corresponds to $|GHZ\rangle_3$. Since the feedback Hamiltonian only performs local operations on each qubit, it is impossible to generate $|GHZ\rangle_3$ from a completely mixed state [66] . A condition for the feedback to generate the state $|GHZ\rangle_3$ is that the initial state has to be defined on the subspace spanned by $|\tilde{0}\rangle = |GHZ\rangle_3$ and $|\tilde{1}\rangle = \frac{1}{\sqrt{6}}(|100\rangle + |010\rangle + |001\rangle + |011\rangle + |110\rangle + |101\rangle)$, where $|\tilde{0}\rangle$ is the target and $|\tilde{1}\rangle$ generates the eigenspace of L with eigenvalue -1. Simulations in figure 4.6 with initial state $\rho_0 = |\psi_0\rangle\langle\psi_0|$, $|\psi_0\rangle = \frac{1}{\sqrt{2}^3}(|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle)$ show the exponential preparation of the target state $|GHZ\rangle_3$. In practice this initial state can be easily prepared with local means.

4.7 Conclusions

We have addressed the stabilization of the eigenstates of a measurement operator for a two-level system. We first considered approaching such eigenstates via proportional output feedback, we can stabilize all pure states on the Bloch sphere, except the QND steady states. This indicates that for the

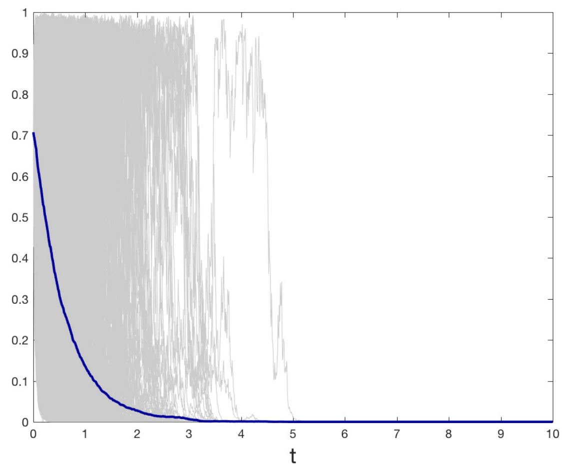


Figure 4.6: Preparation of the state $|GHZ\rangle_3$.

QND eigenstates as targets, one must make a more involved controller in order to bias the stochastic evolution towards the desired extremum. Previous work like [50, 69] has turned to specific stochastic controller designs to achieve this goal, on the basis of a quantum state observer. In this chapter we have shown that just adapting intensity of the noise that drives the control – as a function of the estimated quantum state – is sufficient to obtain global exponential stabilization, something that previous state feedback controls do not achieve or at least could not prove. In our case, the proof is a direct consequence of standard stochastic convergence theorems with a strict Lyapunov function. Interestingly, this controller architecture can be re-interpreted as allowing non-unitary controls, as we can discard the noise driving the closed-loop equation. This idea has been exploited to construct a reduced filter that is just a function of the estimated measurement axis.

Before moving on to more complex systems, we highlight some common underlying questions that will appear in the next chapters:

- While stochastic Lyapunov methods provide a simple and well-proved state feedback law, we have seen that "natural" candidate Lyapunov functions can yield to an under-estimation of the actual closed-loop convergence rate. In the case of the qubit this has been improved by proposing an alternative Lyapunov function in Theorem 4.4.1. Is there a more sistematic way to find more "informative" Lyapunov functions that provide an accurate convergence rate?

- We can discard the noise terms to obtain a reduced order filter that does not need to track coherences between the states. In the case of the qubit this filter only depends on the tracking the qubit QND eigenstates. Is there a systematic way to formulate similar reduced order filters in higher dimensions?

These questions will be addressed in the sequel.

Chapter 5

Exponential stabilization of a quantum non-demolition measurement eigenstate

Dans ce chapitre nous étendons la stratégie de contrôle assistée par bruit à un système QND à plusieurs niveaux, en utilisant un seule contrôle avec un Hamiltonian fixe. L'extension diffère sensiblement du qubit en raison de la présence de multiples états propres orthogonaux à l'état cible, et sur la manière dont l'hamiltonien du contrôle «connecte» les états QND. La stratégie du control est de «secouer» le système lorsqu'il s'approche d'états propres indésirables en y appliquant un bruit fort, tout en s'appuyant sur la dynamique en boucle ouverte pour atteindre progressivement la cible. Pour montrer que le bruit éloigne le système des voisinages des états propres QND non-voulus, nous utilisons des méthodes basiques de la théorie des graphes impliquant l'inversion d'une matrice laplacienne pour montrer la décroissance exponentielle autour desdits états.

Ensuite, l'analyse sur la fonction de Lyapunov montre que, en combinaison avec la dynamique en boucle ouverte, on peut établir une convergence exponentielle de la fonction de Lyapunov en boucle fermée. Cette fonction est une supermartingale avec une décroissance exponentielle, impliquant la stabilité exponentielle globale. Le schéma de contrôle et son analyse de stabilité suggèrent l'utilisation d'un filtre approximatif qui ne fait que suivre les populations des états propres de l'opérateur de mesure. Ceci consiste à remplacer les termes correspondant à la dissipation faite par le bruit par la matrice laplacienne associée au graphe de connectivité induit par le hamiltonien d'actionnement.

5.1 Introduction

This chapter is dedicated to the extension for stabilizing a QND eigenstate of a multi-level quantum system with a single measurement operator under a non-degeneracy condition. We recall the closed-loop SDE

$$d\rho = \mathcal{D}_L(\rho)dt + \sqrt{\eta}\mathcal{M}_L(\rho)dW + \sigma(\rho)^2\mathcal{D}_H(\rho)dt - i\sigma(\rho)[H,\rho]dB, \qquad (5.1)$$

here $L = \sum_{k=1}^{d} \lambda_k \Pi_k$; $\lambda_1, ..., \lambda_d$ are the distinct real eigenvalues of L with corresponding orthogonal rank 1 projectors $\Pi_1, ..., \Pi_d$ resolving the identity, i.e. $\sum_{k=1}^{d} \Pi_k = \boldsymbol{I}$. The feedback Hamiltonian H makes the analysis more involved than in the case of the qubit: indeed, the matrix H specifies the coupling amongst the QND eigenstates of the measurement operator L, fixing the structure of the set of all direct and indirect couplings between eigenstates.

For QND systems, a difficulty appears when trying to transfer the population on a state between two QND eigenstates which are indirectly connected; the closed-loop trajectory must pass trough at least another QND eigenstate, hence $\mathrm{Tr}\,(\rho\Pi_\ell) = 0$ and the image of ρ remains orthogonal to Π_ℓ. This explains why on previous approaches, the control is set to a constant value when the image of ρ is almost orthogonal to the target Π_ℓ [50, 69, 70].

A more precise analysis has to take into account the direct and indirect connnections between the QND eigenstates. The closed-loop Lyapunov function needs to take this into account while being a useful measure of a distance to the target state. The strategy to construct a closed-loop Lyapunov function consists on considering the mapping

$$\{\boldsymbol{p}_k(\rho)\}_{k\in\{1,...,d\}\setminus\{\ell\}} \longmapsto \sum_{s\in\{1,...,d\}\setminus\{\ell\}} \sqrt{\sum_{k\in\{1,...,d\}\setminus\{\ell\}} \alpha_{s,k}\boldsymbol{p}_k(\rho)}.$$

The weighted sum $\sum_{k\in\{1,...,d\}\setminus\{\ell\}} \alpha_{s,k}\boldsymbol{p}_k(\rho)$ is inspired by the stability analysis done in [6]; the role of the weights is to bring out the connectivity structure induced by the feedback Hamiltonian. Indeed, these weights are obtained through an inversion of the Laplacian matrix describing the connectivity graph induced by the feedback Hamiltonian. Elementary results on graph theory allow to use the weights advantageously to deduce exponential decay of the closed-loop Lyapunov function around an undesired QND eigenstate by exploiting the dissipation term induced by the actuation.

Locally around the target Π_ℓ, the behaviour of the square root map defined above is similar to that of the open-loop Lyapunov function (3.3). As said before in the case of the qubit, exponential convergence around the target state holds for a large part of the domain, *except* at the other QND eigenstates. Thus, considering the square root as a way to deduce exponential convergence is a natural choice to use the open-loop dynamics to enforce exponential convergence around the target state.

The control law is an extension of the case treated in the previous chapter: it consists on monitoring the populations of the QND eigenstates. In this setting, the controller will wait until the system is close to one of the few steady-state situations, namely, the state is close to a QND eigenstate $\{\Pi_k\}_{1\leq k\leq d}$. When the population around an undesired eigenstate is large enough, the controller activates the noisy actuator until the state is driven away from the undesired situation. Then the controller deactivates the noisy input so that the system returns to the open-loop regime, and so there is a non-null probability of converging to the target. This procedure is repeated until the state converges to the desired eigenstate. Lyapunov analysis consists in showing that the closed-loop Lyapunov function is a supermartingale with exponential decay, showing exponential convergence towards the desired eigenstate.

The remaining of this chapter is structured as follows: First we remind some elementary notions of graph theory that will be needed in the sequel. Next, we present the control law and its associated stability analysis for exponential stabilization of a QND eigenstate. The stability analysis suggests that we can approximate the quantum filter by filters that only track the populations of the QND eigenstates. To do this, the terms of the filter that are due to the actuation are replaced by the terms corresponding to the associated Laplacian matrix. This endows a nice structure to the filter, the properties of the Laplacian matrix on its rows and columns imply that populations are preserved, hence it is a formulation consisting on classical probabilities. Applications towards stabilizing quantum spin systems and to generate entanglement on multi-qubit systems is discussed. Lastly, simulations suggest that the use of the reduced-filter serves as a good replacement for the quantum filter.

The results of this chapter are based on the article [24], submitted to Automatica.

5.2 Connectivity graph and Laplacian matrix

We associate to the closed-loop system (5.1) an undirected graph $G = (V, E)$ called the *connectivity graph*; V is the set of vertices, consisting on the QND eigenstates and E is the set of edges, consisting of all pairs of eigenstates directly coupled by H, that is

$$\begin{aligned} V &= \{\Pi_k,\ 1 \leq k \leq d\},\\ E &= \{(\Pi_k, \Pi_{k'}) :\ k \neq k', H_{k,k'} = H_{k',k} \neq 0, \}. \end{aligned}$$

A path in a graph is a finite sequence of edges $(\Pi_1, \Pi_2), \ldots (\Pi_s - 1, \Pi_s)$

such that $(\Pi_k, \Pi_{k+1}) \in E$ for all $k \in \{1, \ldots, s\}$. Two vertices $\Pi_j, \Pi_{j'}$ are *connected* if there exists a path joining $\Pi_j, \Pi_{j'}$. The graph $G(H)$ is said to be connected if for all $\Pi_j, \Pi_{j'} \in V$, $j \neq j'$, Π_j and $\Pi_{j'}$ are connected.

Now consider the following $d \times d$ real symmetric matrix with (k, k') elements

$$\Delta_{k,k'} = \mathrm{Tr}\left(\Pi_k \mathcal{D}_H(\Pi_{k'})\right) \tag{5.2}$$

combining the spectral decomposition of $L = \sum_k \lambda_k \Pi_k$ with the actuator Hamiltonian H. Its off-diagonal elements are non negative since $\Delta_{k,k'} = \mathrm{Tr}\left(\Pi_k H \Pi_{k'} H\right) \geq 0$ for $k \neq k'$. Its diagonal elements are non positive since

$$\begin{aligned}\Delta_{k,k} &= \mathrm{Tr}\left(\Pi_k H \Pi_k H\right) - \mathrm{Tr}\left(\Pi_k H^2\right)\\ &= \mathrm{Tr}\left(\Pi_k H \Pi_k H\right) - \mathrm{Tr}\left(\Pi_k H (\Pi_1 + \ldots + \Pi_d) H\right)\\ &= -\sum_{k' \neq k} \mathrm{Tr}\left(\Pi_k H \Pi_{k'} H\right).\end{aligned}$$

Thus Δ is a Laplacian matrix.

Proposition 5.2.1. *Assume G to be connected. For any positive reals β_k, $k \in \{1, \ldots, d\}$, $k \neq \ell$, there exists a vector $\overline{\alpha} \in \mathbb{R}^d$, $\overline{\alpha}_{k \neq \ell} > 0$, $\overline{\alpha}_\ell = 0$, such that $\Delta(H)\overline{\alpha} = \overline{\beta}$ where $\overline{\beta}$ is a vector in $\mathbb{R}^d$ of the form $\overline{\beta}^\intercal = [\beta_1, \ldots, \beta_{\ell-1}, -\sum_k \beta_k, \beta_{\ell+1}, \ldots, \beta_d]$*

Proof. This is a consecuence of known results on graph theory [14, Chapter 4]. The minimum eigenvalue of Δ is always 0, and since G is connected, it has multiplicity one with eigenvector $(1, \ldots, 1)^\intercal$. Thus, any vector orthogonal to $(1, \ldots, 1)^\intercal$ is on the image of $\Delta(H)$. The vector $\overline{\beta}$ as defined above fullfills this requirement. □

5.3 Exponential stabilization via noise-assisted feedback

We state the main convergence theorem. The main idea comes down to using noise to discourage the system to converge towards an eigenstate different from Π_ℓ. In accordance to this, we consider the closed-loop equation (5.1) with the following control:

$$\sigma(\rho) = \bar{\sigma}\ \varphi\left(\frac{\max_{k \neq \ell} \boldsymbol{p}_k(\rho) - p_{\min}}{p_{\max} - p_{\min}}\right) \tag{5.3}$$

where $\sigma \geq 0$ is a smooth saturating function in $[0, 1]$ as in the control law 4.4.1 and with parameters $\bar{\sigma} > 0$ and $1 > p_{\max} > p_{\min} > \frac{1}{2}$. Since $\sum_k \boldsymbol{p}_k = 1$, each $\boldsymbol{p}_k \geq 0$ and $p_{\min} > \frac{1}{2}$, the function $\rho \mapsto \sigma(\rho)$ is smooth despite the use of a max in its definition.

Theorem 5.3.1. *Assume that L is not degenerate, and each projector Π_k is a rank one projector. Consider the closed-loop system* (5.1) *with feedback gain $\sigma(\rho)$ given by* (5.3) *for a given projector Π_ℓ with $\ell \in \{1, \dots, d\}$. Assume that the graph associated to the Laplacian matrix Δ defined in* (5.2) *is connected. Then exists $\overline{p} \in]\frac{1}{2}, 1[$ such that for any choice of parameter $\bar{\sigma} > 0$ and parameters $1 > p_{max} > p_{min} \geq \overline{p}$, the closed-loop trajectories converges exponentially to Π_ℓ in the sense that, exist $\nu > 0$ and $C > 0$ (depending on $\bar{\sigma}$, p_{max}, p_{min}) such that, for any initial state $\rho_0 \in \mathcal{S}$, $\mathbb{E}\left[\sqrt{1 - \boldsymbol{p}_\ell(\rho_t)}\right] \leq Ce^{-\nu t}\sqrt{1 - \boldsymbol{p}_\ell(\rho_0)}$.*

If the graph associated to Δ is not fully connected then exists a partition of $\{1, \dots, d\} = I \cup J$ ($I, J \neq \emptyset$, $I \cap J = \emptyset$) such that $H = \Pi_I H \Pi_I + \Pi_J H \Pi_J$ with $\Pi_I = \sum_{k \in I} \Pi_k$ and $\Pi_J = \sum_{k \in J} \Pi_k$. Then any trajectory ρ_t of (5.1) with any feedback scheme starting from $\mathrm{Tr}\,(\rho_0 \Pi_I) = 0$ (resp. $\mathrm{Tr}\,(\rho_0 \Pi_J) = 0$), satisfies $\mathrm{Tr}\,(\rho_t \Pi_I) = 0$ (resp. $\mathrm{Tr}\,(\rho_t \Pi_J) = 0$) for $t > 0$. Thus, closed-loop convergence to $\boldsymbol{p}_\ell = 1$ with $\ell \in I$ is impossible when $\mathrm{Tr}\,(\rho_0 \Pi_I) = \sum_{k \in I} \boldsymbol{p}_k(\rho_0) = 0$. In this sense the above connectivity condition on the graph of Δ cannot be weakened.

Proof. Construction of the Lyapunov function. We do not prove directly that $\sqrt{1 - \boldsymbol{p}_\ell(\rho)}$ is a closed-loop Lyapunov function. Instead we construct a closed-loop Lyapunov function $V(\rho)$ equivalent to $\sqrt{1 - \boldsymbol{p}_\ell}$ (i.e. $c_* V(\rho) \leq \sqrt{1 - \boldsymbol{p}_\ell(\rho)} \leq c^* V(\rho)$ with $0 < c_* < c^*$) such that $\mathcal{A}V \leq -rV$. More precisely,

$$V_\alpha(\rho) = \sum_{s \in \{1, \dots, d\} \setminus \{\ell\}} \sqrt{\sum_{k \in \{1, \dots, d\} \setminus \{\ell\}} \alpha_{s,k} \boldsymbol{p}_k(\rho)} \tag{5.4}$$

where the positive parameters $\alpha_{s,k}$ are given by solving $d - 1$ linear systems as in Proposition 5.2.1, indexed by s:

$$\sum_{k'} \Delta_{k,k'} \alpha_{s,k'} = -\beta_{s,k}$$

with $\beta_{s,k} > 0$ for $k \neq \ell$ and $\beta_{s,\ell} = -\sum_{k \neq \ell} \beta_{s,k'}$. When the $(d-1) \times d$ matrix β is chosen of maximal rank $d - 1$, then the obtained matrix α is also of maximal rank $d - 1$. Since $\sum_{k \in \{1, \dots, d\} \setminus \{\ell\}} \boldsymbol{p}_k = 1 - \boldsymbol{p}_\ell$, one has

$$c_* V_\alpha(\rho) \leq \sqrt{1 - \boldsymbol{p}_\ell(\rho)} \leq c^* V_\alpha(\rho)$$

where $c_* = \frac{1}{(d-1)\sqrt{\alpha^*}}$ and $c^* = \frac{1}{(d-1)\sqrt{\alpha_*}}$ with

$$\alpha_* = \min_{s,k \in \{1, \dots, d\} \setminus \{\ell\}} \alpha_{s,k} \text{ and } \alpha^* = \max_{s,k \in \{1, \dots, d\} \setminus \{\ell\}} \alpha_{s,k}.$$

Construction of the Markov generator. The rest of the proof consists in showing that for any such choice of maximal rank matrix β, the resulting

V_α becomes an exponential Lyapunov function as soon as $\overline{p}$ is close enough to 1 and $p_{\min} > \overline{p}$. This is based on the following simple but slightly tedious computations of $\mathcal{A}V_\alpha$:

$$\mathcal{A}\mathcal{V}_\alpha(\rho) = \tfrac{\sigma^2(\rho)}{2} f_\alpha(\rho) - \tfrac{\eta}{2} g_\alpha(\rho) - \tfrac{\sigma^2(\rho)}{8} h_\alpha(\rho) \tag{5.5}$$

with

$$f_\alpha(\rho) = \sum_s \frac{\Sigma_k \alpha_{s,k} \operatorname{Tr}(\Pi_k \mathcal{D}_H(\rho))}{\sqrt{\Sigma_k \alpha_{s,k} \boldsymbol{p}_k}}$$

$$g_\alpha(\rho) = \sum_s \left(\frac{\Sigma_k \alpha_{s,k}(\lambda_k - \operatorname{Tr}(L\rho))\boldsymbol{p}_k}{\Sigma_k \alpha_{s,k}\boldsymbol{p}_k} \right)^2 \sqrt{\Sigma_k \alpha_{s,k}\boldsymbol{p}_k}$$

$$h_\alpha(\rho) = \sum_s \frac{\left(\Sigma_k i\alpha_{s,k} \operatorname{Tr}([\Pi_k, H]\rho)\right)^2}{\left(\Sigma_k \alpha_{s,k}\boldsymbol{p}_k\right)^{3/2}}.$$

These expressions rely on the following general formula based on Itō rules ($a_k > 0$ constant)

$$d\sqrt{\sum_k \alpha_k \boldsymbol{p}_k} = \frac{\sum_k \alpha_k d\boldsymbol{p}_k}{2\sqrt{\sum_k \alpha_k \boldsymbol{p}_k}} - \frac{\left(\sum_k \alpha_k d\boldsymbol{p}_k\right)^2}{8\left(\sum_k \alpha_k \boldsymbol{p}_k\right)^{3/2}}$$

with $\mathbb{E}[d\boldsymbol{p}_k] = \sigma^2(\rho) \operatorname{Tr}(\Pi_k \mathcal{D}_H(\rho))\ dt$ and

$$\mathbb{E}[(\Sigma_k \alpha_k d\boldsymbol{p}_k)^2] = \sigma^2 \left(\Sigma_k i\alpha_k \operatorname{Tr}([\Pi_k, H]\rho)\right)^2 dt \\ + 4\eta \left(\Sigma_k \alpha_k (\lambda_k - \operatorname{Tr}(L\rho))\boldsymbol{p}_k\right)^2 dt.$$

The next steps in the proof consist in showing that the Lyapunov function is exponentially converging in the sense of showing an exponential decay inequality for $\mathcal{A}\mathcal{V}_\alpha$. We treat first the regions where feedback is active, and show that this implies exponential decay at a neighborhood of the several maxima of V_α. Next, we show that, with no active feedback, $\mathcal{A}\mathcal{V}_\alpha(\rho)$ is exponentially decreasing *except* when $V_\alpha(\rho)$ is in a maximum. Since we choose a smooth control, this implies that $V(\rho)$ is a supermartingale with exponential decay for all $\rho \in \mathcal{S}$, implying global exponential convergence.

Active feedback contribution For $p \in [0, 1]$ set

$$\mathcal{S}_p \triangleq \left\{ \rho \in \mathcal{S} \mid \exists j \neq \ell,\ \boldsymbol{p}_j(\rho) \geq p \right\}.$$

Take $\rho \in \mathcal{S}_p$ with $p = 1$. Then there exists $j \in \{1, \ldots, d\}/\{\ell\}$ such that $\rho = \Pi_j$, $\sum_k \alpha_{s,k}\boldsymbol{p}_k(\rho) = \alpha_{s,j} > 0$ and

$$\Sigma_k \alpha_{s,k} \operatorname{Tr}(\Pi_k \mathcal{D}_H(\rho)) = -\beta_{s,j} < 0.$$

Consequently $f_\alpha(\rho) = -\sum_{s\in\{1,\ldots,d\}/\{\ell\}} \frac{\beta_{s,j}}{\sqrt{\alpha_{s,j}}} < 0$ with $V_\alpha(\rho) = \sum_s \sqrt{\alpha_{s,j}} > 0$.

By continuity of $f_\alpha \setminus V_\alpha$ on $\mathcal{S}_{1/2}$, exist $\epsilon_f > 0$ and $\overline{p} \in]1/2, 1[$ such that

$$\forall \rho \in \mathcal{S}_{\overline{p}}, \quad f_\alpha(\rho) \leq -\epsilon_f V_\alpha(\rho).$$

Taking $p_{\min} > \bar{p}$, we ensure that the feedback will only be turned on when it contributes a negative term to $\mathcal{A}V_\alpha$.

Open-loop contribution For all $\rho \in \mathcal{S} \setminus \{\Pi_\ell\}$, $\chi(\rho) = g_\alpha(\rho) \setminus V_\alpha(\rho)$ is well defined. Since $\mathrm{Tr}\,(\rho L) = \sum_k \lambda_k \boldsymbol{p}_k(\rho)$, the function $\chi(\rho)$ depends only on the populations $\boldsymbol{p}_k$. Consider the following parametrization exploiting the degree 0 homogeneity of χ versus the populations:

$$r = 1 - \boldsymbol{p}_\ell, \quad x_k = \boldsymbol{p}_k/(1 - \boldsymbol{p}_\ell) \text{ for } k \neq \ell.$$

For $\rho \in \mathcal{S} \setminus \{\Pi_\ell\}$, the function χ admits the following smooth expression with the variables $r \in [0, 1]$ and $x_k \in [0, 1]$ satisfying $\sum_{k \neq \ell} x_k = 1$:

$$\chi(r, x) = \frac{\sum_{s \neq \ell} \left(\frac{\sum_{k \neq \ell} \alpha_{s,k}(\lambda_k - \varpi(r,x)) x_k}{\sum_{k \neq \ell} \alpha_{s,k} x_k} \right)^2 \sqrt{\sum_{k \neq \ell} \alpha_{s,k} x_k}}{\sum_{s \neq \ell} \sqrt{\sum_{k \neq \ell} \alpha_{s,k} x_k}}$$

with $\varpi(r, x) = (1 - r)\lambda_\ell + r \left(\sum_{k \neq \ell} \lambda_k x_k \right)$. Clearly $\chi(r, x) \geq 0$. Consider the solutions (r, x) of $\chi(r, x) = 0$. Necessarily, they satisfy

$$\forall s \neq \ell, \quad \sum_{k \neq \ell} \alpha_{s,k}(\lambda_k - \varpi(r, x)) x_k = 0.$$

Since the $(d - 1) \times d$ matrix $\alpha_{s,k}$ is of maximal rank $d - 1$, $\forall k \neq \ell$ we have $(\lambda_k - \varpi(r, x)) x_k = 0$. Taking the sum versus $k \neq \ell$, one gets $(1 - r) \left(\lambda_\ell - \sum_{k \neq \ell} \lambda_k x_k \right) = 0$ implying two possibilities:

- if $\lambda_\ell = \sum_{k \neq \ell} \lambda_k x_k$, then $\varpi(r, x) = \lambda_\ell$ and $(\lambda_k - \lambda_\ell) x_k = 0$ for $k \neq \ell$. Since $\lambda_k \neq \lambda_\ell$ this implies that $x_k = 0$. But this is not possible since $\sum_{k \neq \ell} x_k = 1$.
- if $r = 1$, then $\varpi(r, x) = \sum_{k' \neq \ell} \lambda_{k'} x_{k'}$. But for $k \neq \ell$ one has

$$\left(\lambda_k - \sum_{k' \neq \ell} \lambda_{k'} x_{k'} \right) x_k = 0.$$

  Since $\lambda_k \neq \lambda_{k'}$ for $k \neq k'$ and $x_k \in [0, 1]$ with $\sum_{k \neq \ell} x_k = 1$, exits necessarily $j \neq \ell$ such that $x_j = 1$ and $x_k = 0$ for $k \neq \ell, j$.

Consequently, the non negative smooth function $\chi(r, x)$ vanishes only at $d - 1$ isolated points, $r = 1$ and $x_k = \delta_{kj}$ labelled by $j \neq \ell$. By continuity for any $p < 1$ ($p > 0$), exists $\theta_p > 0$ such that $\forall \rho \in \mathcal{S} \setminus \mathcal{S}_p$ different of Π_ℓ, $\chi(\rho) \geq \theta_p$. This proves that

$$\forall \rho \in \mathcal{S} \setminus \mathcal{S}_p, \quad g_\alpha(\rho) \geq \theta_p V_\alpha(\rho).$$

To conclude, consider $\mathcal{A}V_\alpha$ given in (5.5). Since $h_\alpha \geq 0$, we have

$$\forall \rho \in \mathcal{S}, \quad \mathcal{A}\mathcal{V}_\alpha(\rho) \leq \tfrac{\sigma^2(\rho)}{2} f_\alpha(\rho) - \tfrac{\eta}{2} g_\alpha(\rho).$$

Consider the feedback gain $\sigma(\rho)$ with $p_{\min} > \overline{p}$. Then

$$\forall \rho \in \mathcal{S}_{p_{\max}}, \quad \mathcal{A}\mathcal{V}_\alpha(\rho) \leq -\tfrac{\bar{\sigma}^2}{2} \epsilon_f V_\alpha(\rho)$$

and

$$\forall \rho \in \mathcal{S} \setminus \mathcal{S}_{p_{\max}}, \quad \mathcal{A}\mathcal{V}_\alpha(\rho) \leq -\tfrac{\eta}{2} \theta_{p_{\max}} V_\alpha(\rho).$$

For all ρ in $\mathcal{S}$, one has $\mathcal{A}\mathcal{V}_\alpha(\rho) \leq -\nu V_\alpha(\rho)$ where ν is equal to $\min\left(\frac{\bar{\sigma}^2 \epsilon_f}{2}, \frac{\eta \theta_{p_{\max}}}{2}\right)$. A direct application of Theorem A.0.1 recalled in Appendix ensures $\mathbb{E}(V_\alpha(\rho_t)] \leq V_\alpha(\rho_0) e^{-\nu t}$.

□

5.4 Approximated quantum filtering

We now address the construction of the reduced-order filter. A first reduction consists in replacing ρ_t in the feedback law by $\widehat{\varrho}_t$ corresponding to the Bayesian estimate of ρ_t knowing ρ_0 and Y_τ for $\tau \in [0, t]$. One discards here the knowledge of B_t. Then, $\widehat{\varrho}_t$ obeys to the following stochastic differential equation:

$$d\widehat{\varrho}_t = \mathcal{D}_L(\widehat{\varrho}_t)dt + \sqrt{\eta}\mathcal{M}_L(\widehat{\varrho}_t)\big(dY_t - 2\sqrt{\eta}\,\mathrm{Tr}\left(\widehat{\varrho}_t L\right) dt\big) \\ + \sigma(\widehat{\varrho}_t)^2 \mathcal{D}_H(\widehat{\varrho}_t)dt. \quad (5.6)$$

This reduced filter (5.6) does not admit a closed-form expression in terms of $\boldsymbol{p}_k(\widehat{\varrho}) = \mathrm{Tr}\left(\widehat{\varrho}\Pi_k\right)$ in general, as $\mathrm{Tr}\left(\mathcal{D}_H(\widehat{\varrho})\Pi_k\right)$, includes coherences of the type $< k|\rho|k' >$. The stability analysis of section 5.3 suggest an approximation.

Let $\boldsymbol{p}_j(\rho) > 1 - \epsilon$, for some $j \neq \ell$ and some $1 - \epsilon > p_{\min}$. Consider a decomposition of ρ into $\rho = \sum_{s=1}^d p_s |s\rangle\langle s|$, $\sum_s p_s = 1$, for some orthonormal basis $\{|s\rangle\}_{s=1}^d$. Enumerate the states as $\tilde{s}$ the $|\tilde{s}\rangle \in \mathrm{Range}(\Pi_j)$, $|s^\perp\rangle \in \mathrm{Kernel}(\Pi_j)$; the constraint on $\mathrm{Tr}\left(\rho\Pi_j\right)$ gives $1 - \epsilon \leq \mathrm{Tr}\left(\Pi_j \rho \Pi_j\right) = \sum_s p_s \mathrm{Tr}\left(\Pi_j |s\rangle\langle s| \Pi_j\right) = \sum_{\tilde{s}} p_{\tilde{s}} \langle j||\tilde{s}\rangle\langle\tilde{s}||j\rangle$ which implies that $\min_{s,j} |\langle\tilde{s}||j\rangle| \sum_{\tilde{s}} p_{\tilde{s}} \geq 1 - \epsilon$, similarly $\sum_{s\perp} p_{s^\perp} \leq \epsilon$. From both it follows that $\|\Pi_j \rho (I - \Pi_j)\|_F^2 = \sum_{\tilde{s}, s^\perp} p_{\tilde{s}} p_{s^\perp} |s^\perp\rangle\langle\tilde{s}| \leq \epsilon$

Write $\rho = \Pi_j\rho\Pi_j + (I-\Pi_j)\rho\Pi_j + \Pi_j\rho(I-\Pi_j) + (I-\Pi_j)\rho(I-\Pi_j)$. Then

$$
\begin{aligned}
&\mathrm{Tr}\,(\mathcal{D}_H(\rho)\Pi_k)\\
&\quad\leq \mathrm{Tr}\left(\mathcal{D}_H\big(\sum_{\tilde{s}} p_{\tilde{s}}\Pi_j|\tilde{s}\rangle\langle\tilde{s}|\Pi_j\big)\Pi_k\right) + (2\sqrt{\epsilon}+\epsilon)\|\mathcal{D}_H(\Pi_k)\|_F\\
&\quad= \sum_{\tilde{s}} p_{\tilde{s}}|\langle\tilde{s}||j\rangle|\Big(\mathrm{Tr}\,(\mathcal{D}_H(\Pi_j)\Pi_k) + (2\sqrt{\epsilon}+\epsilon)\|\mathcal{D}_H(\Pi_k)\|_F\Big)\\
&\qquad\quad= -\sum_{\tilde{s}} p_{\tilde{s}}|\langle\tilde{s}||j\rangle|\Big(\Delta_{k,j}(H) + (2\sqrt{\epsilon}+\epsilon)\|\mathcal{D}_H(\Pi_k)\|_F\Big). \qquad (5.7)
\end{aligned}
$$

Thus, given a small ϵ, the dissipation term $\mathrm{Tr}\,(\mathcal{D}_H(\rho)\Pi_\ell)$ admits an approximation in terms of the eigenstate populations of L. We discard the positive terms of order $\sqrt{\epsilon}$ and consider then the following approximated filter for populations $\widehat{\boldsymbol{p}}_k$ to estimate $\boldsymbol{p}_k(\widehat{\varrho})$:

$$
\begin{aligned}
d\widehat{\boldsymbol{p}}_k = 2\sqrt{\eta}\widehat{\boldsymbol{p}}_k(\lambda_k - \varpi(\widehat{\boldsymbol{p}}))(dY_t - 2\sqrt{\eta}\varpi(\widehat{\boldsymbol{p}})dt)&\\
+ \sigma^2(\widehat{\boldsymbol{p}})\sum_{k'=1}^{d}\Delta_{k,k'}\,\widehat{\boldsymbol{p}}_{k'}\,dt& \qquad (5.8)
\end{aligned}
$$

where $\varpi(\widehat{\boldsymbol{p}}) = \sum_{k'=1}^{d}\lambda_{k'}\widehat{\boldsymbol{p}}_{k'}$ with $\widehat{\boldsymbol{p}} = (\widehat{\boldsymbol{p}}_1,\ldots,\widehat{\boldsymbol{p}}_d)$, where the Laplacian matrix is given by (5.2) and the coherences $\langle k|\widehat{\varrho}|k'\rangle$, $k \neq k'$, are set to zero in (5.8). The above approximate filter requires to store and update in real-time only d real numbers. Moreover it has a nice structure: standard arguments show that, for any measurement trajectory Y_t, the components of $\widehat{\boldsymbol{p}}$ remain non negative and their sum remains equal to one. Moreover, in open-loop ($\sigma \equiv 0$), this population filter is exact.

5.5 Simulations

This section is devoted to numerical estimation of closed-loop convergence rates and investigation of the related robustness on a specific physical quantum systems already considered in the litterature [50, 62, 77].

In this section we consider a quantum spin systems with fixed angular momentum J. This is a system of dimension $n = 2J+1$ where the measurement operator L is the diagonal matrix (measurement along the z-axis)

$$L = \sum_{m=0}^{2J}(J-m)|J\text{-}m\rangle\langle J\text{-}m|$$

and the actuator Hamiltonian H is a tri diagonal matrix (rotation around the y-axis)

$$H = \sum_{m=0}^{2J-1}\frac{\sqrt{(m+1)(2J-m)}}{2i}\left(|J\text{-}m\rangle\langle J\text{-}m\text{-}1| - |J\text{-}m\text{-}1\rangle\langle J\text{-}m|\right)$$

The Hilbert basis is made of the $2J + 1$ ortho-normal vectors $|J\text{-}m\rangle$ for $m = 0, \dots, 2J$.

All the simulations below correspond to $J = 2$ ($n = 5$), detection efficiency $\eta = 8/10$:

$$L = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \text{-1} & 0 \\ 0 & 0 & 0 & 0 & \text{-2} \end{pmatrix}, \quad H = \frac{1}{2i} \begin{pmatrix} 0 & -2 & 0 & 0 & 0 \\ 2 & 0 & -\sqrt{6} & 0 & 0 \\ 0 & \sqrt{6} & 0 & -\sqrt{6} & 0 \\ 0 & 0 & \sqrt{6} & 0 & -2 \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix}.$$

According to lemma (3.1.1), the open-loop convergence rate is $\eta/2 = 0.4$. The control goal is to stabilize the spin-0 state Π_ℓ associated to the zero eigenvalue of L. For the noise-assisted feedback gain (5.3), we take $\bar{\sigma} = \sqrt{5\eta}$, $p_{\max} = p_{\min} + 0.05$, $p_{\min} = 0.9$ or $p_{\min} = 0.6$ with the saturation function $\varphi(s) = \min(1, \max(0, s))$. A closed-loop simulation set is made of 1000 realizations starting from the fully depolarized state $\rho_0 = \boldsymbol{I}/5$. For each simulation set, we get an approximation of the ensemble average trajectory $t \mapsto \sqrt{1 - \mathrm{Tr}\,(\Pi_\ell \rho_t)} \equiv \sqrt{1 - \langle 0|\rho_t|0\rangle}$ and compute an estimate of its exponential converge rate following Theorem 5.3.1 since the Laplacian matrix Δ

$$\Delta = \begin{pmatrix} \text{-1} & 1 & 0 & 0 & 0 \\ 1 & \text{-}\frac{5}{2} & \frac{3}{2} & 0 & 0 \\ 0 & \frac{3}{2} & \text{-3} & \frac{3}{2} & 0 \\ 0 & 0 & \frac{3}{2} & \text{-}\frac{5}{2} & 1 \\ 0 & 0 & 0 & 1 & \text{-1} \end{pmatrix},$$

inherits of the tridiagonal structure of H and thus admits a connected graph.

Figure 5.1 illustrates a first simulation set with $p_{\min} = 0.9$ close to 1 for which one should have, according to Theorem 5.3.1, exponential convergence, estimated here to $\nu \approx 0.04$. This closed-loop rate is much smaller than the open-loop one $\eta/2 = 0.4$. This is in agreement with the rate ν implicitly exhibited by the proof of theorem 5.3.1 assuming $p_{\max}$ close to 1 (here $p_{\max} = 0.95$).

In the second simulation set of figure 5.2, one decreases to $p_{\max} = 0.65$ and observes a much faster convergence rate around 0.2, one half of the open-loop convergence rate. This indicates that such noise-assisted feedback could be tuned to achieve convergence rate similar to the open-loop one.

In the above simulations, the feedback is based on the ideal value ρ_t of the quantum state. In the simulation sets below, the feedback depends on the approximated population vector $\widehat{\boldsymbol{p}}$ obtained by the low-order approximate filter (5.8) where the simulation set of figure 5.3a differs from the one of figure 5.2 just by replacing in the feedback law the ideal populations $\boldsymbol{p}$ by the approximated ones $\widehat{\boldsymbol{p}}$ solutions of (5.8). One observes a reasonable decrease of the convergence rate form 0.19 to 0.12 illustrating the practical

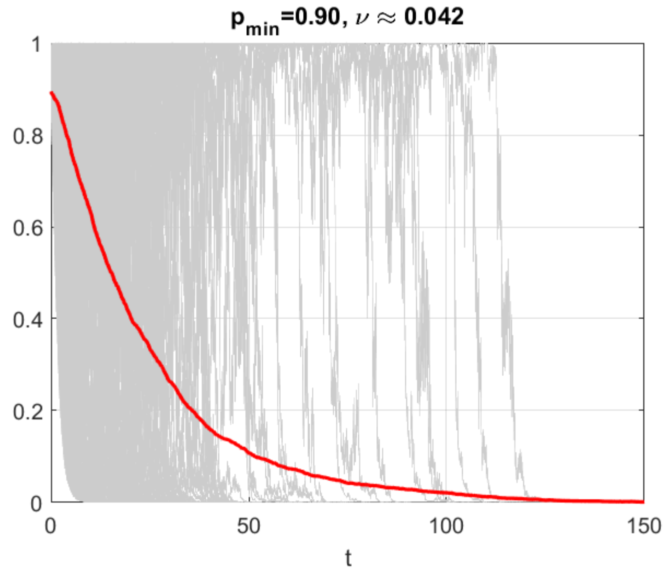


Figure 5.1: Ideal closed-loop simulations with $p_{\min} = 0.9$. In gray: selection of 200 individual trajectories $t \mapsto \sqrt{1 - \mathrm{Tr}\left(\Pi_\ell \rho_t\right)}$; In red: average over 1000 realizations.

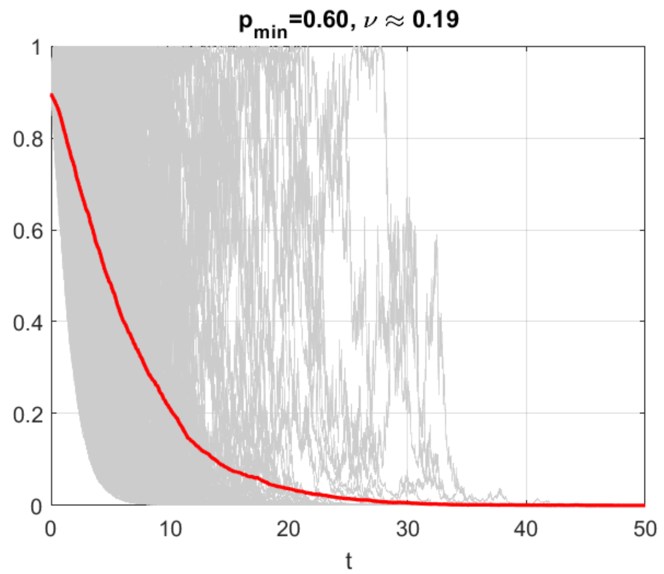


Figure 5.2: Ideal closed-loop simulations with $p_{\min} = 0.6$. In gray: selection of 200 individual trajectories $t \mapsto \sqrt{1 - \mathrm{Tr}\left(\Pi_\ell \rho_t\right)}$; In red: average over 1000 realizations.

interest of such low-dimensional filters. Last simulation set of figure 5.3b evaluates the impact of a feedback delay of 0.5 time unit in the closed-loop simulations of figure 5.3a. Despite a decrease by a factor two of the convergence rate, feedback latency of 1/4 of the open-loop convergence time does not destabilize this feedback scheme which appears here to have promising robustness properties.

5.6 Conclusions

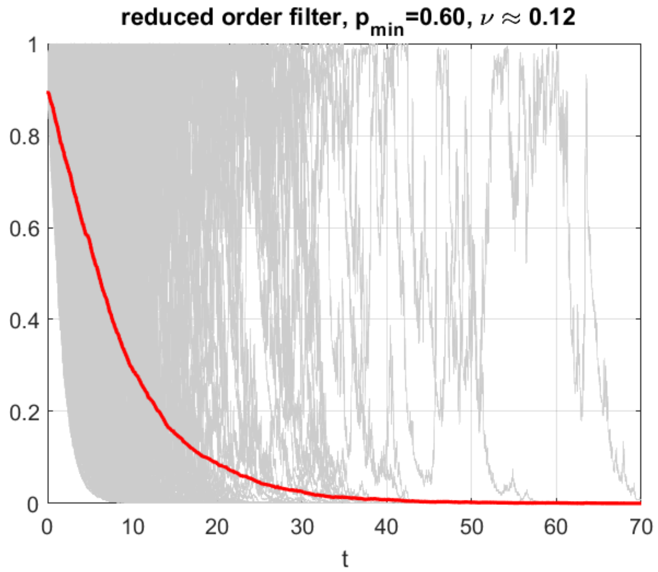
We have shown that the use of Brownian motion to drive the controls allows for exponential stabilization of a prescribed QND eigenstate. A crucial difference on the stability analysis with respect to other chapters, is that of the connectivity graph induced by the feedback Hamiltonian. Lyapunov analysis is more subtle, since it needs to take into account the direct and indirect connections between QND eigenstates to achieve exponential stabilization. Here, we introduced a Lyapunov function which takes into account this connections, while still being related to a measure of a distance. Combining the feedback action with the measurement dynamics, we prove exponential stability of the target eigenstate.

Some themes in this chapter still leave room for improvement, in at least, the following directions:

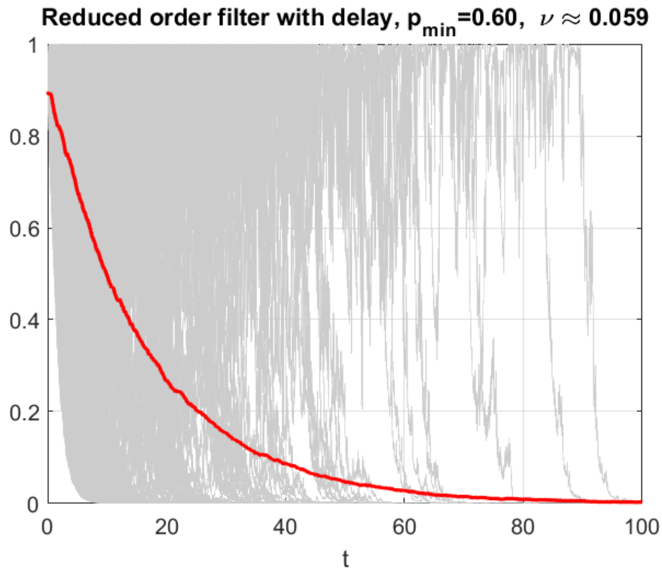
- While our proof of exponential convergence can provide an estimate of the convergence rate, the Lyapunov function nor the other control parameters where not optimized in order to maximize the speed of convergence. We have shown numerically that the closed-loop convergence rate apparently can be made similar to the open-loop convergence rate with our approach, suggesting that a precise analysis of convergence rate can be done.

- In numerical simulations, the reduced approximate filter (5.8) appears good enough to achieve global exponential stabilization. We conjecture that this can be proven, for this filter. In the next Chapter we find a case for which it is possible to formulate an approximate reduced quantum filter that is expressed solely in terms of the eigenstate populations.

- The capability of performing several quantum measurements and of applying different unitary feedback controls on a single quantum system motivates the study of multi-input multi-output (MIMO) quantum feedback schemes. A MIMO version of static output feedback was introduced in [26] and an implementation was made in [21] considering non-commuting measurement operators. In contrast, for QND systems with multiple measurements, these operators must commute



(a) Closed-loop simulations with $p_{\min} = 0.6$ with the approximate population filter (5.8). In gray: selection of 200 individual trajectories $t \mapsto \sqrt{1 - \mathrm{Tr}\,(\Pi_\ell \rho_t)}$; In red: average over 1000 realizations.



(b) Closed-loop simulations with $p_{\min} = 0.6$ with the approximate population filter (5.8) and feedback delay of 0.5. In gray: selection of 200 individual trajectories $t \mapsto \sqrt{1 - \mathrm{Tr}\,(\Pi_\ell \rho_t)}$; In red: average over 1000 realizations.

Figure 5.3: Simulation of the quantum spin system with a reduced filter and subject to imperfections.

so that there exist common eigenspaces. A general MIMO extension of Theorem 5.3.1 should be feasible along the same lines as the present work.

Chapter 6

On continuous-time quantum error correction

Ce chapitre étudie le problème de correction d'erreurs quantiques en temps continu pour le code de correction d'erreurs à trois qubits. Ce code correcteur utilise la redondance pour protéger l'information quantique. Des mesures de parité sur les qubits voisins permettent de localiser les erreurs, et des actions du contrôle unitaires sur chaque qubit individuel permettent de faire la correction. En temps continu, cela implique de rendre une variété cible d'états quantiques globalement attractive. Nous adaptons la loi du contrôle assisté par bruit aux spécificités du problème de la correction d'erreurs. Il est possible de montrer que la dynamique en boucle fermée résultante stabilise la variété cible de manière exponentielle. Néanmoins, il y a quelques différences par rapport aux problémes de stabilisation précédents qui doivent être prises en compte ici: d'abord, ayant une donnée initiale appartenant à la varieté cible, il est nécessaire de montrer qu'elle est protégée des erreurs detectées ; deuxièmement, les actions de contrôle doivent être minimisées pour que des erreurs additionnelles à cause de ladite action soient évitées. En outre, nous présentons une formulation de filtre d'ordre réduit, qui est basé sur des probabilités classiques. L'étude sur la protection induite par la dynamique de correction d'erreur en boucle fermée contre les perturbations est faite numériquement. Nous étudions numéiquement la performance de cette loi de contrôle et du filtre réduit. La procédure de contrôle considerée dans ce chapitre constitue une première extension à plusieurs entrées et sorties — en anglais multi-input multi-output (MIMO) — du contrôle assisté par bruit considéré dans les chapitres précédents.

6.1 Introduction

The development of methods for the protection of quantum information in the presence of disturbances is essential to improve existing quantum technologies [58, 54]. Decoherence is regarded as the major obstacle towards scalable and robust quantum information processing, caused by the interaction between the system of interest and other subsystems present in the environment. Quantum error correction (QEC) codes have emerge as a tool to fight this decoherence, by encoding a logical state into multiple physical states. Similarly to classical error correction, this redundancy allows to protect quantum information from disturbances by stabilizing a *submanifold* of steady states, which represent the nominal logical states [45, 53]. As long as a disturbance does not drive the system out of the basin of attraction of the original nominal state, the logical information remains unperturbed. To stabilize the nominal submanifold in a quantum system, a *syndrome diagnosis* stage performs quantum non-destructive (QND) measurements which extract information about the disturbances without perturbing the encoded data. Based on this information, a *recovery* feedback action restores the corrupted state.

QEC is most often presented as discrete-time operations towards digital quantum computing, see e.g. [53]. Not only the design of the underlying control layer, but also the proposal of analog quantum technologies, like solving optimization problems by quantum annealing, motivate a study of QEC in continuous-time, among them using the tools of *reservoir engineering* and *measurement-based feedback*.

Reservoir engineering techniques consists on coupling the system that we wish to control with a dissipative ancilliary quantum system, such that the entropy introduced by errors on the main system is evacuated through the dissipation of the ancilliary one. Methods of reservoir engineering for autonomous quantum error correction [51, 28, 27] aim towards hardware efficient quantum computing [36]. An advantage of this approach is that there is no need for a complicated external control loop to correct errors. However, the challenge is to implement the specific ancillary system and coupling within experimental constraints.

Advances in experimental methods for performing high-fidelity quantum measurements encourage the consideration of measurement-based feedback strategies for quantum information processing tasks. In the context of QEC, this has been addressed in [1, 2, 60, 47], essentially as proposals illustrated by simulation. The short dynamical time scales encountered in experimental setups represent one of the main obstacles to implement complex feedback laws. Furthermore, the data acquisition and post-processing of the measurement signal leads to a latency in the feedback procedures. This motivates the development of efficiently computable control techniques that are robust against unmodeled dynamics.

The goal of this chapter is to establish analytical results about the convergence rate of QEC systems towards the nominal submanifold, a prerequisite for analytically quantifying the protection of quantum information. It is indeed well known that exponential stability is an indicator of robustness with respect to unmodeled dynamics.

To obtain exponential convergence in a compact space, it is necessary to suppress any spurious unstable equilibria that might remain in the closed-loop dynamics. In the present chapter, in the context of QEC, we propose as well a *noise-assisted* quantum feedback, acting with Brownian noise whose gain is adjusted in real-time. We show via standard stochastic Lyapunov arguments that this new approach renders the target subspace, containing the nominal encoding of quantum information, globally exponentially stable thanks to feedback from syndrome measurements.

Furthermore, our strategy allows to work with a reduced state estimator: while other feedback schemes require to keep track of quantum coherences, our controller only tracks the populations on the various joint eigenspaces of the measurement operators (via classical Bayesian estimation). From a data statistics viewpoint, this corresponds to a reduced-order formulation that depends only on classical probabilities. Since up to quantum noise this information is directly proportional to the measurement outputs, it could open the door towards using even simpler filters in practical setups.

The structure of the rest of this chapter is as follows: In section 6.2 we present the dynamical model of the three-qubit bit-flip code, which is the most basic model in QEC. In section6.3, we review the previous work that has been done on QEC using measurement based feedback. In section 6.4 we introduce our approach to feedback using noise and we prove exponential stabilization of the target manifold of the three-qubit bit-flip code. Section 6.4.4 examines the performance of this feedback to protect quantum information from bit-flip errors.

The results of this chapter are based on the article [23], to appear at Proceedings of the 11th IFAC Symposium on Nonlinear Control Systems.

6.2 Dynamics of the three-qubit bit-flip code

The three-qubit bit-flip code corresponds to a Hilbert space $\mathcal{H} = (\mathbb{C}^2)^{\otimes 3} \simeq \mathbb{C}^8$, where $\otimes$ denotes tensor product (Kronecker product, in matrix representation). We denote $\boldsymbol{I}_n$ the identity operator on $\mathbb{C}^n$ and we write $\boldsymbol{X}_k$, $\boldsymbol{Y}_k$ and $\boldsymbol{Z}_k$ the local Pauli operators acting on qubit k, e.g. $\boldsymbol{X}_2 = \boldsymbol{I}_2 \otimes \boldsymbol{X} \otimes \boldsymbol{I}_2$. We denote $\{|0\rangle, |1\rangle\}$ the usual basis states, i.e. the -1 and +1 eigenstates of the $\boldsymbol{Z}$ operator on each individual qubit [53].

The encoding on this 3-qubit system is meant to counter bit-flip errors, which can map a ± 1 eigenstate of $\boldsymbol{Z}_k$ to a ∓ 1 eigenstate for each $k = 1, 2, 3$. More precisely, the nominal encoding for a logical information 0 (resp. 1) is on the state $|000\rangle$ (resp. $|111\rangle$). A single bit-flip on e.g. the first qubit brings this to $\boldsymbol{X}_1|000\rangle = |100\rangle$ (resp. $|011\rangle$), which by majority vote could be brought back to the nominal encoding.

Bit-flip errors occurring with a probability $\gamma_k \, dt \ll 1$ during a time interval $[t, t+dt]$ are modeled by disturbance channels, with $L_{k+3} = \sqrt{\gamma_k}\,\boldsymbol{X}_k$ and $\eta_{k+3} = 0$, $k = 1, 2, 3$. The measurements needed to implement "majority vote" corrections, so-called syndromes, continuously compare the $\boldsymbol{Z}$ value of pairs of qubits. The associated measurements correspond to $L_k = \sqrt{\Gamma_k}\,\boldsymbol{S}_k$ for $k = 1, 2, 3$, with $\boldsymbol{S}_1 = \boldsymbol{Z}_2\boldsymbol{Z}_3$, $\boldsymbol{S}_2 = \boldsymbol{Z}_1\boldsymbol{Z}_3$, $\boldsymbol{S}_3 = \boldsymbol{Z}_1\boldsymbol{Z}_2$ and Γ_k representing the measurement strength. This yields the following open-loop system with multiple measurement channels:

$$d\rho = \sum_{k=1}^{3} \Gamma_k \mathcal{D}_{\boldsymbol{S}_k}(\rho)dt + \sqrt{\eta_k\Gamma_k}\mathcal{M}_{\boldsymbol{S}_k}(\rho)dW_k + \sum_{s=1}^{3}\gamma_s \mathcal{D}_{\boldsymbol{X}_s}(\rho)dt, \tag{6.1}$$

$$dY_k = 2\sqrt{\eta_k\Gamma_k}\,\mathrm{Tr}\left(\boldsymbol{S}_k\rho\right)dt + dW_k, \quad k \in \{1, 2, 3\}. \tag{6.2}$$

We further define the operators:

$$\Pi_{\mathcal{C}} = \tfrac{1}{4}(\boldsymbol{I}_8 + \sum_{k=1}^{3}\boldsymbol{S}_k)\ ,\ \ \Pi_j := \boldsymbol{X}_j\Pi_{\mathcal{C}}\boldsymbol{X}_j, j \in \{1, 2, 3\}, \tag{6.3}$$

corresponding to orthogonal projectors onto the various joint eigenspaces of the measurement syndromes. The first one $\Pi_{\mathcal{C}}$ projects onto the nominal code $\mathcal{C} := \mathrm{span}(|000\rangle, |111\rangle)$ (+1 eigenspace of all the $\boldsymbol{S}_k$), whereas Π_j projects onto the subspace where qubit j is flipped with respect to the two others. Similarly to the single measurement case, we define for each $k \in \{C, 1, 2, 3\}$,

$$\boldsymbol{p}_{t,k}(\rho) := \mathrm{Tr}\left(\Pi_k\rho_t\right)\ \geq 0$$

the so-called population of subspace k, i.e. the probability that a projective measurement of the syndromes would give the output corresponding to subspace k. By the law of total probabilities we have $\sum_{k\in\{C,1,2,3\}}\boldsymbol{p}_{t,k} = 1$ for all t.

We have the following behavior in absence of feedback actions and disturbances. This is the equivalent, for invariant subspaces, of Lemma 3.1.1.

Lemma 6.2.1. *Consider equation* (6.1) *with* $\gamma_s = 0$ *for* $s \in 1, 2, 3$*.*

(i) For each $k \in \{C, 1, 2, 3\}$*, the subspace population* $\boldsymbol{p}_{t,k}(\rho)$ *is a martingale i.e.* $\mathbb{E}(\boldsymbol{p}_{t,k}(\rho)|p_{0,k}(\rho)) = \boldsymbol{p}_{0,k}(\rho)$ *for all* $t \geq 0$*.*

(ii) For a given ρ_0*, if there exists* $\bar{k} \in \{C, 1, 2, 3\}$ *such that* $\boldsymbol{p}_{0,\bar{k}}(\rho) = 1$ *and* $\boldsymbol{p}_{0,k}(\rho) = 0$ *for all* $k \neq \bar{k}$*, then* ρ_0 *is a steady state of* (6.1)*.*

(iii) The Lyapunov function

$$V(\rho) = \sum_{k\in\{\mathcal{C},1,2,3\}}\sum_{k'\neq k}\sqrt{\boldsymbol{p}_k\boldsymbol{p}_{k'}(\rho)},$$

decreases exponentially as

$$\mathbb{E}[V(\rho_t)] \leq e^{-rt}V(\rho_0)$$

for all $t \geq 0$, with rate

$$r = 4 \min_{k \in \{1,2,3\}} \eta_k \Gamma_k .$$

In this sense the system exponentially approaches the set of invariant states described in point (ii).

Proof. The first two statements are easily verified, following the same lines as in the case with a single measurement (6.2.1). We prove the last one. The variables $\xi_j = \sqrt{\boldsymbol{p}_j}$, $j \in \{1, 2, 3, \mathcal{C}\}$ satisfy the following SDE's:

$$\begin{aligned} d\xi_{\mathcal{C}} = -2\xi_{\mathcal{C}} \Big(\sum_{k \in \{1,2,3\}} \eta_k \Gamma_k (1 - \xi_{\mathcal{C}}^2 - \xi_k^2)^2 \Big)\, dt \\ + 2\xi_{\mathcal{C}} \Big(\sum_{k \in \{1,2,3\}} \sqrt{\eta_k \Gamma_k} (1 - \xi_{\mathcal{C}}^2 - \xi_k^2)\, dW_k \Big) , \end{aligned}$$

$$\begin{aligned} d\xi_{j \neq \mathcal{C}} = -2\xi_j \Big(\eta_j \Gamma_j (1 - \xi_{\mathcal{C}}^2 - \xi_j^2)^2 + \sum_{k \in \{1,2,3\} \setminus j} \eta_k \Gamma_k (\xi_{\mathcal{C}}^2 + \xi_k^2)^2 \Big)\, dt \\ + 2\xi_j \Big(\sqrt{\eta_j \Gamma_j} (1 - \xi_{\mathcal{C}}^2 - \xi_j^2)\, dW_j - \sum_{k \in \{1,2,3\} \setminus j} \sqrt{\eta_k \Gamma_k} (\xi_{\mathcal{C}}^2 + \xi_k^2)\, dW_k \Big) , \end{aligned}$$

while $V = \sum_{k \in \{\mathcal{C},1,2,3\}} \sum_{k' \neq k} \xi_k \xi_{k'}$. Noting that $2(1 - \xi_{\mathcal{C}}^2 - \xi_k^2)$ and $2(\xi_{\mathcal{C}}^2 + \xi_k^2)$ just correspond to $1 \pm \mathrm{Tr}\,(\rho \boldsymbol{S}_k)$, we only have to keep track of $\pm$ signs in the various terms to compute

$$\mathcal{A}V = -2 \sum_{k \in \{\mathcal{C},1,2,3\}} \sum_{j \in \{\mathcal{C},1,2,,3\} \setminus k} \xi_j \xi_k \sum_{l \in \{1,2,3\}} \epsilon_{j,k,l} \eta_l \Gamma_l$$

where, for each pair (j, k), the selector $\epsilon_{j,k,l} \in \{0, 1\}$ equals 1 for two l values, namely $\epsilon_{\mathcal{C},k,l} = \epsilon_{k,\mathcal{C},l} = 1$ if $l \neq k \in \{1, 2, 3\}$ and $\epsilon_{j,k,j} = \epsilon_{j,k,k} = 1$ for $j, k \in \{1, 2, 3\}$. This readily leads to $\mathcal{A}V \leq -4 \min_{k \in \{1,2,3\}} (\eta_k \Gamma_k)\, V$.

We conclude by Theorem A.0.1 and noting that $V = 0$ necessarily corresponds to a state as described in point (ii). □

The above Lyapunov function describes the convergence of the state towards $\mathrm{Tr}\,(\Pi_k \rho) = 1$, for a random subspace $k \in \{\mathcal{C}, 1, 2, , 3\}$ chosen with probability $\boldsymbol{p}_{0,k}$. We now address how to render one particular subspace globally attractive, here the one associated to nominal codewords and with projector $\Pi_{\mathcal{C}}$.

6.3 Some open issues on continuous-time QEC

Quantum error correction in continuous time with measurement-based feedback was first investigated in by Ahn et.al. [1]. Subsequent works [2, 60, 47] have been looking to improve mainly one issue, which is addressing the real-time implementation of the control layer. The use of a full quantum filter seems unnecessary for this application, as the measurements are specifically tailored to localize the errors. Thus, in open loop, it is possible to design reduced filters that only track errors in real time. The challenge comes when considering the feedback action into the filter. Another point is that control design could be improved in order to provide estimates of convergence. We review the existing work on the subject to highlight the challenges; as in previous chapters, it is by changing the actuation strategy that we will address both issues.

The continuous-time closed-loop model considered in previous works [1, 2, 60, 47] considers a deterministic actuation of the form

$$d\rho = -i\sum_{j=1}^{3} u_j(\rho)[\boldsymbol{X}_j,\rho]dt + \sum_{k=1}^{3}\Gamma_k\mathcal{D}_{\boldsymbol{S}_k}(\rho)dt + \sqrt{\eta_k\Gamma_k}\mathcal{M}_{\boldsymbol{S}_k}(\rho)dW_k + \sum_{s=1}^{3}\gamma_s\mathcal{D}_{\boldsymbol{X}_s}(\rho)dt, \quad (6.4)$$

the role of the controls u_j, $j \in \{1,2,3\}$ is to counteract the error terms $\mathcal{D}_{X_s}(\rho)$ present in Eq. (6.4). The control design goal is to minimize the distance to the codespace manifold $V'(\rho) = 1 - \mathrm{Tr}\,(\rho\Pi_{\mathcal{C}})$. In the absence of perturbations, the control law

$$u_j(\rho) = -\lambda\,\mathrm{sign}\big(\,\mathrm{Tr}\,(i[\boldsymbol{X}_j,\rho]\Pi_{\mathcal{C}})\,\big), \quad j \in \{1,2,3\}), \quad \lambda > 0$$

makes

$$\frac{d}{dt}\mathbb{E}[V'(\rho)] = \mathbb{E}[\sum_{j=1}^{3} u_j(\rho)\,\mathrm{Tr}\,(i[\boldsymbol{X}_j,\rho]\Pi_{\mathcal{C}})]$$

negative. This only shows that $\mathrm{Tr}\,(\rho\Pi_{\mathcal{C}})$ is increasing on average. It does not mean that, for *all* individual realizations, $\mathrm{Tr}\,(\rho\Pi_{\mathcal{C}})$ converges to 1. Indeed, the above control law does not prevent convergence towards a state supported on an error eigenspace, since $u_j(\rho) = 0$ for any state that commutes with $\Pi_{\mathcal{C}}$. Other control proposals [60, 2] do not provide similar stabilization statements. Optimal control strategies have also been considered [47], by monitoring the error populations and applying discrete kicks when the population on a particular error subspace is close to one. This could have a negative effect on implementations, as in practice it could excite spurious subsystems that are not usually taken into account in the approximations used to model the nominal system.

Another point is with respect to the issues related to real-time implementation of the feedback law. Implementing the quantum filter verbatim on an experiment would not be without challenges, since in this simplified setting, as the dimension of the Hilbert space grows as 2^d for d qubits. It is fundamental to show that we can reduce the overhead of computing the quantum filter. Towards this end, Mabuchi & Van Handel [47] propose a reduced filter describing the dynamics of the system in open-loop subject to errors. It consists on a formulation based on a Wonham filter that only tracks the populations on the error subspaces $\boldsymbol{p}_j(\rho)$. The formulation comes down to the observation that the measurement operators fix a common eigenbasis, restricting the dynamics of the filter to the diagonal elements of the density matrix in said eigenbasis. With the actuation strategy present in Eq. 6.4, it is not possible to express the terms due to feedback as a linear combination of the error populations. Indeed, it can be seen that for any $j \in \{1, 2, 3\}$, the terms $\mathrm{Tr}\,(i[\boldsymbol{X}_j, \Pi_k]\rho)$ correspond to the symmetric (j, k) off-diagonal elements of the density matrix ρ. Thus the dimension of the filter grows as $\mathcal{O}(d^2)$ in the number of physical qubits d.

Since the information on the population on an error subspace is directly proportional to the measurement outputs — up-to some disturbing noise— the idea of bypassing the filter altogether has been considered as well in Sarovar et. al. [60]. It consists on adding low-pass filters of the form $R_j(t) = \int_{t-T}^{t} e^{-r(t-t')} dY_j(t)$. The control u_j in Eq. (6.4) is implemented as a function of $R_j(t)$. Recalling that $\boldsymbol{S}_1 = \boldsymbol{Z}_2\boldsymbol{Z}_3$, $\boldsymbol{S}_2 = \boldsymbol{Z}_1\boldsymbol{Z}_3$, $\boldsymbol{S}_3 = \boldsymbol{Z}_1\boldsymbol{Z}_2$, the control logic is easy to understand since it follows the logic given by the syndrome measurements

- If $R_3(t) < 0$ and $R_1(t) > 0$, apply correction via $\boldsymbol{X}_1$.
- If $R_3(t) > 0$ and $R_1(t) < 0$, apply correction via $\boldsymbol{X}_3$.
- If $R_3(t) < 0$ and $R_1(t) < 0$, apply correction via $\boldsymbol{X}_2$.
- If $R_3(t) > 0$ and $R_1(t) > 0$, do not apply feedback.

The presence of noise has negative effects on the control, as the control signals become innacurate [60], thus potentially causing the feedback action to be more counterproductive than it should be. Stability analysis of a system with controls computed by similar filters does not seem an easy task (cf. Concluding remarks in Chapter 7). But it is important to emphazise that this kind of controllers, needing only coarse-grained information on the state, are being actively pursued in current experimental setups.

Despite the lack of further study, the motivation behind these works is clear: it is unsatisfactory to use a full quantum filter to do quantum error correction in continuous-time. The intuition on the problem suggests that it is only necessary to track the populations on the error eigenspaces in order to do correction. It is the actuation strategy that poses a challenge to define a closed-loop population filter; under the actuation present in Eq. (6.4), there is the need to track off-diagonal block terms, which define the couplings amongst the eigenspaces.

The objectives pursued in the rest of this chapter are not markedly different than those presented before, and we will use noise-assisted feedback to address them. Namely: on one hand, we have shown that the open-loop system (6.1), the support of ρ_t tends to belong to one of the joint eigenspaces of the syndrome measurements as $t \to \infty$ with exponential speed, but the resulting eigenspace will be at random. The control goal is to show that we can approach a target eigenspace, namely, that corresponding to the codespace, globally exponentially attractive. Furthermore, in the context of QEC, it is clear that tracking the populations on an error eigenstate is the only relevant control variable, as it is directly suggested by the syndrome measurements. Regarding this issue, the strategy is to first show that those control variables are enough to achieve exponential stabilization assuming a nominal quantum filter. Next, it will be shown how the use of noise to drive the controls allows us to bypass the filtering reduction issues in closed-loop that were present on previous actuation strategies, by formulating filters that track the populations on the error subspaces. Contrary to the approximate filter considered in Chapter 5, this formulation is exact.

6.4 Error correction as noise-assisted feedback stabilization

6.4.1 Controller design

Error correction requires to design a control law satisfying two properties:

- *Stabilization of target manifold:* drive any initial state ρ_0 towards a state with support only on the nominal codespace $\mathcal{C} = \text{span}\{|000\rangle, |111\rangle\}$. This comes down to making $\text{Tr}\,(\rho_t \Pi_{\mathcal{C}}) \to 1$ as $t \to \infty$. To address this first point, the strategy —similar to the previous chapters— comes down to defining a Lyapunov function $V(\rho)$ and constructing controls such that it is a supermartingale with exponential decay on $\mathcal{S}$.
- *Protection of information:* for $\text{Tr}\,(\Pi_{\mathcal{C}} \rho_0) = 1$ and in the presence of disturbances $\gamma_s \neq 0$, minimize the distance between ρ_t and ρ_0 for all $t \geq 0$.

We now directly address the first point, the second one will be discussed in the sequel.

As mentioned in the introduction, this problem has already been considered before, yet without proof of exponential convergence. We will extend our noise-assisted feedback scheme to three controls

$$u_j dt = \sigma_j(\rho) dB_j \ ,$$

with $B_j(t)$ Brownian motions independent of any $W_k(t)$. As control Hamiltonians we take $H_j = \boldsymbol{X}_j$, thus rotating back the bit-flip actions. The reasoning on using a noisy input is similar to previous chapters: to destabilize the open-loop equilibria where $\boldsymbol{p}_k(\rho) = 1$ for any $k \in \{1, 2, 3\}$, while progressively approaching the state towards $\boldsymbol{p}_\mathcal{C}(\rho) = 1$. The closed-loop dynamics in Itō sense then writes:

$$d\rho = \sum_{k=1}^{3} \Gamma_k \mathcal{D}_{\boldsymbol{S}_k}(\rho)dt + \sqrt{\eta_k \Gamma_k} \mathcal{M}_{\boldsymbol{S}_k}(\rho)dW_k + \sum_{s=1}^{3} \gamma_s \mathcal{D}_{\boldsymbol{X}_s}(\rho)dt \\ + \sum_{j=1}^{3} -i\sigma_j(\rho)[\boldsymbol{X}_j, \rho]dB_j + \sigma_j(\rho)^2 \mathcal{D}_{\boldsymbol{X}_j}(\rho)dt\,. \quad (6.5)$$

The last term can be viewed as “encouraging” a bit-flip with a rate depending on the value of σ_j and thus on ρ. The remaining task is to design the gains σ_j, which in general can follow some dynamic control logic.

There are many options for designing σ_j — its only essential role is to “shake” the state when it is close to $\mathrm{Tr}\,(\Pi_\mathcal{C}\rho) = 0$, since the open-loop behavior already ensures stochastic convergence to either $\mathrm{Tr}\,(\Pi_\mathcal{C}\rho) = 0$ or $\mathrm{Tr}\,(\Pi_\mathcal{C}\rho) = 1$. The following simple hysteresis-based control law illustrated by figure 6.1 depends only on the variables $p_{t,k}$ and should not be too hard to implement in practice. Select real parameters α_j and β_j such that $\frac{1}{2} < \beta_j < \alpha_j < 1$ for $j \in \{1, 2, 3\}$, and take a constant $c > 0$.

Control Law 6.4.1. *Select $\sigma_j(\rho)$ for each $j \in \{1, 2, 3\}$ as follows*

1. *If $\boldsymbol{p}_j(\rho) \geq \alpha_j$ then take $\sigma_j(\rho) = \sqrt{\frac{6c\eta\Gamma}{2\alpha_1 - 1}}$;*
2. *If $\boldsymbol{p}_j(\rho) \leq \beta_j$ then take $\sigma_j(\rho) = 0$;*
3. *When entering or moving in the hysteresis region, i.e. the values of p_j in $]\beta_j, \alpha_j[$ not covered by the above two cases: keep the previous value of $\sigma_j(\rho)$.*

Contrary to the preceding sections, here we consider a continuous control with hysteresis. The reasoning behind it comes down to being able to chose separately when to turn-on/turn-off the noisy input. Having α_j close to one means that the controller will wait more until it applies a correction, but the error is better localized. On the other hand, having α close to 1/2, yields a larger control gain, improving the convergence rate to the target manifold, but with the risk of manipulating quantum information more than necessary, which could yield to an error induced by feedback. In the sequel it will be discussed how the choice of α_j, and β_j affects the the capability of the controller to protect encoded information.

We can replace the above control law by a smooth control in the same lines as chapters 4 and 5. That is, we consider the control law

$$\sigma_j(\rho) = c\,\varphi\left(\frac{\boldsymbol{p}_j(\rho) - \beta_j}{\alpha_j - \beta_j}\right), \quad j \in \{1, 2, 3\}, \quad (6.6)$$

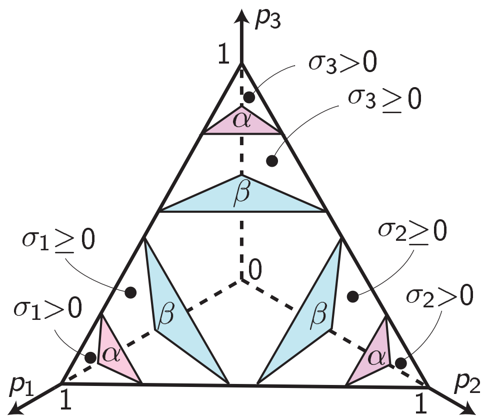


Figure 6.1: for $\alpha_j \equiv \alpha$ and $\beta_j \equiv \beta$, the 6 active feedback zones in the simplex $\left\{(\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3) \mid \boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3 \geq 0,\ \boldsymbol{p}_1 + \boldsymbol{p}_2 + \boldsymbol{p}_3 \leq 1\right\}$.

where again $\varphi \geq 0$ is a smooth saturating function on $[0,1]$, i.e. $\varphi(]-\infty,0]) = \{0\}$ and $\varphi([1,+\infty[) = \{1\}$. The proof of convergence follows the same lines as the control with hysteresis. The main difference is that the hysteresis-based controller can be fine tuned to highlight the capabilities of information protection. Simulations in Figures 6.2a and 6.3) compare both schemes.

6.4.2 Closed-loop exponential stabilization

We propose the closed-loop Lyapunov function:

$$V(\rho) = V_1(\rho) + V_2(\rho) + V_3(\rho) \tag{6.7}$$

with

$$V_1(\rho) = \sqrt{2\boldsymbol{p}_1(\rho) + \boldsymbol{p}_2(\rho) + \boldsymbol{p}_3(\rho)},$$

$$V_2(\rho) = \sqrt{\boldsymbol{p}_1(\rho) + 2\boldsymbol{p}_2(\rho) + \boldsymbol{p}_3(\rho)},$$

and

$$V_3(\rho) = \sqrt{\boldsymbol{p}_1(\rho) + \boldsymbol{p}_2(\rho) + 2\boldsymbol{p}_3(\rho)}.$$

Theorem 6.4.1. *Consider system (6.5) with $\gamma_s = 0$, $s \in \{1,2,3\}$ and feedback gains (σ_j) designed in items 1, 2 and 3 in the control law 6.4.1. Then*

$$\mathbb{E}[V(\rho_t)] \leq V(\rho_0)e^{-rt},\ \forall t \geq 0,$$

with the exponential convergence rate estimated as:

$$r = \left(\min_{j\in\{1,2,3\}} \eta_j \Gamma_j\right) \min\left(c\ ,\ \tfrac{4}{3\sqrt{2}} \min_{(s,x_1,x_2,x_3)\in K} g(s,x_1,x_2,x_3)\right) > 0$$

where the function $g(s, x_1, x_2, x_3)$*, given in Eq.* (6.10)*, is a smooth, strictly positive function defined on the compact set*

$$K = \left\{(s, x_1, x_2, x_3) \in [0,1]^4 \;\middle|\; x_1 + x_2 + x_3 = 1; sx_j \le \alpha_j\right\}$$

For a heuristic estimate of r, take $s = \alpha_j$ with $x_j = 1$ for some j to get

$$r \sim \left(\min_{j\in\{1,2,3\}} \eta_j\Gamma_j\right) \ \min\left(c, \tfrac{8}{\sqrt{2}}(1-\bar{\alpha})^2\right)$$

with $\bar{\alpha} = \max_{j\in\{1,2,3\}} \alpha_j$. Typically one would take $c = 1$ and $\alpha_1 = \alpha_2 = \alpha_3 = \alpha$ close to 1. When $\eta_j\Gamma_j$ are all equal, such a rough estimate simplifies to $r = 4\sqrt{2}(1-\alpha)^2\eta\Gamma$.

Proof. We consider the following partition of the state-space: $\mathcal{Q} := \cup_{j=1}^3\{\rho \in \mathcal{S} \mid \boldsymbol{p}_j(\rho) \ge \alpha_j\}$ and $\mathcal{S}/\mathcal{Q}$. Then we analyze how the diffusion behaves on such a partition, by computing its infinitesimal generator. By design of the hysteresis, well-posedness of the solution then follows from standard arguments on the construction of solutions of SDE's. There remains to check that $\mathcal{A}V(\rho) \le -rV(\rho)$.

From (6.5) compute $\mathcal{A}V(\rho) = \mathbb{E}\left[dV_t \mid \rho_t = \rho\right]/dt$ for any value of the control gain-vector σ. We take $\eta_j \equiv \eta$ and $\Gamma_j \equiv \Gamma$[1]. With $F_1 = 2\Pi_1+\Pi_2+\Pi_3$ and $V_1(\rho) = \sqrt{\mathrm{Tr}\,(F_1\rho)}$, we get

$$\begin{aligned}\mathcal{A}V_1(\rho) = &\frac{2\sigma_1^2(1-f_1) + \sigma_2^2(1-2(\boldsymbol{p}_1(\rho)+\boldsymbol{p}_2(\rho))) + \sigma_3^2(1-2(\boldsymbol{p}_1(\rho)+\boldsymbol{p}_3(\rho)))}{2\sqrt{f_1}} \\ &- \frac{4\eta\Gamma}{f_1\sqrt{f_1}}\Bigg(\big((\boldsymbol{p}_2(\rho)+\boldsymbol{p}_3(\rho))(1-f_1)\big)^2 + \big(\boldsymbol{p}_1(\rho)+(\boldsymbol{p}_1(\rho)+\boldsymbol{p}_3(\rho))(1-f_1)\big)^2 \\ &\qquad + \big(\boldsymbol{p}_1(\rho)+(\boldsymbol{p}_1(\rho)+\boldsymbol{p}_2(\rho))(1-f_1)\big)^2\Bigg) \\ &- \frac{\sigma_1^2\,\mathrm{Tr}^2\,([X_1,\rho]F_1) + \sigma_2^2\,\mathrm{Tr}^2\,([X_2,\rho]F_1) + \sigma_3^2\,\mathrm{Tr}^2\,([X_3,\rho]F_1)}{4f_1\sqrt{f_1}}\end{aligned} \tag{6.8}$$

where $f_1 = \mathrm{Tr}\,(F_1\rho) = 2\boldsymbol{p}_1(\rho)+\boldsymbol{p}_2(\rho)+\boldsymbol{p}_3(\rho)$. Since $\sqrt{f_1} \ge \frac{1}{3\sqrt{2}}V$, we have

$$\begin{aligned}\mathcal{A}V_1(\rho) \le &\frac{2\sigma_1^2(1-f_1) + \sigma_2^2(1-2(\boldsymbol{p}_1(\rho)+\boldsymbol{p}_2(\rho))) + \sigma_3^2(1-2(\boldsymbol{p}_1(\rho)+\boldsymbol{p}_3(\rho)))}{2\sqrt{f_1}} \\ -V\frac{4\eta\Gamma}{3\sqrt{2}}&\frac{\big((\boldsymbol{p}_2(\rho)+\boldsymbol{p}_3(\rho))(1-f_1)\big)^2+\big(\boldsymbol{p}_1(\rho)+(\boldsymbol{p}_1(\rho)+\boldsymbol{p}_3(\rho))(1-f_1)\big)^2+\big(\boldsymbol{p}_1(\rho)+(\boldsymbol{p}_1(\rho)+\boldsymbol{p}_2(\rho))(1-f_1)\big)^2}{f_1^2}.\end{aligned}$$

Via circular permutation and summation, we get

$$\mathcal{A}V(\rho) \le \sum_{j=1}^{3} \sigma_j^2(\rho)g_j(\rho) - \tfrac{4\eta\Gamma}{3\sqrt{2}}g(\rho)V(\rho) \tag{6.9}$$

[1] Otherwise we can take $4\eta\Gamma = 4\min_{j\in\{1,2,3\}} \eta_j\Gamma_j$ and the equality in Eq. (6.8) becomes an inequality, and the computations follow identically.

where

$$g_j(\rho) = \frac{1-f_j}{\sqrt{f_j}} + \frac{1-2(\boldsymbol{p}_j(\rho)+\boldsymbol{p}_{j'}(\rho))}{2\sqrt{f_{j'}}} + \frac{1-2(\boldsymbol{p}_j(\rho)+\boldsymbol{p}_{j''}(\rho))}{2\sqrt{f_{j''}}}$$

with $\{j, j', j''\} = \{1, 2, 3\}$ and

$$\begin{aligned} g(\rho) = \\ &\frac{\big((\boldsymbol{p}_2(\rho)+\boldsymbol{p}_3(\rho))(1-f_1)\big)^2 + \big(\boldsymbol{p}_1(\rho)+(\boldsymbol{p}_1(\rho)+\boldsymbol{p}_3(\rho))(1-f_1)\big)^2 + \big(\boldsymbol{p}_1(\rho)+(\boldsymbol{p}_1(\rho)+\boldsymbol{p}_2(\rho))(1-f_1)\big)^2}{(2\boldsymbol{p}_1(\rho)+\boldsymbol{p}_2(\rho)+\boldsymbol{p}_3(\rho))^2} \\ &+\frac{\big((\boldsymbol{p}_3(\rho)+\boldsymbol{p}_1(\rho))(1-f_2)\big)^2 + \big(\boldsymbol{p}_2(\rho)+(\boldsymbol{p}_2(\rho)+\boldsymbol{p}_1(\rho))(1-f_2)\big)^2 + \big(\boldsymbol{p}_2(\rho)+(\boldsymbol{p}_2(\rho)+\boldsymbol{p}_3(\rho))(1-f_2)\big)^2}{(2\boldsymbol{p}_2(\rho)+\boldsymbol{p}_3(\rho)+\boldsymbol{p}_1(\rho))^2} \\ &+\frac{\big((\boldsymbol{p}_1(\rho)+\boldsymbol{p}_2(\rho))(1-f_3)\big)^2 + \big(\boldsymbol{p}_3(\rho)+(\boldsymbol{p}_3(\rho)+\boldsymbol{p}_2(\rho))(1-f_3)\big)^2 + \big(\boldsymbol{p}_3(\rho)+(\boldsymbol{p}_3(\rho)+\boldsymbol{p}_1(\rho))(1-f_3)\big)^2}{(2\boldsymbol{p}_3(\rho)+\boldsymbol{p}_1(\rho)+\boldsymbol{p}_2(\rho))^2}. \end{aligned}$$

When $\rho \in \mathcal{Q}$, we have $p_j \geq \alpha_j > 1/2$ for a unique $j \in \{1, 2, 3\}$, since $\boldsymbol{p}_1(\rho)+\boldsymbol{p}_2(\rho)+\boldsymbol{p}_3(\rho) \leq 1$. Assume first that $\boldsymbol{p}_1(\rho) \geq \alpha_1$, thus $\sigma_1 = \sqrt{\frac{6c\eta\Gamma}{2\alpha_1-1}}$ and $\sigma_2(\rho) = \sigma_3(\rho) = 0$. Since $g(\rho) \geq 0$, inequality (6.9) implies

$$\mathcal{A}V \leq \frac{6c\eta\Gamma}{2\alpha_1-1}\left(\frac{1-f_1}{\sqrt{f_1}} + \frac{1-2(\boldsymbol{p}_1(\rho)+\boldsymbol{p}_2(\rho))}{2\sqrt{f_2}} + \frac{1-2(\boldsymbol{p}_1(\rho)+\boldsymbol{p}_3(\rho))}{2\sqrt{f_3}}\right).$$

Since $f_1 \geq 2\alpha_1$, $1-2p_1 \leq 0$, $f_1 \leq 2$ and $V \leq 3\sqrt{2}$ we get

$$\mathcal{A}V \leq \frac{6c\eta\Gamma}{2\alpha_1-1}\frac{1-2\alpha_1}{\sqrt{f_1}} = -\frac{6c\eta\Gamma}{V\sqrt{f_1}}V \leq -c\eta\Gamma V.$$

We get a similar inequality when $\boldsymbol{p}_2(\rho) \geq \alpha_2$ or $\boldsymbol{p}_3(\rho) \geq \alpha_3$. Thus

$$\forall \rho \in \mathcal{Q},\ \mathcal{A}V(\rho) \leq -c\eta\Gamma V(\rho).$$

Consider now $\rho \in \mathcal{S}/\mathcal{Q}$. Then, $\boldsymbol{p}_j(\rho) < \alpha_j$ for all j. Since $\sigma_j(\rho) = 0$ when $\boldsymbol{p}_j(\rho) \leq 1/2$ we have $\sigma_j^2(\rho)g_j(\rho) \leq 0$. From (6.9), we have $\mathcal{A}V(\rho) \leq -\frac{4\eta\Gamma}{3\sqrt{2}}g(\rho)V(\rho)$. Let us prove that $g(\rho) \geq r$ for any $\rho \in \mathcal{S}/\mathcal{Q}$. With $s = \boldsymbol{p}_1(\rho)+\boldsymbol{p}_2(\rho)+\boldsymbol{p}_3(\rho)$ and $x_j = \boldsymbol{p}_j(\rho)/s$, g can be seen as a function of (s, x_1, x_2, x_3),

$$\begin{aligned} g(\rho) = g(s, x_1, x_2, x_3) \triangleq \\ &\frac{\big((x_2+x_3)(1-f_1)\big)^2 + \big(x_1+(x_1+x_3)(1-f_1)\big)^2 + \big(x_1+(x_1+x_2)(1-f_1)\big)^2}{(1+x_1)^2} \\ &+\frac{\big((x_3+x_1)(1-f_2)\big)^2 + \big(x_2+(x_2+x_1)(1-f_2)\big)^2 + \big(x_2+(x_2+x_3)(1-f_2)\big)^2}{(1+x_2)^2} \\ &+\frac{\big((x_1+x_2)(1-f_3)\big)^2 + \big(x_3+(x_3+x_2)(1-f_3)\big)^2 + \big(x_3+(x_3+x_1)(1-f_3)\big)^2}{(1+x_3)^2} \end{aligned} \tag{6.10}$$

with $f_j = 1 - s - sx_j$. Here (s, x_1, x_2, x_3) belongs to the compact set $s \in [0, 1]$, $x_j \geq 0$, $\sum_j x_j = 1$ and $sx_j \leq \alpha_j$ for all j. On this compact set, g

is a smooth function. Moreover it is strictly positive since $g = 0$ implies that $s = 1$ and $x_j = 1$ for some $j \in \{1, 2, 3\}$ which would not satisfy $sx_j \leq \alpha_j$ i.e. lie in $\mathcal{Q}$. This means that $\min_{\rho \in \mathcal{S}/\mathcal{Q}} g(\rho) > 0$.

Taking all things together, we have proved that $\mathcal{A}V(\rho) \leq -rV(\rho)$ always holds and we can conclude with Theorem A.0.1. □

6.4.3 Reduced order filtering

We now address the design of a reduced filter that only tracks the error populations. The quantum filter for the closed-loop system (6.5) reads

$$\begin{aligned} d\rho = &\sum_{k=1}^{3} \Gamma_k \mathcal{D}_{\boldsymbol{S}_k}(\rho)dt + \sum_{k=1}^{3} \sqrt{\eta_k \Gamma_k} \mathcal{M}_{\boldsymbol{S}_k}(\rho)\Big(dY_k - 2\sqrt{\eta_k \Gamma_k}\,\mathrm{Tr}\,(\boldsymbol{S}_k \rho)\,dt\Big) \\ &+ \sum_{s=1}^{3} \gamma_s \mathcal{D}_{\boldsymbol{X}_s}(\rho)dt + \sum_{j=1}^{3} -i\sigma_j(\rho)[\boldsymbol{X}_j, \rho]dB_j + \sigma_j(\rho)^2 \mathcal{D}_{\boldsymbol{X}_j}(\rho)dt\,. \quad (6.11) \end{aligned}$$

where $dY_k = 2\sqrt{\eta_k \Gamma_k}\,\mathrm{Tr}\,(\boldsymbol{S}_k \rho)\,dt + dW_k$ is the measurement outcome of syndrome $\boldsymbol{S}_k$, and the random dB_j applied to the system are accessible too a posteriori.

Instead, we can replace the state ρ_t in the feedback law, by $\widehat{\rho}_t$ corresponding to the Bayesian estimate of ρ_t knowing its initial condition ρ_0 and the syndrome measurements Y_k between 0 and the current time $t > 0$, but not the dB_j. The state $\widehat{\rho}_t$ obeys to the SME:

$$\begin{aligned} d\widehat{\rho} = &\sum_{k=1}^{3} \Gamma_k \mathcal{D}_{\boldsymbol{S}_k}(\widehat{\rho})dt + \sum_{k=1}^{3} \sqrt{\eta_k \Gamma_k} \mathcal{M}_{\boldsymbol{S}_k}(\widehat{\rho})(dY_k - 2\sqrt{\eta_k \Gamma_k}\,\mathrm{Tr}\,(\boldsymbol{S}_k \widehat{\rho})\,dt) \\ &+ \sum_{j=1}^{3} (\gamma_s + \sigma_j^2(\widehat{\rho}))\mathcal{D}_{\boldsymbol{X}_j}(\widehat{\rho})dt \quad (6.12) \end{aligned}$$

where $dY_k = 2\sqrt{\eta_k \Gamma_k}\,\mathrm{Tr}\,(\boldsymbol{S}_k \rho)\,dt + dW_k$ with ρ governed by (6.5) where $\sigma_j(\rho)$ is replaced by $\sigma_j(\widehat{\rho})$. Denote $\hat{p}_j = \mathrm{Tr}\,(\Pi_j \widehat{\rho})$ and $\hat{s}_k = \mathrm{Tr}\,(\boldsymbol{S}_k \widehat{\rho})$. Then we have

$$\begin{aligned} d\hat{s}_1 = &-2(\gamma_2 + \sigma_2^2 + \gamma_3 + \sigma_3^2)\hat{s}_1 dt \\ &+ 2\sqrt{\eta_1 \Gamma_1}(1 - \hat{s}_1^2)\big(dY_1 - 2\sqrt{\eta_1 \Gamma_1}\hat{s}_1 dt\big) \\ &+ 2\sqrt{\eta_2 \Gamma_2}(\hat{s}_3 - \hat{s}_1 \hat{s}_2)\big(dY_2 - 2\sqrt{\eta_2 \Gamma_2}\hat{s}_2 dt\big) \\ &+ 2\sqrt{\eta_3 \Gamma_3}(\hat{s}_2 - \hat{s}_1 \hat{s}_3)\big(dY_3 - 2\sqrt{\eta_3 \Gamma_3}\hat{s}_3 dt\big) \quad (6.13) \end{aligned}$$

with $\hat{p}_1 = (1 + \hat{s}_1 - \hat{s}_2 - \hat{s}_3)/4$. The formulae for $d\hat{s}_2$, $d\hat{s}_3$ and $\hat{p}_2$ $\hat{p}_3$ are obtained via circular permutation in $\{1, 2, 3\}$. Since the feedback law depends only on the populations $\hat{p}_j$, it can be implemented with the exact quantum filter reduced to $(\hat{s}_1, \hat{s}_2, \hat{s}_3) \in \mathbb{R}^3$. Contrarily to the full quantum filter (6.11), here the syndrome dynamics $\hat{s}_k$ are independent of any coherences among the different subspaces. Thus, instead of using nine variables by including such coherences, the reduced order formulation through noise-assisted feedback only needs three.

6.4.4 On the protection of quantum information

It is well-known in control theory that exponential stability gives an indication of robustness against unmodeled dynamics. In the present case, this concerns the first control goal, namely stabilization of ρ_t close to the nominal subspace $\mathcal{C}$ in the presence of bit-flip errors $\gamma_s \neq 0$. About the second control goal, namely keeping the dynamics on $\mathcal{C}$ close to zero such that logical information remains protected, the analysis of the previous section is less telling.

We can illustrate both control goals by simulation. As in [1] we set as initial condition $\rho_0 = |000\rangle\langle 000|$ and simulate 1000 closed-loop trajectories under the feedback law of section 6.4.1. We compare the average evolution of this encoded qubit with a single physical qubit subject to a $\boldsymbol{X}$ decoherence of the same strength, since this is the situation that the bit-flip code is meant to improve.

The figures in this section are meant to examine two main situations: Performance of the closed-loop system with respect to changes in the control parameters, and performance of the closed-loop system when the control law is computed via the reduced order filters.

We fix for all simulations the initial state is $\rho_0 = |000\rangle\langle 000|$ and the nominal system parameters of the closed loop system (6.5) are $\Gamma_j = 1$, $\gamma_j = 1/64$, $\eta_j = 0.8$. Control parameters or changes on the nominal system parameters will be specified in the figures.

In all the simulations the colors are: solid red: mean overlap of the logical qubit versus the code space. Solid black: mean fidelity of the logical qubit versus ρ_0. Solid blue: mean correctable fidelity under active quantum feedback. Dashed line: mean fidelity towards $|0\rangle\langle 0|$ for a single physical qubit without measurement/control and subject to bit-flip disturbances with $\gamma = 1/64$.

The observations found in the simulations are the following:

- In Figure 6.2a we consider that the quantum filter perfectly follows (6.5), and set the control parameters $\beta_j = 0.6$, $\alpha_j = 0.95$.

  This Figure corresponds to simulating the ideal situation. Regarding the first control goal, we observe that the controller indeed confines the mean evolution to a small neighborhood of $\mathcal{C}$, for all times, as expected from our analysis. Regarding the second criterion, the distance between ρ_t and ρ_0 cannot be confined to a small value for all times.

Indeed, majority vote can decrease the rate of information corruption but not totally suppress it; as corrupted information is irremediably lost, ρ_t progressively converges towards an equal distribution of logical 0 and logical 1. However, for the protected 3-qubit code, this information loss is much slower than for the single qubit; this indicates that the 3-qubit code with our feedback law indeed improves on its components.

- Figure 6.2b corresponds to the situation where the control parameters arec chosen to maximize the decay rate towards the target manifold. n our feedback design, making α_j closer to $1/2$ would improve the convergence rate estimate in Theorem 6.4.1. Accordingly, we set the control parameters as $\beta_j = 0.52$, $\alpha_j = 0.57$ for this simulation. The simulation indicates that making α_j closer to $1/2$ has a negative effect on the codeword fidelity, since it means that we turn on the noisy drives more often. Indeed, compared to figure 6.2a, in Figure 6.2b the codespace fidelity is slightly improved, at the cost of a faster decay of the codeword fidelity, at almost double the speed. Thus, there is a tradeoff when setting the value where the control input is nonzero. Analytically computing the optimal tradeoff is an open problem, but simulations clearly show that intermediate values of the control parameters delivers better overall results.

- Figure 6.3 changes the control law with the smooth control law (6.6) with $\beta_j = 0.6$, $\alpha_j = 0.95$. Compared to Figure 6.2a, performance under the smooth control law is slightly worse than the one with hysteresis, with a difference of around a five percent loss in the codeword fidelity at $t = 100$. This is probably due to the lack of fine-tuning present in the hysteresis control developed in this chapter. In practice, this loss of fidelity could be offset by the simpler implementation of the smooth control law, compared to the controller with hysteresis.

- Figure 6.4a corresponds to a simulation where the control law is computed via the reduced filter formulated in (6.13). No imperfections such as delays in the feedback loop or modelling erros are considered. We do not observe a noticeable qualitative difference of the codespace and codeword fidelities with respect to the ideal situation of Fig. 6.2a, in agreement to the fact that the filters defined in (6.13) are an exact reduced order formulation of the full quantum filter 6.11.

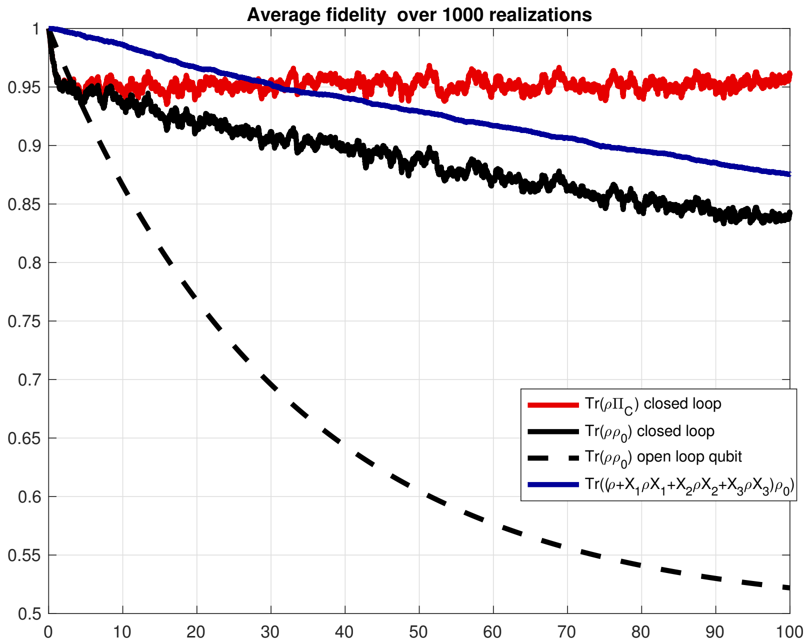
- Figure 6.4b corresponds to a more realistic situation where the same control law relies on the reduced order quantum filter (6.13) corrupted by errors and feedback latency. We observe that these errors contribute the largest to a decrease in the performance.

6.5 Conclusions

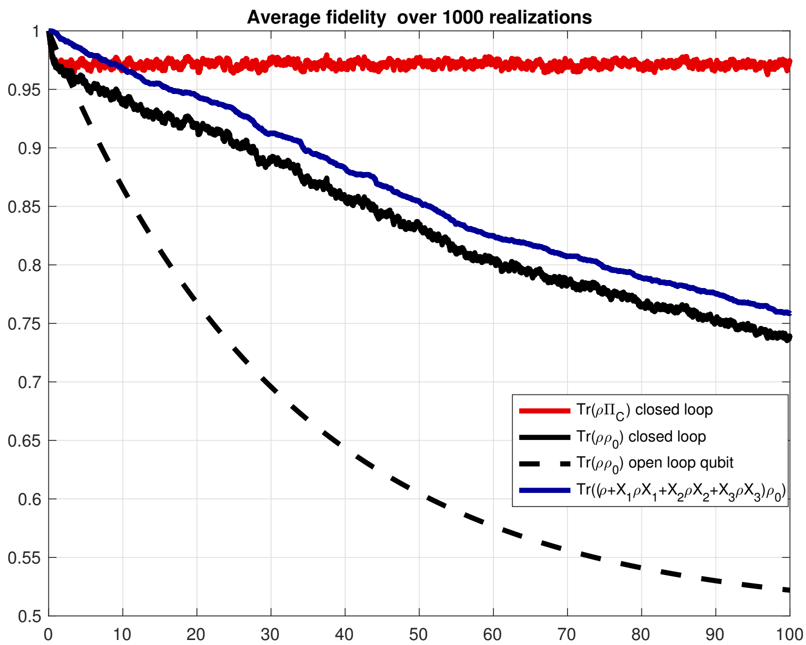
We have approached continuous-time quantum error correction in the same spirit as [1], and showed how introducing Brownian motion to drive control fields yields exponential stabilization of the nominal codeword manifold. The main idea relies on the fact that the SDE in open loop stochastically converges to one of a few steady-state situations, but on the average does not move closer to any particular one. It is then sufficient to activate noise only when the state is close to a bad equilibrium, in order to induce global convergence to the target ones.

In a purely discrete error correction scenario, the syndrome measurement extracts information about the disturbances, while a unitary gate is solely in charge of recovering the encoded state. We have exploted a combination of both the measurements and the noise-assisted feedback to arrive to a nominal eigenspace of the measurements. This general idea can be extended to other systems with this property, and in particular to more advanced error-correcting schemes.

The convergence rate obtained is dependent on our choice of Lyapunov function and on the values of α_j; from parallel investigation it seems possible to get a closed-loop convergence rate arbitrarily close to the measurement rate. However, unlike in classical control problems, the key performance indicator is not how fast we approach the target manifold. Instead, what matters is how well, in presence of disturbances, we preserve the encoded information. Towards this goal, we should refrain from disturbing the system with feedback actions; accordingly, we have noticed that taking α_j closer to 1 can improve the codeword fidelity, despite leading to a slower convergence rate estimate.



(a) Ideal situation where the feedback of subsection 6.4.1 is based on ρ governed by (6.5). Control parameters are $\beta_j = 0.6$, $\alpha_j = 0.95$, $c = 3/2$.



(b) Simulation comparing the effect of changing the control parameters. Control parameters are $\beta_j = 0.52$; $\alpha_j = 0.57$, $c = 3/2$.

Figure 6.2: Closed-loop simulations with direct state feedback.

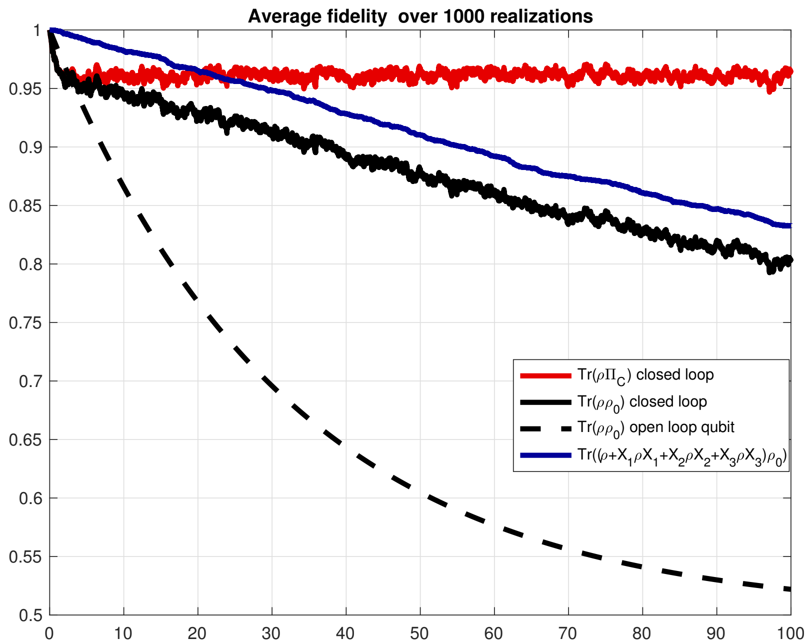
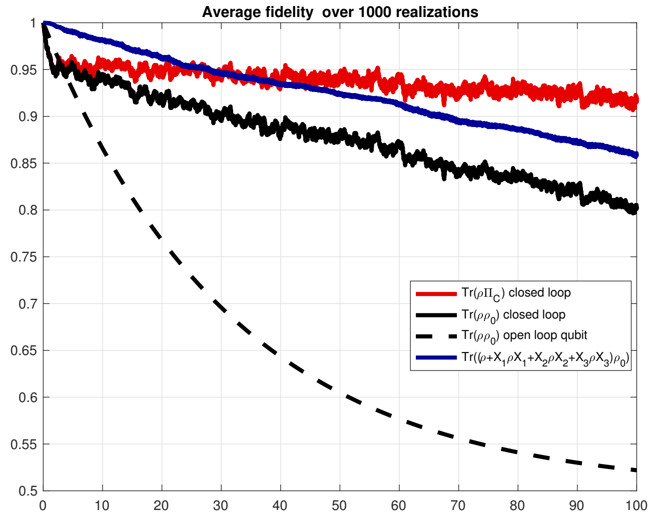
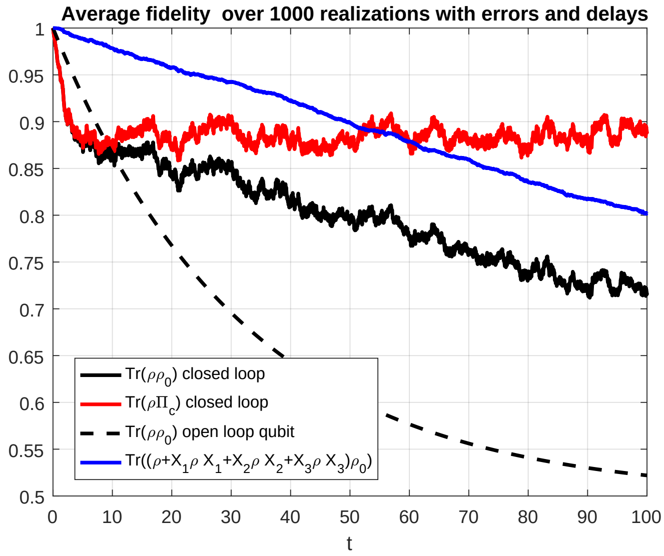


Figure 6.3: Closed loop simulation considering an ideal situation but with the smooth control (6.6).The control parameters are $\beta_j = 0.6$, $\alpha_j = 0.95$



(a) Simulation where the control law 6.4.1 is computed via the reduced filter formulated in . Filter system and control parameters are the same as in figure 6.2a.



(b) Simulation considering a more realistic case where feedback is based on the reduced order filter (6.13) including modeling/measurement errors and feedback latency. Marked with subscript $*$, the parameter values used in (6.13) are as follows: $\gamma_* = 0.8\gamma$, $\Gamma_* = 0.9\Gamma$, $\eta_* = 0.9\eta$; constant measurement bias according to $dY_{*,1} = dY_1 + \frac{\sqrt{\eta\Gamma}}{10}dt$, $dY_{*,2} = dY_2 - \frac{\sqrt{\eta\Gamma}}{10}dt$ and $dY_{*,3} = dY_3 + \frac{\sqrt{\eta\Gamma}}{20}dt$, , and feedback latency of $1/(2\Gamma)$; measurement signals Y_k are based on (6.5) with nominal values identical to simulation of figure 6.2a and control values $\sigma_j(\widehat{\rho})$.

Figure 6.4: Closed-loop simulations assuming quantum filter and reduced order filter

Chapter 7

Concluding remarks and perspectives

La stabilisation d'un état propre QND est un probème de base en contrôle quantique basé sur la mesure, mais tellement different d'un problème du contrôle standard. En effet, il parait que ça suffit que le contrôleur ne fait rien, sauf dans quelques situations. Et la, il est suffisant de secouer l'état pour ensortir le système de cette situation indésirable. Nous avons utilisé des mouvements browniens pour montrer que cette procédure est exponentiellement stable.

Au delà de l'étude de convergence exponentielle, qui fait la partie integrale de cette thèse, plusieurs avenues de recherche peuvent être suivies. Parmi elles:

- *L'optimisation de la vitesse de convergence en utilisant le contrôle assisté par bruit. Cette amelioration à été remarqué de manière empirique.*

- *L'analyse de robustesse du filtre reduit. Plus important, proposer des architechtures du contrôle par retour dynamique de sortie, e.g., filtre passe-bas.*

- *Nous avons étudié le code correcteur d'erreur autant q'une memoire quantique. Le pas suivant est le développement des algorithmes de contrôle en temps continu pour des autres tâches impliquant le traîtement quantique de l'information, e.g., calcul quantique adiabatique avec correction d'erreurs active.*

- *Le développement des outils analytiques pour étudier la protection de l'information quantique en temps continu.*

7.1 Towards robust control methods for quantum information processing

Stabilizing a particular eigenstate of a quantum non-demolition measurement is a basic problem in quantum control, but also remarkably different from a standard control problems. With respect to previous design methodologies, it seems that the best course of action to achieve global exponential stabilization of QND systems is for the controller to do nothing on a large region of the state-space, and only act on the system when the state is sufficiently close to an undesired eigenstate. It suffices then to show that when the control is active, this induces exponential convergence towards the target. In order to prove this apparently simple requirement, it was needed to introduce a new element, namely, Brownian motion to drive the unitary control fields. With the introduction of Brownian noise it was possible for us to provide closed-loop Lyapunov functions that are supermartingales with exponential decay. This yields automatically strong global stochastic stability statements about the closed-loop dynamics, something that was missing in previous works on the topic and that we felt it deserved full attention.

In our current setting, our results still leave room for improvement on the side of convergence analysis. While we have a well-proven convergence analysis with a closed-loop Lyapunov function with exponential decay, an improvement with respect to the current stability analysis is on providing sharp estimates of closed-loop convergence rates. It was shown that even on simple cases, closed-loop design using "natural" measures of a distance, yielded to subestimations on the speed of convergence. Even more, on higher dimensions, simulations suggested that noise could be fine-tuned to improve the convergence speed beyond the estimation given by Lyapunov analysis. Finding useful measures of a distance that provide accurate closed-loop convergence speeds, are fundamental pre-requisite in order to provide sharp estimates of robustness against unmodeled perturbations.

The use of Brownian motion to drive the control fields fits nicely into the QND eigenstate stabilization problem; other quantum control problems could call for different actuators, so one could think as well of controls driven by other types of stochastic processes —like Poisson or Lévy processes— and then adapting the design methodology to the specificities of the problem. An omission in this thesis is that we did not addressed in detail the idea of considering dynamic extensions of the Markovian feedback in the sense of Wiseman [75, 77, 48, 22]; the main reason is because we found that an exogenous Brownian motion was best suited towards QND eigenstate stabilization, which was the base control problem of this thesis. For other problems, like targeting states which are not QND eigenstates, a dynamical version of Markovian feedback could yield improved stability rates with respect to the current situation with static output feedback.

The design of the control layer for quantum information processing tasks has been relatively unexplored in the context of measurement-based feedback. However, as technologies improve, feedback designs should see increased attention as analog quantum information tasks, notably analog quantum simulation and quantum annealing, become more prominent. In this thesis it was explored —even if briefly— the design of the continuous-time control layer for a quantum error correcting scheme. In this context, we have shown that with noise-assisted feedback it is possible to render a target manifold of quantum states globally exponentially attractive. Moreover, the control law requires to compute only the variables corresponding to the population of the state on the joint eigenspaces of the syndrome measurement operators.

Several questions remain open in this direction, mainly with respect to the information protection capabilities of the closed-loop system. Using noise-assisted feedback, the intuition on the control law is that noise encourages bit-flips. As counterintuitive as it may seem, since this control is activated carefully, namely, whenever the state is close to an error subspace, we can show that the codespace is globally attractive. So far it is an open issue to analytically quantify the loss of fidelity between the state subject to perturbations with respect to the initial states, but simulations show that feedback control indeed improves on its individual components. Providing quantitative estimates of this information protection is an open issue that is important to have to show the ultimate utility of quantum feedback. In addition, another open issue is the analysis of robustness against imperfections like delays in the feedback loop, bias in the measurement signal or robustness of the filter against mismatch on the system parameters.

From a broad point of view, Lyapunov functions are a powerful tool to analyze the convergence of nonlinear systems towards an invariant set. Beyond control problems, this could be useful to characterize long term behaviours of other processes, for instance, the rate of purification of a system subject to several, possibly non-commuting, measurements [59, 37].

7.2 Towards dynamical output feedback controllers

The use of noise provided us with insight to propose reduced order filters that depend only on the populations around the QND eigenstates, that is, the diagonal of the state ρ in the eigen-basis of the measurement operator L. It was shown how we could either find closed form expresions of the filters in terms of said populations, or find approximate filters with the same structure. The importance of such reduction is that the reduced filter only grows linearly as a function of the populations as dimension increases. Simulations show that these filters are a good enough approximation for online computations, so their use in experimental implementations is promising. Here we did not adressed the convergence analysis of the closed-loop dynamics of the plant and reduced filter, however, the stability analysis and control methodology developed here makes us believe that closed-loop convergence can be proved.

Measurement-based feedback control problems are fundamentally control problems where only limited knowledge of the system is available. A main initial motivation that drove this thesis was studying dynamical output feedback schemes. So far it seems that closed-loop analysis is challenging on this setup. The experience gained in this thesis and numerical simulations do suggest some candidates. Information on the QND eigenstates can be seen on the asymptotic regime of the measurement output, thus the idea of using a perturbing noise to "shake" the state towards a target state seems close in spirit to what an extremum seeking agorithm would do [7]. We can consider an even simpler case, using a simple low-pass filter to filter out the measurement outcomes and trying to control the system with the knowledge obtained from the measurements. Consider the dynamical system

$$\begin{aligned}
d\rho &= \Gamma\Big(\boldsymbol{Z}\rho\boldsymbol{Z} - \rho\Big)dt - \sqrt{\eta\Gamma}\Big(\boldsymbol{Z}\rho + \rho\boldsymbol{Z} - 2\operatorname{Tr}(\rho\boldsymbol{Z})\,\rho\Big)dW \\
&\quad + \sigma(\xi)^2\Big(\boldsymbol{Z}\rho\boldsymbol{Z} - \rho\Big)dt - i\sigma(\xi)[\boldsymbol{Y},\rho]dB, \\
dY &= 2\sqrt{\eta\Gamma}\operatorname{Tr}(\rho\boldsymbol{Z})\,dt + dW, \\
d\xi &= -2\epsilon\sqrt{\eta\Gamma}\xi dt + \sqrt{\epsilon}dY.
\end{aligned}$$

The quantum system is a qubit system with measurements on the z-axis, and we consider that the actuation is done through via a Brownian motion. The control function $\sigma(\xi)$ is a smooth function similar to previous chapters, but dependent on the simple low-pass filter denoted by the ξ variable. We translate our control design from the previous chapters and implement it on Fig. 7.1. As before, in gray it is plotted 100 realizations and the ensemble average in blue.

Qualitatively speaking, the qubit system converges to the excited state at speed that seems exponential but with a small delay. The delay comes from the timescale set by the parameter ϵ; setting it small yields a less noisy ξ_t, but at the expense of making the filter too slow. The difficulty on analyzing this system comes from the fact that ξ does not have an equilibrium point, but a stationary probability distribution. In the absence of control the dynamical system composed by the (ρ, ξ) variables does converge towards a stationary solution, namely ρ converges to one of the eigenstates of the $\boldsymbol{Z}$ operator; accordingly ξ admits stationary probability distributions corresponding to that of a Ornstein-Uhlenbeck process with drift. Unfortunately, the tools of stochastic Lyapunov stability theory used here does not cover cases like this to construct stabilizing controls.

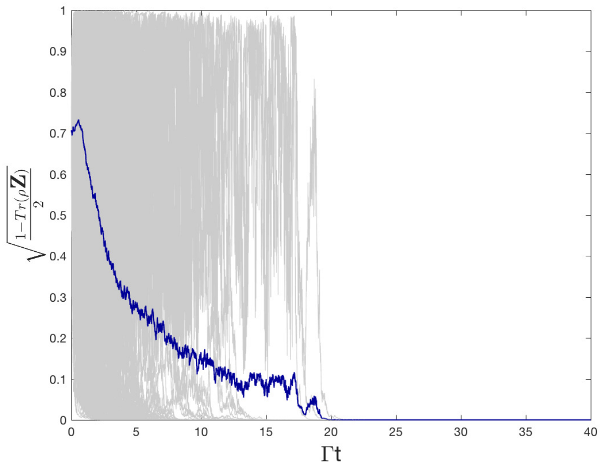


Figure 7.1: Closed-loop simulations of a qubit system where the control depends on a low-pass filter. Here $\eta = 0.5, \Gamma = 1$, $\epsilon = .1$ and $\rho_0 = \boldsymbol{I}/2$, $\xi_0 = 0$. The smooth control $\sigma(\xi) = \bar{\sigma}\varphi\Big(\frac{(1-\xi)/2-p_{\min}}{p_{\max}-p_{\min}}\Big)$ as in 4.4.1 with $p_{\min} = 0.6$ and $p_{\max} = 0.7$.

A more interesting situation for the use of a low-pass filter is on its use for continuous-time quantum error correction, although some additional adjustments are necessary. The closed-loop system describing three qubit bit-flip code and the low-pass filters is

$$\begin{aligned}
d\rho &= \sum_{k=1}^{3} \Gamma_k \mathcal{D}_{\boldsymbol{S}_k}(\rho)dt + \sqrt{\eta_k\Gamma_k}\mathcal{M}_{\boldsymbol{S}_k}(\rho)dW_k + \sum_{s=1}^{3}\gamma_s\mathcal{D}_{\boldsymbol{X}_s}(\rho)dt \\
&\quad + \sum_{j=1}^{3} -i\sigma_j(\xi)[\boldsymbol{X}_j,\rho]dB_j + \sigma_j(\xi)^2\mathcal{D}_{\boldsymbol{X}_j}(\rho)dt \\
dY_k &= 2\sqrt{\eta_k\Gamma_k}\,\mathrm{Tr}\left(\rho\boldsymbol{S}_k\right)dt + dW_k, \qquad k \in \{1,2,3\} \\
d\xi_k &= -2\epsilon_k\sqrt{\eta\Gamma}\xi_k dt + \sqrt{\epsilon_k}dY_k.
\end{aligned}$$

Coarse information on the error populations is then obtained as $\mathfrak{p}_1 = (1+\xi_1-\xi_2-\xi_3)/4$ (similarly for the other variables). From what we learned in Chapter 6, there are two rules of thumb that we should follow: first, errors must be sufficiently localized so that the controller does not cause a logical error; second, the controller should refrain as much as possible from disturbing the system with feedback actions. As the filtered signals ξ_k are noisy, this means that we have to take some extra care in order to avoid the feedback to do a harmful action. A choice for the saturating function $\sigma(\xi)$

is

$$\sigma_k(\mathfrak{p}_k) = \sqrt{c\ \eta_k \Gamma_k}\ \varphi\left(\frac{\mathfrak{p}_k - \beta_k}{\alpha_k - \beta_k}\right) = \sqrt{c\ \eta_k \Gamma_k}\ \frac{\left(\tanh\left(\theta(\frac{\mathfrak{p}_k-\beta_k}{\alpha_k-\beta_k})\right) + 1\right)}{2}, \quad k \in \{1,2,3\},$$

with $\alpha_k > \beta_k > 1/2$. The control input is quickly attenuated as soon as $\xi_k(t) < \alpha_k$. As the function decreases to zero faster than a polynomial, there is a low probability that some spurious noise drives turns the control back on. The gain in the linear region can be modified through the parameter θ.

As for the rate of data acquisition, making ϵ_k small gives a cleaner signal, at the expense that the filter is at slower timescale. Making this parameter larger improves the speed at which the controller can response, but at the expense of having a lower signal-to-noise ratio that could yield to an undesired feedback action.

In the simulations of figures 7.2 and 7.3, we take $\alpha_k = 0.95$, $\beta_k = 0.9$ and $\theta = 30$, so that the controller is turned on as less as possible. Simulations show that the biggest impact on performance comes from the parameter ϵ_k, which sets the timescale of the filtered signals $\mathfrak{p}_k$. In figure 7.3, $\epsilon_k = 0.1$ gives a slow timescale, and the information is degraded quite fast. Remarkably, in figure 7.2, setting $\epsilon_k = 0.5$ yields better overal results. Even with the risk of having a too much noisy control input, it seems that having a faster filter, and a highly damped feedback action, is the better choice. The reasoning on this is that, the filter cannot be too slow to detect an error, that is, its characteristic timescale cannot be too slow compared to the timescale of the errors γ_s. Else, there is a probability of having logical errors in the detection stage that could further damage information. Notice that the closed-loop response under the low-pass filter of figure 7.2 behaves quite similar as the filter with delay in figure 6.4b.

It would be an open issue to show convergence of similar filters that are able to extract information on the state in an efficient manner, and to develop tools to characterize convergence. This would be an important step towards the implementation of continuous-time control laws via analog electronics.

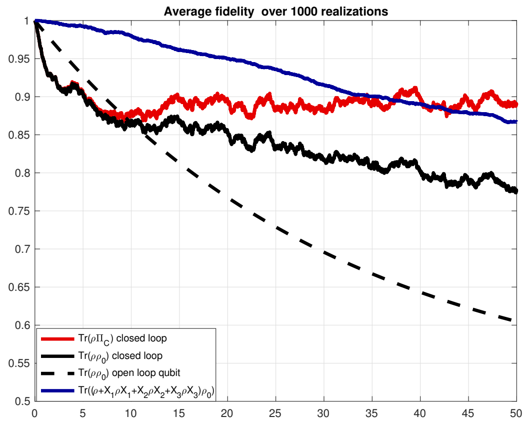


Figure 7.2: Closed-loop simulations of the three qubit bit-flip code. Settings on the quantum system are the same as the ideal situation of Fig. 6.2a. Parameters of the low-pass filter are: $\xi_k(0) = 1$ and $\epsilon_k = 0.5$, $k \in \{1, 2, 3\}$.

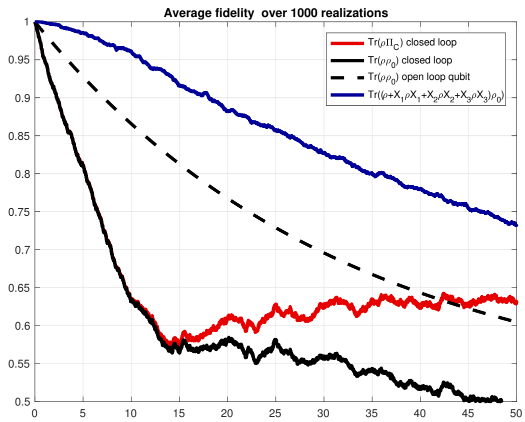


Figure 7.3: Same closed-loop system with a low-pass filter. Here $\epsilon_k = 0.1$, $k \in \{1, 2, 3\}$.

Appendix A

Lyapunov's second method for stochastic stability

We provide some background from stochastic stability and refer the reader to [43, 42, 41] for the proof of these results and for further reference. Consider diffusion processes x_t on $\mathbb{R}^n$, corresponding to solutions of Itō stochastic differential equations of the form

$$dx = \theta(x)dt + \sigma(x)dW, \tag{A.1}$$

where the
theta, σ are regular functions of x that satisfy the usual conditions for existence and uniqueness of SDE's [8]. In this appendix it is assumed that $\mathcal{S}$, a compact subset of $\mathbb{R}^n$ is positively invariant, that is, if $x_0 \in \mathcal{S}$, then $x_t \in \mathcal{S}, \forall t \geq 0$ with unit probability.

For a real-valued twice continuously differentiable function V, Itō's formula [55] yields

$$dV = \Big(\sum_i \theta_i \frac{\partial}{\partial x_i} V + \frac{1}{2}\sum_{i,j} \sigma_i \sigma_j \frac{\partial^2}{\partial x_i x_j} V\Big) dt + \sum_i \sigma_i \frac{\partial}{\partial x_i} V dW_i. \tag{A.2}$$

The Markov generator associated with (A.1) is

$$\mathcal{A}V = \sum_i \theta_i \frac{\partial}{\partial x_i} V + \frac{1}{2}\sum_{i,j} \sigma_i \sigma_j \frac{\partial^2}{\partial x_i x_j} V, \tag{A.3}$$

for any function V in its domain. It is related to (A.2) [55, Chapter 7] by

$$\mathbb{E}[V(x(t))] = V(x(0)) + \mathbb{E}\left[\int_0^t \mathcal{A}V(x(s))ds\right].$$

We now state the Lyapunov criteria for stochastic stability. Let $\mathcal{I} := \{x \in \mathcal{S} : \theta(x) = \sigma(x) = 0\}$ be a set of steady states of Eq. (A.1) and let

$V(x)$ be a nonnegative real-valued twice continuously differentiable function with respect to $x \in \mathcal{S}$ everywhere except possibly for $x \in \mathcal{I}$, $V(x) = 0$ for $x \in \mathcal{I}$.

The stochastic counterpart of Lyapunov's second method provides a sufficient criterion for proving the stability of continuous-time diffusion processes in terms of the expectation of a scalar positive function V. Essentially if $\mathcal{A}V(x) \leq 0$, $\mathcal{A}V(x) < 0$, $\forall x \notin \mathcal{I}$, then $V(x_t)$ decreases in expectation over time; then results on martingale theory imply that it must converge to zero, which in turn implies convergence of x_t towards the set $\mathcal{I}$.

We consider the following specific definition of stability

Definition A.0.1 (Khasminskii [41])**.** *Consider the diffusion process on $\mathcal{S}$ governed by* (A.1) *with $0 \in \mathcal{I}$ and $p > 0$. Let $\|\cdot\|$ be a norm defined on $\mathcal{S}$. The equilibrium solution $x_t = 0$ is said exponentially stable if, for some constants $C > 0$ and $r > 0$*

$$\mathbb{E}(\|x_t\|^p) \leq C\|x_0\|^p e^{-rt}.$$

Theorem A.0.1 (Khasminskii [41])**.** *Asume that there exists a positive constant r such that*

$$\mathcal{A}V(x) \leq -rV(x), \quad \forall x \in \mathcal{S}.$$

then

$$\mathbb{E}[V(x(t))] \leq V(x(0)) \, \exp(-r\,t),$$

i.e., $V(x_t)$ is a supermartingale with exponential decay, and $x(t)$ converges towards the set $\{x : V(x) = 0\}$.

Moreover, if there exist strictly positive constants p, k_1, k_2 and k_3 such that for all $x \in \mathcal{S}$ we have

- $\mathcal{A}V(x) \leq -k_3\|x\|^p$,
- $k_1\|x\|^p \leq V(x) \leq k_2\|x\|^p$.

Then the equilibrium solution $x_t = 0$ is globally exponentially stable *in the sense of definition A.0.1.*

The first part Theorem A.0.1 just says that x

A.1 Lyapunov functions for QND systems

From a stochastic stability point of view, this thesis revolves around defining Lyapunov functions V and constructing controls such that V is a supermartingale with exponential decay in the sense of Theorem A.0.1. The following pages consists on some common observations that appeared on the Lyapunov analysis of the precedent sections, and we add some computations that could be useful for further research.

The Lyapunov functions used in this thesis are of the form

$$V(\rho) = \sum_s \sqrt{\mathrm{Tr}\,(\boldsymbol{Q}_s\rho)} \tag{A.4}$$

where $\boldsymbol{Q}_s = \sum_k \alpha_{s,k}\Pi_k$, with $\alpha_{s,k}$ positive constants and recall that Π_k are the $d-1$ projectors orthogonal to the target Π_ℓ, $1-\Pi_\ell = \sum_{k\neq\ell}\Pi_k$. The $\boldsymbol{Q}_s$ are non-negative Hermitian operators whose kernels intersections coincide with the target eigenspace of L, and $\mathrm{Tr}\,(\boldsymbol{Q}_s\rho) = 0$ only when $V(\rho) = 0$.

With this condition, we can always find c_1, c_2 such that $c_1\sqrt{1-\boldsymbol{p}_\ell} \leq V(\rho) \leq c_2\sqrt{1-\boldsymbol{p}_\ell}$. Itō's formula applied to the concave function $V(\rho)$ yields

$$dV = \frac{1}{2}\sum_k \sqrt{\mathrm{Tr}\,(\boldsymbol{Q}_k\rho)}\left(\frac{\mathrm{Tr}\,(\boldsymbol{Q}_k\, d\rho)}{\mathrm{Tr}\,(\boldsymbol{Q}_k\rho)} - \frac{(\mathrm{Tr}\,(\boldsymbol{Q}_k\, d\rho))^2}{4\,\mathrm{Tr}\,(\boldsymbol{Q}_k\rho)^2}\right). \tag{A.5}$$

To design stabilizing feedback controls, an important observation is that a closed-loop Lyapunov function having a minimum at Π_ℓ does not automatically hint at the fact that the latter is more attractive than other critical points, which are maxima of $V(\rho)$. The role of feedback is to preserve the monotonicity of the square root by making $\mathcal{A}V(\rho)$ strictly negative even when the state ρ is supported on an eigenspace orthogonal to the target, implying that, in expectation, V is monotonically decreasing at maxima.

In the qubit case, with Lyapunov function

$$\sqrt{1-\mathrm{Tr}\,(\rho\Pi_z)}$$

when $\rho = \Pi_{Z_1}$ we have

$$\tfrac{d}{dt}\mathbb{E}[V(\rho)]\big|_{\rho=\Pi_{Z-1}} = -\frac{\eta\Gamma\bar{\sigma}^2}{2}.$$

For the multi-level case with single actuator

$$d\rho = \mathcal{D}_L(\rho)dt + \sqrt{\eta}\mathcal{M}_L(\rho)dW + \sigma(\rho)^2\mathcal{D}_H(\rho)dt - i\sigma(\rho)[H,\rho]dB,$$

the Lyapunov function $\sum_{s\in\{1,\ldots,d\}\setminus\{\ell\}}\sqrt{\sum_{k\in\{1,\ldots,d\}\setminus\{\ell\}}\alpha_{s,k}\boldsymbol{p}_k(\rho)}$ is monotonically decrasing at the extrema thanks to the tuning procedure of Proposition 5.2.1, so

$$\tfrac{d}{dt}\mathbb{E}[V(\rho)]\big|_{\rho=\Pi_j} = -\sum_{s\in\{1,\ldots,d\}/\{\ell\}}\frac{\beta_{s,j}}{\sqrt{\alpha_{s,j}}} < 0, \quad j\neq\ell$$

.

In the case of the three qubit bit-flip code For the Lyapunov function (6.7) on the QEC scheme, it can be equally verified that

$$\tfrac{d^2}{d\sigma^2}\mathbb{E}[V(\rho)] = -\tfrac{\eta\Gamma c}{\sqrt{2}}$$

when $\mathrm{Tr}\,(\rho\Pi_j) = 1$ for an error projector $j \in \{1, 2, 3\}$

Once we are able to establish that V is monotonically decreasing at the extrema, the square root endows the Lyapunov function of the following properties that are used to deduce exponential convergence close to a target eigenstate (3.13), by exploiting positivity of ρ and the geometry fixed by the QND eigenspaces.

1. The Hessian of $V(\rho)$ is negative semidefinite for all ρ.
2. $V(\rho)$ is a positively homogeneous function of the populations $\{\boldsymbol{p}_k(\rho)\}_{k\neq\ell}$, $\boldsymbol{p}_k(\rho) = \mathrm{Tr}\,(\rho\Pi_k)$.

It is convenient to think of the eigenstate populations in terms of the homogeneous coordinates, $\boldsymbol{p}_\ell := 1 - x_\ell$, $\boldsymbol{p}_k := x_\ell x_k$ for each $k \neq \ell$. Then

$$\sqrt{\mathrm{Tr}\,(\rho\boldsymbol{Q}_s)} = \sqrt{\sum_{s,k\neq\ell} \alpha_k \boldsymbol{p}_k} = \sqrt{x_\ell}\sqrt{\sum_{s,k\neq\ell} \alpha_k x_k}, \tag{A.6}$$

Since for any $\rho_0 \in \mathcal{S}$, $\rho_t \geq 0$ for all $t \geq 0$, the Markov generator for the square root process $\mathcal{A}V(\rho)$ would be defined everywhere except possibly when the state ρ is supported on the subspace with projector Π_ℓ. Since $\sigma(\rho) = 0$ when $\mathrm{Tr}\,(\Pi_\ell\rho) = 1$ and the open-loop system is at an equilibrium, the stochastic Lyapunov theory is equiped to deal with the singularity when $\mathrm{Tr}\,(\Pi_\ell\rho) = 1$ [41]. For QND systems, we can go further and show that $\mathcal{A}V(\rho)$ is actually a continuous function of ρ. We show this by exploiting the transformation of the closed-loop generator into the homogeneous coordinates described previously.

Lemma A.1.1. *Consider the closed-loop system* (3.13) *and $V(\rho)$ be of the form* (A.4)*.*

- *Assume that there exist constants c_1, c_2 such that*

$$c_1\sqrt{1 - \mathrm{Tr}\,(\rho\Pi_\ell)} \leq V(\rho) \leq c_2\sqrt{1 - \mathrm{Tr}\,(\rho\Pi_\ell)}.$$

- *The control $\sigma(\rho)$ Eq.* (3.13) *is a smooth and bounded function of ρ and there exists $\epsilon > 0$ such that $\sigma(\rho) = 0$ when $1 - \mathrm{Tr}\,(\rho\Pi_\ell) \leq 1 - \epsilon$.*

Then, $V(\rho_t)$ is an Itō diffusion defined on the compact set

$$\tilde{P} = \{(\boldsymbol{p}_1(\rho), \ldots, \boldsymbol{p}_d(\rho)) \in \mathbb{R}^d \mid \sum_{k\neq\ell} \boldsymbol{p}_k(\rho) = 1 - \boldsymbol{p}_\ell(\rho), 0 \leq \boldsymbol{p}_k(\rho) \leq 1\}$$

for all $t \geq 0$ and the Markov generator $\mathcal{A}V(\rho)$ is a continuous function defined on $\tilde{P}$.

Proof. We do not prove directly that the generator of $V(\rho)$, rather we use the first assumption to show that $\sqrt{1-\mathrm{Tr}\,(\rho\Pi_\ell)}$ is a continuous function on $\tilde{P}$, the conditions on the theorem allow us to conclude. The Markov generator of $\sqrt{1-\mathrm{Tr}\,(\rho\Pi_\ell)}$ is

$$\mathcal{A}\sqrt{1-\mathrm{Tr}\,(\rho\Pi_\ell)} = -\tfrac{1}{2}v(\rho)\left(\tfrac{\sigma(\rho)^2\,\mathrm{Tr}(\mathcal{D}_H(\Pi_\ell)\rho)}{1-\boldsymbol{p}_\ell} + \tfrac{(\lambda_\ell-\sum_k\lambda_k\boldsymbol{p}_k)^2\boldsymbol{p}_\ell^2}{(1-\boldsymbol{p}_\ell)^2} + \tfrac{\sigma(\rho)^2(\mathrm{Tr}([H,\rho]\Pi_\ell))^2}{(1-\boldsymbol{p}_\ell)^2}\right).$$

The assumption on the control implies directly that the terms multiplying σ are bounded and continuous functions on $\tilde{P}$. It suffices then to verify the term

$$\mu(\boldsymbol{p}_1,\dots,\boldsymbol{p}_d) = \frac{(\lambda_\ell - \sum_k\lambda_k\boldsymbol{p}_k)^2\boldsymbol{p}_\ell^2}{(1-\boldsymbol{p}_\ell)^2}.$$

Under the homogeneous change of coordinartes $\boldsymbol{p}_\ell := 1-x_\ell$, $\boldsymbol{p}_k := x_\ell x_k$, $0\le x_\ell\le 1$, with $\sum_{k\neq\ell}x_k = 1$. Under these coordinates

$$\mu(x) = \Big(\sum_s(\lambda_s-\lambda_\ell)x_s\Big)^2(1-x_\ell)^2,$$

$\mu(x)$ is a smooth function of x, since the map $\phi = (x_1,\dots,x_{\ell-1},x_\ell,x_{\ell+1},\dots,x_d)\mapsto (x_\ell x_1,\dots,x_\ell x_{\ell-1},x_\ell x_{\ell+1},\dots,x_\ell x_d)$ is a continuous bijection this implying continuity of $\mathcal{A}\sqrt{1-\mathrm{Tr}\,(\rho\Pi_\ell)}$. □

For the sake of comparison, we consider two Lyapunov functions used in previous works, e.g. [50, 69] : $V_1(\rho) = 1-(\mathrm{Tr}\,(\rho\Pi_\ell))^2$ and $V_2(\rho) = 1-\mathrm{Tr}\,(\rho\Pi_\ell)$. V_1 is concave, so its second derivative is negative, but it is not a homogeneous function of the populations. V_2 is homogeneous, but its second derivative is zero. It appears that having a homogeneus Lyapunov function with non-vanishing second derivative is essential when targeting QND eigenstates. In [17] it is showed exponential convergence of a projector Π_ℓ in terms of $V_2(\rho) = 1-\mathrm{Tr}\,(\Pi_\ell\rho)$, when Π_ℓ *is not* a QND eigenstate, extending the result of Theorem 3.2.1. On deterministic systems, exponential convergence of a function $V(x)$ is related to exponential convergence of $\sqrt{V(x)}$ up-to a numerical factor on the convergence rate. Since the square root does not commute with the expectation, the same thing cannot be said for supermartingales; observe by Jensen's inequality that $\mathbb{E}(\sqrt{V_2(x_t)}) \le \sqrt{\mathbb{E}(V_2(x_t))}$, so showing the exponential convergence of $V_2(x_t)$ as in [17] implies exponential convergence of $\sqrt{V_2(x_t)}$, since here we do the inverse, our approach is weaker.

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RÉSUMÉ

Dans cette thèse, nous développons des méthodes de contrôle pour stabiliser des systèmes quantiques en temps continu sous mesures quantiques non-destructives. En boucle ouverte, ces systèmes convergent vers un état propre de l'opérateur de mesure, mais l'état résultant est aléatoire. Le rôle du contrôle est de préparer un état prescrit avec une probabilité de un. Le nouvel élément pour atteindre cet objectif est l'utilisation d'un mouvement Brownien pour piloter les actions de contrôle. En utilisant la théorie stochastique de Lyapunov, nous montrons stabilité exponentielle globale du système en boucle fermés. Nous explorons aussi la syntèse du contrôle pour stabiliser un code correcteur d'erreurs quantiques en temps continu. Un autre sujet d'intérêt est l'implementation de contrôles efficacement calculables dans un contexte expérimental. Dans cette direction, nous proposons l'utilisation de contrôles et filtres qui calculent seulement les characteristiques classiques du système, correspondant a la base propre de l'opérateur de mesure. La formulation de dites filtres est importante pour adresser les problèmes de scalabilité du filtre posées par l'avancement des technologies quantiques.

MOTS CLÉS

Contrôle quantique, information quantique, correction d'erreurs quantiques, systèmes stochastiques.

ABSTRACT

In this thesis, we develop control methods to stabilize quantum systems in continuous-time subject to quantum non-demolition measurements. In open-loop such quantum systems converge towards a random eigenstate of the measurement operator. The role of feedback is to prepare a prescribed eigenstate with unit probability. The novel element to achieve this is the introduction of an exogenous Brownian motion to drive the control actions. By using standard stochastic Lyapunov techniques, we show global exponential stability of the closed-loop dynamics. We explore as well the design of the control layer for a quantum error correction scheme in continuous-time. Another theme of interest is towards the implementation of efficiently computable control laws in experimental settings. In this direction, we propose the use control laws and of reduced-order filters which only track classical characteristics of the system, corresponding to the populations on the measurement eigenbasis. The formulation of these reduced filters is important to address the scalability issues of the filter posed by the advancement of quantum technologies.

KEYWORDS

Quantum control, quantum information, quantum error correction, stochastic systems.