A topology for labelled metric spaces, application to s-compact random genealogical trees

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Une topologie pour les arbres labellés, application aux arbres aléatoires S-compacts

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Chapter 1

Résumé étendu en français

1.1 L'ensemble des espaces métriques labellés, topologie et propriétés

Une des premières tentatives de comparer deux espaces métriques compacts vient de Gromov [39]. Son idée est de plonger isométriquement les deux espaces à comparer dans un troisième espaces métrique et de comparer les plongements à l'aide de la distance de Hausdorff. La distance de Gromov-Hausdorff est l'infimum pour tous les plongements de la distance de Hausdorff entre les plongements. Comme la distance de Hausdorff n'est définie que pour les compacts, la distance de Gromov-Hausdorff n'est définie que pour les compacts. La distance de Gromov-Hausdorff a été rapidement généralisée aux espaces compacts munis d'une mesure finie par la distance de Gromov-Hausdorff-Prokhorov. Elle est définie comme la distance de Gromov-Hausdorff, mais en remplaçant la distances de Hausdorff par le maximum de la distance de Hausdorff entre les plongement des deux ensembles et la distance de Prokhorov entre les plongement des deux mesures.

Ces deux distances ont été étendues de plusieurs manières, sur des espaces avec toutes sortes de structures supplémentaires, mais toujours avec l'une des contraintes suivantes :

1. les espaces sont compacts (Gromov [39]);
2. les espaces sont complets, pointés et toutes les boules fermées sont compactes (Abraham, Delmas & Hoscheit [1], Khezeli [43]);
3. Les espaces sont complets, séparables et munis d'une mesure finie dont le support est l'espace entier (Aldous [6] and [5], Greven, Pfaffelhuber & Winter [36]).

Dans cette section, on considère une classe d'espaces métriques mesurés munis de fonctions 1-lipschitziennes, dans le but d'obtenir une version plus large de 2, en considérant des espaces qui ne sont ni bornés ni pointés, équipés avec des mesures finies sur tout compact. En échange, on doit remplacer la condition "les boules sont compactes" par "les tranches sont compactes" (une tranche est l'ensemble des points dont les labels sont dans un compact $[-h, h] \subset \mathbb{R}$) qui est un peu plus forte. On appelle espace métrique labellé mesuré tout quatruplet (E, d, H, ν), où (E, d) est un espace métrique complet séparable, ν est une mesure de Borel[1] positive et

[1]Une mesure de Borel est une mesure définie sur l'ensemble des boréliens de (E, d) qui est finie sur tous les compacts de E.

H est une fonction 1-lipschitzienne de E dans $\mathbb{R}$.

Pour tout $h \in \mathbb{R}_+$, on appelle $\mathrm{Slice}_h(E, d, H, \nu,)$ ou simplement E_h l'ensemble $\{x \in E||H(x)| \leq h\}$, muni des restrictions de d, H et ν. On dit que (E, d, H, ν) est S-compact si pour tout $h \in \mathbb{R}_+$, $\mathrm{Slice}_h(E, d, H, \nu)$ est compacte.

On dit que deux espaces métriques labellés mesurés (E, d_E, H_E, ν_E) et (F, d_F, H_F, ν_F) sont équivalents si il existe une isometrie bijective ϕ de (E, d_E) sur (F, d_F) telle que $H_E = H_F \circ \phi$ et la mesure image de ν_E par ϕ est égale à ν_F (c-à-d $\phi(\nu_E) = \nu_F$). C'est une relation d'équivalence sur la classe $\tilde{\mathbb{X}}$ de tous les espaces métriques labellés mesurés, qui préserve la compacité et la S-compacité. On appelle $\mathbb{X}$ l'enemble des classes d'équivalence dans $\tilde{\mathbb{X}}$. On appelle $\mathbb{X}^S$ l'ensemble des classes d'équivalences d'espaces métriques labellés mesurés S-compacts et $\mathbb{X}^K$ l'ensemble des classes d'équivalences d'espaces métriques labellés mesurés compacts. On note abusivement E l'espace métrique labellé mesuré (E, d, H, ν), et on confondra souvent une classe d'équivalence avec n'importe lequel de ses representants.

Pour (E, d_E, H_E, ν_E) et (F, d_F, H_F, ν_F) deux espaces métriques labellés mesurés compacts, (Z, d_Z) un espace métrique séparable, ϕ_E (resp. ϕ_F) une isometrie de E (resp. F) dans Z, on considère le plongement ϕ_E^* (resp. ϕ_F^*) de E (resp. F) dans $Z \times \mathbb{R}$ défini par $\phi_E^*(x) = (\phi_E(x), H_E(x))$ (resp. $\phi_F^*(y) = (\phi_F(y), H_F(y))$. Avec cette construction, on représente E, F et leurs labels dans un même espace $Z^* = Z \times \mathbb{R}$. On munit Z^* de la distance d_Z^* définie par :

$$d_Z^*\big((x, h), (x', h')\big) = d_Z(x, x') \vee |h - h'|.$$

On pose

$$\Delta^Z_{\phi_E,\phi_F}(E, F) = d_{\mathrm{H}}\big(\phi_E^*(E), \phi_F^*(F)\big) \vee d_{\mathrm{P}}\big(\phi_E^*(\nu_E), \phi_F^*(\nu_F)\big),$$

où d_{H} (resp. d_{P}) est la distance de Hausdorff (resp. Prohorov) sur (Z^*, d_Z^*), $\phi_E^*(E)$ est l'image directe de E par ϕ_E^* et $\phi_E^*(\nu_E)$ est la mesure-image de ν_E par ϕ_E^* (on procède de même pour F). Ainsi, le nombre $\Delta^Z_{\phi_E,\phi_F}(E, F)$ tient compte de tous les aspects de E et F. Sur le modèle de la distance de Gromov-Hausdorff-Prohorov pour les espaces compacts sans labels, on définit

$$d_{\mathrm{GHP}}(E, F) = \inf_{Z,\phi_E,\phi_F} \Delta^Z_{\phi_E,\phi_F}(E, F),$$

où l'infimum est pris sur tous les espaces métriques Z et isometries ϕ_E (resp. ϕ_F) de E (resp. F) dans Z.

Pour (E, d_E, H_E, ν_E) et (F, d_F, H_F, ν_F) deux espaces métriques labellés mesurés S-compacts, on définit

$$d_{\mathrm{LGHP}}(E, F) = \int_0^\infty \Big(1 \wedge d_{\mathrm{GHP}}(E_h, F_h)\Big) \mathrm{e}^{-h} dh.$$

De la même manière, on définit d_{GH} et d_{LGH} pour les espaces métriques labellés (sans mesures) par

$$d_{\mathrm{GH}}\big((E, d_E, H_E), (F, d_F, H_F)\big) = d_{\mathrm{GHP}}\big((E, d_E, H_E, 0), (F, d_F, H_F, 0)\big)$$

et

$$d_{\mathrm{LGH}}\big((E, d_E, H_E), (F, d_F, H_F)\big) = d_{\mathrm{LGHP}}\big((E, d_E, H_E, 0), (F, d_F, H_F, 0)\big).$$

On notera que d_{GHP} (resp. d_{LGHP}), qui est défini sur la classes des espaces métriques labellés mesurés S-compact (resp. compact), peut être défini sur $\mathbb{X}^K$ (resp. $\mathbb{X}^S$), puisque la valeur pour deux classes d'équivalence ne dépend pas du représentant choisi. On prouve le résultat suivant :

Proposition 3.1.13 et Théoreme 3.3.1 La fonction d_{LGHP} est une distance sur $\mathbb{X}^S$ et l'espace métrique $(\mathbb{X}^S, d_{\mathrm{LGHP}})$ est polonais.

On définit $\mathbb{X}^C$ l'ensemble des espaces métriques labellés mesurés S-compacts (E, d, H, ν) (à équivalence près) pour lesquels $H(E)$ est connexe (c-à-d. un intervalle). Posons $\mathbb{X}^{C,K} = \mathbb{X}^K \cap \mathbb{X}^C$, on a:

Proposition 3.4.4 and Lemme 3.4.5 Sur $\mathbb{X}^K$, la topologie induite par d_{GHP} est strictement plus fine que la topologie induite par d_{LGHP}. Les distances d_{GHP} et d_{LGHP} induisent la même topologie sur $\mathbb{X}^{C,K}$.

Dans la definition de d_{LGHP}, 0 joue un rôle particulier puisque les tranches Slice_h sont prises entre $-h$ et h. Les changements au niveau des labels proches de 0 sont plus visibles pour la distance que les changements loin de 0. Pour voir si cette différence se voit dans la topologie, on définit une autre distance où les tranches sont centrées autour de $a \in \mathbb{R}$. Pour $a \in \mathbb{R}$, (E, d, H, ν) un espace métrique labellé mesuré S-compact et $h \in \mathbb{R}_+$, on définit $E_h^a = \mathrm{Slice}_h^a(E, d, H, \nu)$ l'ensemble

$$\{x \in E \,|\, |H(x) - a| \leq h\}$$

muni des restrictions de d, H et ν. Pour $a \in \mathbb{R}$, (E, d_E, H_E, ν_E) et (F, d_F, H_F, ν_F) deux espaces métriques labellés mesurés S-compacts, on définit

$$d_{\mathrm{LGHP}}^a(E, F) = \int_0^\infty \Big(1 \wedge d_{\mathrm{GHP}}(E_h^a, F_h^a)\Big) \mathrm{e}^{-h} dh.$$

Cette distance est une version de d_{LGHP} où on a donné à a le rôle particulier qu'avait 0 dans d_{LGHP}. On a

Proposition 3.4.6 Pour tout $a \in \mathbb{R}$, la translation $(E, d, H, \nu) \mapsto (E, d, H + a, \nu)$ est continue. De manière équivalente, d_{LGHP}^a induit la même topologie que d_{LGHP} sur $\mathbb{X}^S$.

1.2 Un nouvel espace d'arbres généalogiques et sa topologie

On rapelle qu'un arbre est un espace de longueur dans lequel chaque paire de point est relié par un unique chemin, qui doit être une géodésique. On cherche à donner un cadre pour des arbres comme les arbrbres aléatoires stationnaires (voir par exemple Chen & Delmas [16]) et la généalogie du processus look-down (Donnelly & Kurtz [22]), qui ne sont pas compacts et dont la mesure naturelle est infinie. De plus, la "racine" de ces arbres est à $-\infty$, donc ils ne tombent pas dans le cas des espaces de longueur pointés décrit dans Abraham, Delmas & Hoscheit [1]. Pour représenter ces arbres, on oublie la notion de racine et on appelle *arbres labellés par la hauteur* tous les éléments $(T, d, H, \nu) \in \mathbb{X}^S$ tels que (T, d) est un arbre et pour tout $x, y \in T$,

$$d(x, y) = H(x) + H(y) - 2h_{\min},$$

où $h_{\min}$ est le minimum de H sur la geodesique de x à y. L'idée est que (T, d) est un arbre généalogique et $H(x)$ represente le temps auquel l'individu x a vécu. La distance entre deux points est alors la somme des temps qui séparent les deux individus de leur plus proche ancêtre commun. On note $\mathbb{T} \subset \mathbb{X}^S$ l'ensemble des arbres labellés par la hauteur, considérés à équivalence près.

L'espace des arbres labellés par la hauteur est muni de d_{LGHP}. On prouve

Théorèmes 4.1.15 L'espace $(\mathbb{T}, d_{\text{LGHP}})$ est un fermé de $\mathbb{X}^S$, donc polonais.

Notons que pour tout arbre labellé par la hauteur (T, d, H, ν), (T, d) est connexe, donc l'image directe $H(T)$ est toujours un intervalle. Ainsi, l'espace des arbres labellés par la hauteur $\mathbb{T} \cap \mathbb{X}^K$ est inclus dans $\mathbb{X}^{C,K}$. On en déduit par le Lemme 3.4.5 que d_{GHP} definit la même topologie que d_{LGHP} sur $\mathbb{T}$.

On donne dans Proposition 4.1.14 une bijection entre arbres labellés par leur hauteur, et les arbres codés (arbres labellés dont la distance a été remplacée par un ordre partiel (l'ordre généalogique)), ce qui fournit une caractérisation alternative des arbres labellés par leur hauteur.

1.3 Quelques operations mesurables sur les arbres

On définit quelques opérations sur les arbres labellés par leur hauteur, et on étudie leur mesurabilité.

Le ε-trimming est défini dans la litterature comme l'ensemble des points d'un arbre qui sont le milieu d'une géodésiqu de longueur au moins 2ε. Dans cette définition, on supprime systématiquement l'extrémité des branches. Cela affecte la hauteur des branches, et rends les tranches plus difficiles à contrôler (on perd la propriété selon laquelle $d_{\text{LGHP}}(T, \text{Trim}_\varepsilon(T)) \leq \varepsilon$). Il est donc plus confortable de redéfinir le ε-trimming $\text{Trim}_\varepsilon(T) = (T^\varepsilon, d^\varepsilon, H^\varepsilon, \nu^\varepsilon)$ d'un arbre labellé par la hauteur (T, d, H, ν) comme un quotient de l'arbre. On considère que deux points $x, y \in T$ sont dans la même classe si $H(x) = H(y)$ et $d(x, y) \leq 2\varepsilon$ (c'est une relation d'équivalece), et on définit T^ε le quotient de T par cette relation. Pour $\mathbf{x}, \mathbf{y} \in T^\varepsilon$, on définit

$$d^\varepsilon(\mathbf{x}, \mathbf{y}) = \big(d(x, y) - 2\varepsilon\big) \vee \big|H(x) - H(y)\big|$$

$$H^\varepsilon(\mathbf{x}) = H(x),$$

où x, y sont deux representants de $\mathbf{x}, \mathbf{y} \in T^\varepsilon$. Les definitions ci-dessus ne dépendent pas du choix des representants x, y. On pose ρ la projection canonique de T dans T^ε,et on définit ν^ε la mesure image de ν par ρ. On prouve que

Lemmes 4.2.7, 4.2.8 & 4.2.10 Pour T un arbre labellé par la hauteur, $\varepsilon > 0$, $\text{Trim}_\varepsilon(T)$ le ε-trimming de T, on a

- $\text{Trim}_\varepsilon(T)$ est bien défini et donne un arbre discret,
- $d_{\text{LGHP}}(T, \text{Trim}_\varepsilon(T)) \leq \varepsilon$,
- $T \mapsto \text{Trim}_\varepsilon(T)$ est 1-lipschitzienne de $\mathbb{T}$ dans $\mathbb{T}$.

Pour (T, d, H, ν) un arbre labellé par la hauteur et $h \in \mathbb{R}$, on définit $\mathrm{Stump}_h(T)$ l'ensemble $\{x \in T | H(x) \leq h\}$ muni des restrictions de d, H et ν. C'est la partie de T sous le niveau h.

Proposition 4.2.11 La fonction $(T, h) \mapsto \mathrm{Stump}_h(T)$ est mesurable de $\mathbb{T} \times \mathbb{R}$ dans T.

On souhaite définir la couronne $\mathrm{Crown}_h(T)$ d'un arbre labellé par la hauteur T comme la forêt non-ordonnée des branches de T au-dessus du niveau h. Pour ce faire, nous devons commencer par construire un espace dans lequel définir les couronnes. On commence par définir l'ensemble des séquences dont on a oublié l'ordre. On pose $\tilde{\mathbb{X}}^S_C \subset (\mathbb{X}^S)^{\mathbb{N}^*}$ l'ensemble des suites convergentes de $\mathbb{X}^S$, et on considère la pseudo-distance définie sur $\tilde{\mathbb{X}}^S_C$ par:

$$d^{\infty}_{\mathrm{LGHP}}((T_n)_{n\in\mathbb{N}^*}, (T'_n)_{n\in\mathbb{N}^*}) = \inf_{\sigma\in\mathfrak{S}(\mathbb{N}^*)} \sup_{n\in\mathbb{N}^*} d_{\mathrm{LGHP}}(T_n, T'_{\sigma(n)}).$$

On définit $\mathbb{X}^S_C$ le quotient de $\tilde{\mathbb{X}}^S_C$ par la relation d'équivalence $d^{\infty}_{\mathrm{LGHP}}(\cdot, \cdot) = 0$. L'espace $(\mathbb{X}^S_C, d^{\infty}_{\mathrm{LGHP}})$ est un espace métrique complet séparable.

On définit ensuite un borélien $\mathbb{X}^S_C$ qui contiendra toutes les couronnes. Pour tout $h \in \mathbb{R}$, on note $0_h = (\{h\}, 0, h, 0)$ l'arbre constitué d'un seul point au niveau h muni de la mesure nulle. On pose $\mathbb{T}_C$ l'ensemble des éléments $(T^n)_{n\in\mathbb{N}^*} \in \mathbb{T}^{\mathbb{N}^*}$ pour lesquels il existe $h \in \mathbb{R}$ tel que

- $\lim_n T_n = 0_h$
- tous les arbres $(T_n)_{n\in\mathbb{N}^*}$ sont enracinés à hauteur h.

Pour un arbre (T, d, H, ν), $h \in H(T)$ et $x_0 \in \mathrm{Skel}(T)$ tel que $H(x_0) = h$, on appelle branche au-dessus de h le sous-arbre $\{x \in T | x_0 \preceq x\}$ muni des restrictions de d, H et ν. Si T est S-compact, l'ensemble de ses branches au-dessus de h est au plus dénombrable. Quand il y a une infinité de branches, on pose $\mathrm{Crown}_h(T)$ une énumération $(T_n)_{n\in\mathbb{N}^*}$ de ses branches au-dessus de h. notons que pour chaque énumération, $\lim_n T_n = 0_h$. Si T a un nombre fini de branches au-dessus de h, on complete la suite avec une succession infinie de 0_h. dans les deux cas, on a $\mathrm{Crown}_h(T) \in \mathbb{T}_C$. On étend la definition de $\mathrm{Crown}_h(T)$ aux cas où $h \notin H(T)$ par $\mathrm{Crown}_h(T) = (0_h)_{n\in\mathbb{N}^*}$ quand il n'y a aucun point strictement au-dessus de h ($T = \emptyset$ ou $\sup_T H \leq h$), et que la couronne ne contient aucune branche, et par $\mathrm{Crown}_h(T) = (T, 0_h, ...)$ quand $\min_T H > h$ (tous les points de T sont au-dessus de h, donc il y a une unique branche enracinée strictement au-dessus de h). Notons que dans ce dernier cas, $\mathrm{Crown}_h(T) \in \mathbb{X}^S_C \setminus \mathbb{T}_C$ car $\min_T H \neq h$. Ainsi, $(h, T) \mapsto \mathrm{Crown}_h(T)$ est défini de $\mathbb{R} \times \mathbb{T}$ dans $\mathbb{X}^S_C$.

Proposition 4.3.11 La fonction $(h, T) \mapsto \mathrm{Crown}_h(T)$ est mesurable.

Notre principal résultat sur les opérations concerne la greffe aléatoire d'une couronne sur un arbre. Comme nos arbres sont définis à ismoétrie près, on ne peux pas indiquer les endroits où greffer les points. Nous n'avons donc pas d'autre choix que de greffer les arbres au hasard, suivant une probabilité sur l'arbre sur lequel on greffe.

Pour comparer plus facilement la greffe de deux couronnes sur deux arbres, il vaut mieux considérer la loi de la greffe aléatoire. Pour un arbre labellé par la hauteur (T, d, H, ν, p) muni d'une mesure de probabilité supplémentaire p concentrée qu niveau $H^{-1}(\{h\})$ pour un certain $h \in H(T)$, T' un autre arbre labellé par la hauteur contenant au moins un point à hauteur h, prenons un représentant $(T_n, d_n, H_n, \nu_n)_{n\in\mathbb{N}^*}$ de $\mathrm{Crown}_h(T')$ et $(X_n)_{n\in\mathbb{N}^*}$ une suite i.i.d de variables aléatoires dans T de loi marginale p. On note $T \star_p T'$ le résultat de la greffe de chaque T_n au point $X_n \in T$, avec d' la distance H' la fonction label. On pose aussi $\nu' = \nu + \sum_n \nu_n$ et on note $P_{T\star_p T'}$ la loi $(T \star_p T', d', H', \nu', p)$. Si on muni l'ensemble des probabilités sur $(\mathbb{X}^S, d_{\mathrm{LGHP}})$ de la distance de Prokhorov, on a

Théoreme 4.4.5 et Proposition 4.4.7 L'opération de mélange $((T,p),T') \mapsto P_{T\star_p T'}$ est bien définie. C'est une mesure de probabilité independende du choix de représentant pour T et $\mathrm{Crown}_h(T')$. Elle est mesurable de son domaine de définition $D \subset \mathbb{T}^{[2]} \times \mathbb{T}$ dans $\mathbb{T}^{[2]}$, où $\mathbb{T}^{[2]}$ est une généalisation de $(\mathbb{T}, d_{\mathrm{LGHP}})$ aux arbres labellés par la hauteur équipés de deux mesures. D est un borélien de $\mathbb{T}^{[2]} \times \mathbb{T}$.

Ces résultats complètent Abraham, Delmas & Hoscheit [2], où la mesurabilité de l'opération de greffe n'était pas démontrée.

1.4 Arbre brownien conditionné par son temps local, une tentative de généralisation

Dans [7], Aldous presente la loi de l'excursion brownienne conditionnée par son temps local. Pour construire cette loi à partir du temps local $(l(h))_{h\geq 0}$ (c'est une densité de probabilité car l'excursion est normalisée pour être sur [0,1]), il crée $n \in \mathbb{N}^*$ feuilles à des hauteurs i.i.d (avec densité $l(h)dh$), et construit un coalescent à partir des feuilles (chaque couple de branche fusionne avec intensité $\frac{1}{l(h)}dh$) pour obtenir un arbre T_n^l. Il montre que la suite $(T_n^l)_{n\in\mathbb{N}^*}$ converge vers un arbre aléatoire T^l qu'il prouve être l'arbre brownien conditionné au temps local (Construction 1 and Theorem 2 respectivement dans Aldous [7]).

On se propose de généraliser ces lois. Plutôt que de les caractériser par un temps local l, on utilise une mesure de coalescence μ (qui joue le rôle de $\frac{1}{l(h)}dh$) et une mesure de masse ν qui décide la répartition de la masse entre les différents étages. Pour définir notre arbre, nous mélangeons des coalescents à taux μ commençant à différents niveaux. Cela ne pose aucun problème tant qu'on ne prend qu'un nobre fini de niveaux, et on utilise une convergence en loi pour obtenir un arbre limite. On prouve dans le Lemme 5.3.7 que l'arbre limite ne dépend pas de la suite des niveaux utilisés pour le construire, tant que cette suite est dense. On prouve une régularité faible dans le Lemme 5.4.3, qui constitue un premier pas vers la construction presque-sure d'une famille de mesures intrinsèques à presque-tout niveau.

Chapter 2

Introduction

2.1 Biological motivations

The objects in this thesis derive from a variety of works describing genealogies and, in a broader sense, the transmission and diffusion of genes in a population. In each cell, the genetic information is encoded in molecules of DNA, one by chromosome. Each information is coded as a sequence of nucleotides (ACGT) at a locus (a segment of the DNA specific to that information). The human genom consists in $6.5 \cdot 10^9$ nucleotides. In a given specie, the loci are in the same position, but will hold different sequences of nucleotides, hence a different information. Those different sequences are called versions of a gene, or alleles. In diploids, the chromosomes are split in pairs (23 for humans). Two chromosomes of a pair will have the same loci, but may carry different alleles. Through meiosis, each of the two parents produces a gamete, a reproductive cell holding one chromosome from each pair. When two gametes meet, the resulting offspring receives two chromosomes for each pair. Since the genetic information is held in the chromosomes, most genetic models will study the chromosomes rather than the parents. A way to study the diffusion of genes in the population is to draw the genealogical tree of the population, or rather of its chromosomes. To simplify matters, we will only consider a single pair of chromosomes in the rest of this section.

This view must still be refined, as we neglected another mechanism. Each parent possesses two chromosomes inherited from its own parents (labelled as grandparents from here on). Sometimes, during gamete production, the two chromosomes of one parent will exchange material, such that the first part of the resulting chromosomes holds material from one grandparent, while the latter part holds material from the other grandparent. This exchange is symmetric, so that the recombined chromosomes still have the same loci as the originals, but a new repartition of alleles. The process is called recombination, illustrated in Figure 2.1. Recombination is a rather frequent occurence, as both Sun & al. [59] and Kong and al.[47] find an order of around 50 recombination events per meiosis in humans, so about 2 per chromosome and reproduction.

When recombinations occur, we note a drastic change in the genealogical tree. As individuals with two parents appear, loops become possible. With recombinations, the natural representation of the genealogy is no longer a loop-free tree and becomes a regular graph. This graph is called the ancestral recombination graph (ARG). On it are represented all the contributors to the chromosomes of the top-most individuals. This means that past individ-

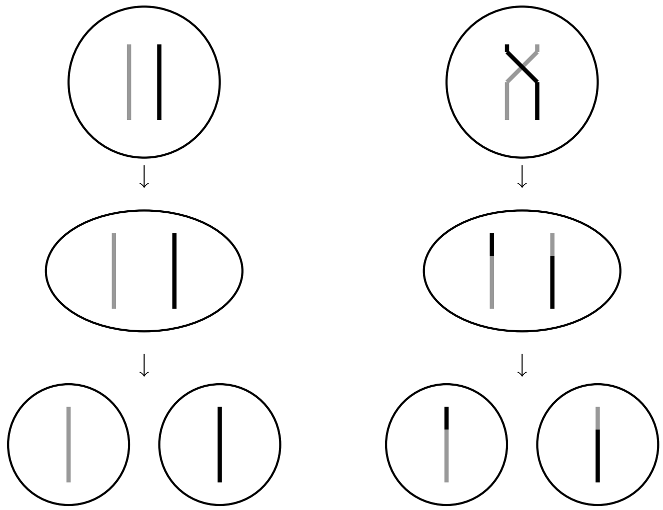

Figure 2.1: On the left, we see the usual process for meiosis (the production of gametes). On the right,we see how recombination may happen during meiosis.

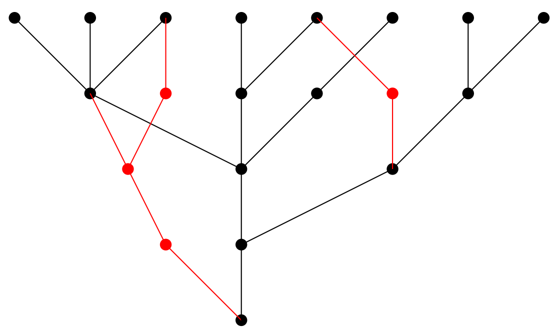

Figure 2.2: In black, a normal genealogical tree. In red and black, the ancestral recombination graph. See that the recombination event gave rise to additionnal ancestors to our initial population, and changed the depth of the latest common ancestor.

uals whose genetic material was not passed on to the current generation are not represented. This is shown in Figure 2.2.

Since chromosomes that where made by recombination have two parents, it is no longer possible to draw a loop-free genealogical tree for the whole chromosome, but we can still do it for a single locus. A locus comes from only one of the parent chromosomes, so it can be considered to have a single parent. This gives us a loop-free genealogical tree for each locus of a chromosome.

A way to look at recombination is to compare the genealogical tree of a locus directly on the left of the point to that of a locus directly to the right of the point. Looking to the left, it is the child of the parent that gave the left side. Looking to the right, it is a child of the parent that provided the right side. Note that this relation of coming from one parent of the other carries for all the offspring of the recombined chromosome. In a recombination event, the genealogical branch of our chromosome and all its descendants, is cut from one parent then grafted to the other. All the recombination events can thus be encoded in the process

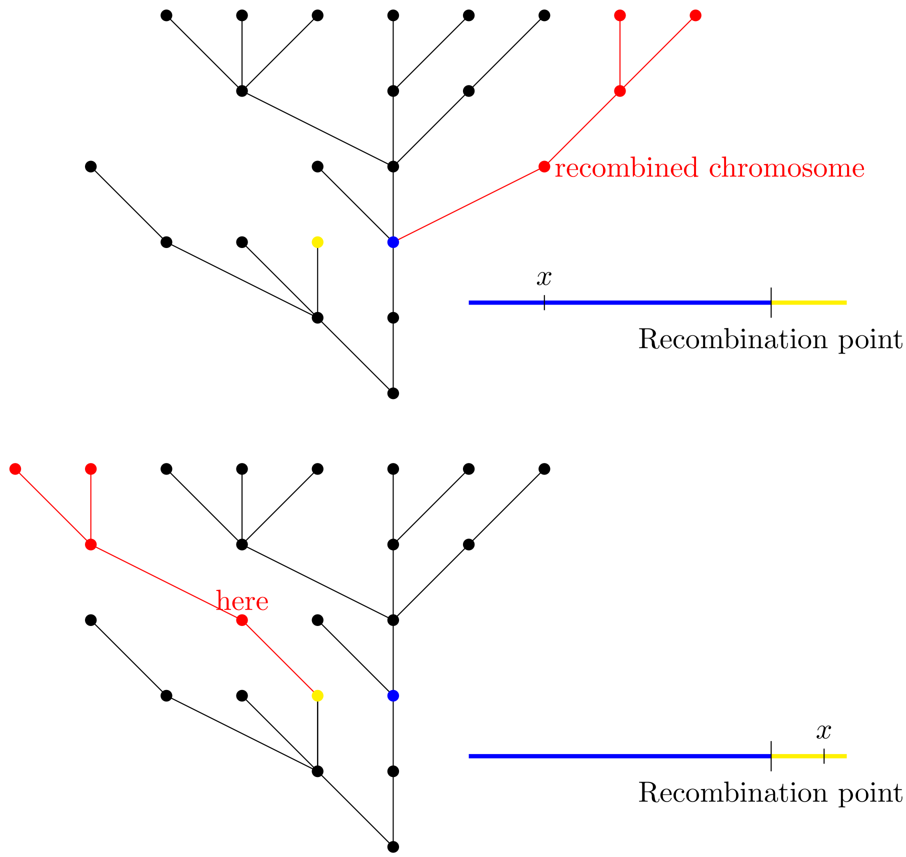

Figure 2.3: This is an exemple of the process with a single recombination event, where we show how the relative position of the locus x to the recombination point affects the genealogical tree for x. The recombination displaces a chromosome and the branch of all its descendants (in red) from one parent (yellow) to the other (blue). Note that all the individuals are represented, not only the ancestors of the current generation.

of the successive genealogical trees read along the chromosome. In this setting, we report all the individuals, past and present on the tree, not only those who contribute genetic material, lest we omit the ancestor of another locus when drawing the genealogical tree of the first locus. An example is shown in Figure 2.3. Most of the biologial considerations can be found in Durret [26], along with a number of phylogenetic models and tests.

2.2 Modelisation of genealogies

2.2.1 Discrete modelisations

To infer a philogenetic tree from the repartition of different alleles in a population (see Givnish & al. [35]), or predict the diffusion of a new allele after its apparition, we need a model for the population, or, more precisely, a model of the genealogy. One of the first models developed to study the diffusion of an allele in a population is the Wright-Fisher model (see for example [26]), where each member of a generation picks its parent independently at random in the previous generation (the generations are non-overlapping). This allows to study the varying proportion of an allele in the population (without additional mutations). This model has since been subjected to many refinements, allowing for mutations. The Moran model (Moran [55]) adds a natural mutation rate between the two alleles, departing from the simplistic long-term behaviour of the Wright-Fisher model, where all but one version of the gene disappear. Another model adding mutations can be seen in Wirtz & Wiehe [61]. Other variations are present in the literature, with different offspring distributions (Cannings [15]) and varying population size, as in the Galton-Watson tree or in more complex models introducing competition (Lambert [48]).

Under the Wright-Fisher model, a particular genealogical structure appears when looking at the genealogical tree of n individuals in a population of size $N \to \infty$ (with proper time rescaling). This structure, shown by Kingman in [46], is called the Kingman coalescent.

Another important mention for population models goes to Galton-Watson trees, which is a good representation for varying size population. A particular attention has been given to critical and sub-critical Galton-Watson trees conditioned to survive.

Other population models explore spatial repartition of individuals. In Kimura & Weiss [45], we see a model with different sites, whereas Etheridge [30] presents a model which accounts for spacial distribution of individuals, with individuals living in very populated neighbourhoods experiencing a drop in fertility. Models with selections are studied for example in Kimura & Otha [44] and Kaplan, Darden & Hudson [41].

2.2.2 Limits for large populations

Models for large populations include the family of superprocesses, that is, measures-valued processes. They can be a limit object for particles systems characterized by a Markov process Π over some space E, a branching process with generating function $\phi^t(x,z) = \sum_n p^t(x) z^n$ ($x \in E, z \in [0,1]$) and a function K from E to $\mathbb{R}$. Each particle moves independently from the other following Π, and dies at rate $K(x)$ (x is the position of the particle). When a particle dies, it gives birth to new particles according to the branching process at the point of its death. A typical example would be the position of the individuals in space or the representation of some trait (height, speed, fitness...). Discussion on superprocesses can be found in Dynkin [28], Perkins [19] and Dawson & Perkins [18]. Note that the mechanics of the patricules model makes this class of superpocesses a limit representation for inhomogenous Galton-Watson processes (the inhomogenous Galton-Watson processes are a generalization of Galton-Watson trees) see Dawson [17].

The Fleming-Viot process is the limiting object for the Moran process with a spatial component, so, when a particle dies, it gives birth to exactily one particle at the location of one of the other particles, chosen uniformly at random. This makes it a model with constant-size population. See Donnelly & Kurtz [22] and two articles from Ethier & Kurtz: [32] and [31].

2.2.3 Continuous coalescent and ARG

After the sucess of Kingman's coalescent, which is the limit of the genealogy in a range of settings (see Durret & Schweinsberg [27], Möhle & Sagitov [53]), many generalizations have been constructed to give limiting genealogy in other models (coalescence events of more than two particles: Pitman [56], many coalescences occurring at the same time: Schweinsberg [58]). We see in Möhle & Sagitov [52] and Eldon & Wakeley [29] the link between the coalescents and a Wright-Fisher model with different exchangeable offspring distributions. In Greven, Pfaffelhuber & Winter [36], we find the condition for a coalescent process to converge to a (locally compact) tree.

Another generalization of the Kingman tree is given in Aldous [8], where the coalescence rate of two clusters depends on their size.

The Ancestral Recombination Graph (ARG) is a variant of the Kingman coalescent which accounts for recombination. We start with n particles, which coalesce at rate 1 (for each pair of particle) and split at rate $r > 0$ (for each particle). By looking at the birth and death process, we see that when there are k particles, the birth rate is kr (recombination event) and the death rate $\frac{k(k-1)}{2}$ (coalescence event). Note that when the process hits 1, the birth rate is r and the death rate 0. Thus, the birth-death process is a recurrent Markov process and the number of particles eventually hits 1. This means that under this model, all current chromosomes of any n individuals descend from an ancestral chromosome that is the sole contributor to their genome. See Griffith & Marjoram [38] for an example.

2.2.4 Real trees as a scaling limit

We have seen with the coalescents that a notion of continuous tree (as opposed to graph) is pertinent when considering large populations over large timescales. The notion of continuous tree was introduced in Aldous [4] to describe the Brownian tree, a limit object for the uniform random ordered binary tree. Other laws exist, like Levy trees Duquesne & Le Gall [24] that provide a limit for critical and sub-critical Galton-Watson trees. See also Haas & Miermont [40].

Random real trees are used in non-biological settings as well. For example, they are instrumental in the construction of the Brownian map in Miermont [54].

2.3 Topologies for spaces of metric spaces

2.3.1 Topology on the space of metric spaces

One of the first attempts to compare two metric spaces comes from Gromov [39]. To compare two metric spaces, the idea is to isometrically embed them in a third metric space and compare their embeddings using the Hausdorff distance. Taking the infimum over all isometrical embeddings in all metric spaces yields the Gromov-Hausdorff distance. The use of the Hausdorff distance means that the Gromov-Hausdorff distance is only defined between compact metric sets. A commonly seen extension, the Gromov-Hausdorff-Prohorov distance, is defined on the space of compact metric spaces equipped with a finite measure. It is defined by taking the max between the Prohorov of the embedded measures and Hausdorff distance of the embedded spaces for each embedding, before taking the infimum of the resulting quantities for all embeddings.

Convergence for this distance has been characterized in many ways, and some of them give rise to topologically equivalent distances. One way is to introduce a correspondence between the two spaces, and measuring how fitting the correspondence is by measuring how much it distorts the distance. The study of correspondences gives rise to a reformulation of the distance, see Evans, Pitman & Winter [33] or Proposition 3.4.1 of the present paper.

A way to define a Gromov distance for non-compact metric spaces is to compare increasingly large compact subsets. This is explored in Abraham, Delmas & Hoscheit [1] to define a Gromov distance on complete locally compact length spaces with a marked point, called the root (this setting is well-suited to the study of real trees). In a complete locally compact length space, the closed balls centred at the root are compact. We take as distance between two rooted metric spaces E and F the integral $\int_{\mathbb{R}_+} (1 \wedge d(B_r, B'_r))\mathrm{e}^{-r} dr$, where B_r and B'_r are the closed balls of radius r centred on the root in E and F respectively, and d is the Gromov-Hausdorff distance for compact sets. There is a similar distance over complete measured locally-compact length spaces, as long as the measure is finite over every compact (still in [1]). These two distances have been extended to the spaces of boundedly-compact[1] pointed metric spaces in Khezeli [42], adding a variety of decorations in Khezeli [43].

To define a distance over complete separable metric spaces with a probability measure, we see in Greven, Pfaffelhuber & Winter [36] the Gromov-Prohorov distance. The resulting topology coincides with the Gromov-weak topology, where a sequence (E_k, d_k, ν_k) of probability metric spaces converges to (E, d, ν) if and only if for every $n \in \mathbb{N}^*$, an i.i.d sequence $(X_i^k)_{i\in\mathbb{N}^*}$ with marginal ν_k and an i.i.d sequence $(X_i)_{i\in\mathbb{N}^*}$, with marginal ν the random matrix of distances $\left(d_k(X_i^k, X_j^k)\right)_{1\leq i,j\leq n}$ converges in law to $\left(d(X_i, X_j)\right)_{1\leq i,j\leq n}$.

See Athreya, Löhr & Winter [9] and Löhr [51], on the relations between different Gromov-like topologies.

2.3.2 Definition of labelled metric spaces, topology and new results

The distances exposed so far all require one of the following conditions :

1. the metric spaces are compact (Gromov [39]);
2. the metric spaces are rooted, complete and boundedly-compact (Abraham, Delmas & Hoscheit [1], Khezeli [43]);
3. the metric spaces are complete, separable, and carry a finite measure or a probability measure, and the metric space is equal to or function of the support of this measure (Aldous [6] and [5], Greven, Pfaffelhuber & Winter [36]).

In this section, we consider a class of measured metric spaces decorated with 1-Lipschitz maps and aim to give a relaxed version of 2, namely, to consider non-compact non-pointed metric spaces equipped with boundedly-finite measures. This comes at a small cost, since we have to replace boundedly-compactness with the slightly stricter condition of S-compactness (see below for a definition of S-compactness; see Remark 3.1.1 for a comparison of S-compactness and boundedly-compactness). We call measured labelled metric spaces any quadruple (E, d, H, ν), where (E, d) is a complete separable metric space, ν is a Borel measure[2] and H is a 1-Lipschitz map from E to $\mathbb{R}$.

[1] Here, a metric space (E, d) is boundedly compact if every closed bounded set is compact, that is if closed balls are compact.

[2] A Borel measure is defined on the Borel sets of (E, d) such that all compact sets of E have finite measure.

For every $h \in \mathbb{R}_+$, we call $\mathrm{Slice}_h(E, d, H, \nu,)$ or simply E_h the set $\{x \in E||H(x)| \leq h\}$, equipped with the restrictions of d, H and ν. We say that (E, d, H, ν) is S-compact if for every $h \in \mathbb{R}_+$, $\mathrm{Slice}_h(E, d, H, \nu)$ is compact.

We say that two labelled metric spaces (E, d_E, H_E, ν_E) and (F, d_F, H_F, ν_F) are equivalent if there exists an isometric bijection ϕ from (E, d_E) to (F, d_F) such that $H_E = H_F \circ \phi$ and the image measure of ν_E by ϕ equals ν_F (that is, $\phi(\nu_E) = \nu_F$). This relation is an equivalence relation on the class $\tilde{\mathbb{X}}$ of all measured labelled spaces, which preserves compactness and S-compactness. We call $\mathbb{X}$ the set of all equivalence classes of measured labelled metric spaces. We call $\mathbb{X}^S$ the set of all equivalence classes of S-compact measured labelled metric spaces and $\mathbb{X}^K$ the set of all equivalence classes of compact measured labelled metric spaces. We will abusively denote by E the measured labelled space (E, d, H, ν), and confuse an equivalence class with any of its representatives when convenient.

For (E, d_E, H_E, ν_E) and (F, d_F, H_F, ν_F) two compact measured labelled metric spaces, (Z, d_Z) a separable metric space, ϕ_E (resp. ϕ_F) an isometry from E (resp. F) to Z. We consider the embedding ϕ_E^* (resp. ϕ_F^*) from E (resp. F) to $Z \times \mathbb{R}$ defined by $\phi_E^*(x) = (\phi_E(x), H_E(x))$ (resp. $\phi_F^*(y) = (\phi_F(y), H_F(y))$. This way, we embed both the metric and the labels of E and F in a single space $Z^* = Z \times \mathbb{R}$. We equip Z^* with the distance d_Z^* defined as follows :

$$d_Z^*\big((x, h), (x', h')\big) = d_Z(x, x') \vee |h - h'|.$$

We set

$$\Delta^Z_{\phi_E,\phi_F}(E, F) = d_{\mathrm{H}}\big(\phi_E^*(E), \phi_F^*(F)\big) \vee d_{\mathrm{P}}\big(\phi_E^*(\nu_E), \phi_F^*(\nu_F)\big),$$

where d_{H} (resp. d_{P}) is the Hausdorff (resp. Prohorov) distance in (Z^*, d_Z^*), $\phi_E^*(E)$ is the direct image of E by ϕ_E^* and $\phi_E^*(\nu_E)$ the push-forward of ν_E by ϕ_E^* (similarly for F). Thus, the number $\Delta^Z_{\phi_E,\phi_F}(E, F)$ takes into account all the aspects of E and F. As in the Gromov-Hausdorff-Prohorov metric for compact metric sets, we set

$$d_{\mathrm{GHP}}(E, F) = \inf_{Z,\phi_E,\phi_F} \Delta^Z_{\phi_E,\phi_F}(E, F),$$

where the infimum is taken on all the metric spaces Z with isometries ϕ_E and ϕ_F from E and F respectively to Z.

For (E, d_E, H_E, ν_E) and (F, d_F, H_F, ν_F) two S-compact measured labelled metric spaces, we define

$$d_{\mathrm{LGHP}}(E, F) = \int_0^\infty \Big(1 \wedge d_{\mathrm{GHP}}(E_h, F_h)\Big)\mathrm{e}^{-h} dh.$$

We define similar quantities d_{GH} and d_{LGH} for labelled metric spaces (without measures), by

$$d_{\mathrm{GH}}\big((E, d_E, H_E), (F, d_F, H_F)\big) = d_{\mathrm{GHP}}\big((E, d_E, H_E, 0), (F, d_F, H_F, 0)\big)$$

and

$$d_{\mathrm{LGH}}\big((E, d_E, H_E), (F, d_F, H_F)\big) = d_{\mathrm{LGHP}}\big((E, d_E, H_E, 0), (F, d_F, H_F, 0)\big).$$

We note that d_{GHP} and d_{LGHP}, that are defined on classes of measured height-labelled trees, can be defined on $\mathbb{X}^K$ and $\mathbb{X}^S$ respectively, as their value for two given equivalence classes does not depend on the choice of representatives. We have the following results :

Proposition 3.1.13 and Theorem 3.3.1 The function d_{LGHP} is a distance over $\mathbb{X}^S$ and the metric space $(\mathbb{X}^S, d_{\mathrm{LGHP}})$ is a Polish space.

Define $\mathbb{X}^C$ the space of measured labelled spaces (E, d, H, ν) (up to equivalence) such that $H(E)$ is connected (i.e. an interval). Define $\mathbb{X}^{C,K} = \mathbb{X}^K \cap \mathbb{X}^C$, we have:

Proposition 3.4.4 and Lemma 3.4.5 On $\mathbb{X}^K$, the topology induced by d_{GHP} is strictly finer than the topology induced by d_{LGHP}. The two distances d_{GHP} and d_{LGHP} induce he same topology on $\mathbb{X}^{C,K}$.

In the definition of d_{LGHP}, 0 plays a special role since the slices Slice_h are taken such that $-h \leq H \leq h$. Thus, changes at height 0 induce bigger change in the distance. To see the effect on topology, we define another distance with slices taken around $a \in \mathbb{R}$. For $a \in \mathbb{R}$, (E, d, H, ν) a measured labelled metric space and $h \in \mathbb{R}_+$, define $E_h^a = \mathrm{Slice}_h^a(E, d, H, \nu)$ as the set

$$\{x \in E | |H(x) - a| \leq h\}$$

equipped with the restrictions of d, H and ν. For $a \in \mathbb{R}$, (E, d_E, H_E, ν_E) and (F, d_F, H_F, ν_F) two S-compact measured labelled metric spaces, we define

$$d_{\mathrm{LGHP}}^a(E, F) = \int_0^\infty \Big(1 \wedge d_{\mathrm{GHP}}(E_h^a, F_h^a)\Big) \mathrm{e}^{-h} dh.$$

This distance is the same as d_{LGHP}, but with a playing a special role instead of 0. We have

Proposition 3.4.6 For every $a \in \mathbb{R}$, the shift application $(E, d, H, \nu) \mapsto (E, d, H + a, \nu)$ is continuous. Equivalently, d_{LGHP}^a induces the same topology as d_{LGHP} on $\mathbb{X}^S$.

2.4 The space of real trees

2.4.1 Real trees and topology on the space of real trees

We call real trees, or simply trees (in the context of this thesis), any acyclic geodesic metric space. A metric space (E, d) is called acyclic if between any two points $x, y \in E$, there exists a unique injective continuous path from x to y, and is also called geodesic if the length of this path is equal to $d(x, y)$. Additional structures or restrictions may be placed on this notion. It is often convenient to restrict ourself to complete, compact or locally compact metric spaces. Trees have been decorated with measures and marked points (most often, a root). See examples of additional decorations on metric space in Depperschmidt, Greven & Pfaffelhuber [20] and of decorations on trees in Donnelly & Kurtz [23]. Aldous considered trees as measures over $\mathbb{L}_1$, while other may consider functions, ultrametric spaces (Greven, Pfaffelhuber & Winter [37]) or equivalence classes in the class of metric spaces.

The setting of Gromov-Hausdorff and Gromov-Hausdorff-Prohorov distance over complete locally compact pointed length spaces developed in Abraham, Delmas & Hoscheit [1] is especially adapted to rooted trees.

Comparisons between the contour functions are a highly convenient way to bound the Gromov-Hausdorff-Prohorov between two trees. This bound allows us to translate convergence of random processes to the convergence of associated trees, but doesn't yield a topologically equivalent distance. See Le Gall [50] for example.

2.4.2 A new space of decorated trees and its topology

In this paper, the trees that interest us include examples like the stationary random tree (as appearing in Chen & Delmas [16]) and the genealogy of the look-down process (Donnelly & Kurtz [22]), which is not compact and has infinite measure. Furthermore, its natural "root" is infinitely ancient, so it won't fall under the "complete boundedly-compact rooted metric space". Representing the trees will need another approach. To this end, we remove the notion of root and and call *height-labelled trees* all the elements $(T, d, H, \nu) \in \mathbb{X}^S$ such that (T, d) is a tree and for every $x, y \in T$,

$$d(x, y) = H(x) + H(y) - 2h_{\min},$$

where $h_{\min}$ is the minimum of H on the geodesic between x and y. The idea is that (T, d) is a genealogical tree, and, for $x \in T$, $H(x)$ represents the time at which the individual x lived, and that the distance between two points be the time to their closest common ancestor and back. We note $\mathbb{T} \subset \mathbb{X}^S$ the set of height-labelled trees up to equivalence.

On the space of height-labelled trees, we use d_{LGHP}. We prove

Theorems 4.1.15 The space $(\mathbb{T}, d_{\mathrm{LGHP}})$ is a closed subset of $\mathbb{X}^S$ and thus Polish.

Note that for any height-labelled tree (T, d, H, ν), (T, d) is connected, so the direct image $H(T)$ is always an interval. Thus, the space of compact height-labelled trees $\mathbb{T} \cap \mathbb{X}^K$ is a subset of $\mathbb{X}^{C,K}$. It follows that, by Lemma 3.4.5, d_{GHP} defines the same topology as d_{LGHP} on the space of compact trees.

We give an alternate characterization of those trees, showing that the distance can be replaced by the genealogical order without loss of information, see Proposition 4.1.14. See Lambert & Bravo [49] to a different use of an order on random trees.

2.5 Operation on trees

2.5.1 Cutting and grafting

In the literature, we see several commonly-used operations on trees. The cutting is the operation of removing a part of the tree, in general the part beyond a cutting point. A variant of the cutting is the truncation, where we remove everithing beyond a certain level (usually measured from the root). The grafting, reverse of the cutting, consists in glueing some tree to another, obtaining a bigger tree. A last operation is the ε-trimming, where we only keep points that are the middle of a geodesic of length at least 2ε. The effect is to "delete" small branches. The trimming operation is very useful, since the Gromov-Hausdorff between a tree and its ε-timming is less than ε for both versions of the distance : compact trees (Gromov [39] or complete locally compact rooted trees (Abraham, Delmas & Hoscheit[1]). The trimming of a locally compact tree is discrete[3]. See Evans, Pitman & Winter [33] (cut

[3] A tree is discrete if its nodes form a discrete set and all have finite degree

and regraft, ε-trimming), Evans & Winter [34] (cut and regraft), Duquesne & Winkel [25] (Bernoulli leaf colouring), Abraham, Delmas & Voisin [3] (cut) and Pitman & Winkel [57] (forest growth by wrapping) for other examples.

2.5.2 Some measurable operations on height-labelled trees

We define some operations on measured height-labelled trees, and study their measurability.

In the ε-trimming as defined in the literature, the extremities are deleted. This affects the maximum height of branches, and makes the slices harder to control (we would typically lose the property that $d_{\text{LGHP}}(T, \text{Trim}_\varepsilon(T)) \leq \varepsilon$). As a result, it is convenient to redefine the ε-trimming $\text{Trim}_\varepsilon(T) = (T^\varepsilon, d^\varepsilon, H^\varepsilon, \nu^\varepsilon)$ of a height-labelled tree (T, d, H, ν) as a quotient of the tree. Consider that two points $x, y \in T$ are in the same class if $H(x) = H(y)$ and $d(x, y) \leq 2\varepsilon$ and name T^ε the quotient. For $\mathbf{x}, \mathbf{y} \in T^\varepsilon$, we define

$$d^\varepsilon(\mathbf{x}, \mathbf{y}) = \big(d(x, y) - 2\varepsilon\big) \vee \big|H(x) - H(y)\big|$$

$$H^\varepsilon(\mathbf{x}) = H(x),$$

where x, y are any two representatives of $\mathbf{x}, \mathbf{y} \in T^\varepsilon$. The above definitions do not depend on the choice of representatives x, y. We set ρ the canonical projection from T to T^ε, and define ν^ε the pushforward of ν by ρ. We prove that

Lemmas 4.2.7, 4.2.8 & 4.2.10 For T a measured height-labelled tree, $\varepsilon > 0$, $\text{Trim}_\varepsilon(T)$ the ε-trimming of T, we have

- $\text{Trim}_\varepsilon(T)$ is well-defined and a discrete height-labelled tree,
- $d_{\text{LGHP}}(T, \text{Trim}_\varepsilon(T)) \leq \varepsilon$,
- $T \mapsto \text{Trim}_\varepsilon(T)$ is 1-Lipschitz from $\mathbb{T}$ to $\mathbb{T}$.

For (T, d, H, ν) a measured height-labelled tree and $h \in \mathbb{R}$, we define $\text{Stump}_h(T)$ the set $\{x \in T | H(x) \leq h\}$ equipped with the restriction of d, H and ν. This corresponds to the part of T below level h.

Proposition 4.2.11 The function $(T, h) \mapsto \text{Stump}_h(T)$ is measurable from $\mathbb{T} \times \mathbb{R}$ to T.

We want to define the crown $\text{Crown}_h(T)$ of a tree as the forest consisting in the infinite unordered collection of its branches above above a certain level h. To do so, we first define a suitable space to contain the crowns. Fistly, we define the set of unordered converging sequences. Set $\tilde{\mathbb{X}}_C^S \subset (\mathbb{X}^S)^{\mathbb{N}^*}$ the set of all converging sequence in $\mathbb{X}^S$, and consider the following pseudo-distance on $\tilde{\mathbb{X}}_C^S$:

$$d_{\text{LGHP}}^\infty((T_n)_{n \in \mathbb{N}^*}, (T'_n)_{n \in \mathbb{N}^*}) = \inf_{\sigma \in \mathfrak{S}(\mathbb{N}^*)} \sup_{n \in \mathbb{N}^*} d_{\text{LGHP}}(T_n, T'_{\sigma(n)}).$$

We define $\mathbb{X}_C^S$ the quotient of $\tilde{\mathbb{X}}_C^S$ by the equivalence relation $d_{\text{LGHP}}^\infty(\cdot, \cdot) = 0$. The space $(\mathbb{X}_C^S, d_{\text{LGHP}}^\infty)$ is a complete separable metric space.

Now, we define a Borel subset of $\mathbb{X}_C^S$ adapted to crowns. For every $h \in \mathbb{R}$, consider $0_h = (\{h\}, 0, h, 0)$ the tree consisting in a single point at height h with null measure. We consider $\mathbb{T}_C$ the set of all sequences $(T^n)_{n \in \mathbb{N}^*} \in \mathbb{T}^{\mathbb{N}^*}$ for which there exists $h \in \mathbb{R}$ such that

- $\lim_n T_n = 0_h$
- all the trees $(T_n)_{n\in\mathbb{N}^*}$ are rooted at height h.

For (T, d, H, ν) a tree, $h \in H(T)$ and $x_0 \in \text{Skel}(T)$ such that $H(x_0) = h$, we call branch above h the subtree $\{x \in T | x_0 \preceq x\}$ equipped with the restrictions of d, H, ν. If T is S-compact, the set of its branches above h is at most countable. When there are countably many branches, we set $\text{Crown}_h(T)$ an enumeration $(T_n)_{n\in\mathbb{N}^*}$ of its branches above h. Note that for any such enumeration, $\lim_n T_n = 0_h$. If T has a finite number of branches above h, we complete the sequence with a succession of 0_h. In both cases, we have $\text{Crown}_h(T) \in \mathbb{T}_C$. We extend the definition of $\text{Crown}_h(T)$ to cases where $h \notin H(T)$ by $\text{Crown}_h(T) = (0_h)_{n\in\mathbb{N}^*}$ when there are no points strictly above h ($T = \emptyset$ or $\sup_T H \le h$), so the crown holds no branches at all and by $\text{Crown}_h(T) = (T, 0_h, ...)$ when $\min_T H > h$ (all the points of T are above h, so there is a single branch which is rooted strictly above level h). Note that in this last case, $\text{Crown}_h(T) \in \mathbb{X}_C^S \setminus \mathbb{T}_C$ because $\min_T H \neq h$. Thus, $(h, T) \mapsto \text{Crown}_h(T)$ is defined from $\mathbb{R} \times \mathbb{T}$ to $\mathbb{X}_C^S$.

Proposition 4.3.11 The function $(h, T) \mapsto \text{Crown}_h(T)$ is measurable.

The main result on operations concerns the grafting of a crown on a tree. Since our trees are defined up to an isometry, we cannot indicate the location of the grafting through a point, so we have to graft the branches of the crown at random according to a probability measure on the receiving tree.

This means that the resulting grafting is the law of a random tree. For (T, d, H, ν, p) a measured height-labelled tree equipped with an additional probability measure p concentrated on $H^{-1}(\{h\})$ for some $h \in H(T)$, T' another measured height-labelled tree containing at least one point at height h, we take a representative $(T_n, d_n, H_n, \nu_n)_{n\in\mathbb{N}^*}$ of $\text{Crown}_h(T')$ and $(X_n)_{n\in\mathbb{N}^*}$ an i.i.d sequence of random variables in T with marginal law p. Note $T \star_p T'$ the grafting of each T_n at the point $X_n \in T$, equipped with distance d' and label function H'. Set $\nu' = \nu + \sum_n \nu_n$ and note $P_{T\star_p T'}$ the law of $(T \star_p T', d', H', \nu', p)$. We equip the set of all probability measures over $(\mathbb{X}^S, d_{\text{LGHP}})$ with the Prohorov distance, and have

Theorem 4.4.5 and Proposition 4.4.7 The mixing operation $((T, p), T') \mapsto P_{T\star_p T'}$ gives a well-defined probability measure, independent of the choice of representative for T and $\text{Crown}_h(T')$, and is measurable from its domain $D \subset \mathbb{T}^{[2]} \times \mathbb{T}$ to $\mathbb{T}^{[2]}$, where $\mathbb{T}^{[2]}$ is the genealization of $(\mathbb{T}, d_{\text{LGHP}})$ to height-labelled trees equipped with two measures. D is a Borel set of $\mathbb{T}^{[2]} \times \mathbb{T}$.

These results complete Abraham, Delmas & Hoscheit [2], where the grafting operation was not proven to be measurable.

2.6 Brownian tree conditioned on its local time

2.6.1 Aldous' construction, using coalescing particles

In [7], Aldous presents a law for the standard Brownian excursion conditioned on its local time. To build this law of the tree for a given local time $(l(h))_{h\ge 0}$ (which must be the density of a probability measure since the excursion is defined on [0,1]), he first creates $n \in \mathbb{N}^*$ leaves at random i.i.d heights (with density $l(h)dh$), and builds a coalescent from them (each couple of branch coalesces at rate $\frac{1}{l(h)}dh$) to obtain a tree T_n^l. He finds that the sequence $(T_n^l)_{n\in\mathbb{N}^*}$ converges to a random tree T^l, then proves it is the Brownian tree conditioned on its local time(Construction 1 and Theorem 2 respectively in Aldous [7]).

This tree can be interpreted as the limit of Wright-Fisher genealogy with varying population size. For constant population, this limit is given by the Fleming-Viot process. Note that this genealogy can be build with a look-down process which is similar to the approach of Aldous. This means that Aldous' construction with varying coalescence rate can be reproduced with a simple time-change of the Fleming-Viot genealogy. See Birkner & al. [13] for results on time-changes of coalescents.

2.6.2 An attempt at generalization

We mean to extend the previous construction to a further class of laws. Rather than characterizing our trees by the local time l, we use a coalescence measure μ (which plays the role of $\frac{1}{l(h)}dh$) and a mass measure ν, whose only purpose is to decide the mass repartition on the tree. To define the tree, we mix coalecents at rate μ, starting at different levels. This poses no problem when mixing a finite number of coalescents, and we use the convergence in law of the random trees to build a limiting tree. We prove that the limiting tree is independent from the sequence used to build it in Lemma 5.3.7. We prove a weak regularity for the measures of the coalescent in Lemma 5.4.3, which is a step toward equipping the tree with an intrinsic probability measure almost-surely at almost-every level.

An approach with time-change may be possible, and a simpler way to derive stronger properties from the original object (continuity in h of the measure at level h for example).

2.7 Motivation and perspectives

In Depperschmidt, Pardoux & Pfaffelhuber [21], we see a process generalizing the ARG from Durett [26] for an infinite number of individuals living at the same time. We aim, in future works, to do the same for entire genealogical trees.

Note that for every locus t on the chromosome, the ARG gives a coalescent tree, so we can also see the ARG as the family of those trees. This approach is easier to generalize to a limit where the sample size is equal to the population size. In [21], such a generalization is given, through a distance on a set of individuals (the leaves of Kingman's coalescent). Our main perspective is to generalize the ARG to encompass all of the genealogy, past and future in a $\mathbb{T}$-valued process $(T_t)_{t\in\mathbb{R}_+}$ (see Figure 2.3 for the discrete version, where the parameter t is the position on the chromosome). Here, we give a sketch of the constructions and proofs.

The construction of the process $(T_t, d_t, H_t, \nu_t)_{t\in\mathbb{R}_+}$, described in the next paragraph, requires an initial random tree equipped with probability measures at almost-every levels (a paper is in progress to provide a Polish space adapted to such an object). To ensure that it stays a tree at every time, we need, in a number of proofs, to have a stationary and reversible process, so the initial law needs to be a stable law. The Brownian tree conditioned on its local time, and our generalization, happen to be stable laws (there may be others).

The idea, conditionally on $(T_0, d_0, H_0, (\nu_h)_{h\in H(T_0)})$, then to code all the jumps of $(T_t)_{t\in\mathbb{R}_+}$ through triplets (u, v, t), for a cut at the point u, regraft at the v, t beeing the time of the jump. We take a Poisson process X on $T_0 \times T_0 \times \mathbb{R}_+$ with intensity $\Lambda(du)\nu_{H(u)}(dv)dt$ at the point (u, v, t), where Λ is the length measure of T_0. A potential difficulty is that Λ is often infinite, so we have to be prepared for infinitely many jumps in any time interval. This is not a problem, as we can directly build the tree T_t from T_0 and the Poisson process X. Since height-labelled trees can be characterized indifferently by the distance or its genealogical order, changing the genealogical order is equivalent to changing the distance (this equivalence is proven in Proposition 4.1.14 of the present thesis). Seen in T_0, the ancestral line of a point jumps each time it meets a cutting point u such that $(u, v, s) \in X$ and $s \leq t$. Almost-surely, this does not happen too much since the distance between two cutting points follows the exponential law of parameter 1, so the ancestral line is well-defined. To get the distance d_t between two points x and y, follow their ancestral lines to their common ancestor $x \wedge y$ and define $d(x, y) = H(x) + H(y) - 2H(x \wedge y)$. This works almost-surely for almost-every point in T_0, but not for all. Thus, T_0 is cut in a jigsaw that leaves a negligible set of points out, and completion is necessary at each time.

At this stage, there are many questions about this process:

pending questions :

- is T_t is a random variable?
- is the process Markov?
- is the process stationary and reversible?
- is almost-surely, T_t stay always connected?
- is the process càdlàg for d_{LGHP}?
- at any time $t \in \mathbb{R}_+$, does each piece of the jigsaw (the connected component of T without the cutting points) have positive measure for $\int_{\mathbb{R}} \nu_h dh$?
- if we take an entire level (or generation) h, and look at their ancestors at height $h - \varepsilon$ in all the trees $(Ts)_{0 \leq s \leq t}$, is the number of ancestors is finite (this has a meaning since almost all points in (T_s) are points of T_0) ?

Our conjecture is that the answer to all those questions is yes.

Chapter 3

Topology on measured labelled metric spaces

In this chapter, we develop a distance on a new class of decorated metric spaces, the measured labelled spaces, which will be needed in Chapter 4. The interest here is to describe and compare metric spaces such as the tree with stationary quadratic branching process, which is not rooted but has a natural time direction. In this sense, we could say that it escapes the scope of [1] and [43]. To encode the time direction, we use a map, called label or height. This chapter deals about general metric spaces, while trees will be looked at in more details in Chapter 4.

3.1 Definitions

3.1.1 Labelled Spaces

We call $\tilde{\mathbb{M}}$ the class of all separable metric spaces. All the elements $(Z, d) \in \tilde{\mathbb{M}}$ are equipped with their Borel σ-field $\mathscr{B}(Z)$. For $(E, d_E), (F, d_F)$ two separable metric spaces, we define $\mathrm{Iso}(E, F)$ the set of all isometries from E to F. The set $\mathrm{Iso}(E, F)$ can be empty and the isometries are not necessarily surjective. If ν is a measure over E, ϕ a measurable function from E to F and f a measurable function from E to $\mathbb{R}_+$, we note $\phi\nu$ or $\phi(\nu)$ the image measure on F of ν through ϕ and $f \cdot \nu$ the measure that has density f with respect to ν. In the case of the indicator function of $A \in \mathscr{B}(E)$, we have for every $B \in \mathscr{B}(E)$ and $\phi \in \mathrm{Iso}(E, F)$ that

$$\nu(B) = \phi\nu(\phi(B)) \ ; \ [1_A \cdot \nu](B) = \nu(A \cap B) \ ; \ \phi[1_A \cdot \nu] = 1_{\phi(A)} \cdot (\phi\nu). \tag{3.1.1}$$

We call labelled metric space any triplet (E, d, H), where (E, d) is a complete separable metric space and H a 1-Lipschitz map from E to $\mathbb{R}$. For (E, d, H) a labelled metric space and for $h \in \mathbb{R}_+$, we set

$$\mathrm{Slice}_h(E, d, H) = \{x \in E \mid |H(x)| \leq h\}.$$

The set $\mathrm{Slice}_h(E, d, H)$ is equipped with the restriction of d to form a labelled metric space. We say that a labelled metric space (E, d, H) is S-compact if for every $h \in \mathbb{R}_+$, $\mathrm{Slice}_h(E, d, H)$ is compact.

We call measured labelled metric space any (E, d, H, ν) where (E, d, H) is a labelled metric space and ν a non-negative measure on $\mathscr{B}(E)$ such that for all $h \in \mathbb{R}_+$:

$$\nu(\mathrm{Slice}_h(E, d, H)) < \infty.$$

Recall that a Borel measure is a measure defined on the Borel σ-field such that all the compact sets have finite measure (as in Definition 25.2 in [10]). Since for every compact $K \subset E$, H is bounded on K, so $\nu(K) < \infty$, we deduce that ν is a Borel measure.

For (E, d, H, ν) a measured labelled metric space and $h \in \mathbb{R}_+$, we define $\mathrm{Slice}_h(E, d, H, \nu)$ by equipping the already defined $\mathrm{Slice}_h(E, d, H)$ with the restriction of ν to form a measured labelled metric space. We will often use the abusive notation E to designate (E, d, H) or (E, d, H, ν). In Sections 3.1 to 4.3, we will use the more convenient notation $E_h = \mathrm{Slice}_h(E, d, H, \nu)$ for every measured labelled space (E, d, H, ν) and $h \in \mathbb{R}_+$.

Remark 3.1.1. Here are some examples of S-compacity.

- If we take (T, d, ν) the Brownian tree with its mass measure, $\omega \in T$ and H the function from T to $\mathbb{R}$ defined by $H(x) = d(\omega, x)$ for $x \in T$, then (T, d, H, ν) is a measured labelled set, and it is S-compact since the Brownian tree is compact.

- There are S-compact spaces that are not compact, like $(\mathbb{R}, d_{\mathbb{R}}, \mathrm{Id}_{\mathbb{R}})$, with $d_{\mathbb{R}}$ the Euclidean distance on $\mathbb{R}$.

- The notion of S-compactness is stronger than local compactness. We give an example of locally compact labelled metric space that is not S-compact. Take $E = (\mathbb{R}_+ \times \{0\}) \cup (\mathbb{N} \times \mathbb{R}_+)$, and define, for every $(x, y), (x', y') \in E$, $H(x, y) = y - x$ and

$$d((x, y), (x', y')) = \begin{cases} |y - y'| & \text{if } x = x' \\ |x - x'| + y + y' & \text{if } x \neq x'. \end{cases}$$

  The function d is sometimes called the comb distance on $\mathbb{R}^2$. The space (E, d) is separable, complete, locally compact and H is 1-Lipschitz, but $\mathrm{Slice}_0(E) = \{(n, n) | n \in \mathbb{N}\}$ is not compact, so E is not S-compact. See Figure 3.1 for a representation of (T, d, H) with some distinguished points of T.

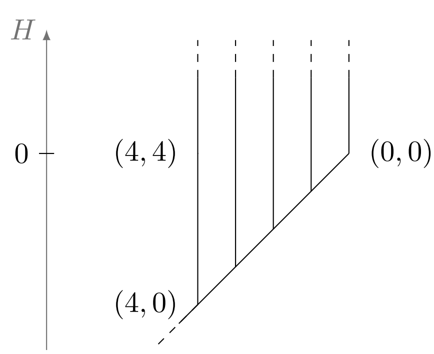

Figure 3.1: This comb-tree is locally compact, but is not S-compact as a labelled metric space.

Definition 3.1.2. *We note $\tilde{\mathbb{X}}$ the class of all measured metric labelled spaces. We say that two measured metric labelled spaces (E, d_E, H_E, ν_E) and (F, d_F, H_F, ν_F) are equivalent if there exists a bijection $\phi \in \mathrm{Iso}(E, F)$ such that $H_E = H_F \circ \phi$ and $\nu_F = \phi\nu_E$. We note $\mathbb{X}$ the set of all equivalence classes in $\tilde{\mathbb{X}}$. We note $\mathbb{X}^0 = \{\bar{E} \in \mathbb{X} | \forall (E, d, H, \nu) \in \bar{E}, \nu(E) = 0\}$ (here only, $\bar{E}$ is the equivalence class containing E) the set of all classes of metric labelled spaces up to equivalence, seen as measured metric labelled spaces with the null measure. We note $\mathbb{X}^K$ and $\mathbb{X}^{0,K}$ the restrictions of $\mathbb{X}$ and $\mathbb{X}^0$ to compact spaces, $\mathbb{X}^S$ and $\mathbb{X}^{0,S}$ the restrictions of $\mathbb{X}$ and $\mathbb{X}^0$ to S-compact spaces.*

3.1.2 Distances

For (Z, d) a metric space, we define the ε-closure and the ε-neighborhood of a set $A \subset Z$ as

$$A^\varepsilon = \{x \in Z | d(x, A) \leq \varepsilon\} \text{ and } A^{\varepsilon^-} = \{x \in Z | d(x, A) < \varepsilon\}.$$

We now introduce the Prohorov distance, which can be found in Section 6 of [12], along with a proof of Lemma 3.1.5 in the special case of probability measures.

Definition 3.1.3 (Prohorov distance)**.** *Let (Z, d) be a separable metric space with $\mathscr{B} = \mathscr{B}(Z)$ its Borel σ-field, and ν, ν' two finite non-negative measures over Z. We define $d_{\mathrm{P}}^{(Z,d)}(\nu, \nu')$ the Prohorov distance between ν and ν' as*

$$\begin{aligned} &d_{\mathrm{P}}^{(Z,d)}(\nu, \nu') \\ &\quad= \inf\{\varepsilon \geq 0 | \forall A \in \mathscr{B}, \nu(A) \leq \nu'(A^{\varepsilon^-}) + \varepsilon\} \vee \inf\{\varepsilon \geq 0 | \forall A \in \mathscr{B}, \nu'(A) \leq \nu(A^{\varepsilon^-}) + \varepsilon\} \\ &\quad= \min\{\varepsilon \geq 0 | \forall A \in \mathscr{B}, \nu(A) \leq \nu'(A^{\varepsilon}) + \varepsilon\} \vee \min\{\varepsilon \geq 0 | \forall A \in \mathscr{B}, \nu'(A) \leq \nu(A^{\varepsilon}) + \varepsilon\}. \end{aligned}$$

When the choice of the underlying metric space (Z, d) is clear, we use the notation d_{P}. Note that for $A \subset B \subset Z$ two Borel sets we have

$$d_{\mathrm{P}}(1_A \cdot \nu, 1_B \cdot \nu) = \nu(B \setminus A). \tag{3.1.2}$$

Remark 3.1.4. In Definition 3.1.3, we give two expressions for the Prohorov distance. The first one is standard and a close look at the second one shows that they are equal. We still need to prove that the minimum exists in the second equality of Definition 3.1.3. Set $\delta = \inf\{\varepsilon \geq 0 | \forall A \in \mathscr{B}, \nu(A) \leq \nu'(A^\varepsilon) + \varepsilon\}$. For every $A \in \mathscr{B}$ and $\varepsilon > \delta$ we have $\nu(A) \leq \nu'(A^\varepsilon) + \varepsilon$. Since A^δ is closed and equal to the intersection $\cap_{\varepsilon > \delta} A^\varepsilon$, we have by dominated convergence that

$$\nu'(A^\delta) = \lim_{\varepsilon \to \delta^+} \nu'(A^\varepsilon),$$

so $\nu(A) \leq \nu'(A^\delta) + \delta$ for every A i.e. $\delta \in \{\varepsilon \geq 0 | \forall A \in \mathscr{B}, \nu(A) \leq \nu'(A^\varepsilon) + \varepsilon\}$, so the minimum exists. □

To avoid proving $\nu(A) \leq \nu'(A^\varepsilon) + \varepsilon$ and $\nu'(A) \leq \nu(A^\varepsilon) + \varepsilon$ each time we need an upper bound of the distance, the next lemma provides a shortcut. We use for $x \in \mathbb{R}$ the notation $(x)^+ = x \vee 0 = \max(x, 0)$.

Lemma 3.1.5. *If ν, ν' are two finite non-negative measures over a metric space (Z, d) such that for some $\varepsilon > 0$ and every Borel set $B \subset Z$ we have $\nu(B) \leq \nu'(B^\varepsilon) + \varepsilon$ then $d_{\mathrm{P}}(\nu, \nu') \leq \varepsilon + (\nu'(Z) - \nu(Z))^+$.*

Proof. First, notice that for $\eta > \varepsilon$ and every Borel set $B \subset Z$ we have $\nu(B) \leq \nu'(B^{\eta^-}) + \eta$. Take $B \in \mathscr{B}(Z)$. We have $B \subset Z \setminus (Z \setminus B^{\eta^-})^{\eta^-}$ so

$$\nu'(B) \leq \nu'(Z) - \nu'((Z \setminus B^{\eta^-})^{\eta^-}) \leq \nu'(Z) - \nu(Z \setminus B^{\eta^-}) + \eta = \nu'(Z) - \nu(Z) + \nu(B^{\eta^-}) + \eta,$$

where we used the hypothesis: $\nu(Z \setminus B^{\eta^-}) \leq \nu'((Z \setminus B^{\eta^-})^{\eta^-}) + \eta$ for the second inequality. This proves that $d_{\mathrm{P}}(\nu, \nu') \leq \eta + (\nu'(Z) - \nu(Z))^+$ for every $\eta > \varepsilon$ so $d_{\mathrm{P}}(\nu, \nu') \leq \varepsilon + (\nu'(Z) - \nu(Z))^+$. □

If $\nu(Z) = \nu'(Z)$, then, using Lemma 3.1.5, we get that the two infimums in the definition of the Prohorov distance are equal and we have with Remark 3.1.4:

$$\begin{aligned} d_{\mathrm{P}}(\nu, \nu') &= \min\{\varepsilon \geq 0 | \forall A \in \mathscr{B}(Z), \nu(A) \leq \nu'(A^\varepsilon) + \varepsilon\} \\ &= \min\{\varepsilon \geq 0 | \forall A \in \mathscr{B}(Z), \nu'(A) \leq \nu(A^\varepsilon) + \varepsilon\}. \end{aligned}$$

The next lemma links the distance of two measures and the distance of their restriction to a smaller set.

Lemma 3.1.6. *Let (Z, d) be a separable metric space, ν, ν' two finite non-negative measures over Z and $\alpha \geq d_{\mathrm{P}}(\nu, \nu')$. Take H a 1-Lipschitz map from Z to $\mathbb{R}$ and set for every $h \in \mathbb{R}_+$ $Z_h = \{z \in Z | |H(z)| \leq h\}$. In this setting, we have for every $h \in \mathbb{R}_+$:*

$$d_{\mathrm{P}}(1_{Z_h} \cdot \nu \ , \ 1_{Z_{h+\alpha}} \cdot \nu') \leq \alpha + (\nu'(Z_{h+\alpha}) - \nu(Z_h))^+.$$

Proof. Take $A \in \mathscr{B}(Z)$ a Borel set. Since H is 1-Lipschitz, $(Z_h)^\alpha \subset Z_{h+\alpha}$ and we have

$$\nu(A \cap Z_h) \leq \nu'((A \cap Z_h)^\alpha) + \alpha \leq \nu'(A^\alpha \cap Z_{h+\alpha}) + \alpha.$$

With Lemma 3.1.5 this gives $d_{\mathrm{P}}(1_{Z_h} \cdot \nu \ , \ 1_{Z_{h+\alpha}} \cdot \nu') \leq \alpha + (\nu'(Z_{h+\alpha}) - \nu(Z_h))^+$. □

Definition 3.1.7 (Hausdorff distance)**.** *Define $d_{\mathrm{H}}(K, K')$ the Hausdorff distance between two compacts sets K, K' of a metric set (Z, d) as*

$$\begin{aligned} d_{\mathrm{H}}(K, K') =& \min\{\varepsilon \geq 0 | K \subset (K')^\varepsilon, K' \subset K^\varepsilon\} \\ =& (\max_{x \in K} \min_{x' \in K'} d(x, x')) \vee (\max_{x' \in K'} \min_{x \in K} d(x', x)). \end{aligned}$$

By convention, we consider that $\emptyset$ is a compact set, that $d_{\mathrm{H}}(\emptyset, \emptyset) = 0$ and that for every non-empty compact K we have $d_{\mathrm{H}}(K, \emptyset) = \infty$. As with the Prohorov distance, we will use the notation $d_{\mathrm{H}}^{(Z,d)}$ when the underlying metric space is not obvious.

For a proof that d_{H} is a distance see Chapter 4 of [60].

For (E, d_E, H_E) a metric labelled space, (Z, d) a metric space and $\phi \in \mathrm{Iso}(E, Z)$, set $\phi \times H$ the function from E to $Z \times \mathbb{R}$ defined by

$$[\phi \times H_E](x) = (\phi(x), H_E(x)).$$

Definition 3.1.8. *Let (E, d_E, H_E) and (F, d_F, H_F) be two compact metric labelled spaces. We define:*

$$d_{\mathrm{GH}}((E, d_E, H_E), (F, d_F, H_F)) = \inf_{\substack{(Z,d_Z)\in\tilde{\mathbb{M}} \\ \phi_E\in\mathrm{Iso}(E,Z) \\ \phi_F\in\mathrm{Iso}(F,Z)}} \Big[\max_{x\in E}\min_{y\in F} \left(d_Z(\phi_E(x), \phi_F(y)) \vee |H_E(x) - H_F(y)|\right) \vee \max_{y\in F}\min_{x\in E} \left(d_Z(\phi_E(x), \phi_F(y)) \vee |H_E(x) - H_F(y)|\right)\Big].$$

This definition is very close to that of the Gromov-Hausdorff distance, with the additional term $|H_E(x)-H_F(y)|$ to check whether the labels of the two spaces are close from one another. Note that if we set d_Z^* the distance on $Z \times \mathbb{R}$ defined by

$$d_Z^*((z, h), (z', h')) = d_Z(z, z') \vee |h - h'|, \tag{3.1.3}$$

we get

$$d_{\mathrm{H}}^{(Z\times\mathbb{R},d_Z^*)}([\phi_E \times H_E](E), [\phi_F \times H_F](F)) = \Big[\max_{x\in E}\min_{y\in F} \left(d_Z(\phi_E(x), \phi_F(y)) \vee |H_E(x) - H_F(y)|\right) \vee \max_{y\in F}\min_{x\in E} \left(d_Z(\phi_E(x), \phi_F(y)) \vee |H_E(x) - H_F(y)|\right)\Big].$$

This provides a more compact formulation for the next definitions:

$$d_{\mathrm{GH}}((E, d_E, H_E), (F, d_F, H_F)) = \inf_{\substack{(Z,d_Z)\in\tilde{\mathbb{M}} \\ \phi_E\in\mathrm{Iso}(E,Z) \\ \phi_F\in\mathrm{Iso}(F,Z)}} d_{\mathrm{H}}^{(Z\times\mathbb{R},d_Z^*)}([\phi_E \times H_E](E), [\phi_F \times H_F](F)).$$

The construction of d_Z^* will occur again on different distances. From now on, adding a star to a distance will always refer to the construction done in (3.1.3).

Definition 3.1.9. *Let (E, d_E, H_E, ν_E), (F, d_F, H_F, ν_F) be two compact measured labelled metric spaces, we define:*

$$d_{\mathrm{GHP}}((E, d_E, H_E, \nu_E), (F, d_F, H_F,\nu_F)) = \inf_{\substack{(Z,d_Z)\in\tilde{\mathbb{M}} \\ \phi_E\in\mathrm{Iso}(E,Z) \\ \phi_F\in\mathrm{Iso}(F,Z)}} \max\Big(d_{\mathrm{H}}^{(Z\times\mathbb{R},d_Z^*)}([\phi_E \times H_E](E), [\phi_F \times H_F](F)), d_{\mathrm{P}}^{(Z\times\mathbb{R},d_Z^*)}([\phi_E \times H_E]\nu_E, [\phi_F \times H_F]\nu_F)\Big).$$

For two compact measured labelled metric spaces $(E, d_E, H_E, 0)$ and $(F, d_F, H_F, 0)$ with the null measure, we have that

$$d_{\mathrm{GHP}}((E, d_E, H_E, 0), (F, d_F, H_F, 0)) = d_{\mathrm{GH}}((E, d_E, H_E), (F, d_F, H_F)).$$

To define the next two symmetric functions from Definitions 3.1.11 and 3.1.12, we need the following lemma, which is proved in Subsection 3.1.3 . Recall that $E_h = \mathrm{Slice}_h(E, d_E, H_E, \nu_E)$.

Lemma 3.1.10. *For (E, d_E, H_E, ν_E), (F, d_F, H_F, ν_F) two S-compact measured labelled metric spaces, the maps*

$$h \mapsto d_{\mathrm{GH}}(E_h, F_h) \ ; \ h \mapsto d_{\mathrm{GHP}}(E_h, F_h)$$

are measurable.

Definition 3.1.11. *Let (E, d_E, H_E) and (F, d_F, H_F) be two S-compact metric labelled spaces, we define:*

$$d_{\mathrm{LGH}}((E, d_E, H_E), (F, d_F, H_E)) = \int_0^\infty (1 \wedge d_{\mathrm{GH}}(E_h, F_h))\, \mathrm{e}^{-h} dh.$$

Definition 3.1.12. *Let (E, d_E, H_E, ν_E), (F, d_F, H_F, ν_F) be two S-compact measured labelled metric spaces, we define:*

$$d_{\mathrm{LGHP}}((E, d_E, H_E, \nu_E), (F, d_F, H_F, \nu_E)) = \int_0^\infty (1 \wedge d_{\mathrm{GHP}}(E_h, F_h))\, \mathrm{e}^{-h} dh.$$

Note that for $(E, d_E, H_E, 0)$ and $(F, d_F, H_F, 0)$ two S-compact measured labelled metric spaces equipped with the null measure, we have $d_{\mathrm{LGHP}}(E, F) = d_{\mathrm{LGH}}(E, F)$.

The purpose of d_{GH}, d_{GHP}, d_{LGH} and d_{LGHP} is to adapt the Gromov-Hausdorff and Gromov-Hausdorff-Prohorov distances introduced in [39] and [36] to compact and S-compact measured labelled metric spaces. This adaptation follows the one developed for rooted length spaces in [1], only replacing the balls (centered on the root) of a rooted length space by our compact slices. This replaces the condition "locally compact rooted length space", by "compact slices", but most of the proof still follows the same logic. Choose $d \in \{d_{\mathrm{GH}}, d_{\mathrm{GHP}}, d_{\mathrm{LGH}}, d_{\mathrm{LGHP}}\}$ and E, E', F, F' four measured labelled spaces such that E and E' are equivalent, F and F' are equivalent and $d(E, F)$ is defined. From the definitions of d_{GH} , d_{GHP}, d_{LGH} and d_{LGHP}, we find that $d(E', F')$ is defined and $d(E, F) = d(E', F')$. This means that d is constant on equivalence classes, so we can consider d_{GH}, d_{GHP}, d_{LGH} and d_{LGHP} as functions on $(\mathbb{X}^{0,K})^2$, $(\mathbb{X}^K)^2$, $(\mathbb{X}^{0,S})^2$ and $(\mathbb{X}^S)^2$ respectively. Moreover, $d_{\mathrm{GH}}(E, F)$ is the restriction of $d_{\mathrm{GHP}}(E, F)$ to $\mathbb{X}^{0,K}$ and $d_{\mathrm{LGH}}(E, F)$ is the restriction of $d_{\mathrm{LGHP}}(E, F)$ to $\mathbb{X}^{0,S}$.

Now, we state one of our main results.

Proposition 3.1.13. *We have that:*

- d_{GH} *is a distance over* $\mathbb{X}^{0,K}$,
- d_{GHP} *is a distance over* $\mathbb{X}^K$,
- d_{LGH} *is a distance over* $\mathbb{X}^{0,S}$,
- d_{LGHP} *is a distance over* $\mathbb{X}^S$.

We will prove Proposition 3.1.13 in Section 3.2.

3.1.3 Proof of Lemma 3.1.10

For (E,d_E,H_E,ν_E) and (F,d_F,H_F,ν_F) two S-compact measured labelled metric spaces, set $\mathbb{D}(E,F)$ the set of all distances d on the disjoint union $E\sqcup F$ such that for every $x,x'\in E$, $y,y'\in F$, $d(x,x')=d_E(x,x')$ and $d(y,y')=d_F(y,y')$. We recall that for every $h\in\mathbb{R}_+$, we note $E_h=\mathrm{Slice}_h(E)$. Note $d_{\mathbb{R}}$ the Euclidean distance on $\mathbb{R}$. For $d\in\mathbb{D}(E,F)$, $h\in\mathbb{R}_+$, we define

$$\Delta_{\mathrm{H}}(E,F,d,h)=d_{\mathrm{H}}^{((E\sqcup F)\times\mathbb{R},d\vee d_{\mathbb{R}})}([\mathrm{Id}_E\times H_E](E_h),[\mathrm{Id}_F\times H_F](F_h))$$

and

$$\Delta_{\mathrm{P}}(E,F,d,h)=d_{\mathrm{P}}^{((E\sqcup F)\times\mathbb{R},d\vee d_{\mathbb{R}})}([\mathrm{Id}_E\times H_E](1_{E_h}\cdot\nu_E),[\mathrm{Id}_F\times H_F](1_{F_h}\cdot\nu_E)).$$

We begin with 3 intermediate lemmas.

Lemma 3.1.14. *If E and F are S-compact measured metric labelled spaces, then we have*

$$d_{\mathrm{GHP}}(E_h,F_h)=\inf_{d\in\mathbb{D}(E,F)}\Delta_{\mathrm{H}}(E,F,d,h)\vee\Delta_{\mathrm{P}}(E,F,d,h).$$

Proof. Let us note $\Delta=d_{\mathrm{GHP}}(E_h,F_h)$ and $\Delta'=\inf_{d\in\mathbb{D}(E,F)}\Delta_{\mathrm{H}}(E,F,d,h)\vee\Delta_{\mathrm{P}}(E,F,d,h)$. If E_h or F_h is empty, we refer to the convention for d_{H} adopted in Definition 3.1.7, and find that we have for every $d\in\mathbb{D}(E,F)$:

$$\begin{cases}\Delta=\Delta_{\mathrm{H}}(E,F,d,h)=\Delta_{\mathrm{P}}(E,F,d,h)=0 & \text{when } E_h=F_h=\emptyset,\\ \Delta=\Delta_{\mathrm{H}}(E,F,d,h)=\infty & \text{when only one is empty.}\end{cases}$$

This proves the lemma in those cases. We suppose from now on that E_h and F_h are non-empty, so that Δ and Δ' are finite. Since for every $d\in\mathbb{D}(E,F)$, $G=(E\sqcup F,d)$ is a separable metric space and Id_{E_h}, Id_{F_h} are isometries of $\mathrm{Iso}(E_h,G)$ and $\mathrm{Iso}(F_h,G)$, we naturally have $\Delta\le\Delta'$.

Choose $\varepsilon>0$. Take $(Z,d_Z)\in\tilde{\mathbb{M}}$, $\phi_E\in\mathrm{Iso}(E_h,Z)$ and $\phi_F\in\mathrm{Iso}(F_h,Z)$ such that

$$d_{\mathrm{H}}^{(Z\times\mathbb{R},d_Z^*)}([\phi_E\times H_E](E_h),[\phi_F\times H_F](F_h))\le\Delta+\varepsilon,$$
$$d_{\mathrm{P}}^{(Z\times\mathbb{R},d_Z^*)}([\phi_E\times H_E](1_{E_h}\cdot\nu_E),[\phi_F\times H_F](1_{F_h}\cdot\nu_F))\le\Delta+\varepsilon.$$

Consider $A=\{(x,y)\in E_h\times F_h|d_Z(\phi_E(x),\phi_F(y))\vee|H_E(x)-H_F(y)|\le\Delta+\varepsilon\}$, and define d' the symmetric function on $E\sqcup F$ such that for every $x,y\in E\sqcup F$,

$$d'(x,y)=\begin{cases}d_E(x,y) & \text{if } x,y\in E\\ d_F(x,y) & \text{if } x,y\in F\\ \inf_{(x',y')\in A}[d_E(x,x')+d_F(y',y)]+\Delta+\varepsilon & \text{if } x\in E,y\in F.\end{cases}$$

The function d' is symmetric definite-positive. Let us prove it satisfies the triangular inequality. If $x,y,z\in E$ or $x,y\in E$ $z\in F$, then we simply obtain $d'(x,z)\le d'(x,y)+d'(y,z)$ from the triangular inequality of d_E and the definition of d'. If $x,z\in E$, $y\in F$, then, using the triangular inequalities of d_E and d_F we get

$$\begin{aligned}
d'(x,y)+d'(y,z) &= \inf_{(x',y'),(x'',y'')\in A} \left[d_E(x,x')+d_E(x'',z)\right]+\left[d_F(y',y)+d_F(y,y'')\right]+2(\Delta+\varepsilon) \\
&\geq \inf_{(x',y'),(x'',y'')\in A} d_E(x,z)-d_E(x',x'')+d_F(y',y'')+2(\Delta+\varepsilon) \\
&= d_E(x,z)+2(\Delta+\varepsilon)-\sup_{(x',y'),(x'',y'')\in A} d(\phi_E(x'),\phi_E(x''))-d(\phi_F(y'),\phi_F(y'')) \\
&\geq d_E(x,z)+2(\Delta+\varepsilon)-\sup_{(x',y'),(x'',y'')\in A} d(\phi_E(x'),\phi_F(y'))+d(\phi_E(x''),\phi_F(y'')) \\
&\geq d_E(x,z).
\end{aligned}$$

The last inequality follows from the definition of A. Using those three cases, the symmetry of d' and the fact that E and F play symmetric roles, we have the triangular inequality. This implies that d' is a distance and $d'\in\mathbb{D}(E,F)$. We deduce that $\Delta'\leq\Delta_{\mathrm{H}}(E,F,d',h)\vee\Delta_{\mathrm{P}}(E,F,d',h)$. Since for every $x\in E$, $y\in F$ we have

$$\Big(d'(x,y)\leq\Delta+\varepsilon\Big)\Leftrightarrow(x,y)\in A\Leftrightarrow\Big(d_Z(\phi_E(x),\phi_F(y))\vee|H_E(x)-H_F(y)|\leq\Delta+\varepsilon\Big),$$

we deduce from the conditions on Z,ϕ_E,ϕ_F that

$$\Delta_{\mathrm{H}}(E,F,d',h)\leq\Delta+\varepsilon \quad\text{and}\quad \Delta_{\mathrm{P}}(E,F,d',h)\leq\Delta+\varepsilon.$$

This shows that $\Delta'\leq\Delta+\varepsilon$. Since $\Delta\leq\Delta'$ and ε is arbitrary, we must have $\Delta=\Delta'$. □

Corollary 3.1.15. *If E and F are S-compact metric labelled spaces, then we have*

$$d_{\mathrm{GH}}(E_h,F_h)=\inf_{d\in\mathbb{D}(E,F)}\Delta_{\mathrm{H}}(E,F,d,h).$$

Proof. Simply note that we can consider E and F as measured labelled spaces equipped with the null measure, and that $d_{\mathrm{GH}}(E_h,F_h)=d_{\mathrm{GHP}}(E_h,F_h)$ for spaces with the null measure. This gives

$$d_{\mathrm{GH}}(E_h,F_h)=d_{\mathrm{GHP}}(E_h,F_h)=\inf_{d\in\mathbb{D}(E,F)}\Delta_{\mathrm{H}}(E,F,d,h)\vee\Delta_{\mathrm{P}}(E,F,d,h)$$

with $\Delta_{\mathrm{P}}(E,F,d,h)=0$. □

Lemma 3.1.16. *If E and F are two S-compact measured metric labelled spaces and d an element of $\mathbb{D}(E,F)$, then $h\mapsto\Delta_{\mathrm{H}}(E,F,d,h)$ and $h\mapsto\Delta_{\mathrm{P}}(E,F,d,h)$ are right-continuous functions.*

Proof. Step 1: We prove that $h\mapsto\Delta_{\mathrm{P}}(E,F,d,h)$ is right-continuous. For $\Delta_{\mathrm{P}}(E,F,d,h)$, we can use the triangular inequality. For $0\leq h<h'$ we have:

$$\begin{aligned}
&|\Delta_{\mathrm{P}}(E,F,d,h)-\Delta_{\mathrm{P}}(E,F,d,h')| \\
&\qquad\leq d_{\mathrm{P}}^{(E\sqcup F\times\mathbb{R},d\vee d_{\mathbb{R}})}([Id_E\times H_E](1_{E_h}\cdot\nu_E),[Id_E\times H_E](1_{E_{h'}}\cdot\nu_E)) \\
&\qquad\quad+d_{\mathrm{P}}^{(E\sqcup F\times\mathbb{R},d\vee d_{\mathbb{R}})}([Id_F\times H_F](1_{F_h}\cdot\nu_F),[Id_F\times H_F](1_{F_{h'}}\cdot\nu_F)) \\
&\qquad=\nu_E(E_{h'}\setminus E_h)+\nu_F(F_{h'}\setminus F_h).
\end{aligned}$$

We used Equation (3.1.2) for the last line. We deduce that $\Delta_{\mathrm{P}}(E,F,d,\cdot)$ is right-continuous.

Step 2: We prove that the function $h \mapsto \Delta_{\mathrm{H}}(E,F,d,h)$ is right-continuous. Consider $h_E = \min_E |H_E| \in [0,\infty]$ and $h_F = \min_F |H_F| \in [0,\infty]$. We have $\Delta_{\mathrm{H}}(E,F,d,h) = 0$ for $h \in [0, h_E \wedge h_F)$ and $\Delta_{\mathrm{H}}(E,F,d,h) = \infty$ for $h \in [h_E \wedge h_F, h_E \vee h_F)$, so $\Delta_{\mathrm{H}}(E,F,d,\cdot)$ is right-continuous on $[0, h_E \vee h_F)$. This covers the case where E or F is empty, so we can assume that $h_E \vee h_F < \infty$. For $x \in E, y \in F$, note $d^*(x,y) = d(x,y) \vee |H_E(x) - H_F(y)|$. We now prove that the function defined by

$$\delta_{E,F,d}(h) = \max_{x\in E_h} \min_{y \in F_h} d^*(x,y)$$

is right-continuous on $[h_E \vee h_F, \infty)$. Set $\varepsilon > 0$. Since the slices of E are compact by definition, E is the union of a non-decreasing sequence of compact sets, so we can choose a locally finite partition $(B_i)_{i\in I}$ of E (that is, such that every bounded subset $A \subset E$ only intersects a finite number of elements of the partition) such that for every $i \in I$, diam $(B_i) \leq \varepsilon$. Now, for every $i \in I$, choose x_i in the adherence of B_i such that $|H_E(x_i)| = \inf_{B_i} |H_E|$. We have $E_h \subset \bigcup_{\substack{i\in I\\ x_i \in E_h}} B_i$. We deduce, with $d^*(x, F_h) = \min_{y\in F_h} d^*(x,y)$, that

$$\delta_{E,F,d}(h) \geq \max_{\substack{i\in I\\ x_i\in E_h}} d^*(x_i, F_h),$$

and

$$\begin{aligned}\delta_{E,F,d}(h) &\leq \max_{\substack{i\in I\\ x_i\in E_h}} \left[d^*(x_i, F_h) + \sup_{x\in B_i} \left[d_E(x, x_i) \vee |H_E(x) - H_E(x_i)| \right] \right] \\ &\leq \max_{\substack{i\in I\\ x_i\in E_h}} d^*(x_i, F_h) + \varepsilon,\end{aligned}$$

where the last inequality comes from the diameter of each of the B_i and the fact that H_E is 1-Lipschitz. The function $h \mapsto d^*(x_i, F_h)$ is càdlàg. The set $\{i \in I | x_i \in E_h\}$ is finite and the map $h \mapsto \{i \in I | x_i \in E_h\}$ is piece-wise constant, right-continuous and non-decreasing. Thus, the function

$$h \mapsto \max_{\substack{i\in I\\ x_i\in E_h}} d^*(x_i, F_h)$$

is right-continuous. It follows that $\delta_{E,F,d}$ is the uniform limit of right-continuous functions, so $\delta_{E,F,d}$ is right-continuous. This implies that $\Delta_{\mathrm{H}}(E,F,d,h) = \delta_{E,F,d}(h) \vee \delta_{F,E,d}(h)$ is right-continuous over $[0,\infty)$. □

The following lemma is similar to Theorem 4 and Proposition 1 in Chapter IV Section 6 of [14], stating respectively that the lower bound of a collection of non-negative continuous functions is upper semi-continuous, and that upper semi-continuous functions are measurable.

Lemma 3.1.17. *Let $(f_i)_{i\in I}$ be a collection of right-continuous functions from an interval $D \subset \mathbb{R}$ to $\mathbb{R}_+$. We set $f = \inf_I f_i$. Then the function f is measurable and for every $h \in I$,*

$$f(h) \geq \limsup_{y\to h^+} f(y).$$

Proof. For $x \in D$, $\varepsilon > 0$, take $i \in I$ such that $f_i(x) - f(x) \leq \varepsilon$. Since f_i is right-continuous and $f \leq f_i$, we have

$$\limsup_{y \to x^+} f(y) \leq \lim_{y \to x^+} f_i(y) = f_i(x) \leq f(x) + \varepsilon$$

so $f(x) \geq \limsup_{y \to x^+} f(y)$.

Now, take $h > 0$ and let us prove that $A = \{x \in D | f(x) < h\}$ is measurable. Suppose that A is non-empty. For every $x \in A$, if x is not the maximum of D we have

$$f(x) \geq \limsup_{y \to x^+} f(y),$$

so there exists $x' > x$ such that for every $y \in [x, x']$, $f(y) < h$. It follows that $[x, x'] \subset A$. We deduce that $A \cap (-\infty, \sup D)$ is a union of disjoint intervals. The union is at most countable, so A is measurable. □

We now give the proof of Lemma 3.1.10. Using Lemma 3.1.14 and its Corollary 3.1.15 we see that for every $h \in \mathbb{R}_+$ we have

$$d_{\mathrm{GHP}}(E_h, F_h) = \inf_{d \in \mathbb{D}(E,F)} \Delta_{\mathrm{H}}(E, F, d, h) \vee \Delta_{\mathrm{P}}(E, F, d, h),$$
$$d_{\mathrm{GH}}(E_h, F_h) = \inf_{d \in \mathbb{D}(E,F)} \Delta_{\mathrm{H}}(E, F, d, h).$$

Lemma 3.1.16 tells us that $h \mapsto \Delta_{\mathrm{H}}(E, F, d, h)$ and $h \mapsto \Delta_{\mathrm{H}}(E, F, d, h) \vee \Delta_{\mathrm{P}}(E, F, d, h)$ are right continuous, so, using Lemma 3.1.17, the functions

$$h \mapsto d_{\mathrm{GH}}(E_h, F_h) \quad \text{and} \quad h \mapsto d_{\mathrm{GHP}}(E_h, F_h)$$

are measurable. □

3.2 Proof of Proposition 3.1.13

We prove in this section that d_{GH}, d_{GHP}, d_{LGH} and d_{LGHP} are distances. The symmetry of d_{GH}, d_{LGH}, d_{GHP} and d_{LGPH} is obvious from the definitions. To complete the proof of Proposition 3.1.13, we shall prove the triangular inequality in Lemma 3.2.2 and that they are positive-definite in Lemma 3.2.4. Since d_{GH} is the restriction of d_{GHP} to $\mathbb{X}^{0,K}$ and d_{LGH} is the restriction of d_{LGHP} to $\mathbb{X}^{0,S}$, we limit ourselves to the study of d_{GHP} and d_{LGHP}.

Remark 3.2.1. (A) We we will prove, on several occasions, results of the type $A \subset B^\varepsilon$ for A, B parts of some metric space (Z, d) and $\varepsilon > 0$. Note that since $\emptyset \subset \emptyset^\varepsilon \subset B^\varepsilon$, we can suppose $A \neq \emptyset$ whenever it suits us. It follows that a proof of the form "Take $x \in A$, ... , we have found $y \in B$ such that $d(x, y) \leq \varepsilon$, so $A \subset B^\varepsilon$" is always valid.

(B) The same holds when proving $d_{\mathrm{H}}(A, B) \leq \varepsilon$, we can apply this remark to both $A \subset B^\varepsilon$ and $B \subset A^\varepsilon$ to obtain the result.

(C) In a more general manner, recall that when proving results of the form "for all $x \in A$, we have..." it does not matter if A is empty.

3.2.1 Triangular inequality

We first introduce a construction to "glue" two metric spaces. The construction is used explicitly in the proof of the triangular inequality, and similar constructions are used in the proof of the positive-definiteness and for the completeness in Section 3.3.

For $(F, d_F), (Z, d_Z), (Z', d_{Z'})$ three separable metric spaces, and $\phi_F : F \to Z$, $\phi'_F : F \to Z'$ two isometries, we set $\tilde{Z}$ the disjoint union of Z and Z' and d the symmetric function from $\tilde{Z}^2$ to $\mathbb{R}_+$ such that

$$d(x,y) = \begin{cases} d_Z(x,y) & \text{if} \quad x, y \in Z \\ d_{Z'}(x,y) & \text{if} \quad x, y \in Z' \\ \inf_{z\in F} d_Z(x, \phi_F(z)) + d_{Z'}(\phi'_F(z), y) & \text{if} \quad x \in Z, y \in Z', \end{cases}$$

with the convention $\inf_{z\in\emptyset}(\cdot) = \infty$ if $F = \emptyset$.

The function d is symmetric and satisfies the triangular inequality. We define Z'' the quotient of $\tilde{Z}$ by the equivalence relation $d(x,y) = 0$ (for $x, y \in \tilde{Z}$) so that (Z'', d) is a separable metric space. We write $Z \sqcup_{\phi_F, \phi'_F} Z' = (Z'', d_{Z''})$. There are two canonical isometric embeddings, from Z to Z'' and from Z' to Z'', which are the projections on the quotient Z'' of the inclusions $Z \subset \tilde{Z}$ and $Z' \subset \tilde{Z}$. This is a classical construction and can be found, in explicit and implicit forms throughout literature.

Lemma 3.2.2. *The triangular inequality holds for d_{GHP} and d_{LGHP}.*

Proof. Let us begin with d_{GHP}. Let $(E, d_E, H_E, \nu_E), (F, d_F, H_F, \nu_F), (G, d_G, H_G, \nu_G)$ be three compact labelled measured metric spaces, $(Z, d_Z), (Z', d_{Z'})$ two separable metric spaces, and four isometries $\phi_E : E \to Z; \phi_F : F \to Z; \phi'_F : F \to Z'; \phi'_G : G \to Z'$. Set $Z'' = Z \sqcup_{\phi_F, \phi'_F} Z'$ and $\rho : Z \to Z''$, $\rho' : Z' \to Z''$ the canonical isometric embeddings of Z and Z' into Z''.

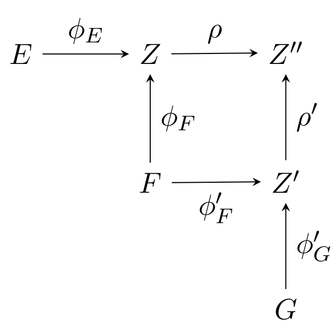

Figure 3.2: Diagram of the embeddings into Z''.

The diagram of Figure 3.2 commutes, as we have $\rho \circ \phi_F = \rho' \circ \phi'_F$ by definition of Z''. Recall the notation of d^* from Equation (3.1.3). Since ρ and ρ' are isometries, we can use the triangular inequality for $d_{\mathrm{H}}^{(Z''\times\mathbb{R},d^*)}$ to obtain

$$\begin{aligned}
d_{\mathrm{H}}^{(Z''\times\mathbb{R},d^*)}&([(\rho\circ\phi_E)\times H_E](E),[(\rho'\circ\phi'_G)\times H_G](G))\\
&\leq d_{\mathrm{H}}^{(Z''\times\mathbb{R},d^*)}([(\rho\circ\phi_E)\times H_E](E),[(\rho\circ\phi_F)\times H_F](F))\\
&\quad + d_{\mathrm{H}}^{(Z''\times\mathbb{R},d^*)}([(\rho'\circ\phi'_F)\times H_F](F),[(\rho'\circ\phi'_G)\times H_G](G))\\
&= d_{\mathrm{H}}^{(Z\times\mathbb{R},d^*_Z)}([\phi_E\times H_E](E),[\phi_F\times H_F](F))\\
&\quad + d_{\mathrm{H}}^{(Z'\times\mathbb{R},d^*_{Z'})}([\phi'_F\times H_F](F),[\phi'_G\times H_G](G)).
\end{aligned}$$

The same holds for $d_{\mathrm{P}}^{(Z''\times\mathbb{R},d^*_{Z''})}$:

$$\begin{aligned}
d_{\mathrm{P}}^{(Z''\times\mathbb{R},d^*_{Z''})}&([(\rho\circ\phi_E)\times H_E]\nu_E,[(\rho'\circ\phi'_G)\times H_G]\nu_G)\\
&\leq d_{\mathrm{P}}^{(Z\times\mathbb{R},d^*_Z)}([\phi_E\times H_E]\nu_E,[\phi_F\times H_F]\nu_F)\\
&\quad + d_{\mathrm{P}}^{(Z'\times\mathbb{R},d^*_{Z'})}([\phi'_F\times H_F]\nu_F,[\phi'_G\times H_G]\nu_G).
\end{aligned}$$

Getting the infimum over Z, Z', ϕ_E, ϕ_F, ϕ'_F and ϕ'_G implies the triangular inequality for d_{GHP}. This in turn implies the triangular inequality for d_{LGHP}. □

3.2.2 Positive-definiteness

We prove that d_{GHP} and d_{LGHP} are positive-definite. Recall the equivalence relation from Definition 3.1.2 and the notation $E_h = \mathrm{Slice}_h(E)$ for any measured labelled metric space E.

Lemma 3.2.3. *Let (E, d_E, H_E, ν_E) and (F, d_F, H_F, ν_F) be two S-compact labelled measured metric spaces. If there exists an increasing sequence $(h_k)_{k\in\mathbb{N}^*}$ of positive real numbers such that*

$$\lim_{k\to\infty} h_k = \infty \text{ and } d_{\mathrm{GHP}}(E_{h_k}, F_{h_k}) = 0,$$

then E and F are equivalent.

Proof. From a sequence of isometric embeddings of E and F, we will build an isometry $\rho^{-1}\circ\phi$ from E to F. We will show that it preserves the labels, then that it is a bijection, then that it preserves the measure.

Step 1: building $\rho^{-1}\circ\phi$ that preserves the labels. Take $(Z_k, d_k)_{k\in\mathbb{N}^*}$ a sequence of separable metric spaces and $(\phi_E^k)_{k\in\mathbb{N}^*}$, (resp. $(\phi_F^k)_{k\in\mathbb{N}^*}$) a sequence of isometries in $\mathrm{Iso}(E_{h_k}, Z_k)$ (resp. $\mathrm{Iso}(F_{h_k}, Z_k)$) such that for every $k \in \mathbb{N}^*$

$$\begin{aligned}
\frac{1}{k} \geq\ & d_{\mathrm{H}}^{(Z_k\times\mathbb{R},d^*_{Z_k})}([\phi_E^k\times H_E](E_{h_k}),[\phi_F^k\times H_F](F_{h_k}))\\
&\vee d_{\mathrm{P}}^{(Z_k\times\mathbb{R},d^*_{Z_k})}([\phi_E^k\times H_E](1_{E_{h_k}}\cdot\nu_E),[\phi_F^k\times H_F](1_{F_{h_k}}\cdot\nu_F)). \quad (3.2.1)
\end{aligned}$$

Now, set Z' the disjoint union of all the Z_k, d the function from $(Z')^2$ to $\mathbb{R}_+$ defined by

$$d(x,y) = \begin{cases} d_k(x,y) & \text{if } \ x,y \in Z_k \\ \inf_{\substack{x' \in E_{h_k} \\ y' \in E_{h_{k'}}}} d_k(x, \phi_E^k(x')) + d_E(x',y') + d_{k'}(\phi_E^{k'}(y'), y) & \text{if } \ x \in Z_k, y \in Z_{k'}. \end{cases}$$

Set Z the quotient of Z' by the equivalence relation $d(x,y) = 0$. The metric space (Z,d) is separable. For every $k \in \mathbb{N}^*$, note ρ_k the canonical embedding of Z_k in Z. For $k \in \mathbb{N}^*$, $x \in E_{h_k} \setminus E_{h_{k-1}}$, we set $\rho(x) = \rho_k \circ \phi_E^k(x)$. For every $k' \geq k$, $x \in E$, we have $d(\phi_E^{k'}(x), \phi_E^k(x)) = 0$ by definition of Z, so $\rho_{k'} \circ \phi_E^{k'}(x) = \rho(x)$. It follows that the restriction of ρ to each E_{h_k} is an isometry, so ρ is an isometry. On Figure 3.3 we see two diagrams summing up the construction.

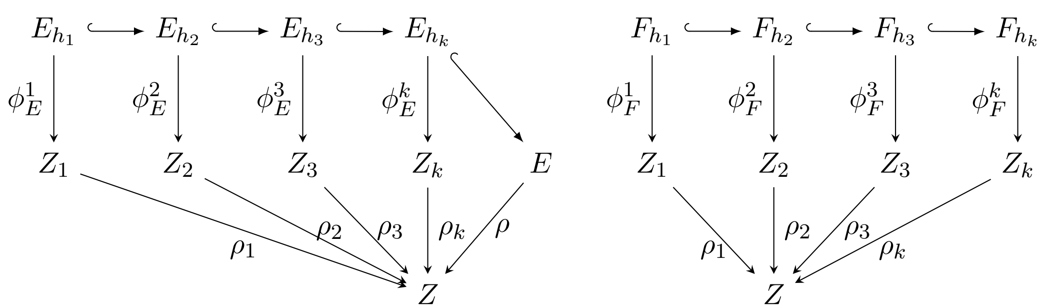

Figure 3.3: The left-hand diagram is commutative thanks to the definition of d, but the right-hand diagram isn't.

To simplify the expressions, we abusively use the notation ϕ_F^k for each embedding $\rho_k \circ \phi_F^k$ of F into Z, resulting in Figure 3.4.

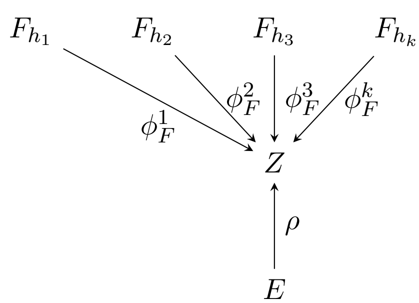

Figure 3.4: The embeddings of $(F_{h_k})_{k \in \mathbb{N}^*}$ and of E into Z.

Now, define for every $h \in \mathbb{R}_+$:

$$K_h = \rho(E_{h+1}) \cup \left(\bigcup_{k \in \mathbb{N}^*} \phi_F^k(F_{h \wedge h_k}) \right) \subset Z. \tag{3.2.2}$$

The image of a compact by a continuous map is also compact, so $\rho(E_{h+1})$ is compact and $\phi_F^k(F_{h \wedge h_k})$ is compact for every $k \in \mathbb{N}^*$. Let us prove that K_h is compact using the Bolzano-Weierstrass characterization. If K_h is empty, it is compact. If not, let $(x_n)_{n \in \mathbb{N}^*}$ be a sequence of elements of K_h. If an infinite number of terms are in $\rho(E_{h+1})$, then we can extract a converging sub-sequence since $\rho(E_{h+1})$ is compact. If not, we can without loss of generality choose $(k_n)_{n\in\mathbb{N}^*}$ a sequence of integers such that $x_n \in \phi_F^{k_n}(F_{h\wedge h_{k_n}})$. If $(k_n)_{n\in\mathbb{N}^*}$ visits some integer k an infinite number of times, then we can extract a converging sequence because $\phi_F^k(F_{h\wedge h_k})$ is compact. If not, k_n goes to infinity and, with $x_n \in F_{h\wedge h_{k_n}}$ and (3.2.1), there exists a sequence $(y_n)_{n\in\mathbb{N}^*}$ of elements of $\rho(E_{h+1})$ such that $d(x_n, y_n) \leq \frac{1}{k_n}$ as soon a $h_{k_n} \geq h$. Since $\rho(E_{h+1})$ is compact, we can extract a converging sub-sequence of $(y_n)_{n\in\mathbb{N}^*}$ and thus of $(x_n)_{n\in\mathbb{N}^*}$. This directly implies that K_h is compact.

The sequence $(\phi_F^k)_{k\in\mathbb{N}^*}$ is equicontinuous and K_h is compact, so the Arzela-Ascoli theorem gives us a sub-sequence that converges uniformly to an isometry over F_h. A diagonal extraction gives us a sub-sequence $(\phi_F^{k_n})_{n\in\mathbb{N}^*}$ that converges uniformly over every compact F_h to an isometry $\phi : F \to Z$.

Since E is a complete set and ρ is an isometry, the set $\rho(E)$ is closed. Adding the fact that $d(\phi_F^k(x), \rho(E)) \leq \frac{1}{k}$ for every $k \in \mathbb{N}^*$ such that $h_k \geq |H_F(x)|$, we find that ϕ is actually an isometry from F to $\rho(E)$. Taking $k \to \infty$ in Equation (3.2.1), we have $H_E \circ \rho^{-1} \circ \phi = H_F$, so $\rho^{-1} \circ \phi$ preserves the labels.

Step 2: prove that $\rho^{-1} \circ \phi$ is a bijective label-preserving isometry. We already know that $\rho^{-1} \circ \phi$ is a label-preserving isometry from F to E. Let us prove that $\rho^{-1} \circ \phi$ is surjective. For every $y \in E$, consider $k_0 \in \mathbb{N}^*$ such that $y \in E_{h_{k_0}}$. With (3.2.1), there exists a sequence $(x_k)_{k\geq k_0}$ of elements of $F_{h_{k_0}+1}$ such that $d(\phi_F^k(x_k), \rho(y)) \leq \frac{1}{k}$. Let us prove that $(x_{k_n})_{n\in\mathbb{N}^*}$ is a Cauchy sequence:

$$\begin{aligned} d_F(x_{k_n}, x_{k_{n'}}) =& d(\phi_F^{k_n}(x_{k_n}), \phi_F^{k_n}(x_{k_{n'}})) \\ \leq& d(\phi_F^{k_n}(x_{k_n}), \rho(y)) + d(\rho(y), \phi_F^{k_{n'}}(x_{k_{n'}})) + d(\phi_F^{k_{n'}}(x_{k_{n'}}), \phi_F^{k_n}(x_{k_{n'}})) \\ \leq& \frac{1}{k_n} + \frac{1}{k_{n'}} + \sup_{x\in F_{h_{k_0}+1}} d(\phi_F^{k_{n'}}(x), \phi_F^{k_n}(x)). \end{aligned}$$

Since $(\phi_F^{k_n})_{n\in\mathbb{N}^*}$ converges uniformly over $F_{h_{k_0}+1}$, the sequence $(x_{k_n})_{k\in\mathbb{N}^*}$ is Cauchy and converges to some limit $x \in F_{h_{k_0}+1}$. Since $(\phi_F^k)_{k\in\mathbb{N}^*}$ is equicontinuous, we have $\phi(x) = \lim_n \phi_F^{k_n}(x_{k_n}) = \rho(y)$, so ϕ is surjective, and $\rho^{-1} \circ \phi$ is a bijective isometry from F to E, preserving the labels.

Step 3: $\rho^{-1} \circ \phi$ preserves the measure. To ease the notations, we let go of the extraction and from now on we suppose without loss of generality that $(\phi_F^k)_{k\in\mathbb{N}^*}$ converges to ϕ uniformly over every compact F_h. Recall (and keep in mind for the rest of the proof) Equation (3.1.1) that will help us to handle the indicator functions and image measures. Take $h \in \mathbb{R}_+$. For $k \in \mathbb{N}^*{}_+$ such that $h_k \geq h+1$, we deduce from Equation (3.2.1) that we have $d_{\mathrm{P}}(\phi_F^k(1_{F_{h_k}} \cdot \nu_F), \rho(1_{E_{h_k}} \cdot \nu_E)) \leq \frac{1}{k}$, so

$$\begin{aligned} & d_{\mathrm{P}}^{(Z,d_Z)}(\phi_F^k(1_{F_h} \cdot \nu_F), \rho(1_{E_h} \cdot \nu_E)) \\ &\leq d_{\mathrm{P}}^{(Z,d_Z)}(\phi_F^k(1_{F_h} \cdot \nu_F), \rho(1_{E_{h+\frac{1}{k}}} \cdot \nu_E)) + d_{\mathrm{P}}^{(Z,d_Z)}(\rho(1_{E_{h+\frac{1}{k}}} \cdot \nu_E), \rho(1_{E_h} \cdot \nu_E)) \\ &\leq \frac{1}{k} + \Big([\rho(1_{E_{h+\frac{1}{k}}} \cdot \nu_E)](Z) - [\phi_F^k(1_{F_h} \cdot \nu_F)](Z)\Big)^+ + d_{\mathrm{P}}^{(E,d_E)}(1_{E_{h+\frac{1}{k}}} \cdot \nu_E, 1_{E_h} \cdot \nu_E) \qquad (3.2.3) \\ &= \frac{1}{k} + \left(\nu_E(E_{h+\frac{1}{k}}) - \nu_F(F_h)\right)^+ + \nu_E(E_{h+\frac{1}{k}} \setminus E_h) \end{aligned}$$

where we used the triangular inequality for the first inequality. In the second inequality, we applied Lemma 3.1.6 to the first term; for the second term, we used the fact that ρ is an isometry. Note from Equation (3.2.1) that for $h \leq h_k$, $[\rho \times H_E](E_h) \subset ([\phi_F^k \times H_F](F_{h_k}))^{\frac{1}{k}}$, so for every $x \in E_h \subset E_{h_k}$, there exists $y \in F_{h_k}$ such that $d(\rho(x), \phi_F^k(y)) \vee |H_E(x) - H_F(y)| \leq \frac{1}{k}$. We have $|H_F(y)| \leq |H_E(x)| + \frac{1}{k} \leq h + \frac{1}{k}$ so $y \in F_{(h+\frac{1}{k}) \wedge h_k}$. Thus, we have

$$[\rho \times H_E](E_h) \subset ([\phi_F^k \times H_F](F_{h+\frac{1}{k}}))^{\frac{1}{k}}. \tag{3.2.4}$$

We also have for $h \leq h_k - \frac{2}{k}$:

$$\begin{aligned}\nu_E(E_{h+\frac{1}{k}}) &= [\rho\nu_E](\rho(E_{h+\frac{1}{k}})) \\ &\leq [\phi_F^k \nu_F]((\rho(E_{h+\frac{1}{k}}))^{\frac{1}{k}}) + \frac{1}{k} \\ &\leq [\phi_F^k \nu_F]((\phi_F^k(F_{h+\frac{2}{k}}))^{\frac{2}{k}}) + \frac{1}{k} \\ &\leq [\phi_F^k \nu_F](\phi_F^k(F_{h+\frac{4}{k}})) + \frac{1}{k} \\ &= \nu_F(F_{h+\frac{4}{k}}) + \frac{1}{k}.\end{aligned}$$

We used (3.1.1) for the first equality, that $d_{\mathrm{P}}(\phi_F^k(1_{F_{h_k}} \cdot \nu_F), \rho(1_{E_{h_k}} \cdot \nu_E)) \leq \frac{1}{k}$ for the first inequality, we derive from Equation (3.2.4) that $\rho(E_{h+\frac{1}{k}}) \subset (\phi_F^k(F_{h+\frac{2}{k}}))^{\frac{1}{k}}$ for the second, and that H_F is 1-Lipschitz for the third. Combining this with Equation (3.2.3) we get

$$d_{\mathrm{P}}^{(Z,d_Z)}(\phi_F^k(1_{F_h} \cdot \nu_F), \rho(1_{E_h} \cdot \nu_E)) \leq \frac{2}{k} + \nu_F(F_{h+\frac{4}{k}} \setminus F_h) + \nu_E(E_{h+\frac{1}{k}} \setminus E_h) \underset{k\to\infty}{\longrightarrow} 0.$$

Since $(\phi_F^k)_{k\in\mathbb{N}^*}$ converges toward ϕ uniformly over F_h, we have

$$\lim_{k\to\infty} d_{\mathrm{P}}^{(Z,d_Z)}(\phi_F^k(1_{F_h} \cdot \nu_F), \phi(1_{F_h} \cdot \nu_F)) = 0.$$

We deduce that $\phi(1_{F_h} \cdot \nu_F) = \rho(1_{E_h} \cdot \nu_E)$. Since ρ is injective, we have $[\rho^{-1} \circ \phi]\nu_F = \nu_E$. The map $\rho^{-1} \circ \phi$ is an isometry from F to E preserving the measure and the labels, so E and F are equivalent. □

Lemma 3.2.4. *The functions d_{GHP} and d_{LGHP} are positive-definite.*

Proof. Using Lemma 3.2.3, we see that d_{GHP} is a positive-definite over $\mathbb{X}^K$. For d_{LGHP}, take (E, d_E, H_E, ν_E) and (F, d_F, H_F, ν_F) two S-compact labelled metric spaces with

$$d_{\mathrm{LGHP}}((E, d_E, H_E, \nu_E), (F, d_F, H_F, \nu_F)) = 0.$$

There exists an increasing sequence of positive real numbers $(h_k)_{k\in\mathbb{N}^*}$, with $\lim_{k\to\infty} h_k = \infty$ such that for every $n \in \mathbb{N}^*$, $d_{\mathrm{GHP}}(E_{h_k}, F_{h_k}) = 0$. Using Lemma 3.2.3, we see that E and F are equivalent, so d_{LGHP} is positive-definite. □

3.3 Polish spaces

The aim of this section is to prove the following main result:

Theorem 3.3.1. *The metric space* $(\mathbb{X}^S, d_{\mathrm{LGHP}})$ *is a Polish space.*

Since d_{LGH} is the restriction of d_{LGHP} to the closed set $\mathbb{X}^{0,S}$, we get as a corollary that $(\mathbb{X}^{0,S}, d_{\mathrm{LGH}})$ is Polish. Theorem 3.3.1 is a direct consequence of the separability proved in Lemma 3.3.3 and the completeness proved in Lemma 3.3.6. The demonstrations of those two lemmas are close to the proof of Theorem 2.9 (ii) in [1], which states the same result for a marginally different distance over the space of rooted locally compact length spaces. We do not prove that $(\mathbb{X}^K, d_{\mathrm{GHP}})$ and $(\mathbb{X}^{0,K}, d_{\mathrm{GH}})$ are polish, although the proofs should be very similar to those for the Gromov-Hausdorff and Gromov-Hausdorff-Prohorov distances in the more classical setting of metric (not labelled) spaces (see [39] and [36] for more details).

3.3.1 Separability

We first prove that $(\mathbb{X}^S, d_{\mathrm{LGHP}})$ is separable with the help of a preliminary lemma. Recall that $E_h = \mathrm{Slice}_h(E)$.

Lemma 3.3.2. *If* (E, d, H, ν) *is a compact measured labelled metric space, then for all* $\varepsilon > 0$ *there exists a measure* ν_X *over a finite set* $X \subset E$ *such that for every* $h \in \mathbb{R}_+$

$$d_{\mathrm{H}}(E_h, X_h) \vee d_{\mathrm{P}}(1_{E_h} \cdot \nu, 1_{X_h} \cdot \nu_X) \leq \varepsilon.$$

Proof. For $E = \emptyset$, $X = \emptyset$ and $\nu_X = 0$ satisfies the condition of the lemma. Since E is compact, $\nu(E)$ is finite. Take $\varepsilon > 0$. For $h \in \mathbb{R}_+$, define $f(h) = \nu(E_h)$. The map f is non-decreasing, càdlàg and bounded by $\nu(E)$, so we can choose $k \in \mathbb{N}^*$ and real numbers $0 = h_0 < ... < h_k = \infty$ such that for every integer $0 \leq j < k$ we have $f(h_{j+1}) - f(h_j) \leq \frac{\varepsilon}{2}$. Now, set $(B_1, ..., B_n)$ a measurable partition of E such that for every $1 \leq i \leq n$,

- diam $(B_i) \leq \frac{\varepsilon}{2}$,
- there exists $0 \leq j < n$ such that $|H|(B_i) \subset [h_j, h_{j+1})$.

Since E is compact, we can choose $(x_1, ..., x_n) \in E$ such that for every i, x_i is in the closure of B_i and $|H(x_i)| = \inf_{B_i} |H|$. Set $X = \{x_1, ..., x_n\}$. We have for every $h \in \mathbb{R}_+$, $X_h \subset E_h \subset (X_h)^{\frac{\varepsilon}{2}}$, so $d_{\mathrm{H}}(E_h, X_h) \leq \frac{\varepsilon}{2}$. Set

$$\nu_X = \sum_{i=1}^{n} \nu(B_i) \cdot \delta_{x_i}.$$

Take $h \in \mathbb{R}_+$ and j such that $h_j \leq h < h_{j+1}$. By choice of $(h_0, ..., h_k)$ and monotony of f, we have

$$\nu_X(E_h) \leq \sup_{t < h_{j+1}} f(t) \leq f(h_j) + \frac{\varepsilon}{2} \leq \nu(E_h) + \frac{\varepsilon}{2}. \tag{3.3.1}$$

For any Borel set $B \subset E_h$ we have $1_{E_h} \cdot \nu(B) \leq 1_{E_h} \cdot \nu_X(B^{\frac{\varepsilon}{2}})$, so using (3.3.1) and Lemma 3.1.5 we finally obtain

$$d_{\mathrm{P}}(1_{E_h} \cdot \nu, 1_{E_h} \cdot \nu_X) \leq \frac{\varepsilon}{2} + \nu_X(E_h) - \nu(E_h) \leq \varepsilon.$$

□

Lemma 3.3.3. *The space* $(\mathbb{X}^S, d_{\mathrm{LGHP}})$ *is separable.*

Proof. Take $(E, d, H, \nu) \in \mathbb{X}^S$. The space E is the limit of E_h when $h \to \infty$. Since E_h is compact, the compact labelled spaces are dense in $\mathbb{X}^S$.

Using Lemma 3.3.2, we find that for every measured labelled compact set (K, d, H, ν) we have a measure ν_X over a finite set $X \subset K$ such that for every $h \in \mathbb{R}$,

$$d_{\mathrm{H}}(K_h, X_h) \vee d_{\mathrm{P}}(1_{K_h} \cdot \nu, 1_{X_h} \cdot \nu_X) \le \varepsilon,$$

so $d_{\mathrm{LGHP}}(K, X) \le \varepsilon$. This proves that the set of finite measured labelled spaces are dense in $\mathbb{X}^S$.

Finite measured labelled metric sets can be approximated by finite sets with rational distance, measure and labels. This provides a countable dense family, and we find that $(\mathbb{X}^S, d_{\mathrm{LGHP}})$ is separable. □

3.3.2 Completeness

In the next two lemmas, we build the limit of a Cauchy sequence in a simpler case where all measured metric spaces $(F^k)_{k\in\mathbb{N}^*}$ are already embedded in a single separable metric space (Z, d) with a common 1-Lipschitz label function H. We will then go on and prove Theorem 3.3.1.

Let (Z, d) be a separable complete metric space, H a 1-Lipschitz map from Z to $\mathbb{R}$ and $(h_k)_{k\in\mathbb{N}^*}$ an increasing sequence of positive real numbers with $h_1 \ge 1$ and limit $+\infty$. Let $(F^k)_{k\in\mathbb{N}^*}$ be a sequence of closed sets of Z, $(\nu^k)_{k\in\mathbb{N}^*}$ a sequence of Borel measures over Z such that Supp $(\nu^k) \subset F^k$. For every $k \in \mathbb{N}^*, h \in \mathbb{R}_+$ set $F^k_h = \mathrm{Slice}_h(F^k)$ and $\nu^k_h = 1_{F^k_h} \cdot \nu^k$. We suppose that for every $k \in \mathbb{N}^*, h \in \mathbb{R}_+$, F^k_h is compact. This makes (F^k, d, H, ν^k) a S-compact measured labelled space. Finally, we suppose that for every $k \in \mathbb{N}^*$ we have

$$d_{\mathrm{H}}(F^k_{h_k}, F^{k+1}_{h_k}) \vee d_{\mathrm{P}}(\nu^k_{h_k}, \nu^{k+1}_{h_k}) \le \frac{1}{2^{k+1}}. \tag{3.3.2}$$

Take $k \in \mathbb{N}^*$ and $h \in [0, h_k - \frac{1}{2^{k+1}}]$, $z \in F^{k+1}_{h+\frac{1}{2^{k+1}}} \subset F^{k+1}_{h_k}$. Since $d_{\mathrm{H}}(F^k_{h_k}, F^{k+1}_{h_k}) \le \frac{1}{2^{k+1}}$, there exists $z' \in F^k_{h_k}$ such that $d(z, z') \le \frac{1}{2^{k+1}}$. Since H is 1-Lipschitz, $|H(z')| \le |H(z)| + \frac{1}{2^{k+1}}$. Thus, we have $z' \in F^k_{h+\frac{1}{2^k}}$, and we conclude that

$$F^{k+1}_{h+\frac{1}{2^{k+1}}} \subset (F^k_{(h+\frac{1}{2^k})\wedge h_k})^{\frac{1}{2^{k+1}}}. \tag{3.3.3}$$

With Remark 3.2.1(A), we see that the inclusion still holds when $F^{k+1}_{h+\frac{1}{2^{k+1}}}$ is empty. Note that this result only supposes that H is 1-Lipschitz and a Hausdorff control for a bigger slice. Similarly, we have for $h \in [0, h_k]$ that

$$F^k_h \subset (F^{k+1}_{(h+\frac{1}{2^{k+1}})\wedge h_k})^{\frac{1}{2^{k+1}}}.$$

By an immediate induction we see that for $h \in [0, h_k], k' \ge k$, we have

$$F^k_h \subset (F^{k'+1}_{(h+\sum_{i=k}^{k'} \frac{1}{2^{i+1}})\wedge h_{k'}})^{\sum_{i=k}^{k'} \frac{1}{2^{i+1}}} \subset (F^{k'}_{(h+\frac{1}{2^k})\wedge h_k})^{\frac{1}{2^k}}. \tag{3.3.4}$$

Lemma 3.3.4. *There exists a closed set $E \subset Z$ such that (E, d, H) is an S-compact labelled metric space and*

$$\lim_{k\to\infty} \int_0^\infty d_{\mathrm{H}}(F_h^k, E_h)\mathrm{e}^{-h} dh = 0.$$

Proof. Consider for $h \geq 0$ the closed set

$$E_{(h)} = \bigcap_{\substack{k\in\mathbb{N}^* \\ h_k \geq h + \frac{1}{2^{k+1}}}} (F^k_{h+\frac{1}{2^k}})^{\frac{1}{2^k}}. \tag{3.3.5}$$

From Equation (3.3.3) we infer that the intersection is monotonic, that is, each term is a subset of the previous. We set $E = \bigcup_{h\in\mathbb{R}_+} E_{(h)}$.

Step 1: we prove that $E_{(h)}$ is compact. We assume for this step that $E_{(h)} \neq \emptyset$, since the empty set is always a compact. Since $F^k_{h+\frac{1}{2^k}}$ is compact, we can choose a finite covering of $F^k_{h+\frac{1}{2^k}}$ using balls of of diameter $\frac{1}{2^k}$. We denote by $(x_1, ..., x_n)$ their centers. Changing their diameter to $\frac{1}{2^{k-1}}$, we get a covering of $E_{(h)}$, so $E_{(h)}$ is totally bounded. Since $E_{(h)}$ is closed and Z is complete, $E_{(h)}$ is compact.

Step 2: we prove that $E_{(h)} = E_h$ and deduce that E is S-compact. From the definition of $E_{(h)}$, we see that $\sup_{E_{(h)}} |H| \leq h$, which proves the inclusion $E_{(h)} \subset E_h$. To prove the other inclusion, take $z \in E_h \subset \bigcup_{\ell\in\mathbb{R}_+} E_{(\ell)}$, set $h'' = |H(z)|$ and take h' such that $z \in E_{(h')}$. We have $h'' \leq \sup_{E_{(h')}} |H| \leq h'$. Take k such that $h_k \geq h' + \frac{1}{2^{k+1}}$. Since $z \in E_{(h')} \subset (F^k_{h'+\frac{1}{2^k}})^{\frac{1}{2^k}}$, there exists $z' \in F^k_{h'+\frac{1}{2^k}}$ such that $d(z, z') \leq \frac{1}{2^k}$. As H is 1-Lipschitz, we have $|H(z')| \leq |H(z)| + \frac{1}{2^k} = h'' + \frac{1}{2^k} \leq h + \frac{1}{2^k}$ and $z' \in F^k_{h''+\frac{1}{2^k}}$. Since $h' \geq h''$ and the intersection in (3.3.5) defining $E_{(h)}$ is monotonic, see (3.3.3), we have

$$z \in \bigcap_{\substack{k\in\mathbb{N}^* \\ h_k \geq h' + \frac{1}{2^{k+1}}}} (F^k_{h''+\frac{1}{2^k}})^{\frac{1}{2^k}} = \bigcap_{\substack{k\in\mathbb{N}^* \\ h_k \geq h'' + \frac{1}{2^{k+1}}}} (F^k_{h''+\frac{1}{2^k}})^{\frac{1}{2^k}} = E_{(h'')} \subset E_{(h)}.$$

We conclude that

$$E_h = E_{(h)}.$$

Since $E_{(h)} = E_h$ is compact by Step 1, (E, d, H) is an S-compact labelled set.

Step 3: we prove that

$$\lim_{k\to\infty} \int_0^\infty d_{\mathrm{H}}(F_h^k, E_h)\mathrm{e}^{-h} dh = 0.$$

Take $0 < \varepsilon < 1$, $h_{\max} = 1 - \log(\varepsilon)$. Using Lemma 3.3.2 we can choose a finite set $X \subset E_{h_{\max}}$ such that for every $h \in [0, h_{\max}]$,

$$d_{\mathrm{H}}(E_h, X_h) \leq \varepsilon.$$

Now, consider $k \in \mathbb{N}^*$ such that $h_{\max} + 1 \leq h_k$, $A_k = \bigcup_{z\in X} [|H(z)| - \frac{1}{2^k}, |H(z)| + \frac{1}{2^k}]$ and $h \in [0, h_{\max} - 1] \setminus A_k$. Let us prove that $d_{\mathrm{H}}(F_h^k, X_h) \leq \varepsilon + \frac{1}{2^k}$. Take $z \in X_h \subset E_{(h)}$. By definition of $E_{(h)}$, there exists $z' \in F^k_{h+\frac{1}{2^k}}$ such that $d(z, z') \leq \frac{1}{2^k}$. Since H is 1-Lipschitz,

$|H(z')| \leq |H(z)| + \frac{1}{2^k}$. Since $z \in X_h$ and $h \notin A_k$, we must have $h - |H(z)| > \frac{1}{2^k}$ so $|H(z')| \leq h$. With this, we have $z' \in F_h^k$. We have proven that $X_h \subset (F_h^k)^{\frac{1}{2^k}}$.

Since $h \in [0, h_{\max}] \subset [0, h_k - 1]$, we have from Equation (3.3.4) that

$$F_h^k \subset (E_{h+\frac{1}{2^k}})^{\frac{1}{2^k}}. \tag{3.3.6}$$

Since $h \leq h_{\max} - 1$, for every $z' \in F_h^k$ we can choose $z'' \in E_{h+\frac{1}{2^k}}$ such that $d(z', z'') \leq \frac{1}{2^k}$ and there exists $z \in X_{h+\frac{1}{2^k}}$ such that $d(z'', z) \leq \varepsilon$, so $d(z', z) \leq \varepsilon + \frac{1}{2^k}$. By definition of A_k and by choice of h, $X_{h+\frac{1}{2^k}} = X_h$ so $z \in X_h$. We have for every $h \in [0, h_{\max} - 1] \setminus A_k$ that

$$d_{\mathrm{H}}(F_h^k, X_h) \leq \frac{1}{2^k} + \varepsilon.$$

With this, we get for every k such that $h_k \geq h_{\max} + 1$ and $h \in [0, h_{\max} - 1] \setminus A_k$ that

$$d_{\mathrm{H}}(F_h^k, E_h) \leq d_{\mathrm{H}}(F_h^k, X_h) + d_{\mathrm{H}}(X_h, E_h) \leq \frac{1}{2^k} + 2\varepsilon.$$

If we take n the cardinality of X, this translates to

$$\begin{aligned}\int_0^\infty \Big(1 \wedge d_{\mathrm{H}}(F_h^k, E_h)\Big) \mathrm{e}^{-h} dh \leq & \int_{[0, h_{\max}-1]\setminus A_k} (\frac{1}{2^k} + 2\varepsilon)\mathrm{e}^{-h} dh + \int_{A_k} dh + \mathrm{e}^{-(h_{\max}-1)} \\ \leq & (\frac{1}{2^k} + 2\varepsilon) + \frac{2n}{2^k} + \varepsilon.\end{aligned}$$

This means that for every $\varepsilon > 0$,

$$\limsup_{k\to\infty} \int_0^\infty \Big(1 \wedge d_{\mathrm{H}}(F_h^k, E_h)\Big) \mathrm{e}^{-h} dh \leq 3\varepsilon.$$

□

Recall that for every measured labelled space (E, d, H, μ) and $h \in \mathbb{R}_+$ we note $\mu_h = 1_{E_h} \cdot \mu$.

Lemma 3.3.5. *There exists a measure μ over E such that*

$$\lim_{k\to\infty} \int_0^\infty d_{\mathrm{P}}(\nu_h^k, \mu_h) \mathrm{e}^{-h} dh = 0.$$

Proof. Step 1: we build a family $(\mu_{(h)})_{h\in\mathbb{R}_+}$ such that

$$\lim_{k\to\infty} \int_0^\infty (1 \wedge d_{\mathrm{P}}(\nu_h^k, \mu_{(h)}))\mathrm{e}^{-h} = 0.$$

Take $h \in \mathbb{R}_+$ and $k_0(h) = \min\{k \in \mathbb{N}^* | h \leq h_k - 1\}$. For every $k \geq k_0(h)$, we have by Equation (3.3.2) that

$$d_{\mathrm{P}}(\nu_{h_k}^k, \nu_{h_k}^{k+1}) \leq \frac{1}{2^{k+1}}.$$

We have

$$\begin{aligned}\nu^{k+1}\left(F^{k+1}_{h+\frac{1}{2^{k+1}}}\right) &\leq \nu^k\left(\left(F^{k+1}_{h+\frac{1}{2^{k+1}}}\right)^{\frac{1}{2^{k+1}}} \cap \operatorname{Supp}\left(\nu^k\right)\right) + \frac{1}{2^{k+1}} \\ &= \nu^k\left(\left(F^{k+1}_{h+\frac{1}{2^{k+1}}}\right)^{\frac{1}{2^{k+1}}} \cap F^k\right) + \frac{1}{2^{k+1}}.\end{aligned}$$

Since H is 1-Lipschitz, for every $y \in \left(F^{k+1}_{h+\frac{1}{2^{k+1}}}\right)^{\frac{1}{2^{k+1}}} \cap F^k$ we have $|H(y)| \leq h + \frac{1}{2^{k+1}} + \frac{1}{2^{k+1}} = h + \frac{1}{2^k}$ so

$$\nu^{k+1}\left(F^{k+1}_{h+\frac{1}{2^{k+1}}}\right) \leq \nu^k\left(\left(F^{k+1}_{h+\frac{1}{2^{k+1}}}\right)^{\frac{1}{2^{k+1}}} \cap F^k\right) + \frac{1}{2^{k+1}} \leq \nu^k\left(F^k_{h+\frac{1}{2^k}}\right) + \frac{1}{2^{k+1}}$$

and we obtain by induction that for $k' \geq k \geq k_0(h)$:

$$\nu^k\left(F^k_{h+\frac{1}{2^k}}\right) + \frac{1}{2^k} \geq \nu^{k'}\left(F^{k'}_{h+\frac{1}{2^{k'}}}\right) + \frac{1}{2^{k'}}.$$

This is equivalent to

$$\nu^k_{h+\frac{1}{2^k}}(Z) + \frac{1}{2^k} \geq \nu^{k'}_{h+\frac{1}{2^{k'}}}(Z) + \frac{1}{2^{k'}}. \tag{3.3.7}$$

The sequence $(\nu^k_{h+\frac{1}{2^k}}(Z) + \frac{1}{2^k})_{k \geq k_0(h)}$ is decreasing and non-negative so it converges for every $h \in \mathbb{R}_+$, we set $M(h)$ its limit. The function $h \mapsto \nu^k_{h+\frac{1}{2^k}}(Z) + \frac{1}{2^k}$ is right-continuous, so using Lemma 3.1.17, we see that for every $h \in \mathbb{R}$,

$$M(h) \geq \limsup_{y \to h^+} M(y).$$

Since M is non-decreasing, M is càdlàg.

Using Lemma 3.1.6, we find that for every $k \geq k_0(h)$ we have

$$\begin{aligned}d_{\mathrm{P}}\left(\nu^{k+1}_{h+\frac{1}{2^{k+1}}}, \nu^k_{h+\frac{1}{2^k}}\right) &\leq \frac{1}{2^{k+1}} + \left(\nu^k_{h+\frac{1}{2^k}}(Z) - \nu^{k+1}_{h+\frac{1}{2^{k+1}}}(Z)\right)^+ \\ &\leq \frac{1}{2^{k+1}} + \left(\left(\nu^k_{h+\frac{1}{2^k}}(Z) + \frac{1}{2^k}\right) - \left(\nu^{k+1}_{h+\frac{1}{2^{k+1}}}(Z) + \frac{1}{2^{k+1}}\right)\right)^+ \\ &= \frac{1}{2^{k+1}} + \left(\nu^k_{h+\frac{1}{2^k}}(Z) + \frac{1}{2^k}\right) - \left(\nu^{k+1}_{h+\frac{1}{2^{k+1}}}(Z) + \frac{1}{2^{k+1}}\right),\end{aligned}$$

where the equality comes from Equation (3.3.7). By induction, this yields, for every $k' > k$

$$d_{\mathrm{P}}\left(\nu^k_{h+\frac{1}{2^k}}, \nu^{k'}_{h+\frac{1}{2^{k'}}}\right) \leq \frac{1}{2^k} + \left(\nu^k_{h+\frac{1}{2^k}}(Z) + \frac{1}{2^k}\right) - M(h) \underset{k \to \infty}{\to} 0.$$

This means that $\left(\nu^k_{h+\frac{1}{2^k}}\right)_{k \in \mathbb{N}^*}$ is a Cauchy sequence. Since (Z, d) is Polish, the space of finite measures over Z is Polish for the Prohorov distance. It follows that $\left(\nu^k_{h+\frac{1}{2^k}}\right)_{k \in \mathbb{N}^*}$ converges to a limit $\mu_{(h)}$ for every $h \in \mathbb{R}_+$, with $\mu_{(h)}(Z) = M(h)$. The map $h \mapsto \nu^k_{h+\frac{1}{2^k}}$ is measurable, so its point-wise limit $h \mapsto \mu_{(h)}$ is measurable as well. By dominated convergence, we have

$$\lim_{k\to\infty} \int_0^\infty \left(1 \wedge d_{\mathrm{P}}\left(\nu^k_{h+\frac{1}{2^k}}, \mu_{(h)}\right)\right) \mathrm{e}^{-h} dh = 0.$$

Since M is càdlàg, we also have the dominated convergence

$$\lim_{k\to\infty} \int_0^\infty \left(1 \wedge \left(M\left(h + \frac{1}{2^k}\right) - M(h)\right)\right) \mathrm{e}^{-h} dh = 0.$$

Using these two limits, we have

$$\begin{aligned}
&\int_0^\infty \left(1 \wedge d_{\mathrm{P}}\left(\nu^k_h, \mu_{(h)}\right)\right) \mathrm{e}^{-h} dh \\
&\qquad \le \frac{1}{2^k} + \int_{\frac{1}{2^k}}^\infty \left(1 \wedge \left(d_{\mathrm{P}}\left(\nu^k_h, \mu_{(h-\frac{1}{2^k})}\right) + d_{\mathrm{P}}\left(\mu_{(h-\frac{1}{2^k})}, \mu_{(h)}\right)\right)\right) \mathrm{e}^{-h} dh \\
&\qquad = \frac{1}{2^k} + \int_0^\infty \left(1 \wedge d_{\mathrm{P}}\left(\nu^k_{h+\frac{1}{2^k}}, \mu_{(h)}\right)\right) \mathrm{e}^{-h} dh \\
&\qquad\qquad + \int_0^\infty \left(1 \wedge \left(M\left(h + \frac{1}{2^k}\right) - M(h)\right)\right) \mathrm{e}^{-h} dh,
\end{aligned}$$

which converges to 0.

Step 2: build μ such that $\mu_h = \mu_{(h)}$ for almost-every $h \in \mathbb{R}_+$. We define $\mu = \sup_{h\in\mathbb{R}} \mu_{(h)}$ which is a measure as the sequence $(\mu_{(h)})_{h\ge 0}$ is non-decreasing. Let us prove that for every $h \in \mathbb{R}_+$, $\mu_h = \mu_{(h)}$. We use notation from the proof of Lemma 3.3.4. From Lemma 3.3.4, we deduce that

$$\liminf_{k\to\infty} d_{\mathrm{H}}(F^k_h, E_h) = 0$$

for almost-every $h \in \mathbb{R}_+$ and Supp $(\nu^k_{h+\frac{1}{2^k}}) \subset F^k_{h+\frac{1}{2^k}}$. Therefore, thanks to (3.3.5), we get Supp $(\mu_{(h)}) \subset E_{(h)} = E_h$. For almost-every $h \in \mathbb{R}^+$, this gives $\mu_{(h)} = 1_{E_h} \cdot \mu_{(h)} \le 1_{E_h} \cdot \mu = \mu_h$.

Conversely, take $h' > h$. Define for every $k \in \mathbb{N}^*$ the non-negative real number $\varepsilon_k = d_{\mathrm{P}}(\nu^k_{h'+\frac{1}{2^k}}, \mu_{(h')})$. By convergence, $\lim_k \varepsilon_k = 0$ a.e. and we can define for a.e. h' the quantity $\varepsilon_{\max} = \max_{k\in\mathbb{N}^*} \varepsilon_k$. Using the definition of d_{P} we have

$$\mu_{(h')}(E_h) \le \inf_{k'\in\mathbb{N}^*} \nu^{k'}((E_h)^{\varepsilon_{k'}}) + \varepsilon_{k'} \le \inf_{k'\in\mathbb{N}^*} \nu^{k'}_{h+\varepsilon_{k'}}(Z) + \varepsilon_{k'}. \tag{3.3.8}$$

Recall $k_0(h) = \min\{k \in \mathbb{N}^* | h \le h_k - 1\}$ from Step 1. Using Equation (3.3.7), we find that for every $k \in \mathbb{N}^*$ such that $k_1 = k_0(h + \varepsilon_{\max}) \le k \le k'$, we have

$$\nu^{k'}_{h+\varepsilon_{k'}}(Z) + \varepsilon_{k'} \le \nu^{k'}_{h+\varepsilon_{k'}+\frac{1}{2^{k'}}}(Z) + \varepsilon_{k'} \le \nu^k_{h+\varepsilon_{k'}+\frac{1}{2^k}}(Z) + \frac{1}{2^k} - \frac{1}{2^{k'}} + \varepsilon_{k'}.$$

Combining the last line with Equation (3.3.8), we get

$$\begin{aligned}
\mu_{(h')}(E_h) &\le \inf_{k'\ge k_1} \min_{k_1\le k\le k'} \left(\nu^k_{h+\varepsilon_{k'}+\frac{1}{2^k}}(Z) + \frac{1}{2^k} - \frac{1}{2^{k'}} + \varepsilon_{k'}\right) \\
&\le \liminf_{k\to\infty} \liminf_{k'\to\infty} \left(\nu^k_{h+\varepsilon_{k'}+\frac{1}{2^k}}(Z) + \frac{1}{2^k} - \frac{1}{2^{k'}} + \varepsilon_{k'}\right) \\
&= \lim_{k\to\infty} \left(\nu^k_{h+\frac{1}{2^k}}(Z) + \frac{1}{2^k}\right) = M(h).
\end{aligned}$$

We deduce that a.e. $\mu_h(Z) = M(h) = \mu_{(h)}(Z)$. Since $\mu_{(h)} \leq \mu_h$ a.e., we get $\mu_{(h)} = \mu_h$ a.e. and

$$\lim_{k\to\infty} \int_0^\infty d_{\mathrm{P}}(\nu_h^k, \mu_h) \mathrm{e}^{-h} dh = 0.$$

□

Lemma 3.3.6. *The space* $(\mathbb{X}^S, d_{\mathrm{LGHP}})$ *is complete.*

Proof. Take $(E^k, d_k, H_k, \mu^k)_{k\in\mathbb{N}^*}$ a sequence of S-compact measured labelled spaces such that for every $k \in \mathbb{N}^*$,

$$d_{\mathrm{LGHP}}(E^k, E^{k+1}) < \frac{\mathrm{e}^{-k}}{2^{k+1}}. \tag{3.3.9}$$

Step 1: we embed $(E^k)_{k\in\mathbb{N}^*}$ in a common space (Z^*, d^*, H^*). For every $h \in \mathbb{R}^+$, note $\mu_h^k = 1_{E_h^k} \cdot \mu^k$. From (3.3.9), we can choose $h_k \in [k-1, k]$ such that

$$d_{\mathrm{GHP}}(E_{h_k}^k, E_{h_k}^{k+1}) < \frac{1}{2^{k+1}}.$$

By definition of d_{GHP}, we can chose (Z_k, d_{Z_k}) a separable metric space, $\phi_k \in \mathrm{Iso}(E_{h_k}^k, Z_k)$, and $\psi_{k+1} \in \mathrm{Iso}(E_{h_k}^{k+1}, Z_k)$ such that

$$d_{\mathrm{H}}(\phi_k(E_{h_k}^k), \psi_{k+1}(E_{h_k}^{k+1})) \vee d_{\mathrm{P}}(\phi_k \nu_{h_k}^k, \psi_{k+1} \nu_{h_k}^{k+1}) \leq \frac{1}{2^{k+1}}. \tag{3.3.10}$$

Set Z' the disjoint union of all the $(Z_k)_{k\in\mathbb{N}^*}$. Set d the symmetric function from $(Z')^2$ to $\mathbb{R}_+$ such that $d(x,y) = d_{Z_k}(x,y)$ if $x, y \in Z_k$ and recursively for $x \in Z_k$ and $y \in Z_{k'}$ with $k < k'$:

$$d(x,y) = \inf_{z_{k'} \in E_{h_{k'-1}}^{k'}} d(x, \psi_{k'}(z_{k'})) + d_{Z_{k'}}(\phi_{k'}(z_{k'}), y).$$

Since $\psi_{k'}(z_{k'}) \in Z_{k'-1}$, d is well-defined.

We call Z the completion of Z' quotiented by the equivalence relation $d(x,y) = 0$. The pair (Z, d) is a separable metric space.

Remark 3.3.7. The idea in the construction of Z is to form a chain by successively gluing the metric spaces $(Z^k, Z^{k+1})_{k\in\mathbb{N}^*}$ along the isometric embeddings of $(E_{h_k}^{k+1})_{k\in\mathbb{N}^*}$. It is very similar to the constructions in Lemmas 3.2.2 and 3.3.4. We sum up the construction of Z in Figure 3.5.

Set $Z^* = Z \times \mathbb{R}$ and d^* the distance over Z^* defined by $d^*((x,h),(x',h')) = d(x,x') \vee |h-h'|$. The space (Z^*, d^*) is complete and separable. Set $F^k = [\phi_k \times H_k](E_{h_k}^k) \subset Z$ and $\nu^k = [\phi_k \times H_k](\mu_{h_k}^k)$ for every $k \in \mathbb{N}^*$. Set H^* the projection from Z^* to $\mathbb{R}$ and ρ the projection from Z^* to Z such that for $z \in Z^*$, $z = (\rho(z), H^*(z))$. The map H^* is 1-Lipschitz. For every $k \in \mathbb{N}^*$, set $Z_h^* = \{x \in Z^* | |H^*(x)| \leq h\}$, $F_h^k = F^k \cap Z_h^*$, $\nu_h^k = 1_{Z_h^*} \cdot \nu^k$.

Step 2: we use Lemma 3.3.4 and Lemma 3.3.5 to obtain a limit in (Z^*, d^*, H^*). By construction, we have

$$d_{\mathrm{H}}(F_{h_k}^k, F_{h_k}^{k+1}) \vee d_{\mathrm{P}}(\nu_{h_k}^k, \nu_{h_k}^{k+1}) \leq \frac{1}{2^{k+1}}.$$

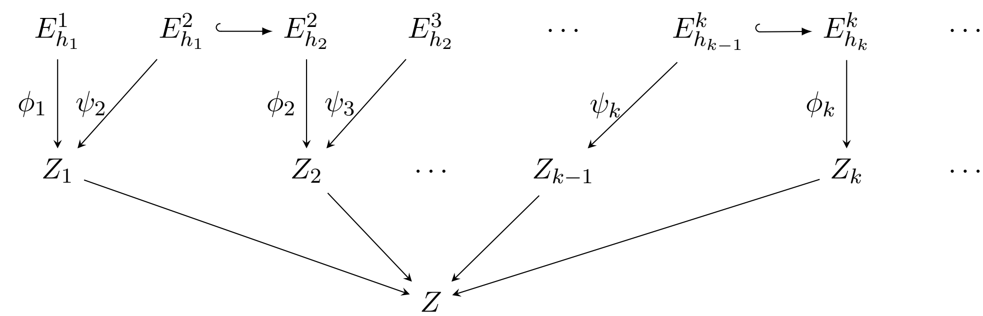

Figure 3.5: The commutative diagram of the construction of Z. The unlabeled arrows to Z are the canonical projections into the quotient.

Using Lemmas 3.3.4 and 3.3.5, we see that there exists $E^* \subset Z^*$ and μ^* a measure over E^* such that

$$\lim_{k\to\infty} \int_0^\infty (1 \wedge (d_{\mathrm{H}}(F^k_h, E^*_h) \vee d_{\mathrm{P}}(\nu^k_h, \mu^*_h)))\mathrm{e}^{-h} dh = 0.$$

We define $E = \rho(E^*)$ and $\mu = \rho(\mu^*)$.

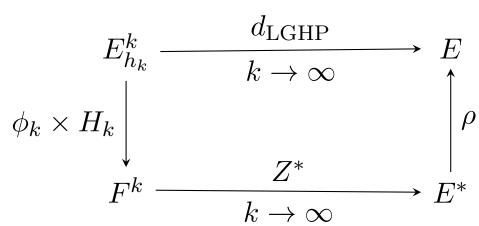

Figure 3.6: We solved the convergence in Z^*, and we want to obtain the topmost convergence. To that end, we want to reformulate the convergence of $(F^k)_{k\in\mathbb{N}^*}$ to E^* into the convergence of $(E^k)_{k\in\mathbb{N}^*}$ to E. According to the diagram, it is enough to write ρ^{-1} in the form $\mathrm{Id}_E \times H$. Note that ρ must be injective for the label function H to be defined on E.

Step 3: we build a map H and prove that (E, d, H, μ) is the limit of $(E^k)_{k\in\mathbb{N}^*}$. For $z = (x, h) \in E^*$, $z' = (x', h') \in E^*$, $\varepsilon > 0$, take $k \in \mathbb{N}^*$ and $z_k, z'_k \in F^k$ such that $d^*(z, z_k) \le \varepsilon$ and $d^*(z', z'_k) \le \varepsilon$. By definition of F^k, there exists $x_k, x'_k \in E^k_{h_k}$ such that $z_k = [\phi_k \times H_k](x_k)$, $z'_k = [\phi_k \times H_k](x'_k)$. Since ϕ_k is an isometry and H_k is 1-Lipschitz, we have that

$$|H^*(z_k) - H^*(z'_k)| = |H_k(x_k) - H_k(x'_k)| \le d_k(x_k, x'_k) = d(\rho(z_k), \rho(z'_k)),$$

so

$$d(x, x') \ge d(\rho(z_k), \rho(z'_k)) - 2\varepsilon \ge |H^*(z_k) - H^*(z'_k)| - 2\varepsilon \ge |h - h'| - 4\varepsilon.$$

Since ε was arbitrary, we have $d(x, x') \ge |h - h'|$ so ρ is bijective from E^* to E and $H = H^* \circ \rho^{-1}$ is 1-Lipschitz. Since ρ is continuous, (E, d, H, μ) is a S-compact labelled metric space. We have $E^* = [\mathrm{Id}_E \times H](E)$ so for every $k \in \mathbb{N}^*$,

$$d_{\mathrm{LGHP}}(E^k_{h_k}, E) \le \int_0^\infty (1 \wedge (d_{\mathrm{H}}(F^k_h, E^*_h) \vee d_{\mathrm{P}}(\nu^k_h, \mu^*_h)))\mathrm{e}^{-h} dh \underset{k\to\infty}{\longrightarrow} 0.$$

Since

$$d_{\mathrm{LGHP}}(E^k, E^k_{h_k}) \leq \mathrm{e}^{-h_k}$$

and $\lim_k h_k = \infty$, we have

$$\lim_{k\to\infty} d_{\mathrm{LGHP}}(E, E^k) = 0.$$

□

Remark 3.3.8. We can see in the proof of Lemma 3.3.6 that if $(E^k)_{k\in\mathbb{N}^*}$ is a sequence of elements of $\mathbb{X}^S$ satisfying

$$d_{\mathrm{GHP}}(E^k_{h_k}, E^{k+1}_{h_k}) \leq \frac{1}{2^{k+1}}$$

for some increasing sequence of positive real numbers $(h_k)_{k\in\mathbb{N}^*}$ going to $+\infty$, then there exists $E \in \mathbb{X}^S$ such that

$$d_{\mathrm{LGHP}}(E^k, E) \underset{k\to\infty}{\longrightarrow} 0.$$

Lemma 3.3.9. *Let $(E^n)_{n\in\mathbb{N}^*}$ be a sequence of elements of $\mathbb{X}^S$. If there exists an increasing sequence of positive real numbers $(h_k)_{k\in\mathbb{N}^*}$ going to $+\infty$ such that the sequence $(E^n_{h_k})_{n\in\mathbb{N}^*}$ is Cauchy in $(\mathbb{X}^K, d_{\mathrm{GHP}})$ for every $k \in \mathbb{N}^*$, then there exists $E \in \mathbb{X}$ such that*

$$d_{\mathrm{LGHP}}(E^n, E) \underset{n\to\infty}{\longrightarrow} 0.$$

Proof. For every $k \in \mathbb{N}^*$, define by induction $n_0 = 0$ and

$$n_k = \min\{n > n_{k-1} | \forall n', n'' \geq n, d_{\mathrm{GHP}}(E^{n'}_{h_k}, E^{n''}_{h_k}) < \frac{1}{2^{k+1}}\}.$$

Note $\mathcal{F}$ the set of all extractions ϕ such that for every $n \in \mathbb{N}^*$ we have $\phi(k) \geq n_k$. For every $\phi \in \mathcal{F}$, $k \in \mathbb{N}^*$, we have $n_k \leq \phi(k) < \phi(k+1)$, so

$$d_{\mathrm{GHP}}(E^{\phi(k)}_{h_k}, E^{\phi(k+1)}_{h_k}) \leq \frac{1}{2^{k+1}}.$$

The sequence $(h_k)_{k\in\mathbb{N}^*}$ is increasing to ∞ so by Remark 3.3.8, there exists $E_\phi \in \mathbb{X}^S$ such that

$$d_{\mathrm{LGHP}}(E^{\phi(k)}, E_\phi) \underset{k\to\infty}{\longrightarrow} 0.$$

Now, take $\phi, \phi' \in \mathcal{F}$, and let us prove that $E_\phi = E'_\phi$. We define by induction an extraction $\phi'' \in \mathcal{F}$ taking its terms alternatively in ϕ and ϕ':

$$\begin{aligned}
\phi''(1) &= \phi(1)\\
\phi''(2k) &= \min\left(\phi'(\mathbb{N}^*) \cap (n_{2k} \vee \phi''(2k-1), +\infty)\right),\\
\phi''(2k+1) &= \min\left(\phi(\mathbb{N}^*) \cap (n_{2k+1} \vee \phi''(2k), +\infty)\right).
\end{aligned}$$

Note that for every $k \in \mathbb{N}^*$, $\phi''(k) > n_k$, so $\phi'' \in \mathcal{F}$. Thus, we have

$$d_{\mathrm{LGHP}}(E^{\phi''(k)}, E_{\phi''}) \underset{k\to\infty}{\longrightarrow} 0.$$

By construction of ϕ'', we have $E_\phi = E_{\phi''} = E_{\phi'}$, so we can call E the common limit.

From every subsequence $(E^{\psi n})_{n\in\mathbb{N}^*}$, we can extract a converging sub-subsequence $(E^{\phi k})_{k\in\mathbb{N}^*}$ converging to E by taking $\phi(k) = \psi(n_k)$, which is in $\mathcal{F}$. Since $\mathbb{X}^S$ is a metric space, this implies that $(E^n)_{n\in\mathbb{N}^*}$ converges to E. □

3.4 ε-correspondences and properties of subspaces of $\mathbb{X}$

3.4.1 ε-correspondences

In this part, we introduce another way to compute/control d_{GHP}.

We call correspondence between two metric spaces E and F any Borel set $A \subset E \times F$ such that

- for every $x \in E$ there exists $y \in F$ such that $(x, y) \in A$,
- for every $y \in F$ there exists $x \in E$ such that $(x, y) \in A$,

In order to ease some proofs, we define a ε-correspondence between two compact labelled sets (E, d_E, H_E, ν_E) and (F, d_F, H_F, ν_F) as any correspondence $A \subset E \times F$ satisfying:

$$\text{for every } (x, y), (x', y') \in A,\ |d_E(x, x') - d_F(y, y')| \leq 2\varepsilon, \tag{3.4.1}$$

$$\text{for every } (x, y) \in A,\ |H_E(x) - H_F(y)| \leq \varepsilon, \tag{3.4.2}$$

$$\text{for every Borel set } B \subset E,\ \nu_E(B) \leq \nu_F(\{y \in F | \exists x \in B, (x, y) \in A)\}) + \varepsilon, \tag{3.4.3}$$

$$\text{for every Borel set } B' \subset F,\ \nu_F(B') \leq \nu_E(\{x \in E | \exists y \in B', (x, y) \in A)\}) + \varepsilon. \tag{3.4.4}$$

Condition (3.4.1) allows us to build a metric d over the disjoint union $Z = E \sqcup F$ such that $(x, y) \in A \Rightarrow d(x, y) \leq \varepsilon$, (3.4.2) controls the labels and (3.4.3), (3.4.4) ensure that the Prohorov distance on (Z, d) between ν_E and ν_F is smaller than ε. The main interest of correspondences is to provide a simpler way to compute Gromov distances, with Proposition 3.4.1.

Proposition 3.4.1. *Let (E, d_E, H_E, ν_E), (F, d_F, H_F, ν_F) be compact measured labelled spaces. We have*

$$d_{\text{GHP}}(E, F) = \inf\{\varepsilon > 0 | \exists A \subset E \times F, A \text{ is an } \varepsilon\text{−correspondence between } E \text{ and } F\}.$$

The proof of the proposition still holds for E and F arbitrary measured labelled spaces, but we only defined d_{H} for compacts and d_{GHP} for compact spaces.

Proof. Step 1: suppose that A is an ε-correspondence between E and F, and let us prove that $d_{\text{GHP}}(E, F) \leq \varepsilon$. Build Z the disjoint union of E and F and set $d(x, y)$ the symmetric function on Z^2 defined by

$$d(x, y) = \begin{cases} d_E(x, y) & \text{if } x, y \in E \\ d_F(x, y) & \text{if } x, y \in F \\ \inf_{(x', y') \in A} d_E(x, x') + \varepsilon + d_F(y', y) & \text{if } x \in E, y \in F. \end{cases}$$

The function d is positive-definite. Let us prove the triangular inequality. Take $u, v, w \in Z$, the most difficult case is to prove $d(u, w) \leq d(u, v) + d(v, w)$ for $u, w \in E$ and $v \in F$ or $u, w \in F$ and $v \in E$. Since E and F play symmetric roles we only prove the former:

$$\begin{aligned} d(u, v) + d(v, w) &= \inf_{(x,y) \in A} d_E(u, x) + \varepsilon + d_F(y, v) + \inf_{(x',y') \in A} d_E(w, x') + \varepsilon + d_F(y', v) \\ &= \inf_{(x,y),(x',y') \in A} d_E(u, x) + d_E(w, x') + d_F(y, v) + d_F(v, y') + 2\varepsilon \\ &\geq d_E(u, w) + \inf_{(x,y),(x',y') \in A} (d_F(y, y') - d_E(x, x') + 2\varepsilon) \\ &\geq d(u, w). \end{aligned}$$

We used the triangular inequality of d_E and d_F at the first inequality and the fact that A is an ε-correspondence to conclude. Thus, (Z,d) is a metric space. Let us prove that $d_{\mathrm{GHP}}(E,F) \leq \varepsilon$. We consider $\phi = \mathrm{Id}_E$ and $\phi' = \mathrm{Id}_F$ the isometric eembeddings of E and F into Z. We have, by definition of a correspondence that

$$d_{\mathrm{H}}^{(Z\times\mathbb{R},d^*)}([\mathrm{Id}_E \times H_E](E), [\mathrm{Id}_F \times H_F](F)) = \varepsilon,$$

where d^* is defined as in (3.1.3). By construction of d, we have for every Borel set $B \subset E$ that

$$\{y \in F | \exists x \in B, (x,y) \in A\} \subset \{y \in Z | \exists x \in B, d(x,y) \vee |H(x) - H(y)| \leq \varepsilon\},$$

so by (3.4.3) and (3.4.4) (and symmetry of E and F), we also have

$$d_{\mathrm{P}}^{(Z\times\mathbb{R},d^*)}([\mathrm{Id}_E \times H_E]\nu_E, [\mathrm{Id}_F \times H_F]\nu_F) = \varepsilon.$$

By definition of d_{GHP}, it follows that

$$d_{\mathrm{GHP}}(E,F) \leq \varepsilon.$$

Step 2: suppose that $d_{\mathrm{GHP}}(E,F) < \varepsilon$, and let us build a ε-correspondence between E and F. Take $(Z,d_Z) \in \tilde{\mathbb{M}}$, $\phi \in \mathrm{Iso}(E,Z)$, $\phi' \in \mathrm{Iso}(F,Z)$ such that

$$d_{\mathrm{H}}([\phi \times H_E](E), [\phi' \times H_F](F)) \vee d_{\mathrm{P}}([\phi \times H_E]\nu_E, [\phi' \times H_F]\nu_F) = \delta < \varepsilon$$

and define $A = \{(x,y) \in E \times F | d_Z(\phi(x), \phi'(y)) \vee |H_E(x) - H_F(y)| < \varepsilon\}$. By definition of d_{H}, A is a correspondence between E and F, satisfying (3.4.1) and (3.4.2). For $B \subset E$ a Borel set, we have

$$\begin{aligned}\nu_E(B) &\leq \nu_F(B^\delta) + \delta \\ &= \nu_F(\{y \in F | d(y,B) \leq \delta\}) + \delta \\ &< \nu_F(\{y \in F | \exists x \in B, d(y,x) < \varepsilon\}) + \varepsilon \\ &= \nu_F(\{y \in F | \exists x \in B, (x,y) \in A\}) + \varepsilon.\end{aligned}$$

We have proven Condition (3.4.3). We can see that E and F play symmetric roles, so we similarly have

$$\nu_F(B) \leq \nu_E(\{x \in E | \exists y \in B, (x,y) \in A\}) + \varepsilon.$$

Condition (3.4.4) is satisfied.

We have proven that A is a ε-correspondence and $\varepsilon > d_{\mathrm{GHP}}(E,F)$ was arbitrary, so

$$d_{\mathrm{GHP}}(E,F) \geq \inf\{\varepsilon > 0 | \exists A \subset E \times F, A \text{ is an } \varepsilon\text{−correspondence between } E \text{ and } F\}.$$

Adding the result of Step 1, we have proven the Proposition. □

3.4.2 Properties of $(\mathbb{X}^S, d_{\mathrm{LGHP}})$ and $(\mathbb{X}^K, d_{\mathrm{GHP}})$

In this section, we prove some useful topological results on $(\mathbb{X}^S, d_{\mathrm{LGHP}})$. We will give convergence criterions for d_{LGHP} and compare topologies. Most results from this section follow from the following technical lemma.

Lemma 3.4.2. *Let $(F^k, d_k, H_k, \nu_k)_{k\in\mathbb{N}^*}$ be a sequence of S-compact measured labelled spaces, (E, d, H, ν) a S-compact measured labelled space and $(h_k)_{k\in\mathbb{N}^*}$ a sequence of positive real numbers with $\lim_k h_k = \infty$. For every $k \in \mathbb{N}^*$, let E^k and G^k be two compact sets such that $F^k_{h_k} \subset G^k \subset F^k$ and $E_{h_k} \subset E^k \subset E$. The set G^k (resp. E^k) is equipped with the restrictions of d_k, H_k and ν_k (resp. d, H, ν).*

If we suppose

$$d_{\mathrm{GHP}}(G^k, E^k) \underset{k\to\infty}{\longrightarrow} 0,$$

then we have

$$d_{\mathrm{LGHP}}(F^k, E) \underset{k\to\infty}{\longrightarrow} 0.$$

Recall the construction model of d^* given in Equation (3.1.3).

Proof. For $\varepsilon \in (0, 1)$, take $h_{\max} = -\log(\varepsilon)$. Using Lemma 3.3.2 on the compact slice $E_{h_{\max}}$, there exists a finite set $X \subset E_{h_{\max}}$ equipped with d, H and some measure ν_X such that for every $h \in [0, h_{\max}]$, $d_{\mathrm{GHP}}(X_h, E_h) \le \varepsilon$. We set $R = \{|H(x)|\}_{x\in X} \subset [0, h_{\max}]$. Now, take $k \in \mathbb{N}^*$ such that $h_k \ge h_{\max}$ and $\delta_k = 2d_{\mathrm{GHP}}(G^k, E^k)$. By definition of d_{GHP}, we can choose $(Z, d_Z) \in \tilde{\mathbb{M}}$, $\phi \in \mathrm{Iso}(E^k, Z)$, $\phi_k \in \mathrm{Iso}(G^k, Z)$ such that

$$d_{\mathrm{H}} \vee d_{\mathrm{P}}([\phi_k \times H_k](G^k), [\phi \times H](E^k)) \le \delta_k. \tag{3.4.5}$$

We will give an upper bound on $d_{\mathrm{LGHP}}(F^k, E)$ depending only on ε, X and δ_k, then immediately use it to conclude.

Step 1: We prove that for $h \in [\delta_k, h_{\max} - \delta_k] \setminus (R)^{\delta_k}$, we have

$$d_{\mathrm{H}}([\phi_k \times H_k](G^k_h), [\phi \times H](E^k_h)) \le 2\varepsilon + \delta_k.$$

For every $h \le h_{\max} - \delta_k$, we have, like in Equation (3.3.3):

$$[\phi \times H](E^k_h) \subset \Big([\phi_k \times H_k](G^k_{h+\delta_k})\Big)^{\delta_k} \quad \text{and} \quad [\phi_k \times H_k](G^k_h) \subset \Big([\phi \times H](E^k_{h+\delta_k})\Big)^{\delta_k}. \tag{3.4.6}$$

For every $h \in [\delta_k, h_{\max} - \delta_k] \setminus (R)^{\delta_k}$, since $d_H(X_h, E_h) \le \varepsilon$, it is enough to prove that $d_{\mathrm{H}}([\phi_k \times H_k](G^k_h), [\phi \times H](X_h)) \le \varepsilon + \delta_k$ and $E_h = E^k_h$. Since $h \notin (R_\varepsilon)^{\delta_k}$, we have $|h - |H(x)|| > \delta_k$ for every $x \in X$, so

$$\forall x \in X,\ \big(\ |H(x)| \le h - \delta_k \Leftrightarrow |H(x)| \le h \Leftrightarrow |H(x)| \le h + \delta_k\ \big),$$

i.e.

$$X_{h-\delta_k} = X_h = X_{h+\delta_k}. \tag{3.4.7}$$

Since $E_{h_k} \subset E^k$ and $h_k \geq h + \delta_k$, $E^k_{h+\delta_k} = E_{h+\delta_k}$, so $d_{\mathrm{H}}(X_{h+\delta_k}, E^k_{h+\delta_k}) \leq \varepsilon$ and $X_h \subset E_h = E^k_h$. Thus, we have

$$\begin{aligned}
[\phi_k \times H_k](G^k_h) &\subset \Big([\phi \times H](E^k_{h+\delta_k})\Big)^{\delta_k} \\
&\subset \Big([\phi \times H](X_{h+\delta_k})\Big)^{\varepsilon+\delta_k} \\
&= \Big([\phi \times H](X_h)\Big)^{\varepsilon+\delta_k} \\
&\subset \Big([\phi \times H](E^k_h)\Big)^{\varepsilon+\delta_k}.
\end{aligned}$$

We used $h + \delta_k \leq h_{\max}$ and the right-hand of (3.4.6) for the first inclusion. The equality comes from the right hand of (3.4.7). Similarly,

$$\begin{aligned}
[\phi \times H](E^k_h) \subset&\Big([\phi \times H](X_h)\Big)^{\varepsilon} \\
=&\Big([\phi \times H](X_{h-\delta_k})\Big)^{\varepsilon} \\
\subset&\Big([\phi \times H](E^k_{h-\delta_k})\Big)^{\varepsilon} \\
\subset&\Big([\phi_k \times H_k](G^k_h)\Big)^{\varepsilon+\delta_k},
\end{aligned}$$

so $d_{\mathrm{H}}([\phi_k \times H_k](G^k_h), [\phi \times H](X_h)) \leq \varepsilon + \delta_k$.

Step 2: We prove that for $h \in [\delta_k, h_{\max} - \delta_k] \setminus (R)^{\delta_k}$, we have

$$d_{\mathrm{P}}([\phi \times H](1_{E^k_h} \cdot \nu), [\phi_k \times H_k](1_{G^k_h} \cdot \nu_k)) \leq 4\varepsilon + 2\delta_k.$$

For $h \in [0, h_{\max} - \delta_k]$, we have $d_{\mathrm{P}}\big([\phi \times H](1_{E^k} \cdot \nu), [\phi_k \times H_k](1_{G^k} \cdot \nu_k)\big) \leq \delta_k$, so using Lemma 3.1.6, we get

$$d_{\mathrm{P}}([\phi \times H](1_{E^k_{h+\delta_k}} \cdot \nu), [\phi_k \times H_k](1_{G^k_h} \cdot \nu_k)) \leq \delta_k + (\nu(E^k_{h+\delta_k}) - \nu_k(G^k_h))^+. \tag{3.4.8}$$

We have $h \in [\delta_k, h_{\max}]$ and the Prohorov control from (3.4.5). Additionally, we have

$$([\phi \times H](E^k_{h-\delta_k}))^{\delta_k} \cap [\phi_k \times H_k](G^k) \subset [\phi_k \times H_k](G^k_h),$$

so we can deduce that

$$\begin{aligned}
\nu(E^k_{h-\delta_k}) =&\Big[[\phi \times H](1_{E^k} \cdot \nu)\Big]\Big([\phi \times H](E^k_{h-\delta_k})\Big) \\
\leq&\Big[[\phi_k \times H_k](1_{G^k} \cdot \nu_k)\Big]\Big(([\phi \times H](E^k_{h-\delta_k}))^{\delta_k}\Big) + \delta_k \\
\leq&\Big[[\phi_k \times H_k](1_{G^k} \cdot \nu_k)\Big]\Big([\phi_k \times H_k](G^k_h)\Big) + \delta_k \\
=&\nu_k(G^k_h) + \delta_k.
\end{aligned}$$

Using this, we can rewrite Equation (3.4.8):

$$d_{\mathrm{P}}([\phi \times H](1_{E^k_{h+\delta_k}} \cdot \nu), [\phi_k \times H_k](1_{G^k_h} \cdot \nu_k)) \leq 2\delta_k + (\nu(E^k_{h+\delta_k}) - \nu(E^k_{h-\delta_k}))^+. \tag{3.4.9}$$

Since $d_{\mathrm{P}}(1_{E_h}\cdot\nu, 1_{X_h}\cdot\nu_X)\le\varepsilon$ for $h\in[0,h_{\max}]$, we have a control on the total masses:

$$|\nu(E_h)-\nu_X(X_h)|\le\varepsilon.$$

For $h\in[\delta_k,h_{\max}-\delta_k]\setminus(R)^{\delta_k}$, recall that $E_h=E_h^k$. This allows us to use Equation (3.4.7) to further simplify Equation (3.4.9).

$$\nu(E^k_{h+\delta_k})-\nu(E^k_{h-\delta_k})\le\nu_X(X^k_{h+\delta_k})-\nu_X(X^k_{h-\delta_k})+2\varepsilon=2\varepsilon. \tag{3.4.10}$$

With this, Equation (3.4.9) finally gives

$$d_{\mathrm{P}}([\phi\times H](1_{E^k_{h+\delta_k}}\cdot\nu),[\phi_k\times H_k](1_{G^k_h}\cdot\nu_k))\le 2\varepsilon+2\delta_k. \tag{3.4.11}$$

With the triangular inequality of d_{P}, we have

$$\begin{aligned}
d_{\mathrm{P}}([\phi\times H](1_{E^k_h}\cdot\nu),[\phi_k\times H_k](1_{G^k_h}\cdot\nu_k))&\\
\le& d_{\mathrm{P}}([\phi\times H](1_{E^k_h}\cdot\nu),[\phi\times H](1_{E_{h+\delta_k}}\cdot\nu))\\
&+d_{\mathrm{P}}([\phi\times H](1_{E^k_{h+\delta_k}}\cdot\nu),[\phi_k\times H_k](1_{G^k_h}\cdot\nu_k))\\
\le&\nu(E^k_{h+\delta_k})-\nu(E^k_h)+2\varepsilon+2\delta_k\\
\le&\nu(E^k_{h+\delta_k})-\nu(E^k_{h-\delta_k})+2\varepsilon+2\delta_k\\
\le&4\varepsilon+2\delta_k.
\end{aligned}$$

For the second inequality, we used Equation (1.1.2) on the first term and Equation (3.4.11) on the second term. We used (3.4.10) for the last inequality.

Step 3: conclusion. Combining Step 1 and 2, we have proven that for every $h\in[\delta_k,h_{\max}]\setminus(R)^{\delta_k}$,

$$d_{\mathrm{GHP}}(G^k_h,E^k_h)\le 4\varepsilon+2\delta_k.$$

Recall that for $h\in[0,h_{\max}]$ and $h_k\ge h_{\max}$, we have $E_h\subset E^k$ and $F^k_h\subset G^k$, so $E_h=E^k_h$ and $F^k_h=G^k_h$. When we set $n=\#(X)$, we obtain:

$$\begin{aligned}
d_{\mathrm{LGHP}}(F^k,E)&=\int_0^\infty(1\wedge d_{\mathrm{GHP}}(F^k_h,E_h)\mathrm{e}^{-h}dh\\
&\le\int_{[\delta_k,h_{\max}-\delta_k]\setminus(R)^{\delta_k}}d_{\mathrm{GHP}}(G^k_h,E^k_h)\mathrm{e}^{-h}dh+\int_{(R\cup\{0\})^{\delta_k}\cup[h_{\max}-\delta_k,\infty)}\mathrm{e}^{-h}dh\\
&\le 4\varepsilon+2\delta_k+\int_{[0,\delta_k)\cup(R)^{\delta_k}\cup[h_{\max}-\delta_k,h_{\max}]}dh+\mathrm{e}^{-h_{\max}}\\
&\le 4\varepsilon+2\delta_k+2(n+1)\delta_k+\varepsilon\\
&=5\varepsilon+2(n+2)\delta_k.
\end{aligned}$$

With our choice of $h_{\max}$ and k, we have proven that for every k such that $h_k\ge-\log(\varepsilon)$, $d_{\mathrm{LGHP}}(E,F^k)\le5\varepsilon+2(n+2)\delta_k$. Since n depends only on X and X depends only on E and ε, we have

$$\limsup_{k\to\infty}d_{\mathrm{LGHP}}(E,F^k)\le5\varepsilon.$$

This concludes the proof since ε was arbitrary. □

Lemma 3.4.2 has many corollaries:

Lemma 3.4.3. *For a sequence $(F^k)_{k\in\mathbb{N}^*}$ of S-compact measured labelled metric spaces, and a S-compact measured labelled metric space E, we have the convergence*

$$\lim_{k\to\infty} d_{\mathrm{LGHP}}(F^k, E) = 0$$

if and only if there exists a sequence $(h_k)_{k\in\mathbb{N}^*}$ of positive real numbers with $\lim_k h_k = \infty$ such that

$$\lim_{k\to\infty} d_{\mathrm{GHP}}(F^k_{h_k}, E_{h_k}) = 0.$$

Proof. For the direct sense, set for every $k \in \mathbb{N}^*$ $\delta_k = d_{\mathrm{LGHP}}(F^k, E)$ and $h_k = -\frac{1}{2}\log(1 \wedge \delta_k)$, so that we have $\mathrm{e}^{-h_k} = 1 \wedge (\delta_k)^{\frac{1}{2}}$. For every k such that $\delta_k < 1$, we have $d_{\mathrm{LGHP}}(F^k, E) = (\delta_k)^{\frac{1}{2}}\mathrm{e}^{-h_k}$. By Definition 3.1.12, there exists $h'_k \geq h_k$ such that $d_{\mathrm{GHP}}(F^k_{h'_k}, E_{h'_k}) \leq (\delta_k)^{\frac{1}{2}}$. We have $\lim_k h'_k = \infty$ and

$$\lim_{k\to\infty} d_{\mathrm{GHP}}(F^k_{h'_k}, E_{h'_k}) \leq \lim_{k\to\infty} (\delta_k)^{\frac{1}{2}} = 0.$$

The converse sense is a special case of Lemma 3.4.2 with $G^k = F^k_{h_k}$, $E^k = E_{h_k}$. □

Proposition 3.4.4. *On $\mathbb{X}^K$, the topology induced by d_{GHP} is strictly finer than the topology induced by d_{LGHP}.*

Proof. Step 1: we prove that on $\mathbb{X}^K$, the topology induced by d_{GHP} is finer than the topology induced by d_{LGHP}. Since our topologies are defined by distances, we can compare them through their converging sequences. Take $(K^k, d_k, H_k, \nu_k)_{k\in\mathbb{N}^*}$ a sequence of compact measured labelled spaces converging for d_{GHP} to a compact measured labelled space (K, d, H, ν). Since H and all the H_k are continuous and defined on compact sets, they are bounded. From the convergence for d_{GHP}, we deduce that

$$\lim_{k\to\infty}\left(\sup_{K^k}|H_k|\right) = \sup_K |H|.$$

Since a converging sequence of real numbers is always bounded, there exists $h \in \mathbb{R}_+$ such that for every $k \in \mathbb{N}^*$, $|H_k|$ is bounded by h. We have for every $k \in \mathbb{N}^*$ that

$$K^k = K^k_h = K^k_{h+k} \quad ; \quad K = K_h = K_{h+k},$$

so

$$d_{\mathrm{GHP}}(K^k_{h+k}, K_{h+k}) = d_{\mathrm{GHP}}(K^k, K) \underset{k\to\infty}{\longrightarrow} 0.$$

With Lemma 3.4.3, we find that $(K^k)_{k\in\mathbb{N}^*}$ also converges to K for d_{LGHP}. Since K and $(K^k)_{k\in\mathbb{N}^*}$ were arbitrary, we find that d_{GHP} defines a finer topology than d_{LGHP}.

Step 2: we prove that on $\mathbb{X}^K$, the topology induced by d_{GHP} and d_{LGHP} are different. We only need to find a sequence that converges for d_{LGHP} but not for d_{GHP}. Take $d_{\mathbb{R}}$ the usual distance on $\mathbb{R}$ and consider the sequence

$$(F^k, d_k, H_k, \nu_k)_{k\in\mathbb{N}} = (\{0, k\}, d_{\mathbb{R}}, \mathrm{Id}, 0)_{k\in\mathbb{N}^*}.$$

The sequence $(F^k)_{k\in\mathbb{N}^*}$ converges to F^0 for d_{LGHP} since we have $d_{\mathrm{LGHP}}(F^k,F^0)=e^{-k}$, but is not a Cauchy sequence for d_{GHP} since we have in this case

$$d_{\mathrm{GHP}}(F^k,F^{k'})\geq |\max_{F^k} H_k-\max_{F^{k'}} H_{k'}|=|k-k'|.$$

□

When we define trees in Chapter 4, we will find that d_{GHP} and d_{LGHP} are topologically equivalent on the set of non-empty compact trees. Here, we give the proof in a more general setting.

We define $\mathbb{X}^C$ the set of measured labelled spaces (E,d,H,ν), up to equivalence, such that $H(E)\subset\mathbb{R}$ is connected (that is, an interval). We also introduce $\mathbb{X}^{C,K}\subset\mathbb{X}^{C,S}\subset\mathbb{X}^C$ the restrictions of $\mathbb{X}^C$ to compact and S-compact spaces respectively.

Lemma 3.4.5. *The set $\mathbb{X}^{C,K}\setminus\{\emptyset\}$ is open in $(\mathbb{X}^{C,S},d_{\mathrm{LGHP}})$, and on $\mathbb{X}^{C,K}\setminus\{\emptyset\}$, d_{GHP} and d_{LGHP} induce the same topology.*

Proof. Since $\mathbb{X}^{C,K}\setminus\{\emptyset\}\subset\mathbb{X}^K$, we already know from Proposition 3.4.4 that the topology defined by d_{GHP} is finer. Take some arbitrary (K,d,H,ν) in $\mathbb{X}^{C,K}\setminus\{\emptyset\}$. For every $\varepsilon>0$, set

$$B_{\mathrm{GHP}}(\varepsilon)=\{K'\in\mathbb{X}^{C,K}|d_{\mathrm{GHP}}(K,K')<\varepsilon\}\ \ ;\ \ B_{\mathrm{LGHP}}(\varepsilon)=\{K'\in\mathbb{X}^{C,S}|d_{\mathrm{LGHP}}(K,K')<\varepsilon\}.$$

Note that, by convention in Definition 3.1.8, $d_{\mathrm{GHP}}(K,\emptyset)=\infty$, so $B_{\mathrm{GHP}}(\varepsilon)\subset\mathbb{X}^{C,K}\setminus\{\emptyset\}$. Set $h_0=\max_K|H|$. To prove that the topologies are equal and that $\mathbb{X}^{C,K}\setminus\emptyset$ is open in $(\mathbb{X}^{C,S},d_{\mathrm{LGHP}})$, it is enough to prove that for $\varepsilon\in(0,1)$, $B_{\mathrm{LGHP}}(\varepsilon\mathrm{e}^{-h_0-\varepsilon})\subset B_{\mathrm{GHP}}(\varepsilon)$.

Take $(K',d',H',\nu')\in B_{\mathrm{LGHP}}(\varepsilon\mathrm{e}^{-h_0-\varepsilon})$. By Definition 3.1.12, there exists $h\geq h_0+\varepsilon$ such that $d_{\mathrm{GHP}}(K_h,K'_h)<\varepsilon$. We can choose $(Z,d_Z)\in\tilde{\mathbb{M}}$, $\phi\in\mathrm{Iso}(K,Z)$, $\phi'\in\mathrm{Iso}(K',Z)$ such that

$$d_{\mathrm{H}}\vee d_{\mathrm{P}}([\phi\times H](K_h),[\phi'\times H'](K'_h))<\varepsilon. \tag{3.4.12}$$

By choice of h we have $K_h=K\neq\emptyset$. Since $d_{\mathrm{GHP}}(K_h,K'_h)<\varepsilon<\infty$, K'_h is non-empty as well. We have

$$\sup_{K'_h}|H'|<\sup_{K_h}|H|+\varepsilon=\sup_K|H|+\varepsilon=h_0+\varepsilon\leq h.$$

We have $\sup_{K'_h}|H'|<h$ and $H'(K')$ is an interval, so $\sup_{K'}|H'|=\sup_{K'_h}|H'|<h$. It follows that we have $K_h=K$ and $K'_h=K'$. Since K' is S-compact, $K'=K'_h$ is compact and we can rewrite Equation (3.4.12) as

$$d_{\mathrm{H}}\vee d_{\mathrm{P}}([\phi\times H](K),[\phi'\times H'](K'))<\varepsilon,$$

which proves that $d_{\mathrm{GHP}}(K,K')<\varepsilon$. We have $B_{\mathrm{LGHP}}(\varepsilon\mathrm{e}^{-h_0-\varepsilon})\subset B_{\mathrm{GHP}}(\varepsilon)$. □

When we defined d_{LGHP}, we distinguished the label 0, and we can ask our-self whether this has any topological implication. We prove in Proposition 3.4.6 that it doesn't.

For $a\in\mathbb{R}$ and (E,d,H,ν), F a S-compact measured labelled space, define

$$\mathrm{Slice}^a_h=\{x\in E||H(x)-a|\leq h\},$$

equipped with the restrictions of d, H and ν. We define the distance d^a_{LGHP} on $\mathbb{X}^S$ by

$$d^a_{\mathrm{LGHP}}(E,F)=\int_0^\infty \left(1\wedge d_{\mathrm{GHP}}(\mathrm{Slice}^a_h(E),\mathrm{Slice}^a_h(F))\right)\mathrm{e}^{-h}dh.$$

For every $a\in\mathbb{R}$, define the application Φ_a from $\mathbb{X}^S$ to itself with

$$\Phi_a(E,d,H,\nu)=(E,d,H+a,\nu),$$

where $H+a$ represents the map $x\mapsto H(x)+a$. Recalling Definition 3.1.9, note that $E\mapsto \Phi_a(E)$ is an isometry on $(\mathbb{X}^K,d_{\mathrm{GHP}})$. We have $d^0_{\mathrm{LGHP}}=d_{\mathrm{LGHP}}$ and for every $a,b\in\mathbb{R}$, Φ_a is a bijective isometry from $(\mathbb{X}^S,d^b_{\mathrm{LGHP}})$ to $(\mathbb{X}^S,d^{b+a}_{\mathrm{LGHP}})$.

Proposition 3.4.6. *For every $a\in\mathbb{R}$, the application Φ_a is continuous from $(\mathbb{X}^S,d_{\mathrm{LGHP}})$ to itself. Furthermore, d_{LGHP} and d^a_{LGHP} define the same topology on $\mathbb{X}^S$.*

Proof. Step 1: continuity. Take $(F^k,d_k,H_k,\nu_k)_{k\in\mathbb{N}^*}$ a sequence of S-compact measured labelled spaces converging to some S-compact measured labelled space (E,d,H,ν) for d_{LGHP}. Using Lemma 3.4.3, there exists a sequence $(h_k)_{k\in\mathbb{N}^*}$ such that $\lim_k h_k=\infty$ and such that $\lim_k d_{\mathrm{GHP}}(F^k_{h_k},E_{h_k})=0$. Take $k_0\in\mathbb{N}^*$ such that for every $k\geq k_0$, $h_k\geq|a|$. Set, for $k\geq k_0$, $G^k=\Phi_a(F^k_{h_k})$, $E^k=\Phi_a(E_{h_k})$ and $h'_k=h_k-|a|$. Since Φ_a doesn't affect the metric of its argument, G^k and E^k are compact as images of the compact sets $F^k_{h_k}$ and E_{h_k}. The map Φ_a preserves d_{GHP}, so $d_{\mathrm{GHP}}(G^k,E^k)=d_{\mathrm{GHP}}(F^k_{h_k},E_{h_k})$, and thus $\lim_k d_{\mathrm{GHP}}(G^k,E^k)=0$. We have $\lim_k h'_k=\infty$,

$$\mathrm{Slice}_{h'_k}(\Phi_a(F^k))\subset G^k\subset\Phi_a(F^k),\qquad \mathrm{Slice}_{h'_k}(\Phi_a(E))\subset E^k\subset\Phi_a(E),$$

so we can apply Lemma 3.4.2 to get that

$$d_{\mathrm{LGHP}}(\Phi_a(F^k),\Phi_a(E))\underset{k\to\infty}{\longrightarrow}0.$$

This means that for every a, Φ_a is continuous from $(\mathbb{X}^S,d_{\mathrm{LGHP}})$ to itself.

Step 2: equivalence of topologies. Let us prove that for every $a,b\in\mathbb{R}$ d^a_{LGHP} and d^b_{LGHP} define the same topology. Take U an open set of $(\mathbb{X}^S,d^a_{\mathrm{LGHP}})$. Since Φ_{b-a} is a bijective isometry from $(\mathbb{X}^S,d^a_{\mathrm{LGHP}})$ to $(\mathbb{X}^S,d^b_{\mathrm{LGHP}})$, it is bi-continuous and thus the direct image $U'=\Phi_{b-a}(U)$ is an open set of $(\mathbb{X}^S,d^b_{\mathrm{LGHP}})$. As seen in Figure 3.7, for every $b\in\mathbb{R}$, Φ_a is continuous from $(\mathbb{X}^S,d^b_{\mathrm{LGHP}})$ to itself, so $U=(\Phi_{b-a})^{-1}(U')$ is still an open set of $(\mathbb{X}^S,d^b_{\mathrm{LGHP}})$. Since a and b are arbitrary, they play symmetric roles, so $(\mathbb{X}^S,d^a_{\mathrm{LGHP}})$ and $(\mathbb{X}^S,d^b_{\mathrm{LGHP}})$ have the same topology. In particular, d^a_{LGHP} induces the same topology as $d^0_{\mathrm{LGHP}}=d_{\mathrm{LGHP}}$. □

3.4.3 Some closed sets of $\mathbb{X}^S$

In the next chapter, we will talk about trees as particular elements of $\mathbb{X}^S$. We would like to know that the space of trees is a closed set. Since trees can be characterized by the so-called four points condition and the (exact) middle-point condition, one way to prove that the set of trees is closed in $\mathbb{X}^S$ would be to prove for each condition that the set of spaces satisfying the condition is closed, that is

$$F_{4-points}=\left\{(E,d,H,\nu)\in\mathbb{X}^S\;\middle|\;\begin{array}{l}\forall x_1,\dots,x_4\in E,\\ d(x_1,x_2)+d(x_3,x_4)\\ \leq(d(x_1,x_3)+d(x_2,x_4))\vee(d(x_1,x_4)+d(x_2,x_3))\end{array}\right\}\tag{3.4.13}$$

$$\begin{array}{ccc} (\mathbb{X}^S, d^b_{\mathrm{LGHP}}) & \xrightarrow{\Phi_a} & (\mathbb{X}^S, d^b_{\mathrm{LGHP}}) \\ \Phi_{-b} \downarrow & & \uparrow \Phi_b \\ (\mathbb{X}^S, d_{\mathrm{LGHP}}) & \xrightarrow{\Phi_a} & (\mathbb{X}^S, d_{\mathrm{LGHP}}) \end{array}$$

Figure 3.7: Here, Φ_b and Φ_{-b} are isometries so continuous, while Φ_a is continuous on $(\mathbb{X}^S, d_{\mathrm{LGHP}})$. Since the diagram commutes, the top arrow $\Phi_a = \Phi_b \circ \Phi_a \circ \Phi_{-b}$ is continuous from $(\mathbb{X}^S, d^b_{\mathrm{LGHP}})$ to itself.

and

$$F_{geo} = \{(E, d, H, \nu) \in \mathbb{X}^S | \forall x_1, x_2 \in E, \exists x_3 \in E, d(x_1, x_3) = d(x_3, x_2) = \frac{1}{2} d(x_1, x_2)\} \quad (3.4.14)$$

are closed. Lastly, to exclude the empty set from our closed sets, we need to check that the set of measured labelled spaces (E, d, H, ν) such that E_h is non-empty is a closed set, that is

$$F_{\exists} = \left\{(E, d, H, \nu) \in \mathbb{X}^S \middle| \exists x_1 \in E, |H(x_1)| \leq h\right\}. \quad (3.4.15)$$

We will prove in three lemmas that well-chosen generalizations of $F_{4-points}$, F_{geo} and $F_{\exists}$ are always closed in $(\mathbb{X}^S, d_{\mathrm{LGHP}})$.

To generalize closed conditions on the distance between points of E and their labels, we introduce a function M^n_E to reduce any n-uple of points of E to their "usable" characteristics. This function resembles (except on the diagonal) those used in [39] and [36] to define the Gromov-weak topology. We note $\mathcal{M}_n(\mathbb{R})$ the set of square real matrices of size n, equipped with the norm $||\cdot||_\infty$. For $n \in \mathbb{N}^*$, we set $M^n_\emptyset$ the empty function. For every non-empty measured labelled metric space (E, d, H, ν), $n \in \mathbb{N}^*$ and $x_1, ..., x_n \in E$, we set

$$M^n_E(x_1, ..., x_n) = \begin{pmatrix} H(x_1) & \frac{1}{2}d(x_1, x_2) & \cdots & \frac{1}{2}d(x_1, x_n) \\ \frac{1}{2}d(x_2, x_1) & H(x_2) & \ddots & \vdots \\ \vdots & \ddots & \ddots & \frac{1}{2}d(x_{n-1}, x_n) \\ \frac{1}{2}d(x_n, x_1) & \cdots & \frac{1}{2}d(x_n, x_{n-1}) & H(x_n) \end{pmatrix}.$$

This first lemma generalizes the example of F_{geo} to other sets with conditions of the type "$\forall\exists$". We find in the proof that a control (f in the lemma) on the height is paramount in the "$\exists$" part to have a closed set.

Lemma 3.4.7. *For every $n, p \in \mathbb{N}^*$, f a continuous function from $\mathcal{M}_n(\mathbb{R})$ to $\mathbb{R}_+$ and $F \subset \mathcal{M}_{n+p}(\mathbb{R})$ a closed set,*

$$A = \left\{ (E, d, H, \nu) \in \mathbb{X}^S \middle| \begin{array}{l} \forall x_1, ..., x_n \in E, \\ \exists x_{n+1}, ..., x_{n+p} \in E_{f(M_E^n(x_1, ... x_n))}, \\ M_E^{n+p}(x_1, ..., x_{n+p}) \in F \end{array} \right\}$$

is a closed set of $(\mathbb{X}^S, d_{\mathrm{LGHP}})$.

In less formal and more legible terms,

$$\{(E, d, H, \nu) \in \mathbb{X}^S | \forall x_1, ..., x_n \in E, \exists x_{n+1}, ..., x_{n+p} \in E_{h(x_1, ..., x_n)}, g(x_1, ..., x_{n+p}) = 0\}$$

is a closed set of $(\mathbb{X}^S, d_{\mathrm{LGHP}})$ if $h(x_1, ..., x_n)$ is a continuous function of $(H(x_i))_{1 \leq i \leq n}$ and of $(d(x_i, x_j))_{1 \leq i < j \leq n}$ and $g(x_1, ..., x_{n+p})$ is a continuous function of $(H(x_i))_{1 \leq i \leq n+p}$ and of $(d(x_i, x_j))_{1 \leq i < j \leq n+p}$.

Remark 3.4.8. As an example, we can apply the lemma to F_{geo} from Equation (3.4.14) to prove that it is closed. Since H is 1-Lipschitz, $d(x_1, x_3) = \frac{1}{2} d(x_1, x_2) \Rightarrow |H(x_3)| \leq |H(x_1)| + \frac{1}{2} d(x_1, x_2)$, so

$$F_{geo} = \left\{ (E, d, H, \nu) \in \mathbb{X}^S \middle| \begin{array}{l} \forall x_1, x_2 \in E, \\ \exists x_3 \in E_{|H(x_1)| + \frac{1}{2} d(x_1, x_2)}, \\ d(x_1, x_3) = d(x_3, x_2) = \frac{1}{2} d(x_1, x_2) \end{array} \right\}.$$

With this new expression, we can apply Lemma 3.4.7, with $f((a_{i,j})_{1 \leq i,j \leq 2}) = |a_{1,1}| + \frac{1}{2} a_{1,2}$ and the closed set of $M_3(\mathbb{R})$:

$$F = \left\{ ((a_{i,j})_{1 \leq i,j \leq 3} \in M_3(\mathbb{R}) \middle| a_{1,3} = a_{2,3} = \frac{1}{2} a_{1,2} \right\}.$$

Equivalently, we can see it as

$$F_{geo} = \{(E, d, H, \nu) \in \mathbb{X}^S | \forall x_1, ..., x_n \in E, \exists x_{n+1}, ..., x_{n+p} \in E_{h(x_1, ..., x_n)}, g(x_1, ..., x_{n+p}) = 0\}$$

with $h(x_1, x_2) = |H(x_1)| + \frac{1}{2} d(x_1, x_2)$ and

$$g(x_1, x_2, x_3) = \left| d(x_1, x_3) - \frac{1}{2} d(x_1, x_2) \right| + \left| d(x_2, x_3) - \frac{1}{2} d(x_1, x_2) \right|.$$

Proof. We have $\emptyset \in A$, so A is non-empty. Take (E^k, d_k, H_k, ν_k) a sequence of elements of A converging for d_{LGHP} to some S-compact measured labelled space (E, d, H, ν). Let us prove that $E \in A$. If E is empty, then we have $E \in A$. We suppose that E is non-empty in the remaining of the proof. Choose $x_1, ..., x_n \in E$, $\varepsilon > 0$, and set $h_0 = \max(|H(x_1)|, ..., |H(x_n)|, f(M_E^n(x_1, ..., x_n)))$. Since f is continuous, there exists a radius $\delta \in (0, \varepsilon)$ such that for every $M \in \mathcal{M}_n(\mathbb{R})$,

$$(||M - M_E^n(x_1, ..., x_n)||_\infty \leq \delta) \Rightarrow (|f(M) - f(M_E^n(x_1, ..., x_n))| \leq \varepsilon). \tag{3.4.16}$$

Now, take $k \in \mathbb{N}^*$ such that $d_{\mathrm{LGHP}}(E^k, E) < \delta \mathrm{e}^{-h_0-\varepsilon}$. Since $\int_{h_0+\varepsilon}^{\infty} \mathrm{e}^{-h} dh = \mathrm{e}^{-h_0-\varepsilon}$, we can choose $h \geq h_0+\varepsilon$ such that $d_{\mathrm{GHP}}(E_h^k, E_h) < \delta$. Thus, there exists $(Z, d_Z) \in \tilde{\mathbb{M}}$, $\phi \in \mathrm{Iso}(E_h, Z)$, $\phi_k \in \mathrm{Iso}(E_h^k, Z)$ such that

$$d_{\mathrm{H}} \vee d_{\mathrm{P}} \Big([\phi_k \times H_k](E_h^k), [\phi \times H](E_h) \Big) \leq \delta.$$

Since $x_1, ..., x_n \in E_h$, there exists $x_1^k, ..., x_n^k \in E_h^k$ such that for every $1 \leq i \leq n$ we have $d_Z(\phi_k(x_i^k), \phi(x_i)) \vee |H_k(x_i^k) - H(x_i)| \leq \delta$. This yields, for $1 \leq i < j \leq n$:

$$\begin{aligned} \left| \frac{1}{2} d_k(x_i^k, x_j^k) - \frac{1}{2} d(x_i, x_j) \right| &= \left| \frac{1}{2} d_Z(\phi_k(x_i^k), \phi_k(x_j^k)) - \frac{1}{2} d_Z(\phi(x_i), \phi(x_j)) \right| \\ &\leq \frac{1}{2} d_Z(\phi_k(x_i^k), \phi(x_i)) + \frac{1}{2} d_Z(\phi_k(x_j^k), \phi(x_j)) \\ &\leq \delta. \end{aligned}$$

It follows that $||M_{E_k}^n(x_1^k, ..., x_n^k) - M_E^n(x_1, ..., x_n)||_\infty \leq \delta$, so, according to (3.4.16), we have

$$f(M_{E_k}^n(x_1^k, ..., x_n^k)) \leq f(M_E^n(x_1, ..., x_n)) + \varepsilon \leq h_0 + \varepsilon \leq h.$$

Since $E^k \in A$, there exists $x_{n+1}^k, ..., x_{n+p}^k \in E^k_{f(M^n_{E^k}(x_1^k, ... x_n^k))} \subset E_h^k$ such that

$$M_{E^k}^{n+p}(x_1^k, ..., x_{n+p}^k) \in F.$$

We can choose $x_{n+1}(\varepsilon), ..., x_{n+p}(\varepsilon) \in E_h$ such that for $n < i \leq n+p$ we have

$$d_Z(\phi_k(x_i^k), \phi(x_i(\varepsilon))) \vee |H_k(x_i^k) - H(x_i(\varepsilon))| \leq \delta \leq \varepsilon.$$

We have $x_{n+1}(\varepsilon), ..., x_{n+p}(\varepsilon) \in E_{f(M_E^n(x_1, ..., x_n))+2\varepsilon}$ and we have

$$||M_{E^k}^{n+p}(x_1^k, ..., x_{n+p}^k) - M_E^{n+p}(x_1, ..., x_n, x_{n+1}(\varepsilon), ..., x_{n+p}(\varepsilon))||_\infty \leq \delta \leq \varepsilon.$$

Since $M_{E^k}^{n+p}(x_1^k, ..., x_{n+p}^k) \in F$, the distance between $M_E^{n+p}(x_1, ..., x_n, x_{n+1}(\varepsilon), ..., x_{n+p}(\varepsilon))$ and F is less than ε.

Since ε was arbitrary, there exists a sequence $(x_{n+1}(\frac{1}{k}), ..., x_{n+p}(\frac{1}{k}))_{k \in \mathbb{N}^*}$ such that for every $k \in \mathbb{N}^*$, $x_{n+1}(\frac{1}{k}), ..., x_{n+p}(\frac{1}{k}) \in E_{f(M_E^n(x_1, ..., x_n)) + \frac{2}{k}} \subset E_{h_0+2}$ and the distance between $M_E^{n+p}(x_1, ..., x_n, x_{n+1}(\frac{1}{k}), ..., x_{n+p}(\frac{1}{k}))$ and F is less than $\frac{1}{k}$. The space E_{h_0+1} is compact, so we can choose a sub-sequence of $(x_{n+1}(\frac{1}{k}), ..., x_{n+p}(\frac{1}{k}))_{k \in \mathbb{N}^*}$ converging to some $x_{n+1}, ..., x_{n+p}$. By continuity of H and M_E^{n+p}, the distance between $M_E^{n+p}(x_1, ..., x_{n+p})$ and F is 0, so $M_E^{n+p}(x_1, ..., x_{n+p}) \in F$ and $x_{n+1}, ..., x_{n+p} \in E_{f(M_E^n(x_1, ..., x_n))}$. Since $x_1, ..., x_n \in E$ were arbitrary, we have $E \in A$ and A is closed by sequential characterization. □

Lemma 3.4.9. *For every $n \in \mathbb{N}^*$ and $F \subset \mathcal{M}_n(\mathbb{R})$ a closed set,*

$$A = \{(E, d, H, \nu) \in \mathbb{X}^S | \forall x_1, ..., x_n \in E, M_E^n(x_1, ..., x_n) \in F\}$$

is a closed set of $(\mathbb{X}^S, d_{\mathrm{LGHP}})$.

In less formal and more legible terms,

$$\{(E,d,H,\nu)\in\mathbb{X}^S|\forall x_1,...,x_n\in E, g(x_1,...,x_n)=0\}$$

is a closed set of $(\mathbb{X}^S, d_{\text{LGHP}})$ if $g(x_1,...,x_{n+p})$ is a continuous function of $(H(x_i))_{1\le i\le n+p}$ and $(d(x_i,x_j))_{1\le i<j\le n+p}$.

Proof. This is a special case of Lemma 3.4.7 for $p=0$. □

Remark 3.4.10. This second lemma generalizes the example of $F_{4-points}$ from Equation (3.4.13) to other sets with conditions of the type "∀". We can apply Lemma 3.4.9 to $F_{4-points}$ to prove that it is closed, using the closed set of $M_3(\mathbb{R})$:

$$F=\{((a_{i,j})_{1\le i,j\le 4}\in M_3(\mathbb{R})|a_{1,2}+a_{3,4}\le(a_{1,3}+a_{2,4})\vee(a_{1,4}+a_{2,3})\}\,.$$

Equivalently, we can see it as

$$F_{geo}=\{(E,d,H,\nu)\in\mathbb{X}^S|\forall x_1,...,x_n\in E, g(x_1,...,x_{n+p})=0\}$$

with

$$g(x_1,x_2,x_3,x_4)=\left(d(x_1,x_2)+d(x_3,x_4)-[(d(x_1,x_3)+d(x_2,x_4))\vee(d(x_1,x_4)+d(x_2,x_3))]\right)^+_.$$

This last lemma lets us consider the set of all spaces with at least one point in some compact range.

Lemma 3.4.11. *For every compact set $K\subset\mathbb{R}$, $h\in\mathbb{R}_+$,*

$$A=\{(E,d,H,\nu)\in\mathbb{X}^S|\exists x\in E, H(x)\in K\}$$

is a closed set of $(\mathbb{X}^S, d_{\text{LGHP}})$.

Proof. Take (E,d,H,ν) in the closure of A. Set $h_0=\max|K|$. For $\varepsilon>0$, there exists $(E',d',H',\nu')\in A$ such that $d_{\text{LGHP}}(E,E')<\varepsilon e^{-h_0}$. There exists $h\ge h_0$ such that $d_{\text{GHP}}(E_h,E'_h)<\varepsilon$. It follows that we can find $(Z,d_Z)\in\tilde{\mathbb{M}}$, $\phi\in\text{Iso}(E_h,Z)$, $\phi'\in\text{Iso}(E'_h,Z)$ such that

$$d_{\text{H}}\left([\phi\times H](E_h),[\phi'\times H'](E'_h)\right)\le\varepsilon.$$

Since $E'\in A$ and $K\subset[-h_0,h_0]\subset[-h,h]$, there exists $x'\in E'_h$ such that $H'(x')\in K$. Thus, we can find $x\in E_h$ such that $d_Z(\phi(x),\phi'(x'))\vee|H(x)-H'(x')|\le\varepsilon$. We automatically have $|H(x)|\le h_0+\varepsilon$, so $x\in E_{h_0+\varepsilon}$. This means that for every $\varepsilon>0$, we can find some $x\in E_{h_0+\varepsilon}$ such that the distance between $H(x)$ and K is less than ε.

Take $(x_k)_{k\in\mathbb{N}^*}$ a sequence of points in E_{h_0+1} such that for every $k\in\mathbb{N}^*$ the distance between $H(x_k)$ and K is less than $\frac{1}{k}$. Since E_{h_0+1} is compact, there exists a sub-sequence $(x_k)_{k\in\mathbb{N}^*}$ converging to some point $x\in E$. By continuity of H and closure of K, $H(x)\in K$. We deduce that $E\in A$. □

3.4.4 Some measurable maps on $(\mathbb{X}^S, d_{\mathrm{LGHP}})$

In this subsection we will prove the continuity or measurability of simple functions of interest. The first of those is the projection of the measure $(E, d, H, \nu) \mapsto H\nu$, defined on $\mathbb{X}^S$. To study its continuity, we equip the space of Borel measures on $\mathbb{R}$ with the local-Prohorov distance

$$d_{\mathrm{LP}}(\mu, \mu') = \int_0^\infty \left(1 \wedge d_{\mathrm{P}}(1_{[-h,h]} \cdot \mu, 1_{[-h,h]} \cdot \mu')\right) \mathrm{e}^{-h} dh. \tag{3.4.17}$$

Lemma 3.4.12. *$(E, d, H, \nu) \mapsto H\nu$ is 1-Lipschitz from $(\mathbb{X}^S, d_{\mathrm{LGHP}})$ to the space of Borel measures on $\mathbb{R}$ equipped with the local Prohorov distance.*

Proof. Consider $(E, d, H, \nu), (E', d', H', \nu') \in \mathbb{X}^S$, ν_h and ν'_h the restrictions of ν and ν' to E_h and E'_h. We have

$$\begin{aligned} d_{\mathrm{LGHP}}(E, E') &\geq \int_0^\infty \left(1 \wedge \inf_{Z,\phi,\phi'} d_{\mathrm{P}}^{(Z\times\mathbb{R}, d_Z^*)}([\phi \times H](\nu_h), [\phi' \times H'](\nu_h))\right) \mathrm{e}^{-h} dh \\ &\geq \int_0^\infty (1 \wedge d_{\mathrm{P}}(H\nu_h, H'\nu'_h))\, \mathrm{e}^{-h} dh. \end{aligned}$$

The last term is exactly $d_{\mathrm{LP}}(H(\nu), H'(\nu'))$. ☐

The next lemma will help us in many measurability questions.

Lemma 3.4.13. *Let (X, d_X) be a separable metric space, (Z, d_Z) a metric space, both equipped with their Borel σ-field, Y a space equipped with some σ-field and f a function from $X \times Y$ to Z. If f is continuous in the first variable and measurable in the second, then f is measurable. If $(X, d_X) = (\mathbb{R}, d_{\mathbb{R}})$, then if f is right-continuous in the first variable and measurable in the second, then f is measurable.*

Proof. The result is obvious if X is at most countable. Since X is separable, there exists a dense sequence $(x_n)_{n\in\mathbb{N}^*}$. Set $X' = \{x_n\}_{n\in\mathbb{N}^*}$. Since X' is countable, the restriction of f to $X' \times Y$ is measurable. For every $n \in \mathbb{N}^*$, $x \in X$, take $\phi_n(x) = \min\{k \in \mathbb{N}^* | d_X(x, x_k) \leq \frac{1}{n}\}$. The application $x \mapsto x_{\phi_n(x)}$ is measurable, so $f_n : (x, y) \mapsto f(x_{\phi_n(x)}, y)$ is measurable. We have $d_X(x, x_{\phi_n(x)}) \leq \frac{1}{n}$ by definition and f is continuous in x, so f_n converges point-wise to f, and f is measurable as limit of measurable functions.

We proceed similarly for the right-continuous case with $\phi_n(x) = \frac{\lceil nx \rceil}{n}$. We have that $\phi_n(x) \downarrow x$ as $n \to \infty$ and f is right-continuous, so f is the point-wise limit of the measurable sequence of functions $\Big((x, y) \mapsto f(\phi_n(x), y)\Big)_{n\in\mathbb{N}^*}$. ☐

Lemma 3.4.14. *The functions*

$$(h, (E, d, H, \nu)) \overset{f}{\mapsto} (E, d, H, 1_{H\leq h} \cdot \nu) \quad \text{and} \quad (h, (E, d, H, \nu)) \overset{g}{\mapsto} (E, d, H, 1_{H>h} \cdot \nu),$$

from $(\mathbb{R} \times \mathbb{X}^S, d_{\mathbb{R}} \vee d_{\mathrm{LGHP}})$ to $(\mathbb{X}^S, d_{\mathrm{LGHP}})$, are measurable with regard to the σ-field $\mathscr{B}(\mathbb{R}) \otimes \mathscr{B}(\mathbb{X}^S)$.

Proof. To get the result, we prove that $f(h, \cdot)$ is measurable as the limit of measurable functions over $\mathbb{X}^S \times \mathbb{R}$. We define for every S-compact labelled space (E, d, H, ν) and real number $\varepsilon > 0$:

$$f_\varepsilon(h, E) = (E, d, H, \lambda_{\varepsilon,h} \cdot \nu)$$

where for every $x \in E$, $\lambda_{\varepsilon,h}(x) = \quad 0 \vee \frac{1}{\varepsilon}(h+\varepsilon - H(x)) \wedge 1$. To study the continuity in E throughout Step 1 and 2, we use (E,d,H,ν) and (E',d',H',ν') two S-compact measured labelled spaces. Note that f_ε and *Slice* commute, and that for every $h \in \mathbb{R}$, $h' \geq 0$ we have

$$\mathrm{Slice}_{h'}(f_\varepsilon(h,E)) = f_\varepsilon(h, E_{h'}).$$

Step 1: for any $h \in \mathbb{R}$, $h' \geq 0$, we bound $d_{\mathrm{GHP}}(f_\varepsilon(h,E_{h'}), f_\varepsilon(h,E'_{h'}))$. For $h' \in \mathbb{R}_+$, $(Z,d_Z) \in \tilde{\mathbb{M}}$, $\phi \in \mathrm{Iso}(E_{h'},Z)$, $\phi' \in \mathrm{Iso}(E'_{h'},Z)$, consider

$$\Delta_{\mathrm{H}} = d_{\mathrm{H}}([\phi \times H](E_{h'}), [\phi' \times H'](E'_{h'})) \quad ; \quad \Delta_{\mathrm{P}} = d_{\mathrm{P}}([\phi \times H](1_{E_{h'}} \cdot \nu), [\phi' \times H'](1_{E'_{h'}} \cdot \nu')).$$

Since we did not change the metric spaces the Hausdorff distance stays the same between $f_\varepsilon(h,E)$ and $f_\varepsilon(h,E')$, so to bound the distance we only have to bound the Prohorov part:

$$\Delta'_{\mathrm{P}} = d_{\mathrm{P}}([\phi \times H](\lambda_{\varepsilon,h} \cdot \nu), [\phi' \times H'](\lambda_{\varepsilon,h} \cdot \nu')).$$

If $E_{h'}$ or $E'_{h'}$ is empty, then $d_{\mathrm{GHP}}(f_\varepsilon(E_{h'},h), f_\varepsilon(E'_{h'},h)) = d_{\mathrm{GHP}}(E_{h'}, E'_{h'})$ (0 if they are both empty, ∞ if exactly one is empty). If both are non-empty we set for every $h'' \in [0,h']$:

$$F_{h''} = [\phi \times H](E_{h''}) \quad \text{and} \quad F'_{h''} = [\phi' \times H'](E'_{h''}).$$

We have for every Borel set $B \subset [\phi \times H](E_{h'})$:

$$\begin{aligned}
&[[\phi \times H](\lambda_{\varepsilon,h} \cdot \nu)](B) \\
&\quad = \int_0^1 [\phi \times H]\nu(B \cap F_{h+\varepsilon t})dt \\
&\quad \leq \int_0^1 [\phi' \times H']\nu'((B \cap F'_{h+\varepsilon t})^{\Delta_{\mathrm{P}}})dt + \Delta_{\mathrm{P}} \\
&\quad \leq \int_0^1 \Big([\phi' \times H']\nu'(B^{\Delta_{\mathrm{P}}} \cap F'_{h+\varepsilon t}) + [\phi' \times H']\nu'(F'_{h+\varepsilon t+\Delta_{\mathrm{P}}} \setminus F'_{h+\varepsilon t})\Big)\, dt + \Delta_{\mathrm{P}} \\
&\quad = [[\phi' \times H'](\lambda_{\varepsilon,h} \cdot \nu')](B^{\Delta_{\mathrm{P}}}) + \Delta_{\mathrm{P}} + \int_0^1 H'\nu'((h+\varepsilon t, h+\varepsilon t + \Delta_{\mathrm{P}}])dt \\
&\quad \leq [\phi'(\lambda_{\varepsilon,h} \cdot \nu')](B^{\Delta_{\mathrm{P}}}) + \Delta_{\mathrm{P}} + \frac{\Delta_{\mathrm{P}}}{\varepsilon} H'\nu'([h, h+\varepsilon+\Delta_{\mathrm{P}}]) \\
&\quad = [\phi'(\lambda_{\varepsilon,h} \cdot \nu')](B^{\Delta_{\mathrm{P}}}) + \left(1 + \frac{1}{\varepsilon} H'\nu'([h, h+\varepsilon+\Delta_{\mathrm{P}}])\right)\Delta_{\mathrm{P}}.
\end{aligned} \tag{3.4.18}$$

For the first equality, we used the Fubini Theorem and the definition of $\lambda_{\varepsilon,h}$. We use the same method to obtain the first term after the second equality, while the second term is obtained with the Fubini Theorem alone. The last inequality is obtained as follows: $\int_0^1 H'\nu'((h+\varepsilon t, h+\varepsilon t + \Delta_{\mathrm{P}}])dt$ is the integral of $[H'\nu'](dh'')dt$ on the domain

$$D \quad : \quad \begin{cases} 0 \leq t \leq 1 \\ h + \varepsilon t \leq h'' \leq h + \varepsilon t + \Delta_{\mathrm{P}}. \end{cases}$$

The system is equivalent to

$$D \quad : \quad \begin{cases} h \leq h'' \leq h + \varepsilon + \Delta_{\mathrm{P}} \\ 0 \vee \frac{h''-h-\Delta_{\mathrm{P}}}{\varepsilon} \leq t \leq 1 \wedge \frac{h''-h}{\varepsilon}. \end{cases}$$

We define a new domain

$$D' \quad : \quad \begin{cases} h \le h'' \le h + \varepsilon + \Delta_{\mathrm{P}} \\ \frac{h''-h-\Delta_{\mathrm{P}}}{\varepsilon} \le t \le \frac{h''-h}{\varepsilon} \end{cases}$$

and note that $D \subset D'$. It follows that

$$\begin{aligned} \int_0^1 H'\nu'((h+\varepsilon t, h+\varepsilon t+\Delta_{\mathrm{P}}])dt &= \int_D [H'\nu'](dh'')dt \\ &\le \int_{D'} [H'\nu'](dh'')dt \\ &= \int_h^{h+\varepsilon+\Delta_{\mathrm{P}}} \left(\frac{h''-h}{\varepsilon} - \frac{h''-h-\Delta_{\mathrm{P}}}{\varepsilon} \right) [H'\nu'](dh'') \\ &= \frac{\Delta_{\mathrm{P}}}{\varepsilon} [H'\nu']([h, h+\varepsilon+\Delta_{\mathrm{P}}]). \end{aligned}$$

From (3.4.18) and by symmetry of E and E', we have

$$\Delta'_{\mathrm{P}} \le \left(1 + \frac{1}{\varepsilon} [H\nu + H'\nu']([h, h+\varepsilon+\Delta_{\mathrm{P}}]) \right) \Delta_{\mathrm{P}}.$$

Recall that the Hausdorff distance isn't affected by f_ε. Taking the infimum on Z, ϕ, ϕ', we have for every $h' \ge 0$ that:

$$\begin{aligned} &d_{\mathrm{GHP}}(f_\varepsilon(h, E_{h'}), f_\varepsilon(h, E'_{h'})) \\ &\qquad \le \left(1 + \frac{1}{\varepsilon} [H\nu + H'\nu']([h, h+\varepsilon+d_{\mathrm{GHP}}(E_{h'}, E'_{h'})]) \right) d_{\mathrm{GHP}}(E_{h'}, E'_{h'}). \end{aligned} \tag{3.4.19}$$

Step 2: prove that f_ε is continuous in E. Take (E, d, H, ν) a S-compact measured labelled space and $(E^n, d_n, H_n, \nu_n)_{n\in\mathbb{N}^*}$ a sequence of S-compact measured labelled spaces converging to E. Noting

$$\Delta_n(h') = d_{\mathrm{GHP}}(E_{h'}, E^n_{h'}),$$

we have, using Equation (3.4.19)

$$\begin{aligned} &d_{\mathrm{LGHP}}(f_\varepsilon(h, E), f_\varepsilon(h, E^n)) \\ &\qquad = \int_0^\infty (1 \wedge d_{\mathrm{GHP}}(f_\varepsilon(h, E_{h'}), f_\varepsilon(h, E^n_{h'})))\, \mathrm{e}^{-h'} dh' \\ &\qquad \le \int_0^\infty \left(1 \wedge [1 + \frac{1}{\varepsilon}(H\nu + H_n\nu_n)([h, h+\varepsilon+\Delta_n(h')])]\, \Delta_n(h') \right) \mathrm{e}^{-h'} dh'. \end{aligned}$$

We know from the convergence of $(E^n)_{n\in\mathbb{N}^*}$ that $\Delta_n \xrightarrow{a.e.} 0$ so $\Delta_\infty(h') = \sup_{n\in\mathbb{N}^*} \Delta_n(h')$ is finite for almost every h'. Since the function $(E, d, H, \nu) \mapsto H\nu$ is continuous, the function $(E, d, H, \nu) \mapsto H\nu(F)$ is upper semi-continuous for every compact interval $F \subset \mathbb{R}$, that is $\limsup(H_n\nu_n(F)) \le H\nu(F)$. When $\Delta_n(h') \to 0$, $\Delta_\infty(h')$ is finite and we have

$$\begin{aligned} \limsup_{n\to\infty} (H\nu + H_n\nu_n)([h, h+\varepsilon+\Delta_n(h')]) &\le \limsup_{n\to\infty} (H\nu + H_n\nu_n)([h, h+\varepsilon+\Delta_\infty(h')]) \\ &\le 2H\nu([h, h+\varepsilon+\Delta_\infty(h')]) < \infty. \end{aligned}$$

This give that for every h' such that $\Delta_n(h') \to 0$,

$$\begin{aligned}\limsup_{n\to\infty} \Delta_n(h')\Big[1 + \frac{1}{\varepsilon}(H\nu + H_n\nu_n)\big((h, h+\varepsilon+\Delta_n(h'))\big)\Big] \\ \leq \big(\limsup_{n\to\infty} \Delta_n(h')\big)\Big[1 + \frac{1}{\varepsilon}\limsup_{n\to\mathbb{N}^*}(H\nu + H_n\nu_n)\big((h, h+\varepsilon+\Delta_n(h'))\big)\Big] \\ = 0.\end{aligned}$$

By dominated convergence, $\limsup d_{\mathrm{LGHP}}(f_\varepsilon(h,E), f_\varepsilon(h,E^n))$ converges to 0 and f_ε is continuous in E.

Step 3: prove that f_ε is measurable. Since f_ε is continuous in E and $\mathbb{R}$ is separable, it is enough, with Lemma 3.4.13, to prove that f_ε is continuous in h. Take (E,d,H,ν) a S-compact measured labelled space. Take $h < h'$ two real numbers. Since $f_\varepsilon(h,E)$ and $f_\varepsilon(h',E)$ are supported by the same labelled metric space (E,d,H), we have

$$d_{\mathrm{LGHP}}(f_\varepsilon(h,E), f_\varepsilon(h',E)) \leq d_{\mathrm{P}}(\lambda_{\varepsilon,h}\cdot\nu, \lambda_{\varepsilon,h'}\cdot\nu).$$

Since $\lambda_{\varepsilon,h}\cdot\nu \leq \lambda_{\varepsilon,h'}\cdot\nu$, we have $d_{\mathrm{P}}(\lambda_{\varepsilon,h}\cdot\nu, \lambda_{\varepsilon,h'}\cdot\nu) \leq \int_{\mathbb{R}}(\lambda_{\varepsilon,h'} - \lambda_{\varepsilon,h})[H\nu](dh)$. By monotonic convergence, we have

$$\begin{aligned}\lim_{h\uparrow h'} d_{\mathrm{LGHP}}(f_\varepsilon(h,E), f_\varepsilon(h',E)) &= 0 \\ \lim_{h'\downarrow h} d_{\mathrm{LGHP}}(f_\varepsilon(h,E), f_\varepsilon(h',E)) &= 0.\end{aligned}$$

We deduce that $h \mapsto f_\varepsilon(h,E)$ is continuous on $\mathbb{R}$. Thus, f_ε is measurable on $\mathbb{R}\times\mathbb{X}^S$.

Step 4: express f and g as limits of measurable functions. We use the same method as in Step 3 to prove that for every (h,E),

$$f(h,E) = \lim_{n\to\infty} f_{\frac{1}{n}}(h,E).$$

To get g, consider the function $\sigma : (E,d,H,\nu) \mapsto (E,d,-H,\nu)$. The function σ clearly is a isometric involution of $\mathbb{X}^S$, so is measurable, and we have by the same method as in Step 3 that

$$g(h,E) = \lim_{n\to\infty}(E,d,H,1_{H\geq h+\frac{1}{n}}\cdot\nu) = \sigma\left(\lim_{n\to\infty} f(-h-\frac{1}{n}, \sigma(E))\right).$$

As limits of measurable functions, f and g are measurable. $\square$

Chapter 4

The space of height-labelled trees

4.1 Height-labelled trees

4.1.1 Definition

Let (E, d) be a metric space. For $x, y \in E$, let $\mathcal{C}(x, y)$ be the set of all continuous maps f from $[0, 1]$ to E such that $f(0) = x$, $f(1) = y$. We say that (E, d) is arc-connected if for all $x, y \in E$, $\mathcal{C}(x, y)$ is non-empty. We say (E, d) is a length space if it is arc-connected and for every $x, y \in E$, $d(x, y) = \inf_{f \in \mathcal{C}(x,y)} L(f)$, where

$$L(f) = \sup_{\substack{n \in \mathbb{N}^* \\ 0 = x_0 < x_1 < \ldots < x_n = 1}} \sum_{i=1}^{n} d(f(x_{i-1}), f(x_i)).$$

We say (E, d) is a geodesic space if it is a length-space and for every $x, y \in E$, $d(x, y) = \min_{f \in \mathcal{C}(x,y)} L(f)$. In this case, for $x, y \in E$, we call geodesic between x and y the image of any path $f \in \mathcal{C}(x, y)$ such that $L(f) = d(x, y)$. We say (E, d) is acyclic if for all $x, y \in E$, there does not exist $f, g \in \mathcal{C}(x, y)$ such that $f([0, 1]) \cap g([0, 1]) = \{x, y\}$.

We call tree any acyclic length space. We recall the so-called four-points condition. A connected metric space (T, d) is a tree if and only if for every four points $x_1, x_2, x_3, x_4 \in T$ the following holds

$$d(x_1, x_2) + d(x_3, x_4) \leq \max(d(x_1, x_3) + d(x_2, x_4), d(x_1, x_4) + d(x_2, x_3)).$$

Trees have been extensively studied and we will only shortly point out some of the properties of a tree (T, d):

- Between two points x, y of T there is always a unique geodesic which we note $[\![x, y]\!]$.

- If $F \subset T$ is connected then (F, d) is a tree and F is called a sub-tree of (T, d). In particular, F is geodesic and for every $x, y \in F$, $[\![x, y]\!] \subset F$.

- For $x \in T$ and F a closed sub-tree of (T, d), we can and will define $\rho(x, F)$ the projection of x on F as the unique point in F such that $d(x, \rho(x)) = d(x, F)$. For every $y \in F$, it satisfies $d(x, y) = d(x, \rho(x, F)) + d(\rho(x, F), y)$.

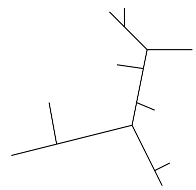

Figure 4.1: A tree.

As usual, we call leaf every point $x \in T$ such that $T \setminus \{x\}$ is connected. We note Skel(T) and call skeleton of T the complementary of the set of the leaves. We call branching point every point $x \in T$ such that $T \setminus \{x\}$ has at least three connected components. For any tree T, we call length measure the measure Λ over the skeleton of T such that for any geodesic $[\![a, b]\!]$, $\Lambda([\![x, y]\!]) = d(x, y)$.

In this chapter, we introduce the height-labelled trees which are a particular class of labelled measured spaces along with a bijective coding of the tree $(T, d, H, 0)$ using the height and a partial order $(T, H, \preceq, 0)$. Additionally, we prove the measurability of some functions of interest.

Definition 4.1.1. *We call height-labelled tree any quadruple (T, d, H, ν) where (T, d) is a tree, H a map from T to $\mathbb{R}$ such that for every $x, x' \in T$,*

$$d(x, x') = H(x) + H(x') - 2 \min_{y \in [\![x, x']\!]} H(y) \tag{4.1.1}$$

and ν is a σ-finite measure such that

$$\nu\Big(\{x \in T | |H(x)| \leq h\}\Big) < \infty \quad \textit{for all } h \in \mathbb{R}_+. \tag{4.1.2}$$

We shall see in Lemma 4.1.3 that H is always 1-Lipschitz, making any complete separable height-labelled tree (T, d, H, ν) a measured labelled space. We set $\mathbb{T}$ the space of S-compact height-labelled trees, up to label- and measure-preserving isometry.

Remark 4.1.2. For every non-empty tree (T, d) there are an infinite number of ways to label it. For example, pick a point $\omega \in T$ as the root of T and $\lambda \in \mathbb{R}$, then set $H(x) = \lambda + d(\omega, x)$. This makes (T, d, H) a height-labelled tree.

To prove this last statement, take $x, x' \in T$ and y the projection of the root ω on $[\![x, x']\!]$, we have $H(y) = \min_{[\![x,x']\!]} H$ and

$$H(x) = \lambda + d(\omega, x) = \lambda + d(\omega, y) + d(y, x) = H(y) + d(y, x).$$

It follows that $d(x, y) = H(x) - H(y)$. Similarly, we have $d(x', y) = H(x') - H(y)$. Since $y \in [\![x, x']\!]$, we have

$$d(x, x') = d(x, y) + d(y, x') = H(x) + H(x') - 2H(y) = H(x) + H(x') - 2 \min_{[\![x,x']\!]} H.$$

This concludes the remark.

Lemma 4.1.3. *Every complete separable height-labelled tree is a measured labelled space.*

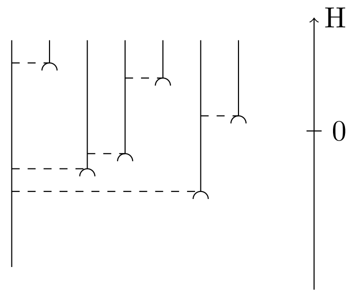

Figure 4.2: A height-labelled tree.

Proof. Take complete separable (T, d, H, ν) a height-labelled tree. Given the definition of a height-labelled tree, we only have to prove that H is 1-Lipschitz. For $x, y \in T$, we have

$$d(x,y) = H(x) + H(y) - 2 \inf_{[\![x,y]\!]} H \geq H(x) + H(y) - 2(H(x) \wedge H(y)) = |H(x) - H(y)|.$$

□

Lemma 4.1.4. *If $(T', d, H, 0)$ is a height-labelled tree, then the completion $(T, d, H, 0)$ (where H is extended by continuity) of T' is still a height-labelled tree.*

Proof. The space $(T, d, H, 0)$ is the completion of the height-labelled tree $(T', d, H, 0)$, so it still satisfies the four-points-condition, it is still connected and the extension of H' is 1-Lipschitz. In particular, $(T, d, H, 0)$ is a tree.

To prove that $(T, d, H, 0)$ is a height-labelled tree, we need to check that 4.1.1 holds. Take $x, y \in T$, there exists $(x_n)_{n \in \mathbb{N}^*}$ and $(y_n)_{n \in \mathbb{N}^*}$ two sequences of T' converging to x and y respectively. Let us prove that $d(x, y) = H(x) + H(y) - 2 \inf_{[\![x,y]\!]} H$. For $\varepsilon > 0$, take $n \in \mathbb{N}^*$ such that $d(x, x_n) \leq \varepsilon$ and $d(y, y_n) \leq \varepsilon$. We have

$$[\![x, y]\!] \subset [\![x, x_n]\!] \cup [\![x_n, y_n]\!] \cup [\![y_n, y]\!].$$

It follows that

$$\begin{aligned}
\inf_{[\![x,y]\!]} H &\geq (\inf_{[\![x,x_n]\!]} H) \wedge (\inf_{[\![x_n,y_n]\!]} H) \wedge (\inf_{[\![y_n,y]\!]} H) \\
&\geq (H(x_n) - \varepsilon) \wedge (\inf_{[\![x_n,y_n]\!]} H) \wedge (H(y_n) - \varepsilon) \\
&\geq \inf_{[\![x_n,y_n]\!]} H - \varepsilon.
\end{aligned}$$

Since (x, y) and (x_n, y_n) play symmetric roles here, we have proven that $(x, y) \mapsto \inf_{[\![x,y]\!]} H$ is continuous. We deduce that

$$\begin{aligned}
d(x,y) &= \lim_{n\to\infty} d(x_n, y_n) \\
&= \lim_{n\to\infty} H(x_n) + H(y_n) - 2 \inf_{[\![x_n,y_n]\!]} H \\
&= H(x) + H(y) - 2 \inf_{[\![x,y]\!]} H.
\end{aligned}$$

This ends the proof of the lemma. □

4.1.2 Coded trees

The aim of this part is to give another characterization of a height-labelled tree, using a partial order function rather than a distance. The main result of this section is Proposition 4.1.14, which states that under sufficient assumptions, a partially ordered set with a label function can be equipped with a distance making it a height-labelled tree.

Lemma 4.1.5. *For (T, d, H, ν) a height-labelled tree and $x, x' \in T$, the minimum of H over $[\![x, x']\!]$ is reached at a single point c and, for every $y \in [\![x, x']\!], H(y) = H(c) + d(c, y)$.*

Proof. The geodesic $[\![x, x']\!]$ is compact and H is continuous, so we can consider $c \in [\![x, x']\!]$ such that $H(c) = \min_{[\![x,x']\!]} H$. Since H is 1-Lipschitz, we have

$$H(x) - H(c) \leq d(x, c) \quad \text{and} \quad H(x') - H(c) \leq d(c, x'), \tag{4.1.3}$$

and since c is on the geodesic,

$$d(x, c) + d(c, x') = d(x, x') = H(x) + H(x') - 2H(c)$$

by definition of a height-labelled tree. From that last line, we deduce that the inequalities in Equation (4.1.3) are equalities, and we have:

$$H(x) - H(c) = d(x, c) \quad \text{and} \quad H(x') - H(c) = d(c, x').$$

Since the length $d(x, c)$ of the segment $[\![x, c]\!]$ is equal to the difference $H(x) - H(c)$, there is exactly one 1-Lipschitz map f from $[\![x, c]\!]$ to $\mathbb{R}$ such that $H(x) = f(x)$ and $H(c) = f(c)$. The function $f : y \mapsto H(c) + d(c, y)$ is 1-Lipschitz and satisfies $f(c) = H(c)$ and $f(x) = H(c)+d(c, x) = H(x)$, so $H = f$. We have the same result on $[\![c, x']\!]$, so $H(y) = H(c)+d(c, y)$ for every $y \in [\![x, x']\!]$. From the last formula, we see that c is the unique point of $[\![x, y]\!]$ where H reaches its minimum. □

Definition 4.1.6. *For (T, d, H, ν) a height-labelled tree, we call most recent common ancestor (MRCA) of x and y the unique point $x \wedge y \in [\![x, y]\!]$ such that $H(x \wedge y) = \min_{[\![x,y]\!]} H$.*

For every $x, y \in T$, we have $d(x, y) = H(x) + H(y) - 2H(x \wedge y)$.

Recall that an order is a relation $\preceq$ that is reflexive ($\forall x, x \preceq x$), transitive ($\forall x, y, z, (x \preceq y$ and $y \preceq z) \Rightarrow x \preceq z$) and anti-symmetric ($\forall x, y, (x \preceq y$ and $y \preceq x) \Rightarrow x = y$). We say that a set E is totally ordered for $\preceq$ if for every $x, y \in E$, x and y are comparable, that is $x \preceq y$ or $y \preceq x$. If E isn't totally ordered, we say that $\preceq$ is a partial order. Note that even for a partial order $\preceq$ over a set E, we can use the notions of minimum and maximum when they apply, the exact formulation being: x is the maximum (resp minimum) of E if $x \in E$ and for every $y \in E$, $y \preceq x$ (resp $x \preceq y$). The minimum is of particular significance in a tree since it represents the root of the tree.

From now on, we write $x \preceq y$ when $d(x, y) = H(y) - H(x)$, and $x \prec y$ when we have $x \preceq y$ and $x \neq y$. The condition $d(x, y) = H(y) - H(x)$ is equivalent to $H(x \wedge y) = H(x)$, which is in turn equivalent to $x = x \wedge y$ by uniqueness of the minimum in Definition 4.1.6. We say in this case that x is an ancestor of y or that y descends from x. We call $\preceq$ the genealogical order on (T, d, H), and we will consider it is canonical.

Lemma 4.1.7. *Let (T, d, H, ν) be a height-labelled tree. Its genealogical order $\preceq$ is an order relation over T.*

Proof. We must prove that $\preceq$ is reflexive, transitive and anti-symmetric. The reflexivity is obvious. Take any three points $x, y, z \in T$. If $x \preceq y \preceq z$ then $H(z)-H(x) = d(x,y)+d(y,z) \geq d(x,z)$ and since H is 1-Lipschitz we have $H(z) - H(x) = d(x,z)$. This means that $x \preceq z$, which yields the transitivity. If $x \preceq y \preceq x$ then $d(x,y) = H(x) - H(y) = -d(y,x)$, so $x = y$ and $\preceq$ is anti-symmetric. □

Lemma 4.1.8. *For $h \in H(T)$ the range of H and $x \in T$ such that $h \leq H(x)$ there exists a unique $x' \in T$ such that $H(x') = h$ and $x' \preceq x$.*

Proof. For $x \in T$, $h \leq H(x)$, take $y \in T$ such that $H(y) = h$. Using Lemma 4.1.5, there is a point $c \in [\![x, y]\!]$ such that $H(c) = \min_{[\![x,y]\!]} H$ and for every $x' \in [\![x, y]\!]$, $H(x') = H(c)+d(c, x')$. Since H is continuous $[\![x, c]\!]$, there exists $x' \in [\![x, c]\!]$ such that $H(x') = h$. We have

$$H(x) - H(x') = H(c) + d(c, x) - H(c) - d(c, x') = d(x, c) - d(x', c) = d(x, x'),$$

so x' is an ancestor of x with height h. To prove the uniqueness, consider x'' another ancestor of x at height h, we have $d(x', x'') = 2h - 2\min_{[\![x',x'']\!]} H$. Now, we use the fact that $[\![x', x'']\!] \subset [\![x', x]\!] \cup [\![x, x'']\!]$ to see that

$$\min_{[\![x',x'']\!]} H \geq \min_{[\![x',x]\!]\cup[\![x,x'']\!]} H = h$$

so $d(x', x'') = 0$ and $x' = x''$. □

Remark 4.1.9. Combining Lemma 4.1.5 and Lemma 4.1.8, we find another equivalent definition of $x \wedge y$. With the uniqueness in Lemma 4.1.8 and the transitivity of $\preceq$, we see that $\{z \in T | z \preceq x, z \preceq y\}$ is totally ordered for $\preceq$ and that $x \wedge y$ can also be characterized as the maximum of $\{z \in T | z \preceq x, z \preceq y\}$.

Remark 4.1.10. In a height-labelled tree, only one branch[1] can go to $-\infty$. Indeed, suppose that a height-labelled tree (T, d, H, ν) satisfies $\inf_T H = -\infty$. For $x \in T$, Lemma 4.1.8 tells us that H induces a bijection from the set $A(x) = \{y \in T | y \preceq x\}$ to $(-\infty, H(x)]$. We know from Remark 4.1.9 that $A(x)$ is totally ordered for $\preceq$, so by definition of $\preceq$, H is an isometry from $A(x)$ to $(-\infty, H(x)]$, and $A(x)$ is a branch going to $-\infty$.

Let us prove that it is the only one. Suppose that $A'(x)$ is another infinite branch starting from x, we prove that it does not go to $-\infty$. Since $A(x)$ and $A'(x)$ are two distinct geodesics starting from x, and since T is acyclic, then $A(x) \cap A'(x) = [\![x, x']\!]$ for some $x' \in T$. Since $A(x)$ is the set of all ancestors of x, we have for every $y \in A'(x)$ that $x \wedge y \in A(x)$. Since $A'(x)$ is connected, we have $x \wedge y \in [\![x, y]\!] \subset A'(x)$, so $x \wedge y \in A(x) \cap A'(x) = [\![x, x']\!]$. It follows that

$$\inf_{y\in A'(x)} H(y) = \inf_{y\in A'(x)} (\min_{[\![x,y]\!]} H) = \inf_{y\in A'(x)} H(x \wedge y) \geq \min_{[\![x,x']\!]} H = H(x \wedge x') > -\infty.$$

For the first equality, we use the fact that $A'(x) = \bigcup_{y\in A'(x)} [\![x, y]\!]$. It follows that $A'(x)$ has a lower bound. This implies that $A(x)$ is the unique branch going to $-\infty$.

[1]Here, a branch is an isometrical embedding ϕ from $[0, a]$ to the tree such that $\phi(0) = x$ and $\phi(a)$ is a leaf, or an isometric embedding from $\mathbb{R}_+$ to the tree such that $\phi(0) = x$, where x is a point fixed in advance.

Definition 4.1.11. *Let T be a set, H a map from T to $\mathbb{R}$ and $\preceq$ an order on T. We say that $(T, H, \preceq)$ is a coded tree if the following conditions are satisfied:*

1. *the direct image $H(T)$ is connected,*
2. *H is strictly increasing for $\preceq$,*
3. *for every $x \in T$ and $h \in H(T)$ with $h \leq H(x)$, there exists a unique $y \preceq x$ with $H(y) = h$,*
4. *for every $x, y \in T$ the set of all common ancestors $\{z \in T | z \preceq x, z \preceq y\}$ has a maximum, denoted by $x \wedge y$.*

Proposition 4.1.14 ensures that any coded tree, equipped with the right distance, is a height-labelled tree. Condition 2 could be derived from 3, but we keep it to avoid an unnecessary lemma. Condition 3 emulates the result from Lemma 4.1.8 for height-labelled trees, while condition 4 ensures that we can define a tree distance, as proven in Proposition 4.1.14. Remark 4.1.9 tells us that the definition of $x \wedge y$ for height-labelled trees agrees with the notation given in Condition 4. Note that $(x \wedge y) \wedge z$ is the maximum of the common ancestors of x, y, z. This characterization means that $\wedge$ is commutative and associative.

Remark 4.1.12. If three points $x, y, z \in T$ of a coded tree $(T, H, \preceq)$ satisfy $y \preceq x$ and $z \preceq x$ then y and z are comparable for $\preceq$.

Indeed, suppose $H(z) \leq H(y)$ and, by Condition 3, consider $z' \preceq y$ with $H(z') = H(z)$. By transitivity, we have $z' \preceq x$. We deduce that $z' = z$ by uniqueness in Condition 3, so $z \preceq y$.

Lemma 4.1.13. *If (T, d, H, ν) is a height-labelled tree and $\preceq$ its genealogical order, then $(T, H, \preceq)$ is a coded tree.*

Proof. We prove all the conditions from Definition 4.1.11.

Condition 1: we have that H is continuous and (T, d) is geodesic.

Condition 2: by definition, $x \preceq y$ and $x \neq y$ implie $H(y) - H(x) = d(x, y) > 0$ for $x, y \in T$.

Condition 3 is exactly Lemma 4.1.8.

Condition 4 is proved by Remark 4.1.9, since the maximum in the condition is exactly $x \wedge y$. □

For every coded tree $(T, H, \preceq)$, we note $\Phi(T, H, \preceq) = (T, d, H, 0)$, with:

$$d(x, y) = H(x) + H(y) - 2H(x \wedge y). \tag{4.1.4}$$

Proposition 4.1.14. *For any coded tree $(T, H, \preceq)$, $\Phi(T, H, \preceq) = (T, d, H, 0)$ is a height-labelled tree, and the genealogical order of $(T, d, H, 0)$ is $\preceq$.*

The transformation Φ, is a bijection from coded trees to height-labelled trees with null measure.

Proof. Step 1: let us prove the four-points condition. Consider four points $x_1, x_2, x_3, x_4 \in T$ and suppose that for every $i \neq j \in \{1, 2, 3, 4\}$, $H(x_i \wedge x_j) \leq H(x_1 \wedge x_2)$. It follows from Remark 4.1.12 that $x_1 \wedge x_3 \preceq x_1 \wedge x_2$, so we have $x_1 \wedge x_3 = (x_1 \wedge x_2) \wedge (x_1 \wedge x_3) = (x_1 \wedge x_2) \wedge x_3$.

Since x_1 and x_2 play similar roles, we have $x_2 \wedge x_3 = (x_1 \wedge x_2) \wedge x_3 = x_1 \wedge x_3$. Similarly, we have $x_1 \wedge x_4 = x_2 \wedge x_4$. This yields

$$\begin{aligned} d(x_1, x_3) + d(x_2, x_4) &= H(x_1) + H(x_3) - 2H(x_1 \wedge x_3) + H(x_2) + H(x_4) - 2H(x_2 \wedge x_4) \\ &= H(x_1) + H(x_4) - 2H(x_1 \wedge x_4) + H(x_2) + H(x_3) - 2H(x_2 \wedge x_3). \end{aligned}$$

That is

$$d(x_1, x_3) + d(x_2, x_4) = d(x_1, x_4) + d(x_2, x_3). \tag{4.1.5}$$

Moreover, $x_1 \wedge x_3$ and $x_1 \wedge x_4$ are comparable for $\preceq$ so we can take the minimum $\min(x_1 \wedge x_3, x_1 \wedge x_4) = x_1 \wedge x_3 \wedge x_4 \preceq x_3 \wedge x_4$. Since H is increasing and $x_1 \wedge x_4 = x_2 \wedge x_4$, we have

$$\begin{aligned} d(x_1, x_2) + d(x_3, x_4) =& H(x_1) + H(x_2) - 2H(x_1 \wedge x_2) + H(x_3) + H(x_4) - 2H(x_3 \wedge x_4) \\ \leq& H(x_1) + H(x_2) - 2H(\max(x_1 \wedge x_3, x_1 \wedge x_4)) \\ &+ H(x_3) + H(x_4) - 2H(\min(x_1 \wedge x_3, x_1 \wedge x_4)) \\ =& H(x_1) + H(x_3) - 2H(x_1 \wedge x_3) + H(x_2) + H(x_4) - 2H(x_1 \wedge x_4) \\ =& d(x_1, x_3) + d(x_2, x_4) \\ =& d(x_1, x_4) + d(x_2, x_3). \end{aligned} \tag{4.1.6}$$

We used Equation (4.1.5) for the last two equalities.

Set $a_1 = d(x_1, x_2) + d(x_3, x_4)$, $a_2 = d(x_1, x_3) + d(x_2, x_4)$ and $a_3 = d(x_1, x_4) + d(x_2, x_3)$. Equations (4.1.5) and (4.1.6) imply that $a_1 \leq a_2 = a_3$, so

$$\begin{cases} a_1 \leq \max(a_2, a_3) \\ a_2 \leq \max(a_3, a_1) \\ a_3 \leq \max(a_1, a_2). \end{cases}$$

With those three inequality, we have proven the four points condition for (T, d).

Step 2: let us prove that d is a distance. The function d is non-negative since H is increasing. If $d(x, y) = 0$ then $H(x) = H(y) = H(x \wedge y)$. We have $x \wedge y \prec x$ and $H(x) = H(x \wedge y)$, so by uniqueness in condition 3 we have $x \wedge y = x$. We find similarly $x \wedge y = y$, so $x = y$. We have proven that d is positive-definite. The triangular inequality of d is implied by the four-points condition: $d(z, z) + d(x, y) \leq d(x, z) + d(z, y)$. We have proven that d is a distance.

Step 3: let us prove that (T, d) is a tree. Given Steps 1-2, we just need to prove that T is connected. We note that for $x \preceq y$, we have $d(x, y) = H(y) - H(x)$. Using conditions 1 and 3 we find that in that case

$$[\![x, y]\!] := \{z \in T | x \preceq z \preceq y\}$$

is a geodesic since for every $y, y' \in [\![x, x']\!]$, $d(y, y') = |H(y) - H(y')|$. For every two points $x, x' \in T$, we use condition 4 and see that

$$[\![x, x']\!] := [\![x, x \wedge x']\!] \cup [\![x \wedge x', x']\!]$$

is a geodesic between x and x' so (T, d) is connected. It satisfies the four points condition so (T, d) is a tree. We see that $\min_{[\![x,x']\!]} H = H(x \wedge x')$ so by (4.1.1) and the definition of d, $(T, d, H, 0)$ is a height-labelled tree.

Step 4: We prove $\preceq$ is the canonical order of $(T, d, H, 0)$ and that Φ is bijective. Let us first prove that $\preceq$ is the canonical order of $(T, d, H, 0)$. We have already seen that if $x \preceq y$ then $d(x, y) = H(y) - H(x)$. Conversely, if $d(x, y) = H(y) - H(x)$ then $H(x \wedge y) = H(x)$ so $x = x \wedge y$ and $x \preceq y$.

Since we can express $(T, H, \preceq)$ as a function of $\Phi(T, H, \preceq)$, we deduce that Φ is injective.

To prove that Φ is bijective, take $(T, d, H, 0)$ a height-labelled tree $\preceq$ its canonical order. By Lemma 4.1.13, $(T, H, \preceq)$ is a coded tree. By Definition 4.1.6, we have $d(x, y) = H(x) + H(y) - 2H(x \wedge y)$. Using Remark 4.1.9 and the construction of Φ, we have $\Phi(T, H, \preceq) = (T, d, H, 0)$.

We have proven $\preceq$ is the canonical order of $(T, d, H, 0)$ and that Φ is bijective.

With the conclusions of Steps 3 and 4, we have proven our proposition. □

4.1.3 Induced topology on $\mathbb{T}$

We prove Theorem 4.1.15, stating that the space of all S-compact height-labelled trees $\mathbb{T}$ equipped with the distance d_{LGHP} is Polish.

Theorem 4.1.15. *The space* $(\mathbb{T}, d_{\mathrm{LGHP}})$ *is Polish.*

Proof. Since $(\mathbb{X}^S, d_{\mathrm{LGHP}})$ is Polish, it is enough to prove that $\mathbb{T}$ is closed in $(\mathbb{X}^S, d_{\mathrm{LGHP}})$. A S-compact measured labelled space (T, d, H, ν) is a tree if and only if (T, d) is a geodesic space satisfying the four-points condition. It is a height-labelled tree if for every $x, y \in T$, $d(x, y) = H(x) + H(y) - 2 \min_{[\![x,y]\!]} H$.

The set $F_{4-points}$ of all S-compact measured labelled space satisfying the four-points condition, defined in (3.4.13), is closed in $(\mathbb{X}^S, d_{\mathrm{LGHP}})$ as seen in Remark 3.4.10. The set F_{geo} of geodesic spaces is closed, by Remark 3.4.8. For the last condition, we want to prove that the set of trees (T, d) equipped with a 1-Lipschitz map $H : T \to \mathbb{R}$ that satisfy the condition

$$\forall x_1, x_2 \in T, \quad d(x_1, x_2) = H(x_1) + H(x_2) - 2 \min_{x_3 \in [\![x_1, x_2]\!]} H(x_3) \tag{4.1.7}$$

is closed in $(\mathbb{X}^S, d_{\mathrm{LGHP}})$. To this end, we will find equivalent formulations of Condition (4.1.7) so that we can apply Lemma 3.4.7.

Note that for $x_3 \in [\![x_1, x_2]\!]$, we have $d(x_1, x_3) + d(x_3, x_2) = d(x_1, x_2)$. Since H is 1-Lipschitz, this implies $H(x_1) + H(x_2) - 2H(x_3) \leq d(x_1, x_3) + d(x_3, x_2) = d(x_1, x_2)$. It follows that

$$\begin{aligned} d(x_1, x_2) = H(x_1) + H(x_2) - 2 \min_{x_3 \in [\![x_1, x_2]\!]} H(x_3) \\ \iff \exists x_3 \in [\![x_1, x_2]\!], \quad H(x_1) + H(x_2) - 2H(x_3) = d(x_1, x_2). \end{aligned}$$

We can reformulate (4.1.7) as

$$\forall x_1, x_2 \in T, \quad \exists x_3 \in [\![x_1, x_2]\!], d(x_1, x_2) = H(x_1) + H(x_2) - 2H(x_3)$$

i.e.

$$\forall x_1, x_2 \in T, \quad \exists x_3 \in T, d(x_1, x_3) + d(x_3, x_2) = d(x_1, x_2) = H(x_1) + H(x_2) - 2H(x_3).$$

which is equivalent to

$$\forall x_1, x_2 \in T, \quad \exists x_3 \in T, \begin{cases} H(x_3) = \frac{1}{2}(H(x_1) + H(x_2) - d(x_1, x_2)), \\ d(x_1, x_3) + d(x_3, x_2) = d(x_1, x_2) = H(x_1) + H(x_2) - 2H(x_3). \end{cases}$$

It follows that the set of trees satisfying (4.1.7) is $F_{4-points} \cap F_{geo} \cap F_H$, with

$$F_H = \left\{ (T, d, H, \nu) \in \mathbb{X}^S \middle| \begin{array}{l} \forall x_1, x_2 \in T, \exists x_3 \in \text{Slice}_{|\frac{1}{2}(H(x_1)+H(x_2)-d(x_1,x_2))|}(T), \\ d(x_1, x_3) + d(x_3, x_2) = d(x_1, x_2) = H(x_1) + H(x_2) - 2H(x_3) \end{array} \right\},$$

and F_H is closed thanks to Lemma 3.4.7. The set F_H of S-compact measured labelled spaces satisfying (4.1.7) is closed in $(\mathbb{X}^S, d_{\text{LGHP}})$.

We have $\mathbb{T} = F_{4-points} \cap F_{geo} \cap F_H$, so $\mathbb{T}$ is closed in $(\mathbb{X}^S, d_{\text{LGHP}})$. □

Remark 4.1.16. By definition, $H(T)$ is an interval for every height-labelled tree (T, d, H, ν). We deduce from Lemma 3.4.5 that d_{GHP} and d_{LGHP} induce the same topology on the space of non-empty compact trees, and that this set is open in $(\mathbb{T}, d_{\text{LGHP}})$ as the trace on $\mathbb{T}$ of the open set $\mathbb{X}^{C,K} \setminus \{\emptyset\}$. In particular, this means that for T a compact tree, $\varepsilon > 0$, there exists $\delta > 0$ such that

- for every compact height-labelled tree T', if $d_{\text{GHP}}(T, T') \le \delta$ then $d_{\text{LGHP}}(T, T') \le \varepsilon$;
- for every S-compact height-labelled tree T', if $d_{\text{LGHP}}(T, T') \le \delta$ then T' is compact and $d_{\text{GHP}}(T, T') \le \varepsilon$.

Note that for $(T, d, H, \nu) \in \mathbb{T}$, T is compact if and only if $H(T)$ is compact.

Definition 4.1.17. *For every $h \in \mathbb{R}$, we define $\mathbb{T}_h$ the set of trees $(T, d, H, \nu) \in \mathbb{T}$ such that $\min_T H = h$. We define $\mathbb{T}_{-\infty}$ the set of trees $(T, d, H, \nu) \in \mathbb{T}$ such that $T = \emptyset$ or $\inf_T H = -\infty$.*

We can rewrite

$$\mathbb{T}_{-\infty} = \{(T, d, H, \nu) \in \mathbb{T} | \forall x_1 \in T, \exists x_2 \in T_{|H(x_1))|+1}, H(x_2) = H(x_1) - 1\}$$

and

$$\mathbb{T}_h = \{(T, d, H, \nu) \in \mathbb{T} | \forall x_1 \in T, H(x_1) \ge h\} \cap \{(T, d, H, \nu) \in \mathbb{T} | \exists x \in T, H(x) = h\}.$$

With this expression of $\mathbb{T}_{-\infty}$, we can use Lemma 3.4.7 to see that $\mathbb{T}_{-\infty}$ is closed. The set $\mathbb{T}_h$ is the intersection of two sets. The first set is closed by Lemma 3.4.9 and the second by Lemma 3.4.11, so $\mathbb{T}_h$ is closed for every h.

4.2 Some measurable maps over $\mathbb{T}$

We will study in this section some measurable maps of interest defined over $\mathbb{T}$.

4.2.1 Number of balls in a level

For (T, d, H, ν) a height-labelled tree, $h \in H(T)$, the set $H^{-1}(\{h\})$, that we call level h of T, is ultra-metric when equipped with d. This implies that for every $\varepsilon > 0$, the closed balls of diameter 2ε form a partition of $H^{-1}(\{h\})$.

Definition 4.2.1. *For $h < h' \in \mathbb{R}$, (T, d, H, ν) a height-labelled tree, we call $n^{h,h'}(T) \in \mathbb{N} \cup \{\infty\}$ the number of closed balls of diameter $2(h' - h)$ at level h' of T. In the case where T has no point at level h', then we consider that $n^{h,h'}(T) = 0$.*

We set

$$D = \{(T, h, h') \in \mathbb{T} \times \mathbb{R}^2 | h < h'\}$$

the domain of definition of the function $(T, h, h') \mapsto n^{h,h'}(T)$. Notice D is an open set.

Remark 4.2.2. Note that for $h' \in H(T)$ and every ball $B \subset H^{-1}(h')$ of radius $2(h' - h)$, the MRCA of B (that is the maximal element of $\{z \in T |\, z \preceq x, \forall x \in B\}$) exists in T, and its height is above h. If $B' \subset H^{-1}(h')$ is another ball of radius $2(h' - h)$, then for $x \in B$, $x' \in B'$, $H(x \wedge x')$ is strictly below h. It follows that when $h \in H(T)$, $n^{h,h'}(T)$ is the number of points at height h that are ancestors of at least one point at height h'.

Lemma 4.2.3. *A height-labelled tree (T, d, H, ν) is S-compact if and only if it is complete and $\forall a < b \in \mathbb{R},\ n^{a,b}(T) < \infty$.*

Proof. Step 1: $\Rightarrow$ Suppose that T is S-compact. For every Cauchy sequence $(x_n)_{n\in\mathbb{N}^*} \in T^{\mathbb{N}^*}$, the sequence $(|H(x_n)|)_{n\in\mathbb{N}^*}$ is Cauchy also as H is 1-Lipschitz. It follows that $(|H(x_n)|)_{n\in\mathbb{N}^*}$ is bounded from above by some $h \in \mathbb{R}_+$, so $(x_n)_{n\in\mathbb{N}^*}$ is actually a Cauchy sequence in T_h, which is compact, so $(x_n)_{n\in\mathbb{N}^*}$ converges. As $(x_n)_{n\in\mathbb{N}^*}$ was arbitrary, T is complete.

For every $h' \in \mathbb{R}$, $T_{|h'|}$ is compact and d is ultra-metric over $T_{|h'|}$, so for every $h < h'$ the closed balls of diameter $2(h' - h)$ form a partition of $T_{|h'|}$ of cardinal $n^{h,h'}(T)$. Since $T_{|h'|}$ is ultra-metric, the closed balls are open sets. Thus, they form a minimal covering[2] of $T_{|h'|}$ with open sets. Since $T_{|h'|}$ is compact, the minimal covering is finite. We have proved that $n^{h,h'}(T)$ is finite.

Step 2: $\Leftarrow$ Suppose that T is complete and $\forall a < b, n^{a,b}(T) < \infty$. Take $h_0 \in \mathbb{R}_+$, $(x_n)_{n\in\mathbb{N}^*}$ a sequence of points of T_{h_0}. We want to prove that $(x_n)_{n\in\mathbb{N}^*}$ has a converging sub-sequence. The sequence $(H(x_n))_{n\in\mathbb{N}^*}$ has its terms in the compact space $[-h_0, h_0]$, so we can find an extraction ϕ such that $(H(x_{\phi(n)}))_{n\in\mathbb{N}^*}$ converges to some $h_\infty \in [-h_0, h_0]$.

If $h_\infty = \inf_T H$, then we have

$$\begin{aligned} d(x_{\phi(n)}, x_{\phi(n+p)}) &= H(x_{\phi(n)}) + H(x_{\phi(n+p)}) - 2H(x_{\phi(n)} \wedge x_{\phi(n+p)}) \\ &\leq H(x_{\phi(n)}) + H(x_{\phi(n+p)}) - 2h_\infty \\ &\underset{n\to\infty}{\longrightarrow} 0. \end{aligned}$$

This proves that $(x_{\phi(n)})_{n\in\mathbb{N}^*}$ is a Cauchy sequence, so it converges.

If $h_\infty > \inf_T H$, then for every $h < h' \in H(T) \cap [\inf_T H, h_\infty)$, we can set $y_1, ..., y_{n^{h,h'}(T)} \in T$ the ancestors at height h of level h'. Take $n_0 \in \mathbb{N}^*$ such that for all $n \geq n_0$, $|H(x_{\phi(n)}) - h_\infty| \leq$

[2] A covering is minimal if its only sub-covering is itself.

$h_\infty - h'$. We have $H(x_{\phi(n)}) \geq h' \in H(T)$, so $x_{\phi(n)}$ has an ancestor at height h', and by definition of $n^{h,h'}(T)$ there is an index i such that $1 \leq i \leq n^{h,h'}(T)$ and $y_i \preceq x_{\phi(n)}$. If $x_{\phi(n')}$ has the same ancestor y_i for some $n' \geq n_0$, we have $y_i \preceq x_{\phi(n)} \wedge x_{\phi(n')}$ so

$$d(x_{\phi(n)}, x_{\phi(n')}) \leq H(x_{\phi(n)}) + H(x_{\phi(n')}) - 2H(y_i) \leq 2(h_\infty + (h_\infty - h')) - 2h \leq 4(h_\infty - h).$$

We have proven that the set $\{x_{\phi(n)}\}_{n\in\mathbb{N}^*}$ is covered by at most $n_0 + n^{h,h'}(T)$ balls of diameter $4(h_\infty - h)$. Since $(h_\infty - h)$ can be arbitrarily small, and $n_0 + n^{h,h'}(T)$ is finite, we have proven that $\{x_{\phi(n)}\}_{n\in\mathbb{N}^*}$ is precompact. Since T is complete, the closure of $\{x_{\phi(n)}\}_{n\in\mathbb{N}^*}$ is compact and there exists a converging sub-sequence. Since T_{h_0} is closed, the closure of $\{x_{\phi(n)}\}_{n\in\mathbb{N}^*}$ is a subset of T_{h_0}.

We have proven that $(x_n)_{n\in\mathbb{N}^*}$ has a converging sub-sequence in T_{h_0}. Since $(x_n)_{n\in\mathbb{N}^*}$ was an arbitrary sequence of T_{h_0}, we have proven that T_{h_0} is compact. Since h_0 was arbitrary, T is S-compact. □

Lemma 4.2.4. *The map $(T, h, h') \mapsto n^{h,h'}(T)$ is measurable on D when $\mathbb{T}$ and $\mathbb{R}$ are equipped with their Borel σ-fields.*

Proof. Step 1: we prove that $D_0 = \{(T, h, h') \in D | n^{h,h'}(T) = 0\}$ is open. We have

$$\begin{aligned} D_0 &= \{(T, h, h') \in \mathbb{T} \times \mathbb{R}^2 | h < h', n^{h,h'}(T) = 0\} \\ &= \left\{ ((T, d, H, \nu), h, h') \in \mathbb{T} \times \mathbb{R}^2 \,\middle|\, \begin{array}{l} h < h', \\ \forall x \in T, H(x) \neq h' \end{array} \right\}. \end{aligned}$$

The space T is S-compact, so the image $H(T) \cap [-r, r]$ is compact for every $r \in \mathbb{R}_+$. It follows that $H(T)$ is always a closed set of $\mathbb{R}$, so $(\forall x \in T, H(x) \neq h')$ is equivalent to $(\inf_{x\in T} |H(x) - h'| > 0)$. From this, we deduce that

$$\begin{aligned} D_0 &= \left\{ ((T, d, H, \nu), h, h') \in \mathbb{T} \times \mathbb{R}^2 \,\middle|\, \begin{array}{l} \exists q < p' < q' \in \mathbb{Q}, \\ h < q,\ p' < h' < q', \\ \forall x \in T, H(x) \notin [p', q'] \end{array} \right\}. \\ &= \bigcup_{q<p'<q'\in\mathbb{Q}} U^{p',q'} \times (-\infty, q) \times (p', q'), \end{aligned}$$

where

$$U^{p',q'} = \left\{ (T, d, H, \nu) \in \mathbb{T} \,\middle|\, \forall x \in T, H(x) \notin [p', q'] \right\}.$$

The set $[p', q']$ is compact so the complement of $U^{p',q'}$ is closed by Lemma 3.4.11. This means that $U^{p',q'}$ is open, so D_0 is an open set as the reunion of open sets.

Step 2: we prove that $D_0' = \{(T, h, h') \in D | \min_T H = h'\}$ is measurable. Notice that $n^{h,h'}(T) = 1$ on D_0'. We can write:

$$D_0' = \bigcap_{n\in\mathbb{N}^*} \bigcup_{q<q'\in\mathbb{Q}} V^{q',n} \times (-\infty, q) \times (q', q' + \frac{1}{n}),$$

with

$$V^{q',n} = \left\{ (T,d,H,\nu) \in \mathbb{T} \,\middle|\, \begin{array}{l} \forall x \in T, H(x) \geq q' \\ \exists y \in T, q' \leq H(y) \leq q' + \frac{1}{n} \end{array} \right\}.$$

Use Lemma 3.4.9 and Lemma 3.4.11 to get that $V^{q',n}$ is closed. This implies that D'_0 is measurable.

Step 2: we prove that the auxiliary map

$$f : (T,h,h') \mapsto 1 \vee \sup_{h''>h'} n^{h,h''}(T) \tag{4.2.1}$$

is measurable on D. For every $(T,d,H,\nu) \in \mathbb{T}$, $k \geq 2$, $h < h' \in \mathbb{R}$ we have $n^{h,h'}(T) \geq k$ if and only if there exists $x_1,...,x_k \in T$ such that $H(x_1) = ... = H(x_k) = h'$ and

$$\min_{1 \leq i < j \leq n} d(x_i,x_j) > 2(h'-h).$$

It follows that

$$\begin{aligned} U_k =& \{((T,d,H,\nu),h,h') \in D | \exists h'' > h', n^{h,h''}(T) \geq k\} \\ =& \left\{ ((T,d,H,\nu),h,h') \in \mathbb{T} \times \mathbb{R}^2 \,\middle|\, \begin{array}{l} h < h', \\ \exists h'' > h', \exists x_1,...,x_k \in T, \\ H(x_1) = ... = H(x_k) = h'', \\ \forall 1 \leq i < j \leq n, d(x_i,x_j) > 2(h''-h) \end{array} \right\}. \end{aligned}$$

We have:

$$U_k = \left\{ ((T,d,H,\nu),h,h') \in \mathbb{T} \times \mathbb{R}^2 \,\middle|\, \begin{array}{l} h < h', \\ \exists x_1,...,x_k \in T, \\ \min_{1 \leq i \leq k} H(x_i) > h', \\ \forall 1 \leq i < j \leq n, d(x_i,x_j) > H(x_i) + H(x_j) - 2h \end{array} \right\}.$$

In this equality, the inclusion ($\subset$) is obvious. We now prove the inclusion ($\supset$). Take $((T,d,H,\nu),h,h')$ in the right-hand set and $x_1,...,x_k \in T$ such that $\min_{1 \leq i \leq k} H(x_i) > h'$ and $\forall 1 \leq i < j \leq n, d(x_i,x_j) > H(x_i) + H(x_j) - 2h$, we can take $h'' = \min_{1 \leq i \leq k} H(x_i) > h'$ and $y_1,...,y_k$ the ancestors of $x_1,...,x_k$ at height h''. For every $1 \leq i < j \leq n$, we have $d(x_i,x_j) > H(x_i) + H(x_j) - 2h$, so $H(x_i \wedge x_j) < h$. Since $H(x_i \wedge x_j) < h < h' < h'' = H(y_i) = H(y_j)$, we have $y_i \wedge y_j = x_i \wedge x_j$, so we have $H(y_i \wedge y_j) < h$. It follows that $d(y_i,y_j) = 2(h'' - H(y_i \wedge y_j)) > 2(h''-h)$. We have found $h'' > h'$, $y_1,...,y_k \in T$ such that $H(y_1) = ... = H(y_k) = h''$ and $\forall 1 \leq i < j \leq n, d(y_i,y_j) > 2(h''-h)$, so $((T,d,H,\nu),h,h')$ is in the left-hand set. This proves the inclusion ($\supset$), so the equality holds.

We can reformulate the last expression for U_k:

$$U_k = \left\{ ((T,d,H,\nu),h,h') \in \mathbb{T}\times\mathbb{R}^2 \;\middle|\; \begin{array}{l} \exists p,q \in \mathbb{Q}, \\ p < h < h' < q, \\ \exists x_1,...,x_k \in T, \\ \min_{1\le i\le k} H(x_i) > q, \\ \forall 1 \le i < j \le n, d(x_i,x_j) > H(x_i)+H(x_j)-2p \end{array} \right\}$$
$$= \bigcup_{p<q\in\mathbb{Q}} U_k^{p,q} \times \{(h,h') \in \mathbb{R}^2 | p < h < h' < q\},$$

where we set

$$U_k^{p,q} = \left\{ (T,d,H,\nu) \in \mathbb{T} \;\middle|\; \begin{array}{l} \exists x_1,...,x_k \in T, \\ \min_{1\le i\le k} H(x_i) > q, \\ \forall 1 \le i < j \le n, d(x_i,x_j) > H(x_i)+H(x_j)-2p \end{array} \right\}.$$

Now, let us look at the complement of $U_k^{p,q}$:

$$\mathbb{T}\setminus U_k^{p,q} = \left\{ (T,d,H,\nu) \in \mathbb{T} \;\middle|\; \begin{array}{l} \forall x_1,...,x_k \in T, \\ \min_{1\le i\le k} H(x_i) \le q \text{ or} \\ \exists 1 \le i < j \le n, d(x_i,x_j) \le H(x_i)+H(x_j)-2p \end{array} \right\}.$$

It is closed by Lemma 3.4.9, so $U_k^{h,h'}$ is open. We have proven that the map $f : (T,h,h') \mapsto 1 \vee \sup_{h''>h'} n^{h,h''}(T)$ is measurable on D.

Step 3: conclusion. Set

$$D_{\neq 0} = D \setminus D_0.$$

The set $D_{\neq 0}$ is the set of all triplets (T,h,h') such that $n^{h,h'}(T) > 0$. Not that this is equivalent to $h' \in H(T)$. It is a Borel set since D_0 is a Borel subset of D (Step 1). Note that for every S-compact T, the map $h' \mapsto n^{h,h'}(T)$ is non-increasing, piece-wise constant and left-continuous on $H(T) \cap (h,\infty)$. It follows that if $n^{h,h'}(T) \neq 0$, then $h' \in H(T)$ and one of two cases arises.

- If $h \le \min_T H$, then $h' \mapsto n^{h,h'}(T) = 1$ for every $h' \in H(T)$ and 0 everywhere else. By definition of f, we have $1 \le f(T,h,h') \le \sup_{h''>h} n^{h,h''}(T) = 1$ for every $h' > h$, so $n^{h,h'}(T) = 1 = \lim_{h''\uparrow h'} f(T,h,h'')$.
- If $h' > h > \min_T H$, then, since the map $h' \mapsto n^{h,h'}(T)$ is non-increasing, piece-wise constant and left-continuous on $H(T)\cap(h,\infty)$, there is a non-empty interval $[h'',h')$ on which $n^{h,\cdot}(T)$ is constant equal to $n^{h,h'}(T)$. Since the map $h' \mapsto n^{h,h'}(T)$ is non-increasing, we have $f(T,h,\cdot) = n^{h,h'}(T) \ge 1$ on $[h'',h')$, so $n^{h,h'} = \lim_{h''\uparrow h'} f(T,h,h'')$.

It follows that $(T,h,h') \mapsto n^{h,h'}(T)$ is the point-wise limit of the sequence

$$\left((T,h,h') \mapsto \mathbf{1}_{D_{\neq 0}}(T,h,h') \cdot f\left(T,h,\frac{1}{n}h + (1-\frac{1}{n})h'\right)\right)_{n\in\mathbb{N}^*}.$$

With Step 2 and since $D_{\neq 0}$ is a Borel set, each term is a measurable map, so the point-wise limit $(T,h,h') \mapsto n^{h,h'}(T)$ is measurable on D. □

4.2.2 Trimming

In this subsection, we adapt the ε-trimming from real trees to our height-labelled trees. The ε-trimming is a powerful tool for approximation and will play an extensive role in the following sections.

For $(T, d, H, \nu) \in \mathbb{T}$ and $\varepsilon > 0$, define for every $x, y \in T$:

$$d^\varepsilon(x, y) = \max(|H(x) - H(y)|, d(x, y) - 2\varepsilon).$$

We will prove in Lemma 4.2.6 that d^ε is a pseudo-distance, so we can define T^ε as the quotient of T by the equivalence relation $d^\varepsilon(x, y) = 0$ and $\rho : T \to T^\varepsilon$ the canonical projection. Note that by definition of d^ε, H is constant on each equivalence class, so H is still defined on the quotient T^ε and $H(\rho(x)) = H(x)$ for all $x \in T$.

Definition 4.2.5. *For* $(T, d, H, \nu) \in \mathbb{T}$ *and* $\varepsilon > 0$, *we define* $\mathrm{Trim}_\varepsilon(T) = (T^\varepsilon, d^\varepsilon, H, \rho\nu)$ *the* ε*-trimming of* T.

See Figure 4.3 for an instance of T and $\mathrm{Trim}_\varepsilon(T)$.

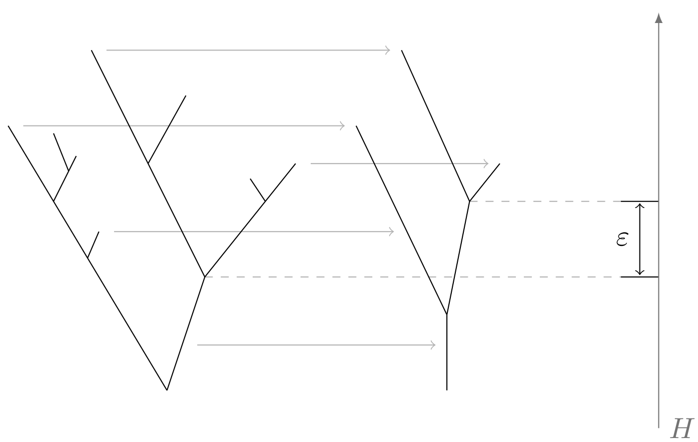

Figure 4.3: Example of an ε-trimming. The left-hand tree represents T, the right-hand one T^ε. The general shape is preserved, as the height of the root. The branching points are elevated by ε, as shown by the dotted lines. The branches shorter than ε disappear. Represented in gray are the projections of some chosen points.

Lemma 4.2.6. $\mathrm{Trim}_\varepsilon(T)$ *is a well-defined height-labelled tree of* $\mathbb{T}$.

Proof. Let us prove that d^ε satisfies the triangle inequality. Take $x, x', x'' \in T$ and suppose without loss of generality that $x \wedge x' \preceq x' \wedge x''$. We then have $x \wedge x' = x \wedge x' \wedge x'' \preceq x \wedge x''$ and

$$\begin{aligned} d^\varepsilon(x, x'') &= \max(|H(x) - H(x'')|, H(x) + H(x'') - 2H(x \wedge x'') - 2\varepsilon) \\ &\le \max(|H(x) - H(x'')|, H(x) + H(x'') - 2H(x \wedge x') - 2\varepsilon) \\ &\le \max(|H(x) - H(x')|, H(x) + H(x') - 2H(x \wedge x') - 2\varepsilon) + |H(x') - H(x'')| \\ &\le d^\varepsilon(x, x') + d^\varepsilon(x', x''). \end{aligned}$$

Thus, the function d^ε is symmetric, non-negative and satisfies the triangle inequality, so T^ε is well-defined and d^ε is a distance over T^ε. The function H is well defined and 1-Lipschitz on T^ε.

For $y, y' \in T^\varepsilon$ we note $y' \preceq^\varepsilon y$ when $d^\varepsilon(y, y') = H(y) - H(y')$. To prove that $\preceq^\varepsilon$ is a order, the reflexivity and anti-symmetry are straightforward, so let us check the transitivity. Since H is 1-Lipschitz for d_ε, so, for $x \preceq^\varepsilon y \preceq^\varepsilon z \in T^\varepsilon$, we have

$$d_\varepsilon(x, z) \geq H(z) - H(x) = H(z) - H(y) + H(y) - H(x) = d_\varepsilon(z, y) + d_\varepsilon(y, x) \geq d_\varepsilon(x, z)$$

so $d_\varepsilon(x, z) = H(z) - H(x)$, and we have $x \preceq^\varepsilon z$.

Set $\preceq$ the canonical order of T. Let us prove that $(T^\varepsilon, H, \preceq^\varepsilon)$ is a coded tree. Recall Conditions 1-4 in Definition 4.1.11. The proof of 1. and 2. are directly included.

1. The image $H(T^\varepsilon)$ is connected as $H(T^\varepsilon) = H(T)$.
2. H is strictly increasing by definition for $\preceq^\varepsilon$.
3. For every $y \in T^\varepsilon$ and $h \in H(T^\varepsilon)$ such that $h \leq H(y)$, there exists a unique $y' \in T^\varepsilon$ such that $y' \preceq^\varepsilon y$ and $H(y') = h$.
4. For $y, y' \in T^\varepsilon$, there exists a point $y \wedge^\varepsilon y' = \max\{y'' \in T^\varepsilon | y'' \preceq^\varepsilon y, y'' \preceq^\varepsilon y'\}$. We shall also check that for $y, y' \in T^\varepsilon$:

$$d^\varepsilon(y, y') = H(y) + H(y') - 2H(y \wedge^\varepsilon y'). \tag{4.2.2}$$

Proof of 3.: take $y \in T^\varepsilon$, $h \in H(T^\varepsilon) = H(T)$ such that $h \leq H(y)$. Choose x an antecedent of y by ρ and let us find y'. Take $x' \in T$ the only point such that $H(x') = h$ and $x' \preceq x$. Set $y' = \rho(x')$. Since $d(x, x') = H(x) - H(x')$ we have $d^\varepsilon(x, x') = H(x) - H(x') = H(x) - h$, so $y' \preceq^\varepsilon y$.

Now, for the uniqueness, take $y'' \preceq^\varepsilon y$ such that $H(y'') = h$ and $x'' \in \rho^{-1}(\{y''\})$, and let us prove that $y'' = y'$. The point x'' satisfies $H(x'') = H(y'') = h$ and $d^\varepsilon(x, x'') = d^\varepsilon(y, y'') = H(x) - h$. We have $H(x'') = H(x')$ and $d(x, x'') \geq d^\varepsilon(x, x'') = d(x, x')$, so $x \wedge x'' \preceq x \wedge x' = x'$. This implies $x \wedge x'' = x \wedge x' \wedge x'' = x' \wedge x''$ and we have

$$\begin{aligned} d(x'', x') &= H(x'') + H(x') - 2H(x'' \wedge x') \\ &= H(x'') + H(x) - H(x) + H(x') - 2H(x'' \wedge x) \\ &= d(x'', x) - (H(x) - h) \\ &\leq d^\varepsilon(x'', x) + 2\varepsilon - (H(x) - h) = 2\varepsilon. \end{aligned}$$

We conclude that $d^\varepsilon(y'', y') = d^\varepsilon(x'', x') = 0$ so y' is the only ancestor of y at height h.

Proof of 4. Figure 4.4 should help to visualize the following proof. Take $y, y' \in T^\varepsilon$. If $d^\varepsilon(y, y') = |H(y) - H(y')|$ then y and y' are comparable so $\min(y, y')$ is the MRCA that is $\max\{z \in T^\varepsilon | z \preceq^\varepsilon y, z \preceq^\varepsilon y'\}$. If not, then $d^\varepsilon(y, y') > |H(y) - H(y')|$. Use Figure 4.4 for reference. Take $x, x' \in T$ respective antecedents of y and y' by ρ. Since

$$d^\varepsilon(x, x') = d^\varepsilon(y, y') > |H(y) - H(y')| = |H(x) - H(x')|,$$

we have $d(x, x') - 2\varepsilon = d^\varepsilon(x, x') > |H(x) - H(x')|$. Consider $h = H(x \wedge x') + \varepsilon$, we have

$$\begin{aligned} H(x) + H(x') - 2\min(H(x), H(x')) &= |H(x) - H(x')| \\ &< d(x, x') - 2\varepsilon \\ &= H(x) + H(x') - 2(H(x \wedge x') + \varepsilon), \end{aligned}$$

so $h = H(x \wedge x') + \varepsilon < \min(H(x), H(x'))$. Take $x'' \preceq x$ and $x''' \preceq x'$ the two points of $[\![x, x']\!]$ such that $H(x'') = H(x''') = h$. Note that $x'' \wedge x''' = x \wedge x'$. We have $d(x'', x''') = H(x'') + H(x''') - 2H(x'' \wedge x''') = 2(h - H(x \wedge x')) = 2\varepsilon$ so $d^\varepsilon(x'', x''') = 0$. We set $y'' = \rho(x'') = \rho(x''')$. We want to prove that $y'' \preceq^\varepsilon y$, $y'' \preceq^\varepsilon y'$ and that if $z \in T^\varepsilon$ satisfies $z \preceq^\varepsilon y$ and $z \preceq^\varepsilon y'$, then we have $z \preceq^\varepsilon y''$.

We have

$$d^\varepsilon(y, y'') = \max(|H(x) - H(x'')|, d(x, x'') - \varepsilon) = H(x) - H(x'') = H(y) - H(y'').$$

So we deduce that $y'' \preceq^\varepsilon y$ and similarly $y'' \preceq^\varepsilon y'$.

Let $z \in T^\varepsilon$ be such that $z \preceq^\varepsilon y$ and $z \preceq^\varepsilon y'$. We have:

$$d^\varepsilon(y, y') \leq d^\varepsilon(y, z) + d^\varepsilon(z, y') = H(y) + H(y') - 2H(z).$$

This implies that $H(z) \leq H(y'')$. According to 3., there exists $z' \in T^\varepsilon$ such that $z' \preceq^\varepsilon y''$ and $H(z') = H(z)$. Thus, we have $z' \preceq y'' \preceq^\varepsilon y$, $z \preceq^\varepsilon y$ and $H(z') = H(z)$. This implies that $z' = z$. Thus $z \preceq y''$, and so $y'' = \max\{y'' \in T^\varepsilon | z \preceq^\varepsilon y, z \preceq^\varepsilon y'\}$, that is by convention $y'' = y \wedge^\varepsilon y'$. We also have that (4.2.2) holds as:

$$\begin{aligned}
d^\varepsilon(y, y') &= d(x, x') - 2\varepsilon \\
&= d(x, x'') + d(x'', x'' \wedge x''') + d(x''', x'' \wedge x''') + d(x', x''') - 2\varepsilon \\
&= d(x, x'') + d(x''', x') \\
&= H(x) - H(x'') + H(x') - H(x''') \\
&= H(y) + H(y') - 2H(y \wedge^\varepsilon y').
\end{aligned}$$

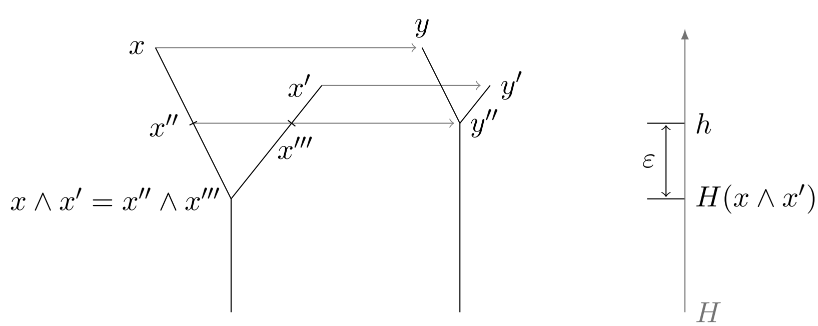

Figure 4.4: Position of x, x', x'', x''', y, y' and y'' on a simple example. The left-hand tree represents T, the right-hand one T^ε. The dashed arrows represent ρ_ε. x'' and x''' have the same image y''.

We have proven Conditions 1-4, so $(T^\varepsilon, H, \preceq^\varepsilon)$ is a coded tree. Using Proposition 4.1.14 with (4.2.2), we see that $\mathrm{Trim}_\varepsilon(T)$ is a height-labelled tree with its genealogical order $\preceq^\varepsilon$. Since $d^\varepsilon \leq d$, ρ is continuous, so for every $h \in \mathbb{R}_+$, $\mathrm{Slice}_h(T^\varepsilon)$ is the continuous image of the compact $\mathrm{Slice}_h(T)$, hence $\mathrm{Slice}_h(T^\varepsilon)$ is compact. It follows that $\mathrm{Trim}_\varepsilon(T)$ is S-compact. For every $h \in \mathbb{R}_+$, $\rho\nu(\mathrm{Slice}_h(T^\varepsilon)) = \nu(\mathrm{Slice}_h(T)) < \infty$; We get that $\mathrm{Trim}_\varepsilon(T) \in \mathbb{T}$. □

The next lemma assert that $\mathrm{Trim}_\varepsilon(T)$ is an approximation of T.

Lemma 4.2.7. *Let $T \in \mathbb{T}$ and $\varepsilon > 0$. We have:*

$$d_{\text{LGHP}}(T, \text{Trim}_\varepsilon(T)) \leq \varepsilon.$$

Proof. Consider the Borel subset of $T \times T^\varepsilon$: $A = \{(x, \rho(x))\}_{x \in T}$. The projection ρ is surjective by definition of T^ε, so A is a correspondence. Note that, since ρ preserves H, the restriction $A_h = \{(x, \rho(x))\}_{x \in T_h}$ is a correspondence between $T_h = \text{Slice}_h(T)$ and $T_h^\varepsilon = \text{Slice}_h(T^\varepsilon)$. Recall Conditions (3.4.1) to (3.4.4) to be a ε-correspondence between T_h and T_h^ε, and let us check them for A_h, using the properties of ρ.

Condition (3.4.1). We have for every $(x, y), (x', y') \in A_h$: $|d(x, x') - d^\varepsilon(y, y')| \leq 2\varepsilon$.

Condition (3.4.2). We have for every $(x, y) \in A_h$: $H(x) = H(y)$.

Condition (3.4.3). We have $\nu(B) \leq \rho\nu(\rho(B))$ for every Borel set $B \subset T_h$.

Condition (3.4.4).We have $\rho\nu(B') = \nu(\rho^{-1}(B'))$ for every Borel set $B' \subset T_h^\varepsilon$.

We get that A_h is a ε-correspondence. So Proposition 3.4.1 gives that $d_{\text{GHP}}(T_h, T_h^\varepsilon) \leq \varepsilon$ for every $h \in \mathbb{R}_+$ so $d_{\text{LGHP}}(T, \text{Trim}_\varepsilon(T)) \leq \varepsilon$. □

Lemma 4.2.8. *The map $T \mapsto \text{Trim}_\varepsilon(T)$ defined on $\mathbb{T}$ is 1-Lipschitz, hence measurable.*

Proof. Consider $(T, d, H, \nu), (T', d', H', \nu') \in \mathbb{T}$, $h \in \mathbb{R}_+$, $T_h = \text{Slice}_h(T)$ and $T'_h = \text{Slice}_h(T')$. Set $(U, \delta, H, \mu) = \text{Trim}_\varepsilon(T)$ and $(U', \delta', H', \mu') = \text{Trim}_\varepsilon(T')$, $U_h = \text{Slice}_h(U)$, $U'_h = \text{Slice}_h(U')$. We note ρ and ρ' the projections $T \mapsto U$ and $T' \mapsto U'$. By Proposition 3.4.1, we can choose A a η_0-correspondence between T_h and T'_h with $\eta_0 > d_{\text{GHP}}(T_h, T'_h)$. We set $A' = \{(\rho(x), \rho'(x'))\}_{(x,x') \in A}$. Let us prove that A' is a η_0-correspondence between U_h and U'_h. By construction of A and A', A' is a correspondence satisfying Condition (3.4.2) (3.4.3) and (3.4.4) with ε replaced by η_0. So we only have to prove Condition (3.4.1). Take $(u, u'), (v, v') \in A'$ and $(x, x'), (y, y') \in A$ some respective antecedents by ρ and ρ'. Using the fact that A is a δ_0-correspondence, we have

$$\begin{aligned}
&|\delta(u, v) - \delta'(u', v')| \\
&\qquad = |\max(|H(x) - H(y)|, d(x, y) - 2\varepsilon) - \max(|H'(x') - H'(y')|, d'(x', y') - 2\varepsilon)| \\
&\qquad \leq \max\left(||H(x) - H(y)| - |H'(x') - H'(y')||, |(d(x, y) - 2\varepsilon) - (d'(x', y') - 2\varepsilon)|\right) \\
&\qquad \leq \max\left(||H(x) - H'(x')| + |H(y) - H'(y')||, |d(x, y) - d'(x', y')|\right) \\
&\qquad \leq 2\eta_0.
\end{aligned}$$

We used Condition (3.4.2) and condition (3.4.1) for A in the last equality. We have proven that A' is a η_0-correspondence. Using Proposition 3.4.1 we see that $d_{\text{GHP}}(U_h, U'_h) \leq \eta_0$. Since $\eta_0 > d_{\text{GHP}}(T_h, T'_h)$ was arbitrary, we have $d_{\text{GHP}}(U_h, U'_h) \leq d_{\text{GHP}}(T_h, T'_h)$. By Definition 3.1.12 we see that $d_{\text{LGHP}}(U, U') \leq d_{\text{LGHP}}(T, T')$, so $T \mapsto T^\varepsilon$ is 1-Lipschitz on $(\mathbb{T}, d_{\text{LGHP}})$. □

Definition 4.2.9. *We call discrete tree any tree $T \in \mathbb{T}$ satisfying the following conditions:*

(i) For every $h \in \mathbb{R}$, T has only finitely many points at height h;

(ii) For every compact interval I, T has only a finitely many leaves with heights in I.

Any slice of a discrete tree T only contains finitely many branching points. For any leaf $x \in T$, take $y(x)$ the closest branching point in T, we call $B(x) = [\![x, y]\!]$ the external branch of x.

Lemma 4.2.10. *For every* $T \in \mathbb{T}$, $\varepsilon > 0$, $\mathrm{Trim}_\varepsilon(T)$ *is a discrete tree.*

Proof. The number of points at any height $h \in \mathbb{R}$ in $\mathrm{Trim}_\varepsilon(T)$ is simply $n^{h-\varepsilon,h}(T)$ which is finite by Lemma 4.2.3 since T is S-compact.

Take E the set of all leaves of $\mathrm{Trim}_\varepsilon(T)$ except, if $\inf_T H > -\infty$ the unique leaf at height $\inf_T H$ (which is usually called the root of T). For every $x \neq y \in E$, we can take $x', y' \in T$ some antecedents of x and y by ρ, the canonical projection from T to $\mathrm{Trim}_\varepsilon(T)$. Since x and y are leaves, they are distinct maximal elements for $\preceq^\varepsilon$, so they are not comparable for $\preceq^\varepsilon$. By definition of $\preceq^\varepsilon$, this means that $d^\varepsilon(x,y) > |H(x) - H(y)|$, so $d^\varepsilon(x,y) = d(x',y') - 2\varepsilon > |H(x') - H(y')|$. It follows that $d(x',y') > 2\varepsilon$. Let $h > 0$. Take E' a set of elements of T such that ρ is one-to-one from E' to E. Since $\inf_{x',y' \in E', x' \neq y'} d(x',y') \geq 2\varepsilon$ and T_h is compact, E' has only a finite number of elements in T_h. Since ρ is a height-preserving one-to-one map between E' and E, $\mathrm{Trim}_\varepsilon(T)$ only has a finite number of leaves with height in $[-h, h]$. □

4.2.3 Stump

We define the stump below h of a height-labelled tree (T, d, H, ν) as the sub-tree $\mathrm{Stump}_h(T) = \{x \in T | H(x) \leq h\}$. $\mathrm{Stump}_h(T)$ is equipped with the restriction of d, H and ν to $\mathrm{Stump}_h(T)$. The function Stump can easily be extended to measured labelled spaces. Note that Stump commutes with Slice and Trim.

Lemma 4.2.11. *The map* $(T, h) \mapsto \mathrm{Stump}_h(T)$ *is measurable on* $(\mathbb{T} \times \mathbb{R})$.

Proof. We write $S_h(T) = \mathrm{Stump}_h(T)$ for simplicity.

Step 1: we prove that for every $h \in \mathbb{R}$, the map $T \mapsto S_h(T)$ is measurable. Recall the measurable map $f(h, (T, d, H, \nu)) = (T, d, H, 1_{H \leq h} \cdot \nu)$ from Lemma 3.4.14, and write $f_h(T) = f(h, T)$. Note that $S_h(T) = S_h(f(h, T))$. Recall that $\mathbb{T}$ is closed in $(\mathbb{X}^S, d_{\mathrm{LGHP}})$ according to Theorem 4.1.15. Since $T \mapsto f_h(T)$ is measurable, we only need to prove that S_h is measurable on the direct image

$$f_h(\mathbb{T}) = \{(T, d, H, \nu) \in \mathbb{T} | H\nu((h, \infty)) = 0\}.$$

Since the application $\mu \mapsto \mu((h, +\infty)) \in [0, +\infty]$ defined on the set of Borel measures (*i.e.* measures which are finite on compact sets) on $\mathbb{R}$ is measurable, we deduce using Lemma 3.4.12 and Theorem 4.1.15 that $f_h(\mathbb{T})$ is a Borel subset of $\mathbb{T}$. We will prove that $D_h^\emptyset = \{(T, d, H, \nu) \in \mathbb{T} | S_h(T) = \emptyset\}$ is a Borel set, then we will prove that on $D_h = f_h(\mathbb{T}) \setminus D_h^\emptyset$, the map $T \mapsto S_h(T)$ is 2-Lipschitz in T.

Step 1.1: we prove that $D_h^\emptyset$ is a Borel set. We have $D_h^\emptyset = \{(T, d, H, \nu) \in \mathbb{T} | \forall x, H(x) \in (h, \infty)\}$. Set

$$F = \{(T, d, H, \nu) \in \mathbb{T} | \forall x, H(x) \in [h, \infty)\} \quad \text{and} \quad U = \{(T, d, H, \nu) \in \mathbb{T} | \forall x, H(x) \neq h\},$$

so that we have $D_h^\emptyset = F \cap U$. The set F is closed in $\mathbb{T}$ by Lemma 3.4.9 and since $\mathbb{T}$ is closed in $\mathbb{X}^S$, while

$$\mathbb{T} \setminus U = \{(T, d, H, \nu) \in \mathbb{T} | \exists x, H(x) = h\}$$

is closed in $\mathbb{T}$ by Lemma 3.4.11. This makes $D_h^\emptyset$ the intersection of a closed set and an open set, thus a Borel subset of $\mathbb{T}$.

Step 1.2: we prove that on D_h, the map $T \mapsto S_h(T)$ is 2-Lipschitz in T. Take $h \in \mathbb{R}$, (T, d, H, ν), (T', d', H', ν') two S-compact height-labelled trees of D_h, i.e. such that $H\nu((h,\infty)) = H'\nu'((h,\infty)) = 0$ and $S_h(T) \neq \emptyset \neq S_h(T')$. For $r \in \mathbb{R}_+$, take (Z, d_Z) a separable metric space, $\phi \in \mathrm{Iso}(T_r, Z)$, $\phi' \in \mathrm{Iso}(T'_r, Z)$. Note

$$\Delta_{\mathrm{H}} = d_{\mathrm{H}}([\phi \times H](T_r), [\phi' \times H'](T'_r)) \quad \text{and} \quad \Delta_{\mathrm{P}} = d_{\mathrm{P}}([\phi \times H](1_{T_r} \cdot \nu), [\phi' \times H'](1_{T'_r} \cdot \nu')).$$

We are looking at upper bounds for

$$\begin{aligned} \Delta'_{\mathrm{H}} &= d_{\mathrm{H}}([\phi \times H](S_h(T_r)), [\phi' \times H'](S_h(T'_r))) \\ \Delta'_{\mathrm{P}} &= d_{\mathrm{P}}([\phi \times H](1_{S_h(T_r)} \cdot \nu), [\phi' \times H'](1_{S_h(T'_r)} \cdot \nu')). \end{aligned}$$

We obviously have $\Delta'_{\mathrm{P}} = \Delta_{\mathrm{P}}$. Let us prove that when $\Delta_{\mathrm{H}} < \infty$,

$$[\phi \times H](S_h(T_r)) \subset \left([\phi' \times H'](S_h(T'_r))\right)^{2\Delta_{\mathrm{H}}}.$$

Recall Remark 3.2.1 (A). For any $x \in S_h(T_r)$, there exists $y \in T'_r$ such that

$$d_Z(\phi(x), \phi'(y)) \vee |H(x) - H'(y)| \leq \Delta_{\mathrm{H}}.$$

If $y \in S_h(T'_r)$ then we are done. If not, we have $h < H'(y)$. Since $S_h(T') \neq \emptyset$, $\inf_{T'} H' \leq h$, so we can take y' the ancestor of y in T' at height h. Since $x \in S_h(T_r)$, and $y \in T'_r \setminus S_h(T')$, we have $-r \leq H(x) \leq h < H'(y) \leq r$. Since $H'(y') = h$, we have $y' \in S_h(T'_r)$. Let us check that y' is close to x. We have

$$\begin{aligned} d_Z(\phi(x), \phi'(y')) &\leq d_Z(\phi(x), \phi'(y)) + d'(y, y') \\ &\leq \Delta_{\mathrm{H}} + H'(y) - h \\ &\leq \Delta_{\mathrm{H}} + H'(y) - H(x) \\ &\leq 2\Delta_{\mathrm{H}} \end{aligned}$$

and $|H'(y') - H(x)| = h - H(x) \leq H'(y) - H(x) \leq \Delta_{\mathrm{H}}$. We have proven that

$$[\phi \times H](S_h(T_r)) \subset \left([\phi' \times H'](S_h(T'_r))\right)^{2\Delta_{\mathrm{H}}}.$$

Since T and T' hold symmetric roles, we have $\Delta'_{\mathrm{H}} \leq 2\Delta_{\mathrm{H}}$.

Taking the infimum in Z, ϕ, ϕ', we have for every $r \in \mathbb{R}_+$ that

$$d_{\mathrm{GHP}}(S_h(T_r), S_h(T'_r)) \leq 2 d_{\mathrm{GHP}}(T_r, T'_r),$$

so $d_{\mathrm{LGHP}}(S_h(T), S_h(T')) \leq 2 d_{\mathrm{LGHP}}(T, T')$, so $T \mapsto S_h(T)$ is 2-Lipschitz on D_h. This implies that $S_h \mathbf{1}_{D_h}$ is measurable. Then notice that

$$S_h(T) = S_h \circ f_h(T) = S_h(f_h(T)) \mathbf{1}_{f_h(T) \in D_h} + \emptyset \mathbf{1}_{f_h(T) \in D_h^{\emptyset}}.$$

We deduce that $S_h = (S_h \mathbf{1}_{D_h}) \circ f_h + \emptyset \mathbf{1}_{D_h^{\emptyset}} \circ f_h$ and thus S_h is measurable for every h.

Step 2: we prove that for every $T \in \mathbb{T}$, the map $h \mapsto S_h(T)$ is right-continuous on $\mathbb{R}$. Take (T, d, H, ν) a S-compact height-labelled tree and $h \in \mathbb{R}$. If T is empty, then S is constant. Suppose T not empty.

Case 1. Assume $S_h(T)$ is empty. This implies that $h < \min_T H$. Since $h \mapsto S_h(T)$ is constant on $(-\infty, \min_T H)$, we get that $S_h(T)$ is right-continuous at h.

Case 2. Assume that $S_h(T)$ is not empty. We will prove intermediary result, then make a second disjunction between sub-cases 2.1 and 2.2 to prove the right-hand continuity on $\mathbb{R}_+$, then on $\mathbb{R}$. For every $h' > h$ we can use the inclusion $S_h(T) \subset S_{h'}(T)$ as a particular embedding to give an upper bound for $d_{LGHP}(S_h(T), S_{h'}(T))$. For $r \in \mathbb{R}_+$, note

$$\Delta_{\mathrm{H}}(r) = d_{\mathrm{H}}([\mathrm{Id} \times H](S_h(T_r)), [\mathrm{Id} \times H](S_{h'}(T_r)))$$

and

$$\Delta_{\mathrm{P}}(r) = d_{\mathrm{P}}([\mathrm{Id} \times H](1_{H \leq h, |H| \leq r} \cdot \nu), [\mathrm{Id} \times H](1_{H \leq h', |H| \leq r} \cdot \nu)).$$

For every r, we have $S_h(T_r) \subset S_{h'}(T_r)$.

Let us prove that when $|h| \leq r$ we have

$$\Delta_{\mathrm{H}}(r) \leq h' - h \quad \text{and} \quad \Delta_{\mathrm{P}}(r) \leq H\nu((h, h']). \tag{4.2.3}$$

We first prove that $S_{h'}(T_r) \subset (S_h(T_r))^{h'-h}$. Recalling Remark 3.2.1 (A), take $x \in S_{h'}(T_r)$. If $x \in S_h(T_r)$, we are done. If not, we have $H(x) \in (h, h']$. Since $S_h(T)$ is non-empty by hypothesis, we can take x' the ancestor of x at height h. Since $-r \leq h < r$, we have $x' \in S_h(T_r)$ and $d(x, x') = H(x) - H(x') \leq h' - h$. Since $S_h(T) \subset S_{h'}(T)$, we have proven that $\Delta_{\mathrm{H}}(r) \leq h' - h$. Since $1_{S_h(T_r)}\nu \leq 1_{S_{h'}(T_r)}\nu$, we have with Lemma 3.1.5 that

$$\Delta_{\mathrm{P}}(r) \leq \nu(S_{h'}(T_r)) - \nu(S_h(T_r)) = H\nu((h, h'] \cap [-r, r]) \leq H\nu((h, h']).$$

We have proven Equation (4.2.3).

Case 2.1: if $h \geq 0$ and since $S_h(T)$ is non-empty, then for $r \in [0, h]$, we have

$$T_r = S_h(T_r) = S_{h'}(T_r),$$

so $\Delta_{\mathrm{H}}(r) = \Delta_{\mathrm{P}}(r) = 0$. For $r > h$, Equation (4.2.3) holds. By Definitions 3.1.9 and 3.1.12, this gives

$$\begin{aligned} d_{\mathrm{LGHP}}(S_h(T), S_{h'}(T)) &\leq \int_0^\infty \Big(1 \wedge (\Delta_{\mathrm{H}} \vee \Delta_{\mathrm{P}})\Big) \mathrm{e}^{-r} dr \\ &\leq \int_0^h 0\mathrm{e}^{-r} dr + \int_h^\infty (|h - h'| \vee H\nu((h, h'])) \mathrm{e}^{-r} dr \\ &\leq (|h - h'| \vee H\nu((h, h'])) \mathrm{e}^{-h} \\ &\xrightarrow[h' \downarrow h]{} 0, \end{aligned}$$

so, with the eventual addition of Case 1, $h \mapsto S_h(T)$ is right-continuous on $\mathbb{R}_+$.

Case 2.2: we treat the last case $h < 0$. We shall only consider that $h' \in (h, 0)$. For $r \in [0, -h')$, we have $S_h(T_r) = S_{h'}(T_r) = \emptyset$. For $r \in [-h', -h)$ we know nothing as Δ_{H} may be infinite. For $r \geq -h$, Equation (4.2.3) holds. By Definitions 3.1.9 and 3.1.12, this gives

$$\begin{aligned} d_{\mathrm{LGHP}}(S_h(T), S_{h'}(T)) &\leq \int_0^\infty \Big(1 \wedge (\Delta_{\mathrm{H}} \vee \Delta_{\mathrm{P}})\Big) \mathrm{e}^{-r} dr \\ &\leq 0 + \int_{-h'}^{-h} 1 \cdot \mathrm{e}^{-r} dr + \int_{-h}^{\infty} (|h-h'| \vee H\nu((h,h']))\mathrm{e}^{-r} dr \\ &= \mathrm{e}^{h'} - \mathrm{e}^{h} + (|h-h'| \wedge H\nu((h,h']))\mathrm{e}^{-h} \\ &\underset{h' \downarrow h}{\longrightarrow} 0. \end{aligned}$$

This concludes Step 2, as we have proven that $h \mapsto S_h(T)$ is right-continuous on $\mathbb{R}$.

Using Lemma 3.4.13, the measurability in T and right-continuity in h imply the measurability of $(T, h) \mapsto S_h(T)$. □

4.2.4 Measurability of the ancestral process

In this section, we give a parametrization of some trees which will be used in the next chapter. We define the vertical deformation of a tree, and give its action on the parametrization.

Definition 4.2.12. *For (T, d, H, ν) a height-labelled tree such that $\nu(T)$ is finite and f a non-decreasing continuous function from $\mathbb{R}$ to $\mathbb{R}$, we call vertical deformation of T by f the 4-uple (T', d', H', ν'), where*

- *$H'' = f \circ H$,*
- *$d'(x, y) = H''(x) + H''(y) - 2H''(x \wedge y)$,*
- *T'' is the quotient of T by the relation $d'(\cdot, \cdot) = 0$,*
- *T' is the completion of T'' for d',*
- *$\rho(x)$ is the natural projection of T into T',*
- *$\nu' = \rho \circ \nu$,*
- *H' is the 1-Lipschitz extension of H'' over T'.*

Remark 4.2.13. The hypothesis $\nu(T) < \infty$ is only used when f is bounded, to prevent the accumulation of an infinite measure at finite height for ν'. We could alternatively suppose that f is surjective on $\mathbb{R}$. Similarly, the completion step in the definition of T' comes into play when f is bounded, and allows T' to be S-compact (this is proven in Lemma 4.2.14).

Lemma 4.2.14. *The vertical deformation $(T', d', H', \rho \circ \nu)$ of a S-compact tree (T, d, H, ν), with $\nu(T)$ finite, by a non-decreasing continuous function f is still a S-compact height-labelled tree.*

Proof. Step 1: we prove that $(T'', d, H', 0)$ is a height-labelled tree. Set $\preceq'$ the relation such $x' \preceq' y' \Leftrightarrow d'(x', y') = H'(y') - H'(x')$. With Proposition 4.1.14, we just have to prove that $T'' = (\rho(T), H', \preceq')$ is a coded tree. Let us prove the four conditions in Definition 4.1.11 and that $\preceq'$ is an order.

Condition 1: f is continuous, so $H'(T'') = f \circ H(T)$ is an interval.

Condition 2: if $x' \preceq' y'$ and $x' \neq y'$, then $H'(y) - H'(x) = d'(x', y') > 0$, so H' is increasing.

We now prove that the relation $\preceq'$ is an order on T'. The reflexivity is obvious since $d'(x', x') = H'(x) - H'(x') = 0$; Condition 2 implies the anti-symmetry; and for $x' \preceq' y'$ and $y' \preceq' z'$, we use the fact that H' is 1-Lipschitz for d' to prove that

$$\begin{aligned} H'(z') - H'(x') &\leq d'(z', x') \\ &\leq d'(z', y') + d'(y', x') \\ &= H'(z') - H'(y') + H'(y') - H'(x') \\ &= H'(z') - H'(x'), \end{aligned}$$

and deduce that $z' \preceq' x'$, so $\preceq'$ is transitive. We have proven that $\preceq'$ is an order on T''.

Condition 3: take $x' \in T''$ and $h' \in H'(T'')$ such that $h' \leq H'(x')$ and let us prove that there exists $y' \preceq' x'$ such that $H'(y') = h'$. Choose $x \in T$, $h \in H(T)$ such that $\rho(x) = x'$ and $f(h) = h'$. If $h' = H'(x')$ then the result is obvious. If $h' < H'(x')$, then $h < H(x)$, so there exists $y \preceq x$ such that $H(y) = h$. Set $y' = \rho(y)$, we have $H'(y') = h'$. Since $y = y \wedge x$, we have $d'(x', y') = H'(x') + H'(y') - 2H'(\rho(y \wedge x)) = H'(x') - H'(y')$, so $y' \preceq' x'$. We have proven the existence. Now, suppose that there exists $y_1', y_2' \preceq' x'$ with $H'(y_1') = H'(y_2')$. Take $x, y_1, y_2 \in T$ some respective antecedents of x', y_1' and y_2' by ρ. $x \wedge y_1$ and $x \wedge y_2$ are ancestors of x, so they are comparable by Remark 4.1.12. Suppose without loss of generality that $x \wedge y_1 \preceq x \wedge y_2$. This means that $x \wedge y_1$ is a common ancestor of y_1 and y_2 and thus $H(x \wedge y_1) \leq H(y_1 \wedge y_2)$. We have

$$\begin{aligned} d'(y_1', y_2') &= d'(y_1, y_2) \\ &= H'(y_1) + H'(y_2) - 2H'(y_1 \wedge y_2) \\ &\leq H'(y_1) + H'(y_2) - 2H'(y_1 \wedge x) \\ &= H'(y_2) + d'(x, y_1) - H'(x) \\ &= H'(y_2') - H'(x') + d'(x', y_1') \\ &= H'(y_2') - H'(x') + H'(x') - H'(y_1') \\ &= 0, \end{aligned}$$

where for the first and fourth equality, we used the fact that ρ preserves d' and H'; the fifth equality comes from the fact that $y_1' \preceq' x'$. We have proven Condition 3.

Condition 4: let us prove that for every $x', y' \in T''$, the set of all points $z' \in T''$ such that $z' \preceq' x'$ and $z' \preceq' y'$ has a maximal element. With Condition 2, if we find z' maximizing $H'(z')$, z' is automatically maximal. Take x, y antecedents of x', y' for ρ. For any z' such that $z' \preceq' x'$ and $z' \preceq' y'$, we have $H'(x') - 2H'(z') + H'(y') = d'(x', z') + d'(z', y') \geq d'(x', y') = H'(x') - 2H'(x \wedge y) + H'(y')$, so $H'(z') \leq H'(x \wedge y)$. It follows that taking $z' = \rho(x \wedge y)$ provides a maximal element.

We have proven that $(T'', H', \preceq')$ is a coded tree, so with Proposition 4.1.14, $(T'', d', H', 0)$ is a height-labelled tree.

Step 2: We prove that (T', d', H', ν') is an S-compact height-labelled tree. The space $(T', d', H', 0)$ is the completion of $(T'', d', H', 0)$ which is a height-labelled tree, so it is a height-labelled tree by Lemma 4.1.4. We also have $\nu'(T') = \nu'(T'') = \nu(T) < \infty$, so (T', d', H', ν') is a height-labelled tree. To prove that (T', d', H', ν') is S-compact, we will use Lemma 4.2.3. Take $h'_1 < h'_2 \in \mathbb{R}$ and $0 < \varepsilon < \frac{1}{2}(h'_2 - h'_1)$. Suppose that $n^{h'_1,h'_2}(T') \geq k$ for some integer $k > 1$. Since $H'(T')$ is the closure of the interval $H'(T)$ and $h'_1 < h'_1 + \varepsilon < h'_2 - \varepsilon < h'_2$, there exists $h_1(\varepsilon) < h_2(\varepsilon) \in H(T)$ such that $f(h_1(\varepsilon)) = h'_1 + \varepsilon$ and $f(h_2(\varepsilon)) = h'_2 - \varepsilon$. By definition, there exists $x'_1, ..., x'_k \in T'$ such that for every $i \neq j$, $H'(x'_i) = h'_2$ and $d'(x'_i, x'_j) > 2(h'_2 - h'_1)$. Take $\varepsilon > 0$. Since $T'' = \rho(T)$ is dense in T', there exists $x_1, ..., x_k \in T$ such that for every $i \neq j$, $|H'(x_i) - h'_2| < \varepsilon$ and $d'(x_i, x_j) > 2(h'_2 - h'_1)$. Since $d'(x_i, x_j) = H'(x_i) + H'(x_j) - 2H'(x_i \wedge x_j)$, we have

$$H'(x_i \wedge x_j) = \frac{1}{2}\left(H'(x_i) + H'(x_j) - d'(x_i, x_j)\right) < h'_2 + \varepsilon - (h'_2 - h'_1) = h'_1 + \varepsilon.$$

It follows that for every $i \neq j$, $H'(x_i) > h'_2 - \varepsilon$ and $H'(x_i \wedge x_j) < h'_1 + \varepsilon$. Recalling the choice of $h_1(\varepsilon)$, $h_2(\varepsilon)$ and since f is non-decreasing, we deduce that for every $i \neq j$, $H(x_i) > h_2(\varepsilon)$ and $H(x_i \wedge x_j) < h_1(\varepsilon)$. This means that $n^{h_1(\varepsilon),h_2(\varepsilon)}(T) \geq k$.

We have proven that for for every $k > 1$, $n^{h'_1,h'_2}(T') \geq k \Rightarrow n^{h_1(\varepsilon),h_2(\varepsilon)}(T) \geq k$, so we have $n^{h'_1,h'_2}(T)' \leq 1 \vee n^{h_1(\varepsilon),h_2(\varepsilon)}(T)$. Since T is S-compact, so with Lemma 4.2.3, we get that $n^{h_1(\varepsilon),h_2(\varepsilon)}(T)$ is finite. We have proven that $n^{h'_1,h'_2}(T')$ is finite with arbitrary $h_1 < h_2$. Since T' is complete, we know from Lemma 4.2.3 that T' is S-compact. □

Now, we give a construction similar to the ancestral processes defined in [8], that is a tree with all the leaves at the same height, a measure concentrated on the leaves and a characterization of the tree by the coalescence times.

Set $\mathbb{R}^{\mathbb{N}^*}_{+,0}$ the set of non-increasing sequences of non-negative real numbers converging to 0, and $\mathbb{R}^{\omega}_{+,0} \subset \mathbb{R}^{\mathbb{N}^*}_{+,0}$ the set of non-increasing sequences of non-negative real numbers containing only a finite number of positive terms. We set

$$D = \{(u_n)_{n\in\mathbb{N}^*} \in (0,1)^{\mathbb{N}^*} | \forall i < j, u_i \neq u_j\}.$$

The spaces $\mathbb{R}^{\omega}_{+,0}$ and $\mathbb{R}^{\mathbb{N}^*}_{+,0}$ are equipped with the norm $||\cdot||_\infty$ of uniform convergence, for which $\mathbb{R}^{\omega}_{+,0}$ is dense in $\mathbb{R}^{\mathbb{N}^*}_{+,0}$. The space D equipped with the topology of the pointwise convergence, for which it is a Borel set of the Polish space $[0,1]^{\mathbb{N}^*}$.

Definition 4.2.15. *For $h \in \mathbb{R}$, $(\zeta_n)_{n\in\mathbb{N}^*} \in \mathbb{R}^{\mathbb{N}^*}_{+,0}$ and $(u_n)_{n\in\mathbb{N}^*} \in D$, we define $E = (0,1) \times (-\infty, h]$, and for $(x,y), (x',y') \in E$, $H(x,y) = y$,*

$$d((x,y),(x',y')) = y + y' - 2(y \wedge y' \wedge \inf_{x \leq u_n < x'} (h - \zeta_n)),$$

with the convention that $\inf \emptyset = +\infty$. Let ν be the 1-dimensional Lebesgue measure on $(0,1) \times \{h\}$ and T the quotient of E by the relation $d(\cdot,\cdot) = 0$. We call $\tau(h, (\zeta_n)_{n\in\mathbb{N}^*}, (u_n)_{n\in\mathbb{N}^*})$ the space (T', d', H', ν'), where (T', d') is the completion of (T, d), H' the 1-Lipschitz extension of H to T, and ν' the projection of ν onto the quotient.

Lemma 4.2.16. *Let $h \in \mathbb{R}$, $(\zeta_n)_{n\in\mathbb{N}^*} \in \mathbb{R}^{\mathbb{N}^*}_{+,0}$, and $(u_n)_{n\in\mathbb{N}^*} \in D$. The space $\tau(h, (\zeta_n)_{n\in\mathbb{N}^*}, (u_n)_{n\in\mathbb{N}^*}))$ is well-defined, and is a S-compact tree.*

Proof. In the proof, we use the notations from Definition 4.2.15. Let us define the relation $\preceq$, such that for every $(x,y)(x',y') \in E = (0,1) \times (-\infty, h]$ we have

$$(x,y) \preceq (x',y) \preceq (x',y') \Leftrightarrow y \leq y' \wedge \inf_{x\wedge x' \leq u_n < x \vee x'} (h - \zeta_n).$$

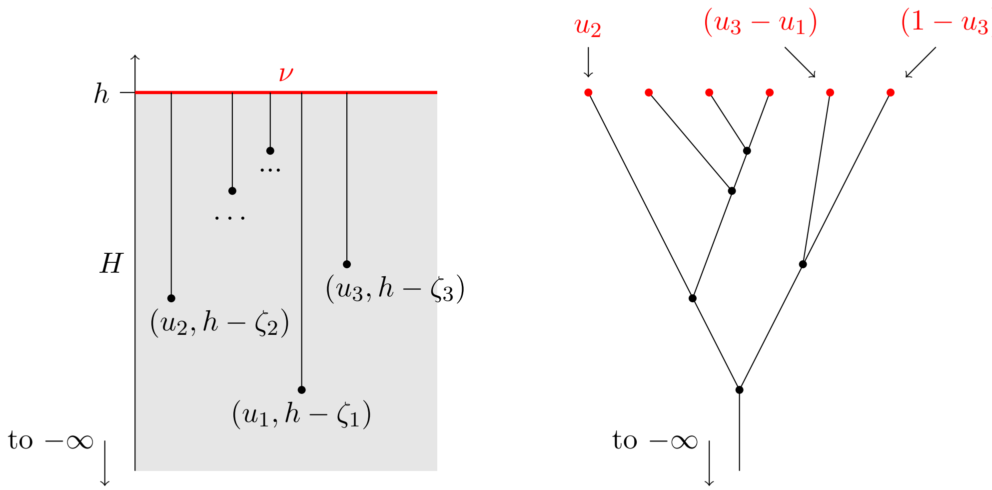

Figure 4.5: Example of the construction.

Step 1. We will prove that $d(\cdot, \cdot) = 0$ is an equivalence relation, and that $\preceq$ is an order on the quotient of E by $d(\cdot, \cdot) = 0$. The relation $\preceq$ is reflexive. Let us prove that it is transitive. Suppose $(x'', y'') \preceq (x', y') \preceq (x, y)$. We need to consider the order of the abscissas, there are three cases to consider, depending on which abscissa is in the middle. Since the demonstration is the same in each case, we only do the case $x'' \leq x' \leq x$. In this case, we have $x'' \leq x$ and

$$\begin{aligned} y \wedge (\inf_{x'' \leq u_n < x} (h - \zeta_n)) &= y \wedge (\inf_{x' \leq u_n < x} (h - \zeta_n)) \wedge (\inf_{x'' \leq u_n < x'} (h - \zeta_n)) \\ &\geq y' \wedge (\inf_{x'' \leq u_n < x'} (h - \zeta_n)) \\ &\geq y'', \end{aligned}$$

where we used the definition of $(x', y') \preceq (x, y)$ for the first inequality and $(x'', y'') \preceq (x', y')$ for the second. We have proven that $(x'', y'') \preceq (x, y)$, so $\preceq$ is transitive. Finally, let us prove that $(x', y') \preceq (x, y)$ and $(x, y) \preceq (x', y')$ if and only if $d((x, y), (x', y')) = 0$. Suppose without loss of generality that $x \leq x'$. We see easily that

$$\begin{aligned} \left\{ \begin{array}{l} (x', y') \preceq (x, y) \\ (x, y) \preceq (x', y') \end{array} \right. &\Leftrightarrow \left\{ \begin{array}{l} y' \leq y \wedge (\inf_{x \leq u_n < x'} (h - \zeta_n)) \\ y \leq y' \wedge (\inf_{x \leq u_n < x'} (h - \zeta_n)) \end{array} \right. \\ &\Leftrightarrow y = y' \leq \inf_{x \leq u_n < x'} (h - \zeta_n) \\ &\Leftrightarrow y + y' - 2(y \wedge y' \wedge (\inf_{x \leq u_n < x'} (h - \zeta_n))) = 0 \\ &\Leftrightarrow d((x, y), (x', y')) = 0. \end{aligned}$$

Since $\preceq$ is transitive and reflexive, we have proven that $d(\cdot, \cdot) = 0$ is an equivalence relation, so the quotient T of E by this relation is well-defined. This means that $\preceq$ is defined without ambiguity and is anti-symmetric on T, so $\preceq$ is an order on T.

In particular, note that if $d((x, y), (x', y')) = 0$, then $y = y'$, so H is also defined on T. This means that (T, d, H) is well-defined.

Step 2. Let us prove that $(T, H, \preceq)$ is a coded tree, by checking the four conditions from Definition 4.1.11.

Condition 1: the direct image $H(T)$ is equal to $(-\infty, h]$ by choice of E, so it is an interval.

Condition 2: by definition of $\preceq$, the map $H : (x, y) \mapsto y$ is strictly increasing.

Condition 3: we work on E. Take $(x_0, y_0) \in E$, $h_0 \in (-\infty, y_0]$, so that the point (x_0, h_0) satisfies $(x_0, h_0) \preceq (x_0, y_0)$ and $H(x_0, h_0) = h_0$. Let us prove that for every (x, y) such that $(x, y) \preceq (x_0, y_0)$ and $H(x, y) = h_0$, we have $d((x, y), (x_0, h_0)) = 0$. Suppose that $x_0 \leq x$ (the demonstration is the same for $x < x_0$), we have

$$y \leq y_0 \wedge (\inf_{x_0 \leq u_n < x} (h - \zeta_n))$$

by definition of $\preceq$. It follows that, by definition of d,

$$\begin{aligned} d((x_0, h_0), (x, y)) &= h_0 + y - 2(h_0 \wedge y \wedge (\inf_{x_0 \leq u_n < x} (h - \zeta_n))) \\ &= h_0 + y - 2(h_0 \wedge y). \end{aligned}$$

Since we have $y = H(x, y) = h_0$, the last line gives that $d((x_0, h_0), (x, y)) = 0$. Since ρ (the projection from E to T) is surjective onto T, we have proven that $\rho(x_0, h_0)$ is the only ancestor of $\rho(x_0, y_0)$ at height h_0, so (T, d, H) satisfies Condition 3.

Condition 4: once again we work on E. Take (x, y), $(x', y') \in E$. From Step 1 and the proof of Condition 3, we see that (x'', y'') is an ancestor of (x, y) at height y'' if and only if it is equivalent to (x, y'') and $y'' \leq y$. It follows that (x'', y'') is a common ancestor of (x, y) and (x', y') if and only if $d((x, y''), (x'', y'')) = 0$, $d((x', y''), (x'', y'')) = 0$ and $y'' \leq y \wedge y'$. This means that there exists a common ancestor of (x, y) and (x', y') at height y'' if and only if $d((x, y''), (x', y'')) = 0$ and $y'' \leq y \wedge y'$. Supposing without loss of generality that $x \leq x'$, we have

$$\begin{aligned} d((x, y''), (x', y'')) = 0 &\Leftrightarrow y'' + y'' - 2(y'' \wedge y'' \wedge (\inf_{x \leq u_n < x'} (h - \zeta_n))) = 0 \\ &\Leftrightarrow y'' \leq (\inf_{x \leq u_n < x'} (h - \zeta_n)). \end{aligned} \tag{4.2.4}$$

It follows that $(x, y \wedge y' \wedge \inf_{x \leq u_n < x'} (h - \zeta_n))$ is the MRCA of (x, y) and (x', y'). Noticed that:

$$\begin{aligned} d((x, y), (x', y') &= y + y' - 2(y \wedge y' \wedge \inf_{x \leq u_n < x'} (h - \zeta_n)) \\ &= H((x, y)) + H(x,', y')) - 2H((x, y) \wedge (x', y')). \end{aligned}$$

Thus (4.1.4) holds. We have proven that (T, d, H) is a coded tree, so with Proposition 4.1.14, $(T, d, H, 0)$ is a height-labelled tree.

Step 3 : Using Lemma 4.2.3, we prove that (T', d', H', ν') is S-compact height-labelled tree. Thanks to Lemma 4.1.4, $(T', d', H', 0)$ is a complete height-labelled tree, and ν' is finite, so, with Lemma 4.2.3 we only have to prove that $n^{h_1, h_2}(T') < \infty$ for every $h_1 < h_2 \in \mathbb{R}$. For every $h_1 < h_2 \leq h$ we have $n^{h_1, h}(T') \leq n^{h_1, h_2}(T')$, so it is enough to prove it for $h_2 \neq h$. We have $H'(T') = H(T) = (-\infty, h]$, so, if $h_2 > h$, we have $n^{h_1, h_2}(T') = 0 < \infty$. We can suppose $h_2 < h$ without loss of generality.

Set $n(h_2)$ the number of integers $n \in \mathbb{N}^*$ such that $h - \zeta_n < h_2$. Since $\lim_n \zeta_n = 0$, $n(h_2)$ is finite, and Equivalence (4.2.4) implies that T has exactly $n(h_2)+1$ points $(z_1, ..., z_{n(h_2)+1})$ at height h_2. In E, every point (x,y) with $y \leq h_2$ satisfies $(x,y) \preceq (x,h_2)$. This means that $\{z \in T | H(z) \leq h_2\}$ is covered by the sets $\{z \in T | z \preceq z_i\}$ for $1 \leq i \leq n(h_2)+1$. For every i, H is an bijective isometry from $\{z \in T | z \preceq z_i\}$ to $(-\infty, h_2]$, which is complete, so $\{z \in T | \exists i, z \preceq z_i\}$ is a closed set of T'. It follows that $\{x \in T | H(x) \leq h_2\}$ is a closed subset of T', so all the points of $T' \setminus T$ are at height at least h_2. In particular, the points at height h_1 in T' are all in T, so they are all ancestors of $\{z_1, ..., z_{n(h_2)}\}$. By Condition 3, there are at most $n(h_2)+1$ points at height h_1. Since $n^{h_1,h_2}(T')$ is the number of ancestors at height h_1 of the points at level h_2. We get $n^{h_1,h_2}(T') \leq n(h_2)+1 < \infty$. Since the restriction $h_2 \neq h$ was done without loss of generality, we have proven that for all $h_1 < h_2 \leq h$, $n^{h_1,h_2}(T') < \infty$, so (T', d', H', ν') is a S-compact height-labelled tree. □

Lemma 4.2.17. *Let $(h, (\zeta_n)_{n\in\mathbb{N}^*}, (u_n)_{n\in\mathbb{N}^*}) \in \mathbb{R} \times \mathbb{R}_{+,0}^{\mathbb{N}^*} \times D$ and f a continuous non-decreasing map from $\mathbb{R}$ to $\mathbb{R}$. Set $(T,d,H,\nu) = \tau(h, (\zeta_n)_{n\in\mathbb{N}^*}, (u_n)_{n\in\mathbb{N}^*})$. Then $(f(h) - f(h-\zeta_n))_{n\in\mathbb{N}^*}$ belongs to $\mathbb{R}_{+,0}^{\mathbb{N}^*}$ and the vertical deformation of (T,d,H,ν) by f is equal to the only non-$0_{h'}$ term of*

$$\mathrm{Crown}_{h'}\tau(f(h), (f(h) - f(h-\zeta_n))_{n\in\mathbb{N}^*}, (u_n)_{n\in\mathbb{N}^*}),$$

where $h' = \lim_{r\to-\infty} f(r)$.

Proof. It suffices to carefully consider the height of the leaves (they are all at height $f(h)$) and of the branching points (they were at height $(h-\xi_n)_{n\in\mathbb{N}^*}$ in T, so they are at height $(f(h-\xi_n))_{n\in\mathbb{N}^*}$ in T'). □

Lemma 4.2.18. *The function $(h, (\zeta_n)_{n\in\mathbb{N}^*}, (u_n)_{n\in\mathbb{N}^*}) \mapsto \tau(h, (\zeta_n)_{n\in\mathbb{N}^*}, (u_n)_{n\in\mathbb{N}^*})$ from $\mathbb{R} \times \mathbb{R}_{+,0}^{\mathbb{N}^*} \times D$ to $\mathbb{T}$ is measurable.*

Proof. Step 1: we prove that for every $(\zeta_n)_{n\in\mathbb{N}^*} \in \mathbb{R}_0^{\mathbb{N}^*}$ and $(u_n)_{n\in\mathbb{N}^*} \in D$, the map $h \mapsto \tau(h, (\zeta_n)_{n\in\mathbb{N}^*}, (u_n)_{n\in\mathbb{N}^*})$ is continuous. Take $h \in \mathbb{R}$, $(T,d,H,\nu) = \tau(h, (\zeta_n)_{n\in\mathbb{N}^*}, (u_n)_{n\in\mathbb{N}^*})$ and $(\delta_k)_{k\in\mathbb{N}^*}$ a sequence of real numbers converging to 0 such that $\sup_{k\in\mathbb{N}^*} |\delta_k| \leq 1$. Note that replacing h by $h+\delta_k$ only introduce a shift in H, such that we have the simple relation $\tau(h+\delta_k, (\zeta_n)_{n\in\mathbb{N}^*}, (u_n)_{n\in\mathbb{N}^*}) = (T, d, H+\delta_k, \nu)$, where $H+\delta_k$ represents the map $x \mapsto H(x)+\delta_k$. We shall now prove that $(T,d,H+\delta_k,\nu)$ converges to (T,d,H,ν) when δ_k goes to 0.

For all $k \in \mathbb{N}^*$, let us set $E^k = \mathrm{Slice}_{|h|+k+1}(T,d,H,\nu)$ and $G^k = \mathrm{Slice}_{|h|-\delta_k+k+1}(T,d,H+\delta_k,\nu)$. We have $-(|h|+k+1) < h = \max_T H < |h|+k+1$, so E^k is non-empty and we have $E^k = \{x \in T | H(x) \geq -(|h|+k+1)\}$. Since $k \geq 1$, we have

$$-(|h|-\delta_k+k+1) \leq -(|h|+k) \leq h+\delta_k = \max_T(H+\delta_k) \leq |h|+k \leq |h|-\delta_k+k+1.$$

This implies that G^k is non-empty and that

$$G^k = \{x \in T | H(x)+\delta_k \geq -(|h|-\delta_k+k+1)\} = \{x \in T | H(x) \geq -(|h|+k+1)\}.$$

This means that G^k is just the shift in height of E^k, so

$$d_{\mathrm{GHP}}(E^k, G^k) \leq |\delta_k| \underset{k\to\infty}{\longrightarrow} 0.$$

The sets E^k and G^k are both compact, and we have $\min(|h|+k+1, |h|-\delta_k+k+1) \geq |h|+k$, so $\mathrm{Slice}_{|h|+k}(T,d,H,\nu) \subset E^k \subset (T,d,H,\nu)$ and $\mathrm{Slice}_{|h|+k}(T,d,H+\delta_k,\nu) \subset G^k \subset (T,d,H+\delta_k,\nu)$. The sequence $(|h|+k)_{k\in\mathbb{N}^*}$ goes to ∞, so, by Lemma 3.4.2,

$$d_{\mathrm{LGHP}}((T,d,H+\delta_k,\nu),(T,d,H,\nu)) \underset{k\to\infty}{\longrightarrow} 0.$$

We have proven that the sequence of trees $(\tau(h+\delta_n,(\zeta_n)_{n\in\mathbb{N}^*},(u_n)_{n\in\mathbb{N}^*}))_{k\in\mathbb{N}^*}$ converges to the tree $\tau(h,(\zeta_n)_{n\in\mathbb{N}^*},(u_n)_{n\in\mathbb{N}^*})$ for d_{LGHP}. The choice h and $(\delta_k)_{k\in\mathbb{N}^*} \in [-1,1]^{\mathbb{N}^*}$ was arbitrary, so $h \mapsto \tau(h,(\zeta_n)_{n\in\mathbb{N}^*},(u_n)_{n\in\mathbb{N}^*})$ is continuous by sequential characterization.

Step 2: we prove that $(\zeta_n)_{n\in\mathbb{N}^*} \mapsto \tau(h,(\zeta_n)_{n\in\mathbb{N}^*},(u_n)_{n\in\mathbb{N}^*})$ is 1-Lipschitz, that is:

$$d_{\mathrm{LGHP}}\left(\tau(h,(\zeta_n)_{n\in\mathbb{N}^*},(u_n)_{n\in\mathbb{N}^*}),\tau(h,(\zeta'_n)_{n\in\mathbb{N}^*},(u_n)_{n\in\mathbb{N}^*})\right) \leq \sup_{n\in\mathbb{N}^*}|\zeta_n-\zeta'_n|. \tag{4.2.5}$$

Set $\delta = \sup_{n\in\mathbb{N}^*}|\zeta_n-\zeta'_n|$. Take $E=[0,1]\times(-\infty,h]$, equipped with ν_E the 1-dimensional Lebesgue measure on $[0,1]\times\{h\}\subset E$. We note $(T,d,H,\nu)\subset\tau(h,(\zeta_n)_{n\in\mathbb{N}^*},(u_n)_{n\in\mathbb{N}^*})$ the quotient of E by the pseudo-distance

$$d((x,y),(x',y')) = y+y'-2(y\wedge y'\wedge \inf_{x\leq u_n<x'} h-\zeta_n),$$

and $(T',d',H',\nu')\subset\tau(h,(\zeta'_n)_{n\in\mathbb{N}^*},(u_n)_{n\in\mathbb{N}^*})$ the quotient of E by the pseudo-distance

$$d'((x,y),(x',y')) = y+y'-2(y\wedge y'\wedge \inf_{x\leq u_n<x'} h-\zeta'_n).$$

Let us call ρ the projection of E to the quotient T, and ρ' the projection of E to the quotient T'. Note that $\tau(h,(\zeta_n)_{n\in\mathbb{N}^*},(u_n)_{n\in\mathbb{N}^*})$ is the completion of (T,d,H,ν) and that ρ is surjective from E to T. We set $A=\{(\rho(x,y),\rho'(x,y))\}_{(x,y)\in E}$.

Take $r\in\mathbb{R}_+$, and let us prove that A induces a δ-correspondence between $\mathrm{Slice}_r(T)$ and $\mathrm{Slice}_r(T')$. For every $(x,y)\in E$, we have $H(\rho(x,y))=H'(\rho'(x,y))$, so $\rho(x,y)\in\mathrm{Slice}_r(T)\Leftrightarrow\rho'(x,y)\in\mathrm{Slice}_r(T')$. Since ρ and ρ' are surjective, A induces a correspondence between $\mathrm{Slice}_r(T)$ and $\mathrm{Slice}_r(T')$. Recall Conditions (3.4.1)-(3.4.4) to be a δ-correspondence, and let us prove that they are satisfied by A. Take $(x,y),(x',y')\in E$, and let us compute the distortion of A. We have

$$\begin{aligned}
&|d(\rho(x,y),\rho(x',y'))-d'(\rho'(x,y),\rho'(x',y'))|\\
&\qquad=\big|[y+y'-2(y\wedge y'\wedge \inf_{x\leq u_n<x'} h-\zeta_n)]-[y+y'-2(y\wedge y'\wedge \inf_{x\leq u_n<x'} h-\zeta'_n)]\big|\\
&\qquad=2\big|(y\wedge y'\wedge \inf_{x\leq u_n<x'} h-\zeta_n)-(y\wedge y'\wedge \inf_{x\leq u_n<x'} h-\zeta'_n)\big|\\
&\qquad\leq 2\big|(\sup_{x\leq u_n<x'}\zeta_n)-(\sup_{x\leq u_n<x'}\zeta'_n)\big|\\
&\qquad\leq 2\sup_{x\leq u_n<x'}|\zeta_n-\zeta'_n|\leq 2\delta.
\end{aligned}$$

This implies that A satisfies Condition (3.4.1) with ε replaced by δ. For $(x,y)\in E$, we have $H(\rho(x,y))=H'(\rho'(x,y))$, so A satisfies Condition (3.4.2). Finally, for $B\subset T$ a Borel set, we define $B'=\{\rho'(x,y),\ \rho(x,y)\in B\}=\rho'\circ\rho^{-1}(B)$. We have

$$\nu(B)=\nu_E(\rho^{-1}(B))\leq\nu_E(\rho'^{-1}(B'))=\nu'(B').$$

This proves Condition (3.4.3). Since T and T' play symmetric roles here, we have Condition (3.4.4) as well. We have proven that A induces a δ-correspondence between $\mathrm{Slice}_r(T)$ and $\mathrm{Slice}_r(T')$ for every $r \in \mathbb{R}$.

The spaces $T'' = \tau(h, (\zeta_n)_{n\in\mathbb{N}^*}, (u_n)_{n\in\mathbb{N}^*})$ and $T''' = \tau(h, (\zeta'_n)_{n\in\mathbb{N}^*}, (u_n)_{n\in\mathbb{N}^*})$ are the respective completions of T and T'. Set A' the closure of A in $T'' \times T'''$, and let us prove that A' is a δ-correspondence. For every $x \in T''$, there exists a sequence $(x_n)_{n\in\mathbb{N}^*}$ of elements of T converging to x. Since A is a correspondence, there exists a sequence $(x'_n)_{n\in\mathbb{N}^*}$ of elements of T' such that $(x_n, x'_n) \in A$. By definition of A, we have $H'(x'_n) = H(x_n)$, so $\lim_n H(x'_n) = H(x)$. Since T''' is S-compact, it follows that $(x'_n)_{n\in\mathbb{N}^*}$ has an adherence value $x' \in T'''$. By choice of x', we have immediately $H'(x') = H(x)$ and $(x, x') \in A'$. Since T'' and T''' have symmetric roles, we have proven that A' is an height-preserving correspondence. Since $T'' \setminus T$ is ν-negligible and $T''' \setminus T'$ is ν'-negligible, A' still satisfies Conditions (3.4.1)-(3.4.4). For every $r \in \mathbb{R}_+$, since A' is an height preserving δ-correspondence, it induces a δ-correspondence between $\mathrm{Slice}_r(T'')$ and $\mathrm{Slice}_r(T''')$. With Proposition 3.4.1, we have

$$d_{\mathrm{GHP}}(\mathrm{Slice}_r(T''), \mathrm{Slice}_r(T''')) \leq \delta.$$

By definition of d_{LGHP}, this yields

$$d_{\mathrm{LGHP}}(T'', T''') \leq \delta.$$

Since T'' and T''' are the completions of T and T', we have $T'' = \tau(h, (\zeta_n)_{n\in\mathbb{N}^*}, (u_n)_{n\in\mathbb{N}^*})$ and $T''' = \tau(h, (\zeta'_n)_{n\in\mathbb{N}^*}, (u_n)_{n\in\mathbb{N}^*})$. Since $\delta = ||(\zeta_n)_{n\in\mathbb{N}^*} - (\zeta'_n)_{n\in\mathbb{N}^*}||_\infty$, we have proven that the application $(\zeta_n)_{n\in\mathbb{N}^*} \mapsto \tau(h, (\zeta_n)_{n\in\mathbb{N}^*}, (u_n)_{n\in\mathbb{N}^*})$ is 1-Lipschitz.

Step 3: we prove that if $(\zeta_n)_{n\in\mathbb{N}^*} \in \mathbb{R}^\omega_{+,0}$, $(u_n)_{n\in\mathbb{N}^*} \mapsto \tau(h, (\zeta_n)_{n\in\mathbb{N}^*}, (u_n)_{n\in\mathbb{N}^*})$ is continuous from D with the pointwise convergence topology to $\mathbb{T}$. Take $\varepsilon > 0$ and $n_0 \in \mathbb{N}^*$ such that for every $n > n_0$, $\zeta_n = 0$. Set $\delta = \frac{\varepsilon}{2(n_0+1)} \wedge (\frac{1}{2}\min_{1\leq i<j\leq n_0} |u_i - u_j|)$, and take $(u'_n)_{n\in\mathbb{N}^*} \in D$ such that $\max_{n\leq n_0} |u_n - u'_n| \leq \delta$. We set

$$\begin{aligned} T =& \tau(h, (\zeta_n)_{n\in\mathbb{N}^*}, (u_n)_{n\in\mathbb{N}^*}), \\ T' =& \tau(h, (\zeta_n)_{n\in\mathbb{N}^*}, (u'_n)_{n\in\mathbb{N}^*}), \end{aligned}$$

and ρ (resp. ρ') the projection from $(0,1) \times (-\infty, h]$ to T (resp. T'). Note that since $(\zeta_n)_{n\in\mathbb{N}^*} \in \mathbb{R}^\omega_{+,0}$, the projections $\rho(E)$ and $\rho'(E)$ are complete discrete trees. This implies that $T = \rho(E)$ and $T' = \rho'(E)$. Consider σ the permutation such that $u_{\sigma(1)} < u_{\sigma(2)} < ... < u_{\sigma(n_0)}$. Since $\delta \leq \frac{1}{2}\min_{1\leq i<j\leq n} |u_i - u_j|$, we have $u'_{\sigma(1)} < u'_{\sigma(2)} < ... < u'_{\sigma(n_0)}$ as well. Set

$$A_E = \left\{ ((x,y),(x',y')) \in E^2 \,\middle|\, \begin{array}{l} y = y', \\ \exists 0 \leq i \leq n_0, \\ u_{\sigma(i)} < x \leq u_{\sigma(i+1)}, \\ u'_{\sigma(i)} < x' \leq u'_{\sigma(i+1)} \end{array} \right\},$$

where $E = (0,1) \times (-\infty, h]$, and we set by convention $u_{\sigma(0)} = u'_{\sigma(0)} = 0$ and $u_{\sigma(n_0+1)} = u'_{\sigma(n_0+1)} = 1$. Now, consider $A = \{\rho(x,y), \rho'(x',y')\}_{(x,y)\in A_E}$, and let us prove that A is a ε-correspondence. Since for every i, $u_{\sigma(i)} < u_{\sigma(i+1)}$ and $u'_{\sigma(i)} < u'_{\sigma(i+1)}$, A_E is a correspondence. By surjectivity of ρ and ρ', A is a correspondence as well. Now, we find a simpler expression of

d and d'. For $(z_1, z_1'), (z_2, z_2') \in A$, there exists $((x_1, y_1), (x_1', y_1')), ((x_2, y_2), (x_2', y_2')) \in A_E$ such that $(z_1, z_1') = (\rho(x_1, y_1), \rho'(x_1', y_1')) \in A$ and $(z_2, z_2') = (\rho(x_2, y_2), \rho'(x_2', y_2'))$. By definition of A_E, there exists i_1, i_2 between 0 and n_0 such that for $j \in \{1, 2\}$, $u_{\sigma(i_j)} < x_j \leq u_{\sigma(i_j+1)}$. Suppose $x_1 \leq x_2$, we have

$$\begin{aligned} d(z_1, z_2) &= y_1 + y_2 - 2(y_1 \wedge y_2 \wedge (\inf_{\substack{n \in \mathbb{N}^* \\ x_1 \leq u_n < x_2}} h - \zeta_n)) \\ &= y_1 + y_2 - 2(y_1 \wedge y_2 \wedge (\min_{\substack{1 \leq i \leq n_0 \\ x_1 \leq u_{\sigma(i)} < x_2}} h - \zeta_{\sigma(i)}) \wedge (\inf_{\substack{n > n_0 \\ x_1 \leq u_n < x_2}} h - \zeta_n)) \\ &= y_1 + y_2 - 2(y_1 \wedge y_2 \wedge (\min_{i_1 < i \leq i_2} h - \zeta_{\sigma(i)})). \end{aligned}$$

For the last equality, we used the fact that for $n > n_0$, $h - \zeta_n = h \geq y_1$, and the fact that since for $j \in \{1, 2\}$, $u_{\sigma(i_j)} < x_j \leq u_{\sigma(i_j+1)}$, we have $u_{\sigma(i)} < x_j$ if and only if $i \leq i_j$. Note that the distance depends only on (i_1, y_1) and (i_2, y_2), so we have

$$d(z_1, z_2) = y_1 + y_2 - 2(y_1 \wedge y_2 \wedge (\min_{i_1 < i \leq i_2} h - \zeta_{\sigma(i)}))$$

as soon as $i_1 \leq i_2$. Similarly, we have, supposing that $i_1 \leq i_2$, that:

$$d'(z_1', z_2') = y_1' + y_2' - 2(y_1' \wedge y_2' \wedge (\min_{i_1 < i \leq i_2} h - \zeta_{\sigma(i)})).$$

By definition of A_E, we have $y_1 = y_1'$ and $y_2 = y_2'$, so we have proven that $d(z_1, z_2) = d'(z_1', z_2')$ and $H(z_1) = H'(z_1')$. This means that A induces a height-preserving isometry between T and T', and satisfies Conditions (3.4.1) and (3.4.2).

We set, for all $0 \leq i \leq n_0$, $B_i = \{h\} \times (u_{\sigma(i)}, u_{\sigma(i+1)}]$ and $B_i' = \{h\} \times (u_{\sigma(i)}', u_{\sigma(i+1)}']$. For every $0 \leq i \leq n_0$, $\nu(B_i) = u_{\sigma(i+1)} - u_{\sigma(i)}$ and $\nu(B_i') = u_{\sigma(i+1)}' - u_{\sigma(i)}'$. According to our previous calculation, ρ is constant on each B_i and ρ' on each B_i', and the B_i form a partition of $\mathrm{Supp}(\nu)$. Take $B \subset T$. Since all the mass is at height h and A preserves the height, we can neglect the part of B situated strictly under height h. Since ρ is constant on the B_i, $\rho^{-1}(B)$ is of the form $\bigcup_{i \in I} B_i$ for I some subset of $\{0, ..., n_0\}$. Set $B' = \{z' \in T' \mid \exists z \in B, (z, z') \in A\}$. By definition of A, it means that $B' = \rho'(\bigcup_{i \in I} B_i')$. It follows that

$$\begin{aligned} \nu(B) &= \sum_{i \in I} \nu(B_i) \\ &= \sum_{i \in I} (u_{\sigma(i+1)} - u_{\sigma(i)}) \\ &\leq \sum_{i \in I} (u_{\sigma(i+1)}' - u_{\sigma(i)}' + 2\delta) \\ &= \sum_{i \in I} \nu(B_i') + 2(n_0 + 1)\delta \\ &\leq \nu(B') + \varepsilon. \end{aligned}$$

We have proven Condition (3.4.3) for A. By symmetry of the roles for T and T', we have Condition (3.4.4) as well, so A is a ε-correspondence. Since A preserves the labels, A induces a ε-correspondence between $\mathrm{Slice}_r(T)$ and $\mathrm{Slice}_r(T')$ for every $r \in \mathbb{R}_+$. Using Proposition

3.4.1, we have $d_{\mathrm{GHP}}(\mathrm{Slice}_r(T), \mathrm{Slice}_r(T')) \leq \varepsilon$ for every $r \in \mathbb{R}_+$, so by Definition 3.1.12 we have $d_{\mathrm{LGHP}}(T, T') \leq \varepsilon$. Since ε was arbitrary in $\mathbb{R}_+^*$ and $(u'_n)_{n\in\mathbb{N}^*}$ was arbitrary in a ball of center $(u_n)_{n\in\mathbb{N}^*}$ and radius $\delta > 0$, we have proven that if $(\zeta_n)_{n\in\mathbb{N}^*} \in \mathbb{R}_{+,0}^{\omega}$, the map $(u_n)_{n\in\mathbb{N}^*} \mapsto \tau(h, (\zeta_n)_{n\in\mathbb{N}^*}, (u_n)_{n\in\mathbb{N}^*})$ is continuous.

Conclusion: Since $\tau(h, (\zeta_n)_{n\in\mathbb{N}^*}, (u_n)_{n\in\mathbb{N}^*})$ is 1-Lipschitz in $(\zeta_n)_{n\in\mathbb{N}^*}$ and $\lim_{n\in\mathbb{N}^*} \zeta_n = 0$, the map $(u_n)_{n\in\mathbb{N}^*} \mapsto \tau(h, (\zeta_n)_{n\in\mathbb{N}^*}, (u_n)_{n\in\mathbb{N}^*})$ is the uniform limit of

$$(u_n)_{n\in\mathbb{N}^*} \mapsto \tau(h, (1_{n\leq n_0}\zeta_n)_{n\in\mathbb{N}^*}, (u_n)_{n\in\mathbb{N}^*})$$

when n_0 goes to ∞. With the result of Step 3, $(u_n)_{n\in\mathbb{N}^*} \mapsto \tau(h, (\zeta_n)_{n\in\mathbb{N}^*}, (u_n)_{n\in\mathbb{N}^*})$ is the uniform limit of continuous functions, hence is continuous.

The map $\tau(h, (\zeta_n)_{n\in\mathbb{N}^*}, (u_n)_{n\in\mathbb{N}^*})$ is continuous in $(u_n)_{n\in\mathbb{N}^*}$ and 1-Lipschitz in $(\zeta_n)_{n\in\mathbb{N}^*}$ over its domain, so it is continuous in $((\zeta_n)_{n\in\mathbb{N}^*}, (u_n)_{n\in\mathbb{N}^*})$. The map

$$\Big(h, \big((\zeta_n)_{n\in\mathbb{N}^*}, (u_n)_{n\in\mathbb{N}^*}\big)\Big) \mapsto \tau(h, (\zeta_n)_{n\in\mathbb{N}^*}, (u_n)_{n\in\mathbb{N}^*})$$

is continuous in its two variables h and $((\zeta_n)_{n\in\mathbb{N}^*}, (u_n)_{n\in\mathbb{N}^*})$, and $\mathbb{R}$ is separable, so, using Lemma 3.4.13 the map τ is measurable from $\mathbb{R} \times \mathbb{R}_{+,0}^{\mathbb{N}^*} \times D$ to $(\mathbb{T}, d_{\mathrm{LGHP}})$. □

4.3 Crown of a tree

4.3.1 Unordered forest Topology

The aim of this subsection is to define and study unordered forests of height-labelled trees. The main result of this chapter is Theorem 4.3.13, giving a filtration $(\mathscr{S}_h)_{h\in\mathbb{R}}$ of $(\mathbb{T}, d_{\mathrm{LGHP}})$ adapted to growth process (for example in the case of Galton-Watson and Lévy trees), and a filtration $(\mathscr{C}_{-h})_{h\in\mathbb{R}}$ of $(\mathbb{T}, d_{\mathrm{LGHP}})$ adapted to coalescent processes (for example in the case of Kingman's or Λ-coalescent).

For every $h \in \mathbb{R}$, we set $0_h = (\{h\}, d_{\{h\}}, h, 0) \in \mathbb{T}$, where $d_{\{h\}}$ is the only distance over the singleton $\{h\}$. We consider $\tilde{\mathbb{T}}_C$ the set of all sequences $(T_n, d_n, H_n, \nu_n)_{n\in\mathbb{N}^*}$ of S-compact height-labelled trees such that $\lim_n d_{\mathrm{LGHP}}(T_n, 0_h) = 0$ for some $h \in \mathbb{R}$ and for all $n \in \mathbb{N}^*$ but possibly one, $\min_{T_n} H_n = h$ and if there is a n_0 such that $\min_{T_{n_0}} H_{n_0} \neq h$ then $T_{n_0} = \emptyset$ or $\min_{T_{n_0}} H_{n_0} > h$. We define, for $(T_n)_{n\in\mathbb{N}^*}$, $(T'_n)_{n\in\mathbb{N}^*} \in \tilde{\mathbb{T}}_C$,

$$d_{\mathrm{LGHP}}^{\infty}((T_n)_{n\in\mathbb{N}^*}, (T'_n)_{n\in\mathbb{N}^*}) = \inf_{\sigma\in\mathfrak{S}(\mathbb{N}^*)} \sup_{n\in\mathbb{N}^*} d_{\mathrm{LGHP}}(T_n, T'_{\sigma(n)}), \tag{4.3.1}$$

where $\mathfrak{S}(\mathbb{N}^*)$ is the set of all permutations of $\mathbb{N}^*$. The function $d_{\mathrm{LGHP}}^{\infty}$ is non-negative and satisfies the triangular inequality, so it is a pseudo-distance over $\tilde{\mathbb{T}}_C$. We define $\mathbb{T}_C$ the quotient of $\tilde{\mathbb{T}}_C$ by the equivalence relation $d_{\mathrm{LGHP}}^{\infty}((T_n)_{n\in\mathbb{N}^*}, (T'_n)_{n\in\mathbb{N}^*}) = 0$, and call crowns the equivalence classes. Note that $(\mathbb{T}_C, d_{\mathrm{LGHP}}^{\infty})$ is separable, but not complete. The space $\mathbb{T}_C$ is Polish though, since the distance

$$((T_n)_{n\in\mathbb{N}^*}, (T'_n)_{n\in\mathbb{N}^*}) \mapsto d_{\mathrm{LGHP}}^{\infty}((T_n)_{n\in\mathbb{N}^*}, (T'_n)_{n\in\mathbb{N}^*}) \vee |h - h'|,$$

where 0_h and $0_{h'}$ are the respective limits of $(T_n)_{n\in\mathbb{N}^*}$ and $(T'_n)_{n\in\mathbb{N}^*}$ induces the same topology and makes $\mathbb{T}_C$ complete.

Remark 4.3.1. We give an intuition of the equivalence relation on $\tilde{\mathbb{T}}_C$. Take two sequences $(T_n)_{n\in\mathbb{N}^*}$ and $(T'_n)_{n\in\mathbb{N}^*}$ elements of $\tilde{\mathbb{T}}_C$. We have $d^\infty_{\mathrm{LGHP}}((T_n)_{n\in\mathbb{N}^*},(T'_n)_{n\in\mathbb{N}^*})=0$ if and only if $(T_n)_{n\in\mathbb{N}^*}$ and $(T'_n)_{n\in\mathbb{N}^*}$ have the same limit 0_h and the terms different from 0_h are the same in both sequences. For example, for $(T_n)_{n\in\mathbb{N}^*}$ and $0_h=\lim_n T_n$, the following sequences:

- $(0_h, T_1, T_2, T_3, T_4, T_5, ...)$,
- $(0_h, T_1, 0_h, T_2, 0_h, T_3, ...)$,
- $(T_{\phi(n)})_{n\in\mathbb{N}^*}$, for ϕ a permutation of $\mathbb{N}^*$,

are all in the equivalence class of $(T_n)_{n\in\mathbb{N}^*}$. But adding or removing any term from $(T_n)_{n\in\mathbb{N}^*}$ different from 0_h would change the equivalence class.

An element of $\mathbb{T}_C$ can always be represented as the class of a sequence $(T_n, d_n, H_n, \nu_n)_{n\in\mathbb{N}^*}$ such that $(\sup_{T_n} H_n)_{n\in\mathbb{N}^*}$ is a non-increasing sequence of elements converging to some $h\in\mathbb{R}$. To ease of notation, we will abusively confuse the classes of $\mathbb{T}_C$ with the representents in $\tilde{\mathbb{T}}_C$. If the terms of $(T_n)_{n\in\mathbb{N}^*}$ and $(T'_n)_{n\in\mathbb{N}^*}\in\mathbb{T}_C$ are all compact trees, then we can define

$$d^\infty_{\mathrm{GHP}}((T_n)_{n\in\mathbb{N}^*},(T'_n)_{n\in\mathbb{N}^*}) = \inf_{\sigma\in\mathfrak{S}(\mathbb{N}^*)}\sup_{n\in\mathbb{N}^*} d_{\mathrm{GHP}}(T_n, T'_{\sigma(n)}).$$

Note that, like d^∞_{LGHP}, the function d^∞_{GHP} is a pseudo-distance, and that d^∞_{GHP} has the same zeros as d^∞_{LGHP} thanks to Remark 4.1.16. It follows that d^∞_{GHP} defines the same quotient as d^∞_{LGHP}, so d^∞_{GHP} is a distance on the set of all crowns containing only compact terms.

For $h\in\mathbb{R}$, $r>|h|$ and $(T_n)_{n\in\mathbb{N}^*}\in\mathbb{T}_C$ with $\lim_n T_n = 0_h$, we define

$$\mathrm{Slice}_r((T_n)_{n\in\mathbb{N}^*}) = (\mathrm{Slice}_r(T_n))_{n\in\mathbb{N}^*}.$$

Lemma 4.3.2. *For $h\in\mathbb{R}$, $r>|h|$ and $(T_n)_{n\in\mathbb{N}^*}\in\mathbb{T}_C$ with $\lim_n T_n=0_h$, we have*

$$\mathrm{Slice}_r((T_n)_{n\in\mathbb{N}^*})\in\mathbb{T}_C.$$

Proof. Let $(T_n, d_n, H_n, \nu_n)_{n\in\mathbb{N}^*}$ be an element of $\mathbb{T}_C$. Take $n\in\mathbb{N}^*$ such that $\min_{T_n} H_n\in[-r,r]$. It follows immediately that $\mathrm{Slice}_r(T_n)$ contains the root of T_n, so $\mathrm{Slice}_r(T_n)$ is non-empty and $\min_{\mathrm{Slice}_r(T_n)} H_n = \min_{T_n} H_n$. Since $\min_{T_n} H_n\in[-r,r]$, we have $H_n(T_n)\cap[-r,r] = H_n(T_n)\cap(-\infty,r]$. It follows that

$$\mathrm{Slice}_r(T_n) = \{x\in T_n | H_n(x)\le r\}.$$

For every $x,y\in\mathrm{Slice}_r(T_n)$, we have $\max_{[\![x,y]\!]} H_n = H_n(x)\vee H_n(y)\le r$, so $[\![x,y]\!]\subset\mathrm{Slice}_r(T_n)$. We have proven that if $\min_{T_n} H_n\in[-r,r]$, then $\mathrm{Slice}_r(T)$ is a tree and $\min_{\mathrm{Slice}_r(T_n)} H_n = \min_{T_n} H_n$.

If $\min_{T_n} H_n > r$ or if $T_n=\emptyset$ then $\mathrm{Slice}_r(T_n)=\emptyset$.

From these two results, we see that for every $n\in\mathbb{N}^*$ such that $\min_{T_n} H_n = h\in[-r,r]$, $\mathrm{Slice}_r(T_n)$ is a tree and $\min_{\mathrm{Slice}_r(T_n)} H_n = \min_{T_n} H_n = h$. If $\min_{T_n} H_n > h$ or $T_n=\emptyset$, which happens for at most one index, then we either have $\min_{\mathrm{Slice}_r(T_n)} H_n = \min_{T_n} H_n > h$ or $\mathrm{Slice}_r(T_n)=\emptyset$ and in the former case $\mathrm{Slice}_r(T_n)$ is a tree with $\min_{\mathrm{Slice}_r(T_n)} H_n > h$.

We deduce that $\mathrm{Slice}_r((T_n)_{n\in\mathbb{N}^*}$ is in $\mathbb{T}_C$. □

To help prove convergences, we adapt Lemma 3.4.3 to $\mathbb{T}_C$ and prove the following result.

Lemma 4.3.3. *Let* $(T_n, d_n, H_n, \nu_n)_{n\in\mathbb{N}^*}$ *and* $((T_n^k, d_n^k, H_n^k, \nu_n^k)_{n\in\mathbb{N}^*})_{k\in\mathbb{N}^*}$ *be elements of* $\mathbb{T}_C$*. Take* $h \in \mathbb{R}$ *(resp* $h_k \in \mathbb{R}$*) such that* $\lim_n T_n = 0_h$ *(resp.* $\lim_n T_n^k = 0_{h_k}$*) and* $(r_k)_{k\in\mathbb{N}^*}$ *a sequence of positive real numbers such that* $\lim_k r_k = \infty$ *and for every* $k \in \mathbb{N}^*$*,* $r_k > |h| \vee |h_k|$*. If*

$$d_{\mathrm{GHP}}^{\infty}(\mathrm{Slice}_{r_k}((T_n)_{n\in\mathbb{N}^*}), \mathrm{Slice}_{r_k}((T_n^k)_{n\in\mathbb{N}^*})) \underset{k\to\infty}{\longrightarrow} 0, \tag{4.3.2}$$

then, we have:

$$d_{\mathrm{LGHP}}^{\infty}((T_n)_{n\in\mathbb{N}^*}, (T_n^k)_{n\in\mathbb{N}^*}) \underset{k\to\infty}{\longrightarrow} 0.$$

Proof. Take $\varepsilon \in (0,1)$ and h the real number such that $\lim_n T_n = 0_h$. The tree 0_h is compact, so according to Remark 4.1.16 there exists $\varepsilon' \in (0,1)$ such that for every compact height-labelled tree T,

$$d_{\mathrm{GHP}}(T, 0_h) \le \varepsilon' \Rightarrow d_{\mathrm{LGHP}}(T, 0_h) \le \frac{\varepsilon}{2} \tag{4.3.3}$$

and $\varepsilon'' \in (0,1)$ such that for every S-compact height-labelled tree T,

$$d_{\mathrm{LGHP}}(T, 0_h) \le \varepsilon'' \Rightarrow (T \text{ is compact and } d_{\mathrm{GHP}}(T, 0_h) \le \frac{\varepsilon'}{2}). \tag{4.3.4}$$

By definition of $d_{\mathrm{GHP}}^{\infty}$ and d_{GHP}, there exists a sequence of permutations of $\mathbb{N}^*$: $(\sigma_k)_{k\in\mathbb{N}^*}$, such that for every k,

$$\sup_{n\in\mathbb{N}^*} d_{\mathrm{GHP}}(\mathrm{Slice}_{r_k}(T_n), \mathrm{Slice}_{r_k}(T_{\sigma_k(n)}^k)) \le d_{\mathrm{GHP}}^{\infty}(\mathrm{Slice}_{r_k}((T_n)_{n\in\mathbb{N}^*}), \mathrm{Slice}_{r_k}((T_n^k)_{n\in\mathbb{N}^*})) + \frac{\varepsilon'}{8}. \tag{4.3.5}$$

Take $n_0 \in \mathbb{N}^*$ such that for every $n \ge n_0$, $d_{\mathrm{LGHP}}(T_n, 0_h) \le \varepsilon'' \wedge \frac{\varepsilon}{2}$. Thanks to Equation (4.3.4), we have

$$d_{\mathrm{GHP}}(T_n, 0_h) \le \frac{\varepsilon'}{2}. \tag{4.3.6}$$

From the $\Leftarrow$ direction of Lemma 3.4.3, we have that for every $n < n_0$,

$$\lim_{k\to\infty} d_{\mathrm{LGHP}}(T_{\sigma_k(n)}^k, T_n) = 0. \tag{4.3.7}$$

Combining (4.3.7) line with $\lim_k r_k = \infty$ and (4.3.2), we can take k_0 such that for every $k \ge k_0$, we have

$$\begin{cases} \forall n < n_0, d_{\mathrm{LGHP}}(T_n, T_{\sigma_k(n)}^k) \le \varepsilon \\ r_k > |h| + \varepsilon' \\ d_{\mathrm{GHP}}^{\infty}(\mathrm{Slice}_{r_k}((T_n)_{n\in\mathbb{N}^*}), \mathrm{Slice}_{r_k}((T_n^k)_{n\in\mathbb{N}^*})) \le \frac{\varepsilon'}{4}. \end{cases}$$

Let us prove that for $k \ge k_0$, $d_{\mathrm{LGHP}}^{\infty}((T_n)_{n\in\mathbb{N}^*}, (T_n^k)_{n\in\mathbb{N}^*}) \le \varepsilon$. By choice of k_0, we already have $d_{\mathrm{LGHP}}(T_n, T_{\sigma_k(n)}^k) \le \varepsilon$ for $n < n_0$. For $n \ge n_0$, we have by choice of σ_k and (4.3.5) that

$$\sup_{n\ge n_0} d_{\mathrm{GHP}}(\mathrm{Slice}_{r_k}(T_n), \mathrm{Slice}_{r_k}(T_{\sigma_k(n)}^k)) \le 2 d_{\mathrm{GHP}}^{\infty}(\mathrm{Slice}_{r_k}((T_n)_{n\in\mathbb{N}^*}), \mathrm{Slice}_{r_k}((T_n^k)_{n\in\mathbb{N}^*})) \le \frac{\varepsilon'}{2}. \tag{4.3.8}$$

From Equation (4.3.6), we have $d_{\mathrm{GHP}}(T_n, 0_h) \le \frac{\varepsilon'}{2}$, so the label function of T_n takes its values in $[h - \frac{\varepsilon'}{2}, h + \frac{\varepsilon'}{2}]$. Since $r_k > |h| + \varepsilon'$ and the labels of a tree always span an interval, it follows that $T_n = \mathrm{Slice}_{r_k}(T_n)$. Using (4.3.8), we find that the label function of $\mathrm{Slice}_{r_k}(T^k_{\sigma_k(n)})$ takes its values in $[h - \varepsilon', h + \varepsilon']$, so $\mathrm{Slice}_{r_k}(T^k_{\sigma_k(n)}) = T^k_{\sigma_k(n)}$. This gives, using (4.3.5) again, that:

$$d_{\mathrm{GHP}}(T^k_{\sigma_k(n)}, 0_h) \le d_{\mathrm{GHP}}(T_n, T^k_{\sigma_k(n)}) + d_{\mathrm{GHP}}(T_n, 0_h) \le \varepsilon'.$$

With Equation (4.3.3), we have $d_{\mathrm{LGHP}}(T^k_{\sigma_k(n)}, 0_h) \le \frac{\varepsilon}{2}$. By definition of n_0, we also have $d_{\mathrm{LGHP}}(T_n, 0_h) \le \frac{\varepsilon}{2}$, so $d_{\mathrm{LGHP}}(T^k_{\sigma_k(n)}, T_n) \le \varepsilon$.

We have for $k \ge k_0$:

$$d^{\infty}_{\mathrm{LGHP}}((T_n)_{n\in\mathbb{N}^*}, (T^k_n)_{n\in\mathbb{N}^*}) \le \sup_{n\in\mathbb{N}^*} d_{\mathrm{LGHP}}(T^k_{\sigma_k(n)}, T_n) \le \varepsilon.$$

We conclude the proof as ε is arbitrary. □

4.3.2 Crown of a tree

For (T, d, H, ν) a S-compact height-labelled tree, $h \in H(T)$, we define the elements of the skeleton at level h

$$\mathcal{I}_h(T) = \{x \in \mathrm{Skel}(T) | H(x) = h\} \tag{4.3.9}$$

and the collection $\mathrm{Cr}_h(T)$ of sub-trees of T above level h as

$$\mathrm{Cr}_h(T) = (C^h_i(T))_{i\in\mathcal{I}_h(T)},$$

with for $i \in \mathcal{I}_h(T)$, $C^h_i(T) \in \mathbb{T}$ defined by:

$$C^h_i(T) = (\{x \in T | x \succeq i\}, d, H, 1_{H>h} \cdot \nu). \tag{4.3.10}$$

For $h' > h$, recall $n^{h,h'}(T)$ of Definition 4.2.1, and note with remark 4.2.2 that $n^{h,h'}(T)$ is the number of indices $i \in \mathcal{I}_h(T)$ such that $C_i(T)$ reaches height h'. Lemma 4.2.3 tells us that $n^{h,h'}(T)$ is finite. If $C_i(T)$ doesn't reach h', then $H(C_i(T)) \subset [h, h')$. By definition, it follows that for such an index i, its total measure is less than $H\nu((h, h'))$, so

$$d_{\mathrm{GHP}}(C_i(T), 0_h) \le (h' - h) \vee H\nu((h, h')) \xrightarrow{h'\downarrow h} 0.$$

This means that $\mathcal{I}_h(T)$ is at most countable, and that if $\mathcal{I}_h(T)$ is infinite then, thanks to Lemma 3.4.5,

$$\lim_n d_{\mathrm{GHP}}(C_{i_n}(T), 0_h) = 0 \quad \text{and} \quad \lim_n d_{\mathrm{LGHP}}(C_{i_n}(T), 0_h) = 0$$

for every enumeration $(i_n)_{n\in\mathbb{N}^*}$ of the elements of $\mathcal{I}_h(T)$. This allows us to define an object in $\tilde{\mathbb{T}}_C$ similar to Cr:

$$\mathrm{Crown}_h(T) = \begin{cases} (0_h)_{n\in\mathbb{N}^*} & \text{if } T = \emptyset \text{ or } \sup_T H \le h; \\ (T, 0_h, ...) & \text{if } \min_T H > h; \\ (C_{i_1}(T), ..., C_{i_n}(T), 0_h, ...) & \text{if } \mathcal{I}_h(T) = \{i_1, ..., i_n\} \text{ with distinct } i_1, ..., i_n; \\ (C_{i_n}(T))_{n\in\mathbb{N}^*} & \text{if } \mathcal{I}_h(T) = \{i_n\}_{n\in\mathbb{N}^*} \text{ with distinct } (i_n)_{n\in\mathbb{N}^*}. \end{cases}$$

Note that with this definition, $\mathrm{Crown}_h(T)$ belongs to $\tilde{\mathbb{T}}_C$ since the sequences converge to 0_h. We shall denote by $\mathrm{Crown}_h(T)$ its equivalence class in $\mathbb{T}_C$ whose definition does not depend on the choice of the enumeration of $\mathcal{I}_h(T)$. So the map Crown is defined on $\mathbb{R} \times \mathbb{T}$ and takes values in $\mathbb{T}_C$.

As T is S-compact, then with Lemma 4.2.3, all the $(n^{h,h'}(T))_{h'>h}$ are finite, so we can order $(C_i^h(T))_{i\in\mathcal{I}(T)}$ in some order of non-increasing height, and it will converge to 0_h.

Our aim for the rest of this section is to prove, after a series of technical lemmas, that the function $(h,T) \mapsto \mathrm{Crown}_h(T)$ is measurable from $(\mathbb{R}\times\mathbb{T}, d_\mathbb{R}\times d_{\mathrm{LGHP}})$ to $(\mathbb{T}_C, d^\infty_{\mathrm{LGHP}})$, see Proposition 4.3.11. We set $D_{\neq}$ the set of all $(h,T)\in\mathbb{R}\times\mathbb{T}$ such that $\mathrm{Crown}_h(T)\neq(0_h)_{n\in\mathbb{N}^*}$. Looking at the definition of Crown, this is equivalent to "there exists $x \in T$ such that $H(x) > h$".

Lemma 4.3.4. *The set $D_{\neq}$ is open in $\mathbb{R}\times\mathbb{T}$.*

Proof. Take $(h,(T,d,H,\nu)) \in D_{\neq}$. By definition of $D_{\neq}$ and Crown_h, there exists $x \in T$ such that $H(x) > h$. Take any element $(h',(T',d',H',\nu')) \in \mathbb{R}\times\mathbb{T}$ that satisfies $|h'-h| < \frac{1}{2}(H(x)-h)$ and

$$d_{\mathrm{LGHP}}(T,T') < \frac{1}{2}(1\wedge(H(x)-h))\mathrm{e}^{-|H(x)|}.$$

Since $\frac{1}{2}(1\wedge(H(x)-h)) < 1$ there exists $r > |H(x)|$ such that

$$d_{\mathrm{GHP}}(\mathrm{Slice}_r(T),\mathrm{Slice}_r(T')) < \frac{1}{2}(H(x)-h)$$

by definition of d_{LGHP}. Note that x is in $\mathrm{Slice}_r(T)$. By Proposition 3.4.1, there exists a $\frac{1}{2}(H(x)-h)$-correspondence A between $\mathrm{Slice}_r(T)$ and $\mathrm{Slice}_r(T')$. Take $x' \in \mathrm{Slice}_r(T')$ such that $(x,x') \in A$. We have

$$\begin{aligned} H'(x')-h' &\geq H(x) - |H'(x')-H(x)| - h - |h'-h| \\ &> H(x) - h - 2\,\frac{1}{2}(H(x)-h) \\ &= 0. \end{aligned}$$

We have $H'(x') > h'$. By definition of Crown, $(h'_k, T^k) \in D_{\neq}$. Since h' and T' were arbitrary in a small ball, we have proven that $D_{\neq}$ is open. □

Lemma 4.3.5. *Let $(h,(T,d,H,\nu)),(h',(T',d',H',\nu')) \in D_{\neq}$, $r\in\mathbb{R}_+$, $h''\in\mathbb{R}$ and $\delta > 0$ be such that $h'' > h\vee h'$, $0<\delta<h''-h'$ and $r > |h|\vee|h'|\vee|h''|$. Set $T_r = \mathrm{Slice}_r(T)$, $T'_r = \mathrm{Slice}_r(T')$. If $A \subset T_r\times T'_r$ is a δ-correspondence between T_r and T'_r such that for every $(x,x'),(y,y')\in A$,*

$$H(x)\wedge H(y) \geq h'' \Rightarrow \Big(H(x\wedge y)\geq h \Leftrightarrow H'(x'\wedge y')\geq h'\Big),$$

then

$$\begin{aligned} &d^\infty_{\mathrm{GHP}}(\mathrm{Slice}_r(\mathrm{Crown}_h(T)),\mathrm{Slice}_r(\mathrm{Crown}_{h'}(T'))) \\ &\qquad\qquad \leq 2(h''-(h\wedge h')) + H\nu([h\wedge h'-\delta, h''+2\delta]). \end{aligned} \tag{4.3.11}$$

Proof. Note that $\mathrm{Slice}_r(\mathrm{Crown}_h(T,d,H,\nu)) = \mathrm{Slice}_r(\mathrm{Crown}_h(T,d,H,1_{T_r}\cdot\nu))$ and that

$$[H(1_{\mathrm{Slice}_r(T)}\cdot\nu)]([h\wedge h'-\delta,h''+2\delta]) = H\nu([h\wedge h'-\delta,h''+2\delta]).$$

In the Inequality (4.3.11), if we replace (T,d,H,ν) and $(T',d,'H',\nu')$ by $(T,d,H,1_{T_r}\cdot\nu)$ and $(T',d,'H',1_{T'_r}\cdot\nu')$, we do not change the left-hand term, and we replace the right-hand term by a smaller (or equal) upper bound. This means that proving the lemma with the additional assumption $[H\nu\vee H'\nu']((-\infty,-r)\cup(r,\infty)) = 0$ is sufficient to prove the lemma. Thus, we assume $[H\nu\vee H'\nu']((-\infty,-r)\cup(r,\infty)) = 0$ in the rest of the proof.

Step 1: we handle all the non-empty terms of $\mathrm{Slice}_r(\mathrm{Crown}_h(T))$ reaching level h'' (if any). Take $n\in\mathbb{N}$ and $C_1,...,C_n$ the terms of $\mathrm{Slice}_r(\mathrm{Crown}_h(T))$ reaching at least level h''. If $n=0$, that is if $\sup_T H < h''$, there are none and we can directly go to Step 2. In the rest of Step 1, suppose that $n>0$, that is $\sup_{T_r} H \geq h''$. Take $x\in T_r$ with $H(x)\geq h''$, and the index i such that $x\in C_i$. Take $x'\in T'_r$ such that $(x,x')\in A$. Since A is a δ-correspondence, $H'(x')\geq H(x)-\delta\geq h''-\delta > h'$ so there exists a sub-tree $C'_{i'}$ of $\mathrm{Crown}_{h'}(T')$ such that $x'\in C_i$. Take $(y,y')\in A$ with $H(y)\geq h''$, we find that y' is in the same sub-tree as x' if and only $H'(x'\wedge y')\geq h'$, which is equivalent by hypothesis to $H(x\wedge y)\geq h$, that is if and only if x and y are in the same sub-tree C_i. Thus, we find that $C'_{i'}$ is defined independently from the choice of (x,x') and that for $1\leq i,j\leq n$, $C_i = C_j \Leftrightarrow C'_{i'} = C'_{j'}$. Therefore, we shall keep the same index i and write C'_i instead of $C'_{i'}$.

Now, we try to build a correspondence between C_i and C'_i. Consider

$$A_i = \{(x,x')\in A | x\in C_i, H(x)\geq h''\}\cup\{x\in C_i|H(x)<h''\}\times\{x'\in C'_i|H'(x')<h''+\delta\}.$$

Let us prove that A_i is a correspondence between C_i and C'_i. For $x\in C_i$, it is straightforward to see that there is $x'\in C'_i$ such that $(x,x')\in A_i$. Reciprocally for $x'\in C'_i$, if $H'(x') < h''+\delta$ then x' is in correspondence with elements of C_i thanks to the second term of A_i. If $H'(x')\geq h''+\delta$, then there exists $x\in T_r$ such that $(x,x')\in A$, and $H(x)\geq H'(x')-\delta\geq h''$, so $x\in C_i$ by definition of C'_i, and thus (x,x') is in the first term of A_i. So A_i is a correspondence between C_i and C'_i. We compute the distortion of A_i. We find that

$$\sup_{(x,x')\in A_i} |H(x)-H'(x')| \leq \max(\delta, h''-h, h''+\delta-h')) \leq \delta + h'' - (h\wedge h') \leq 2(h''-(h\wedge h')).$$

Take $(x,x'),(y,y')\in A_i$, we have three cases to check. If $H(x)\geq h''$ and $H(y)\geq h''$ then $(x,x'),(y,y')\in A$ and we have $|d(x,y)-d'(x',y')|\leq 2\delta$. If $H(x)<h''$ and $H(y)<h''$, then the distortion is at most

$$\begin{aligned}|d(x,y)-d'(x',y')| &\leq \mathrm{Diam}\big(\{x\in C_i|H(x)<h''\}\big)\vee\mathrm{Diam}\big(\{x'\in C'_i|H'(x')<h''+\delta\}\big)\\ &\leq 2(h''+\delta-(h\wedge h')),\end{aligned}$$

since the diameter of a tree is at most twice its height. In the last case, suppose $H(y)<h''\leq H(x)$. We have $h\leq H(x\wedge y)\leq H(y)<h''$, so

$$\begin{aligned}d(x,y) &= H(x)+H(y)-2H(x\wedge y)\\ &\in [H(x)-H(y), H(x)+H(y)-2h]\\ &\subset (H(x)-h'', H(x)+h''-2h).\end{aligned}$$

Similarly, we have $h' \leq H'(x' \wedge y') \leq H'(y') < h'' + \delta$, so

$$\begin{aligned} d'(x', y') &= H'(x') + H'(y') - 2H'(x' \wedge y') \\ &\in [H'(x') - H'(y'), H'(x') + H'(y') - 2h'] \\ &\subset (H'(x') - h'' - \delta, H'(x') + h'' + \delta - 2h'). \end{aligned}$$

From those two intervals and the fact that $|H(x) - H'(x')| \leq \delta$, we deduce that

$$\begin{aligned} &|d(x, y) - d'(x', y')| \\ &\quad < \big((H(x) + h'' - 2h) - (H'(x') - h'' - \delta)\big) \vee \big((H'(x') + h'' + \delta - 2h') - (H(x) - h'')\big) \\ &\quad \leq |H(x) - H'(x')| + (\delta + 2h'' - 2h) \vee (\delta + 2h'' - 2h') \\ &\quad \leq 2\delta + 2h'' - 2(h \wedge h'). \end{aligned}$$

In the three cases the distortion is less than $2(\delta + h'' - (h \wedge h')) \leq 4(h'' - (h \wedge h'))$.

Finally, let us control the measures. Set ν_i the measure of C_i and ν_i' the measure of C_i'. For any Borel set $B_0 \subset T$, $A_0 \subset T \times T'$, we note $(B_0)^{\overrightarrow{A_0}} = \{x' \in T' | \exists x \in B_0, (x, x') \in A_0\}$. Take $B \subset C_i$ a measurable set, $B_{\geq} = B \cap H^{-1}([h'', \infty))$, $B_{<} = B \cap H^{-1}((-\infty, h''))$. Using the fact that A is a δ-correspondence, we have:

$$\nu_i(B) \leq \nu_i'((B_{\geq})^{\overrightarrow{A}}) + \delta + \nu_i(B_{<}) = \nu_i'((B_{\geq})^{\overrightarrow{A_i}}) + \delta + H\nu_i([h, h'')).$$

For any Borel $B_0' \subset T'$, we note $(B_0')^{\overleftarrow{A}} = \{x \in T | \exists x' \in B_0', (x, x') \in A\}$ and $(B_0')^{\overleftarrow{A_i}} = \{x \in C_i | \exists x' \in B_0', (x, x') \in A_i\}$. Take $B' \subset C_i'$ a measurable set, $B_{\geq}' = B' \cap (H')^{-1}([h'' + \delta, \infty))$, $B_{<}' = B' \cap (H')^{-1}((-\infty, h'' + \delta))$. Using the fact that A is a δ-correspondence, we have:

$$\nu_i'(B') \leq \nu_i((B_{\geq}')^{\overleftarrow{A}}) + \delta + \nu_i'(B_{<}') = \nu_i((B_{\geq}')^{\overleftarrow{A_i}}) + \delta + H'\nu_i'([h', h'' + \delta)).$$

We find that A_i is a δ_i-correspondence, with

$$\begin{aligned} \delta_i &= \max\Big(2(h'' - (h \wedge h')), \frac{1}{2} \cdot 4(h'' - (h \wedge h')), \delta + H\nu_i([h, h'')), \delta + H'\nu_i'([h', h'' + \delta))\Big) \\ &= \max\Big(2(h'' - (h \wedge h')), \delta + H\nu_i([h, h'')), \delta + H'\nu_i'([h', h'' + \delta))\Big). \end{aligned}$$

Since A_i is a δ_i-correspondence, we get with Proposition 3.4.1 that $d_{\mathrm{GHP}}(C_i, C_i') \leq \delta_i$. We set

$$\delta' = \max\Big(2(h'' - (h \wedge h')), \delta + H\nu([h, h'')), \delta + H'\nu'([h', h'' + \delta))\Big).$$

We shall use later on that since $h'' \geq h \vee h'$, we have

$$|h - h'| \leq h \vee h' - h \wedge h' \leq h'' - h \wedge h' \leq \delta'/2. \tag{4.3.12}$$

Notice that δ' depends neither on i nor on n. It follows that

$$\begin{aligned} \max_{1 \leq i \leq n} d_{\mathrm{GHP}}(C_i, C_i') &\leq \max_{1 \leq i \leq n} \delta_i \\ &\leq \max_{1 \leq i \leq n} \Big(\max\Big(2(h'' - (h \wedge h')), \delta + H\nu_i([h, h'')), \delta + H'\nu_i'([h', h'' + \delta))\Big)\Big) \\ &\leq \delta'. \end{aligned}$$

Step 2: we prove the result when there exists an term B_0 of $\mathrm{Crown}_h(T)$ or of $\mathrm{Crown}_{h'}(T')$ such that $\mathrm{Slice}_r(B_0) = \emptyset$. For convenience, assume that $B_0 = (T_n, d_n, H_n, \nu_n)$ is a term of $\mathrm{Crown}_h(T)$. By Lemma 4.3.2, $\mathrm{Slice}_r(\mathrm{Crown}_h(T)) \in \mathbb{T}_C$, so $\mathrm{Slice}_r(T_n)$ is the only empty term. Since there can't be empty terms in $\mathrm{Crown}_h(T)$ by definition, and terms of $\mathrm{Crown}_h(T)$ rooted at height h have at least a point at height $h \in [-r, r]$, we have $\min_{T_n} H_n > h$. A second look at the definition of $\mathrm{Crown}_h(T)$ immediately tells us that $T_n = T$.

We have $T_r = \mathrm{Slice}_r(T_n) = \emptyset$, and $d_{\mathrm{GHP}}(T_r, T'_r) \leq \delta < h'' - h' < \infty$, so $T'_r = \emptyset$. By hypothesis, $\mathrm{Crown}_{h'}(T') \neq (0_{h'})_{n\in\mathbb{N}^*}$, so T' has at least a point at height $> h'$. Since $h' \in [-r, r]$ and $T'_r = \emptyset$, T' does not have a point at height h'. Since $H'(T')$ is an interval containing a point above $h'' > h'$ but none at height h', we have $\min_{T'} H' > h'$, so $\mathrm{Crown}_{h'}(T') = (T', 0_{h'}, 0_{h'}, ...)$ and we have

$$\begin{aligned} d^{\infty}_{\mathrm{GHP}}(\mathrm{Slice}_r(\mathrm{Crown}_h(T)), \mathrm{Slice}_r(\mathrm{Crown}_{h'}(T'))) &= d^{\infty}_{\mathrm{GHP}}((\emptyset, 0_h, \ldots), (\emptyset, 0_{h'}, \ldots)) \\ &= |h - h'| \\ &\leq \delta'. \end{aligned}$$

Step 3: control of the short sub-trees. Suppose that there are non trivial elements in $\mathrm{Slice}_r(\mathrm{Crown}_h(T))$ or $\mathrm{Slice}_r(\mathrm{Crown}_{h'}(T'))$. Recall $C_1, ..., C_n, C'_1, ..., C'_n$ from Step 1. Set $(C_i)_{i>n}$ and $(C'_i)_{i>n}$ the rest of the sub-trees. None of the sub-trees $(C_i)_{i>n}$ reache h'' while none of the sub-trees $(C'_i)_{i>n}$ reache $h'' + \delta$. For $i > n$, we take $A_i = C_i \times C'_i$. A_i is a correspondence between C_i and C'_i satisfying

$$\sup_{(x,x')\in A_i} |H(x) - H'(x')| \leq \delta + h'' - (h \wedge h') \leq \delta'.$$

Its distortion is less than

$$\mathrm{Diam}(C_i) \vee \mathrm{Diam}(C'_i) \leq 2(h'' + \delta - (h \wedge h')) \leq 2\delta'.$$

We have that for every measurable sets $B \subset C_i$, $B' \subset C'_i$

$$\begin{aligned} |\nu(B) - \nu'(B')| &\leq \max\left(\nu(C_i), \nu'(C'_i)\right) \\ &\leq \max\Big(H\nu([h, h'')), H'\nu'([h', h'' + \delta)\Big) \\ &\leq \delta'. \end{aligned}$$

We deduce that A_i is a δ'-correspondence. We have proven that for $i \in \mathbb{N}^*$, $d_{\mathrm{GHP}}(C_i, C'_i) \leq \delta'$ so

$$d^{\infty}_{GHP}(\mathrm{Slice}_r(\mathrm{Crown}_h(T)), \mathrm{Slice}_r(\mathrm{Crown}_{h'}(T'))) \leq \sup_{i\in\mathbb{N}^*} d_{\mathrm{GHP}}(C_i, C'_i) \leq \delta'.$$

Step 4: conclusion. We only need to prove $\delta' \leq 2(h'' - (h \wedge h')) + H\nu([h \wedge (h' - \delta), h'' + 2\delta])$. Recall that we have either $[h \wedge h', h'' + \delta] \subset [-r, r]$ or $[h \wedge h', h'' + \delta] \cap [-r, r] = \emptyset$. Recall the assumption that $[H\nu \vee H'\nu']((-\infty, -r) \cup (r, \infty)) = 0$, and

$$\delta' = \max\Big(2(h'' - (h \wedge h')), \delta + H\nu([h, h'')), \delta + H'\nu'([h', h'' + \delta))\Big).$$

If $[h \wedge h', h''+\delta] \cap [-r,r] = \emptyset$, then $\delta + H\nu([h,h'')) = \delta + H'\nu'([h', h''+\delta)) = \delta \leq h'' - h' \leq h'' - h \wedge h'$, so $\delta' = 2(h'' - (h \wedge h'))$ and we are done.

Now, suppose $[h \wedge h', h''+\delta] \subset [-r,r]$. Since A is a δ-correspondence between T_r and T'_r, we have $|H(x) - H'(x')| \leq \delta$ for all $(x,x') \in A$, so

$$(\{x' \in T' | H'(x') \in [h', h''+\delta)\})^{\overleftarrow{A}} \subset \{x \in T | H(x) \in [h'-\delta, h''+2\delta]\}.$$

Since A is a δ-correspondence between T_r and T'_r, we have, with the last inclusion:

$$\begin{aligned} H'\nu'([h', h''+\delta)) &= \nu'(\{x' \in T' | H'(x') \in [h', h''+\delta)\}) \\ &\leq \nu\Big((\{x' \in T' | H'(x') \in [h', h''+\delta)\})^{\overleftarrow{A}}\Big) + \delta \\ &\leq \nu\Big(\{x \in T | H(x) \in [h'-\delta, h''+2\delta]\}\Big) + \delta \\ &= H\nu([h'-\delta, h''+2\delta]) + \delta. \end{aligned}$$

As $\delta' \leq h'' - h \wedge h'$, this means that

$$\begin{aligned} \delta' &\leq \max(2(h'' - (h \wedge h')), 2\delta + H\nu([h \wedge (h'-\delta), h''+2\delta])) \\ &\leq 2(h'' - (h \wedge h')) + H\nu([h \wedge h' - \delta, h''+2\delta]). \end{aligned}$$

This ends the proof of the lemma. □

The next lemma uses Lemma 4.3.5 to give a sufficient (very technical) criterion for the $d^{\infty}_{\mathrm{LGHP}}$-convergence of the crown of a sequence of trees.

Lemma 4.3.6. *Take $(h, (T,d,H,\nu)) \in D_{\neq}$ with $H\nu(\{h\}) = 0$, $(T^k, d_k, H_k, \nu^k)_{k \in \mathbb{N}^*}$ a sequence of elements of $\mathbb{T}$, $(h'_k)_{k\in\mathbb{N}^*}$, $(h''_k)_{k\in\mathbb{N}^*}$ two sequences of real numbers converging to h and satisfying $h''_k > h \vee h'_k$, $(r_k)_{k\in\mathbb{N}^*}$ a sequence of positive real numbers with limit ∞, $(\delta_k)_{k\in\mathbb{N}^*}$ a sequence of positive real numbers with $\delta_k < h''_k - h'_k$. If for every $k \in \mathbb{N}^*$, there exists a δ_k-correspondence A^k between $\mathrm{Slice}_{r_k}(T)$ and $\mathrm{Slice}_{r_k}(T^k)$ such that for every $(x,x'), (y,y') \in A^k$,*

$$H(x) \wedge H(y) \geq h''_k \Rightarrow \Big(H(x \wedge y) \geq h \Leftrightarrow H_k(x' \wedge y') \geq h'_k\Big), \tag{4.3.13}$$

then

$$d^{\infty}_{\mathrm{LGHP}}(\mathrm{Crown}_h(T), \mathrm{Crown}_{h'_k}(T^k)) \underset{k\to\infty}{\longrightarrow} 0.$$

Proof. Recall that by Lemma 4.3.4, $D_{\neq}$ is open in $\mathbb{R} \times \mathbb{T}$. As $\lim_k \delta_k = 0$, we see by Lemma 3.4.3 that

$$\lim_{k\to\infty} d_{\mathrm{LGHP}}(T^k, T) = 0.$$

Since $\lim_k h'_k = h$ and $(h,T) \in D_{\neq}$, there exists k_0 such that for every $k \geq k_0$, $(h'_k, T^k) \in D_{\neq}$. By hypothesis, $\lim_k |h'_k| = \lim_k |h''_k| = |h|$, so

$$\sup_{k\in\mathbb{N}^*} |h'_k| \vee |h''_k| < \infty.$$

Since $\lim_k r_k = \infty$, there exists $k'_0 \geq k_0$ such that for every $k \geq k'_0$,

$$r_k > |h| \vee \sup_{k'\in\mathbb{N}^*} |h'_{k'}| \vee |h''_{k'}| \geq |h| \vee |h'_k| \vee |h''_k|.$$

Using Lemma 4.3.5, we have for every $k \geq k_0'$ that

$$d_{\mathrm{GHP}}^{\infty}(\mathrm{Slice}_{r_k}(\mathrm{Crown}_h(T)), \mathrm{Slice}_{r_k}(\mathrm{Crown}_{h_k'}(T^k)))$$
$$\leq 2(h_k'' - (h \wedge h_k')) + H\nu([h \wedge h_k' - \delta_k, h_k'' + 2\delta_k]) \underset{k\to\infty}{\longrightarrow} H\nu(\{h\}) = 0.$$

Using Lemma 4.3.3 we have $d_{\mathrm{LGHP}}^{\infty}(\mathrm{Crown}_h(T), \mathrm{Crown}_{h_k'}(T^k)) \underset{k\to\infty}{\longrightarrow} 0.$ □

The next three lemmas apply Lemma 4.3.6 to obtain some form of continuity for the application $(h,T) \mapsto \mathrm{Crown}_h(T)$ in three specific settings.

Lemma 4.3.7. *For every $(h,T) \in D_{\neq}$ such that $H\nu(\{h\}) = 0$, we have:*

$$d_{LGHP}^{\infty}(\mathrm{Crown}_h(T), \mathrm{Crown}_{h+\frac{1}{n}}(\mathrm{Trim}_{\frac{1}{n}}(T))) \underset{n\to\infty}{\longrightarrow} 0.$$

Proof. Take $(h,(T,d,H,\nu)) \in D_{\neq}$. Set, for $k \in \mathbb{N}^*$, $(T^k, d_k, H, \nu_k) = \mathrm{Trim}_{\frac{1}{k}}(T)$ (see Definition 4.2.5), $\rho_{\frac{1}{k}}$ the projection from T to T^k and $A_{\frac{1}{k}} = \{(x, \rho_{\frac{1}{k}}(x)\}_{x\in T}$ the canonical correspondence. According to the proof of Lemma 4.2.7, for all $r \geq 0$ the restriction of $A_{\frac{1}{k}}$ to $\mathrm{Slice}_r(T) \times \mathrm{Slice}_r(\mathrm{Trim}_{\frac{1}{k}}(T))$ provides a $\frac{1}{k}$-correspondence between $\mathrm{Slice}_r(T)$ and $\mathrm{Slice}_r(\mathrm{Trim}_{\frac{1}{k}}(T))$. For every $k \in \mathbb{N}^*$, set $h_k' = h + \frac{1}{k}$, $h_k'' = h_k' + 2\frac{1}{k}$ and $\delta_k = \frac{1}{k}$. Thus, $A_{\frac{1}{k}}$ is a δ_k-correspondence.

Consider $(x,x'), (y,y') \in A$ with $H(x) \geq h_k''$ and $H(y) \geq h_k''$. Let us prove that x and y are in the same sub-tree of $\mathrm{Crown}_h(T)$ if and only if x' and y' are in the same sub-tree of $\mathrm{Crown}_{h_k'}(\mathrm{Trim}_{\frac{1}{k}}(T))$. We have $x' = \rho_{\frac{1}{k}}(x)$ and $y' = \rho_{\frac{1}{k}}(y)$, so

$$\begin{aligned} 2H(x' \wedge y') &= H(x') + H(y') - d_k(x',y') \\ &= H(x) + H(y) - \max(|H(x) - H(y)|, d(x,y) - 2\frac{1}{k}) \\ &= \min(H(x) + H(y) - |H(x) - H(y)|, H(x) + H(y) - d(x,y) + 2\frac{1}{k}) \\ &= 2\min(H(x) \wedge H(y), H(x \wedge y) + \frac{1}{k}), \end{aligned}$$

where, at the second line, we used Definition 4.2.5. If x and y are in the same sub-tree of $\mathrm{Crown}_h(T)$ then $H(x \wedge y) \geq h$, so we have

$$H(x' \wedge y') = \min(H(x) \wedge H(y), H(x \wedge y) + \frac{1}{k}) \geq \min(h_k'', h + \frac{1}{k}) = h_k'$$

so x' and y' are in the same sub-tree of $\mathrm{Crown}_{h_k'}(T^k)$.

If x and y are not in the same sub-tree, then $H(x \wedge y) < h$, so we have

$$H(x' \wedge y') = \min(H(x) \wedge H(y), H(x \wedge y) + \frac{1}{k}) \leq H(x \wedge y) + \frac{1}{k} < h + \frac{1}{k} = h_k'$$

so x' and y' are not in the same sub-tree of $\mathrm{Crown}_{h_k'}(T^k)$. This means that Condition (4.3.13) holds.

We have $h < h_k' < h_k'' \to_{k\to\infty} h$ and $0 < \delta_k = \frac{1}{k} < h_k'' - h_k'$. Let $(r_k)_{k\in\mathbb{N}^*}$ be any sequence of positive real numbers converging to infinity. $(h,T) \in D_{\neq}$, so we can apply Lemma 4.3.6 with parameters $(T_k)_{k\in\mathbb{N}^*} = (\mathrm{Trim}_{\frac{1}{k}}(T))_{k\in\mathbb{N}^*}$, $h_k' = h + \frac{1}{k}$, $h_k'' = h + \frac{3}{k}$ and $\delta_k = \frac{1}{k}$. This gives the result. □

Lemma 4.3.8. *Take* $(h,(T,d,H,\nu)) \in D_{\neq}$ *such that* $H\nu(\{h\})=0$, *we have*

$$d^{\infty}_{LGHP}(\mathrm{Crown}_h(T),\mathrm{Crown}_{h'}(T)) \underset{h'\uparrow h}{\longrightarrow} 0.$$

Proof. Take $(h'_k)_{k\in\mathbb{N}^*}$ a non-decreasing sequence of real numbers in $(-\infty,h)$ converging to h. The set $A=\{(x,x)\}_{x\in T}$ is a δ-correspondence between $T_r=\mathrm{Slice}_r(T)$ and T_r for every $r\geq 0$ and $\delta>0$. Let us find a sequence $(h''_k)_{k\in\mathbb{N}^*,k\geq k_0}$ such that $h''_k\downarrow_k h$ and for every $x,y\in T$,

$$H(x)\wedge H(y)\geq h''_k \Rightarrow \Big(H(x\wedge y)\geq h \Leftrightarrow H(x\wedge y)\geq h'_k\Big).$$

Finding such a sequence will immediately solve our problem, by taking $\delta_k=\frac{1}{2}(h''_k-h)$, $r_k=k+\max(|h'_k|,|h''_k+\delta_k|)$, and by applying Lemma 4.3.6.

Take $h''>h$. The application $h'\mapsto n^{h',h''}(T)$ is left-continuous on $(-\infty,h'')$ and has integer values, so there exists $h'_0(h'')<h$ such that the map $h'\mapsto n^{h',h''}(T)$ is constant on $[h'_0(h''),h]$. From this and the convergence of $(h'_k)_{k\in\mathbb{N}^*}$, we can define $k_{h''}$ the smallest positive integer such that for every $k\geq k_{h''}$, $n^{h'_k,h''}(T)=n^{h,h''}(T)$.

For $k\geq k_{h+1}$, set $E_k=\{n\in\mathbb{N}^*|\, n\vee k_{h+\frac{1}{n}}\leq k\}$ and $n_k=\max(E_k)$. The sequence $(n_k)_{k\geq k_{h+1}}$ is well-defined since E_k is bounded and non-empty (it contains 1). By definition of $k_{h+\frac{1}{n}}$, we see that $k\mapsto\{n\in\mathbb{N}^*|\, n\vee k_{h+\frac{1}{n}}\leq k\}$ is non-decreasing, and that $\lim_k n_k=\infty$. For $k\geq k_{h+1}$ set $h''_k=h+\frac{1}{n_k}$. We have $h'_k\leq h<h''_k$ and $n^{h'_k,h''_k}(T)=n^{h,h''_k}(T)$. So the definition of $k_{h''_k}=k_{h+\frac{1}{n_k}}\leq k$ is consistent with the first part of the proof.

Set $\delta_k=\frac{1}{2}(h''_k-h)$ and $r_k=k+\max(|h'_k|,|h''_k+\delta_k|)$. The set $A=\{(x,x)\}_{x\in\mathrm{Slice}_{r_k}(T)}$ is a δ_k-correspondence between $\mathrm{Slice}_{r_k}(T)$ and itself. Let us prove that for all $x,y\in T_r$ with $H(x)\wedge H(y)\geq h''_k$ we have

$$H(x\wedge y)\geq h\Leftrightarrow H(x\wedge y)\geq h'_k. \tag{4.3.14}$$

If there are no points in T_r above level h''_k, then (4.3.14) is true. If there are points in $\mathrm{Slice}_r(T)$ above level h''_k but no point in T at level h''_k, then $\min_T H>h''_k\geq h\geq h'_k$, so (4.3.14) is true in this case as well. In the remaining case, T has at least one point at level h''_k. Reasoning on the ancestors of x and y at level h''_k (which might be equal), we can suppose without loss of generality that $H(x)=H(y)=h''_k$. The distance d is ultra-metric on level h''_k and the sub-trees for $\mathrm{Crown}_h(T)$ (resp $\mathrm{Crown}_{h'_k}(T)$) are the equivalence classes of the relation $R_h: d(x,y)\leq 2(h''_k-h)$ (resp. $R_{h'_k}: d(x,y)\leq 2(h''_k-h'_k)$). Since $h'_k\leq h$ we naturally have $xR_hy\Rightarrow xR_{h'_k}y$. It follows that the partition induced by R_h is finer than the partition induced by $R_{h'_k}$. The partition induced by R_h consists of $n^{h,h''_k}(T)$ equivalence classes, and the partition induced by $R_{h'_k}$ consists of $n^{h'_k,h''_k}(T)$ equivalence classes. By choice of h''_k, they have the same number of classes, and one is finer, so the relations are equal. We have

$$H(x\wedge y)\geq h\Leftrightarrow H(x\wedge y)\geq h'.$$

Since $(h,T)\in D_{\neq}$, we get by Lemma 4.3.6 that:

$$\lim_{k\to\infty} d^{\infty}_{\mathrm{LGHP}}(\mathrm{Crown}_h(T),\mathrm{Crown}_{h'_k}(T))=0.$$

The sequence $(h'_k)_{k\in\mathbb{N}^*}$ was arbitrary, so we have the continuous limit by sequential characterization:

$$\lim_{h'\uparrow h} d^{\infty}_{\mathrm{LGHP}}(\mathrm{Crown}_h(T), \mathrm{Crown}_{h'}(T)) = 0.$$

□

Lemma 4.3.9. *Let $(h,(T,d,H,\nu)) \in D_{\neq}$ be such that $H\nu(\{h\}) = 0$. If T has no branching point at height h, then the map $(h',T') \mapsto \mathrm{Crown}_{h'}(T')$, taking its values in $(\mathbb{T}_C, d^{\infty}_{\mathrm{LGHP}})$, is continuous at the point $(h,(T,d,H,\nu)) \in \mathbb{R}\times\mathbb{T}$.*

Proof. Take $(h'_k)_{k\in\mathbb{N}^*}$ a sequence of real numbers converging to h, $(T^k, d_k, H_k, \nu_k)_{k\in\mathbb{N}^*}$ a sequence of S-compact height-labelled trees converging to T for d_{LGHP}. Thanks to Lemma 3.4.3 and Proposition 3.4.1, there exists $(r_k)_{k\in\mathbb{N}^*}$ and $(\delta_k)_{k\in\mathbb{N}^*}$ two sequences of positive real numbers such that $\lim_k r_k = \infty$, $\lim_k \delta_k = 0$ and for every $k \in \mathbb{N}^*$, there exists A_k a δ_k-correspondence between $\mathrm{Slice}_{r_k}(T)$ and $\mathrm{Slice}_{r_k}(T^k)$. We shall find an integer k_0 and a sequence $(h''_k)_{k\in\mathbb{N}^*, k\geq k_0}$ such that for every $k \in \mathbb{N}^*$, $h''_k > (h'_k \vee h) + \delta_k$, $\lim_k h''_k = h$ and for every $(x,x'),(y,y') \in A_k$,

$$H(x)\wedge H(y) \geq h''_k \Rightarrow (H(x\wedge y) \geq h \Leftrightarrow H_k(x'\wedge y') \geq h'_k).$$

Then, we shall use Lemma 4.3.6 to end the proof.

Take $h'' > h$. Set $K_{h''} = \{x \in T | H(x) = h''\}$. Since T is S-compact, $K_{h''}$ is compact. We set $\delta(h'') = \frac{1}{2}\inf_{x,y\in K_{h''}} |d(x,y) - 2(h''-h)|$ if $K_{h''}$ is non empty, else take $\delta(h'') = |h''-h|$. Let us prove that we still have $0 < \delta(h'') \leq h''-h$. It is true by definition when $K_{h''}$ is empty so we only prove the case $K_{h''}$ is non-empty. Take $x \in K_{h''}$, and we see that by definition, $\delta(h'') \leq \frac{1}{2}|d(x,x) - 2(h''-h)| = |h''-h|$. A continuous map on a non-empty compact set reaches its minimum, so there exists $x_0, y_0 \in K_{h''}$ such that $\delta(h'') = \frac{1}{2}|d(x_0,y_0) - 2(h''-h)|$. We have

$$\begin{aligned}\delta(h'') &= \frac{1}{2}|d(x_0,y_0) - 2(h''-h)| \\ &= \frac{1}{2}|H(x_0) + H(y_0) - 2H(x_0\wedge y_0) - 2(h''-h)| \\ &= \frac{1}{2}|2h'' - 2H(x_0\wedge y_0) - 2h'' + 2h| \\ &= |h - H(x_0\wedge y_0)|.\end{aligned}$$

Since T has no branching points at height h, $H(x_0\wedge y_0) \neq h$ and $\delta(h'') > 0$. This means that whether $K_{h''}$ is empty or not, $0 < \delta(h'') \leq h''-h$. From the convergence of $(h'_k)_{k\in\mathbb{N}^*}$ and $(\delta_k)_{k\in\mathbb{N}^*}$, for all $h'' > h$, we can define $k_{h''}$ the smallest positive integer such that for every $k \geq k_{h''}$, $\delta(h'') > 2\delta_k + |h'_k - h|$.

For $k \geq k_{h+1}$, set $E_k = \{n \in \mathbb{N}^* | n \vee k_{h+\frac{1}{n}} \leq k\}$ and $n_k = \max(E_k)$. The sequence $(n_k)_{k\geq k_{h+1}}$ is well-defined since E_k is bounded and non-empty (it contains 1). By definition of $k_{h+\frac{1}{n}}$, we see that $k \mapsto \{n \in \mathbb{N}^* | n \vee k_{h+\frac{1}{n}} \leq k\}$ is non-decreasing, and that $\lim_k n_k = \infty$. For $k \geq k_{h+1}$ set $h''_k = h + \frac{1}{n_k}\cdot$ By construction $k_{h''_k} \leq k$. Thus, we have $\delta(h''_k) > 2\delta_\ell + |h'_\ell - h|$ for all $\ell \geq k$ as $k_{h''_k} = k_{h+\frac{1}{n_k}} \leq k$. So taking $\ell = k$ gives:

$$\delta(h''_k) > 2\delta_k + |h'_k - h|$$

and, as $\delta(h'') \le h'' - h$, we have $h''_k \ge h + \delta(h''_k) > h + |h'_k - h| + 2\delta_k \ge (h \vee h'_k) + \delta_k$.

Let us prove that for $(x, x_k), (y, y_k) \in A_k$, we have

$$H(x) \wedge H(y) \ge h''_k \Rightarrow \Big(H(x \wedge y) \ge h \Leftrightarrow H_k(x_k \wedge y_k) \ge h'_k\Big).$$

If T doesn't reach above the level h''_k, that is if $\sup_T H < h''_k$, there is nothing to do. If it does, let us take $(x, x_k), (y, y_k) \in A_k$ such that $H(x) \wedge H(y) \ge h''_k$. We have :

$$\begin{aligned}
&|(H_k(x_k \wedge y_k) - h'_k) - (H(x \wedge y) - h)| \\
&\qquad \le \frac{1}{2}|2H_k(x_k \wedge y_k) - 2H(x \wedge y)| + |h'_k - h| \\
&\qquad = \frac{1}{2}|d_k(x_k, y_k) - H_k(x_k) - H(y_k) - (d(x, y) - H(x) - H(y))| + |h'_k - h| \\
&\qquad \le \frac{1}{2}(|d_k(x_k, y_k) - d(x, y)| + |H_k(x_k) - H(x)| + |H(y_k) - H(y))|) + |h'_k - h| \\
&\qquad \le \frac{1}{2}(2\delta_k + \delta_k + \delta_k) + |h'_k - h| \\
&\qquad = 2\delta_k + |h'_k - h| \\
&\qquad < \delta(h''_k),
\end{aligned}$$

where for the third inequality, we used the fact that $(x, x_k), (y, y_k) \in A_k$ and that A_k is a δ_k-correspondence.

Let us prove that $\delta(h''_k) \le |H(x \wedge y) - h|$. If $H(x \wedge y) \ge h''_k$ then we have

$$|H(x \wedge y) - h| \ge h''_k - h \ge \delta(h''_k).$$

If $H(x \wedge y) < h''_k$, set x' and y' the respective ancestors of x and y at level h''_k. We have $H(x' \wedge y') = H(x \wedge y)$ and $x', y' \in K_{h''_k}$, and thus $|H(x \wedge y) - h| = |H(x' \wedge y') - h| \ge \delta(h''_k)$ by definition of $\delta(h''_k)$. Applying this to the upper bound on $|(H_k(x_k \wedge y_k) - h'_k) - (H(x \wedge y) - h)|$, we find

$$\big|(H_k(x_k \wedge y_k) - h'_k) - (H(x \wedge y) - h)\big| < \delta(h''_k) \le |H(x \wedge y) - h|.$$

For every $a, b \in \mathbb{R}$, $|b - a| < |a|$ implies that a and b have the same sign, so $H_k(x_k \wedge y_k) - h'_k$ and $H(x \wedge y) - h$ have the same sign. We have proven that

$$H(x) \wedge H(y) \ge h''_k \Rightarrow \Big(H(x \wedge y) \ge h \Leftrightarrow H_k(x_k \wedge y_k) \ge h'_k\Big).$$

$(h, T) \in D_{\neq}$, so by Lemma 4.3.6, we get

$$\lim_{k \to \infty} d^{\infty}_{\mathrm{LGHP}}(\mathrm{Crown}_{h'_k}(T^k), \mathrm{Crown}_h(T)) = 0.$$

The sequences $(h'_k)_{k \in \mathbb{N}^*}$ and $(T^k)_{k \in \mathbb{N}^*}$ were arbitrary, so we have the continuity of Crown at (h, T) by sequential characterization. □

Definition 4.3.10. *For $B \subset \mathbb{R} \times \mathbb{T}$, $h \in \mathbb{R}$, $T \in \mathbb{T}$ and $(T_n)_{n \in \mathbb{N}^*}$ a sequence of elements of $\mathbb{T}$, we note*

$$1_{(h,T) \in B} \cdot (T_n)_{n \in \mathbb{N}^*} = \begin{cases} (0_h)_{n \in \mathbb{N}^*} & \text{if } (h, T) \notin B \\ (T_n)_{n \in \mathbb{N}^*} & \text{if } (h, T) \in B. \end{cases}$$

Note that $(h,T) \mapsto (0_h)_{n\in\mathbb{N}^*}$ is a 1-Lipschitz application from $\mathbb{R}\times\mathbb{T}$ to $\mathbb{T}_C$, so for any map f from $\mathbb{R}\times\mathbb{T}$ to $\mathbb{T}_C$, and Borel set B, the application $(h,T) \mapsto 1_{(h,T)\in B}\cdot f(h,T)$ is measurable if and only if f is measurable on B. Recall that $\mathrm{Crown}_h(T)$ is an unordered sequence of elements of $\mathbb{T}$. Recall the measurable map g defined on $\mathbb{R}\times\mathbb{T}$ by $g(h,(T,d,H,\nu)) = (T,d,H,1_{H>h}\cdot\nu)$ from Lemma 3.4.14 which will be useful in the proof of the next Proposition.

Proposition 4.3.11. *The map $(h,T) \mapsto \mathrm{Crown}_h(T)$ defined on $(\mathbb{R}\times\mathbb{T}, d_{\mathbb{R}}\times d_{\mathrm{LGHP}})$ taking values in $(\mathbb{T}_C, d^{\infty}_{\mathrm{LGHP}})$ is measurable.*

Proof. Step 1: we prove that $f_1 : (h,T) \mapsto 1_{(h,T)\in D_1}\cdot \mathrm{Crown}_h(T)$ is measurable, for D_1 the measurable (we will prove it in a moment) set of all $(h,(T,d,H,\nu)) \in D_{\neq}$ such that T has no branching point at height h. By definition, the terms of $\mathrm{Crown}_h(T)$ do not have positive measure at their root, so we have $\mathrm{Crown}_h(T) = \mathrm{Crown}_h(g(h,T))$. Since the measure of $g(h,T)$ has no mass at height h, we can apply Lemma 4.3.9 to find that for $(h,T)\in D_1$, Crown is continuous at $(h,g(h,T))$. It follows that on D_1, Crown is measurable as the composition of a measurable function by a continuous function. Let us prove that D_1 is a Borel set. We have

$$
\begin{aligned}
&(\mathbb{R}\times\mathbb{T})\setminus D_1\\
&\qquad = D_= \cup \{(h,(T,d,H,\nu)) \in \mathbb{R}\times\mathbb{T} | \exists x_1,x_2\in T, H(x_1)\wedge H(x_2) > h, H(x_1\wedge x_2) = h\}\\
&\qquad = D_= \bigcup_{\varepsilon\in\mathbb{Q}^*_+} F_\varepsilon,
\end{aligned}
$$

where $D_= = (\mathbb{R}\times\mathbb{T})\setminus D_{\neq}$ and

$$
F_\varepsilon = \left\{ (h,(T,d,H,\nu)) \in \mathbb{R}\times\mathbb{T} \;\middle|\; \begin{array}{l} \exists x_1,x_2\in T, \\ H(x_1)\geq h+\varepsilon, H(x_2)\geq h+\varepsilon, H(x_1\wedge x_2) = h \end{array} \right\}.
$$

Let us prove that for $\varepsilon\in\mathbb{Q}^*_+$, the set F_ε is a closed set. Take $(h_k,(T^k,d_k,H_k,\nu_k))$ a sequence of elements of F_ε converging to some $(h,T)\in\mathbb{R}\times\mathbb{T}$. By hypothesis, we can find a sequence $(x_k,y_k)_{k\in\mathbb{N}^*}$ such that $x_k,y_k\in T^k$, $H_k(x_k)\geq h_k+\varepsilon$, $H_k(y_k)\geq h_k+\varepsilon$ and $H_k(x_k\wedge y_k) = h_k$. For every $k\in\mathbb{N}^*$, $\inf_{T^k} H_k \leq H(x_k\wedge y_k) = h_k$, so we can take x'_k and y'_k the respective ancestors of x_k and y_k at height $h+\varepsilon$. Since T is the limit of $(T^k)_{k\in\mathbb{N}^*}$, by Lemma 3.4.3 and Proposition 3.4.1, there exists two sequence $(r_k)_{k\in\mathbb{N}^*}$, $(\delta_k)_{k\in\mathbb{N}^*}$ of positive integers such that $\lim_k r_k = \infty$ and $\lim_k \delta_k = 0$ and for every $k\in\mathbb{N}^*$ some A_k is a δ_k correspondence between $\mathrm{Slice}_{r_k}(T)$ and $\mathrm{Slice}_{r_k}(T^k)$. Since $(h_k)_{k\in\mathbb{N}^*}$ converges, $r_k\geq |h_k+\varepsilon|$ for k above some k_0. For $k\geq k_0$, we choose $x''_k, y''_k\in T$ such that $(x''_k,x'_k),(y''_k,y'_k)\in A_k$. By choice, we have $H(x''_k), H(y''_k)\in[h_k+\varepsilon-\delta_k, h_k+\varepsilon+\delta_k]$. Since $(\delta_k)_{k\in\mathbb{N}^*}$ converges to 0, $(x''_k)_{k\geq k_0}$ and $(y''_k)_{k\geq k_0}$ are bounded in H. As T is S-compact, up to considering a sub-sequence, we can assume that $(x''_k,y''_k)_{k\geq k_0}$ converges to some $(x,y)\in T\times T$. By continuity, we have

$$
H(x) = H(y) = \lim_{k\to\infty} H(x''_k) = \lim_{k\to\infty} H(y''_k) = h+\varepsilon
$$

and

$$\begin{aligned} d(x,y) &= \lim_{k\to\infty} d(x''_k, y''_k) \\ &= \lim_{k\to\infty} d_k(x'_k, y'_k) \\ &= \lim_{k\to\infty} H_k(x'_k) + H_k(y'_k) - 2H(x'_k \wedge y'_k) \\ &= \lim_{k\to\infty} 2(h_k+\varepsilon) - 2h_k \\ &= 2\varepsilon. \end{aligned}$$

It follows that $H(x\wedge y) = \frac{1}{2}(H(x)+H(y)-d(x,y)) = h$. Since $H(x)\wedge H(y) \geq h+\varepsilon$, we deduce that $(h,T)\in F_\varepsilon$. We have proven that F_ε is closed, and $D_{\neq}$ is open by Lemma 4.3.4, so $D_1 = D_{\neq}\bigcap_{\varepsilon\in\mathbb{Q}_+^*} F_\varepsilon^c$ is a Borel set. This implies that f_1 is measurable.

Step 2: we prove that $f_2 : (h,T)\mapsto 1_{(h,T)\in D_2}\cdot \mathrm{Crown}_h(T)$ is measurable, for D_2 the measurable (we will prove it in a moment) set of all $(h,(T,d,H,\nu))\in D_{\neq}$ such that T is a discrete tree, see Definition 4.2.9. We prove first that D_2 is a Borel set. Take $(T,d,H,\nu)\in\mathbb{T}$. For every $r\in\mathbb{R}_+$, we set $E_r(T)$ the set of leaves with height in $[-r,r]$ and points at height r. Set $T_r = \mathrm{Slice}_r(T)$. Let us prove that for every $x\in T_r$, there exists $y\in E_r(T)$ such that $x\preceq y$. Take $x\in T_r$, the set $\{y\in T_r | x\preceq y\}$ is a closed set. Since T_r is compact, there exists $y_0\in T_r$ with $x\preceq y_0$ such that $H(y_0)=\max H(\{y\in T_r|x\preceq y\})$. This means that y_0 is maximal for $\preceq$ in $\{y\in T_r|x\preceq y\}$, so y_0 is maximal in T_r and $y_0\in E_r(T)$.

By definition, T is a discrete tree if and only if for every $r\in\mathbb{R}_+$, $E_r(T)$ is finite. Note that if $0\leq r' < r$ and $E_r(T)$ is finite then $E_{r'}(T)$ is finite, so T is a discrete tree if and only if for every $r\in\mathbb{N}^*$, $E_r(T)$ is finite. Note that for $\preceq$, $E_r(T)$ is the set of all maximal points of T_r. For every $x_1,...,x_n$, we can take $y_1,...,y_n\in E_r(T)$ such that for all i, $x_i\preceq y_i$. If $n\leq \#(E_r(T))$, taking $y_1,...,y_n$ distinct elements in $E_r(T)$ provides a family of non-comparable elements since they are maximal. If $n > \#(E_r(T))$, there necessarily exists $i\neq j$ such that $x_i\preceq y_i = y_j \succeq x_j$. By Remark 4.1.12, $x_i\preceq x_j$ or $x_j\preceq x_i$. This means that $E_r(T)$ is finite if and only if for every $r\in\mathbb{N}^*$, there exists $n\in\mathbb{N}^*$ such that

$$\forall x_1,...,x_n\in T_r, \exists 1\leq i,j\leq n, (i\neq j \text{ and } x_i\preceq x_j).$$

This means that we have

$$D_2 = D_{\neq}\cap\left(\mathbb{R}\times\bigcap_{r\in\mathbb{N}^*}\bigcup_{n\in\mathbb{N}^*}\left\{(T,d,H,\nu)\in\mathbb{T}\;\middle|\; \begin{array}{l}\forall x_1,...,x_n\in T_r,\\ \exists 1\leq i,j\leq n, (i\neq j \text{ and } x_i\preceq x_j)\end{array}\right\}\right)$$

and

$$\begin{aligned}
&\bigcap_{r\in\mathbb{N}^*}\bigcup_{n\in\mathbb{N}^*}\left\{(T,d,H,\nu)\in\mathbb{T}\,\middle|\,\begin{array}{l}\forall x_1,...,x_n\in T_r,\\ \exists 1\le i,j\le n,(i\neq j\text{ and }x_i\preceq x_j)\end{array}\right\}\\
&=\bigcap_{r\in\mathbb{N}^*}\bigcup_{n\in\mathbb{N}^*}\left\{(T,d,H,\nu)\in\mathbb{T}\,\middle|\,\begin{array}{l}\forall x_1,...,x_n\in T,\\ \max(|H(x_1)|,...,|H(x_n)|)\le r\Rightarrow\\ \exists 1\le i<j\le n,d(x_i,x_j)=|H(x_i)-H(x_j)|\end{array}\right\}\\
&=\bigcap_{r\in\mathbb{N}^*}\bigcup_{n\in\mathbb{N}^*}\left\{(T,d,H,\nu)\in\mathbb{T}\,\middle|\,\begin{array}{l}\forall x_1,...,x_n\in T,\\ \max(|H(x_1)|,...,|H(x_n)|)< r\Rightarrow\\ \exists 1\le i<j\le n,d(x_i,x_j)=|H(x_i)-H(x_j)|\end{array}\right\}\\
&=\bigcap_{r\in\mathbb{N}^*}\bigcup_{n\in\mathbb{N}^*}F_{r,n},
\end{aligned}$$

where

$$F_{r,n}=\left\{(T,d,H,\nu)\in\mathbb{T}\,\middle|\,\begin{array}{l}\forall x_1,...,x_n\in T,\\ \exists 1\le i<j\le n,\\ (|H(x_i)|\ge r\text{ or }d(x_i,x_j)=|H(x_i)-H(x_j)|)\end{array}\right\}.$$

The condition $\max(|H(x_1)|,...,|H(x_n)|)<r$ is not equivalent to $\max(|H(x_1)|,...,|H(x_n)|)\le r$, but the third equality holds anyway thanks to the intersection over $r\in\mathbb{N}^*$. The set $F_{r,n}$ is closed thanks to Lemma 3.4.9 and $D_{\neq}$ is an open set by Lemma 4.3.4, so D_2 is a Borel set.

Let us prove that f_2 is measurable. Recall from the beginning of Step 1 that $\mathrm{Crown}_h(T)=\mathrm{Crown}_h(g(h,T))$ and the measure of $g(h,T)$ has no mass at height h. With Lemma 4.3.8, we see that for every h and T,

$$\mathrm{Crown}_h(g(h,T))=\lim_{n\to\infty}\mathrm{Crown}_{h-\frac{1}{n}}(g(h,T)).$$

For $(h,T)\in D_2$, T is discrete, so, T only has a finite number of leaves with height in $[h-1,h+1]$, as well as a finite number of points at height $h+1$. It follows that T only has a finite number of branching points with heights in $[h-1,h]$. The tree $g(h,T)$ has the same branching points as T, so for all but a finite number of $h'\in[h-1,h]$, $g(h,T)$ doesn't have a branching point at height h'. Recall that (h',T) is in the set $D_{\neq}$ if and only if there exists $x\in T$ such that $H(x)>h'$. Since $(h,T)\in D_{\neq}$, we have $(h',g(h,T))\in D_{\neq}$ for all $h'\le h$, so $(h',g(h,T))\in D_1$ for all but a finite number of $h'\in[h-1,h)$. It follows that

$$\begin{aligned}
1_{(h,T)\in D_2}\cdot\mathrm{Crown}_h(T)&=1_{(h,T)\in D_2}\cdot\left(\lim_{n\to\infty}\left(1_{(h-\frac{1}{n},g(h,T))\in D_1}\cdot\mathrm{Crown}_{h-\frac{1}{n}}(g(h,T))\right)\right)\\
&=1_{(h,T)\in D_2}\cdot\left(\lim_{n\to\infty}f_1\left(h-\frac{1}{n},\ g(h,T)\right)\right).
\end{aligned}$$

This proves that f_2 is the point-wise limit of $\left((h,T)\mapsto f_1\left(h-\frac{1}{n},\ g(h,T)\right)\right)_{n\in\mathbb{N}^*}$ on D_2. Since D_2 is a Borel set and f_1 is measurable, we have proven that f_2 is measurable on $\mathbb{R}\times\mathbb{T}$.

Step 3: Conclusion. Recall $\mathbf{1}_{(h,T)\in B}\cdot(T_n)_{n\in\mathbb{N}^*}$ from Definition 4.3.10. From Lemma 4.2.8, $\mathrm{Trim}_{\frac{1}{n}}$ is measurable. Since f_2 and g are measurable and $D_{\neq}$ is a Borel set, it is enough to prove that

$$\mathrm{Crown}_h(T)=\lim_{n\to\infty}\left(\mathbf{1}_{(h,T)\in D_{\neq}}\cdot f_2(h+\frac{1}{n},\mathrm{Trim}_{\frac{1}{n}}(g(T)))\right).$$

If $(h,T) \notin D_{\neq}$, the result is trivial since for every $(T_n)_{n\in\mathbb{N}^*} \in \mathbb{T}_C$, $\mathbf{1}_{(h,T)\in D_{\neq}} \cdot (T_n)_{n\in\mathbb{N}^*}$ is constant equal to $(0_h)_{n\in\mathbb{N}^*} = \mathrm{Crown}_h(T)$. Now, assume that $(h,T) \in D_{\neq}$. According to Lemma 4.3.7, we have

$$\mathrm{Crown}_h(T) = \mathrm{Crown}_h(g(h,T)) = \lim_{n\to\infty} \mathrm{Crown}_{h+\frac{1}{n}}(\mathrm{Trim}_{\frac{1}{n}}(g(h,T))).$$

Since $D_{\neq}$ is open, and $(h+\frac{1}{k}, \mathrm{Trim}_{\frac{1}{k}}(T))$ converges to $(h,T) \in D_{\neq}$, we can choose n_0 such that for every $n \geq n_0$, $(h+\frac{1}{k}, \mathrm{Trim}_{\frac{1}{k}}(T)) \in D_{\neq}$. Thanks to Lemma 4.2.10, $\mathrm{Trim}_{\frac{1}{n}}(g(h,T))$ is a discrete tree, so, for every $n \geq n_0$, we have $(h+\frac{1}{k}, \mathrm{Trim}_{\frac{1}{k}}(T)) \in D_2$. We deduce that, in the metric space $(\mathbb{T}_C, d^{\infty}_{\mathrm{LGHP}})$:

$$\begin{aligned}
\mathrm{Crown}_h(T) &= \lim_{n\to\infty} \mathrm{Crown}_{h+\frac{1}{n}}(\mathrm{Trim}_{\frac{1}{n}}(g(h,T))) \\
&= \lim_{\substack{n\to\infty \\ n\geq n_0}} \left(1_{(h+\frac{1}{n}, \mathrm{Trim}_{\frac{1}{n}}(g(h,T)))\in D_2} \cdot \mathrm{Crown}_{h+\frac{1}{n}}(\mathrm{Trim}_{\frac{1}{n}}(g(h,T))) \right) \\
&= \lim_{n\to\infty} f_2(h+\frac{1}{n}, \mathrm{Trim}_{\frac{1}{n}}(g(h,T))).
\end{aligned}$$

We have proven that for every $(h,T) \in \mathbb{R}\times\mathbb{T}$,

$$\mathrm{Crown}_h(T) = \lim_{n\to\infty} \left(\mathbf{1}_{(h,T)\in D_{\neq}} \cdot f_2\big(h+\frac{1}{n}, \mathrm{Trim}_{\frac{1}{n}}(T)\big) \right).$$

The map Crown is the point-wise limit of a sequence of measurable functions, so Crown is measurable. □

We recall that $\mathscr{B}(E)$ denote the Borel σ-field of a metric space E.

Definition 4.3.12. *For every $\mathbb{T}$-valued random variable T, we call $(\mathrm{Stump}_h(T))_{h\in\mathbb{R}}$ the growth process associated with T, and $(\mathrm{Crown}_{-h}(T))_{h\in\mathbb{R}}$ the coalescent process associated with T.*

We also define the filtrations $\mathscr{S} = (\mathscr{S}_h)_{h\in\mathbb{R}} = (\mathrm{Stump}_h^{-1}(\mathscr{B}(\mathbb{T})))_{h\in\mathbb{R}}$ and $\mathscr{C} = (\mathscr{C}_{-h})_{h\in\mathbb{R}} = (\mathrm{Crown}_{-h}^{-1}(\mathscr{B}(\mathbb{T}_C)))_{h\in\mathbb{R}}$.

Theorem 4.3.13. *The families $\mathscr{S}$ and $\mathscr{C}$ are filtrations on $(\mathbb{T}, \mathcal{B}(\mathbb{T}))$. The growth process is adapted for $\mathscr{S}$, and the coalescent process is adapted for $\mathscr{C}$.*

Proof. By Lemma 4.2.11, $T \mapsto \mathrm{Stump}_h(T)$ is measurable for every $h \in \mathbb{R}$, so the family $(\mathscr{S}_h)_{h\in\mathbb{R}}$ consists of σ-sub-algebra of $\mathcal{B}(\mathbb{T})$. For $h < h'$, $T \in \mathbb{T}$, we have $\mathrm{Stump}_h(T) = \mathrm{Stump}_h(\mathrm{Stump}_{h'}(T))$, so $\mathscr{S}_h \subset \mathscr{S}_{h'}$. This proves that $(\mathscr{S}_h)_{h\in\mathbb{R}}$ is a filtration.

By Proposition 4.3.11, $T \mapsto \mathrm{Crown}_h(T)$ is measurable for every $h \in \mathbb{R}$, so the family $(\mathscr{C}_h)_{h\in\mathbb{R}}$ consists of σ-sub-algebra of $\mathcal{B}(\mathbb{T})$. For $h' < h$, $T \in \mathbb{T}$, we have $\mathrm{Crown}_h(T) = \mathrm{Crown}_h(\mathrm{Crown}_{h'}(T))$, so $\mathscr{C}_h \subset \mathscr{C}_{h'}$, and we have proven that $(\mathscr{C}_h)_{h\in\mathbb{R}}$ is the time-reversal of a filtration. □

For T a random variable on $\mathbb{T}$, the process $(\mathrm{Stump}_h(T))_{h\in\mathbb{R}}$ corresponds to the growth process adapted to $(\mathscr{S}_h)_{h\in\mathbb{R}}$. If T is a Lévy tree for example, the growth process is Markov for $(\mathscr{S}_h)_{h\in\mathbb{R}}$.

For T a random variable on $\mathbb{T}$, the process $(\mathrm{Crown}_h(T))_{h\in\mathbb{R}}$ corresponds to a coalescence process adapted to $(\mathscr{C}_h)_{h\in\mathbb{R}}$. If T is a Kingman or Λ-coalescent tree for example, the coalescence process is Markov for $(\mathscr{C}_h)_{h\in\mathbb{R}}$.

4.4 Mixing and exchangeability

The aim of this section is to define relations between the stump and the crown of a tree. The first step is to build a random tree from the stump of a tree and the crown of another. For measurability reasons, we will mainly consider the law of this random tree, rather than the random tree itself. The procedure is to take a height-labelled tree $T = (T, d, H, \nu)$ decorated with an additional probability measure p on T whose support is concentrated on some level, say $h \in H(T)$ of T (that is, p is a probability measure and $p(H^{-1}(\{h\})) = 1)$) and another height-labelled tree $T' = (T', d', H', \nu')$ with at least a point at height h. We give an enumeration $(T_n = (T_n, d_n, H_n, \nu_n))_{n\in\mathbb{N}^*}$ of the crown $\mathrm{Crown}_h(T')$ (which is infinite by definition). Note that for every $n \in \mathbb{N}^*$, $\min_{T_n} H_n = h$ and $\sum_{n\in\mathbb{N}^*} H_n\nu_n$ is a Borel measure. We take an sequence $(X_n)_{n\in\mathbb{N}^*}$ of independent random variables on T distributed as p, and graft each T_n on $\mathrm{Stump}_h(T)$ at X_n. We mixing of T' onto T according to p the probability distribution on $\mathbb{T}$ of the resulting random tree, and note it $P_{T\star_p T'}$, see Theorem 4.4.5. Then, we will define notions that are of use in the next chapter. We say that a random decorated height-labelled tree (T, d, H, ν, p) is exchangeable at level $h \in \mathbb{R}$ with respect to p if a.s. $h \in H(T)$, p is a probability measure over T concentrated at level h, and the probability measure $P_{T\star_p T}$ is equal to the distribution of T. When p is atomless, this property is designed to be an adaptation of the discrete exchangeability of arrays, found for example in de Finetti's representation theorem. We will define properly the mixing operation, prove all relevant measurability results to finally assert that the concept of exchangeability is properly defined, see Definition 4.4.8.

We shall define decorated height-labelled trees in the next remark.

Remark 4.4.1. Let $n \in \mathbb{N}^*$. We can generalize Definition 3.1.2 to n-measured metric labelled spaces $(E, d_E, H_E, (\nu_E^i)_{1\le i\le n})$, where ν_E^i are non-negative measures on $\mathcal{B}(E)$ such that $\nu_E^i(\mathrm{Slice}_h(E, d, H))$ is finite for all $h \in \mathbb{R}_+$ and $i \in \{1, \dots, n\}$. Then, the distance d_{GHP} given in Definition 3.1.9 is generalized by extending the Prohorov distance $d_{\mathrm{P}}^{(Z,d)}$ from Definition 3.1.3 between two measures ν and μ on a metric space (Z, d) by the distance between two family of n measures $(\nu^i)_{1\le i\le n}$ and $(\mu^i)_{1\le i\le n}$ on (Z, d) by:

$$d_{\mathrm{P}}^{(Z,d)}((\nu^i)_{1\le i\le n}, (\mu^i)_{1\le i\le n}) = \max_{1\le i\le n} d_{\mathrm{P}}^{(Z,d)}(\nu^i, \mu^i). \tag{4.4.1}$$

The generalization of local Gromov Hausdorff Prohorov distance d_{LGHP} from Definition 3.1.12 between S-compact measured labelled metric spaces to S-compact n-measured labelled metric spaces is immediate, and the extensions of Proposition 3.1.13 and Theorem 3.3.1 are straightforward. We keep the same notation d_{LGHP} for the corresponding local Gromov Hausdorff Prohorov distance. Then, following Definition 4.1.1 we say that $(T, d, H, (\nu^i)_{1\le i\le n})$, where (T, d) is a tree, H is a map from T to $\mathbb{R}$ such that (4.1.1) holds and ν^i are σ-finite measure satisfying (4.1.2). We define $\mathbb{T}^{[n]}$ as the set of S-compact n-height-labelled trees with the corresponding local Gromov Hausdorff Prohorov distance. An immediate extension of Theorem 4.1.15 gives that $(\mathbb{T}^{[n]}, d_{\mathrm{LGHP}})$ is Polish. Notice that $\mathbb{T}^{[1]} = \mathbb{T}$. The results on the measurability of the various functions defined on $\mathbb{T}$ from the two first chapters can be extended to the analogue functions defined on $\mathbb{T}^{[n]}$.

We could go further in this generalization by considering a space E with a countable family of measures $(E, d_E, H_E, (\nu_E^i)_{i\in\mathbb{N}^*})$ and replacing the distance in (4.4.1) by

$$d_{\mathrm{P}}^{(Z,d)}((\nu^i)_{i\in\mathbb{N}^*}, (\mu^i)_{i\in\mathbb{N}^*}) = \sum_{i\in\mathbb{N}^*} 2^{-i}\,(1 \wedge d_{\mathrm{P}}^{(Z,d)}(\nu^i, \mu^i)). \tag{4.4.2}$$

With evident notations, we would get that $(\mathbb{T}^{[\infty]}, d_{\mathrm{LGHP}})$ is Polish.

Recall $\mathbb{T}_C$ defined in Section 4.3.2. Recall that a Borel measure is a measure defined on the Borel σ-field and finite on every compact set.

Definition 4.4.2. *We define $\mathbb{T}_C^{\mathrm{Borel}}$ the set of all $(T_n, d_n, H_n, \nu_n)_{n\in\mathbb{N}^*} \in \mathbb{T}_C$ such that*

- *the measure $\sum_{n\in\mathbb{N}^*} H_n\nu_n$ is a Borel measure over $\mathbb{R}$, i.e. such that $\sum_{n\in\mathbb{N}^*}[H_n\nu_n]([-k,k]) < \infty$ for every $k \in \mathbb{N}^*$,*
- $\left[\sum_{n\in\mathbb{N}^*} H_n\nu_n\right](\{h\}) = 0$,
- *for all $n \in \mathbb{N}^*$, $\min_{T_n} H_n = h$,*

where h is the height of $0_h = \lim_n T_n$.

For (T, d, H, ν) an element of an equivalence class belonging to $\mathbb{T}$, $h \in H(T)$, $(x_n)_{n\in\mathbb{N}^*}$ a sequence of points in T at height h and $(T_n, d_n, H_n, \nu_n)_{n\in\mathbb{N}^*}$ an element of an equivalence class belonging to $\mathbb{T}_C^{\mathrm{Borel}}$ such that, for every $n \in \mathbb{N}^*$, $\min_{T_n} H_n = h$, we define the $\mathbb{T}$-valued map

$$\chi(T, (x_n)_{n\in\mathbb{N}^*}, (T_n)_{n\in\mathbb{N}^*}) = (T', d', H', \nu') \in \mathbb{T}, \tag{4.4.3}$$

where T', d', H' and ν' are defined as follows. We shall check in Lemma 4.4.3 that χ is indeed $\mathbb{T}$-valued. For convenience, we suppose without loss of generality that the $(x_n)_{n\in\mathbb{N}^*}$ are the respective roots of $(T_n)_{n\in\mathbb{N}^*}$ and that the sets T and $(T_n \setminus \{x_n\})_{n\in\mathbb{N}^*}$ are disjoint. Set

$$T' = T \cup \Big(\bigcup_{n\in\mathbb{N}^*} T_n\Big)$$

and d' the only symmetric real-valued function defined on $T' \times T'$ such that

$$d'(x,y) = \begin{cases} d(x,y) & \text{if } x, y \in T, \\ d_n(x,y) & \text{if } x, y \in T_n, \\ d_n(x,x_n) + d(y,x_n) & \text{if } x \in T_n, y \in T \\ d_n(x,x_n) + d(x_n,x_{n'}) + d_{n'}(y,x_{n'}) & \text{if } x \in T_n, y \in T_{n'}, n \neq n'. \end{cases}$$

We set H' to be H on T and, for all n, H_n on T_n. There are no conflicts in this definition, since $H_n(x_n) = \min_{T_n} H_n = h = H(x_n)$. We define $\nu' = \nu + \sum_{n\in\mathbb{N}^*} \nu_n$. By hypothesis, $H'\nu'$ is a Borel measure. This concludes the definition of χ.

Lemma 4.4.3. *For (T, d, H, ν) an element of an equivalence class belonging to $\mathbb{T}$, $h \in H(T)$, $(x_n)_{n\in\mathbb{N}^*}$ a sequence of points in T at height h and $(T_n, d_n, H_n, \nu_n)_{n\in\mathbb{N}^*}$ an element of an equivalence class belonging to $\mathbb{T}_C^{\mathrm{Borel}}$ such that for every $n \in \mathbb{N}^*, \min_{T_n} H_n = h$, the tuple $\chi(T, (x_n)_{n\in\mathbb{N}^*}, (T_n)_{n\in\mathbb{N}^*})$ is a S-compact tree.*

Proof. In the whole proof, we keep the notations from (4.4.3). Let us first prove that (T', d', H') is a height-labelled tree. For each property of an height-labelled tree, there are many cases to consider, with pretty much the same demonstration. Since they are rather straightforward, we only prove one case for each property.

The function d' is symmetric and positive-definite. Let us prove the triangular inequality $d'(x,z) \leq d'(x,y) + d'(y,z)$ in the case $x \in T_n$, $y \in T_{n'}$ and $z \in T$ for $n \neq n'$. We have

$$\begin{aligned} d'(x,y) + d'(y,z) &= (d_n(x,x_n) + d(x_n,x_{n'}) + d_{n'}(x_{n'},y)) + (d_{n'}(y,x_{n'}) + d(x_{n'},z)) \\ &\geq d_n(x,x_n) + d(x_n,x_{n'}) + d(x_{n'},z) \\ &\geq d_n(x,x_n) + d(x_n,z) \\ &= d'(x,z). \end{aligned}$$

The first inequality comes from the positivity of $d_{n'}$, and the second from the triangular inequality for d; the two equalities come from the definition of d'. The other cases make similar uses of those two properties. This proves that d' is a distance.

Let us prove that T' is acyclic and geodesic. We prove the uniqueness of the injective path between x and y and the existence of the geodesic in the special case $x \in T_n$ and $y \in T$. By definition of d', any injective path from x to y must contain a single instance of x_n. For every $n' \neq n$, the path can't meet $T_{n'} \setminus \{x_{n'}\}$ because its boundary in T' consists in a single point at most. It follows that the injective path must be the concatenation of an injective path from x to x_n and an injective path from x_n to y. Those are unique in T_n and T respectively. Thus, there is at most one single injective path from x to y. Since $d'(x,y) = d'(x,x_n) + d'(x_n,y)$, $[\![x,x_n]\!] \cup [\![x_n,y]\!]$ is a geodesic from x to y. This means that T' is a tree.

Now, we prove that T' is a height-labelled tree, that is for every $x, y \in T'$,

$$d'(x,y) = H'(x) + H'(y) - 2 \min_{z \in [\![x,y]\!]} H'(z). \tag{4.4.4}$$

We only consider the case $x \in T_n$ and $y \in T$. Since x_n is the root of T_n, we have $H(x_n) = h = H_n(x_n)$, so

$$\begin{aligned} H'(x) + H'(y) - 2 \min_{z \in [\![x,y]\!]} H'(z) &= H_n(x) + H(y) - 2\big(\min_{z \in [\![x,x_n]\!]} H_n(z) \wedge \min_{z \in [\![x_n,y]\!]} H(z)\big) \\ &= \big[H_n(x) - h\big] + \big[h + H(y) - 2(h \wedge \min_{z \in [\![x_n,y]\!]} H(z))\big] \\ &= d_n(x,x_n) + d(x_n,y) \end{aligned}$$

since

$$h + H(y) - 2(h \wedge \min_{z \in [\![x_n,y]\!]} H(z)) = H(x_n) + H(y) - 2 \min_{z \in [\![x_n,y]\!]} H(z) = d(x_n,y).$$

We therefore obtain that (4.4.4) holds in this case. The other cases being similar, we deduce that T' is a height-labelled tree.

Let us prove that T' is S-compact. Take $r \in \mathbb{R}_+$ and $(y_k)_{k \in \mathbb{N}^*}$ a sequence of points of T' such that $|H'(y_k)| \leq r$ for $n \in \mathbb{N}^*$. If $(y_k)_{k \in \mathbb{N}^*}$ has an infinite number of points in T or T_n (for some $n \in \mathbb{N}^*$), then $(y_k)_{k \in \mathbb{N}^*}$ has an adherence point by S-compacity. If not, without loss of generality, we can assume that $y_k \notin T$ for all $k \in \mathbb{N}^*$. Then, we define a sequence $(n_k)_{k \in \mathbb{N}^*}$ by setting $n_k = n$ where $y_k \in T_n \setminus \{x_n\}$. The sequence $(x_{n_k})_{k \in \mathbb{N}^*}$ is bounded in the S-compact space T since $H(x_n) = h$. So we can take x an adherence point of $(x_{n_k})_{k\in\mathbb{N}^*}$. We have $d'(y_k, x_{n_k}) = H'(y_k) - h \leq \sup_{T_{n_k}} H_{n_k} - h$ for k large enough. By assumption, $\lim_k n_k = \infty$ and $\lim_n d_{\mathrm{LGHP}}(T_n, 0_h) = 0$, so $\lim_k \sup_{T_{n_k}} H_{n_k} - h = 0$. It follows that x is an adherence point for $(y_k)_{k\in\mathbb{N}^*}$. By the Bolzano-Weierstrass characterization, T' is S-compact. Since $H'\nu'$ is Borel over $\mathbb{R}$ and $[-r, r]$ is compact, we deduce that ν' is finite on $\mathrm{Slice}_r(T')$ for every $r \geq 0$. Thus, ν' is a Borel measure. □

For (Z, d) a metric space, $(x_n)_{n\in\mathbb{N}^*}, (y_n)_{n\in\mathbb{N}^*} \in Z^{\mathbb{N}^*}$, we define

$$d^Z_{\mathrm{PW}}((x_n)_{n\in\mathbb{N}^*}, (y_n)_{n\in\mathbb{N}^*}) = \sup_{n\in\mathbb{N}^*} \frac{1}{n} \wedge d(x_n, y_n). \tag{4.4.5}$$

The function d^Z_{PW} is a distance metrizing the point-wise convergence for the sequences in Z. We write d_{PW} when there is no ambiguity on Z.

Recall d_{LP} the local-Prohorov distance on the space of Borel measures on $\mathbb{R}$ defined by (3.4.17) in Section 3.4.4. In the following technical lemma, we prove that χ is continuous (in some specified sense).

Lemma 4.4.4. *Take (Z, d) a metric space, H a 1-Lipschitz map on Z, E a closed set of Z and μ a Borel measure on E such that $(E, d, H, \mu) \in \mathbb{T}$ and $(x_n)_{n\in\mathbb{N}^*}$ a sequence of elements of E such that the sequence $(H(x_n))_{n\in\mathbb{N}^*}$ is constant equal to some $h \in H(E)$. For $k \in \mathbb{N}^*$, take $(F^k)_{k\in\mathbb{N}^*}$ closed sets of Z and μ^k a Borel measure on F^k such that $(F^k, d, H, \mu^k) \in \mathbb{T}$, and $(x^k_n)_{n\in\mathbb{N}^*}$ a sequence of elements of F^k such that the sequence $(H(x^k_n))_{n\in\mathbb{N}^*}$ is constant equal to some $h_k \in H(F^k)$. Take $(T_n, d_n, H_n, \nu_n)_{n\in\mathbb{N}^*}$ a sequence of non-empty trees such that $\min_{T_n} H_n = h$, $H_n\nu_n(\{h\}) = 0$, $\lim_n d_{\mathrm{LGHP}}(T_n, 0_h) = 0$ and $\sum_{n\geq 1} H_n\nu_n$ is a Borel measure on $\mathbb{R}$. For $k \in \mathbb{N}^*$, take $(T^k_n, d^k_n, H^k_n, \nu^k_n)_{n\in\mathbb{N}^*}$ a sequence of non-empty trees such that $\min_{T^k_n} H^k_n = h_k$, $H^k_n\nu^k_n(\{h_k\}) = 0$, $\lim_n d_{\mathrm{LGHP}}(T^k_n, 0_{h_k}) = 0$ and $\sum_{n\geq 1} H^k_n\nu^k_n$ is a Borel measure on $\mathbb{R}$.*

Assume that:

$$d_{\mathrm{PW}}((x^k_n)_{n\in\mathbb{N}^*}, (x_n)_{n\in\mathbb{N}^*}) \underset{k\to\infty}{\longrightarrow} 0 \tag{4.4.6}$$

$$\sup_{n\in\mathbb{N}^*} d_{\mathrm{LGHP}}(T^k_n, T_n) \underset{k\to\infty}{\longrightarrow} 0,$$

$$d_{\mathrm{LP}}\Big(\sum_{n\geq 1} H^k_n\nu^k_n, \sum_{n\geq 1} H_n\nu_n\Big) \underset{k\to\infty}{\longrightarrow} 0, \tag{4.4.7}$$

and there exists a sequence $(r_k)_{k\in\mathbb{N}^*}$ of positive real numbers such that $\lim_{k\to\infty} r_k = +\infty$ and

$$\Big(d_{\mathrm{H}}(\mathrm{Slice}_{r_k}(F^k), \mathrm{Slice}_{r_k}(E)) \vee d_{\mathrm{P}}(1_{|H|\leq r_k} \cdot \mu^k, 1_{|H|\leq r_k} \cdot \mu)\Big) \underset{k\to\infty}{\longrightarrow} 0.$$

Then we have

$$d_{\mathrm{LGHP}}\Big(\chi(F^k, (x^k_n)_{n\in\mathbb{N}^*}, (T^k_n)_{n\in\mathbb{N}^*}), \chi(E, (x_n)_{n\in\mathbb{N}^*}, (T_n)_{n\in\mathbb{N}^*})\Big) \underset{k\to\infty}{\longrightarrow} 0.$$

Proof. Lemma 3.4.2 is a key ingredient of the proof.

Step 1: We prove the following conditions: for every $\varepsilon \in (0, 1)$ and $r > |h| + \varepsilon$, there exists $n_0, k_0 \in \mathbb{N}^*$ such that

1. $\forall k \geq k_0, |h_k - h| \leq \varepsilon$,
2. $\forall k \geq k_0, r_k \geq r$,
3. $\forall k \geq k_0, d_{\mathrm{H}}(\mathrm{Slice}_{r_k}(F^k), \mathrm{Slice}_{r_k}(E)) < \varepsilon$,
4. $\forall k \geq k_0, d_{\mathrm{P}}(1_{|H|\leq r_k} \cdot \mu^k, 1_{|H|\leq r_k} \cdot \mu) < \varepsilon$,
5. $\forall k \geq k_0, \sup_{n\in\mathbb{N}^*} d_{\mathrm{LGHP}}(T_n^k, T_n) < \frac{\varepsilon}{n_0}\mathrm{e}^{-r}$,
6. $\forall k \geq k_0, \max_{1\leq n\leq n_0} d(x_n^k, x_n) \leq \varepsilon$,
7. $\forall n > n_0, H_n(T_n) \subset [h, h+\varepsilon]$,
8. $\forall n > n_0, k \geq k_0, H_n^k(T_n^k) \subset [h_k, h_k + \varepsilon]$,
9. $\sum_{n>n_0} \nu_n(T_n) \leq \varepsilon$,
10. $\forall k \geq k_0, \sum_{n>n_0} \nu_n^k(T_n^k) \leq \varepsilon$.

Note that under the hypothesis of the lemma, $\lim_k h_k = h$, by continuity of H and the convergence of $(x_1^k)_{k\in\mathbb{N}^*}$ towards x_1. This gives 1 for k_0 large enough. Note that when n_0 is given, Conditions 2 to 6 are straightforward by hypothesis, so we first focus on Conditions 7 to 10. The measure $\sum_{n>0} H_n\nu_n$ is Borel and $\sum_{n>0} H_n\nu_n(\{h\}) = 0$, so there exists $\delta \in (0, \frac{\varepsilon}{2})$ such that

$$\sum_{n>0} H_n\nu_n([h, h+2\delta]) \leq \frac{\varepsilon}{2}. \tag{4.4.8}$$

With Remark 4.1.16, there exists $\delta' > 0$ such that for $T \in \mathbb{T}$,

$$d_{\mathrm{LGHP}}(T, 0_h) \leq \delta' \Longrightarrow (T \text{ is compact and } d_{\mathrm{GHP}}(T, 0_h) \leq \delta).$$

Take $n_0 \in \mathbb{N}^*$ such that for $n > n_0$, $d_{\mathrm{LGHP}}(T_n, 0_h) \leq \frac{\delta'}{2}$. This condition will imply 7 and 9. Thanks to (4.4.7), we can take $k_1 \in \mathbb{N}^*$ such that for $k \geq k_1$,

$$d_{\mathrm{LP}}\Big(\sum_{n\geq 1} H_n^k\nu_n^k, \sum_{n\geq 1} H_n\nu_n\Big) \leq \delta\mathrm{e}^{-(|h|+\delta)} \tag{4.4.9}$$

and $\sup_{n\in\mathbb{N}^*} d_{\mathrm{LGHP}}(T_n^k, T_n) \leq \frac{\delta'}{2}$. The first condition will imply 10, the second will imply 8.

Note that we have $\sup_{n>n_0} d_{\mathrm{LGHP}}(T_n^k, 0_h) \leq \delta'$ and $\sup_{n>n_0} d_{\mathrm{LGHP}}(T_n, 0_h) \leq \delta'$. By choice of δ', this implies that for $n > n_0$ we have $d_{\mathrm{GHP}}(T_n, 0_h) \leq \delta$ as well as $d_{\mathrm{GHP}}(T_n^k, 0_h) \leq \delta$ for $k \geq k_1$. Since $d_{\mathrm{GHP}}(T_n, 0_h) \leq \delta$, we deduce that $H_n(T_n) \subset [h, h+\delta]$ and thus 7 holds. Since $d_{\mathrm{GHP}}(T_n^k, 0_h) \leq \delta$, we deduce that $H_n^k(T_n^k) \subset [h-\delta, h+\delta] \bigcap [h_k, +\infty)$. Use that $\delta \in (0, \varepsilon/2)$ to get that 8 for $k \geq k_1$.

We also have:

$$\sum_{n>n_0} \nu_n(T_n) \leq \sum_{n>n_0} H_n\nu_n([h, h+\delta]) \leq \sum_{n>n_0} H_n\nu_n([h, h+2\delta]) \leq \frac{\varepsilon}{2} \leq \varepsilon,$$

where we used that $d_{\mathrm{GHP}}(T_n, 0_h) \leq \delta$ implies $H_n(T_n) \subset [h, h+\delta]$ for the first inequality and the definition of δ for the last. This gives 9.

Using the definition of d_{LP}, see (3.4.17), and the Markov inequality, we deduce from (4.4.9) that for $k \geq k_1$, there exists $r'_k \geq |h| + \delta$ such that:

$$d_{\mathrm{P}}\left(1_{[-r'_k, r'_k]} \cdot \sum_{n>0} H_n^k \nu_n^k, 1_{[-r'_k, r'_k]} \cdot \sum_{n>0} H_n \nu_n\right) \leq \delta. \tag{4.4.10}$$

This implies that for every $k \geq k_1$:

$$\begin{aligned}\sum_{n>n_0} \nu_n^k(T_n^k) &\leq \sum_{n>n_0} H_n^k \nu_n^k([h-\delta, h+\delta]) \\ &\leq \sum_{n\geq 1} H_n^k \nu_n^k([h-\delta, h+\delta]) \\ &\leq \sum_{n\geq 1} H_n \nu_n([h-2\delta, h+2\delta]) + \delta \\ &= \sum_{n\geq 1} H_n \nu_n([h, h+2\delta]) + \delta \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon,\end{aligned}$$

where we used $H_n^k(T_n^k) \subset [h-\delta, h+\delta]$ for the first inequality, (4.4.10) and the definition of d_{P} for the third inequality, that $\sum_{n>n_0} H_n \nu_n$ has its support on $[h, \infty)$ for the first equality, and the definition of δ, see (4.4.8), for the last inequality. This gives 10.

By hypothesis, we can find $k_0 \geq k_1$ satisfying Conditions 1 to 6 for our chosen n_0. This concludes Step 1.

Step 2: We define some metric spaces that will be of use in the rest of the proof. For convenience, we can suppose without loss of generality that $(x_n)_{n\in\mathbb{N}^*}$, $(x_n^k)_{n,k\in\mathbb{N}^*}$ are the respective roots of $(T_n)_{n\in\mathbb{N}^*}$, $(T_n^k)_{n,k\in\mathbb{N}^*}$, and that the sets Z, $(T_n^k \setminus \{x_n^k\})_{n,k\in\mathbb{N}^*}$ and $(T_n \setminus \{x_n\})_{n\in\mathbb{N}^*}$ are disjoint. The following construction is the same as the construction of χ in (4.4.3). Set

$$Z' = Z \cup \Big(\bigcup_{n\in\mathbb{N}^*} \Big(T_n \cup \Big(\bigcup_{k\in\mathbb{N}^*} T_n^k \Big)\Big)\Big) = Z \sqcup \Big(\bigsqcup_{n\in\mathbb{N}^*} \Big((T_n \setminus \{x_n\}) \sqcup \Big(\bigsqcup_{k\in\mathbb{N}^*} (T_n^k \setminus \{x_n^k\}) \Big)\Big)\Big)$$

and d' the only symmetric function on $Z' \times Z'$ such that

$$d(x,y) = \begin{cases} d(x,y) & \text{if } x,y \in Z, \\ d_n(x,y) & \text{if } x,y \in T_n, \\ d_n(x,x_n) + d'(y,x_n) & \text{if } x \in T_n, y \notin T_n, \\ d_n^k(x,y) & \text{if } x,y \in T_n^k, \\ d_n^k(x,x_n^k) + d'(y,x_n^k) & \text{if } x \in T_n^k, y \notin T_n^k. \end{cases}$$

Note that d' is a distance on Z'. For $x \in T_n$ (resp. T_n^k), we extend H to be $H(x) = H(x_n) + d'(x,x_n)$ (resp. $H(x) = H(x_n^k) + d'(x,x_n^k)$). Note that since x_n is the root of T_n, we have $H_n(x) = h_n + d_n(x,x_n) = H(x_n) + d'(x,x_n) = H(x)$, so H and H_n coincide on T_n. Similarly, H and H_n^k coincide on T_n^k. The function H is 1-Lipschitz on Z', by extension and definition of d'.

For every $k \geq k_0$ and every $n \in \mathbb{N}^*$, we know that $d_{\mathrm{LGHP}}(T_n^k, T_n) < \frac{\varepsilon}{n_0}\mathrm{e}^{-r}$. Since $\varepsilon < 1$, $\frac{\varepsilon}{n_0} < 1$, so there exists[3] $r_n^k \geq r$ such that

$$d_{\mathrm{GHP}}(\mathrm{Slice}_{r_n^k}(T_n^k), \mathrm{Slice}_{r_n^k}(T_n)) < \frac{\varepsilon}{n_0}\cdot \tag{4.4.11}$$

We set

$$G_k' = \mathrm{Slice}_{r_k}(F^k) \cup \Big(\bigcup_{n=1}^{n_0} \mathrm{Slice}_{r_n^k}(T_n^k) \Big) \quad \text{and} \quad G_k = G_k' \cup \Big(\bigcup_{n>n_0} T_n^k \Big), \tag{4.4.12}$$

$$E_k' = \mathrm{Slice}_{r_k}(E) \cup \Big(\bigcup_{n=1}^{n_0} \mathrm{Slice}_{r_n^k}(T_n) \Big) \quad \text{and} \quad E_k = E_k' \cup \Big(\bigcup_{n>n_0} T_n \Big). \tag{4.4.13}$$

We define $\nu^k = \mu^k + \sum_{n\in\mathbb{N}^*} \nu_n^k$ and $\nu = \mu + \sum_{n\in\mathbb{N}^*} \nu_n$. By hypothesis, ν_k and ν are both Borel measures. Set:

$$T^k = \chi(F^k, (x_n^k)_{n\in\mathbb{N}^*}, (T_n^k)_{n\in\mathbb{N}^*}) \quad \text{and} \quad T = \chi(E, (x_n)_{n\in\mathbb{N}^*}, (T_n)_{n\in\mathbb{N}^*}). \tag{4.4.14}$$

Step 3: We prove that for $k \geq k_0$, E_k and G_k are compact sets such that

$$\mathrm{Slice}_r(T^k) \subset G_k \subset T^k \quad \text{and} \quad \mathrm{Slice}_r(T) \subset E_k \subset T, \tag{4.4.15}$$

as well as

$$d_{\mathrm{GHP}}(G^k, E^k) \leq 7\varepsilon.$$

Take $k \geq k_0$. Since we can define T^k to be equal to $F_k \cup \Big(\bigcup_{n\in\mathbb{N}^*} T_n^k) \Big)$ equipped with d', H and ν_k. We naturally have, as $r_k \geq r$ and $r_n^k \geq r$, that:

$$\mathrm{Slice}_r(T) \subset G_k \subset T^k.$$

Note that $|h_k| \leq |h| + \varepsilon \leq r$, so $x_n^k \in G_k$ for every n. Let us prove that G_k is closed in T^k. Take $x \in T^k \setminus G_k$. If $x \in F^k$, then we have that $x \in F^k \setminus \mathrm{Slice}_{r_k}(F^k)$. Since H is 1-Lipschitz, the ball of center x with radius $|H(x) - r_k| > 0$ is a subset of $F^k \setminus \mathrm{Slice}_{r_k}(F^k) \subset T^k \setminus G_k$. If $x \in T_n^k$ for some $n \in \mathbb{N}^*$, then we have that $x \in T_n^k \setminus \mathrm{Slice}_{r_n^k}(T_n^k)$ for some $n \leq n_0$. Since H is 1-Lipschitz, the ball of center x with radius $|H(x) - r_n^k| > 0$ is a subset of $T_n^k \setminus \mathrm{Slice}_{r_n^k}(T_n^k) \subset T^k \setminus G_k$. We have proven that G_k is closed in T^k.

Now, we prove that G_k is compact. We have

$$\sup_{G_k} H = r_k \vee \sup_{n>n_0} (\sup_{T_n^k} H_n) \vee \sup_{1\leq n\leq n_0} r_n^k \leq r_k \vee (|h_k| + \varepsilon) \vee \max_{1\leq n\leq n_0} r_n^k < \infty.$$

[3] We can extract r_n^k for each n with the Markov inequality because we have a control on

$$\sup_{n\in\mathbb{N}^*} d_{\mathrm{LGHP}}(T_n^k, T_n) = \sup_n \int_{\mathbb{R}_+} \big(1 \wedge d_{\mathrm{GHP}}(\mathrm{Slice}_s(T_n^k), \mathrm{Slice}_s(T_n))\big)\mathrm{e}^{-s} ds \leq \frac{\varepsilon}{n_0}\mathrm{e}^{-r},$$

while a control on $\int_{\mathbb{R}_+} \sup_n \big(1 \wedge d_{\mathrm{GHP}}(\mathrm{Slice}_s(T_n^k), \mathrm{Slice}_s(T_n))\big)\mathrm{e}^{-s} ds$ would allow us to extract a common r_k'.

By Lemma 4.4.3, T^k is S-compact. Since G_k is closed in T^k and H is bounded on G_k, G_k is compact.

We similarly prove that E_k is compact and that

$$\mathrm{Slice}_r(T) \subset E_k \subset T.$$

Recall the definition of G'_k and E'_k given in (4.4.12) and (4.4.13). Now, we prove that $d_{\mathrm{GHP}}(G_k, G'_k) \leq \varepsilon$ and $d_{\mathrm{GHP}}(E_k, E'_k) \leq \varepsilon$. We have $d_{\mathrm{GHP}}(G_k, G'_k) \leq d_{\mathrm{H}}(G_k, G'_k) \vee d_{\mathrm{P}}(1_{G_k} \cdot \nu_k, 1_{G'_k} \cdot \nu_k)$. Since $G'_k \subset G_k$, we have

$$\begin{aligned} d_{\mathrm{H}}(G_k, G'_k) &= \sup_{x \in G_k} d'(x, G'_k) \\ &\leq \sup_{n > n_0} \sup_{x \in T^k_n} d'(x, x^k_n) \\ &= \sup_{n > n_0} \sup_{x \in T^k_n} H(x) - H(x^k_n). \end{aligned}$$

As $H(x) - H(x^k_n) = H^k_n(x) - h_k$ and $H^k_n(T^k_n) \subset [h_k, h_k + \varepsilon]$, we have

$$\forall n > n_0,\ \forall x \in T^k_n,\ H(x) - H(x^k_n) \leq h_x + \varepsilon - h_k = \varepsilon,$$

which gives the inequality $d_{\mathrm{H}}(G_k, G'_k) \leq \varepsilon$.

By Equation (3.1.2), we have

$$d_{\mathrm{P}}(1_{G_k} \cdot \nu_k, 1_{G'_k} \cdot \nu_k) = \nu_k(G_k \setminus G'_k) \leq \sum_{n > n_0} \nu^k_n(T^k_n) < \varepsilon.$$

It follows that $d_{\mathrm{GHP}}(G_k, G'_k) \leq \varepsilon$. Similarly, we find that $d_{\mathrm{GHP}}(E_k, E'_k) \leq \varepsilon$.

Let us find a 5ε-correspondence between E'_k and G'_k. By Conditions 2 and 3 from Step 1,

$$d_{\mathrm{GHP}}(\mathrm{Slice}_{r_k}(F^k), \mathrm{Slice}_{r_k}(E)) < \varepsilon$$

and, for $n \leq n_0$, we have by (4.4.11) that $d_{\mathrm{GHP}}(\mathrm{Slice}_{r^k_n}(T^k_n), \mathrm{Slice}_{r^k_n}(T_n)) < \frac{\varepsilon}{n_0}$. By Proposition 3.4.1, there exists for every $n \in \mathbb{N}^*$ a $\frac{\varepsilon}{n_0}$-correspondence A_n between $\mathrm{Slice}_{r^k_n}(T^k_n)$ and $\mathrm{Slice}_{r^k_n}(T_n)$. Set

$$A = \Big\{(x, y) \in \mathrm{Slice}_{r_k}(F^k) \times \mathrm{Slice}_{r_k}(E) \Big| d(x, y) \leq \varepsilon\Big\}.$$

By 2-3 from Step 1, the fact that H is 1-Lipschitz and definition of d_P, A is a ε-correspondence between $\mathrm{Slice}_{r_k}(F^k)$ and $\mathrm{Slice}_{r_k}(E)$. The set $A' = A \cup \left(\bigcup_{1 \leq n \leq n_0} A_n\right)$ is a correspondence between G'_k and E'_k, let us prove that it is a 5ε-correspondence.

Take $(x, y), (x', y') \in A'$. We can restrict ourselves to one of the following cases:

1. $(x, y), (x', y') \in A$,
2. $(x, y), (x', y') \in A_n$,
3. $(x, y) \in A, (x', y') \in A_n$,
4. $(x, y) \in A_n, (x', y') \in A_{n'}, n \neq n'$.

Let us bound $\Delta = |d(x,x') - d(y,y')|$ in each case. In Case 1, $\Delta \leq 2\varepsilon$ because A is a ε-correspondence. In Case 2, $\Delta \leq 2\frac{\varepsilon}{n_0}$ because A_n is a $\frac{\varepsilon}{n_0}$-correspondence. For Cases 3 and 4, we first prove that for every $n \leq n_0$,

$$\sup_{(x,y)\in A_n} |d(x,x_n^k) - d(y,x_n)| \leq 4\varepsilon. \tag{4.4.16}$$

Take $y' \in \mathrm{Slice}_{r_n^k}(T_n)$ such that $(x_n^k, y') \in A_n$. We have, for every $(x,y) \in A_n$ that

$$\begin{aligned}|d(x,x_n^k) - d(y,x_n)| &\leq |d(x,x_n^k) - d(y,y')| + |d(y,y') - d(y,x_n)|\\ &\leq 2\frac{\varepsilon}{n_0} + d(y',x_n)\\ &= 2\frac{\varepsilon}{n_0} + H(y') - H(x_n)\\ &\leq 2\frac{\varepsilon}{n_0} + (h_k + \frac{\varepsilon}{n_0}) - h\\ &\leq 3\frac{\varepsilon}{n_0} + |h_k - h|\\ &\leq 4\varepsilon,\end{aligned}$$

where we used the fact that A_n is a $\frac{\varepsilon}{n_0}$-correspondence for the second inequality, that $H'(y') \leq H'(x_n^k) + \frac{\varepsilon}{n_0} = h_k + \frac{\varepsilon}{n_0}$ for the third inequality and Condition 1 from Step 1 for the last inequality. In Case 3, note that, since $x_n^k \in \mathrm{Slice}_r(F^k)$, $x_n \in \mathrm{Slice}_r(E)$ and $d(x_n^k, x_n) \leq \varepsilon$, we have $(x_n^k, x_n) \in A$. Since $d(x,x') = d(x,x_n^k) + d(x_n^k, x')$ and $d(y,y') = d(y,x_n) + d(x_n,y')$ and It follows that

$$\Delta \leq |d(x,x_n^k) - d(y,x_n)| + |d(x',x_n^k) - d(y',x_n)| \leq 2\varepsilon + 4\varepsilon = 6\varepsilon,$$

where, for the second inequality, we find that the first term corresponds to Case 1, and the second to Equation (4.4.16). For Case 4, we have

$$\Delta \leq |d(x,x_n^k) - d(y,x_n)| + |d(x',x_n^k) - d(y',x_n)| \leq 4\varepsilon + 6\varepsilon = 10\varepsilon,$$

using Equation (4.4.16) on the first term and Case 3 on the second. We have proven Condition (3.4.1).

For every $(x,y) \in A'$, $|H(x) - H(y)| \leq \varepsilon \vee \frac{\varepsilon}{n_0} \leq 5\varepsilon$, which proves Condition (3.4.2).

For $B \subset X$ and $A \subset X \times Y$, recall the notation $B^{\overrightarrow{A}} = \{y \in Y \,|\, \exists x \in B \text{ s.t. } (x,y) \in A\}$ for the set of all elements in correspondence with B for A. Notice that the restriction of ν^k on $F^k \bigcup_{n=1}^{n_0} T_n^k$ is equal to $\mu^k + \sum_{n=1}^{n_0} \nu_n^k$ and that $\mu + \sum_{n=1}^{n_0} \nu_n \leq \nu$. For $B \subset G'_k$ a Borel set, we have

$$\begin{aligned}\nu^k(B) &= \mu^k(B \cap F^k) + \sum_{n=1}^{n_0} \nu_n^k(B \cap T_n^k)\\ &\leq \mu((B\cap F^k)^{\overrightarrow{A}}) + \varepsilon + \sum_{n=1}^{n_0}\left(\nu_n((B\cap T_n^k)^{\overrightarrow{A_n}}) + \frac{\varepsilon}{n_0}\right)\\ &\leq \mu(B^{\overrightarrow{A'}}) + \sum_{n=1}^{n_0}\left(\nu_n(B^{\overrightarrow{A'}})\right) + 2\varepsilon\\ &\leq \nu(B^{\overrightarrow{A'}}) + 5\varepsilon,\end{aligned}$$

which proves Condition (3.4.3). Similarly, we prove Condition (3.4.4).

We have proven Conditions (3.4.1)-(3.4.4), so A' is a 5ε-correspondence between G'_k and E'_k. Using Proposition 3.4.1, we have $d_{\text{GHP}}(G'_k, E'_k) \leq 5\varepsilon$. It follows that

$$d_{\text{GHP}}(G_k, E_k) \leq d_{\text{GHP}}(G_k, G'_k) + d_{\text{GHP}}(G'_k, E'_k) + d_{\text{GHP}}(E'_k, E_k) \leq \varepsilon + 5\varepsilon + \varepsilon = 7\varepsilon.$$

This concludes Step 3.

Conclusion: We have proven in Steps 1-3 that for every $\varepsilon \in (0,1)$, $r > |h| + \varepsilon$, there exists $k_0(r,\varepsilon) \in \mathbb{N}^*$ such that for every $k \geq k_0(r,\varepsilon)$, there exists two compact sets $G^k(r,\varepsilon)$, $E^k(r,\varepsilon)$ such that

$$\text{Slice}_r(T^k) \subset G^k(r,\varepsilon) \subset T_k,$$

$$\text{Slice}_r(T) \subset E^k(r,\varepsilon) \subset T$$

and

$$d_{\text{GHP}}(G^k(r,\varepsilon), E^k(r,\varepsilon)) \leq 7\varepsilon.$$

From this, we deduce that there exists a sequence $(r_k, \varepsilon_k)_{k \geq k_0(|h|+1, \frac{1}{2})}$ in $(|h|+1, +\infty) \times (0, 1/2]$ with $\lim_k r_k = \infty$ and $\lim_k \varepsilon_k = 0$ such that for all $k \geq k_0(|h|+1, \frac{1}{2})$, we have $k \geq k_0(r_k, \varepsilon_k)$.

This means that for every $k \geq k_0(|h|+1, \frac{1}{2})$, there exists two compact sets $G^k(r_k, \varepsilon_k)$, $E^k(r_k, \varepsilon_k)$ such that

$$\text{Slice}_{r_k}(T^k) \subset G^k(r_k, \varepsilon_k) \subset T_k, \quad \text{Slice}_{r_k}(T) \subset E^k(r_k, \varepsilon_k) \subset T$$

and

$$d_{\text{GHP}}(G^k(r_k, \varepsilon_k), E^k(r_k, \varepsilon_k)) \leq 7\varepsilon_k.$$

By Lemma 3.4.2, we have $\lim_{k\to\infty} d_{\text{LGHP}}(T^k, T) = 0$. This and (4.4.14) gives the result. $\square$

Recall $\mathbb{T}^{[n]}$ defined in Remark 4.4.1. For $h \in \mathbb{R}$, we denote by δ_h the Dirac mass at h. Let $\mathbb{T}_{\text{mix}}$ be the subset of $\mathbb{T}^{[2]} \times \mathbb{T}$ of all $((T,p) = (T,d,H,\nu,p), T' = (T',d',H',\nu')) \in \mathbb{T}^{[2]} \times \mathbb{T}$ such that p is a probability measure satisfying $Hp = \delta_h$ for some $h \in H(T)$ (that is the support of p is in $H^{-1}(\{h\})$), additionally satisfying $h \in H'(T')$. Since the map $(T,p) \mapsto Hp$ from $\mathbb{T}^{[2]}$ to $\mathcal{M}(\mathbb{R})$ is 1-Lipschitz by Lemma 3.4.12, this implies that the possible values for the first component in $\mathbb{T}_{\text{mix}}$ form a Borel subset A of $\mathbb{T}^{[2]}$ on which the map $f : ((T,p),T') \mapsto (h, T')$ is continuous, so the map $f : ((T,p),T') \mapsto (h,T')$ is continuous from $A \times \mathbb{T}$ to $\mathbb{R} \times \mathbb{T}$. The set B of all $(h, T') \in \mathbb{R} \times \mathbb{T}$ such that $h \in H(T)$ forms a closed set, so the domain $\mathbb{T}_{\text{mix}} = A \cap f^{-1}(B)$ is closed in $\mathbb{T}^{[2]} \times \mathbb{T}$. We have the following main result which informally states that for $((T,p),T') \in \mathbb{T}_{\text{mix}}$ and $(X_n)_{n\in\mathbb{N}^*}$ a sequence of independent $\mathbb{T}$-valued random variables with distribution p, the probability distribution $P_{T\star_p T'}$ on $\mathbb{T}$ of $\chi(T, (X_n)_{n\in\mathbb{N}^*}, \text{Crown}_h(T'))$ is well defined. The random tree $\chi(T, (X_n)_{n\in\mathbb{N}^*}, \text{Crown}_h(T'))$ corresponds to grafting at level h according to the sampling distribution p the crown of T' on the stump of T.

Theorem 4.4.5. *Let $((T,p),T') \in \mathbb{T}_{\text{mix}}$. The probability measure $P_{T\star_p T'}$ on $\mathbb{T}$ of*

$$\chi(\tilde{T}, (X_n)_{n\in\mathbb{N}^*}, \tau_h),$$

where χ is defined by (4.4.3), $(\tilde{T}, \tilde{p})$ is an element of the equivalence class of (T,p) in $\mathbb{T}^{[2]}$, $(X_n)_{n\in\mathbb{N}^*}$ is a sequence of independent random variables on $\tilde{T}$ with the same distribution $\tilde{p}$ and

τ_h is an element in $\tilde{\mathbb{T}}_C$ of the equivalence class of $\mathrm{Crown}_h(T')$*, is well defined. Furthermore, the probability measure $P_{T\star_p T'}$ does not dependent on choice of the element $(\tilde{T},\tilde{p})$ in the equivalence class of (T,p) nor on the choice of the element τ_h in the equivalence class of* $\mathrm{Crown}_h(T')$*. So the probability measure $P_{T\star_p T'}$ is uniquely defined for $((T,p),T')\in\mathbb{T}_{\mathrm{mix}}$.*

Proof. We have to prove that $P_{T\star_p T'}$ is a probability distribution on $\mathbb{T}$, and that if the height-labelled trees (T_1,p_1) and (T_2,p_2) are in the same equivalence class of $T\in\mathbb{T}^{[2]}$ and if τ_h^1 and τ_h^2, elements of $\tilde{\mathbb{T}}_C$, are in the same equivalence class of $\mathrm{Crown}_h(T')$, then $\chi(T_1,(X_n^1)_{n\in\mathbb{N}^*},\tau_h^1)$ and $\chi(T_2,(X_n^2)_{n\in\mathbb{N}^*},\tau_h^2)$, have the same distribution, where $(X_n^i)_{n\in\mathbb{N}^*}$ are independent T_i-valued random variables with distribution p_i, for $i\in\{1,2\}$.

Recall the distance d_{PW} metrizing the point-wise convergence for the sequences defined in (4.4.5). For given S-compact 2-height-labelled tree $(\tilde{T},\tilde{p})$ and $\tau_h\in\tilde{\mathbb{T}}_C$, we deduce from Lemma 4.4.4 (taking $E=F^k=\tilde{T}$ and $(T_n,d_n,H_n,\nu_n)_{n\in\mathbb{N}^*}=(T_n^k,d_n^k,H_n^k,\nu_n^k)_{n\in\mathbb{N}^*}=\tau_h$ for all $k\in\mathbb{N}^*$) that the map from $(\tilde{T}^{\mathbb{N}^*},d_{\mathrm{PW}})$ to $\mathbb{T}$ defined by:

$$(x_n)_{n\in\mathbb{N}^*}\mapsto\chi(\tilde{T},(x_n),\tau_h)$$

is continuous. Therefore, the probability measure $P_{T\star_p T'}$ on $\mathbb{T}$ is well defined as the push-forward of the probability measure $\nu^{\mathbb{N}^*}$ on $T^{\mathbb{N}^*}$.

As (T_1,p_1) and (T_2,p_2) are in the same equivalence class of $T\in\mathbb{T}^{[2]}$, there exists a bijective isometric map ϕ from T_1 onto T_2 which preserves the labels and the measures p_i. Write $\tau_h^i=(\tilde{T}_n^i=(\tilde{T}_n^i,d_n^i,H_n^i,\nu_n^i))_{n\in\mathbb{N}^*}$ for $i\in\{1,2\}$. Let $\varepsilon>0$, which will be chosen later. According to the definition (4.3.1) of d_{LGHP}^∞, as τ_h^1 and τ_h^2 are in the same equivalence class of $\mathrm{Crown}_h(T')$, there exists a permutation $\sigma\in\mathfrak{S}(\mathbb{N}^*)$ such that:

$$\sup_{n\in\mathbb{N}^*}d_{\mathrm{LGHP}}(\tilde{T}_n^1,\tilde{T}_{\sigma(n)}^2)\leq\varepsilon.$$

Let $(X_n^1)_{n\in\mathbb{N}^*}$ be independent T_1-valued random variables with distribution p_1. By construction, notice that $(X_n^2=\phi(X_{\sigma(n)}^1))_{n\in\mathbb{N}^*}$ are independent T_2-valued random variables with distribution p_2. Thanks to Lemma 4.4.4 (taking $E=\phi(T_1)$, $F^k=T_2$, $(T_n)_{n\in\mathbb{N}^*}=(\tilde{T}_n^1)_{n\in\mathbb{N}^*}$ and $(T_n^k)_{n\in\mathbb{N}^*}=(\tilde{T}_n^2)_{n\in\mathbb{N}^*}$ for all $k\in\mathbb{N}^*$), we deduce, since $\sum_{n\geq1}H_n^i\nu_n^i$ does not depend on i, that for any $\delta>0$, taking $\varepsilon>0$ small enough, we have:

$$d_{\mathrm{LGHP}}\Big(\chi(T_2,(X_n^2)_{n\in\mathbb{N}^*},(\tilde{T}_n^2)_{n\in\mathbb{N}^*}),\chi(\phi(T_1),(\phi(X_n^1))_{n\in\mathbb{N}^*},(\tilde{T}_n^1)_{n\in\mathbb{N}^*})\Big)\leq\delta.$$

By construction of χ see (4.4.3), we have that the trees $\chi\big(\phi(T_1),(\phi(X_n^1))_{n\in\mathbb{N}^*},(\tilde{T}_n^1)_{n\in\mathbb{N}^*}\big)$ and $\chi(T_1,(X_n^1)_{n\in\mathbb{N}^*},(\tilde{T}_n^1)_{n\in\mathbb{N}^*}))$ are equal in $\mathbb{T}$. Notice that the distribution of the random tree $\chi(T_2,(X_n^2)_{n\in\mathbb{N}^*},(\tilde{T}_n^2)_{n\in\mathbb{N}^*})$ does not depend on δ or ε. Since $\delta>0$ is arbitrary, we deduce that the random trees $\chi(T_2,(X_n^2)_{n\in\mathbb{N}^*},(\tilde{T}_n^2)_{n\in\mathbb{N}^*})$ and $\chi(T_1,(X_n^1)_{n\in\mathbb{N}^*},(\tilde{T}_n^1)_{n\in\mathbb{N}^*})$ have the same distribution. This means that the probability distribution $P_{T\star_p T'}$ does not depend on the choice of the elements in the equivalence classes of T and of $\mathrm{Crown}_h(T')$. □

By convention, we shall say that $P_{T\star_p T'}$ is the probability distribution of the random tree $\chi(T,(X_n)_{n\in\mathbb{N}^*},\mathrm{Crown}_h(T'))$, where $(X_n)_{n\in\mathbb{N}^*}$ is a sequence of independent T-valued random variables with probability distribution p (it is assumed that $Hp=\delta_h$). If $(T',d',H',\nu',p')\in$

$\mathbb{T}^{[2]}$ and p' is a probability measure such that $H'p' = \delta_{h'}$ for some $h' \in \mathbb{R}$, then we shall consider the push-forward measure $\tilde{p}$ of p' on $(\tilde{T}, \tilde{d}, \tilde{H}, \tilde{\nu}) = \chi(T, (x_n)_{n\in\mathbb{N}^*}, \mathrm{Crown}_h(T'))$ by the canonical projection:

$$T' \to \mathrm{Crown}_h(T') \to \chi(T, (x_n)_{n\in\mathbb{N}^*}, \mathrm{Crown}_h(T')). \tag{4.4.17}$$

If $h' > h$, it is easy to check that $\tilde{p}$ is a probability measure on $\tilde{T}$ such that $\tilde{H}\tilde{p} = \delta_{h'}$. It is now possible to iterate the grafting procedure.

Remark 4.4.6. Using a similar approach, for $n \geq 2$, $h_1 < \ldots < h_n$ and $(T_i, d_i, H_i, \nu_i, p_i)_{1\leq i\leq n}$ a sequence of $\mathbb{T}^{[2]}$ such that $h_i \in H_i(T_i)$ and $H_i\, p_i = \delta_{h_i}$, we define

$$P_{T_1 \star_{p_1} \cdots \star_{p_{n-1}} T_n} \tag{4.4.18}$$

as the probability distribution on $\mathbb{T}$ of $\tilde{T}_n$, where $(\tilde{T}_1, \tilde{p}_1), \ldots, (\tilde{T}_n, \tilde{p}_n)$ are defined recursively by:

$$\tilde{T}_{i+1} = \chi(\tilde{T}_i, (X^i_n)_{n\in\mathbb{N}^*}, \mathrm{Crown}_{h_i}(T_{i+1})) \quad \text{for } 1 \leq i < n,$$

with $(X^i_n)_{n\in\mathbb{N}^*}$ independent $\tilde{T}_i$-valued random variables with distribution $\tilde{p}_i$, and $\tilde{p}_{i+1}$ the push-forward probability measure on $\tilde{T}_{i+1}$ of p_{i+1} by the canonical projection (4.4.17), but for $\tilde{p}_1$ which is taken to be equal to p_1. We shall not give a more formal description of $P_{T_1 \star_{p_1} \cdots \star_{p_{n-1}} T_n}$.

In Definition 4.4.8 and more generally in Chapter 5, we will use the distribution $P_{T\star_p T'}$ with random trees T and T'. To assure that this is meaningful, we prove in Proposition 4.4.7 that the measure $P_{T\star_p T'}$ is a measurable map of $((T,p),T') \in \mathbb{T}_{\mathrm{mix}}$.

Proposition 4.4.7. *The map $((T,p),T') \mapsto P_{T\star_p T'}$ is measurable from $\mathbb{T}_{\mathrm{mix}}$ equipped with the distance $d_{\mathrm{LGHP}} \vee d_{\mathrm{LGHP}}$ (and the associated σ-field) to the set of probability measures over $\mathbb{T}$, equipped with the Prohorov distance.*

Proof. We decompose the mixing operation as follows:

$$(T,d,H,\nu,p),(T',d',H',\nu') \overset{\phi}{\mapsto} \begin{pmatrix} h, \text{ (such that } Hp = \delta_h) \\ \mathrm{Stump}_h(T,d,H,\nu,p) \\ \mathrm{Crown}_h(T',d',H',\nu') \\ 1_{(h,\infty)} \cdot [H'\nu'], \end{pmatrix} \overset{\psi}{\mapsto} P_{T\star_p T'}, \tag{4.4.19}$$

where $\mathrm{Im}(\phi) \subset \mathbb{R} \times \mathbb{T}^{[2]} \times \mathbb{T}_C \times \mathcal{M}_{\mathrm{Borel}}(\mathbb{R})$ is equipped with the distance $d_{\mathbb{R}} \vee d_{\mathrm{LGHP}} \vee d^{\infty}_{\mathrm{LGHP}} \vee d_{LP}$ (here, $\mathcal{M}_{\mathrm{Borel}}(\mathbb{R})$ is the set of all Borel measures over $\mathbb{R}$). We will prove that ϕ is measurable, and that ψ is continuous on $\mathrm{Im}(\phi)$.

Let us prove that ϕ is measurable. We argue component by component. For the first component h, we first recall that the application $(T,d,H,\nu,p) \mapsto Hp$ is 1-Lipschitz, see Lemma 3.4.12 (with $\mathbb{T}$ replaced by $\mathbb{T}^{[2]}$). Since $((T,p),T') \in \mathbb{T}_{\mathrm{mix}}$, Hp is a Dirac measure δ_h for some $h \in \mathbb{R}$. The application $\delta_h \mapsto h$ is continuous, so $(T,d,H,\nu,p) \mapsto h$ is continuous, hence measurable. This component is used as a measurable parameter for the next two components. Since Crown and Stump are measurable, see Lemma 4.2.11 and Proposition 4.3.11, the corresponding components are measurable as functions of T, T' and h. For the fourth, we use the continuity of $(T',d',H',\nu') \mapsto H'\nu'$ and the measurability in h of the map $(h, H'\nu') \mapsto 1_{(h,\infty)} \cdot [H'\nu']$, see Lemma 3.4.12. We have proven that ϕ is measurable.

Now, we shall use the setting of Lemma 4.4.4 to prove the continuity of ψ over $\phi(\mathbb{T}\times\mathbb{T}^2)$.

Step 1: We build a large separable space (Z,d_Z) equipped with a 1-Lipschitz map H, in which we have the convergence of $\mathrm{Stump}_{h_k}(T_k)$ to $\mathrm{Stump}_h(T)$. Take $\big((T,d,H,\nu,p),(T',d',H',\nu')\big)\in\mathbb{T}_{\mathrm{mix}}$. Take $\big((T_k,d_k,H_k,\nu_k,p_k),(T'_k,d'_k,H'_k,\nu'_k)\big)_{k\in\mathbb{N}^*}$ a sequence of elements of $\mathbb{T}_{\mathrm{mix}}$ such that the image by ϕ of its terms converges to $\phi\big((T,d,H,\nu,p),(T',d',H',\nu')\big)$. By convergence of the second component of ϕ, we have:

$$d_{\mathrm{LGHP}}\big(\mathrm{Stump}_{h_k}(T_k,d_k,H_k,\nu_k,p_k),\mathrm{Stump}_h(T,d,H,\nu,p)\big)\underset{k\to\infty}{\longrightarrow}0.$$

It follows from Lemmas 3.4.3 and 3.4.5, that there exists a sequence $(r_k)_{k\in\mathbb{N}^*}\in(\mathbb{R}_+)^{\mathbb{N}^*}$ with $\lim_{k\to\infty}r_k=+\infty$ such that

$$d_{\mathrm{LGHP}}\big(\mathrm{Slice}_{r_k}(\mathrm{Stump}_{h_k}(T_k,d_k,H_k,\nu_k,p_k)),\mathrm{Slice}_{r_k}(\mathrm{Stump}_h(T,d,H,\nu,p))\big)\underset{k\to\infty}{\longrightarrow}0. \tag{4.4.20}$$

We note δ_k the left-hand side of (4.4.20). Recall that, for (E,d_E), (F,d_F) two metric spaces, $\mathbb{D}(E,F)$ is the set of all distances on $E\sqcup F$ whose restrictions are d_E on E and d_F on F. By Lemma 1.1.14, there exists, for every $k\in\mathbb{N}^*$, some $d''_k\in\mathbb{D}(\mathrm{Stump}_{h_k}(T_k),\mathrm{Stump}_h(T))$ such that

- $\forall k\in\mathbb{N}^*,x,y\in\mathrm{Stump}_h(T),d''_k(x,y)=d(x,y)$ (this is true by definition of $\mathbb{D}$);
- $\forall k\in\mathbb{N}^*,x,y\in\mathrm{Stump}_{h_k}(T_k),d''_k(x,y)=d_k(x,y)$ (this is true by definition of $\mathbb{D}$);
- $\forall k\in\mathbb{N}^*,x\in\mathrm{Slice}_{r_k}(\mathrm{Stump}_h(T)),\exists y\in\mathrm{Slice}_{r_k}(\mathrm{Stump}_{h_k}(T_k))$, $d''_k(x,y)\vee|H(x)-H_k(y)|\le\delta_k+\frac{1}{k}$;
- $\forall k\in\mathbb{N}^*,x\in\mathrm{Slice}_{r_k}(\mathrm{Stump}_{h_k}(T_k)),\exists y\in\mathrm{Slice}_{r_k}(\mathrm{Stump}_h(T))$, $d''_k(x,y)\vee|H(y)-H_k(x)|\le\delta_k+\frac{1}{k}$;
- $d_{\mathrm{P}}(1_{|H_k|\le r_k}\cdot\nu_k,1_{|H|\le r_k}\cdot\nu)\vee d_{\mathrm{P}}(1_{H\le r_k}\cdot p_k,1_{H\le r_k}\cdot p)\le\delta_k+\frac{1}{k}$.

We set $Z=\mathrm{Stump}_h(T)\sqcup\big(\bigsqcup_{k\in\mathbb{N}^*}\mathrm{Stump}_{h_k}(T_k)\big)$ and d'_Z the symmetric function such that

$$d'_Z(x,y)=\begin{cases}d(x,y) & \text{if } x,y\in\mathrm{Stump}_h(T)\\ d''_k(x,y) & \text{if } x\in\mathrm{Stump}_h(T),y\in\mathrm{Stump}_{h_k}(T_k)\\ \inf_{z\in\mathrm{Stump}_h(T)}d''_k(x,z)+d''_{k'}(z,y) & \text{if } x\in\mathrm{Stump}_{h_k}(T_k),y\in\mathrm{Stump}_{h_{k'}}(T_{k'}).\end{cases}$$

The function d'_Z is a distance over Z that satisfies

1. $\forall x,y\in\mathrm{Stump}_h(T),d'_Z(x,y)=d(x,y)$;
2. $\forall k\in\mathbb{N}^*,x,y\in\mathrm{Stump}_{h_k}(T_k),d'_Z(x,y)=d_k(x,y)$;
3. $\forall k\in\mathbb{N}^*,x\in\mathrm{Slice}_{r_k}(\mathrm{Stump}_h(T)),\exists y\in\mathrm{Slice}_{r_k}(\mathrm{Stump}_{h_k}(T_k))$, $d'_Z(x,y)\vee|H(x)-H_k(y)|\le\delta_k+\frac{1}{k}$;
4. $\forall k\in\mathbb{N}^*,x\in\mathrm{Slice}_{r_k}(\mathrm{Stump}_{h_k}(T_k)),\exists y\in\mathrm{Slice}_{r_k}(\mathrm{Stump}_h(T))$, $d'_Z(x,y)\vee|H(y)-H_k(x)|\le\delta_k+\frac{1}{k}$;
5. $d_{\mathrm{P}}^{(Z,d'_Z)}(\nu_k,\nu)\vee d_{\mathrm{P}}^{(Z,d'_Z)}(p_k,p)\le\delta_k+\frac{1}{k}$.

We also define

$$H_Z(x) = \begin{cases} H(x) & \text{if } x \in \mathrm{Stump}_h(T), \\ H_k(x) & \text{if } x \in \mathrm{Stump}_{h_k}(T_k) \end{cases}$$

and the distance $d_Z(x,y) = d'_Z(x,y) \vee |H_Z(x) - H_Z(y)|$. Since H and $(H_k)_{k\in\mathbb{N}^*}$ are 1-Lipschitz, we can replace d'_Z with d_Z in 1-2. The same can be done for 3-4. For 5, we can make the change as well since $d'_Z \leq d_Z$. Furthermore, H_Z is 1-Lipschitz on (Z, d_Z).

Step 2: We couple the measures and order the crowns. We have $d_{\mathrm{P}}^{(Z,d_Z)}(p_k, p) \underset{k\to\infty}{\longrightarrow} 0$, so by Skorokhod's representation Theorem (see [12] p. 70), there exists a random sequence $(X_k)_{k\in\mathbb{N}^*}$ with marginals $(p_k)_{k\in\mathbb{N}^*}$ such that X_k converges a.s. to some random variable X with law p. We note $((X_k^n)_{k\in\mathbb{N}^*})_{n\in\mathbb{N}^*}$ a sequence of independent random variables distributed as $((X_k)_{k\in\mathbb{N}^*})$, and, for every $n \in \mathbb{N}^*$, X^n the a.s. limit of $(X_k^n)_{k\in\mathbb{N}^*}$ when $k \to \infty$.

Take $(T'^n, d'^n, H'^n, \nu'^n)_{n\in\mathbb{N}^*}$ an enumeration of $\mathrm{Crown}_h(T')$. By hypothesis, we have that $\mathrm{Crown}_{h_k}(T'_k)$ converges to $\mathrm{Crown}_h(T')$ for $d_{\mathrm{LGHP}}^{\infty}$. Thus, there exists enumerations $(T_k'^n, d_k'^n, H_k'^n, \nu_k'^n)_{n\in\mathbb{N}^*}$ of $\mathrm{Crown}_{h_k}(T'_k)$ such that

$$\sup_{n\in\mathbb{N}^*} d_{\mathrm{LGHP}}(T_k'^n, T'^n) \underset{k\to\infty}{\longrightarrow} 0.$$

Step 3: Conclusion. Since $(X_n)_{n\in\mathbb{N}^*}$ and $(X_k^n)_{n\in\mathbb{N}^*}$ are sequences of independent and identically distributed random variables, the random trees

$$\chi(\mathrm{Stump}_h(T), (X_n)_{n\in\mathbb{N}^*}, \mathrm{Crown}_h(T')) \text{ and } \chi(\mathrm{Stump}_{h_k}(T_k), (X_k^n)_{n\in\mathbb{N}^*}, (T_k'^n)_{n\in\mathbb{N}^*})$$

are indeed distributed according to $P_{T\star_p T'}$ and $P_{T_k\star_{p_k} T'_k}$ respectively. The space (Z, d_Z) is metric and H_Z is 1-Lipschitz on Z; the subsets $\mathrm{Stump}_h(T)$, $\mathrm{Stump}_{h_k}(T_k)$ are S-compact. Recall the definition (4.4.5) of d_{PW}^Z the distance metrizing the point-wise convergence for the sequences in Z. We also have that:

$$d_{\mathrm{PW}}((X_n^k)_{n\in\mathbb{N}^*}, (X_n)_{n\in\mathbb{N}^*}) \underset{k\to\infty}{\longrightarrow} 0 \quad \text{a.s.},$$

$$\sup_{n\in\mathbb{N}^*} d_{\mathrm{LGHP}}(T_k'^n, T'^n) \underset{k\to\infty}{\longrightarrow} 0,$$

$$d_{\mathrm{LP}}\Big(\sum_{n\geq 1} H_k'^n \nu_k'^n, \sum_{n\geq 1} H'^n \nu'^n\Big) = d_{\mathrm{LP}}\big(1_{(h_k,+\infty)} \cdot \nu'_k, 1_{(h,+\infty)} \cdot \nu'\big) \underset{k\to\infty}{\longrightarrow} 0,$$

and there exists a sequence $(r_k)_{k\in\mathbb{N}^*}$ of positive real numbers such that $\lim_{k\to\infty} r_k = +\infty$ and

$$\Big(d_{\mathrm{H}}(\mathrm{Slice}_{r_k}(\mathrm{Stump}_{h_k}(T_k)), \mathrm{Slice}_{r_k}(\mathrm{Stump}_h(T))) \vee d_{\mathrm{P}}\big(1_{|H|\leq r_k} \cdot \nu_k, 1_{|H|\leq r_k} \cdot \nu\big)\Big) \underset{k\to\infty}{\longrightarrow} 0.$$

By Lemma 4.4.4, we have that a.s.:

$$d_{\mathrm{LGHP}}\Big(\chi(\mathrm{Stump}_{h_k}(T_k), (X_k^n)_{n\in\mathbb{N}^*}, (T_k'^n)_{n\in\mathbb{N}^*}), \chi(\mathrm{Stump}_h(T), (X^n)_{n\in\mathbb{N}^*}, (T'^n)_{n\in\mathbb{N}^*})\Big) \underset{k\to\infty}{\longrightarrow} 0.$$

From this coupling, we deduce the following convergence for the Prohorov distance on $\mathbb{T}$:

$$d_{\mathrm{P}}^{\mathbb{T}}(P_{T_k\star_{p_k} T'_k}, P_{T\star_p T'}) \underset{k\to\infty}{\longrightarrow} 0.$$

This proves that the map ψ is continuous over $\mathrm{Im}(\phi)$. Thus the map $\psi \circ \phi$ defined on $\mathbb{T}_{\mathrm{mix}}$ taking values in the set of probability measures over $\mathbb{T}$ by $\psi \circ \phi((T,p), T') = P_{T\star_p T'}$ is measurable. □

We end this section with a definition which will be very useful in Chapter 5.

Definition 4.4.8. *Let $h \in \mathbb{R}$, and (T, d, H, ν, p) be a $\mathbb{T}^{[2]}$-valued random tree with probability distribution Λ such that a.s. $Hp = \delta_h$. We say that T (or the probability distribution Λ) is exchangeable at level h with respect to p if $P_{T \star_p T} = \Lambda$, that is if the mix of T onto itself with respect to p has the same law as T.*

Remark 4.4.9. Let $((T, p), T') \in \mathbb{T}_{\text{mix}}$ and $h \in \mathbb{R}$ be such that $Hp = \delta_h$. Then, if τ is a $\mathbb{T}$-valued random variable distributed as $P_{T \star_p T'}$, then as $\text{Stump}_h(\tau) = \text{Stump}_h(T)$, we can see p as a probability measure on τ, so that (τ, p) is a $\mathbb{T}^{[2]}$-valued random variable. By construction, the random tree (τ, p) is exchangeable at level h with respect to p.

Chapter 5

An exchangeable random tree

In this chapter, we use the results of the previous chapter to build a family of random trees endowed with a measure ν_h at every level h that are exchangeable with respect to ν_h at every level h (recall Definition 4.4.8 of exchangeability). This construction is based on mixing vertical deformations of Kingman's coalescent (which is briefly reintroduced in Section 5.1). Intuitively, by the "cut and grafting at the same level" construction of the ancestral recombination graph (ARG) process, the distributions of these trees should form a whole family of reversible laws for this process.

We decided to present the results of this chapter which are the motivation of the two previous chapters, even if, by lack of time, its redaction is yet not complete.

5.1 Kingman's coalescent

The n-coalescent is the ancestral tree of n individuals in a large population with proper height scaling (see [26]). It is the representation of a continuous-time Markov process (for decreasing heights). The evolution of the process depends solely on the number k of clusters at a given time: a coalescence will occur at rate $k(k-1)/2$. When a coalescence occurs, two clusters chosen uniformly at random merge and the process continues with the remaining $k-1$ clusters. This corresponds to the coalescence of each pair of clusters at rate 1. The n-coalescent is obtained by starting the process with n clusters. If, for $1 < k \leq n$, we call h_k the first height at which there are only k clusters left, then the stump of the n-coalescent below h_k is a k-coalescent independent from the crown of the n-coalescent above h_k. If we choose k clusters from the n initial clusters and consider the sub-tree generated by those clusters, we find a tree with the same law as the k-coalescent. We define a measure on the n-coalescent, by putting a mass $\frac{1}{n}$ at each leaf.

Kingman's coalescent tree $(\mathcal{T}^{\mathrm{K}}, d, H, \nu^{\mathrm{K}})$, introduced in [46], is the limit in distribution with respect to the Gromov-Hausdorff-Prohorov distance of the n-coalescent as n goes to infinity. It contains the n-coalescent, in the sense that the sub-tree generated by n leaves taken independently with distribution ν^{K} has the law of the n-coalescent.

We recall now the construction of the Kingman's coalescent of [8] using our setting. For this construction, recall the function τ introduced in Definition 4.2.15, and its domain $\mathbb{R}\times\mathbb{R}_{+,0}^{\mathbb{N}^*}\times D$. It is a measurable function (Lemma 4.2.18) from $\mathbb{R}\times\mathbb{R}_{+,0}^{\mathbb{N}^*}\times D$ to $\mathbb{T}$. Let $(U_n)_{n\in\mathbb{N}^*}$ be a sequence of independent random variables uniformly distributed on [0,1], $(X_n)_{n\in\mathbb{N}^*}$ a sequence of independent exponential random variables of parameter 1, so that $(X_n)_{n\in\mathbb{N}^*}$ and $(U_n)_{n\in\mathbb{N}^*}$ are independent. We set $R_n = \sum_{k=n}^{\infty} \frac{2}{k(k+1)} X_k$. Since R_1 is a.s. finite, we can define the random S-compact labelled tree

$$(\mathcal{T}^{\mathrm{K}}, d, H, \nu^{\mathrm{K}}) = \tau\big(0, (R_n)_{n\in\mathbb{N}^*}, (U_n)_{n\in\mathbb{N}^*}\big).$$

In what follows, the labelled tree $(\mathcal{T}^{\mathrm{K}}, d, H, \nu^{\mathrm{K}})$ will be called Kingman's coalescent.

Recall that (see Definition 4.2.1), for $h < h'$, $n^{h,h'}(\mathcal{T}^{\mathrm{K}})$ is the number of points of $\mathcal{T}^{\mathrm{K}}$ at level h that have descendants at level h'. In particular, by construction, if $(\mathcal{T}^{\mathrm{K}}, d, H, \nu^{\mathrm{K}})$ is a Kingman's coalescent, for every $\varepsilon > 0$, $n^{-\varepsilon,0}(\mathcal{T}^{\mathrm{K}})$ is just the number of points at height $-\varepsilon$ in $\mathcal{T}^{\mathrm{K}}$.

For $h < h'$ and $T \in \mathbb{T}$, we set $(C_i^{h,h'}(T))_{1\leq i\leq n^{h,h'}(T)}$ the family of trees of $\mathrm{Crown}_h(T)$ that reach height h'.

Lemma 5.1.1. *If $(\mathcal{T}^{\mathrm{K}}, d, H, \nu^{\mathrm{K}})$ is a Kingman's coalescent, then for $\varepsilon > 0$,*

$$\mathbb{E}\left[\sum_{i=1}^{n^{-\varepsilon,0}(\mathcal{T}^{\mathrm{K}})} \left(\nu^{\mathrm{K}}\left(C_i^{-\varepsilon,0}(\mathcal{T}^{\mathrm{K}})\right)\right)^2\right] = 1 - \mathrm{e}^{-\varepsilon}.$$

Proof. Let $\varepsilon > 0$. Conditionally given $\mathcal{T}^{\mathrm{K}}$, if we take two independent random points $X, Y \in \mathcal{T}^{\mathrm{K}}$ with distribution ν^{K}, we have

$$\mathbb{P}(d(X,Y) \leq 2\varepsilon\ |\mathcal{T}^{\mathrm{K}}) = \sum_{i=1}^{n^{-\varepsilon,0}(\mathcal{T}^{\mathrm{K}})} \mathbb{P}\left(X, Y \in C_i^{-\varepsilon,0}(\mathcal{T}^{\mathrm{K}})|\mathcal{T}^{\mathrm{K}}\right) = \sum_{i=1}^{n^{-\varepsilon,0}(\mathcal{T}^{\mathrm{K}})} \nu^{\mathrm{K}}\left(C_i^{-\varepsilon,0}(\mathcal{T}^{\mathrm{K}})\right)^2.$$

Without conditioning, this yields

$$\mathbb{P}(d(X,Y) \leq 2\varepsilon) = \mathbb{E}\left[\sum_{i=1}^{n^{-\varepsilon,0}(\mathcal{T}^{\mathrm{K}})} \nu^{\mathrm{K}}\left(C_i^{-\varepsilon,0}(\mathcal{T}^{\mathrm{K}})\right)^2\right].$$

Since $\mathbb{P}(d(X,Y) \leq 2\varepsilon)$ is the probability that the ascendancy of two leaves taken at random coalesce before ε, we have also $\mathbb{P}(d(X,Y) \leq 2\varepsilon) = 1 - \mathrm{e}^{-\varepsilon}$. □

A classical result given in [11] states that a.s. $\lim_{\varepsilon\to 0} \varepsilon \cdot n^{-\varepsilon,0}(\mathcal{T}^{\mathrm{K}}) = 2$. We need also to control the expectation $\mathbb{E}[n^{-\varepsilon,0}(\mathcal{T}^{\mathrm{K}})]$.

Lemma 5.1.2. *If $(\mathcal{T}^{\mathrm{K}}, d, H, \nu^{\mathrm{K}})$ is a Kingman's coalescent, then for $\varepsilon > 0$, we have*

$$\mathbb{E}[n^{-\varepsilon,0}(\mathcal{T}^{\mathrm{K}})] \leq \frac{1}{1 - \exp\left(-\frac{\varepsilon}{2}\right)}.$$

Proof. For every $i \in \mathbb{N}^*$, set $\mathcal{T}^{\mathrm{K}}(i)$ the subtree of $\mathcal{T}^{\mathrm{K}}$ generated by i leaves picked independently at random with respect to ν^{K}. By definition of Kingman's coalescent, $\mathcal{T}^{\mathrm{K}}(i)$ is a i-coalescent. Let us set $N^\varepsilon(i) = n^{-\varepsilon,0}(\mathcal{T}^{\mathrm{K}}(i))$ the number of points in $\mathcal{T}^{\mathrm{K}}(i)$ at height $-\varepsilon$. Using Kolmogorov's equation and Jensen's inequality, we have the following inequality:

$$\frac{d}{d\varepsilon}\mathbb{E}[N^\varepsilon(i)] = -\mathbb{E}\left[\frac{N^\varepsilon(i)(N^\varepsilon(i)-1)}{2}\right] \leq -\frac{\mathbb{E}[N^\varepsilon(i)](\mathbb{E}[N^\varepsilon(i)]-1)}{2}.$$

For all $i \in \mathbb{N}^*$, $(\mathbb{E}[N^\varepsilon(i)])_{\varepsilon>0}$ is bounded from above by $(f_i(\varepsilon))_{\varepsilon>0}$ where f_i is the solution of the differential equation $f'(x) = -f(x)(f(x)-1)/2$ with initial condition $f(0) = i$. Solving this equation gives

$$f_i(\varepsilon) = \frac{1}{1 - \left(1 - \frac{1}{i}\right)\exp\left(-\frac{\varepsilon}{2}\right)} \leq \frac{1}{1 - \exp\left(-\frac{\varepsilon}{2}\right)}.$$

Since for all ε, $n^{-\varepsilon,0}(\mathcal{T}^{\mathrm{K}})$ is the limit of the non-decreasing sequence $(N^\varepsilon(i))_{i\in\mathbb{N}^*}$, we have

$$\mathbb{E}[n^{-\varepsilon,0}(\mathcal{T}^{\mathrm{K}})] = \lim_{i\to\infty} \mathbb{E}[N^\varepsilon(i)] \leq \frac{1}{1 - \exp\left(-\frac{\varepsilon}{2}\right)}.$$

□

5.2 The aim

The laws we intend to build in this section are an extension of the law of the Brownian tree conditioned on its local time given in [7]. In Aldous' construction, for $\ell(h)$ the local time of a normalized Brownian excursion, $\ell(h)dh$ is the density of leaves at height h, and $\frac{1}{\ell(h)}dh$ the rate of coalescence in the corresponding tree. To have a more general setting, we decorrelate the two and take two distinct measures. In the remaining of the chapter, we will call m the repartition of the mass at different heights, playing the role of $\ell(h)dh$, and μ the measure in charge of the coalescence, playing the role of $\frac{1}{\ell(h)}dh$.

To describe informally the construction, let us start with two remarks.

- If we apply a vertical deformation (see Definition 4.2.12) to a Kingman's coalescent, we obtain again a coalescent tree but with a different coalescence rate (which may depend on the level h).
- Take two independent versions $(\mathcal{T}^{\mathrm{K}}, d, H, \nu^{\mathrm{K}})$, $(\mathcal{T}^{\mathrm{K}'}, d', H', \nu'_K)$ of Kingman's coalescent, and $h \in \mathbb{R}_+$. If we shift downwards $\mathcal{T}^{\mathrm{K}}$ by replacing its height H with $\tilde{H} : x \mapsto H(x) - h$, $(\mathcal{T}^{\mathrm{K}'}, d', H', \nu'_K)$ can be mixed at height $-h$ onto $(\mathcal{T}^{\mathrm{K}}, d, \tilde{H}, \nu^{\mathrm{K}})$ according to ν_K. The corresponding distribution is the law of a random tree that can be described as follows:
  - At level 0, there are a countable number of leaves that perform a coalescent at rate 1.
  - At level h, a countable number of leaves is added to the $n^{-h,0}(\mathcal{T}^{\mathrm{K}'})$ remaining points of $\mathcal{T}^{\mathrm{K}}$, this new collection of particles performs again a coalescent at rate 1.

The strategy to construct the looked after tree is first to perform a downwards shift and a vertical deformation on Kingman's coalescents so that the trees start at different levels and the coalescent rate is now given by μ, and then perform recursively the mixing of the crown of the tree on another Kingman's coalescent along a countable dense sequence $(h_n)_{n\in\mathbb{N}^*}$ of levels.

First, let us precise the assumptions we must set on m and μ. Let I be a closed interval. Let m be a positive Radon measure on $\mathbb{R}$ which satisfies $m(\mathbb{R} \setminus I) = 0$, and μ a positive measure on $\mathbb{R}$, satisfying the following conditions.

C1. For all $a < b$ with $b \in I$ and $a \notin I$, we have $\mu([a,b)) = \infty$, and for all $a \in I$,

$$\int_{-\infty}^{a} dh\, \mathrm{e}^{-\mu([h,a))} < \infty.$$

C2. For all $a < b \in I$, $\mu((a,b)) > 0$, where by convention $a < b \in I$ means $a \in I, b \in I$ and $a < b$.

C3. For all $a \in \mathbb{R}$, $\mu(\{a\}) = \infty$ or there exists $b < a$ such that $\mu([b,a)) < \infty$.

We will see that Condition C2 ensures that the tree is locally compact. Notice that if $\inf I$ is finite, then $\mu(\{\inf I\}) = \infty$ thanks to C1 and C3. For $h \in I$, we define the function f_h on $(-\infty, 0]$ by

$$f_h(x) = \sup\{a \in I |\ \mu([a,h)) > -x\}.$$

We can see with Condition C1 that f_h takes its values in $I \cap (-\infty, h]$. The function is trivially non-decreasing and Condition C2 ensures that f_h is continuous.

Recall μ is a positive measure on $\mathbb{R}$ satisfying C1-3. Let $h \in I$. Informally, we define the probability distribution King_h^μ on $\mathbb{T}$ as the law of the vertical deformation of Kingman's coalescent $(\mathcal{T}^{\mathrm{K}}, d, H, \nu^{\mathrm{K}})$ by the function f_h, with, if $\inf f_h$ is finite, a semi-infinite branch $(-\infty, \inf f_h]$ added at the root of the vertical deformation of the Kingman's coalescent.

More formally, by definition, we have $\mathcal{T}^{\mathrm{K}} = \tau\big(0, (R_n)_{n\in\mathbb{N}^*}, (U_n)_{n\in\mathbb{N}^*}\big)$. Since f_h is continuous, we find by Lemma 4.2.17 that the vertical deformation of $\mathcal{T}^{\mathrm{K}}$ by f_h is given by $\mathcal{T}'$, the only non-$0_{h'}$ term of $\mathrm{Crown}_{h'}\Big(\tau\big(f_h(0), (f_h(0) - f_h(-R_n))_{n\in\mathbb{N}^*}, (U_n)_{n\in\mathbb{N}^*}\big)\Big)$ with $h' = \inf f_h$. Note that the operation of taking the only non-$0_{h'}$ term in an element of $\mathbb{X}_C^S$ is 1-Lipschitz from its domain (the closed set of all elements of $\mathbb{X}_C^S$ with at most 1 term $\neq 0_{h'}$) to $\mathbb{S}^S$. Since τ is measurable, the vertical deformation $\mathcal{T}'$ is a $\mathbb{T}$-valued random variable. If $h' = -\infty$, we set $\mathcal{T} = \mathcal{T}'$. Otherwise, we define $\mathcal{T}$ as the mixing of $\mathcal{T}'$ onto the tree $T = ((-\infty, h'], d_{\mathbb{R}}, \mathrm{Id}, 0, p = \delta_{h'})$ (an half-line tree with a Dirac mass at its top). Notice the distribution $P_{T\star_p\mathcal{T}'}$ is conditionally on $\mathcal{T}'$ a Dirac mass, thus $\mathcal{T}$ is well defined as a measurable deterministic function of $\mathcal{T}^{\mathrm{K}}$. The distribution of the $\mathbb{T}$-valued random variable $\mathcal{T}$ is denoted King_h^μ. A random tree with law King_h^μ can be interpreted as the coalescent tree with coalescent rate μ and an infinite number of leaves at height h.

Our aim is to build, for I an interval of $\mathbb{R}$ and μ a measure over I satisfying Conditions C1-3 and for every Borel measure m of $\mathbb{R}$ with support in I, a random S-compact tree $(\mathcal{T}_I, d, H, \nu_I)$ and a family $(\nu_h)_{h\in I}$ of measures on $\mathcal{T}_I$ satisfying the following conditions.

- The map $h \mapsto \nu_h$ is measurable on I. (5.2.1)
- $\forall h \in I, \mathbb{P}(H\nu_h = \delta_h) = 1.$ (5.2.2)
- $\nu_I = \int_I \nu_h\, m(dh).$ (5.2.3)
- $\forall h \in I, (\mathrm{Stump}_h(\mathcal{T}_I), (\nu_{h'})_{h'\leq h})$ and $(\mathrm{Crown}_h(\mathcal{T}_I), (\nu_{h'})_{h'>h})$ are independent. (5.2.4)
- For every $h \in I$, $\mathcal{T}_I$ is exchangeable with respect to ν_h. (5.2.5)
- $\forall h \in I,$ the random set $\{x \in \mathcal{T}_I |\ \exists y \in T, H(y) = h, x \preceq y\}$ follows the law of King_h^μ when equipped with d, H and ν_h. (5.2.6)

The next two sections are devoted to the proofs or conjectures related to this aim. In Section 5.3, we build the metric tree $(\mathcal{T}_I, d, H)$, see Lemma 4.2.3, so that it is S-compact. Then in Section 5.4 we give ideas for the proof of the existence and properties of the family $(\nu_h)_{h\in I}$.

5.3 Construction of the tree $\mathcal{T}_I$

Let $E = \{h_1 < \cdots < h_n\} \subset I$. Let $(\mathcal{T}_{h_i}, d_i, H_i, \nu_{h_i})_{1\leq i\leq n}$ be n independent random weighted trees of respective distributions $\mathrm{King}^{\mu}_{h_i}$ for $1 \leq i \leq n$. Recall the definition of the probability measure $P_{T_1\star_{p_1}\cdots\star_{p_{n-1}}T_n}$, see Equation (4.4.18) in Remark 4.4.6 and the measurability of the mixing operation in Proposition 4.4.7. Let $\tilde{\mathcal{T}}_E$ be a random tree distributed according to $P_{\mathcal{T}_{h_1}\star_{\nu_{h_1}}\cdots\star_{\nu_{h_{n-1}}}\mathcal{T}_{h_n}}$. To keep track of the measures ν_{h_i} in $\tilde{\mathcal{T}}_E$, we can either modify the proof of Theorem 4.4.5 and consider the restriction of those measures to the crowns which are grafted and denote by $\tilde{\nu}_{h_i}$ the corresponding probability measure on $\tilde{\mathcal{T}}_E$, or use the intrinsic definition of the uniform probability measure ν^{K} on the leaves for the Kingman's coalescent at level 0 as the limit of the uniform probability measures on the (finite) ancestors living at time $-\varepsilon$ when ε goes down to zero and transpose this intrinsic construction through the vertical deformation and the downward shift.

Lemma 5.3.1. *For every $1 \leq i \leq n$, we set*

$$\tilde{\mathcal{T}}_{h_i} = \{x \in \tilde{\mathcal{T}}_E |\ \exists y \text{ s.t. } x \preceq y \text{ and } H(y) = h_i\}. \tag{5.3.1}$$

that we endow with the distance, and height induced by those on $\tilde{\mathcal{T}}_E$, and measure $\tilde{\nu}_{h_i}$. Then, for all $1 \leq i \leq n$, $\tilde{\mathcal{T}}_{h_i}$ has distribution $\mathrm{King}^{\mu}_{h_i}$.

Proof. By definition, $\tilde{\mathcal{T}}_{h_1} \overset{(d)}{=} \mathcal{T}_{h_1}$ which is distributed according to $\mathrm{King}^{\mu}_{h_1}$.

Again, by definition, $\mathrm{Stump}_{h_2}(\tilde{\mathcal{T}}_E)$ has distribution $P_{\mathcal{T}_{h_1}*_{\nu_{h_1}}\mathcal{T}_{h_2}}$ and $\tilde{\mathcal{T}}_{h_2}$ is the tree generated by the leaves of $\mathrm{Stump}_{h_2}(\tilde{\mathcal{T}}_E)$ at level h_2. Let $(\mathcal{T}^{\mathrm{K}}, d, H, \nu^{\mathrm{K}})$ and $(\mathcal{T}^{\mathrm{K}\prime}, d', H', \nu^{\mathrm{K}\prime})$ be two independent Kingman's coalescents and let $h > 0$. If we set for every $x \in \mathcal{T}^{\mathrm{K}\prime}$, $H''(x) = H'(x) - h$ and $(\mathcal{T}^{\mathrm{K}\prime\prime}, d'', H'', \nu^{\mathrm{K}\prime\prime}) = (\mathcal{T}^{\mathrm{K}\prime}, d', H'', \nu^{\mathrm{K}\prime})$ (*i.e.* we shift downwards the tree $\mathcal{T}^{\mathrm{K}\prime}$ by height h), then we can define a tree $\tilde{\mathcal{T}}$ which has distribution $P_{\mathcal{T}^{\mathrm{K}\prime\prime}*_{\nu^{\mathrm{K}\prime\prime}}\mathcal{T}^{\mathrm{K}}}$. Conditionally given $n^{-h,0}(\mathcal{T}^{\mathrm{K}})$, $\mathrm{Stump}_{-h}(\mathcal{T}^{\mathrm{K}})$ is a $n^{-h,0}(\mathcal{T}^{\mathrm{K}})$-coalescent independent of $\mathcal{T}^{\mathrm{K}\prime\prime}$, hence the tree

$$\{x \in \tilde{\mathcal{T}} |\ \exists y \in \tilde{\mathcal{T}} \text{ s.t. } x \preceq y \text{ and } H(y) = 0\}$$

is again a Kingman's coalescent. Applying this property with the vertical deformations implies that, if $\mu([h_1, h_2)) < \infty$, then $\tilde{\mathcal{T}}_{h_2}$ has distribution $\mathrm{King}^{\mu}_{h_2}$. If $\mu([h_1, h_2)) = \infty$, then below the level h_2, $\mathcal{T}_{h_1}$ is just a simple line and the result is obvious.

An easy induction then gives the result for every $i \leq n$. □

For all finite set E, this construction provides a random tree $\tilde{\mathcal{T}}_E$ and a family of measures $(\tilde{\nu}_h)_{h\in E}$ such that the family $(\tilde{\mathcal{T}}_h, d_h, H, \tilde{\nu}_h)_{h\in E}$ defined by (5.3.1) satisfy the following conditions.

C4 For all $h \in E$, $\tilde{\mathcal{T}}_h$ has distribution King^{μ}_h.

C5 For all $h_0 \in E$, the family $(\mathrm{Crown}_{h_0}(\tilde{\mathcal{T}}_h), \tilde{\nu}_h)_{h\in E, h>h_0}$ and the family $(\tilde{\mathcal{T}}_h, \tilde{\nu}_h)_{h\in E, h\leq h_0}$ are independent.

The tree $(\tilde{\mathcal{T}}_E, d, H, (\tilde{\nu}_h)_{h\in E})$ can be recovered from the family $(\tilde{\mathcal{T}}_h)_{h\in E}$ by setting

$$– \ \tilde{\mathcal{T}}_E = \bigcup_{h\in E} \tilde{\mathcal{T}}_h. \tag{5.3.2}$$

– $\forall x, y \in \tilde{\mathcal{T}}_E, x \preceq y$ if and only if : $\exists h \in E, x \in \tilde{\mathcal{T}}_h, y \in \tilde{\mathcal{T}}_h, x \preceq_h y$.

– $\forall x, y \in \tilde{\mathcal{T}}_E, d(x, y) = H(x) + H(y) - 2H(x \wedge y)$.

In $(\tilde{\mathcal{T}}_h)_{h\in E}$, H is the same function for all the elements, which means that $H(x)$ does not depend on the choice of any particular tree containing x.

Remark 5.3.2. As a consequence of C5 and (5.3.2), for all $h \in E$, the crown of $\tilde{\mathcal{T}}_E$ above h is independent from its stump below h.

Remark 5.3.3. By construction, the tree $\tilde{\mathcal{T}}_E$ is exchangeable (see Definition 4.4.8) at all the levels $h \in E$ with respect to the measure $\tilde{\nu}_h$, see Remark 4.4.9.

Remark 5.3.4. Take $E \subset E' \subset E'' \subset I$ three finite sets, and consider $\tilde{\mathcal{T}}_E$, $\tilde{\mathcal{T}}_{E'}$ and $\tilde{\mathcal{T}}_{E''}$. For $h \in E''$, define

$$\tilde{\mathcal{T}}_h^{(E'')} = \{x \in \tilde{\mathcal{T}}_{E''} |\ \exists y \text{ s.t. } x \preceq y \text{ and } H(y) = h\}.$$

Similarly, define $(\tilde{\mathcal{T}}_h^{(E')})_{h\in E'}$. We set

$$\tilde{\mathcal{T}}_{E'}^{(E'')} = \bigcup_{h\in E'} \tilde{\mathcal{T}}_h^{(E'')} \quad ; \quad \tilde{\mathcal{T}}_{E}^{(E'')} = \bigcup_{h\in E} \tilde{\mathcal{T}}_h^{(E'')} \quad ; \quad \tilde{\mathcal{T}}_{E}^{(E')} = \bigcup_{h\in E} \tilde{\mathcal{T}}_h^{(E')}.$$

We have $\tilde{\mathcal{T}}_E^{(E')} \stackrel{d}{=} \tilde{\mathcal{T}}_E$ and $(\tilde{\mathcal{T}}_E^{(E')}, \tilde{\mathcal{T}}_{E'}) \stackrel{d}{=} (\tilde{\mathcal{T}}_E^{(E'')}, \tilde{\mathcal{T}}_{E'}^{(E'')})$. Note that $(\tilde{\mathcal{T}}_E^{(E')}, \tilde{\mathcal{T}}_{E'})$ provides a coupling of $\tilde{\mathcal{T}}_E$ and $\tilde{\mathcal{T}}_{E'}$ in which $\tilde{\mathcal{T}}_E^{(E')} \subset \tilde{\mathcal{T}}_{E'}$.

Lemma 5.3.5. *If a random tree $(\mathcal{T}, d, H, \nu)$ has distribution* King_h^μ *for some $h \in I$ and some measure μ satisfying C1-3, then, almost surely, the identity is the only height- and measure-preserving isometry from $\mathcal{T}$ to $\mathcal{T}$.*

Proof. From the definition of King_h^μ, the support of ν is a.s. equal to the set F of all the leaves of $\mathcal{T}$: $F = \{x \in \mathcal{T} | H(x) = h\}$. For $n \in \mathbb{N}^*$, set R_n the relation on F such that for all $x, y \in F$, xR_ny if and only if $h - H(x \wedge y) \leq \frac{1}{n}$. Note that R_n is an equivalence relation. Set $F_n = \{y \in \mathcal{T} | H(y) = h - \frac{1}{n}\}$, F_n is a.s. finite and the equivalence classes of R_n are $(\{x \in F | x \geq y\})_{y\in F_n}$. The repartition of the masses between the different classes has the same law as the masses of the sub-trees above level $-\mu([-\frac{1}{n}, 0))$ in Kingman's coalescent, so we have a.s. that for any two classes C, C' of R_n, $\nu(C) \neq \nu(C')$ and $\nu(C) > 0$ as well as $\nu(C') > 0$.

Now, we work conditionally on $\mathcal{T}$, assuming that for every $n \in \mathbb{N}^*$, F_n is finite and for every C, C', distinct classes of R_n, we have $\nu(C) \neq \nu(C')$ and $\nu(C) > 0$ as well as $\nu(C') > 0$. Set ϕ an height- and measure-preserving isometry from $\mathcal{T}$ to $\mathcal{T}$ such that $\phi\nu = \nu$ and $H \circ \phi = H$. Since $d(x, y) = H(x)+H(y)-2H(x\wedge y)$, we actually have that xR_ny if and only if $d(x, y) \leq \frac{2}{n}$, so ϕ preserves R_n. We have the following equivalences:

- $\forall x \in \mathcal{T}$, $(x \in F \Leftrightarrow \phi(x) \in F)$ since ϕ preserves H,

- $\forall x, y \in F$, $(xR_ny \Leftrightarrow \phi(x)R_n\phi(y))$ since ϕ preserves R_n,
- $\forall n \in \mathbb{N}^*, C \subset F$, C is a class of R_n if and only if $\phi(C)$ is a class of R_n.

For $n \in \mathbb{N}^*$, take C a class of R_n. As $\phi(C)$ is a class of R_n and $\nu(\phi(C)) = \nu(C)$, we have under our assumptions that $\phi(C) = C$. This means that for all $x \in F, n \in \mathbb{N}^*$ we have $xR_n\phi(x)$, i.e. $d(x, \phi(x)) \leq \frac{2}{n}$. It follows that for every $x \in F$, $\phi(x) = x$.

Now, take $y \in \mathcal{T}$, we can choose a leaf x such that $y \preceq x$. We have

$$\begin{aligned} d(\phi(x), \phi(y)) &= H(\phi(x)) + H(\phi(y)) - 2H(\phi(x) \wedge \phi(y)) \\ &= H(x) + H(y) - 2H(x \wedge \phi(y)) \end{aligned}$$

as ϕ preserves H and $\phi(x) = x$. As ϕ is an isometry, we also have

$$d(\phi(x), \phi(y)) = d(x, y) = H(x) - H(y)$$

and therefore, $H(y) = H(x \wedge \phi(y))$, which implies that $y \preceq \phi(y)$ by uniqueness of the ancestor of x at some fixed level. As $H(y) = H(\phi(y))$, we eventually get $y = \phi(y)$. □

For $E \subset I$ a finite set and $r \in \mathbb{R}_+$, we define

$$\delta_r(E) = \max\{\delta \in \mathbb{R}_+ | \exists x \in [-r \vee \inf I, (r \wedge \sup I) - \delta], E \cap (x, x + \delta) = \emptyset\}.$$

In the case where $I \cap [-r, r] = \emptyset$, we note $\delta_r(E) = 0$. Note that $\delta_r(E)$ is always defined, as the set in the right-hand is actually a closed interval containing 0 and bounded from above by $2r$. The quantity $\delta_r(E)$ measures the biggest gap without elements of E in $I \cap [-r, r]$. Note that for $(h_n)_{n \in \mathbb{N}^*}$ a sequence of elements of $I \cap [-r, r]$ and $E_n = \{h_i\}_{1 \leq i \leq n}$, the sequence $(\delta_r(E_n))_{n \in \mathbb{N}^*}$ converges to 0 if and only if $(h_n)_{n \in \mathbb{N}^*}$ is dense in $I \cap [-r, r]$.

Lemma 5.3.6. *For $E \subset E' \subset I$ two finite sets and $r \in \mathbb{R}_+ \cap \big(-E \cup [-\inf I, +\infty)\big)$. Recall $\tilde{\mathcal{T}}_{E'}$ and $\tilde{\mathcal{T}}_E^{(E')} \subset \tilde{\mathcal{T}}_{E'}$ from Remark 5.3.4. We a.s. have*

$$d_{\mathrm{GH}}(\mathrm{Slice}_r(\tilde{\mathcal{T}}_E^{(E')}), \mathrm{Slice}_r(\tilde{\mathcal{T}}_{E'})) \leq \delta_r(E).$$

Proof. We have $\tilde{\mathcal{T}}_E^{(E')} \subset \tilde{\mathcal{T}}_{E'}$, so the case $\mathrm{Slice}_r(\tilde{\mathcal{T}}_{E'}) = \emptyset$ is trivial as $\mathrm{Slice}_r(\tilde{\mathcal{T}}_E^{(E')}) = \emptyset$ as well, so the distance $d_{\mathrm{GH}}(\mathrm{Slice}_r(\tilde{\mathcal{T}}_E^{(E')}), \mathrm{Slice}_r(\tilde{\mathcal{T}}_{E'}))$ is 0. In the rest of the proof, consider $x \in \mathrm{Slice}_r(\tilde{\mathcal{T}}_{E'})$. If there is at least one element of E in $[-r, H(x)]$, then take h the biggest possible. We have $H(x) \in [-r, r] \cap I$, so $0 \leq H(x) - h \leq \delta_r(E)$ by definition of the latter. Take y the ancestor of x at height h, we have $y \in \tilde{\mathcal{T}}_E^{(E')}$ and $d(x, y) = H(x) - h \leq \delta_r(E)$.

If there are no elements of E in $[-r, H(x)]$, then $-r \notin E$, so $-r \in (-\infty, \inf I]$. By definition of $\delta_r(E)$, we have $H(x) - \inf I \leq \delta_r(E)$. By definition of $\tilde{\mathcal{T}}_E$ and $\tilde{\mathcal{T}}_{E'}$, they share the only point y at height $\inf I$, which is the common ancestor of all the tree. By definition of $\delta_r(E)$ and since there are no elements of E below $H(x)$, we have $d(x, y) = H(x) - \inf I \leq \delta_r(E)$.

Since $\tilde{\mathcal{T}}_E^{(E')} \subset \tilde{\mathcal{T}}_{E'}$, we have proven that $d_{\mathrm{GH}}(\mathrm{Slice}_r(\tilde{\mathcal{T}}_E^{(E')}), \mathrm{Slice}_r(\tilde{\mathcal{T}}_{E'})) \leq \delta_r(E)$. □

Let $E \subset I$ be a countable dense set in I and μ a measure satisfying conditions C1-3. Our objective is to build a $\mathbb{T}$-valued random variable $\tilde{\mathcal{T}}_E$ for which we conjecture that the sub-trees $\tilde{\mathcal{T}}_h$ satisfies C4-5 for all $h \in E$, where

$$\tilde{\mathcal{T}}_h = \{x \in \tilde{\mathcal{T}}_E | \ \exists y \text{ s.t. } x \preceq y \text{ and } H(y) = h\}. \tag{5.3.3}$$

Recall $\mathbb{X}^{0,S}$ from Definition 3.1.2 as the set of all elements of $\mathbb{X}^S$ with null measure.

Lemma 5.3.7. *For any dense sequence $(h_n)_{n\in\mathbb{N}^*}$ in I, the law of $\tilde{\mathcal{T}}_{\{h_1,\dots,h_n\}}$ equipped with the null measure converges to the law of a random tree $\tilde{\mathcal{T}}_I$ for the Prokhorov distance over $\mathbb{X}^{0,S}$. Moreover, the limit is independent from the choice of the dense sequence.*

Remark 5.3.8. Legitimated by this lemma, we shall denote by $\mathcal{T}_I$ any $\mathbb{T}$-valued random variable distributed as the limit where E is any countable dense subset of I.

Proof. We start with a special case. Let $E \subset E' \subset I$ be two countable dense sets in I. Let $(h_i)_{i\in\mathbb{N}^*}$ be an enumeration of the elements of E and $(h'_i)_{i\in\mathbb{N}^*}$ an enumeration of the elements of E'. Set $E_n = \{h_i\}_{0<i\leq n}$, $E'_n = \{h'_i\}_{0<i\leq n}$ and for all $n \in \mathbb{N}^*$, $\phi(n) = \min\{k \in \mathbb{N}^* | E_n \subset E'_k\}$. We want to build a random sequence alternating between $\tilde{\mathcal{T}}_{E_n}$ and $\tilde{\mathcal{T}}_{E'_n}$. Proving the convergence of this hybrid sequence will prove that its two subsequences converge and have the same limit. The proof of the convergence consists in the construction and the study of a particular coupling of those laws. For every $m \in \mathbb{N}^*$, consider the tree $\tilde{\mathcal{T}}_{E'_{\phi(m)}}$, and for $0 < k \leq m$, define

$$\tilde{\mathcal{T}}^{(m)}_{2k-1} = \tilde{\mathcal{T}}^{(E'_{\phi(m)})}_{E_k} \quad \text{and} \quad \tilde{\mathcal{T}}^{(m)}_{2k} = \tilde{\mathcal{T}}^{(E'_{\phi(m)})}_{E'_k}$$

as in Remark 5.3.4. We obtain a family $(\tilde{\mathcal{T}}^{(m)}_1, \cdots, \tilde{\mathcal{T}}^{(m)}_{2m})$, such that the subsequences of the odd-numbered terms and even-numbered terms are non-decreasing for the inclusion. It is clear that the distribution of that family is consistent from m to $m+1$ *i.e.*

$$(\tilde{\mathcal{T}}^{(m)}_n, 1 \leq n \leq 2m) \overset{d}{=} (\tilde{\mathcal{T}}^{(m+1)}_n, 1 \leq n \leq 2m).$$

Since $\mathbb{X}^S$ is Polish, we can use Kolmogorov extension theorem, so there exists a standard probability space $(\Omega, \mathbb{P})$ and a sequence $(\tilde{\mathcal{T}}^n)_{n\in\mathbb{N}^*}$ of random variables $\omega \mapsto \tilde{\mathcal{T}}^n_\omega \in \mathbb{T}$ such that for every $m \in \mathbb{N}^*$, $(\tilde{\mathcal{T}}^n)_{1\leq n\leq 2m} \overset{d}{=} (\tilde{\mathcal{T}}^{(m)}_n)_{1\leq n\leq 2m}$.

Take $n \in \mathbb{N}^*$, $r \in -E_n \cup (-\inf I, +\infty)$. For all $k \geq n$, we have $E_n \subset E_k$, so by Lemma 5.3.6 and Remark 5.3.4 we have

$$\begin{aligned} d_{\text{GHP}}(\text{Slice}_r(\tilde{\mathcal{T}}^{2n-1}), \text{Slice}_r(\tilde{\mathcal{T}}^{2k-1})) &\overset{d}{=} d_{\text{GHP}}(\text{Slice}_r(\tilde{\mathcal{T}}^{(E'_{\phi(k)})}_{E_n}), \text{Slice}_r(\tilde{\mathcal{T}}^{(E'_{\phi(k)})}_{E_k})) \\ &\overset{d}{=} d_{\text{GHP}}(\text{Slice}_r(\tilde{\mathcal{T}}^{(E_k)}_{E_n}), \text{Slice}_r(\tilde{\mathcal{T}}_{E_k})) \\ &\leq \delta_r(E_n) \text{ a.s.} \end{aligned}$$

Similarly for $k \geq \phi(n)$ we have $E_n \subset E'_k$, so a.s.

$$d_{\text{GHP}}(\text{Slice}_r(\tilde{\mathcal{T}}^{2n-1}), \text{Slice}_r(\tilde{\mathcal{T}}^{2k})) \leq \delta_r(E_n).$$

We have $\phi(n) = \#(E'_{\phi(n)}) \geq \#(E_n) = n$, so for every $i \geq 2\phi(n)$,

$$d_{\text{GHP}}(\text{Slice}_r(\tilde{\mathcal{T}}^{2n-1}), \text{Slice}_r(\tilde{\mathcal{T}}^{i})) \leq \delta_r(E_n).$$

This implies that a.s. for every $i, j \geq 2\phi(n)$,

$$d_{\text{GHP}}(\text{Slice}_r(\tilde{\mathcal{T}}^{i}), \text{Slice}_r(\tilde{\mathcal{T}}^{j})) \leq 2\delta_r(E_n).$$

Since E is dense, we have

$$\lim_{n\to\infty} \delta_r(E_n) = 0$$

for all $r \in -E \cup [-\inf I, +\infty)$. Note that if I has a finite lower bound, $[-\inf I, +\infty) \neq \emptyset$ doesn't have an upper bound. If I has no lower bound, neither does E since it is dense in I, so $\sup(-E) = \infty$. In both cases, we can take $(r_k)_{k\in\mathbb{N}^*}$ an increasing sequence of elements of $-E\cup [-\inf I, +\infty)$. We can deduce that a.s. for every $k \in \mathbb{N}^*$, the sequence $(\mathrm{Slice}_{r_k}(\tilde{\mathcal{T}}^n))_{n\in\mathbb{N}^*}$ is Cauchy in $(\mathbb{X}^{0,K}, d_{\mathrm{GHP}})$. When this is true, then by Lemma 3.3.9, $(\mathrm{Slice}_{r_k}(\tilde{\mathcal{T}}^n))_{n\in\mathbb{N}^*}$ converges in $(\mathbb{X}^S, d_{\mathrm{LGHP}})$ to a random measured labelled space $\tilde{\mathcal{T}}_I$. Since $\mathbb{T}$ and $\mathbb{X}^{0,S}$ are closed in $\mathbb{X}^S$, $\tilde{\mathcal{T}}_I$ is a random tree with null measure. The a.s. convergence of $(\tilde{\mathcal{T}}^n)_{n\in\mathbb{N}^*}$ to $\tilde{\mathcal{T}}$ in this coupling implies the convergence of their laws for the Prokhorov distance.

The sequence $(\tilde{\mathcal{T}}^n)_{n\in\mathbb{N}^*}$ converges in law to $\tilde{\mathcal{T}}$, so its subsequences converge as well. We have proven the lemma in the special case $E \subset E'$. Note that this covers the case $E = E'$ with two different enumerations. For enumerations of E and E' dense countable subsets in the general case, we can use the special case by going through $E'' = E \cup E'$. This concludes the lemma. □

At this stage, we have the following conjecture.

Conjecture 5.3.9. Let μ be a measure satisfying conditions C1-3 and let E be a countable subset of I. The trees defined by (5.3.3) for all $h \in E$ satisfy C4-5.

The idea of the proof of this conjecture is to consider a non-decreasing (for the inclusion) sequence of representatives of $(\tilde{\mathcal{T}}^n)_{n\in\mathbb{N}^*}$, take the completion of its limit. It will be distributed as $\tilde{\mathcal{T}}_I$. Since the trees defined by (5.3.3) (with $\tilde{\mathcal{T}}^{(n)}$ instead of $\tilde{\mathcal{T}}_I$ for some large n) satisfy C4-5, it is reasonable to conjecture that the trees defined by (5.3.3) for all $h \in E$ satisfy C4-5. However, one has to check that the completion of the limit a.s. does not change the definition of $\tilde{\mathcal{T}}_h$ when one replaces $\tilde{\mathcal{T}}^{(n)}$ by $\tilde{\mathcal{T}}_E$ in (5.3.3).

5.4 Construction of the measures $(\nu_h)_{h\in I}$

Let E be a dense subset of I and $(h_i)_{i\in\mathbb{N}^*}$ an enumeration of E. The tree $\tilde{\mathcal{T}}^{(n)}$ introduced in the previous Section is naturally endowed with a family $(\tilde{\nu}_{h_i}, 1 \leq i \leq n)$. If we fix $k \in \mathbb{N}^*$, and consider the tree $(\tilde{\mathcal{T}}^{(n)}, (\tilde{\nu}_{h_i}, 1 \leq i \leq k))$ as an element of $\mathbb{T}^{[k]}$ which is still a Polish space, similar arguments as in the previous Section gives a limiting tree endowed with k measures $(\mathcal{T}_I, (\nu_{h_i}, 1 \leq i \leq k))$. We could make rigorous the construction of $(\mathcal{T}_I, (\nu_h)_{h\in E})$ using the last part of the Remark 4.4.1 and considering it as a $\mathbb{T}^{[\infty]}$-valued random variable. We shall not provide of proof of this fact, but simply conjecture its existence. In particular, this implies that ν_h is a probability measure and that $H\nu_h = \delta_h$ for all $h \in E$. Since $\tilde{\mathcal{T}}^{(n)}$ is exchangeable at every level $h_i, 1 \leq i \leq k$ with respect to ν_{h_i} respectively by Remark 5.3.3 for all $n \geq k$, we also conjecture the same holds for $\mathcal{T}_I$.

Conjecture 5.4.1. Let μ be a measure satisfying conditions C1-3 and let $E = \{h_i | i \in \mathbb{N}^*\}$ be a countable dense subset of I. The $\mathbb{T}^{[\infty]}$-valued random variable $(\mathcal{T}_I, (\nu_h)_{h\in E})$ is well defined as the limit in distribution of $(\tilde{\mathcal{T}}^{(n)}, (\tilde{\nu}_{h_1}, \ldots, \tilde{\nu}_{h_n}, 0, 0, \ldots)$ as n goes to infinity. Furthermore, $(\mathcal{T}_I, (\nu_h)_{h\in E})$ is exchangeable at level h with respect to ν_h, for all $h \in E$.

The next step would then be to extend this family by constructing additional measures $(\nu_h)_{h\in I}$ on $\mathcal{T}_I$ and proving that the law of $(\mathcal{T}_I, (\nu_h)_{h\in I})$ does not depend on the choice of E. We did not perform this program. However, we state, as a first step toward this goal, in the next lemmas some regularity property for the measures ν_h.

Recall Definition 3.1.3 of the Prohorov distance $d_{\mathrm{P}}(\nu,\nu')$ between two probability measures ν,ν' over a metric space (F,d). For any $\varepsilon>0$, we set $n_\varepsilon(F)$ the minimal cardinality of a partition of F using only Borel sets of diameters smaller than ε:

$$n_\varepsilon(F)=\min\left\{k\in\mathbb{N}^*\,\middle|\,\begin{array}{l}\exists(B_1,...,B_k)\in(\mathscr{B}(F))^k,\\ \forall i,\ \mathrm{diam}\ (B_i)\leq\varepsilon,\\ \forall x\in F,\exists!1\leq i\leq\ k,x\in B_i\end{array}\right\}.$$

Lemma 5.4.2. *Let (F,ν) be a compact metric probability space, $(X_n)_{n\in\mathbb{N}^*}$ an i.i.d. sequence of F-valued random variables with distribution ν and $(c_n)_{n\in\mathbb{N}^*}$ a sequence of non-negative real numbers such that $\sum_{n=1}^\infty c_n=1$, then*

$$\mathbb{P}\left(d_{\mathrm{P}}\Big(\sum_n c_n\delta_{X_n},\nu\Big)>\varepsilon\right)\leq\frac{n_\varepsilon(F)}{4\varepsilon^2}\sum_n c_n^2,$$

.

Proof. Let $B_1,...,B_{n_\varepsilon(F)}$ be $n_\varepsilon(F)$ Borel sets of diameter at most ε forming a partition of F and $\mathscr{B}=\sigma(B_1,...,B_{n_\varepsilon(F)})$. For convenience, note $\nu_n=\sum_n c_n\delta_{X_n}$. We note, for any Borel set $A\subset F$,

$$B(A)=\bigcup_{\substack{1\leq i\leq n_\varepsilon(F)\\ B_i\cap A\neq\emptyset}}B_i.$$

For all A, we have that $B(A)\in\mathscr{B}$. We have the following inclusions $A\subset B(A)\subset A^\varepsilon$. This immediately yields

$$\sup_A(\nu_n(A)-\nu(A^\varepsilon))\leq\sup_A(\nu_n(B(A))-\nu(B(A)))=\sup_{B\in\mathscr{B}}\left[\nu_n-\nu\right](B).$$

Consider the probability measures ν and ν_n restricted to $\mathscr{B}$. We have

$$\sup_{B\in\mathscr{B}}\left[\nu_n-\nu\right](B)=\frac{1}{2}\sum_{1\leq i\leq n_\varepsilon(F)}\left|\left[\nu_n-\nu\right](B_i)\right|.$$

Now, we recall an application of the Cauchy-Schwartz inequality to the comparison of the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ in dimension $n_\varepsilon(F)$:

$$\forall(a_1,...,a_{n_\varepsilon(F)})\in\mathbb{R}^{n_\varepsilon(F)},\sum_{1\leq i\leq n_\varepsilon(F)}|a_i|\leq\left(n_\varepsilon(F)\sum_{1\leq i\leq n_\varepsilon(F)}|a_i|^2\right)^{\frac{1}{2}}.$$

Using this, we have

$$\sup_A(\nu_n(A)-\nu(A^\varepsilon))\leq\frac{1}{2}\sum_{1\leq i\leq n_\varepsilon(F)}\left|\left[\nu_n-\nu\right](B_i)\right|\leq\frac{1}{2}\left(n_\varepsilon(F)\sum_{1\leq i\leq n_\varepsilon(F)}(\left[\nu_n-\nu\right](B_i))^2\right)^{\frac{1}{2}}.$$

Using Lemma 3.1.5 (recall that ν and $\nu_n = \sum_n c_n \delta_{X_n}$ are probability measures) and the previous inequalities, we get

$$\begin{aligned}
\mathbb{P}\left(d_{\mathrm{P}}\left(\nu_n, \nu\right) > \varepsilon\right) &= \mathbb{P}\left(\sup_A \left(\nu_n(A) - \nu(A^\varepsilon)\right) > \varepsilon\right) \\
&\leq \mathbb{P}\left(\sum_{1\leq i\leq n_\varepsilon(F)} \left(\left[\nu_n - \nu\right](B_i)\right)^2 > \frac{4\varepsilon^2}{n_\varepsilon(F)}\right) \\
&\leq \frac{n_\varepsilon(F)}{4\varepsilon^2} \mathbb{E}\left[\sum_{1\leq i\leq n_\varepsilon(F)} \left(\left[\nu_n - \nu\right](B_i)\right)^2\right] \\
&= \frac{n_\varepsilon(F)}{4\varepsilon^2} \sum_{1\leq i\leq n_\varepsilon(F)} \mathrm{Var}\left(\nu_n(B_i)\right) \\
&= \frac{n_\varepsilon(F)}{4\varepsilon^2} \sum_{1\leq i\leq n_\varepsilon(F)} \nu(B_i)(1-\nu(B_i)) \sum_n c_n^2 \\
&\leq \frac{n_\varepsilon(F)}{4\varepsilon^2} \sum_n c_n^2,
\end{aligned}$$

where we used the Markov inequality for the second inequality, that $\mathbb{E}\left(\left[\nu_n - \nu\right](B_i)\right) = 0$ for the second equality and that $\sum_{i=1}^{n_\varepsilon(F)} \nu(B_i) = 1$ for the last inequality. □

Recall notations from the beginning of Section 4.3.2. For $E \subset I$ finite or countable, $h, h' \in E$ with $h < h'$, recall $(C_i^h(\mathcal{T}_I))_{i\in\mathcal{I}_h(\mathcal{T}_I)}$ from (4.3.9) and (4.3.10), and $n^{h,h'}(\mathcal{T}_I)$ from Definition 4.2.1. We set

$$\{C_i^{h,h'}(\mathcal{T}_I),\ 1 \leq i \leq n^{h,h'}(\mathcal{T}_I)\} = \{C_j^h(\mathcal{T}_I),\ j \in \mathcal{I}_h(\mathcal{T}_I),\ H(C_j^h(\mathcal{T}_I)) > h'\}.$$

Let $\mathcal{T}^{\mathrm{K}}$ be a random tree distributed as Kingman's coalescent. The random number $n^{h,h'}(\mathcal{T}_I)$ has the same law as $n^{-\mu([h,h')),0}(\mathcal{T}^{\mathrm{K}})$ by definition of $\mathcal{T}_I$, Conjecture 5.3.9 and Lemma 5.3.1 as well as by definition of the probability distribution King_h^μ and condition C2. We now give some regularity on the measure $(\nu_h, h \in E)$. Notice the next Lemma is in fact stated for the random tree $\tilde{\mathcal{T}}^{(n)}$ for n large enough so that h and h' belongs to $\{h_i | 1 \leq i \leq n\}$, and it holds for $\mathcal{T}_I$ if Conjecture 5.3.9 holds.

Lemma 5.4.3. *For $h, h' \in E$ such that $h < h'$, we have:*

$$\mathbb{P}(d_{\mathrm{P}}(\nu_h, \nu_{h'}) > \varepsilon + h' - h) \leq \frac{\mu([h,h'))}{4\varepsilon^2\left(1 - \exp\left(-\frac{1}{2}\mu([h - \frac{\varepsilon}{2}, h))\right)\right)}.$$

Proof. Let $h < h'$. From Lemma 5.3.1, we can assume that $E = \{h, h'\}$. Consider $(\mathcal{T}_h, \mathcal{T}_{h'})$ two independent random trees with respective distributions King_h^μ and $\mathrm{King}_{h'}^\mu$, and $\tilde{\mathcal{T}}_E$ the corresponding mixed tree. For every $x \in \tilde{\mathcal{T}}_E$ such that $H(x) \geq h$, we denote by $\rho_h(x)$ the unique ancestor of x at height h. We have $d(x, \rho_h(x)) = H(x) - h$ which implies

$$d_{\mathrm{P}}(\nu_{h'}, \rho_h \nu_{h'}) \leq h' - h.$$

Using the triangular inequality for d_{P}, we obtain

$$\mathbb{P}(d_{\mathrm{P}}(\nu_{h'},\nu_h) > \varepsilon + h' - h) \leq \mathbb{P}(d_{\mathrm{P}}(\nu_h, \rho_h\nu_{h'}) > \varepsilon).$$

So we have to prove that

$$\mathbb{P}(d_{\mathrm{P}}(\nu_h, \rho_h\nu_{h'}) > \varepsilon) \leq \frac{\mu([h,h'))}{4\varepsilon^2\left(1-\exp\left(-\frac{1}{2}\mu([h-\frac{\varepsilon}{2},h))\right)\right)}.$$

From the exchangeability at level h of the tree $\tilde{\mathcal{T}}_E$ with respect to ν_h, the support of $(\rho_h\nu_{h'})$ consists in $n^{h,h'}(\tilde{\mathcal{T}}_E)$ points. Conditionally given $n^{h,h'}(\tilde{\mathcal{T}}_E)$ and $\tilde{\mathcal{T}}_h$, these points are independent with distribution ν_h and are also independent from the family $\mathcal{A}(\tilde{\mathcal{T}}_E) = \left(\nu_{h'}\left(C_i^{h,h'}(\tilde{\mathcal{T}}_E)\right)\right)_{1\leq i\leq n^{h,h'}(\tilde{\mathcal{T}}_E)}$. Denote $F = \{x \in \tilde{\mathcal{T}}_E | H(x) = h\}$. Using Lemma 5.4.2, we have for all $\varepsilon > 0$

$$\mathbb{P}(d_{\mathrm{P}}(\nu_h, \rho_h\nu_{h'}) > \varepsilon | \tilde{\mathcal{T}}_h, \tilde{\mathcal{T}}_{h'}) \leq \frac{n_\varepsilon(F)}{4\varepsilon^2}\sum_{i=1}^{n^{h,h'}(\tilde{\mathcal{T}}_E)}\left(\nu_{h'}\left(C_i^{h,h'}(\tilde{\mathcal{T}}_E)\right)\right)^2.$$

and so

$$\mathbb{P}(d_{\mathrm{P}}(\nu_h, \rho_h\nu_{h'}) > \varepsilon) \leq \mathbb{E}\left[\frac{n_\varepsilon(F)}{4\varepsilon^2}\sum_{i=1}^{n^{h,h'}(\tilde{\mathcal{T}}_E)}\left(\nu_{h'}\left(C_i^{h,h'}(\tilde{\mathcal{T}}_E)\right)\right)^2\right].$$

We recall that $n_\varepsilon(F)$ is the smallest number of Borel sets of diameter less than ε partitioning F. We have $n_\varepsilon(F) = n^{h-\frac{\varepsilon}{2},h}(\tilde{\mathcal{T}}_E)$, which has the same law as $n^{-\mu([h-\frac{\varepsilon}{2},h)),0}(\mathcal{T}^{\mathrm{K}})$ where $(\mathcal{T}^{\mathrm{K}},\nu^{\mathrm{K}})$ is distributed as a Kingman's coalescent. The family $\mathcal{A}(\tilde{\mathcal{T}}_E)$ has the same law as $\left(\nu^{\mathrm{K}}\left(C_i^{-\mu([h,h')),0}(\mathcal{T}^{\mathrm{K}})\right)\right)_{1\leq i\leq n^{-\mu([h,h')),0}(\mathcal{T}^{\mathrm{K}})}$. Since $\mathcal{A}(\tilde{\mathcal{T}}_E)$ is a function of the crown of $\tilde{\mathcal{T}}_h$, it is independent from $n^{h-\frac{\varepsilon}{2},h}(\tilde{\mathcal{T}}_E) = n^{h-\frac{\varepsilon}{2},h}(\tilde{\mathcal{T}}_h)$. We deduce that:

$$\begin{aligned}&\mathbb{P}(d_{\mathrm{P}}(\nu_h, \rho_h\nu_{h'}) > \varepsilon)\\ &\qquad\leq \frac{1}{4\varepsilon^2}\mathbb{E}\left[n^{-\mu([h-\frac{\varepsilon}{2},h)),0}(\mathcal{T}^{\mathrm{K}})\right]\mathbb{E}\left[\sum_{i=1}^{n^{-\mu([h,h')),0}(\mathcal{T}^{\mathrm{K}})}\left(\nu^{\mathrm{K}}\left(C_i^{-\mu([h,h')),0}(\mathcal{T}^{\mathrm{K}})\right)\right)^2\right].\end{aligned}$$

Using Lemma 5.1.2, we have $\mathbb{E}\left[n^{-\mu([h-\frac{\varepsilon}{2},h)),0}(\mathcal{T}^{\mathrm{K}})\right] \leq \frac{1}{1-\exp\left(-\frac{1}{2}\mu([h-\frac{\varepsilon}{2},h))\right)}$. Using Lemma 5.1.1, we have

$$\mathbb{E}\left[\sum_{i=1}^{n^{-\mu([h,h')),0}(\mathcal{T}^{\mathrm{K}})}\left(\nu^{\mathrm{K}}\left(C_i^{-\mu([h,h')),0}(\mathcal{T}^{\mathrm{K}})\right)\right)^2\right] = 1 - \mathrm{e}^{-\mu([h,h'))} \leq \mu([h,h')).$$

Combining the two inequality gives:

$$\mathbb{P}(d_{\mathrm{P}}(\nu_{h'}, \rho_h\nu_{h'}) > \varepsilon) \leq \frac{\mu([h,h'))}{4\varepsilon^2\left(1-\exp\left(-\frac{1}{2}\mu([h-\frac{\varepsilon}{2},h))\right)\right)}.$$

□

Then, we believe that this regularity result is a corner stone to extend by continuity the family of probability measure $(\nu_h)_{h\in E}$ to a family $(\nu_h)_{h\in I}$ with for a.e. h that $H\nu_h = \delta_h$, see Properties (5.2.1) and (5.2.2). Then Property (5.2.3) can be seen as a definition of ν_I. We now explain how to prove Property (5.2.4).

Conjecture 5.4.4. Let μ be a measure satisfying conditions C1-3. Then, for all $h \in I$,

$$S_h = (\mathrm{Stump}_h(\mathcal{T}_I), (\nu_{h'})_{h'\in(-\infty,h]}) \quad \text{and} \quad C_h = (\mathrm{Crown}(h, \mathcal{T}_I), (\nu_{h'})_{h'\in(h,\infty)})$$

are independent.

Idea of the proof. If $h \in E$, then the result is stated in Remark 5.3.2 for $\tilde{\mathcal{T}}^{(n)}$ for large n, and we conjecture it holds at the limit for $\mathcal{T}_I$. Take $h \in I \setminus E$. For every $h' \in E \cap (-\infty, h)$, we can express C_h as a measurable function of $C_{h'}$, so C_h is independent from $S_{h'}$, so C_h is independent from $(S_{h'})_{h'\in(-\infty,h)}$. Since $\mathrm{Stump}_h(\mathcal{T}_I)$ is the Local-Gromov-Hausdorff limit of $\mathrm{Stump}_{h'}(\mathcal{T}_I)$ when $h' \in E \cap (-\infty, h]$ goes to h^- and ν_h is conjectured to be a measurable function of $(\nu_{h'})_{h'\in E\cap(-\infty,h)}$, so we can express S_h as a measurable function of $(S_{h'})_{h'\in E\cap(-\infty,h)}$ which is independent from C_h. □

Because of the Conjectures 5.3.9 and 5.4.1, Properties (5.2.5) and (5.2.6) hold if h belongs to the dense subset E. Then using Lemma 5.3.8, one can always consider $E \cup \{h\}$ instead of E and deduce that Properties (5.2.5) and (5.2.6) hold.

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Résumé

Dans cette thèse, nous développons un nouvel espace pour l'étude des espaces métriques labellés et mesurés, dans l'optique de décrire des arbres généalogiques dont la racine est infiniment ancienne. Dans ces arbres, le temps est représenté par une fonction label qui est 1-Lipschitz. On appelle espace métrique labellé S-compact et mesuré tout espace métrique E équipé d'une mesure ν et d'une fonction-label 1-Lipschitz de E dans $\mathbb{R}$, avec la condition supplémentaire que chaque tranche (l'ensemble des points de E dont le label appartient à un compact de $\mathbb{R}$) doit être compact et avoir mesure finie. On note $\mathbb{X}^S$ l'ensemble des espaces métriques labellés mesurés S-compacts, considérés à isométries près. Sur $\mathbb{X}^S$, on définit une distance de type Gromov d_{LGHP} qui compare les tranches. Il s'ensuit une étude de l'espace $(\mathbb{X}^S, d_{\text{LGHP}})$, dont on montre qu'il est polonais.

De cette étude, on déduit les propriétés de l'ensemble $\mathbb{T}$ des éléments de $\mathbb{X}^S$ qui sont des arbres continus dont les labels décroissent à vitesse 1 quand on se déplace vers la "racine" (qui peut être infiniment loin). Chaque valeur possible de la fonction label représente une génération de l'arbre généalogique. On montre que $(\mathbb{T}, d_{\text{LGHP}})$ est aussi polonais. On définit ensuite quelques opérations mesurables sur $\mathbb{T}$, dont le recollement aléatoire d'une forêt sur un arbre.

On utilise enfin cette dernière opération pour construire un arbre aléatoire qui est un bon candidat pour généraliser l'arbre brownien conditionné par son temps local (construction due à Aldous).

Abstract

In this thesis, we develop a new space for the study of measured labelled metric spaces, ultimately designed to represent genealogical trees with a root at generation $-\infty$. The time in the genealogical tree is represented by a 1-Lipschitz label function. We define the notion of S-compact measured labelled metric space, that is a metric space E equipped with a measure ν and a 1-Lipschitz label function from E to $\mathbb{R}$, with the additional condition that each slice (the set of points with labels in a compact of $\mathbb{R}$) must be compact and have finite measure. On the space $\mathbb{X}^S$ of measured labelled metric spaces (up to isometry), we define a distance d_{LGHP} by comparing the slices and study the resulting metric space, which we find to be Polish.

We proceed with the study of the subset $\mathbb{T} \subset \mathbb{X}^S$ of all elements of $\mathbb{X}^S$ that are real tree in which the label function decreases at rate 1 when we go toward the "root" (which can be infinitely far). Each possible value of the label function corresponds to a generation in the genealogical tree. We prove that $(\mathbb{T}, d_{\text{LGHP}})$ is Polish as well. We define a number of measurable operation on $\mathbb{T}$, including a way to randomly graft a forest on a tree.

We use this operation to build a particular random tree generalizing Aldous' Brownian motion conditioned on its local time.