Cooperative games and stable matchings in networks

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Jeux Coopératifs et d'Appariements Stables dans les Réseaux

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Before starting the PhD, I met a few people to gather advices, opinions and feedbacks about this postgraduate formation that was supposed to make me a researcher. At this time, I thought *"So, when they say it is a unique experience they probably mean that its a lot of work because of the papers to be published and all these things I will have to learn."* . This was true but highly incomplete. How could I have imagine a three years period of my life that would actually heavily depend on skills and people I did not know about?

I discovered that the PhD is more than doing science. It is also about people. Those that you already know and those that you do not but discover. You interact with them, learn from them and sometimes work or cooperate with them. In any case, it is unique because every person is.

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Résumé

Jeux Coopératifs et d'Appariements Stables dans les Réseaux

Dans cette thèse, nous proposons des solutions à plusieurs problèmes d' allocation de ressources et d'associations dans les réseaux. Pour cela, nous employons les jeux coopératifs, particulièrement les jeux d'appariements stables, classiquement utilisés en économie pour l'analyse de marchés bifaces et la conception de leurs mécanismes d'allocations. Dans la première partie, nous introduisons les jeux de négociation et d'appariements stables. Dans une seconde partie, nous proposons un nouveau mécanisme stable d'association des utilisateurs en WiFi réduisant l'impact de l'anomalie du protocole. Nous présentons également une analyse d'un problème de stockage de videos et un nouvel algorithme d'énumération de structures stables. Dans une troisième partie, nous analysons des conditions pour la stabilité de certains schémas d'équité connus en termes de mesures d'aversion au risque. Dans une quatrième partie, nous analysons la stabilité d'une place de marché biface de crowdsourcing avec contraintes d'ordonnancement de tâches. La classique propriété de substitution des biens n'étant pas satisfaite, nous introduisons des nouvelles conditions et montrons l'existence d'appariements stables. Nous proposons également une résolution du problème par une formulation non-coopérative en forme extensive.

MOTS-CLEFS: réseaux, réseaux sans fil, allocation de ressources, appariements, théorie des jeux, jeux coopératifs, stabilité, marchés bifaces

Abstract

Cooperative Games and Stable Matchings in Networks

In this thesis, we propose new solutions to matching problems in networks. We use cooperative games, particularly stable matchings, classically used in economy to analyze two-sided markets and design matching mechanisms. In the first part, we introduce bargaining and stable matching games. In the second part, we propose a new stable matching mechanism for user association in WiFi reducing the impact of the anomaly in the protocol. Furthermore, we analyze a video caching problem and show a new algorithm enumerating stable structures. In the third part, we analyze conditions for the stability of some fairness schemes in terms of risk aversion indicators. In the fourth part, we analyze the stability of a two-sided crowdsourcing marketplace with scheduling constraints on the tasks. The classical substitutability condition does not hold in this case. We introduce new conditions and show the existence of stable matchings. We also solve the crowdsourcing problem as a non-cooperative game in extensive form.

KEY-WORDS: networks, wireless networks, resource allocation, matchings, game theory, stability, cooperation, two-sided markets

Contents

Résumé détaillé en français

0.1 Introduction

Au cours des dernières années, le développement des réseaux et l'innovation dans les services connectés ont donné aux entreprises et utilisateurs de nouvelles opportunités de création de valeur, consommation et de communications. Les plateformes en lignes ont émergé comme places de marchés virtuelles où plusieurs millions d'utilisateurs peuvent s'échanger des biens ou des services. Dans ces systèmes, l'offre et la demande forment un marché biface permettant une compétition en ligne des agents sur les ressources. Beaucoup de ces marchés sont régulés par des règles et mécanismes (tels les enchères) qui définissent la façon dont les agents participent à la compétition. Ces systèmes bénéficient de la connectivité globale d'Internet et des réseaux de communication hautes performances (fixes ou sans-fils) qui permettent un accès aux services à de nombreux agents et une diffusion à grande échelle de l'information. Les modèles économiques, les habitudes des consommateurs et les technologies ont changé ensemble pour se combiner dans de nouvelles solutions et systèmes plaçant les réseaux au coeur d'une révolution. De nouveaux problèmes sont apparus tels que la gestion d'énormes volumes de données, la pertinence des informations transmises ou l'automatisation de la prise de décision. Cependant, en dépit de leur nombre croissant, les places de marchés et leurs mécanismes ne sont pas nécessairement nouveaux. Beaucoup ont déjà été étudiés en économie, théorie des jeux et réseaux. Une approche couronnée de succès, fondée sur le théorie des jeux coopératifs, est appelée théorie des appariements stables. Les résultats développés ont révélés des propriétés insoupçonnées de mécanismes d'allocation de ressources et les recherches ont développé une méthodologie puissante pour les étudier et en concevoir de nouveaux.

0.2 Contributions et plan

Dans cette thèse, on étudie le lien entre certains problèmes des réseaux et les analyses des marchés bifaces par la théorie des jeux. Plus particulièrement, on traite quatre problèmes en utilisant la théorie des appariements stables. Dans le chapitre 2, on donne une brève introduction à la coopération en théorie des jeux en se concentrant sur le problème de marchandage et la solution de Nash à ce problème. Dans le chapitre 3, on introduit la théorie des appariements stables, utilisée tout au long de cette thèse pour développer des solutions aux problèmes posés. Dans le chapitre 4, on montre que dans certaines conditions, le protocole WiFi peut être modélisé comme un marchandage de Nash. Dans le chapitre 5, on étudie le problème de la gestion des connexions WiFi. On montre que le système peut être modélisé comme une place de marché biface avec un schéma coopératif d'allocation de ressources tels que traités par les jeux. On propose un nouveau mécanisme stable d'appariements réduisant l'impact de la congestion et de l'anomalie du protocole, problèmes bien connus induit par l'impact mutuel des agents communicants. Dans le chapitre 6, on applique certains éléments de la solution précédente au problème de mise en cache de vidéos dans les réseaux. On propose un nouveau mécanisme de stockage entre vidéos d'un fournisseur de contenu et les serveurs d'un opérateur. La solution prend en compte l'impact mutuel des vidéos et les gains de qualité induits par des serveurs différenciés. Dans notre modèle, on suppose un marchandage de Nash sur les revenus générés par le stockage et on considère un jeu général de formation de coalitions avec potentiel. On définit un nouvel algorithme énumératif de structures core stables. Pour améliorer la compréhension du lien entre allocation de ressources et stabilité, en chapitre 7, on étudie l'alpha-équité proportionnelle généralisée. On utilise des résultats récents de la théorie des jeux pour montrer que les conditions de concavité sur les fonctions d'utilités des agents pour l'existence de structures core stables peuvent être simplement formulées en termes d'indicateurs d'aversion au risque. Dans le chapitre 8, on étudie un marché biface de production participative (crowdsourcing) avec contrats et contraintes d'ordonnancement de tâches. On suppose la stabilité pair-à-pair comme concept de solution et on fournit des conditions suffisantes d'existence d'appariements stables. On mène également une analyse via une reformulation par les jeux non-coopératifs. Ce problème est important pour la conception de plateformes de production participative plus complexes et complètes permettant plus de coordination dans la distribution des tâches au sein des entreprises participantes ainsi qu'entre elles. Finalement, au chapitre 9 on expose des problèmes ouverts.

0.3 Coopération et négociation

Dans ce chapitre, on introduit la théorie des jeux coopératifs. On se concentre sur les problèmes de marchandages et la solution de Nash qui sera utilisée avec les jeux d'appariements dans les chapitres 4 et 5 pour étudier l'allocation de ressource WiFi et développer un mécanisme d'association ainsi que dans le chapitre 6 pour développer un mécanisme de mise en cache de vidéos.

Dans un jeu coopératif, les joueurs peuvent faire des choix stratégiques non-coopératifs mais ont aussi des opportunités conjointes introduites par les ententes mutuelles. Dans ce cadre, on peut s'attendre à l'apparition de nouvelles formes de stabilité et concepts de solution dans la prise de décision. Un point particulièrement important est que la coopération, par l'ensemble des nouvelles possibilités stratégiques qu'elle introduit, induit un problème de sélection d'équilibre. Ce problème est à l'origine de la théorie de la négociation (arbitrage, marchandage) qui est en fait une théorie de la sélection coopérative d'équilibres. Un exemple simple est proposé par le jeu du partage des dollars. Supposons deux joueurs devant se partager cent dollars. Si la demande des deux est inférieure ou égale à cent dollars, alors chacun obtient la part demandée. Si la demande dépasse les cent dollars, alors les joueurs ne reçoivent rien. Dans ce cas, toute demande telle que la somme des montants (positifs) est égale à cent dollars est un équilibre de Nash. Parmi ces équilibres, donner cinquante dollars à chacun est un partage équitable. Il s'agit sans doute du partage que suggérerait un arbitre impartial, ce qui confère à cette allocation une propriété spécifique d'attraction que les autres équilibres n'ont pas.

0.3.1 Le marchandage de Nash

Il existe de nombreux schémas et processus possibles d'arbitrage. Le modèle de marchandage de Nash repose sur l'hypothèse que le résultat d'un marchandage entre agents devrait être une fonction de l'ensemble des opportunités conjointement atteignables mesurées en utilité et de menaces qui bloqueraient le processus de négociation en cas de désaccord en garantissant à chaque joueur une quantité d'utilité connue. L'arbitrage proposé par Nash repose sur cinq axiomes. Le premier axiome requiert la faisabilité et la Pareto-efficience de l'allocation. Le second demande une rationalité individuelle: aucun joueur ne devrait accepter moins que l'utilité qu'il peut garantir par application de sa menace. Le troisième requiert une invariance de la solution par transformation affine positive de l'utilité alors que le quatrième requiert une conservation de la solution à la suppression des allocations qui ne sont pas des équilibres. Finalement, le dernier axiome requiert une symétrie telle que l'allocation soit la même pour des agents identiques en termes d'utilité. Soit $\mathrm{B} \subset \mathbb{R}^2$ l'ensemble compacte [1] convexe[2] des allocations d'utilités conjoitement atteignables par les joueurs, appelé *ensemble faisable*. Soit $\mathbf{t} = (t_1, t_2)$ un vecteur de $\mathbb{R}^2$ appelé vecteur de *menaces* et

[1] Fermé et borné

[2] $\forall \mathbf{x}, \mathbf{y} \in \mathrm{B}, \forall \lambda \in [0,1], \lambda x + (1-\lambda) y \in \mathrm{B}$

Φ la fonction associant à tout jeu de marchandage $(B, \mathbf{t})$ une allocation de l'utilité dans B. Nash a montré qu'il existe une unique fonction de solution $\Phi(.,.)$ satisfaisant les cinq axiomes appelés *axiomes de Nash.*

$$\Phi(B, \mathbf{t}) \in \arg\max_{\mathbf{u}\in B, \mathbf{u}\geq \mathbf{t}} (u_1 - t_1)(u_2 - t_2) \tag{1}$$

Ce résultat fondamental a part la suite été généralisé aux jeux de marchandage à n joueurs. Une solution relâchant l'axiome de symétrie par introduction de puissances de négociation a également été proposée.

0.4 Appariements stables

Dans ce chapitre, on introduit la théorie des appariements stables, une branche fructueuse de la théorie des jeux utilisée pour l'analyse de certains marchés bifaces et leurs mécanismes. Il s'agit du principal outil de théorie des jeux utilisé dans cette thèse. Les résultats présentés dans ce chapitre seront utilisés en chapitre 5 pour concevoir un mécanisme d'appariement stable contrôlé pour le problème d'association en WiFi, en chapitre 6, pour concevoir un mécanisme d'appariement stable pour la mise en cache de vidéos, en chapitre 7, pour analyser les conditions de concavité requises pour la stabilité de l'allocation alpha-équitable généralisée et en chapitre 8 pour concevoir un mécanisme d'appariement stable pour plateforme de production participative avec contraintes d'ordonnancement.

Depuis les travaux fondateurs de Gale et Shapley sur les marriage stables et le problème d'admission au collège, les jeux d'appariements stables ont été largement étudiés. La contribution initiale a été développée pour couvrir un grand nombre d'applications. Cet outil est particulièrement adapté à l'analyse des places de marchés pour lesquelles la participation des agents et leurs incitations dépendent du mécanisme d'appariement utilisé. Si le mécanisme n'est pas stable, les agents peuvent considérer des solution alternatives pour s'associer entre eux. Ce phénomène est appelé désagrégation du marché.

0.4.1 Le Problème du marriage stable

Le problème du marriage stable a été analysé par Gale et Shapley en 1962. Ils ont montré l'existence systématique de marriages stables dans un marché biface d'hommes et femmes en utilisant un algorithme connu sous le nom de Deferred Acceptance Algorithm (DAA).

Considérons un ensemble d'hommes $\mathscr{M}$ de cardinalité M et un ensemble de femmes $\mathscr{W}$ de cardinalité W. Chaque homme a des préférences données par une relation d'ordre sur les femmes, et chaque femme a des préférences données par une relation d'ordre sur les hommes. Le problème du marriage est le jeu d'association entre hommes et femmes avec préférences. Un équilibre particulier a été définit pour ce jeu: la stabilité pair-à-pair. Un marriage (ou appariement) $\mu : \mathscr{M} \cup \mathscr{W} \Rightarrow \mathscr{M} \cup \mathscr{W}$ entre hommes et femmes est une function d'association telle que tout joueur soit associé à son partenaire (ou lui même s'il reste seul)

- $\mu(m) \in \mathscr{W} \cup \{m\}$ pour tout $m \in \mathscr{M}$
- $\mu(w) \in \mathscr{M} \cup \{w\}$ pour tout $w \in \mathscr{W}$
- $\mu(m) = w$ ssi $\mu(w) = m$

On dit d'un appariement qu'il est stable pair-à-pair si aucun participant ne préfère quitter son conjoint pour rester seul ou s'il n'existe pas un homme et une femme non-mariés ensembles qui préfèreraient tous deux quitter leurs conjoints pour se marier ensemble. Formellement, un appariement μ est stable pair-à-pair si,

- Il est individuellement rationel: $\mu(i) \succ_i i$ pour tout $i \in \mathscr{M} \cup \mathscr{W}$
- Il n'y a pas de paire bloquante: $\nexists (m, w) \in \mathscr{M} \times \mathscr{W}$ t.q. $\mu(m) \neq w$, $w \succ_m \mu(m)$ et $m \succ_w \mu(w)$

Le résultat fondamental de Gale et Shapley est le suivant,

Theorem. *Il existe un mariage stable pour tout jeu de mariage.*

En d'autres termes, dans tout jeu de mariage (donc quelques soient les préférences) il est possible de marier les participants tel que personne ne veuille divorcer. De plus, il a également été démontré que l'utilisation de l'algorithme DAA résulte en un appariement tel que tout postulant est au moins aussi bien (selon ses préférences) dans l'appariement résultant du DAA qu'il ne le serait dans tout autre marriage stable. Il s'agit d'un autre résultat fort qui montre que la procédure DAA proposée est optimale pour une des faces du marché.

0.4.2 Le problème d'admission au collège

Le problème d'admission au collège est une extension naturelle du problème d'association un-à-un de marriages stables au cadre plusieurs-à-un où chaque étudiant est appairé à un collège mais chaque collège peut être associé à plusieurs étudiants dans la limite de son quota. Dans la version la plus élémentaire du problème, chaque étudiant a des préférences sur les collèges et les collèges ont des préférences sur les groupes d'étudiants qu'ils peuvent recevoir. Dans des configurations plus avancées, les préférences des joueurs peuvent dépendre de façon plus générale de l'appariement du marché ou d'éléments additionnels tels que des rémunérations (cas des marchés de l'emplois entre travailleurs et entreprises). Par exemple, les étudiants peuvent avoir des préférences sur leurs collègues. Nous réservons ces cas (d'externalités) pour les sections suivantes. Les préférences des collèges sur les groupes peuvent également répondre à des règles simples de composition de préférences sur les étudiants individuels.

Soit un ensemble de collèges $\mathscr{C}$ de cardinalité C et un ensemble d'étudiants $\mathscr{S}$ de cardinalité S. Un appariement plusieurs-à-un faisable μ est une fonction de $\mathscr{C} \cup \mathscr{S}$ dans les familles non-ordonnées de $\mathscr{C} \cup \mathscr{S}$ tel que:

- $|\mu(s)| = 1$ pour chaque étudiant s and $\mu(s) = 1$ si $\mu(s) \not\in \mathscr{C}$;
- $|\mu(c)| = q_c$ pour chaque collège c, et si le nombre d'étudiants dans $\mu(c)$, disons r, est inférieur ou égal à q_c, alors $\mu(c)$ contient $q_c - r$ copies de c;
- $\mu(s) = c$ si et seulement si $s \in \mu(c)$

Dans le problème d'admission au collège, on peut encore considérer la stabilité pair-à-pair définie précédemment comme notion d'équilibre. Néanmoins, la notion de groupe est devenue importante par introduction de préférences sur les groupes pour les collèges. Si ces préférences sont telles qu'en fait elles peuvent être réduites aux préférences individuelles, alors la stabilité pair-à-pair peut être suffisante. Néanmoins, dans le cas général il est nécéssaire de considérer les déviations de groupes de joueurs ou coalitions. En ce sens, deux autres définitions d'équilibre ont été proposées, la core stabilité et la stabilité de groupe. On présente la core stabilité.

Definition. *Un appariement* μ' *domine un autre appariement* μ *via une coalition* C *dans* $\mathscr{C} \cup \mathscr{S}$ *si pour tout étudiant s et collège c dans* C,

- *Si* $c' = \mu'(s)$ *alors* $c' \in \mathrm{C}$
- *Si* $s' \in \mu'(c)$ *alors* $s' \in \mathrm{C}$
- $\mu'(s) \underset{s}{\succ} \mu(s)$
- $\mu'(s) \underset{s}{\succ} \mu(s)$

Definition. *Le core,* $\mathrm{C}(\mathbf{P})$*, d'un jeu est l'ensemble des appariements non-dominés par un autre appariement.*

La notion de core faible requiert la relaxation de préférences strictes de tout joueur de la coalition C à une préférence faible pour tout joueur de C et stricte pour au moins un étudiant et au moins un collège de C. Le core faible est contenu dans le core et on peut montrer qu'avec certaines préférences le core faible correspond aux matchings stables pair-à-pair.

0.4.3 Le modèles des entreprises et travailleurs: fonction de choix, subtituabilité et salariés

Dans cette section, on considère le modèles d'entreprises et travailleurs. Il permet entre autres l'introduction de rémunérations versées par les entreprises aux travailleurs en échange de l'exécution de tâches. Dans cette section, on considère les fonctions de choix comme formulation des préférences alternatives aux relations d'ordres et on présente la substituabilité comme condition suffisante d'existence d'appariements stables.

Considérons le problème d'admission au collège de la section précédente mais transformons désormais les étudiants $\mathscr{S}$ en travailleurs $\mathscr{W}$ et les collèges $\mathscr{C}$ en entreprises $\mathscr{F}$. Supposons que chaque paire de la forme (entreprise,travailleur) soit caractérisée par un rémunération versée au travailleur. Les préférences des joueurs sont formulés à l'aide de fonctions de choix. Par exemple, la fonction de choix d'une firme associe à tout groupe de travailleurs son préféré.

Definition. *Pour tout sous-ensemble* $S \subseteq \mathscr{W}$, *l'ensemble choisi par* f *est* $S' = Ch_f(S)$ *tel que* $S' \subseteq S$ *et* $S' \succeq_f S''$ *pour tout* $S'' \subseteq S$.

Les préférences ou choix des firmes sur les groupes peuvent être définies par des complémentarités complexes entre travailleurs. Dans certains cas, elles peuvent aussi limiter ces complémentarités. Un exemple de ce type est donné par la propriété de substituabilité.

Definition. *Les préférences d'une entreprise* f *sur les ensembles de travailleurs a la propriété de substituabilité si, pour tout ensemble* S *contenant les travailleurs* w *and* w', *si* w *est dans* $r_f(S \setminus w')$ *alors* w *est dans* $r_f(S)$.

On a le résultat suivant :

Theorem. *Quand les préférences des firmes satisfont la propriété de substituabilité, l'ensemble des appariements stables est toujours non-vide.*

0.4.4 Complémentarités et effets de pairs

Dans de nombreux cas les hypothèses telles que la propriété de substituabilité des préférences ne sont pas vérifiées. De plus, un modèle plus complexe doit tenir compte des d'effets de pairs où les travailleurs peuvent avoir des préférences sur leurs collègues. Un travailleur n'émet plus seulement des préférences sur les entreprises ou les tâches à exécuter mais sur les groupes de joueurs composés d'autres travailleurs et d'une entreprise. Ces effets de pairs sont un cas particulier d'externalités qui considèrent l'impact de l'appariement courant (état du marché) sur les préférences des individus. De récentes analyses du probléme de formation de coalitions ou d'appariements stables avec complémentarités et effets de pairs ont mis en évidence de nouvelles conditions suffisantes (sur les préférences) d'existence d'appariements core stables. Pour exemple, si l'ensemble des coalitions satisfait une certaine régularité (trois conditions de régularité C1-C2-C3), que les préférences des agents sont alignées pair-à-pair sur un riche profile de préférences (trois conditions de richesses R1-R2-R3) alors il existe des appariements core stables.

Theorem. *Supposons que la famille de coalition* $\mathscr{C}$ *satisfasse les conditions* C1 *and* C2, *et que le domaine de préférences* $\mathbf{R}$ *satisfasse* R1. *Si tous les profiles de préférences de* $\mathbf{R}$ *sont alignés pair-à-pair, alors (i) tout* $\underset{\mathscr{N}}{\preceq} \in \mathbf{R}$ *admet une structure stable de coalitions et (ii) la structure stable de coalitions est unique pour tout profile de préférences strictes* $\underset{\mathscr{N}}{\preceq} \in \mathbf{R}$ *qui est aligné pair-à-pair sur la grande coalition.*

Dans les cas où au sein d'une coalition les joueurs se partagent une valeur ou une ressource selon une règle de partage donnée ces résultats peuvent aussi être déclinés. On a le résultat suivant,

Corollary. *Supposons que la famille de coalitions $\mathscr{C}$ satisfasse C1-C3 et que la règle de partage* D *soit régulière. Il y a une structure stable de coalitions pour tout profile de préférences induit part la règle de partage si et seulement si il existe des fonctions d'utilités $u_i : \mathbb{R}^+ \to \mathbb{R}^+, i \in \mathscr{N}$ croissantes, différentiables, et strictement log-concave, telles que $\frac{u_i}{u_i'}(0) = 0$ et*

$$(\mathrm{D}_{i,\mathrm{C}}(y))_{i\in\mathrm{C}} = \underset{\sum_{i\in\mathrm{C}} s_{i,\mathrm{C}} \leq v(\mathrm{C})}{\arg\max} \prod_{i\in\mathrm{C}} u_i(s_i), \quad y \in \mathbb{R}^+, \mathrm{C} \in \mathscr{C}\backslash\{\mathscr{N}\} \tag{2}$$

Il s'agit d'un résultat particulièrement important que nous utilisons au chapitre 5 pour montrer l'existence d'appariements stables dans les réseaux WiFi. En effet, on montre au chapitre 4 que le protocole WiFi induit des débits saturés qui résultent d'un marchandage de Nash équivalent satisfaisant les conditions requises par le corollaire précédent pour l'existence d'appariements core stables. En chapitre 7, on analyse les conditions de concavité sur les utilités pour le cas particulier de l'allocation alpha-équitable généralisée. On montre que dans ce cas, ces conditions peuvent être analysées en termes de mesures d'aversion au risque.

0.4.5 Appariements avec contrats et externalités

Il est possible d'enrichir les appariements en transformant la relation binaire d'association en contrats qui paramètres l'appariement entre les joueurs. Ainsi, pour une même paire *(travailleur,entreprise)* l'appariement entre les deux joueurs peut être défini différemment selon les contrats. La formulation avec contrats généralise les modèles de marriage, étudiants et collèges, travailleurs et entreprises et même certains modèles d'enchères. Ils couvrent également les cas d'association plusieurs-à-plusieurs où travailleurs et entreprises peuvent tous signer plusieurs contrats.

Considérons les ensemble finis des firmes $\mathscr{F}$ et des travailleurs $\mathscr{W}$. Définissons l'ensemble des contracts entre travailleurs et entreprises comme l'ensemble des accords pair-à-pair qui peuvent être signés. Un contrat stipule, un travailleur, une entreprise et des termes additionnels tels une ou plusieurs tâches, une rémunération, des responsabilités, des contraintes d'execution ou une pénalité en cas de non-exécution. Par définition, un appariement est un ensemble de contrats. Les fonctions de choix des agents sont désormais définies sur les contrats. Une définition possible est la suivante.

Definition. *On construit le choix de l'agent i dans tout ensemble* X *étant donné* μ, $c_i(\mathrm{X}|\mu)$, *tel que:*

$$c_i(\mathrm{X}|\mu) \cup \mu_{-i} \succeq_i \mathrm{X}_i' \cup \mu_{-i} \text{ pour tout } \mathrm{X}_i' \subseteq \mathrm{X}_i \tag{3}$$

On a la définition suivante de la stabilité pair-à-pair d'un appariement μ.

Definition. *Un travailleur i et une firme j forment une paire bloquante pour l'appariement* μ *si il existe un contrat $x \in \mathscr{X}_i \cap \mathscr{X}_j$ tel que $x \notin \mu$ et $x \in c_i\left(\mu \cup x|\mu\right) \cap c_j\left(\mu \cup x|\mu\right)$. Un appariement* μ *est stable pair-à-pair si,*

- *il est individuellement rationnel pour chaque agent: $c_i(\mu|\mu) = \mu_i$ pour tout $i \in \mathscr{N}$,*
- *il n'y a pas de paire bloquante.*

Pour montrer l'existence d'appariements stables plusieurs-à-plusieurs avec contrats et externalités, deux conditions fondamentales ont été proposées dans la littérature, la non-pertinence des contrats rejetés et la substituabilité des contrats. On a les définitions suivantes.

Definition. *Une fonction de choix c_i satisfait la non-pertinence des contrats rejetés si pour tous $\mathrm{X}, \mathrm{X}' \subseteq \mathscr{X}$, on a*

$$c_i(\mathrm{X}') \subseteq \mathrm{X} \subseteq \mathrm{X}' \Rightarrow c_i(\mathrm{X}') = c_i(\mathrm{X}) \tag{4}$$

Definition. *La fonction de choix* C^θ *telle que* $C^\theta(X|\mu) = \bigcup_{i\in\theta} c_i(X_i|\mu_{-i})$ *satisfait la propriété de substituabilité si pour tous* $X, X', \mu, \mu' \subseteq \mathscr{X}$,

$$X' \supseteq X \ \& \ \mu' \succsim^\theta \mu \Rightarrow R^\theta(X'|\mu') \supseteq R^\theta(X|\mu) \tag{5}$$

où $\succsim^\theta$ *est un préordre consistent avec* C^θ.

On a le théorème d'existence suivant.

Theorem. *Supposons que les fonctions de choix satisfassent la subtituabilité et la non-pertinence des contrats rejetés. Alors l'algorithme converge, sa sortie est stable et*

$$\mu^F(T) = \mu^W(T) = A^F(T) \cap A^W(T) \tag{6}$$

où $\mu^F(T)$, $\mu^W(T)$, $A^F(T)$, $A^W(T)$ *sont des ensembles résultants des choix et rejets des joueurs tels que décrits dans l'algoritme.*

Comme pour l'existence des marriages stables, la démonstration du théorème utilise un algorithme similaire au DAA appelé Modified Deferred Acceptance Algorithm. Ce résultat est étudié de façon extensive au chapitre 8 dans lequel nous présentons les conditions de convergence vers un équilibre stable d'un algorithme pour place de marché de production participative avec contraintes d'ordonnancement de tâches par la théorie des appariements stables avec contrats et externalités. Plus particulièrement, on propose de définir la substituabilité contrainte et on montre un théorème d'existence d'appariements stables similaire au précédent.

Algorithm 1: Le Modified Deferred Acceptance Algorithm

Data: $\Gamma = (\mathscr{F}, \mathscr{W}, \{c_i\}, \mathscr{X})$
Result: μ

Phase 1: Construction d'un appariement auxiliaire μ^* tel que $\mu^* \geq^F C^F(\mathscr{X}|\mu)$
-Définir $\mu_0 = \emptyset$;
while $\nexists l < k$ *tel que* $\mu_l = \mu_k$ **do**
  -k = k+1;
  -$\mu_k = C^F(\mathscr{X}|\mu_{k-1})$
-Définir $\mu_* = \mu_k$;
Phase 2: Construction d'un appariement stable
-Définir $A^F(1) = \mathscr{X}$, $A^W(1) = \emptyset$, $\mu^F(1) = \mu^*$ et $\mu^W(1) = \emptyset$;
while $A^F(k) \neq A^F(k-1), A^W(k) \neq A^W(k-1), \mu^F(k) = \mu^F(k-1), \mu^W(k) = \mu^W(k-1)$ **do**
  - k = k+1;

$$A^F(k) = \mathscr{X} \setminus R^W(A^W(k-1)|\mu^W(k-1)) \tag{7}$$
$$A^W(k) = \mathscr{X} \setminus R^F(A^F(k-1)|\mu^F(k-1)) \tag{8}$$
$$\mu^F(k) = C^F(A^F(k-1)|\mu^F(k-1)) \tag{9}$$
$$\mu^W(k) = C^W(A^W(k-1)|\mu^W(k-1)) \tag{10}$$

0.5 Négociation de Nash pour allocation de ressources en WiFi

Dans ce chapitre, on montre que le protocole IEEE 802.11 (consistant en une compétition entre noeuds hétérogènes pour un accès au medium) peut être modélisé comme un marchandage de Nash satisfaisant les conditions d'existence d'appariements core stables entre noeuds. Plus particulièrement, les débits individuels sont obtenus comme solution de Nash correspondant au point

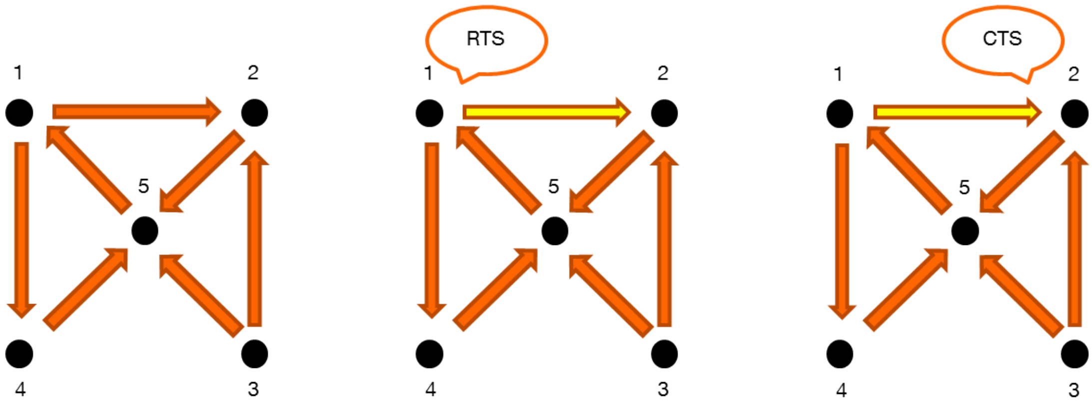


Figure 1: Protocole WiFi. **Gauche:** Un ensemble de nœuds (noirs) et leurs flux (flèches). Par exemple, le nœud 1 a un flux à transmettre au nœud 2. **Milieu:** Le compteur (*backoff*) du nœud 1 est écoulé. Le nœud envoie une demande de transmission RTS (Request to Send) au nœud 2. **Droite:** Le nœud 2 la reçoit et décode la demande RTS. Il envoie un message d'accord pour transmission CTS (Clear To Send).

de marchandage dans un espace d'utilité induit par des fonctions d'utilités paramétrées par les flux à transmettre par les agents du système.

Ces fonctions d'utilité associent l'espace de négociation en utilités à un simplexe de débits défini par la fonction caractéristique du WiFi associant à toute coalition son débit total saturé. Chaque joueur du système WiFi a une fonction d'utilité u_i telle que

$$u_i(s_{i,C}) = s_{i,C}^{\alpha_i} \tag{11}$$

où $\alpha_i \in [0;1]$ est défini selon

$$\alpha_i = \frac{\sum_{j=1}^{|\mathscr{B}_i|} p_{ij} L_{ij}}{\sum_{i \in \mathscr{N}} \sum_{j=1}^{|\mathscr{B}_i|} p_{ij} L_{ij}} \tag{12}$$

Le marchandage se fait dans l'espace d'utilité induit par l'ensemble des allocations faisables en débits,

$$\mathscr{S} = \left\{ s \in \mathbb{R}^{|C|} | \sum_{i \in C} s_{i,C} = \left[\sum_{i \in C} \sum_{j=1}^{|\mathscr{B}_i|} p_{ij} L_{ij} \right] \kappa(C) \;\&\; s_{i,C} \geq 0 \; \forall i \in C \right\} \tag{13}$$

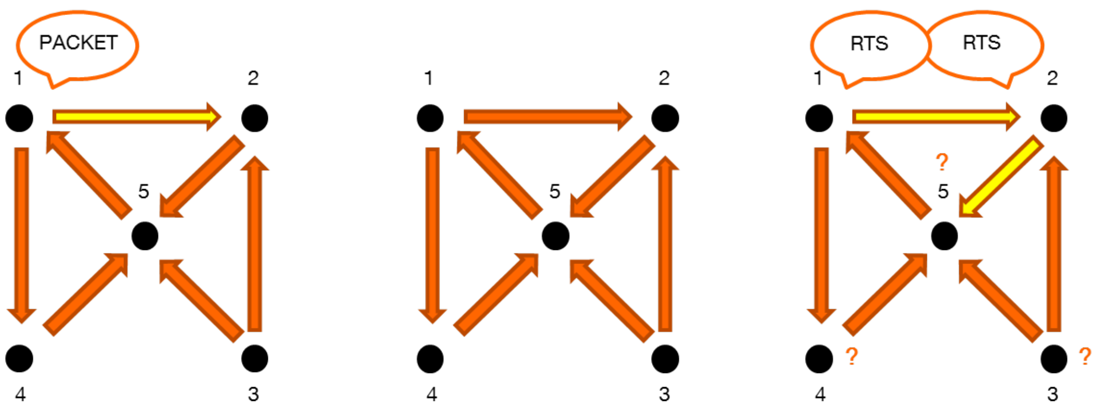


Figure 2: Protocole WiFi. **Gauche:** Le nœud 1 envoie un paquet à son débit. **Milieu:** Le paquet du nœud 1 a été transmis. Les compteurs de *backoff* dcomptent le temps. **Droite:** Les compteurs du nœud 1 et du nœud 2 sont écoulés. Les deux nœud envoient une demande RTS et il y a collision de ces demandes. Aucun nœud ne peut dcoder les demandes RTS. Aucun accusé CTS n'est émis.

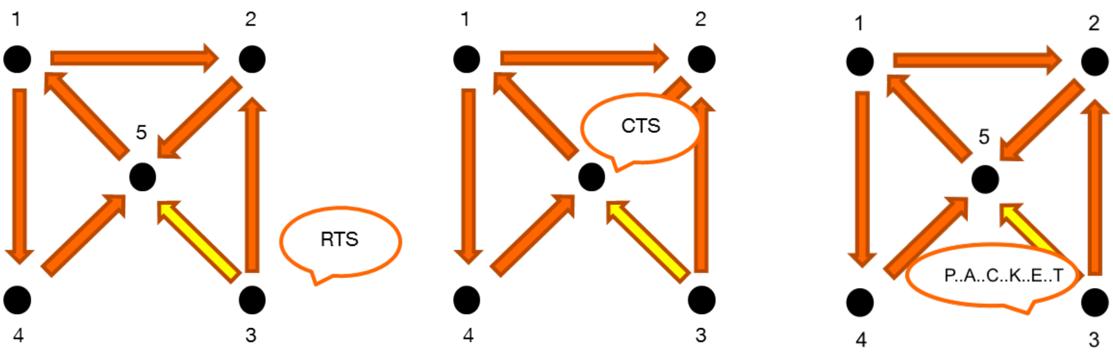


Figure 3: Protocole WiFi. **Gauche:** Le compteur de *backoff* du nœud 3 est écoulé. Le nœud envoie une demande RTS. **Milieu:** Le nœud 5 décode et répond par un CTS. **Droite:** Le nœud 3 transmet un paquet à son débit. Ce débit est plus faible que le débit du nœud 1. L'émission du paquet du nœud 3 prend plus de temps que celle du paquet du nœud 1. Dans ce temps additionnel, aucun noeud ne peut émettre.

où $\kappa(C)$ est une fonction des joueurs de la coalition et des paramètres du système. Ce résultat est utilisé en chapitre 5 pour définir un mécanisme d'appariement contrôlé pour le problème d'association WiFi.

0.6 Une analyse du WiFi par les jeux coopératifs

Dans les réseaux WiFi, la méthode classique d'association fondée sur la meilleure puissance reçue et l'anomalie du protocol MAC peuvent mener à des points d'accès surchargés et des performances inégales ou mauvaises. Dans ce chapitre, nous proposons une approche alternative de l'association fondée sur la théorie des jeux coopératifs. On modélise l'allocation de ressources et l'association des utilisateurs aux points d'accès comme un jeu d'appariements avec des joueurs (utilisateurs ou points d'accès) rationnels maximisant leurs débits individuels. En utilisant les résultats des chapitres 3 et 4, on montre que le protocol WiFi IEEE 802.11 en implémentation avec Fonction de Coordination Distribuée (DCF - Distributed Coordination Function) fait partie des schémas d'allocation de ressources qui induisent la core stabilité des appariements. Le jeu utilise de façon extensive le marchandage de Nash modélisant l'allocation WiFi et certaines de ses propriétés pour contrôler les incitations des joueurs pour des appariements aux propriétés intéressantes en termes de partage de charge dans le réseau. On montre que le mécanisme proposé peut effectivement améliorer l'efficacité du WiFi et réduire les effets de pairs tels que l'anomalie de la couche MAC. Ce mécanisme d'association peut être implémenté sans modification de cette dernière.

Definition (Jeu d'Allocation de Ressources et d'Association d'Utilisateurs)**.** *Le jeu d'allocation de ressources et d'associations est défini comme un jeu d'appariement plusieurs-à-un à N joueurs en forme caractéristique avec la régle de partage* D *et des débits physiques* $\boldsymbol{\theta} = \{\theta_{wf}\}_{(w,f)\in\mathcal{W}\times\mathcal{F}}$ *:* $\Gamma = (\mathcal{W}\cup\mathcal{F}, v, \mathbb{R}^{+\mathrm{N}}, \mathrm{D}, \boldsymbol{\theta})$. *Chaque paire de joueurs de la forme* $(w, f) \in \mathcal{W}\cup\mathcal{F}$ *est dotée d'un débit physique* θ_{wf} *d'un espace de débit* $\Theta = \{\theta^1, \ldots, \theta^m\}$. *Pour ce jeu, on définit l'ensemble des coalitions* $\mathscr{C}$ *:*

$$\mathscr{C} = \{\{f\}\cup \mathrm{J},\ f \in \mathcal{F}, \mathrm{J} \subseteq \mathcal{W}, |\mathrm{J}| \leq q_f\} \cup \{\{w\},\ w \in \mathcal{W}\}. \tag{14}$$

Le jeu d'appariement que nous considérons est caractérisé par des complémentarités dans le sens où les points d'accès ont des préférences sur les groupes d'appareils utilisateurs qui leurs sont connectés et des effets de pairs dans le sens où les appareils utilisateurs se préoccupent des autres appareils connectés au même point d'accès qu'eux (i.e. dans la même cellule ou coalition). Tous les joueurs émettent des préférences sur des groupes. En effet, par définition de l'implémentation DCF du protocole IEEE 802.11, le débit d'un utilisateur ne dépend pas seulement de son débit physique mais aussi de son groupe de connectivité par la taille et la composition. Il s'agit du classique problème d'association mais avec des joueurs (appareils utilisateurs et points d'accès)

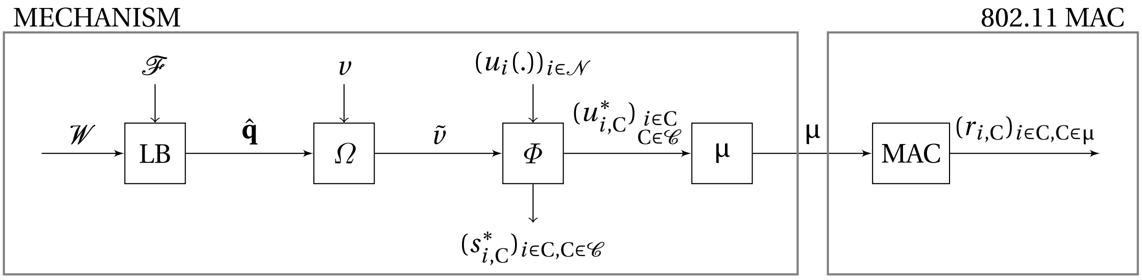


Figure 4: Diagramme bloc du mécanisme dans sa forme la plus générale. Les points d'accès partagent la charge dans le bloc **LB** qui attribue aux points d'accès les objectifs $\hat{\mathbf{q}}$. La fonction caractéristique v du jeu de coalition original est contrôlée dans Ω qui donne la fonction caractéristique contrôlée $\tilde{v}$. La négociation de Nash Φ est jouée dans chaque coalition pour l'allocation de la valeur de la coalition. Les joueurs émettent leurs préférences sur les coalitions à partir de leurs allocations et prennent part à un mécanisme d'appariement stable dans le bloc μ. Ce bloc donne une association μ entre points d'accès et utilisateurs. Finalement, dans le bloc **MAC** les noeuds transmettent leurs paquets selon le protocole IEEE 802.11 MAC non-modifié.

égoïstes et rationnels cherchant à maximiser leurs propres débits. On montre dans ce problème l'existence d'appariements core stable, par définition de la règle de partage (donnant la performance en espérance) induite par le protocole à accès aléatoire et multiple IEEE 802.11 DCF. Néanmoins, l'utilisation de la core stabilité en tant que concept de solution donne des appariements avec un grand nombre d'appareils utilisateurs non-connectés. Cet effet est appelé problème de chômage. Pour le contrer et donner aux noeuds incitation à s'associer les uns aux autres, on conçoit un mécanisme décentralisé en trois étapes. L'objectif est de contrôler l'ensemble des appariements stables. En d'autres termes, on manipule le core de façon à le transposer sur un ensemble d'appariements aux propriétés intéressantes en termes d'équilibrage de charge et de nombre de joueurs connectés.

Dans la première étape, les points d'accès se partagent la charge. La règle de partage est supposée générique et résulte en une charge cible par point d'accès définie en nombre de connexions qui devraient être réalisées à l'équilibre.

Dans la seconde étape, le jeu de coalition (modélisant l'interaction des noeuds par le protocole, résultant en un débit total et un vecteur d'allocation par coalition) est contrôlé pour inciter les agents à réaliser l'équilibrage de charge. Le contrôle est fondé sur une mesure d'aversion au risque appelée *peur de la ruine*. Le contrôle est appliqué à la fonction caractéristique du jeu et on obtient les résultats suivants.

Proposition. *Soit un jeu de coalitions $\Gamma = (\mathscr{F}\cup\mathscr{W}, v, \{u_i\}_{i\in\mathrm{N}})$ en forme caractéristique ayant pour règle de partage le marchandage de Nash sur $v(\mathrm{C})$ dans toute coalition C de $\mathscr{C}$. De plus, supposons des fonctions d'utilités strictement croissantes et concaves* [3] *$u_i : \mathbb{R}^{\cdot+} \to \mathbb{R}^+, i \in \mathscr{N}$. L'ensemble des transformations Ω, de l'ensemble des fonctions caractéristiques dans lui-même, qui donne incitation aux joueurs pour un sous-ensemble de coalitions $\mathscr{C}'$ dans $\mathscr{C}$ doit satisfaire:*

$$\mathrm{F}_{\mathrm{C}'}\circ\Omega(v)(\mathrm{C}') < \mathrm{F}_{\mathrm{C}}\circ\Omega(v)(\mathrm{C}) \quad \forall \mathrm{C}'\in\mathscr{C}', \forall\mathrm{C}\in\mathscr{C}\backslash\mathscr{C}' \tag{15}$$

où $\mathrm{C}'\cap\mathrm{C}\neq\emptyset$, $\mathrm{F}_{\mathrm{C}} = \left(\sum_{i\in\mathrm{C}}\left(\frac{u_i'}{u_i}\right)^{-1}\right)^{-1}$ *et $\circ$ est la fonction de composition.*

Corollary. *Soit un jeu de coalitions $\Gamma = (\mathscr{F}\cup\mathscr{W}, v, \{u_i\}_{i\in\mathrm{N}})$ en forme caractéristique avec le marchandage de Nash sur $v(\mathrm{C})$ pour règle de partage dans chaque coalition $\mathrm{C}\in\mathscr{C}$. De plus, supposons des fonctions d'utilités $u_i : \mathbb{R}^+ \to \mathbb{R}^+, i\in\mathscr{N}$ strictement croissantes et concaves. L'ensemble des transformations Ω de l'ensemble des fonctions caractéristiques dans lui-même qui induit des préférences à sommet unique*[4] *(sommet à $\hat{q}_f$) en cardinalités sur les coalitions contenant un AP $f\in\mathscr{F}$ doivent*

[3] De telles fonctions d'utilité sont bijectives, donc injectives. Le Théorème 10 s'applique.

[4] *"single-peaked preferences"*

satisfaire:

$$\max_{\substack{C\in\mathscr{C}_f \\ s.t.|C|=q}} F_C \circ \Omega(v)(C) < \min_{\substack{C\in\mathscr{C}_f \\ s.t.|C|=q+1}} F_C \circ \Omega(v)(C), \quad \forall q \geq \hat{q}_f \tag{16}$$

et

$$\max_{\substack{C\in\mathscr{C}_f \\ s.t.|C|=q}} F_C \circ \Omega(v)(C) < \min_{\substack{C\in\mathscr{C}_f \\ s.t.|C|=q-1}} F_C \circ \Omega(v)(C), \quad \forall q \leq \hat{q}_f \tag{17}$$

où $F_C = \left(\sum_{i\in C}\left(\frac{u_i'}{u_i}\right)^{-1}\right)^{-1}$.

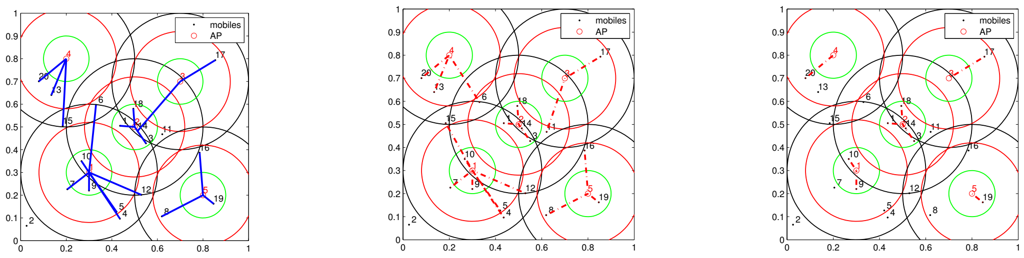
Dans la troisième étape, les joueurs jouent le jeu d'appariement stable avec des préférences induites par les paiements contrôlés (débits) obtenus à l'issue de la deuxième étape. L'appariement core stable donnant l'association finale est obtenu par un algorithme décentralisé. Nous proposons une version modifiée du DAA, appelée Backward Deferred Acceptance Algorithm (BDAA), pour les jeux d'appariements avec complémentarités et effets de pairs. Dans notre problème, BDAA converge vers l'unique appariement core stable et comme pour DAA, la complexité de BDAA est polynomiale en le nombre de propositions.

Algorithm 2: Backward Deferred Acceptance

Data: Pour chaque AP: L'ensembles de utilisateurs acceptables (couverts) et les débits physiques AP-utilisateurs.
Pour chaque utilisateur: L'ensemble des APs acceptables (couvrant).
Result: Un appariement core stable $\mathscr{S}$

begin
- *Etape 1: Initialisation*;
  - **Etape 1.a:** Tous les APs et utilisateurs sont marqués *non-engagés*. $L(f) = L^*(f) = \emptyset, \forall f$;
  - **Etape 1.b:** Chaque AP f évalue les coalitions possibles avec ses utilisateurs acceptables, les paiements utilisateurs correspondants et émet sa liste de préférences $P^{\#}(f)$;
  - **Etape 1.c:** Chaque AP f transmet à ses utilisateurs acceptables le débit le plus élevé qu'ils peuvent obtenir dans les coalitions contenant f;
  - **Etape 1.d:** Chaque utilisateur w émet sa liste de préférences réduites $P'(w)$;
- *Etape 2 (BDAA)*;
  - **Etape 2.a, Proposition des mobiles:** Selon $P'(w)$, chaque utilisateur non-engagé w propose à son AP acceptable préféré parmi ceux auquels il n'a pas déjà proposé. Si cet AP est déjà engagé dans une coalition, tous les joueurs de cette coalition sont marqués *non-engagés*;
  - **Etape 2.b, Mise-à-jour des listes:** Chaque AP f met à jour sa liste avec l'ensemble de ses proposants: $L(f) \longleftarrow L(f) \cup \{\text{proposants}\}$ and $L^*(f) \longleftarrow L(f)$;
  - **Etape 2.c, Contre-propositions:** Chaque AP f évalue l'ensemble des coalitions avec les utilisateurs de sa liste dynamique $L^*(f)$ et contre-propose aux utilisateurs de sa coalition préférée selon $P^{\#}(f)$;
  - **Etape 2.d, Acceptations/Rejets:** En se basant sur les contre-propositions reçues et sur les meilleurs paiements atteignables envoyés en Etape 1.c par les APs auxquels ils n'ont pas encore proposé, les utilisateurs acceptent ou rejettent les contre-propositions;
    - **Etape 2.e:** Si tous les utilisateur de la coalition préférée acceptent la contre-proposition d'un AP f, tous ces utilisateurs et f quittent leurs précédentes coalitions;
    - tous les joueurs de ces coalitions sont marqués *non-engagés*;
    - les utilisateurs ayant acceptés la contre-proposition et f sont marqués *engagés dans cette nouvelle coalition*;
    - **Etape 2.f:** Chaque AP f non-engagé met à jour sa liste dynamique en supprimant les utilisateurs ayant rejeté sa contre-proposition et étant engagés à un autre AP:
    - $L^*(f) \longleftarrow L^*(f) \setminus \{\text{rejetants engagés}\}$;
  - **Etape 2.g:** Retour en Etape 2.c tant que la liste dynamique L^* d'au moins un AP a diminué strictement (au sens de l'inclusion) en Etape 2.f;
  - **Etape 2.h:** Retour en Etape 2.a tant qu'il y a des utilisateurs non-engagés qui peuvent proposer;
  - **Etape 2.i:** Tous les joueurs engagés dans une coalition sont appariés.

On montre par simulations numériques que notre mécanisme ne garantit pas seulement la stabilité, mais réduit aussi l'impact de l'anomalie. En effet, l'équilibre de l'association repose sur les incitations des agents à contrer les effets collatéraux négatifs du protocole. Les gains sont significatifs, en particulier dans les scénarios de congestion où les débits individuels peuvent être



(a) Appariement stable résultant du mécanisme avec contrôle par coûts gaussiens et BDAA.

(b) Optimum global avec contrôle par coûts coûts Gaussiens.

(c) Optimum global sans contrôle.

Figure 5: Un appariement stable dans un scénario de déploiement. Comparaison des associations obtenues avec (a) BDAA, (b) une recherche de l'optimum global avec contrôle par coûts gaussiens de variance $\sigma = 0.2$, (c) une recherche de l'optimum global sans contrôle.

multipliés par trois. On observe un coût de contrôle et de stabilité qui peuvent atteindre 50% dans certains scénarios. Néanmoins, le coût de la stabilité est faible en valeurs de débits modifiés. Ces travaux constituent une première dans le champ des jeux de coalitions contrôlés pour les appariements core stables en réseaux sans fils distribués.

0.7 Mise en cache de vidéos et algorithme enumératif basé sur les cliques

Dans ce chapitre, nous analysons un problème de mise en cache de vidéos entre un créateur de contenus et un fournisseur de service en utilisant les jeux d'appariements. On donne un nouvel algorithme énumératif de structure core stables dans les jeux de coalitions à potentiels. Cet algorithme utilise le graphe d'intersection des coalitions et peut donner à tout instant (si arrêté) une énumération des appariements (ou structures si l'on considère le problème général de formation de coalitions) core stables partiellement construits.

0.7.1 Jeux de coalitions à potentiels

Soit $\Gamma = (\mathcal{N}, v, \{u_i\}_{i\in\mathcal{N}})$ un jeu de coalitions en forme caractéristique où $\mathcal{N}$ désigne l'ensemble des joueurs de cardinalité N, $\{u_i\}_{i\in\mathcal{N}}$ dénote l'ensemble de leurs fonctions d'utilité et $v : \mathcal{N} \rightarrow \mathbb{R}$ est la fonction caractéristique du jeu. On définit l'ensemble de coalitions $\mathcal{C}$. La fonction de potentiel Φ est un potentiel ordinal pour le jeu Γ si pour tout joueur $i \in \mathcal{N}$,

$$u_i(\mathrm{C}) > u_i(\mathrm{C}') \text{ ssi } \Phi(\mathrm{C}) > \Phi(\mathrm{C}'), \text{ pour chaque } \mathrm{C}, \mathrm{C}' \in \mathcal{C} \tag{18}$$

où $u_i(\mathrm{C})$ est l'utilité du joueur i pour son paiement dans la coalition C.
Un jeu de coalitions en forme caractéristique avec joueurs $\mathcal{N}$ admettant un potentiel ordinal Φ est appelé jeu de coalitions à potentiel ordinal en forme caractéristique et dénoté

$$\Gamma = (\mathcal{N}, v, \{u_i\}_{i\in\mathcal{N}}, \Phi) \tag{19}$$

À titre d'exemple, on considère un modèle de mise en cache de vidéos entre un créateur de contenus et un fournisseur de service basé sur le marchandage de Nash pour le partage des revenus générés. On tient compte de la différenciation de qualité de service des serveurs du fournisseur et de l'impact mutuel des vidéos (notamment induit par le système de recommandations). Ce jeu d'appariement est un jeu de coalitions à potentiel ordinal en forme caractéristique. Dans ce cas, le potentiel est une mesure d'aversion au risque appelée peur de la ruine.

Algorithm 3: Algorithme basé sur les cliques pour la recherche de structures stables.

Data: Le graph pondéré de compatibilité de coalitions $(\mathscr{G}, \Phi)$
Result: L'ensemble des structures stables $\mathscr{S}$
begin
  Step 1 (Initialisation);
  $\mathscr{S} := \Sigma_1$ (Σ_1 ensemble de cliques maximales de $\mathscr{G}_1$);
  $p(\mathscr{S}) = 0$ (vecteur de taille $|\mathscr{S}|$, les structures dans $\mathscr{S}$ n'ont pas encore été visitées;
  Step 2 (Formation de la forêt);
  while $\exists \mathrm{S} \in \mathscr{S}$ *t.q.* $p(\mathrm{S}) = 0$ **do**
    prendre S t.q. $p(\mathrm{S}) = 0$;
    $p(\mathrm{S}) = 1$;
    if $\mathscr{W} \subset \mathrm{S}$ *ou* $\mathscr{F} \subset \mathrm{S}$ *(si la structure inclut l'un ou les deux ensembles de joueurs, aucune coalition ne peut êtrue ajoutée)* **then**
      break;
    if $\mathscr{C}_{\mathrm{S}}^{\max} = \emptyset$ *($\mathscr{C}_{\mathrm{S}}^{\max}$ ensemble de coalitions compatibles avec* S *de valeurs maximales* **then**
      break;
    $\mathscr{S} := \mathscr{S} \backslash \mathrm{S}$;
    for $\mathrm{S}' \in \Sigma'_{\mathrm{S}}$ *(Σ'_{S} ensemble de cliques maximales dans $\mathscr{C}_{\mathrm{S}}^{\max}$)* **do**
      $\mathscr{S} := \mathscr{S} \cup \{\mathrm{S}' \cup \mathrm{S}\}$ (complète la structure par l'enfant $\mathrm{S}' \cup \mathrm{S}$);
      $p(\mathrm{S}' \cup \mathrm{S}) = 0$;

0.7.2 Un algorithme énumératif

Dans cette section, on propose un nouvel algorithme énumératif de structures core stables pour les jeux de coalitions à potentiel ordinal. Le fonctionnement de cet algorithme repose sur l'énumération de cliques du graphe d'intersections de coalitions et de matrice d'adjacence $\mathbf{A} = (a_{ij})_{(i,j)\in\mathscr{C}^2}$ telle que

$$a_{ij} = \begin{cases} 1 & \text{if } \mathrm{C}_i \neq \mathrm{C}_j \text{ and } \mathrm{C}_i \cap \mathrm{C}_j = \{\emptyset\} \\ 0 & \text{if } \mathrm{C}_i = \mathrm{C}_j \text{ or } \mathrm{C}_i \cap \mathrm{C}_j \neq \{\emptyset\} \end{cases}$$

L'algorithme construit de façon itérative une forêt telle que chaque arbre est enraciné par un noeud correspondant à une clique maximale obtenue à l'initialisation. Tout noeud est une structure stable. Un noeud qui n'est pas une feuille de l'arbre est une structure stable d'un sous-jeu (propriété d'énumération à tout instant de solutions partielles) et une feuille est une structure stable du jeu. L'ensemble des feuilles constitue les sorties de l'algorithme. On montre que cet algorithme converge en un nombre fini d'itérations et énumère l'ensemble des structures stables.

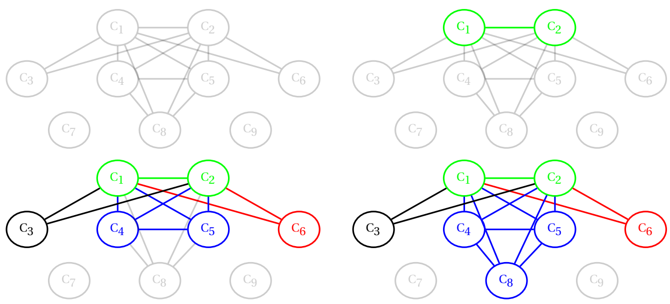


Table 1: L'algorithme de cliques maximales. L'ensemble des coalitions Σ est représenté par les noeuds du graphe. Le placement vertical d'une coalition associée au noeud dans le graphe est en accord avec le poids ω de ce noeud i.e. $[\omega_{C_1}, \omega_{C_2}] > [\omega_{C_3}, \omega_{C_4}, \omega_{C_5}, \omega_{C_6}] > [\omega_{C_7}, \omega_{C_8}, \omega_{C_9}]$. L'algorithme construit itérativement les cliques maximales depuis les poids élevés jusqu'aux plus faibles.

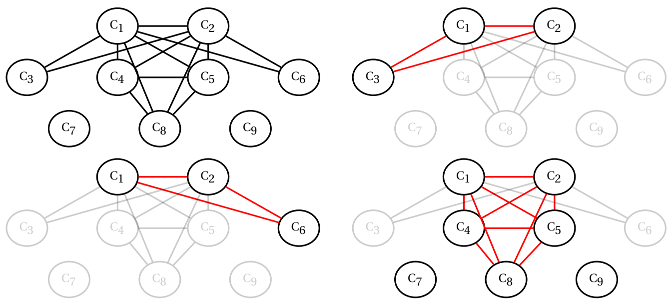


Table 2: Les cliques maximales (en rouge) construites par l'algorithme, i.e. les structures stables de coalitions.

0.8 Peur de la ruine et conditions de concavité dans les jeux de coalitions avec allocation de ressources α-equitable généralisée

Dans ce chapitre, on étudie un système multi-agents générique où les joueurs peuvent former des coalitions. La règle de partage de la valeur ou des ressources au sein d'un groupe est fixée de façon exogène. On se concentre sur l'allocation alpha-équitable généralisée définie comme la solution du problème d'optimisation suivant

$$\max_{x \in X} \nu(x) = \begin{cases} \frac{1}{1-\alpha} \sum_{i=1}^{n} \pi_i (f_i(x_i))^{1-\alpha}, & \alpha \neq 1 \\ \sum_{i=1}^{n} \pi_i \log(f_i(x_i)) & , \alpha = 1 \end{cases}$$

où $\alpha \in [0, +\infty]$, f_i sont des fonctions croissantes, concaves à valeur dans $[0, +\infty)$ et $0 \leq \pi_i \leq 1$. On analyse les conditions de concavité requises sur les fonctions d'utilité des agents pour que la règle d'allocation d'alpha-équité généralisée induise l'existence de structures ou appariements core stables. En particulier, on donne une interprétation de ces conditions en utilisant certaines mesures d'aversion au risque telles que la peur de la ruine déjà utilisée en chapitre 5 et 6. Ce chapitre a pour objectif de renforcer (en complément des chapitres 5 et 6) les liens existant entre des schémas d'équité connus dans les réseaux et l'analyse de l'allocation de ressources menée en théorie des jeux.

Dans les jeux, il est connu que l'existence d'équilibres et leurs propriétés dépendent des préférences des agents ou de leurs fonctions d'utilité. Certaines de ces propriétés ont des interprétations comportementales. Par exemples, une fonction d'utilité croissante concave modélise un joueur attiré par les gains mais averse au risque. En ce sens, plusieurs mesures d'aversion au risque ont été définies pour quantifier les comportements dans des processus de prise de décision en présence d'incertitude. On se concentre sur l'aversion au risque, l'audace, la peur de la ruine et la pure peur de la ruine. On montre que ces indicateurs peuvent être utilisés dans l'analyse de schémas d'équité connus. En effet, de tels indicateurs apparaissent naturellement dans l'analyse des conditions de concavité de l'alpha-équité généralisée pour l'existence d'appariements core stables. Cette étude établit le premier lien entre l'allocation alpha-équitable généralisée, les appariements stables et les mesures d'aversion au risque. On montre également une nouvelle interprétation de la peur de la ruine d'un joueur en tant que limite d'une séquence de ratios de probabilités de paris à gains décroissants.

La peur de la ruine est définie comme l'inverse de l'audace qui est la limite pour un gain allant vers zéro de la probabilité de ruine par unité de gain qui laisse indifférente le joueur entre entrer dans un pari asymétrique avec risque de tout perdre ou gagner ce gain. Plus cette probabilité est grande, plus le joueur est prêt à tout perdre, d'où la notion d'audace. Plus formellement, on a les définitions suivantes de la peur de la ruine et de la pure peur de la ruine.

Definition. *La peur de la ruine du joueur i de fonction d'utilité f_i au point d'allocation x est définie telle que:*

$$\mathrm{FoR}_{f_i}(x) = \frac{f_i(x)}{f_i'(x)} \tag{20}$$

Definition. *La pure peur de la ruine du joueur i de fonction d'utilité f_i au point d'allocation x est définie telle que :*

$$\mathrm{PFoR}_{f_i}(x) = -\frac{f_i''(x) f_i(x)}{(f_i'(x))^2} \tag{21}$$

On observe la relation suivante $\mathrm{FoR}'_{f_i}(x) = 1 + \mathrm{PFoR}_{f_i}(x)$, d'où

$$\mathrm{FoR}_{f_i}(x) = \int_0^x \mathrm{PFoR}_{f_i}(s)\,ds + \mathrm{FoR}_{f_i}(0) \tag{22}$$

Les conditions de concavité pour l'existence de structures core stables appliquées à l'alpha-équité généralisée donnent les résultats suivants.

Proposition. *La fonction d'utilité u_i de tout joueur $i \in \mathcal{N}$ est strictement log-concave dans la forme produit équivalente du problème d'optimisation de l'alpha-équité généralisée si:*

$$\mathrm{PFoR}_{f_i} > -\alpha \tag{23}$$

Proposition. *La fonction d'utilité u_i de tout joueur $i \in \mathcal{N}$ est croissante dans la forme produit équivalente du problème d'optimisation de l'alpha-équité généralisée pour tout* α.

Proposition. *La fonction d'utilité u_i de tout joueur $i \in \mathcal{N}$ est concave dans la forme produit équivalente du problème d'optimisation de l'alpha-équité généralisée si:*

$$\mathrm{PFoR}_{f_i}(x_i) \geq \pi_i f_i(x_i)^{1-\alpha} - \alpha \tag{24}$$

Ainsi, ces conditions appliquées au cas considéré s'expriment simplement en terme d'une mesure d'aversion au risque. Ces mesures quantifient l'aversion pour un joueur à entrer dans une situation où il y a une probabilité de perte de ressources. En particulier, on donne une nouvelle interprétation de la peur de la ruine selon le résultat suivant.

Proposition. *Supposons un joueur i de fonction d'utilité u_i et de ressource x.*

$$\mathrm{FoR}_i(x) = \lim_{n\to+\infty} x\left(1 + 2\sum_{k=1}^{n} \frac{1}{n}\frac{p_{k,n}}{q_{k,n}}\right) + \mathrm{FoR}_i(0) \tag{25}$$

où $p_{k,n}$ is la prime en probabilité dans un pari symétrique $\{-g(x,k,n), +g(x,k,n)\}$ à $k\frac{x}{n}$, $q_{k,n}$est la probabilité de ruine qui rend le joueur indifférent entre un pari où il risque la ruine et peut gagner $+g(x,k,n)$ à $k\frac{x}{n}$ et $g : \mathbb{R}^+ \times \mathbb{N} \times \mathbb{N} \to \mathbb{R}^+$ une fonction telle que, $\forall n \in \mathbb{N}^*, \forall k \in \{1,\dots,n\}$:

$$g(x,n,k) \leq (k-1)\frac{x}{n} \;\; et \;\; \lim_{n\to+\infty} g(x,n,k) = 0 \tag{26}$$

Une interprétation de ce résultat peut être donnée en terme d'investissement ou coût cumulé sur le chemin de la ruine.

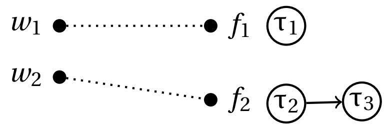


Figure 6: Un marché biface de production participative avec contrats et contraintes d'ordonnancement internes. Les lignes pointillées entre deux agents montrent qu'il existe un contrat entre eux dans l'ensemble des contrats.

0.9 Jeux d'appariement et crowdsourcing

Dans ce chapitre, on analyse par les jeux d'appariement stables (plusieurs-à-plusieurs avec contrats et externalités) une place de marché de production participative (crowdsourcing) avec externalités et contraintes de planification ou ordonnancement (interne et externe) de tâches pour les entreprises. On introduit la propriété de stabilité dans cette place de marché comme concept de solution pour l'allocation des tâches. La stabilité est une propriété essentielle au maintien de la participation à long terme des agents dans les mécanismes de marchés bifaces tel quel l'admission des internes dans les hôpitaux. La résolution de ce problème permet la conception de plateformes de production participative plus riches, et complètes en termes d'opportunités pour les agents (planification pour les entreprises) et d'incitation des agents à la participation par respect de leurs préférences. On montre que le problème considéré ne peut être traité par les résultats de la théorie des jeux d'appariements stables. On propose donc de nouveaux résultats complétant la théorie pour le cadre des appariements contraints. En particulier, on introduit la notion de substituabilité contrainte pour traiter le problème de la non-substituabilité des préférences au sens classique. On montre également les conditions d'existence d'un appariement stable pair-à-pair obtenu comme le point fixe d'un algorithme récemment donné dans la littérature, le Modified Deferred Acceptance Algorithm. On définit également d'autres stabilités adaptées à ce problème et on propose une approche visant à unifier les jeux coopératifs d'appariements stables et les jeux non-coopératifs. Ces derniers résultats reposent sur une transformation du problème initial en un jeu non-coopératif en forme normale ou extensive.

On considère les ensembles finis des entreprises $\mathscr{F}$, travailleurs $\mathscr{W}$, tâches $\mathscr{T}$ et les contraintes de planification ou ordonnancement pour les firmes données par un graphe dirigé $\mathscr{G} = (\mathscr{T}, \mathbf{A})$ où $\mathbf{A}$ est la matrice d'adjacence. L'ensemble des contrats est $\mathscr{X}$ et on note X_i l'ensemble des contrats de $X \subseteq \mathscr{X}$ impliquant le joueur i.

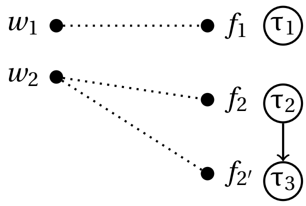


Figure 7: Un marché biface de production participative avec contrats et contraintes d'ordonnancement externes. Les lignes pointillées entre deux agents montrent qu'il existe un contrat entre eux dans l'ensemble des contrats.

On définit les notions de faisabilité d'un ensemble de contrats et d'ensemble maximal de contrats faisables qui sont utilisées pour définir les fonctions de choix des entreprises.

Definition. *Soit une entreprise $f \in \mathscr{F}$, un ensemble de contrats $X \subseteq \mathscr{X}_f$ et un appariement μ, l'ensemble de contrats X est dit faisable pour f à μ si pour tout contrat $x \in X$, il existe un sous-ensemble de contrats $X' \subseteq X$ avec $x \in X'$ et un contrat par tâche dans $\mathscr{T}(X')$ tel que les prédécesseurs de toute tâche dans X' sont dans $\mathscr{T}(X' \cup \mu_{-f})$.*

Definition. *Soit une firme $f \in \mathscr{F}$, un ensemble de contrats $X \subseteq \mathscr{X}_f$ et un appariement μ, l'ensemble de contrats X est appelé ensemble maximal de contrats faisables pour f at μ si X est faisable et de* *cardinalité maximale. L'ensemble maximal de contrats faisables pour* f *dans* $X \subseteq \mathscr{X}_f$ *à* μ *est dénoté* $X(\mu)$.

Une tâche $\tau \in \mathscr{T}$ est dite *faisable* dans X à μ si il existe un contrat faisable $x \in X$ à μ tel que $\mathscr{T}(x) = \tau$.

Definition. *On définit* $\underline{X}_f(\mu)$, *l'ensemble des contrats de salaires minimaux dans* $X_f(\mu)$ *pour chaque tâche faisable de* $\mathscr{T}(X_f(\mu))$.

Dans ce chapitre, on suppose des entreprises à la recherche d'une maximization des profits par minimisation des coûts d'execution et donc des rémunérations. Il s'agit d'une hypothèse classique de la littérature. Etant donné un ensemble de contrats, l'entreprise choisit pour chaque tâche faisable le contrat de salaire minimum. On a la définition suivante.

Definition 1. *La fonction de choix de toute entreprise* $f \in \mathscr{F}$ *est définie de la façon suivante,*

$$c_f(X|\mu) = \bigcup_{\tau \in \mathscr{T}(X_f(\mu))} \underset{\substack{x \in X_f(\mu) \\ t.q. \tau(x) = \tau}}{\arg\min}\, s(x) \tag{27}$$

$$= \underline{X}_f(\mu) \tag{28}$$

On a donc la fonction de choix des entreprises C^F telle que,

$$C^F(X|\mu) = \bigcup_{f \in \mathscr{F}} \underline{X}_f(\mu) \tag{29}$$

0.9.1 Substituabilité contrainte et existence d'appariements stables

Dans cette section on montre qu'il existe un appariement stable pair-à-pair dans le problème d'appariement pour production participative avec contrats, externalités et contraintes de planifiication si les fonctions de choix des agents satisfont certaines conditions. Plus particulièrement, on se concentre sur la propriété de substituabilité et on définit la substituabilité contrainte qui, en plus d'une amélioration des conditions de marché, requiert une structure spécifique des ensembles de contrats faisables. L'intuition est la suivante: un contrat rejeté par une entreprise f dans un ensemble X à μ doit continuer à être rejeté dans un ensemble X' contenant X à μ' (avec plus de faisabilité pour l'entreprise f) si (i) la tâche n'est pas faisable dans X à μ et dans X' à μ' où est faisable dans X à μ mais n'est pas de salaire minimum pour la tâche correspondante (et n'est donc pas choisie par définition des fonction de choix des entreprises).

Definition. *La fonction de choix* C^F *satisfait la propriété de substituabilité contrainte si pour tout* $X, X', \mu, \mu' \subseteq \mathscr{X}$, *tels que*
C1. $X \subseteq X'$
C2. $\mu' \succeq^F \mu$
C3. $\forall f \in \mathscr{F}, \mathscr{T}^f_{\substack{X \to X' \\ \mu \to \mu'}} = \emptyset$ *ou* $X_f(\mathscr{T}^f_{\substack{X \to X' \\ \mu \to \mu'}}) \not\subseteq \underline{X}'_f(\mu')$
alors,

$$R^F(X'|\mu') \supseteq R^F(X|\mu) \tag{30}$$

où, $\mathscr{T}^f_{\substack{X \to X' \\ \mu \to \mu'}} = \left[\mathscr{T}(X_f) \backslash \mathscr{T}(X_f(\mu))\right] \cap \mathscr{T}(X'_f(\mu'))$.

On peut montrer que la fonction de choix C^F définie précédemment satisfait la substituabilité contrainte et la propriété de non-pertinence des contrats rejetés.

Pour montrer l'existence d'appariements stables dans ce problème, on utilise un algorithme connu appelé Modified Deferred Acceptance Algorithme. Cet algorithme est partiellement défini par l'application itérée d'une fonction $f : 2^{\mathscr{X}} \times 2^{\mathscr{X}} \times 2^{\mathscr{X}} \times 2^{\mathscr{X}} \to 2^{\mathscr{X}} \times 2^{\mathscr{X}} \times 2^{\mathscr{X}} \times 2^{\mathscr{X}}$ telle que pour tout quadruplet $A^F, A^W, \mu^F, \mu^W \subseteq \mathscr{X}$

$$f(A^F, A^W, \mu^F, \mu^W) = (\mathscr{X} \backslash R^W(A^W|\mu^W), \mathscr{X} \backslash R^F(A^F|\mu^F), C^F(A^F|\mu^F), C^W(A^W|\mu^W)) \tag{31}$$

où $R^{\theta}(.|.)$ dénote la fonction de rejet des joueurs de la face $\theta \in \{\mathscr{F}, \mathscr{W}\}$ du marché. Afin de garantir que la condition C3 de la substituabilité contrainte est satisfaite à toute itération de f au cours de l'algorithme on définit la condition C4 suivante,

C4. Pout tous les ensembles $A^F, A^W, \mu^F, \mu^W \subseteq \mathscr{X}$, l'image $f(A^F, A^W, \mu^F, \mu^W) = (\tilde{A}^F, \tilde{A}^W, \tilde{\mu}^F, \tilde{\mu}^W)$ est telle que $A^F, \tilde{A}^F, \mu^F, \tilde{\mu}^F$ satisfait

$$\mathscr{T}^{f}_{\substack{\tilde{A}^F \to A^F \\ \tilde{\mu}^F \to \mu^F}} = \emptyset \text{ ou } \tilde{A}^F_f(\mathscr{T}^{f}_{\substack{\tilde{A}^F \to A^F \\ \tilde{\mu}^F \to \mu^F}}) \not\subseteq \underline{A}^F_f(\mu^F) \tag{32}$$

Par définition de f, cette condition implique que les fonctions de choix des agents doivent être telles que les ensembles générés par f satisfont C3. On montre le résultat d'existence d'appariements stables suivant,

Theorem 2. *Supposons que la fonction de choix* C^W *satisfasse les conditions de substitutabilité et de non-pertinence des contrats rejetés. Supposons que les fonctions de choix des firmes* $\{c_f\}_{f \in \mathscr{F}}$ *soient définies telles qu'en Definition 27. Finalement, supposons que* f *satisfasse la condition C4 à toute itération de l'algorithme. Alors l'algorithme converge, sa sortie est stable et*

$$\mu^F(T) = \mu^W(T) = A^F(T) \cap A^W(T) \tag{33}$$

0.9.2 Le Problème de crowdsourcing en formes non-coopératives

On considère deux formulations alternatives du problème. Une première en forme normale et une seconde en forme extensive. L'objectif de ces travaux est d'explorer le lien entre le théorie des jeux non-coopérative et les appariements stables pour l'étude des marchés bifaces. Cette question a déjà été abordée dans la littérature et il existe des jeux de congestions spécifiques dont les équilibres non-coopératifs sont des mariages stables.

Le premier axe d'analyse que nous proposons transforme le problème d'allocation du marché biface en un jeu à un coup en forme normale de proposition de contrats avec les entreprises en tant que joueurs et les travailleurs comme receveurs des propositions. Les travailleurs sont implicitement modélisés par une fonction de réponse Ψ du marché.

Definition. *La fonction de réponse des travailleurs est la fonction* $\Psi : \times_{i \in \mathscr{N}} \mathscr{S}_i \to 2^{\mathscr{X}}$ *de l'ensemble des profils de stratégies des firmes (propositions de contrats) dans l'ensemble des sous-ensembles de contrats (ensemble des appariements) induits par la fonction de choix* C^W *des travailleurs telle que, pour tout profil de stratégies* $s \in \mathscr{S}$, $\psi(s \in \mathscr{S})$ *est la limite de la séquence,*

$$\mu^{(0)} = \emptyset \tag{34}$$

$$\mu^{(1)} = C^W(s|\mu^{(0)}) = \bigcup_{w \in \mathscr{W}} c_w(s|\mu^{(0)}) \tag{35}$$

$$\mu^{(2)} = C^W(s|\mu^{(1)}) \bigcup_{w \in \mathscr{W}} c_w(s|\mu^{(1)}) \tag{36}$$

$$\vdots \tag{37}$$

$$\Psi(s) = \mu^{(k)} = C^W(s|\mu^{(k-1)}) = \bigcup_{w \in \mathscr{W}} c_w(s|\mu^{(k-1)}) \tag{38}$$

On montre en particulier qu'il peut ne pas exister d'équilibre de Nash en stratégies pures dans ce jeu mais qu'il en existe toujours en stratégie mixte par application directe du théorème d'existence de Nash. L'interprétation peut être faite en termes d'appariements fractionnels en temps ou probabilités, concept connu et étudié en théorie des appariements stables. À l'équilibre (de Nash) de ce jeu, il n'y a pas d'entreprise qui puisse générer une séquence de choix des travailleurs convergeant vers une réponse (et donc un matching fractionnaire) qui améliorerait strictement l'utilité espérée. Intuitivement, on comprend que cette formulation par fonction de réponse résultant d'une séquence de choix des travailleurs introduit une forme d'anticipation par les entreprises.

Le deuxième axe d'analyse définit le problème de production participative avec contrats et externalités comme un jeu non-coopératif en forme extensive. L'idée est d'introduire une prise de décision séquentielle dans la forme normale précédente. L'intérêt du jeu séquentiel réside entre autres dans le concept d'équilibre parfait en sous-jeux qui rend non-pertinentes les stratégies auto-pénalisantes. On peut trouver des exemples d'appariements stables pour lesquelles de tels choix sont l'origine de déviations et d'instabilités. Dans la transformation en forme extensive que nous proposons, les joueurs sont les entreprises et les travailleurs sont modélisés par une fonction de réponse Ψ (comme dans la formulation en forme normale précédente), l'information est parfaite et l'ordre de jeu est défini par des priorités induites par les contraintes de planification $\mathscr{G} = (\mathscr{T}, \mathbf{A})$. La transformation en forme extensive n'est a priori pas équivalente au problème d'appariement stable original mais possède des propriétés intéressantes en terme d'existence et caractéristiques des équilibres. En effet, un résultat connu montre l'existence systématique d'équilibres parfaits en sous-jeux dans notre problème. Il est ainsi possible de garantir l'existence de certains appariements d'équilibres pour une prise de décision séquentielle à information parfaite. Ces équilibres ne permettant pas l'utilisation de stratégies auto-pénalisantes et de déviations non-crédibles.

Chapter 1

Introduction

In recent years, the development of networks and innovations in connected services have given companies and users the opportunity to create value, consume and communicate in new ways. As examples, online search engines (e.g. Google, Qwant) allow users to efficiently find what they were looking for in an permanently growing number of webpages or online resources and social networks (e.g. Facebook, LinkedIn) allow users to communicate in a rich environment through global and numerous interactions such as basic text communications, content sharing (photos, videos) as well as online gaming. Such systems have become ubiquitous and their impact is so important that firms or politics have introduced them in their communication and marketing strategies. One of the main consequences is the emergence of an online advertising activity creating worth from the sell of empty spaces and the users' clicks or views in the browsed webpages. As another example, online platforms have emerged as a new form of virtual marketplaces where millions of users can trade goods or services. More generally, business opportunities have appeared, allowing the economic agents to enter new Business-to-Business (B2B, e.g. online advertising), Business-to-Consumer (B2C, e.g. Amazon) or Consumer-to-Consumer (C2C, e.g. Airbnb) interactions and commercial transactions.

In these systems, demand and supply create two-sided markets allowing for an online competition among the agents over the resources. Many of these markets are regulated by rules and mechanisms defining the way the agents compete. As an example of a well-known marketplace consider eBay where buyers and sellers meet and compete through a time-limited auction mechanism. At the deadline, the agent with highest bid wins the auction and pays the bid. Such marketplaces have become not only ubiquitous but also preferred to traditional ones by many economic agents because of their simplicity (any user can buy a proposed good or service from any place in the world at any time). In fact, online systems benefit from the worldwide connectivity of the internet and high performance communication networks (fixed or wireless) since these allow the use of the services by many agents and a fast diffusion of a massively demanded information. Economic models, consumers habits and technology have jointly changed and combined in new solutions and systems placing networks at the heart of a revolution. As a consequence, new problems have emerged such as the management of huge volumes of datas, the relevance of the transmitted information or the automation of decision-taking by machine learning.

However, even though there is an increasing amount of marketplaces, these and some of their mechanisms are not new and have been extensively studied in the economic, game-theoretic and networking literature. A successful cooperative game theoretic approach, called theory of stable matchings, was originated by Gale and Shapley and further extended by A.E. Roth and other contributors. The results revealed unknown properties of existing allocation mechanisms and led to a powerful methodology to study and develop new ones. Thus, the impact is important both from a theoretical point of view and from a practical one because they have been used to design existing mechanisms. As examples, consider matching mechanisms assigning students to colleges in Boston and New-York, interns to hospitals (National Resident Matching Program) or the national kidney exchange program in the US providing hospitals the incentive to pool the organs to create exchange cycles increasing the number of transplants.

In this thesis, we study the link between some network problems and the game-theoretic analysis of two-sided markets. Particularly, we tackle four network problems using the theory of stable matchings. In chapter 2, we give a brief introduction to cooperation in game theory with a particular focus on the bargaining problem and Nash's solution. In chapter 3, we introduce the theory of stable matchings, or matching games, used throughout this thesis to tackle the network problems. In chapter 4, we show that under some conditions, the WiFi protocol can be modeled as a Nash bargaining. In chapter 5, we study the WiFi connectivity management problem. We show that this system can actually be formulated as a two-sided marketplace with a cooperative resource allocation scheme and naturally falls in the scope of the game-theoretic analysis. Using the results of stable matchings, we propose a new stable matching mechanism reducing the impact of congestion and the anomaly in the protocol, well-known issues in performances due to a mutual impact of the communicating agents over each others. A control step is introduced in the mechanism to provide the agents the incentives for coalitions of well-defined cardinalities, thus manipulating the preferences over one-to-one and many-to-one matchings. In chapter 6, we apply the previous framework to caching. We propose a new stable caching mechanism in networks between a content provider's videos and an operator's servers. The mechanism takes into account both the mutual impact of the videos over each others and the gain in quality induced by differentiated servers. In the proposed model, we assume a Nash bargaining over the generated revenues and consider general coalition potential games. We define a new core stable anytime enumerative algorithm giving the set of core stable structures. To go further in the understanding of the link between resource allocation and stability, in chapter 7, we study the generalized alpha-fair allocation scheme. We use recent game-theoretic results to show that the concavity conditions on the players' utility functions for the existence of core stable matchings can be simply formulated in terms of risk aversion indicators. Among other results, this study has led to new interpretations of the fear-of-ruin. In chapter 8, we study a two-sided crowdsourcing market with contracts and scheduling constraints over the tasks. As in the analysis of the firms and workers stable hiring problem in classical stable matchings, we assume a pairwise stability in the problem as solution concept. We show sufficient conditions for the existence of stable matchings and analyze the problem using a non-cooperative reformulation of the matching game. This problem is of fundamental importance in the design of more complex and complete crowdsourcing platforms allowing coordination in and among firms in the distribution of the tasks. Finally, in chapter 9, we show some open questions and problems.

Chapter 2

Cooperation and Bargainings

In this chapter, we introduce cooperative game theory. We focus on bargaining problems and Nash's solution that will be used in conjunction with matching games in chapters 4 and 5 to study the WiFi resource allocation and user association problem as well as in chapter 6 to develop a video caching mechanism.

2.1 Cooperation, Negotiation and Arbitration

The bargaining problem is a long-date topic that has attracted much attention since its game-theoretic formalization by Nash in [1] and [2]. Basically, a bargaining can be considered as a process solving a cooperative competition between the rational players (utility maximizers): even though each seeks for maximizing his payoff, they all consider joint opportunities. Compared to the non-cooperative competition between rational decision-makers, the notion of cooperative competition induces the ideas of the mutual profits, fairness, communication and agreement.

More generally, when taking part in a cooperative game, the players keep on considering the strategic non-cooperative options but are also provided a set of new strategic opportunities that allow for mutual agreements. This transformation is called a *cooperative transformation* of the game. Based on this interpretation of cooperation, one may expect alternative or new solution concepts to be defined. This is not mandatory because the cooperative transformation only consists in the introduction of additional strategic options. The fact that these new opportunities induce a kind of synchronization or agreement among some agents does not change the agents' decision-taking problem : given the game, which strategy to play?

Such transformation from isolated decision-takers to a richer framework with cooperative strategies does not necessarily go along with the development of new solution concepts. The Nash equilibrium can still be considered as the adapted solution concept in the cooperatively transformed game (see [3], pp.371 and references therein). Nevertheless, because of the introduction of these new strategic options the set of Nash equilibria may change. Particularly, as shown by Myerson in [3] (pp.371), in a game with contract-signing any individually rational correlated equilibrium is a Nash equilibrium. There appears the fact that cooperation, because of its additional set of strategic opportunities, induces an equilibrium selection problem. This problem underlies the negotiation (arbitration, bargaining) theory which actually is the theory of cooperative equilibrium selection.

When a single equilibrium exists, it is clear that the players expect each others to implement the corresponding strategies because there is no alternative choice satisfying the constraints. In other situations, the players may face the existence of many equilibria and the following questions may be raised: among these, which equilibrium do the players expects the others to play? More precisely, is there a particular equilibrium that any player expect the others to play? This attractiveness property for some equilibrium point has been studied by Schelling who defined the notion of *focal-point effect* and *focal equilibrium*.

Definition 3 (Focal-point effect, [3], pp.108)**.** *In a game with multiple equilibria, the focal-point effect is the property that anything that tends to focus the players' attention on one equilibrium may make them all expect it and hence fulfill it (like a self-fulfilling prophecy).*

Definition 4 (Focal-point equilibrium, [3], pp.108)**.** *A focal-equilibrium is an equilibrium that has some property that conspicuously distinguishes it from all the other equilibria. According to the focal-point effect, if there is one focal equilibrium in a game, then we should expect to observe that equilibrium.*

This focality property may be contained in the game itself (e.g. in the utilities by some specific fairness, welfare, equity), in its environment (e.g. cultural traditions) or it may be controlled by an entity or an individual called focal arbitrator.

Definition 5 (Focal arbitrator, [3], pp.111)**.** *An individual is a focal arbitrator if he can determine the focal equilibrium in a game by publicly suggesting to the players that they should all implement this equilibrium.*

By definition, there is no bidding force in the recommendation of the focal arbitrator. His power lies in the ability to make each player expect that the others would follow his suggestion. In such case, because the recommendation is an equilibrium, then the players would play the advised strategy. As an example of focal-point effect induced by fairness, we consider the following game called Divide the Dollars.

Example 6 (Divide the Dollars)**.** *Player* 1 *and player* 2 *demand for an amount of money between* 0$ *and* 100$*. If the sum is inferior or equal to* 100 *each gets his demand. If not, both get* 0$*. The pure strategy sets are,*

$$C_1 = C_2 = [0, 100] \tag{2.1}$$

Any pure strategy of the form $(x, 100 - x)$ *is a Nash equilibrium. In fact, if* 1 *plays* $x' > x$ *(and* 2 *keeps on playing* $100 - x$*) then the payoff is null and if* 1 *plays* $x' < x$ *then* 1 *receives* x'*, less than* x*. One can show the same for* 2 *with the symmetric strategies* $x' < x$ *and* $x' > x$*. Thus, at the decision point* $(x, 100 - x)$*, none of the player has the incentive to change. Among these, the equilibrium* $(50, 50)$ *is fair and equitable (*100$ *equally divided among the two identical players). If there were an impartial arbitrator in the game he would surely suggest this equilibrium as a solution. This makes* $(50, 50)$ *very specific with respect to (w.r.t.) the other equilibriums and the fairness property.*

Communication is of fundamental importance in cooperation, particularly in the equilibrium selection process. In case of existence of an arbitrator (whatever out of the game or a player) the focal equilibrium is obtained by a communication from this arbitrator to the players. If such entity does not exist, then either the incentive for the focal equilibrium is individually experienced by the players or results from a communication among them. Such communication is intended to happen before the players play the game because the focality must be intrinsically contained in the corresponding equilibrium at the decision-taking epoch. This preplay communication process is called *focal negotiation*. Finally, game theorists have defined an *equity hypothesis* asking for an equivalence between focal negotiation and impartial arbitration.

An example of such equilibrium selection process has been developed by Nash who defined the set of axioms an impartial arbitration function (mapping each cooperative transformation to a solution) should satisfy. Nash's arbitration is defined by specific fairness criteria, that surprisingly correspond to the proportional fair allocation later on developed in networks by Kelly in [4]. Since then, many works have been devoted to the generalization to n-players of Nash's result and to the derivation of alternative solutions. Other works study the bargaining problem with incomplete information and in repeated settings. Finally, learning algorithms such as fictitious play have been proposed as dynamic bargaining schemes, see [3] and [5].

2.2 The Nash Bargaining Solution

2.2.1 Model

Nash's model of bargaining has been built on the assumption that the result of the negotiation among players should be a function of the set of jointly achievable opportunities measured in utility and some threats that would block the bargaining process in case of disagreement and guarantee each player a given known amount of utility.

Let 1 and 2 be two players and $B \subset \mathbb{R}^2$ be a compact[1] convex[2] set of jointly achievable utility points, called *feasible set* or prospect space in [6]. Let $\mathbf{t} = (t_1, t_2)$ be a vector in $\mathbb{R}^2$ called the *threat vector* or *disagreement point*. The set B and the vector $\mathbf{t}$ are such that,

$$B \cap \{\mathbf{x} \in \mathbb{R}^2 | x_1 \geq t_1, x_2 \geq t_2\} \neq \emptyset \tag{2.2}$$

The pair $(B, \mathbf{t})$ defined the two-person bargaining problem.

2.2.2 Solution Concept

In [1] and [2], Nash developed an axiomatic formulation of the solution to the two-players bargaining problem. The intuition is that a focal negotiation (impartial arbitration) among players should satisfy a set of rules that would in some sense correspond to a fair share of the utility among the

[1] Closed and bounded

[2] $\forall \mathbf{x}, \mathbf{y} \in B, \forall \lambda \in [0, 1], \lambda x + (1 - \lambda) y \in B$

players. Let define the solution function Φ mapping any game $(\mathrm{B},\mathbf{t})$ to a utility vector in $\mathrm{B}\cup\mathbf{t}$. The proposed axioms are the following,

- **Axiom 1: (Strong Efficiency)**
  $\Phi(\mathrm{B},\mathbf{t})$ is an allocation in B, and, for any $\mathbf{u}$ in B, if $\mathbf{u}\geq\Phi(\mathrm{B},t)$, then $\mathbf{u}=\Phi(\mathrm{B},\mathbf{t})$.

- **Axiom 2: (Individual Rationality)**
  $\Phi(\mathrm{B},\mathbf{t})\geq\mathbf{t}$.

- **Axiom 3: (Scale Invariance)**
  For any numbers λ_1, λ_2, γ_1, and γ_2 such that $\lambda_1>0$ and $\lambda_2>0$, if $\mathrm{G}=\{(\lambda_1 u_1+\gamma_1,\lambda_2 u_2+\gamma_2)|(u_1,u_2)\in\mathrm{B}\}$ and $\omega=(\lambda_1 t_1+\gamma_1,\lambda_2 t_2+\gamma_2)$, then

$$\Phi(\mathrm{G},\omega)=(\lambda_1\Phi_1(\mathrm{B},\mathbf{t})+\gamma_1,\lambda_2\Phi_2(\mathrm{B},\mathbf{t})+\gamma_2). \tag{2.3}$$

- **Axiom 4: (Independence of Irrelevant Alternatives)**
  For any closed convex set G, if $\mathrm{G}\subseteq\mathrm{B}$ and $\Phi(\mathrm{B},t)\in\mathrm{G}$, then $\Phi(\mathrm{G},t)=\Phi(\mathrm{B},t)$.

- **Axiom 5: (Symmetry)**
  If $v_1=v_2$ and $\{(u_2,u_1)|(u_1,u_2)\in\mathrm{B}\}=\mathrm{B}$, then $\Phi_1(\mathrm{B},\mathbf{t})=\Phi_2(\mathrm{B},\mathbf{t})$.

Axiom 1 asks for feasibility and Pareto efficiency. Axiom 2 asks for individual rationality, no player should be recommended to play in a way that gives him less that what he is able to guarantee for himself (his threat). The following axioms 3, 4 and 5 are called axioms of fairness. Axiom 3 asks for the invariance of the solution under an affine (positive) transformation of the utility. In other words, the solution function is left unchanged if the performance objectives are scaled linearly. This requirement is built on the decision-theoretic equivalence between a utility scale and an affine transformation of this. Axiom 4 asks for an invariance of the solution point to the elimination of non-equilibrium allocation points. This property has attracted a lot of attention and alternative axioms to Axiom 4 have been proposed. A well-known alternative is the Kalai-Smorodinsky solution, which considers the individual monotonicity property instead of the independence of irrelevant alternatives. Axiom 5 asks for a symmetric allocation point if the game is symmetric. In fact, in an symmetric setting, it seems reasonable that an impartial arbitrator chooses a symmetric equilibrium. In other words, no player should dominate another (in payoffs) if the cooperative strategies do not exhibit any form of dominance. Nash has shown that there exists a unique solution satisfying Axioms 1 to Axiom 5. Furthermore, he has shown that this solution solves a max-product optimization program, called *Nash product*.

Theorem 7 ([3])**.** *There is a unique solution function $\Phi(.,.)$ that satisfies Axioms 1 through 5 above. This solution satisfies, for every two-person bargaining problem $(\mathrm{B},\mathbf{t})$,*

$$\Phi(\mathrm{B},\mathbf{t})\in\arg\max_{\mathbf{u}\in\mathrm{B},\mathbf{u}\geq\mathbf{t}}(u_1-t_1)(u_2-t_2) \tag{2.4}$$

This result has been generalized to n-dimensions (players) where only the grand coalition can benefit from cooperation (see [7], pp.34).

Theorem 8. *There is a unique solution function $\Phi(.,.)$ that satisfies Axioms 1 through 5 above. This solution satisfies, for every n-players bargaining problem $(\mathrm{B},\mathbf{t})$,*

$$\Phi(\mathrm{B},\mathbf{t})\in\arg\max_{\mathbf{u}\in\mathrm{B},\mathbf{u}\geq\mathbf{t}}\prod_{i=1}^{\mathrm{N}}(u_i-t_i) \tag{2.5}$$

A mathematically not surprising but historically and technologically interesting result is that Nash's solution is proportional fair. In fact,

$$\log\max_{\mathbf{u}\in\mathrm{B},\mathbf{u}\geq\mathbf{t}}\prod_{i=1}^{\mathrm{N}}(u_i-t_i)=\max_{\mathbf{u}\in\mathrm{B},\mathbf{u}\geq\mathbf{t}}\log\left(\prod_{i=1}^{\mathrm{N}}(u_i-t_i)\right)=\max_{\mathbf{u}\in\mathrm{B},\mathbf{u}\geq\mathbf{t}}\sum_{i=1}^{\mathrm{N}}\log(u_i-t_i)) \tag{2.6}$$

The interpretation of this equivalence is that this so-called equilibrium selection process allocates the utility among the players in a proportional fair way. A detailed analysis of this unified approach between this game-theoretic allocation scheme and fairness schemes used in networks can be found in [8],[9] and references therein. In Chapter 7, we will use this result to analyze the link between the Nash bargaining, stable matchings and the generalized α-fair allocation.

Even though involving a basic log-transformation, this result establishes that Kelly's well-known, celebrated and implemented proportional fair allocation can be interpreted as an emulated Nash equilibrium selection in an underlying cooperative game. Such connection between the game theoretic bargaining problem, related solution concepts and developed fairness schemes in networks appear is very interesting. Even though known and studied both on the game-theoretic and networking communities, the two approaches can probably still mutually benefit from each others in many other directions that the existing approaches. How could repeated bargaining be used in networks? Why not using the existing learning schemes as basis of the protocols for resource allocation? Could allocation schemes developed for stochastic networks be formalized in the game-theoretic framework as equilibrium selection process with cooperative stochastic games?

The previous results have considered the transformation of a non-cooperative game where the players receive a utility from playing a strategy profile to an extended structure with new strategic opportunities. In such cooperative game, Nash's formalization considers a set of jointly achievable utility points. We now focus on a particular subset of these games where this joint set is induced by the mapping of the players' utility functions $(u_i)_{i\in\mathcal{N}}$ from a set of jointly achievable payoff allocation (e.g. money) S in $\mathbb{R}^N$. Such mapping defines the set of joint achievable utility points. Because the utility functions are assumed concave and increasing, there is a one-to-one mapping between Nash's solution in the utility space and the payoff allocation.

Consider a compact and convex payoff space $\mathcal{S}$ and the set of concave and upper-bounded utility functions $\{u_i\}$. Let define the function $f:\mathbb{R}^n \to \mathbb{R}^n$ as,

$$f(s) = (u_i(s_i))_{i\in\{1,\dots,n\}} = (u_1(s_1),\dots,u_n(s_n)),\ \forall s \in \mathcal{S} \tag{2.7}$$

The image $\mathcal{U}$ of $\mathcal{S}$ by f is called the *set of jointly achievable utilities,*

$$\mathcal{U} = \left\{u \in \mathbb{R}^n | s \in \mathcal{S}, u = f(s) = (u_1(s_1),\dots,u_n(s_n))\right\} \tag{2.8}$$

The point u^* in $\mathcal{U}$ satisfying Nash's axioms is called the *Nash Bargaining Point* and the image of u^* by the inverse of f, formally $f^{-1}(u^*)$, is the set of *Nash Bargaining Solutions.*

Example 9. *A basic example of such setting is the n-players Dividing the Dollars game where n players share a unit of money with a null disagreement (no player can guarantee itself a strictly positive amount). The feasible domain is,*

$$\mathcal{S} = \left\{ s = (s_1,\dots,s_n) | s_i \geq 0\ \forall i \in \{1,\dots,n\}, \sum_{i=1}^{n} x_i = 1 \right\} \tag{2.9}$$

with $\mathbf{t} = \mathbf{0}$. *The utility domain* $\mathcal{U}$ *is,*

$$\mathcal{U} = \{u = (u_1,\dots,u_n) | u = (u_1(s_1),\dots,u_n(s_n)),\ s \in \mathcal{S}\} \tag{2.10}$$

From [10], we have,

Theorem 10 ([10])**.** *Let $u_i(.) : \mathcal{S} \to \mathbb{R}$, $i = 1,\dots,n$ be concave upper-bounded functions defined on $\mathcal{S}$ which is a convex and compact subset of $\mathbb{R}^n$. Let $f(s) = (u_1(s_1),\dots,u_n(s_n))$.*
Let $\mathcal{U} = \{u \in \mathcal{R}^n : \exists s \in \mathcal{S}$ s.t. $f(s) \geq u\}$. Denote by $\mathcal{S}(u) = \{s \in \mathcal{S} : f(s) \geq u\}$ and $\mathcal{S}_\mathbf{t} = \mathcal{S}(\mathbf{t})$ the subset of strategies that enable the users to achieve at least their disagreement point (threat).
Then there exists a bargaining solution and a unique bargaining point u^*. Moreover the set of bargaining solutions $(f^{-1}(u^*))$ is determined as follows,

Let J *be the set of users able to achieve a utility strictly superior to their threat,*

$$J = \{ j \in \{1,\dots,N\} : \exists s \in \mathscr{S}_{\mathbf{t}}, u_j(s) > t_j \} \tag{2.11}$$

Each vector s *in the bargaining solution set verifies* $f_j(s) > \mathbf{t}_j$ *and solve the following maximization problem:*

$$\max \prod_{j\in J} \left(u_j(s) - t_j\right), \quad s \in \mathscr{S}_{\mathbf{t}} \tag{2.12}$$

As shown in the following theorem, if each function u_j, $j \in J$, is injective on $\mathscr{S}_{\mathbf{t}}$, then the bargaining solution set is reduced to a singleton and there exists a unique Nash Bargaining Solution in $\mathscr{S}$. We have the following theorem,

Theorem 11 ([10])**.** *In addition to the assumption of the previous theorem, let* $\{u_j\}_{j\in J}$ *be injective on* $\mathscr{S}_{\mathbf{t}}$.
Consider the two maximization problems (P_J) *and* $(P_{J'})$*:*

$$(P_J) \quad \max \prod_{j\in J} \left(u_j(s) - t_j\right), \quad s \in \mathscr{S}_{\mathbf{t}} \tag{2.13}$$

and,

$$(P_{J'}) \quad \max \sum_{j\in J} \ln\left(u_j(s) - t_j\right), \quad s \in \mathscr{S}_{\mathbf{t}} \tag{2.14}$$

Then:

1. (P_J) *has a unique solution; the bargaining solution set is a singleton.*
2. $(P_{J'})$ *is a convex program and has a unique solution.*
3. (P_J) *and* $(P_{J'})$ *are equivalent. Hence, the unique solution of* $(P_{J'})$ *is the bargaining solution.*

Thus, the bargaining point u^* solving program (2.14) is the Nash Bargaining Solution in the utility space. If the threat vector is null, the allocation vector u^* is proportionally fair. Nevertheless, It may not be the case for the Nash Bargaining Solution s^* in $\mathscr{S}_{\mathbf{t}}$ which can thus be interpreted as the resource or monetary allocation that corresponds to the utility allocation resulting from the impartial arbitration.

We now show and solve an example of such game.

Example 12 (Dividing the Dollars)**.** *Consider a generalization of the Divide the Dollar bargaining problem such that the players of the set* C *compete over the resource* $v(C)$*. The payoff space* $\mathscr{S}$ *is defined as,*

$$\mathscr{S} = \left\{ s \in \mathbb{R}^{|C|} \mid \sum_{i\in C} s_i = v(C) \;\&\; x_i \geq 0 \;\forall i \in C \right\} \tag{2.15}$$

The utility function u_i *of any player* i *in* $\{1,\dots,d\}$*,*

$$u_i(x_i) = x_i^{\alpha_i} \tag{2.16}$$

where $\alpha_i \in [0;1]$*. These utilities are concave, upper-bounded on* $\mathscr{S}$ *and bijective. Assume a null threat vector* $\mathbf{t}$*, where* t_i *is the threat of player* i*.*
As stated in Theorem 10, in such case, there exists a Nash Bargaining Point u^* *in the utility domain induced by the set of these utilities* $\{u_i\}$ *and a Nash Bargaining Solution* $s^* \in \mathscr{S}$ *obtained as the solution of the following optimization problem,*

$$\begin{aligned} \underset{\mathbf{x}}{\text{maximize}} \quad & \prod x_i^{\alpha_i} \\ \text{subject to} \quad & \sum_{i=1}^{d} x_i = v(C) \\ & 0 \leq x_i,\; i = 1,\dots,d. \end{aligned} \tag{2.17}$$

By bijectivity (thus injectivity) of the utilities, the Nash bargaining solution also solves the following equivalent problem,

$$\begin{aligned} \underset{\mathbf{x}}{\textit{minimize}} \quad & -\sum_i \alpha_i \log(x_i) \\ \textit{subject to} \quad & \sum_{i=1}^{d} x_i = v(\mathrm{C}) \\ & 0 \le x_i,\ i = 1,\dots,d. \end{aligned} \tag{2.18}$$

This is a convex optimization problem (objective and inequality constraint functions are convex and the equality constraint is affine). To solve the problem we use the following well-known convex optimization results: if Slater's condition[3] *is satisfied, a point is optimal if and only if there exist Lagrange multipliers that satisfy the K.K.T. conditions.*

We obtain,

$$x_i = \frac{\alpha_i}{\sum_j \alpha_j} v(\mathrm{C}) \tag{2.19}$$

Up to this point, we have considered threats as exogenous parameters, nevertheless it is clear from the axiomatic definition and the corresponding Nash's program that the Nash bargaining solution is a function of the emitted threats. If a single player i increases his threat from t_i to $t_i' > t_i$ then his payoff will be weakly increased (assuming that the set of rational payoffs is not reduced to $\mathbf{t}$). Any player has the incentive to control his threat or influence others' threats so as to maximize his payoff at $\Phi(\mathrm{B}, \mathbf{t})$. Based on this consideration, one can conclude that the choice of a disagreement point must result from a rational decision taking. Even though there could be many ways to define the players' threats, there are three common solutions to this problem. The first solution is to define the threats from a minimax strategy. The second alternative is to develop a theory of rational threats where the threats are defined as the Nash equilibrium of a threat game Γ^*. The third solution is to define the threats as the Nash equilibrium of the original non-cooperative game.

We now enter the last part of this brief introduction to the Nash bargaining and show other interesting properties of the Nash's solution. These are related to the interpersonal comparisons of weighted utilities. As will be shown later on in the document these results will allow us to develop and interpret some elements on the control of incentives of players when they belong to bargaining subsets of players. As already discussed, alternatives to the covariance axiom have been proposed in the literature by game theorists. Some alternatives propose to constrain the solution in terms of relative or global performance of the allocation while relaxing the covariance requirement, hence the idea of interpersonal comparisons. As example, the λ-egalitarian solution of a bargaining problem ask for efficiency and a λ-weighted symmetric share of the gains w.r.t. the disagreement point,

Definition 13. *Given any* $\lambda_1, \lambda_2 > 0$*, the* λ*-egalitarian solution of* $(\mathrm{B}, \mathbf{t})$ *is the unique efficient point in* B *such that,*

$$\lambda_1(x_1 - t_1) = \lambda_2(x_2 - t_2) \tag{2.20}$$

As another example, the λ-utilitarian solution should maximize the λ-weighted social welfare of the players:

Definition 14. *Given any* $\lambda_1, \lambda_2 > 0$*, the* λ*-utilitarian solution of* $(\mathrm{B}, \mathbf{t})$ *is the unique efficient point in* B *such that,*

$$\lambda_1 x_1 + \lambda_2 x_2 = \max_{y \in \mathrm{B}} \lambda_1 y_1 + \lambda_2 y_2 \tag{2.21}$$

As observed by Myerson in [3], the λ-egalitarian and utilitarian correspond to an application of the equal-gains principle or greatest good principle when the players' payoff are compared with λ-scaled (thus decision-theoretic equivalent) utilities. We have the following theorem,

[3]Strong duality holds for a convex problem if it is strictly feasible, i.e., it exists a feasible point s.t. the inequality constraints are strictly satisfied. For a complete introduction to convex optimization, see [11].

Theorem 15 (Natural scale factors and Nash bargaining, [3], pp.383)**.** *Let* $(\mathrm{B}, \mathbf{t})$ *be an essential two-person bargaining problem, and let* x *be an allocation vector such that* $x \in \mathrm{B}$ *and* $x \geq \mathbf{t}$*. Then* x *is the Nash bargaining solution for* $(\mathrm{B}, \mathbf{t})$*, iff there exist strictly positive numbers* λ_1 *and* λ_2*, called natural scale factors, such that*

$$\lambda_1 x_1 - \lambda_1 t_1 = \lambda_1 x_1 - \lambda_2 t_2 \tag{2.22}$$

$$\lambda_1 x_1 + \lambda_2 x_2 = \max_{y \in \mathbf{F}} \lambda_1 y_1 - \lambda_2 y_2 \tag{2.23}$$

We now show an example of the Divide the Dollars game that naturally shows the scaling factor λ that makes the solution λ-egalitarian. This example will be used in chapter 5 to show some results on the controllability of stable matchings with Nash bargaining.

Example 16 (Asymmetric Dividing the Dollar)**.** *Consider a* d*-players bargaining problem over the simplex* $\mathrm{B} = \Delta^{d-1} = \{\mathbf{x} \in [0,1]^d : \sum_{i=1}^d x_i = 1\}$ *with null disagreement,* $\mathbf{t} = \mathbf{0}$*. Assume each player* i *has a concave increasing utility function* $u_i : \mathbb{R} \to \mathbb{R}$*. Nash's solution solves,*

$$\begin{aligned} \underset{\mathbf{x}}{\textit{maximize}} \quad & \prod u_i(x_i) \\ \textit{subject to} \quad & \sum_{i=1}^{d} x_i = 1 \\ & 0 \leq x_i,\ i = 1, \ldots, d. \end{aligned} \tag{2.24}$$

Deriving the K.K.T. conditions, one immediately obtains

$$\frac{u_i'(x_i)}{u_i(x_i)} = \lambda, \quad \forall i \tag{2.25}$$

Thus, for any pair i, j *of different bargainers, we have*

$$\frac{u_i'(x_i)}{u_i(x_i)} = \frac{u_j'(x_j)}{u_j(x_j)} \tag{2.26}$$

equivalently,

$$\frac{u_i(x_i)}{u_i'(x_i)} = \frac{u_j(x_j)}{u_j'(x_j)} \tag{2.27}$$

We thus have that Nash's solution is λ*-egalitarian in the utility space with* $\lambda_i = \frac{1}{u_i'(x_i)}$ *where* x_i *is the money allocation at Nash's solution. For the bargaining solution to be egalitarian (equal gains), player i's utility needs to be scaled by the inverse of the derivative of its utility function at the solution point in the original bargaining. In view of the results of chapter 5 and 7, observe that the product* $\frac{u_i(x_i)}{u_i'(x_i)}$ *is a known measure of risk aversion called fear-of-ruin of player* i *at* x_i*. For now, we do not enter the fear-of-ruin into more details and defer the description and analysis of risk aversion indicators to chapter 7 where we will consider composite games composed of a bargaining and stable matching game.*

2.3 The Generalized Nash Bargaining

In [12], Roth extensively studies the bargaining problem. Among other contributions, he shows that the relaxation of the symmetry axiom (axiom 5) allows to introduce an asymmetry in the bargaining solution and abilities of the players. The abilities are taken into account by defining for any player in the game, an extended utility function obtained by the composition of the player's original utility with a player-specific power function. Any player's power exponent is called *bargaining power*. Let denote $\alpha_i > 0$ the bargaining power of player i and α the vector of bargaining powers. A bargaining solution Φ is a nonsymmetric Nash bargaining solution iff there exists a vector α of strictly positive bargaining powers such that for any n-players bargaining problem $(B, \mathbf{t})$, $\Phi(B, \mathbf{t})$, the solution solves equation(2.28),

$$\Phi(B, \mathbf{t}) \in \arg\max_{\mathbf{u} \in B, \mathbf{u} \geq \mathbf{t}} \prod_{i=1}^{N} (u_i - t_i)^{\alpha_i} \tag{2.28}$$

For more details, see [3] (pp.390), [7] (pp.35) and [12].

Example 17 (Asymmetric Dividing the Dollar)**.** *Consider a d-players bargaining problem over the simplex $\Delta^{d-1} = \{\mathbf{x} \in [0,1]^d : \sum_{i=1}^{d} x_i = 1\}$. The utility function any player i in $\{1, \ldots, d\}$ is,*

$$u_i(x_i) = x_i \tag{2.29}$$

Assume a null threat vector $\mathbf{t}$, where t_i is the threat of player i.
Roth's solution to the Generalized Nash Bargaining with individual bargaining powers solves,

$$\begin{aligned} \underset{\mathbf{x}}{\text{maximize}} \quad & \prod u_i(x_i)^{\alpha_i} \\ \text{subject to} \quad & \sum_{i=1}^{d} x_i = 1 \\ & 0 \leq x_i,\ i = 1, \ldots, d. \end{aligned} \tag{2.30}$$

where α_i is called the bargaining power of player i.

From the previous example, the resulting allocation asymmetrically allocates the money such that, for any player i (taking $v(C) = 1$),

$$x_i = \frac{\alpha_i}{\sum_j \alpha_j} \tag{2.31}$$

2.4 References

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Chapter 3

Stable Matchings

In this chapter we introduce the theory of stable matchings, a well-developed and successful branch of game theory commonly used to analyze two-sided matching markets and their mechanisms. This is the main game-theoretic tool of this thesis. The results shown in this chapter will be used in chapter 5 to design a controlled stable matching mechanism for the WiFi association problem, in chapter 6 to design a stable matching mechanism for video caching , in chapter 7 to analyze the concavity conditions required for the generalized α-fair resource allocation scheme to induce the existence of stable matchings and in chapter 8 to design a stable matching mechanism for a crowdsourcing platform with scheduling constraints.

3.1 Stable Matchings

Since Gale and Shapley's seminal work [1] on the stable marriage and college admission problems, matching games have been widely studied. As will be shown throughout this chapter, the original stable marriage problem has been successively extended and generalized so as to cover numerous applications. This tool is of particular importance in some marketplaces where the participation and incentives of the agents depends on the implemented matching mechanism. As an example, it was shown by A.E. Roth (see [2] and references therein) that the so-called stability is necessary to maintain the participation of the agents in the matching market assigning medical interns to hospitals. If the matching mechanism is not stable the agents may turn to alternative solutions to associate with each others. This phenomenon is known in the name of *unraveling*. As an introduction, we will show Gale and Shapley's well-known result on the marriage problem and its straightforward generalization the college admission problem. We will then turn to a more complex setting where the agents not only care about their mate but also on the matching of others. Such settings are called matching with externalities. These have been assessed by introducing fundamental assumptions on the agents preferences : responsiveness and substitutability. We will then turn to the generalization to the many-to-many setting (each player can be matched to many others) with contracts and externalities generalizing previous works and partially covering a subset of other well-known problems such as auctions.

3.2 The Stable Marriage Problem

In their seminal paper [1], Gale and Shapley study the so-called stable marriage problem. They show the existence of stable matchings by using a (regret-free) stable matching mechanism known in the name of Deferred Acceptance Algorithm (DAA). Furthermore, they show that there exists a structure in the interests of the agents over the set of such equilibriums and extend the result to a the more general college-admission problem.

Consider a set of men $\mathscr{M}$ of cardinality M and a set of women $\mathscr{W}$ of cardinality W. Assume $\mathscr{M}$ and $\mathscr{W}$ are disjoint. Each man $m \in \mathscr{M}$ has preferences P_m given by the order relation $\underset{m}{\succeq}$ over the women in $\mathscr{W}$ and each woman $w \in \mathscr{W}$ has preferences P_w given by the order relation $\underset{w}{\succeq}$ over the men in $\mathscr{M}$. Let denote $\mathbf{P}$ the list of preferences,

$$\mathbf{P} = \{\mathrm{P}_{m_1}, \ldots, \mathrm{P}_{m_\mathrm{M}}, \mathrm{P}_{w_1}, \ldots, \mathrm{P}_{w_\mathrm{W}}\}$$

Example 18. *As an example take $\mathscr{M} = \{m_1, m_2\}$, $\mathscr{W} = \{w_1, w_2\}$ and the preferences list $\mathbf{P}$ as,*

$$\mathrm{P}_{m_1} : w_1 \underset{m_1}{\succeq} w_2 \underset{m_1}{\succeq} m_1 \tag{3.1}$$

$$\mathrm{P}_{m_2} : w_1 \underset{m_2}{\succeq} w_2 \underset{m_2}{\succeq} m_2 \tag{3.2}$$

$$\mathrm{P}_{w_1} : m_1 \underset{w_1}{\succeq} w_1 \underset{w_1}{\succeq} m_2 \tag{3.3}$$

$$\mathrm{P}_{w_2} : m_2 \underset{w_2}{\succeq} m_1 \underset{w_2}{\succeq} m_1 \tag{3.4}$$

$$\tag{3.5}$$

Literally, it is said that man m_1 either strictly prefers woman w_1 to woman w_2 or is indifferent between the two and either strictly prefers woman w_2 to being alone or is indifferent between the two. The preferences of w_1, m_2 and w_2 can be written the same way.

If the preferences are strict, we denote $\underset{i}{\succ}$ the preferences of player $i \in \mathscr{M} \cup \mathscr{W}$. In such case, there are no indifferences. Strict preferences can be obtained from non-strict ones by the use of tie-breaking rules that explicitly describe how to break the indifference between alternatives. As an example of such rule, a man may break an indifference using the alphabetical order, the age, the size, etc.

The marriage problem is defined as the game $\Gamma = (\mathcal{M}, \mathcal{W}, \mathbf{P})$. To solve this game it is not sufficient to define a set of players and their individual preferences over (or, utility of) alternatives. One needs to define an appropriate equilibrium concept that is intended to show attractive properties in terms of agents' decision-taking w.r.t. the conflict situation being studied. In the particular case of the marriage game where the players want to marry with each others according to their individual preferences, an appropriate question may be the following: *Given a set of men, women and their preferences, does there exist a set of marriages such that a man (women) has the incentive to divorce from his (her) mate to stay alone or re-marry with a woman (man) that would have the same incentive?*

The equilibrium concept of *pairwise stability* formalizes such stability of marriages. First let define a matching $\mu : \mathcal{M} \cup \mathcal{W} \to \mathcal{M} \cup \mathcal{W}$ as a one-to-one mapping function that associates any player i in the game to its mate (including himself if the player stays alone).

- $\mu(m) \in \mathcal{W} \cup \{m\}$ for all $m \in \mathcal{M}$
- $\mu(w) \in \mathcal{M} \cup \{w\}$ for all $w \in \mathcal{W}$
- $\mu(m) = w$ iff $\mu(w) = m$

Thus, given any man $m \in \mathcal{M}$, either he is matched to a woman w and w is also matched to m, or m remains single. Observe that each matching function μ defines a unique matching (denoted μ).

A matching μ is said to be *pairwise stable* if at μ no player prefers being single and there does not exist a man and a woman not matched with each other at μ who would prefer leaving their mates to match with each other. Formally, μ is pairwise stable if,

- It is individual rationality: $\mu(i) \underset{i}{\succ} i$ for all $i \in \mathcal{M} \cup \mathcal{W}$
- There are no blocking pairs: $\nexists (m, w) \in \mathcal{M} \times \mathcal{W}$ s.t. $\mu(m) \neq w$, $w \underset{m}{\succ} \mu(m)$ and $m \underset{w}{\succ} \mu(w)$

The *individual rationality* asks for the ability of any player to reject a mate if it prefers to stay alone rather than being matched to this mate. The third condition asks for a robustness to cooperative deviations of pairs (coalitions of a man and a woman). This cooperative interpretation in terms of coalitions may appear artificial at this point but we use it in view of other stabilities (given in the next section) that generalize the pairwise stability.

In [3], Vande Vate has shown that the set of stable matchings is actually the set of integer points of a convex polytope. Let define $\mathbf{x} \in \{0, 1\}^{\mathrm{M} \times \mathrm{W}}$ be the configuration matrix of a matching μ such that, for any pair $(m, w) \in \mathcal{M} \times \mathcal{W}$, $x_{mw} = 1$ if $\mu(m) = w$ and $x_{mw} = 0$ otherwise. Relaxing the definition of the configuration matrix so that $\mathbf{x} \in [0, 1]^{\mathrm{M} \times \mathrm{W}}$ allows to interpret the coefficients x_{mw} as the probability for the pair (m, w) to be matched or the fraction of time they spend matched with each others. We have,

Theorem 19 ([3])**.** *A matching is stable if and only if its configuration x is an integer matrix of dimension* $\mathrm{M} \times \mathrm{W}$ *satisfying the following set of constraints:*

1. $\sum_j x_{mj} \leq 1 \quad \forall m \in \mathcal{M}$,
2. $\sum_i x_{iw} \leq 1 \quad \forall w \in \mathcal{W}$,
3. $\sum_j x_{mj} + \sum_i x_{iw} + x_{mw} \geq 1 \quad \forall (m, w) \in \mathcal{M} \times \mathcal{W}$,
4. $x_{mw} \leq 0 \quad \forall (m, w) \in \mathcal{M} \times \mathcal{W}$.

The first and second set of constraints (1) ask for each man and woman to be matched with probability at most one, or at most one hundred percent of his time. The third constraint asks for the non-existence of blocking pairs. In fact, given a pair (m, w), assuming that (4) is not satisfied for the pair (i.e. $\sum_j x_{mj} + \sum_i x_{iw} + x_{mw} \leq 1$) then both spend less than 100% of their cumulated time with each others or partners they prefer to each others. Both of them could at least increase the share of time they spend with each others by reducing the amount they spend with less preferred mates. In such case, (m, w) is blocking. One can show that an integer feasible solution is equivalent to an integer configuration matrix (non-relaxed definition).

Example 20.

$$\mathrm{P}_{m_1} : w_1 \underset{m_1}{\succeq} w_2 \underset{m_1}{\succeq} m_1 \tag{3.6}$$

$$\mathrm{P}_{m_2} : w_1 \underset{m_2}{\succeq} w_2 \underset{m_2}{\succeq} m_2 \tag{3.7}$$

$$\mathrm{P}_{w_1} : m_1 \underset{w_1}{\succeq} w_1 \underset{w_1}{\succeq} m_2 \tag{3.8}$$

$$\mathrm{P}_{w_2} : m_2 \underset{w_2}{\succeq} m_1 \underset{w_2}{\succeq} m_1 \tag{3.9}$$

$$\tag{3.10}$$

Consider example (18) and the configuration matrix,

$$\mathbf{x} = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$$

The corresponding stable matching polytope is defined by the following set of equations,

$$x_{11} + x_{12} \leq 1 \tag{3.11}$$

$$x_{21} + x_{22} \leq 1 \tag{3.12}$$

$$x_{11} + x_{21} \leq 1 \tag{3.13}$$

$$x_{12} + x_{22} \leq 1 \tag{3.14}$$

$$x_{11} \geq 1 \tag{3.15}$$

$$x_{11} + x_{22} + x_{12} \geq 1 \tag{3.16}$$

$$x_{11} + x_{21} \geq 1 \tag{3.17}$$

$$x_{21} + x_{22} \geq 1 \tag{3.18}$$

$$x_{11} \geq 0, x_{12} \geq 0, x_{21} \geq 0, x_{22} \geq 0 \tag{3.19}$$

The unique integer feasible solution is $x_{11} = 1$, $x_{12} = 0$, $x_{21} = 0$ and $x_{22} = 1$. Thus, the unique stable matching is,

$$\mu = \begin{pmatrix} m_1 & m_2 \\ w_1 & w_2 \end{pmatrix}$$

The existence of pairwise stable matchings (equivalently, non-emptiness of the set of integer points in the stable matching polytope) in the game $\Gamma = (\mathcal{M}, \mathcal{W}, \mathbf{P})$ is not straightforward. Based on the intuition, one may say that the existence of such result may depend on the preferences emitted by the players. In fact, it seems reasonable to think that *if the preferences are very different from one player to another, then given any matching there may always exists a individual or a pair with the incentive to deviate.* Furthermore, by definition of the polytope of stable matchings, one may think that by appropriately changing the number of participants and preferences (i.e. choosing the right matching game) one may change the set of hyperplanes so as to make the set of integer feasible points empty or non-empty.

As shown by Gale and Shapley in [1], such intuition does not hold true and there actually exists a stable matching in any marriage problem $\Gamma = (\mathcal{M}, \mathcal{W}, \mathbf{P})$, i.e. for any sets $\mathcal{M}$ and $\mathcal{W}$ and any list of preferences $\mathbf{P}$. We give this theorem and the original proof that was of quite a singular form at the time it was published. In Gale and Shapley's setting there are n men, n women and all the players are mutually acceptable (any player prefers being matched than being single). Nevertheless the results hold in non-squared markets with non-acceptable players (as assumed in the following formalized algorithm).

Theorem 21 ([1])**.** *There always exists a stable set of marriage.*

[1]. " We shall prove the existence by giving an iterative procedure for actually finding a stable set of marriages.

To start, let each boy propose to his favorite girl. Each girl who receives more than one proposal rejects all but her favorite from among those who have proposed to her. However, she dos not accept him yet, but keeps him on a string to allow for the possibility that someone better may come along later.

We are now ready for the second stage. Those boys who were rejected now propose to their second choices. Each girl receiving proposals chooses her favorite from the group consisting of new proposers and the boy on her string, if any. She rejects all the rest and again keeps the favorite in suspense.

We proceed in the same manner. Those who are rejected at the second stage propose to their next choices, and the girls again reject all but the best proposal they have had so far.

Eventually (in fact, in at most $n^2 - 2n + 2$ stages[1]) every girl will have received a proposal, for as long as any girl has not been proposed to there will be rejections and new proposals, but since no boy can propose to the same girl more than once, every girl is sure to get a proposal in due time. As soon as the last girl gets her proposal the *courtship* is declared over, and each girl is now required to accept the boy on her string. We assert that this set of marriages is stable. Namely, suppose John and Mary are not married to each other but John prefers Mary to his own wife. Then John must have proposed to Mary at some stage and subsequently been rejected in favor of someone that May liked better. It is now clear that Mary must prefer her husband to John and there is no instability. " □

The process of proposal and acceptances or rejections described in the proof is known in the name of the Deferred Acceptance Algorithm (DAA) with men-proposing. In a more conventional algorithmic form and assuming that some players may not be acceptable to others, we have the DAA as,

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Algorithm 4: The deferred acceptance algorithm with men proposing.

Data: $\Gamma = (\mathcal{M}, \mathcal{W}, \mathbf{P})$
Result: μ_M
-Each man proposes to his favorite woman;
-Each woman selects her most preferred man among those having proposed to her;
-The chosen men are tagged *engaged* by the corresponding women;
while *some men are rejected* **do**
    -Each man rejected at the previous step proposes to his favorite (and acceptable) woman among those not having rejected him yet;
    -Each woman receiving new proposals chooses her preferred man among the acceptable new proposers and the man she has kept engaged;
-Each man is matched to the woman he is engaged to, other players stay single;

Before going further, we show to apply the DAA in the previous simple example.

Example 22.

$$P_{m_1} : w_1 \underset{m_1}{\succeq} w_2 \underset{m_1}{\succeq} m_1 \tag{3.20}$$

$$P_{m_2} : w_1 \underset{m_2}{\succeq} w_2 \underset{m_2}{\succeq} m_2 \tag{3.21}$$

$$P_{w_1} : m_1 \underset{w_1}{\succeq} w_1 \underset{w_1}{\succeq} m_2 \tag{3.22}$$

$$P_{w_2} : m_2 \underset{w_2}{\succeq} m_1 \underset{w_2}{\succeq} m_1 \tag{3.23}$$

(3.24)

[1] where $n = M = W$ in Gale and Shapley's setting

- *Step 1: m_1 and m_2 propose to w_1*
- *Step 1': w_1 selects m_1 and rejects m_2*
- *Step 2: m_2 proposes to w_2*
- *Step 2': w_2 selects m_2*
- *Since no man is rejected, the algorithm ends*

The stable matching obtained by DAA with men proposing is,

$$\mu = \begin{pmatrix} m_1 & m_2 \\ w_1 & w_2 \end{pmatrix}$$

Another interesting algorithmic result was shown by Roth and Vande Vate: given any matching μ of a marriage game Γ, there exists a path of blocking pairs to stability. Formally,

Theorem 23 ([2]). *Let μ be an arbitrary matching for $(\mathcal{M}, \mathcal{W}, \mathbf{P})$. Then, there exists a finite sequence of matchings $\mu_1, \ldots, \mu_k$ such that $\mu = \mu_1$, μ_k is stable, and for each $i = 1, \ldots, k-1$, there is a blocking pair (m_i, w_i) for μ_i such that μ_{i+1} is obtained from μ_i by satisfying the blocking pair (m_i, w_i).*

A corollary of this theorem is that given an arbitrary matching μ, a random process selecting blocking pairs must eventually converge with probability one to a stable matching provided each blocking pair has a probability to be selected bounded away from zero. This result also shows the non-emptiness of the set of stable matchings.

The convergence to stable matchings for any preference profile is not the only interesting property of the algorithm. Whether men or women propose is of particular importance (when the set of stable matchings is not reduced to a singleton) because proposers reach optimality.

Theorem 24 ([1]). *Every applicant is at least as well off undertake assignment given by the deferred acceptance procedure as he would be under any other stable assignment.*

In other words, every man (woman) likes the matching from DAA with men proposing at least as well as any other stable matching. In case of strict preferences, one can show that such side-optimal stable matching is unique (see [2], pp.32). Let denote μ_M the men-optimal (M-optimal) stable matching and μ_W the women-optimal (W-optimal) stable matching. This is a surprising result. In fact, despite of eventual divergences in the men (women) preferences over the set of assignments, all men (women) agree on their preferred one μ_M (μ_W) in the subset of stable ones. So, there is a common interest over the set of stable matchings among the players of a side. Even more surprisingly, as shown by the two following theorems, the two sided have opposite common interests (the commonly agreed men-optimal stable matching μ_M is the commonly agreed worst stable matching for women, and vice versa) and the stable matchings form a lattice. Let us begin with Knuth's result on opposite interests and then turn to Conway's result on the lattice structure of the set of stable matchings,

Theorem 25 ([2], pp.33). *When all agents have strict preferences, the common preferences of the two sides of the market are opposed on the set of stable matchings; if μ and μ' are stable matchings, then all men like μ at least as well as μ' if and only if all women like μ' t least as well as μ.*

In order to state the following result on the lattice structure of stable matchings, it is required to define the so-called pointing functions $\underset{M}{\vee}$ and $\underset{M}{\wedge}$ such that,

- $\lambda(m) = \mu(m) \underset{M}{\vee} \mu'(m) \begin{cases} \mu(m) \text{ if } \mu(m) \underset{m}{>} \mu'(m) \\ \mu'(m) \text{ otherwise} \end{cases}$

- $\lambda(w) = \mu(w) \underset{M}{\vee} \mu'(w) \begin{cases} \mu(w) \text{ if } \mu(m) \underset{w}{\prec} \mu'(w) \\ \mu'(w) \text{ otherwise} \end{cases}$

similarly,

- $\nu(m) = \mu(m) \underset{M}{\wedge} \mu'(m) \begin{cases} \mu(m) \text{ if } \mu(m) \underset{m}{\prec} \mu'(m) \\ \mu'(m) \text{ otherwise} \end{cases}$
- $\lambda(w) = \mu(w) \underset{M}{\wedge} \mu'(w) \begin{cases} \mu(w) \text{ if } \mu(m) \underset{w}{\succ} \mu'(w) \\ \mu'(w) \text{ otherwise} \end{cases}$

Thus, $\underset{M}{\vee}$ assigns each man $m \in \mathscr{M}$ its most preferred woman in $\{\mu(m), \mu'(m)\}$ and each woman $w \in \mathscr{W}$ its least preferred man in $\{\mu(m), \mu'(m)\}$. The pointing function $\underset{M}{\wedge}$ works the opposite mapping each man to his less preferred woman and each woman to her preferred man. We have the following theorem,

Theorem 26 ([2], pp.36)**.** *When all preferences are strict, if* μ *and* μ' *are stable matchings, then the function* $\lambda = \mu \underset{M}{\vee} \mu'$ *and* $\eta = \mu \underset{M}{\wedge} \mu'$ *are both matchings. Furthermore, they are both stable.*

The pairwise stability is based on the non-existence of individuals or pairs having the incentive and the power to enforce respectively unilateral (individual rationality) or bilateral deviations (blocking pair). Is it possible to extend to more general subsets than singletons and pairs? Is there a way to consider re-matchings that would involve more than two (an agent leaves its mate to stay single) to four players (two unmarried agents leave their mates to match with each others)? In other words, can we consider blocking subsets bigger than just pairs? Actually, this generalization already exists in cooperative game theory and is known in the name of *core*. Consider a cooperative game with a set of feasible outcomes, individual preferences defined over the set of outcomes and a set of rules defining the set of feasible cooperative behaviors such as the set of feasible solutions (this is of particular importance in section 3.5 where there exists constraints on the set of feasible coalitions for stable matchings to exist). A general definition of the *core* is proposed by Roth,

Definition 27 ([2], pp. 54)**.** *For any two feasible outcomes x and y, x dominates y if and only if there exists a coalition of players* C *such that,*

- *every member of the coalition* C *prefers x to y; and*
- *the rules of the game give the coalition* S *the power to enforce x over y.*

Definition 28 ([2], pp.)**.** *The core of a game is the set of undominated outcomes.*

In case of the marriage market, we have thus have the following definition of domination,

Definition 29 ([2], pp. 54)**.** *A matching* μ' *dominates another matching* μ *if and only if there exists a coalition* C *contained in* $\mathscr{M} \cup \mathscr{W}$*, such that, for all men m and women $w \in$ C,*

- $\mu'(m) \in C$
- $\mu'(w) \in C$
- $\mu'(m) \underset{m}{\succ} \mu(m)$
- $\mu'(w) \underset{w}{\succ} \mu(w)$

A matching is dominated by another one if there is a subset of men and women with the incentive to deviate to either stay alone or match with each others and the power to enforce it. The ability to enforce a new outcome comes from the fact that such deviation involves only decision-taking by the agents in the subset and that any of these is in favor of the deviation. One may see that the agents in the blocking subset have aligned preferences over the two matchings. All the agents prefer the same matching.

Example 30. *Consider the matching*

$$\mu = \begin{pmatrix} m_1 & m_2 & m_3 & m_4 \\ w_1 & w_2 & w_3 & w_4 \end{pmatrix}$$

As an example, this matching is not core stable if $\{m_2, m_3, w_2, w_3\}$ *strictly prefer*

$$\mu' = \begin{pmatrix} m_1 & m_2 & m_3 & m_4 \\ w_1 & w_3 & w_2 & w_4 \end{pmatrix}$$

Observe that the coalition $\{m_2, m_3, w_2, w_3\}$ *can be decomposed in two disjoint sub-coalitions* $\{m_2, w_3\}$, $\{m_3, w_2\}$, *each blocking* μ. *This decomposition principle is used to show Theorem 31.*

In the stable marriage problem, another nice result is that the set of pairwise and core stable matchings are the same.

Theorem 31 ([2], pp.55)**.** *The core of the marriage market equals the set of stable matchings.*

Qualitatively, the proof shows the following: (i)if a matching is core stable then it is pairwise stable because a pair is a coalition, (ii) if a matching is not in the core then there exists a blocking pair (intuition provided by the previous example). The contraposition of (ii) is, if there exists no blocking pair (the matching is pairwise stable) then a matching is in the core. This shows the equivalence.

3.3 The College Admission Problem

The college admission problem is the natural extension of the one-to-one stable marriage problem to the many-to-one setting where the players of a given side can be matched to a finite number of the opposite one. In the basic college admission problem, all the players have preferences (taken as primitives) over individuals of the opposite side as in the marriage problem and the colleges' preferences over groups of students are naturally induced by the preferences over individuals. In more complex problems, both colleges and students have preferences over groups of students (students care about who else is admitted in the college), or there may exist payments and salaries inducing the preferences. Let us start by showing the results on the basic problem that can be quite straightforwardly solved using the previous results, and then turn to the more complex ones.

3.3.1 Basic Setting

Consider a set of colleges $\mathscr{C}$ of cardinality C and a set of students $\mathscr{S}$ of cardinality S. Assume $\mathscr{C}$ and $\mathscr{S}$ are disjoint. Each college $c \in \mathscr{C}$ has a quota q_c, i.e. it can admit a finite number q_c of students. As in the marriage problem, each college $c \in \mathscr{C}$ has preferences P_c given by the order relation $\underset{c}{\succeq}$ over the students in $\mathscr{S}$ and each student $s \in \mathscr{S}$ has preferences P_s given by the order relation $\underset{s}{\succeq}$ over the colleges in $\mathscr{C}$. Let denote $\mathbf{P}$ the list of preferences,

$$\mathbf{P} = \{\mathrm{P}_{s_1}, \ldots, \mathrm{P}_{s_S}, \mathrm{P}_{c_1}, \ldots, \mathrm{P}_{c_C}\}$$

In such a setting where the colleges can admit groups, it is natural to embed in the model preferences over groups of students. Let define $\mathrm{P}_c^{\#}$ the preferences of college c over groups of students. One may also interpret $\mathrm{P}_c^{\#}$ as c's preferences over matchings but with limited care to its own set of matched students.

Example 32. *As an example, take* $\mathscr{C} = \{c_1, c_2, c_3\}$ *and* $\mathscr{S} = \{s_1, s_2, s_3\}$. *Assume the following preferences for student* s_1 *and college* c_1 *with quota* $q_1 = 2$*:*

$$\mathrm{P}_{s_1} : c_1 \underset{s_1}{\succeq} c_2 \underset{s_1}{\succ} c_3 \underset{s_2}{\succeq} s_1 \tag{3.25}$$

$$\mathrm{P}_{c_1} : s_1 \underset{c_1}{\succ} s_2 \underset{c_1}{\sim} c_1 \underset{s}{\succeq} s_3 \tag{3.26}$$

$$\mathrm{P}_{c_1}^{\#} : \{s_1, s_2\} \underset{c_1}{\succ} s_1 \underset{c_1}{\succ} s_2 \underset{c_1}{\sim} c_1 \underset{s}{\succeq} s_3 \tag{3.27}$$

Student s_1 likes at least as well (strictly prefers or is indifferent) being admitted in college c_1 than in c_2 and strictly prefers being admitted in c_2 rather than in c_3. Similarly, college c_1 strictly prefers student s_1 to s_2, is indifferent between admitting s_2 or not and considers s_3 as not acceptable. Finally, the preferences over groups $\mathrm{P}^{\#}_{c_1}$ show that c_1 prefers admitting both s_1 and s_2 than only one of them as previously given in the preferences P_{c_1}.

In the many-to-one matching problems, the players on the colleges' side can be associated to many students. It is often admitted that each college has a finite quota, i.e. admits a finite number of students. A feasible many-to-one matching μ is a function from the set $\mathscr{C} \cup \mathscr{S}$ into the set of unordered families of elements of $\mathscr{C} \cup \mathscr{S}$ such that:

- $|\mu(s)| = 1$ for every student s and $\mu(s) = 1$ if $\mu(s) \notin \mathscr{C}$;
- $|\mu(c)| = q_c$ for every college c, and if the number of students in $\mu(c)$, say r, is less than q_c, then $\mu(c)$ contains $q_c - r$ copies of c;
- $\mu(s) = c$ if and only if $s \in \mu(c)$

In [1], Gale and Shapley have shown that the set of pairwise stable matchings is always non-empty in the college admission problem with preferences over individual:

$$\mathbf{P} = \{\mathrm{P}_{s_1}, \ldots, \mathrm{P}_{s_S}, \mathrm{P}_{c_1}, \ldots, \mathrm{P}_{c_C}\} \tag{3.28}$$

To show this, one just need to slightly modify the DAA (used to show the existence of stable marriages) so that at any round (with students proposing) every college $c \in \mathscr{C}$ accepts its q_c most preferred agents among those already admitted, those currently proposing and itself (i.e. in case of unacceptable students, the college prefers not admitting the rather than admitting). In this setting, each college opportunistically admits its most preferred students according the its preferences over individuals. Thus, each student is assumed independent from the others and is admitted or rejected following one-to-one comparisons w.r.t. other students. When rejected, a student is replaced by a more preferred one without any regret in subsequent rounds, as in the marriage setting.

3.3.2 Responsive Preferences over Groups of Students and Group Stability

The previous results have shown that there exists stable matchings in college admission problems with preferences over individuals on both students' and firms' sides. We now consider the more complex problem where every college $c \in \mathscr{C}$ has preferences $\mathrm{P}^{\#}_c$ as already defined. In order to exhibit the difficulty raised by the introduction of groups, consider the previous DAA with students proposing. Assume that a college c chooses according to $\mathrm{P}^{\#}_c$ its most preferred subset of students among those admitted and those proposing according to $\mathrm{P}^{\#}_c$. Following this choice, a student (potentially, many students) may be rejected because of low complementarities but may be regretted in a subsequent round because of new proposals that make the rejected student now belong to c's most preferred subset. In such case, both c and the rejected students would be matched with each others.

This opens the way toward the introduction of new stability concepts where groups matter and the introduction of assumptions on the structure of the preferences that may induce regret-free deferred acceptance-like processes and guarantee the existence of stable matchings.

A natural way of constructing any college's preferences over groups of students is to compare the groups differing by one student based on the preferences over the individuals. Such preferences over groups are called *responsive* preferences. We have the following formal definition,

Definition 33 ([2], pp.128)**.** *The preference relation $\mathrm{P}^{\#}_c$ over sets of students is responsive to the preferences P_c over individual students if, whenever $\mu'(c) = \mu(c) \cup \{s\} \backslash \{\sigma\}$ for σ in $\mu(c)$, then c prefers $\mu'(c)$ to $\mu(c)$ under $\mathrm{P}^{\#}_c$ if and only if c prefers s to σ under P_c.*

We now define two stability solution concepts, that involve subsets of players bigger than just pairs. These stabilities allow not only individuals or pairs (of the form (man,woman) or (student,college)) to block a matching but subsets possibly composed of several men and women or students and colleges that can re-organize the associations so as to benefit from the deviation. Each of these stability concept is differentiated from the other through the beneficiaries of the deviation and their enforcement power. The first solution concept has actually already been defined, it is the core. In the framework of many-to-one markets, we have the following definition of domination:

Definition 34 ([2], pp.166)**.** *A matching* μ' *dominates another matching* μ *via a coalition* C *contained in* $\mathscr{C} \cup \mathscr{S}$ *if for all students s and colleges c in* C*,*

- *If* $c' = \mu'(s)$ *then* $c' \in C$
- *If* $s' \in \mu'(c)$ *then* $s' \in C$
- $\mu'(s) \underset{s}{\succ} \mu(s)$
- $\mu'(s) \underset{s}{\succ} \mu(s)$

Remember that the core is the set of undominated matching.

Definition 35 ([2], pp.130)**.** *The core,* $C(\mathbf{P})$*, of a game is the set of matchings that are not dominated by an other matching.*

A many-to-one-matching is *core stable* if there is no subset of players that would all prefer be matched with each others exclusively[2].

Example 36. *Consider the matching*

$$\mu = \begin{pmatrix} c_1 & c_2 \\ s_1 s_2 & s_3 s_4 \end{pmatrix}$$

This matching is not core stable and dominated by

$$\mu' = \begin{pmatrix} c_1 & c_2 \\ s_1 s_2 s_3 & s_4 \end{pmatrix}$$

if any agent in $\{c_1, s_1, s_2, s_3\}$ *strictly prefers* μ' *to* μ*.*

A weakened version of the core, called *weak core* is defined by relaxing strictness in the domination:

Definition 37 ([2], pp.166)**.** *A matching* μ' *weakly dominates another matching* μ *via a coalition* C *contained in* $\mathscr{C} \cup \mathscr{S}$ *if for all students s and colleges c in* C*,*

- *If* $c' = \mu'(s)$ *then* $c' \in C$
- *If* $s' \in \mu'(c)$ *then* $s' \in C$
- $\mu'(s) \underset{s}{\succeq} \mu(s)$
- $\mu'(s) \underset{s}{\succeq} \mu(s)$
- $\mu'(s) \underset{s}{\succ} \mu(s)$ *for some* $s \in C$*, or*
- $\mu'(s) \underset{s}{\succ} \mu(s)$ *for some* $c \in C$

The corresponding core is called *core defined by weak domination*:

[2] No other player can be matched to them.

Definition 38 ([2], pp.130)**.** *The core defined by weak domination,* $C_W(\mathbf{P})$*, of a game is the set of matchings that are not weakly dominated by any other matching.*

Because domination implies weak domination we have:

$$C_W(\mathbf{P}) \subseteq C(\mathbf{P})$$

Furthermore, we have equivalence between the weak core and the set of pairwise stable matchings in many-to-one matching markets with responsive strict preferences.

Theorem 39 ([2], pp. 167)**.** *When preferences are responsive and strict in a many-to-one matching market,* $S(\mathbf{P}) = C_W(\mathbf{P})$.

The other solution concept that we consider in matching markets with group deviations is the group stability. It is based on the notion of blocking coalition that weakens domination by allowing the new matching to be preferred to the original one only by the players with the incentive to deviate. Formally:

Definition 40 ([2], pp.130)**.** *A matching* μ *is blocked by a coalition* C*, if there exists another matching* μ' *and a coalition* C*, which might consist of multiple students and/or colleges, such that for all students* $s \in C$ *and for all colleges* $c \in C$*,*

- $\mu'(s) \in C$
- $\mu'(s) \underset{s}{\succ} \mu(s)$
- $\sigma \in \mu'(c)$ *implies* $\sigma \in C \cup \mu(c)$
- $\mu'(c) \underset{c}{\succ} \mu(c)$

Definition 41 ([2], pp.130)**.** *A group stable matching is one that is not blocked by any coalition.*

In the core stability, it is required that any student (college) matched to a deviating college (student) at μ' must be involved in the deviating coalition and thus prefers μ' to μ. The domination results from a joint agreement among all the players involved except those left aside. In the group stability, it is only required that those who actually deviate prefer μ' to μ. This is shown in the following example (compare to example 36 for the difference w.r.t. the core).

Example 42. *Consider the matching*

$$\mu = \begin{pmatrix} c_1 & c_2 \\ s_1 s_2 & s_3 s_4 \end{pmatrix}$$

This matching is not group stable and blocked by $\{c_1, s_3\}$ *if*

$$\mu' = \begin{pmatrix} c_1 & c_2 \\ s_1 s_2 s_3 & s_4 \end{pmatrix}$$

is preferred to μ *by* c_1 *and* s_3*.*

In addition to the previous equivalence result between the weak core and the set of pairwise stable matchings, we have the following one between the set of group stable matchings and the set of pairwise stable ones,

Theorem 43 ([2], pp. 130)**.** *When preferences are responsive, a many-to-one matching is group stable if and only if it is pairwise stable.*

3.4 The Firms and Workers Model: Choice Functions, Subtitutability and Salaries

In this section, we introduce the substitutability property, another sufficient condition for the existence of stable matchings (but weaker that the responsive property). Furthermore, we introduce the choice functions formalism that will be used in the general models of matching with contracts.

Consider the previous college-admission problem, turn the students into workers from a set of workers $\mathscr{W} = \{w_1, \ldots, w_m\}$, the colleges into firms from a set $\mathscr{F} = \{f_1, \ldots, f_n\}$ and assume that each pair (firm,worker) is characterized by a salary to be paid by the firm to the worker. This setting is known in the name of the firms and workers problem. For the sake of homogeneity w.r.t. [2] and simplicity, assume that any firm has quota m. As in the college admission problem, a matching μ is a function from the set $\mathscr{F} \cup \mathscr{W}$ into the set of all subsets of $\mathscr{F} \cup \mathscr{W}$ such that

1. $|\mu(w)| = 1$ for every worker w and $\mu(w) = w$ if $\mu(w) \not\subseteq \mathscr{F}$;
2. $|\mu(f)| \leq m$ for every firm f and $\mu(f) = \emptyset$ if f isn't matched to any workers;
3. $\mu(w) = f$ if and only if w is in $\mu(f)$.

The analytical formalism developed for the marriage and college-admission problems is based on order relations called preferences that are used to define the models and concepts or show the results. The notations were defined in terms of classical order-relations such as $\underset{i}{>}$ or $\underset{i}{\geq}$. In the 1980's, this formalism has been slightly changed for an alternative one based on choice functions that allow to define a new condition, called substitutability, in a convenient way. This notation is used in the most general matching models such as matching with contracts [4] and matching with contracts and externalities [5]. Nevertheless, the classical notation is not left aside and keep on being used when adapted to the problem (as an example of a recent paper using the order-based notation see [6]).

Definition 44 ([2], pp.172)**.** *For any subset* $S \subseteq \mathscr{W}$, *f's choice set is* $S' = Ch_f(S)$ *such that* $S' \subseteq S$ *and* $S' \geq_f S''$ *for all* $S'' \subseteq S$.

In the literature, one can also find the equivalent notation $c_f(S)$ to denote the choice set of f in S. If the preferences are strict, then the choice set in any subset S is unique. As stated, choice functions are used in the definition of the substitutability property that is another sufficient condition for the existence of stable matching. This condition asks for the workers not to be complements for firms. Thus, if a worker w is chosen from a set, removing another worker from this set should not make w less worthable. In other words, if a worker w is rejected (not chosen) from a set of workers by a firm f, then it should be rejected by f from any superset. In fact, the non-complementarity should not make the rejected w worthable when being jointly considered with new workers. Formally, we have the following definition

Definition 45 ([2], pp.173)**.** *A firm f's preferences over sets of workers has the property of substitutability if, for any set* S *that contains workers w and w', if w is in* $Ch_f(S)$ *then w is in* $Ch_f(S \setminus w')$.

Defining the reject of f in S as $R_f(S) = S \setminus Ch_f(S)$ (equivalent notation $r_f(S)$) and using the contraposal: if w is not in $Ch_f(S \setminus w')$ (thus w is rejected from $S \setminus w'$) then w is not in $Ch_f(S)$ (thus w is rejected from S), we have the following equivalent definition

Definition 46 ([2], pp.173)**.** *A firm f's preferences over sets of workers has the property of substitutability if, for any set* S *that contains workers w and w', if w is in* $r_f(S \setminus w')$ *then w is in* $r_f(S)$.

The substitutability property is weaker than the responsive one and if preferences are responsive then they satisfy the substitutability property. We now show a simple example of a firms and workers matching problem with firms' choice functions not satisfying the substitutability property (a generalized version of this problem is assessed in chapter 8).

Example 47. *Assume that each firm f has a set of tasks $\mathcal{T}_f$ to be performed by a worker and the execution of the tasks in $\mathcal{T}_f$ is constrained by a scheduling. Practically, a task can be executed only if the set of its predecessors is executed. In such case, a firm chooses a worker w to execute a task τ only if it has other workers to execute the predecessors of τ. Assume that S contains w and all the workers for the predecessors of τ except one, then w is rejected by the firm (since it is hiring w for a task that cannot be executed) while w is not rejected if the worker for the only one missing predecessor is added to S. In such case, f's choice function does not satisfy the substitutability condition.*

As in the marriage and college-admission problems, we define blocking individuals, pairs and the stability. A matching μ is blocked by an individual worker if $w \underset{w}{\succ} \mu(w)$ and by an individual firm f if $\mu(f) \neq Ch_f(\mu(f))$. Furthermore, μ is blocked by a worker-firm pair (w, f) if $\mu(w) \neq f$ and if $f \underset{w}{\succ} \mu(w)$ and $w \in Ch_f(\mu(f) \cup w)$.

A matching is stable if

- it is not blocked by any individual agent
- it is not blocked by any worker-firm pair

We have the following existence theorem,

Theorem 48 ([2], pp.175)**.** *When firms have substitutable preferences, the set of stable matchings is always non-empty.*

This theorem can be shown using a modified deferred acceptance algorithm with firms proposing such that at each round, each firm proposes to its most preferred set of workers that includes all of those workers whom it previously proposed to and who have not yet rejected it, but does not include any worker who have previously rejected it. The convergence of the algorithm to a stable matching relies on the fact that it is a no-regret procedure for firms to keep on proposing to workers not having rejected (since any worker chosen from a set keep on being chosen in a subset by repeated application of the definition of substitutability).

One could also show that the firms proposing version converges to a firm-optimal matching and that the workers-proposing version converges to a worker-optimal matching (as in the marriage and college admission problem).

Finally, we give the last important result of the section: the weak core equals the set of pairwise stable matchings in case of substitutability on the firms' side.

Theorem 49 ([2], pp.175)**.** *When firms have substitutable preferences (and all preferences are strict),* $S(\mathbf{P}) = C_W(\mathbf{P})$.

3.5 Complementarities and Peer Effects

As explained in the previous section, the substitutability property of the firms' choice functions is a sufficient condition for the existence of stable matchings in matching games with preferences over groups on the firms' side. The substitutability property induces that the workers are considered as substitutes rather that complements by the firms. In other words, either there are no complementarities or they are so weak that they cannot turn rejections into choices.

There are many settings and two-sided markets where such assumptions cannot hold and the workers can be considered by the firms as complements.

Example 50. *As an example, assume that a firm (company) wants to hire a team of three experts to create a new product: one from mechanical engineering, one from electronics and one from computer science. If one is missing, then the product cannot be created because some skill are missing and none of the other two can be chosen by the firm.*

$$c_f(\{\textit{mechanics, electronics, computer science}\}) = \{\textit{mechanics, electronics, computer science}\} \tag{3.29}$$

$$c_f(\{\textit{mechanics, computer science}\}) = \emptyset \tag{3.30}$$

Chosen workers are not maintained in the subset, the choice function does not satisfy the substitutability condition).

As another example, consider the firms as buyers and the workers as sellers.

Example 51. *Assume one buyer and two sellers, one selling home video games console and the other selling a video game. If the buyer can buy both the console and the game, then the buyer buys both. Else the buyer doesn't buy any good,*

$$c_f(\{console, game\}) = \{console, game\} \tag{3.31}$$
$$c_f(\{console\}) = \emptyset \tag{3.32}$$

Chosen goods are not maintained in the subset, the choice function does not satisfy the substitutability condition).

A more complex model considers not only complementarities on the firms' side but also *peer effects* on the workers' side. In such setting, workers may care not only about the firms they work for but also about their colleagues. Such peer effects are a particular case of externalities that consider the impact of the state of the market over the individuals and their preferences. The related branch of matching games is called matching with externalities. The profile of preferences is $\mathbf{P} = (\mathrm{P}^{\#}_{w_1}, \ldots, \mathrm{P}^{\#}_{w_m}, \mathrm{P}^{\#}_{f_1}, \ldots, \mathrm{P}^{\#}_{f_n})$. This is quite a common problem in college admission where the parents care not only about the college (or the class) there child will be admitted in but also who else is admitted (e.g. some want friends to be in their class). As another example consider a firm with a production line producing good sold on a market.

Example 52. *For simplicity assume that the production is a function of the hired workers and that any produced good is sold on the market. The revenues (produced worth) is then shared among the firm and the workers proportionally to the revenues. First, if there are too few workers, then the production level is low and the revenues are low. Both the firm and the workers want to increase the capacity of the line by increasing the numbers of workers. On the opposite, if there are too many workers there may be congestion effects both on the line because the individual tasks cannot be executed properly (or the production activity must be decided into too many sub-tasks that induce a global reduced productivity) and on the sharing scheme since one may improve the individual salaries by rejecting some workers. Second, if there are slow or non-qualified workers reducing the capacity of the production line, then the production is limited to the slowest one (this worker is the bottleneck of the chain) and the revenue is low. In such case, fast workers would prefer another fast worker in the line rather than a slow one while the slow one would only ask for workers not slower than himself. In this production line example there are strong complementarities across workers on the firms' side and peer effects on the workers' side because of both congestion and capacity effects. For an interesting (and successful) novel on operations management, bottlenecks and the theory of constraints, see [7].*

In chapter 5, we will analyze a similar problem in the framework of the WiFi system and its association problem. Below we show small example of such wireless communications system between users' devices and Access Points (APs, nodes that allow the mobile devices to communicate with each others, access to the internet, etc.).

Example 53. *Assume two APs f_1 and f_2 and three mobile users w_1, w_2, w_3 such that $\theta_{11} = 300$ Mbps, $\theta_{12} = \theta_{22} = 54$ Mbps, $\theta_{21} = \theta_{32} = 1$ Mbps. Assuming saturated regime and equal packet size, we can show that $\mathrm{P}^{\#}(w_1) = f_1 \succ f_2 \succ \{w_3; f_1\} \succ \{w_2; f_2\} \succ \{w_2; f_1\} \succ \{w_3; f_2\}$, which is not responsive. One may also have the following preferences for mobile device w_1, $\mathrm{P}^{\#}(w_1) = \{w_3; f_1\} \succ \{w_2; f_2\} \succ \{w_2; f_1\} \succ \{w_3; f_2\} \succ f_1 \succ f_2$. Considering $\mathrm{S} = \{w_2, w_3; f_1, f_2\}$, we have $\mathrm{C}h_{w_1}(\mathrm{S}) = \{w_3; f_1\}$, while $\mathrm{C}h_{w_1}(\mathrm{S} \setminus w_3) = \{w_2; f_2\}$. Preferences are thus not substitutable.*

On the workers' side, complementarities also arise in the many-to-many setting where a worker can work for different firms and thus jointly consider his set of jobs. As an example, a professional car driver may consider working as a UBER driver as a complement to his main driving job hours but, if the professional driver activity stopped, he would rather prefer working as a taxi driver and stop working for UBER.

Complementarities and peer effect in matchings have attracted a lot of attention. Even though it is a difficult problem, game theorists have proposed solutions. The many-to-one matching problem has algorithmically been assed by Echenique and Yenmen in [8] who propose a fixed-point formulation and an algorithm to enumerate the set of stable matchings. It is known, that there is no guarantee that this set is non empty if the individual preferences over groups are not of a particular form. The problem of complementarities and peer effects in matchings has also been analytically tackled by Pycia in [6]. In this section, we show Pycia's result in view of the results of Section 5 that make use of these. Pycia's work is not restricted to the two-sided structure and holds for the general coalition formation problem where individually rational agents seek for forming coalitions based on their preferences over groups.

The general problem of analysis of the formation of coalitions or groups of players is a long date topic. Even though the first formalization in the game-theoretic framework is due to von Neuman and Morgenstern in their well-known Theory of Games and Economic Behaviors [9], the underlying concepts have been naturally used in the framework of the human activities or in the nature developments (whatever environmental or animal). The reason for this group activities to happen may be various and may be observed at many scales such as the microscopic one with the particle mutual attractions or repulsions for increased entropy or energy minimization in physics and chemistry. It may also be observed at the macroscopic one with animals or humans merging in herds, societies or binding country agreements for individual security improvement, increased individual food intake or more generally increased revenues. Closing the loop of scales, this coalition formation phenomenon may be observed at much larger scales in the universe with the formation of planets.

A partitioning of the players in groups from the set of coalitions $\mathscr{C}$ is called a structure [10], see Figure 3.1 for a structure in the coalition formation problem and Figure 3.2 for a structure in the matching problem.

(b)

(a)

Figure 3.1: Examples of structures in the coalition formation problem. In (a), the structure is a single coalition made of all the players and called the grand coalition. In (b), the structure is composed of three coalitions.

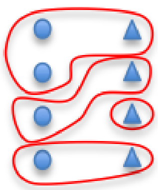

Figure 3.2: Example of a matching with players belonging to two disjoint sets: the set of circles and the set of triangles.

A structure is core stable if no subset of agents has the incentive to leave and form a new coalition. In other words, a structure S is core stable if no unformed group (i.e. no coalition $C' \in \mathscr{C} \backslash S$)

is such that all the players would strictly prefer forming C' rather than being in their respective groups in S.

In [6], Pycia studies the coalition formation problem (including two-sided markets with complementarities and peer effects as a special case) and shows the existence of core stable structures if the set of coalitions is regular and the preferences of the agents satisfy a new condition called pairwise-alignment of the preferences over rich preference profiles. Other results show the set of sharing rules inducing such preferences in a model of coalition game in characteristic form with individual utility functions. Surprisingly, the Nash bargaining falls in the scope of such sharing rules. This shows another hidden property of this cooperative equilibrium. Finally, similar results have been shown in supply chain networks in [11].

We now show Pycia's model and results. Consider a set of agents $\mathcal{N}$. By definition of the cooperative setting, agents can form coalitions. Nevertheless, as already observed, the set of coalitions $\mathcal{C}$ may be unconstrained (thus $\mathcal{C} = 2^{\mathcal{N}}$) or there may exist constraints that allow the agents to form groups of particular form. Other groups are assumed infeasible by definition of the rules of the game. Let define a particular set of coalitions, called *regular* set of coalitions, such that,

Definition 54 ([6])**.** *A set of coalitions is regular if there is a partition of the set of agents $\mathcal{N}$ into two disjoint, possibly empty, subsets $\mathcal{F}$ and $\mathcal{W}$ that satisfy the following three assumptions:*
C1. *For any two different players, there exists a coalition containing them if and only if at least one of the players is a player of $\mathcal{W}$.*
C2. *For any players a_1, $a_2 \in \mathcal{W}$ and player a_3, there exist proper ($\neq \mathcal{N}$) coalitions $C_{1,2}, C_{2,3}, C_{3,1}$ such that $a_k, a_{k+1} \in C_{k,k+1}$ and $C_{1,2} \cap C_{2,3} \cap C_{3,1} \neq \emptyset$.*
C3. *(i) For any player $w \in \mathcal{W}$ and player a, if $\{a, w\}$ is not a coalition then there are two different players $f_1, f_2 \in \mathcal{F}$ such that $\{f_1, a, w\}$ and $\{f_2, a, w\}$ are coalitions. (ii) No coalition, which is different from $\mathcal{N}$ contains $\mathcal{W}$.*

The interpretation of these assumptions is the following. Condition C1 requires that any coalition does not contain more than one firm. Condition C2 requires that for any triplet of the form (w_1, w_2, a_3) where $a_3 \in \mathcal{F} \cup \mathcal{W}$, if $a_3 \in \mathcal{W}$, either it exists coalitions with these three workers or it exists coalitions with firms and at least two workers from the set. Else, if $a_3 \in \mathcal{F}$ then it exists coalitions containing both this firm and the two workers. Note that this condition does not require any coalition to contain at most one firm. This is obtained when jointly considering C1 and C2 which also requires that the quota of any firm is at least two. The condition C3(i) requires that if a subset of player of the form (w_1, a_2) where $a_2 \in \mathcal{F} \cup \mathcal{W}$ is not a coalition then, the pair can be embedded in two different coalitions of cardinality three where the additional players are firms. Thus, if it exists one subset of two players that is not a coalition, then the set of firms $\mathcal{F}$ must be at least of size two. The assumptions C1 and C3(i) complete each others in the sense that C1 allows for any pair of agents containing a worker to be contained in a coalition and C3(i) allows for any pair of agents containing a worker and not forming a coalition in itself to be embedded in two coalitions containing two different firms. If the agent a_2 is a worker, then C3(i) imposes that it exists two firms with quotas at least two. If the agent a_2 is a firm then C1 and C3(i) imposes that (w_1, a_2) must be a coalition otherwise the is a contradiction between the two. One may also observe that C3(i) is weaker than C2 for quotas in the sense that it only imposes that at least two firms must have a quota superior of equal to two whereas C2 imposes that all firms have a quota superior or equal to two. The condition C3(ii) requires that it does not exist a coalition different from the grand coalition that contains the set of workers $\mathcal{W}$. Thus, no coalition with any subset of firms and workers can contain the set of workers except if the subset of firms is $\mathcal{F}$. The joint consideration of C1 and C3(ii) requires that it cannot exist coalitions with a single firm and $\mathcal{W}$. Thus, the quota of any firm is at most $W-1$. Finally, the joint consideration of C1, C3(i) and C3(ii) requires the quota of at least two firms must be superior or equal to two and the the quotas of all firms must be inferior or equal to $W-1$.

As a conclusion to the interpretation of these conditions over the set of coalitions, consider the following two examples. The set of coalitions $\mathcal{C} = \{\{f\} \cup J,\ f \in \mathcal{F},\ J \subseteq \mathcal{W}, |J| \leq q_f\} \cup \{\{w\},\ w \in \mathcal{W}\}$

defined in the many-to-one case is regular if $q_f \in \{2,\ldots,W-1\}$ and $F \geq 2$. The set of subset of players (set of coalitions) $\mathscr{C} = 2^{\mathscr{N}} \setminus \{\emptyset\}$ is regular.

Example 55. *As a practical example in wireless networks, consider Device-to-Device (D2D) communication systems where qualitatively, any device (users' devices, network operators' devices) can connect to any other to communicate. The set of coalitions is the set of subsets of devices since any subset of node may form a connectivity group. This last point may be interpreted as arising from either the loss of the bipartite structure of the set of players or the additional ability of agents of the workers' side $\mathscr{W}$ to form coalitions with each others without players from $\mathscr{F}$ only.*

The responsive and substitutability property of preferences or choice functions focus on the individuals choice functions. Each agent in the game with preferences over groups must have preferences satisfying these conditions. It is only required that the preferences satisfy these conditions. In Pycia's analysis, the condition is no more on the individual preferences but on the polarization of the interest among pairs of agents (namely, pairwise alignment of the preferences) and across a specific domain of preference profiles, called rich domain of preference profiles. Such condition induces a structure in the agents' individual preferences such that core stable structures exist.

Definition 56 ([6])**.** *Preferences are pairwise aligned if for all agents $a, b \in \mathscr{N}$ and proper coalitions C, C' that contain a, b, we have: $C \underset{a}{\succeq} C' \Leftrightarrow C \underset{b}{\succeq} C'$.*

Definition 57 ([6])**.** *A domain of preference profiles $\mathbf{R}$ is called rich if it satisfies the following assumptions:*

- *(R1) For any profile $\underset{\mathscr{N}}{\preceq} = (\underset{i}{\preceq})_{i \in \mathscr{N}} \in \mathbf{R}$, any agent i, and any three different coalitions C_0, C, C_1, if $C_0 \underset{i}{\preceq} C_1$ and $i \in C$, then there is a profile $\underset{\mathscr{N}}{\preceq'} \in \mathbf{R}$ such that $C_0 \underset{i}{\preceq'} C \underset{i}{\preceq'} C_1$ and all agents $\underset{\mathscr{N}}{\preceq'}$-preferences between pairs of coalitions not including C are the same as their $\underset{\mathscr{N}}{\preceq}$-preferences.*
- *(R2) (i) For any $\underset{\mathscr{N}}{\preceq} \in \mathbf{R}$ and two different coalitions C, C_1, there is a profile $\underset{\mathscr{N}}{\preceq'} \in \mathbf{R}$ such that $C \underset{i}{\prec'} C_1$ for all $i \in C \cap C_1$ and all agents $\underset{\mathscr{N}}{\preceq'}$ -preferences between pairs of coalitions not including C are the same as their $\underset{\mathscr{N}}{\preceq}$-preferences.*

  (ii) For any $\underset{\mathscr{N}}{\preceq} \in \mathbf{R}$, any agents i, j, and any three different coalitions C_0, C, C_1, if $C_0 \underset{i}{\prec} C \underset{j}{\sim} C_1$, then there is a profile $\underset{i}{\preceq'} \in \mathbf{R}$ such that $C_0 \underset{i}{\prec'} C \underset{j}{\prec'} C_1$ and all agents $\underset{N}{\preceq'}$-preferences between pairs of coalitions not including C are the same as their $\underset{N}{\preceq}$-preferences.

We have the following theorem,

Theorem 58 ([6])**.** *Suppose that the family of coalitions $\mathscr{C}$ satisfies C1 and C2, and that the preference domain $\mathbf{R}$ satisfies R1. If all preference profiles in $\mathbf{R}$ are pairwise aligned, then (i) all $\underset{\mathscr{N}}{\preceq} \in \mathbf{R}$ admit a stable coalition structure and (ii) the stable coalition structure is unique for any profile of strict preferences $\underset{\mathscr{N}}{\preceq} \in \mathbf{R}$ that is pairwise aligned over the grand coalition.*

3.5.1 Coalition Games in Characteristic Form and Stability Inducing Sharing Rules

Assume that there exists a function $v : \mathscr{C} \to \mathbb{R}$, called characteristic function of the game, mapping any coalition $C \in \mathscr{C}$ to a real value $v(C)$, called worth of C. Originally, $v(C)$ is C's minimax worth when the players in C strategically play against those in $\mathscr{N} \setminus C$ to maximize the worth $v(C)$ (see [12] for more details and [9] for the original definition). By definition, the worth $v(C)$ of any coalition $C \in \mathscr{C}$ depends on the structure only through coalition C and is independent of the partitioning of the players in $\mathscr{N} \setminus C$. The impact of the rest of the agents over the group is obtained as the result of the conflict between C and $\mathcal{N}\setminus$C. Nevertheless, the definition of the characteristic function can be generalized to any mapping from the set of coalitions in $\mathbb{R}$. In Example 52, the characteristic function of a matching game with firms hiring workers may map any production line to its revenues. In Example 53, the characteristic function of a matching game with APs connecting to users' devices may map any connectivity group to its the total throughput assuming no interferences (no externalities) or a maximum level of interferences from the rest of the network.

Given a coalition C and its worth $v(\mathrm{C})$, any player i in C obtains a *payoff* $s_{i,\mathrm{C}}$. Agent i has utility $u_i(s_{i,\mathrm{C}})$ of $s_{i,\mathrm{C}}$ where u_i is i's $u_i : \mathbb{R} \to \mathbb{R}$. The preferences $\succeq_i$ of any agent i over the coalitions are induced by i's utility of the corresponding payoffs: $\mathrm{C} \succeq_i \mathrm{C}'$ iff $u_i(s_{i,\mathrm{C}}) \geq u_i(s_{i,\mathrm{C}'})$. The set of functions $\mathrm{D} = (\mathrm{D}_{i,\mathrm{C}} : \mathbb{R} \to \mathbb{R})_{\mathrm{C}\in\mathscr{C},i\in\mathrm{C}}$ mapping the coalitions' worths to the individual payoffs is called sharing rule.

Definition 59 ([6])**.** *A sharing rule is a collection of functions* $\mathrm{D}_{i,\mathrm{C}} : \mathbb{R}^+ \to \mathbb{R}^+$*, one for each coalition* C *and each of its members* $i \in \mathrm{C}$*, that maps the worth* $v(\mathrm{C})$ *of* C *into the share of output obtained by player* i*. We denote the sharing rule given by functions* $\mathrm{D}_{i,\mathrm{C}}$ *as* $\mathrm{D} = (\mathrm{D}_{i,\mathrm{C}})_{\mathrm{C}\in\mathscr{C},i\in\mathrm{C}}$.

By definition, i's payoff in C is $s_{i,\mathrm{C}} = \mathrm{D}_{i,\mathrm{C}}(v(\mathrm{C}))$.

As an example, equal sharing is a sharing rule: the worth of a coalition is equally shared among the players of the coalition. As another example, Nash's solution to the bargaining problem is another sharing rule. We now define the class of regular charing rules.

Definition 60 ([6])**.** *A sharing rule* D *is regular if:*

- *(i) It is pairwise aligned over the grand coalition: either* $\mathcal{N} \notin \mathscr{C}$*, or* $\mathcal{N} \in \mathscr{C}$ *and the pairwise alignment equivalence is true whenever* C *or* C$'$ *equals* $\mathcal{N}$
- *(ii) All functions* $\mathrm{D}_{i,\mathrm{C}}$ *are strictly increasing, continuous and* $\lim_{y\to+\infty} \mathrm{D}_{i,\mathrm{C}}(y) = +\infty$

The following results show that under regularity conditions over the set of coalitions and monotonicity of the sharing rules there is an equivalence between the existence of a core stable structure (matching in particular cases) and the pairwise alignment of the sharing rule. This first corollary (from [6]) results from the application of the two main theorems of [6] to the particular case of strictly increasing, continuous and unbounded sharing rules. These theorems generalize the results we exhibit here in the sense that they hold for the very general case of preferences as a self-sufficient ordinal approach. Particularly, the following corollary shows that under the assumption that the set of coalitions is regular and the individual payoffs are strictly increasing and continuous in the coalition worth and that the payoff goes to infinity if the worth does, then there is equivalent between the existence of a stable coalition structure for any vector of coalitions worths **v** and the pairwise alignment of the sharing rule.

Corollary 61. *Suppose that the family of coalition* $\mathscr{C}$ *satisfies C1-C3 and the functions* $\mathrm{D}_{i,\mathrm{C}}$ *are strictly increasing and continuous, and* $\lim\limits_{y\to+\infty} \mathrm{D}_{i,\mathrm{C}}(y) = +\infty$ *for all* $\mathrm{C} \in \mathscr{C}, i \in \mathrm{C}$*. Then there is a stable coalition structure for each profile of outputs iff the sharing rule* D *is pairwise aligned.*

This results shows that under some assumptions on the way of sharing the resource, it is necessary and sufficient that this allocation induces pairwise aligned preferences (for all outputs) for a stable structure to exist. The next result shows that under the assumption that the set of coalitions is regular and the the individual payoffs are strictly increasing and continuous in the coalition worth and that the payoff goes to infinity if the worth does, there is an equivalence between the set of pairwise aligned sharing rule and some max-product allocation schemes.

Proposition 62. *Suppose that the family of coalitions* $\mathscr{C}$ *satisfies C1 and C2, and the functions* $\mathrm{D}_{i,\mathrm{C}}$ *are strictly increasing and continuous, and* $\lim\limits_{y\to+\infty} \mathrm{D}_{i,\mathrm{C}}(y) = +\infty$ *for all* $\mathrm{C} \in \mathscr{C}, i \in \mathrm{C}$*. The sharing rule* D *is pairwise aligned and efficient iff there exist increasing, differentiable, and strictly log-concave functions* $u_i : \mathbb{R}^+ \to \mathbb{R}^+, a \in \mathcal{N}$*, such that* $\frac{u_i}{u_i'} = 0$ *and*

$$(\mathrm{D}_{i,\mathrm{C}}(y))_{i\in\mathrm{C}} = \underset{\sum_{i\in\mathrm{C}} s_{i,\mathrm{C}} \leq v(\mathrm{C})}{\arg\max} \prod_{i\in\mathrm{C}} u_i(s_i), \quad y \in \mathbb{R}^+, \mathrm{C} \in \mathscr{C}\setminus\mathcal{N} \tag{3.33}$$

In this thesis, we will particularly make use of the following corollary (see chapter 5 for the application of this result) showing the equivalence between Nash-bargaining like sharing rules and the existence of core stable structures.

Corollary 63. *Suppose that the family of coalition $\mathscr{C}$ satisfies C1-C3 and the sharing rule* D *is regular. There is a stable coalition structure for each preference profile induced by the sharing rule iff there exist increasing, differentiable, and strictly log-concave functions $u_i : \mathbb{R}^+ \to \mathbb{R}^+, i \in \mathscr{N}$, such that $\frac{u_i}{u_i'} = 0$ and*

$$(\mathrm{D}_{i,\mathrm{C}}(y))_{i\in\mathrm{C}} = \underset{\sum_{i\in\mathrm{C}} s_{i,\mathrm{C}} \leq v(\mathrm{C})}{\arg\max} \prod_{i\in\mathrm{C}} u_i(s_i), \quad y \in \mathbb{R}^+, \mathrm{C} \in \mathscr{C}\backslash\{\mathscr{N}\} \tag{3.34}$$

A surprising and very interesting observation is that assuming increasing concave utilities one has the equivalence between the Nash bargaining over a simplex and the core stability inducing sharing rules. Furthermore, knowing that the Nash bargaining achieves a generalized proportional fairness[3], one can see that there actually exists a strong natural link between some known resource allocation schemes in networks, the game theoretical negotiation-arbitration processes and the coalition formation problem or matching games. In chapter 4 and chapter 5, we go further in the analysis of this link and exploit it to derive a new association mechanism for WiFi.

3.6 Matching with Contracts and Externalities

Simultaneously to the introduction of the choice functions, game theorists and economists have introduced contracts in the theory to allow for the binary association (matched or not matched) to be completed by additional terms. Matching with contracts generalize the classical formulation [1][2] of the stable matching theory and incorporates the well-known college admission problem, the Kelso-Crawford labor market matching model and some ascending packet auctions [4]. A recent paper [5] extends matching with contracts to allow for externalities in the many-to-many settings. In this section, we show the model and some of the results of this work in view of the matching problem with contracts, externalities and scheduling constraints assessed in chapter 8. Most of the definitions are given in terms of the recent general model of matching with contracts and externalities defined by Pycia and Yenmez in [5] but for the sake of completeness and to show how the generalization is obtained we also give some definitions in terms of the models of matching with contracts without externalities as defined by Hatfield and Milgrom in [4] and for the sake of homogeneity w.r.t. chapter 8 we define the model in terms of firms and workers despite the original formulation is in terms of buyers and sellers. To the best of out knowledge, this work along with those on trading networks constitute the more advanced analysis in static matching games with complete informations (observe that some results are related to the dynamic setting such as the vacancy chain dynamic, see [5]).

3.6.1 Model and Results

In this model, the agents interact with each others through bilateral contracts.

Consider a finite sets of firms $\mathscr{F}$ and workers $\mathscr{W}$. Let define the finite set of *contracts* $\mathscr{X}$ between workers in $\mathscr{W}$ and firms in $\mathscr{F}$ as the set of bilateral pairwise agreements that the workers and the firms can sign with each others. A contract $x \in \mathscr{X}$ specifies a worker, a firm and additional terms such as a task to be executed, a wage, the execution constraints or the penalties for non-execution. For simplicity, one may assume the set of contracts $\mathscr{X}$ as a subset of $\mathscr{F} \times \mathscr{W} \times \mathscr{T} \times \mathscr{S}$, where $\mathscr{T}$ is a finite set of tasks and $\mathscr{S}$ is a finite set of salaries $\mathscr{S} = \{\underline{s}, \dots, \overline{s}\}$ (see [13] and [4]) paid to the worker for the execution of the task. This reduces the set of contracts to a subset of the set of four-tuple of the form (f, w, τ, s) where is a firm, w is a worker, τ is a task in $\mathscr{T}$ and s is a salary.

[3]Including as a special case the well-known and commonly-used proportional fairness

In the general case, when considering additional endogenously fixed terms such as the slackness in delays, the adjustable terms of the contracts form a n-uplet that is called *generalized salary* (see [13]).

For any subset of contracts $X \subseteq \mathscr{X}$, X_i denotes the maximal set of contracts in X involving i,

$$X_i = \{x \in X | i \in \{f(x), w(x)\}\} \tag{3.35}$$

A matching μ between workers and tasks is defined as a set of contracts $\mu \subseteq \mathscr{X}$. This definition used in matching with contracts generalizes the one used in the stable matching theory without contracts in the sense that the association binary relation between the players of a pair (*matched* or *unmatched*) is completed by the terms of the signed contract (among those that can be signed between the two) and that more than one contract can be signed simultaneously between any two players. If the set of contracts that can be signed between the players of a pair is reduced to a singleton, then two matched players are engaged through the single existing contract between them. Observe that if the set of contracts that can be signed between the players of any pair is reduced to a singleton then the problem falls in the class of the college admission problem.
We define player i's choice function,

$$c_i : 2^{\mathscr{X}} \times 2^{\mathscr{X}} \rightarrow 2^{\mathscr{X}} \tag{3.36}$$

such that $c_i(X|\mu) = c_i(X_i|\mu)$ is the choice set of i in X_i knowing $\mu = (\mu_i, \mu_{-i})$. It is the set of contracts that i chooses from X_i given the set μ of contracts signed. As an example, consider the choice functions without prediction as in the following definition.

Definition 64 ([5]). *We construct the choice of agent i given* μ *from any set* X, $c_i(X|\mu)$, *as follows:*

$$c_i(X|\mu) \cup \mu_{-i} \succeq_i X'_i \cup \mu_{-i} \textit{ for every } X'_i \subseteq X_i \tag{3.37}$$

This definition takes the agents' choice functions and preferences of the players as primitives of the model and does not explicitly assumes utility functions for the players.

Nevertheless, it is interesting to see how choice and utility functions depend on each others. The following definition from [14] defines the choice set of i in $Y \subseteq \mathscr{X}$ as the collection of sets of contracts in Y maximizing the utility $u_i : 2^{\mathscr{X}} \times 2^{\mathscr{X}} \rightarrow \mathbb{R} \cup \{-\infty\}$ of i where $-\infty$ is used for sets of contracts that are not feasible, i.e. violating some constraints (given by definition of the game).

$$c_i(Y|\mu) = \{Z \subseteq Y : Z \text{ is feasible}\ ; \forall \text{ feasible } Z' \subseteq Y, u_i(Z|\mu) \geq u_i(Z'|\mu)\} \tag{3.38}$$

Given any subset of players $\mathscr{N} \subseteq \mathscr{W} \cup \mathscr{F}$, the set of the players' choice functions $\{c_i\}_{i \in N}$ is called a choice function profile. The set of contracts that i rejects from X_i given the set μ of contracts signed is,

$$r_i(X_i|\mu) = X_i \backslash c_i(X_i|\mu) \tag{3.39}$$

The set of contracts that players of type $\theta \in \{W, \mathscr{F}\}$ choose in $\mathscr{X}$ knowing μ is defined as,

$$C^{\theta}(X|\mu) = \bigcup_{i \in \theta} c_i(X_i|\mu_{-i}) \tag{3.40}$$

and the set of contracts that they reject from $\mathscr{X}$ knowing μ as,

$$R^{\theta}(X|\mu) = \bigcup_{i \in \theta} r_i(X_i|\mu_{-i}) \tag{3.41}$$

Let define the matching problem with contracts as $(\mathscr{F}, \mathscr{W}, \mathscr{X}, \{c_i\}_{i \in \mathscr{W} \cup \mathscr{F}})$. We have the following definitions of individual rationality and pairwise stability of a matching μ.

Definition 65 ([5]). *A matching* μ *is individually rational for agent i if* $c_i(\mu|\mu) = \mu_i$.

Definition 66 ([5]). *A worker i and a firm j form a blocking pair for matching* μ *if there exists a contract* $x \in \mathscr{X}_i \cap \mathscr{X}_j$ *such that* $x \notin \mu$ *and* $x \in c_i(\mu \cup x|\mu) \cap c_j(\mu \cup x|\mu)$. *A matching* μ *is pairwise stable if,*

- *it is individually rational for all agents,*
- *there are no blocking pairs.*

We now turn two the characterization of choice functions in terms of variations in chosen sets and rejects when considering changes in the current market (matching) μ or in the sets. We particularly consider the irrelevance of rejected contracts and substitutability conditions that have recurrently been shown in the literature to be of fundamental importance (in the successive generalizations of the analysis of matching markets) as sufficient conditions for the existence of stable matchings (as examples, see [4] for matching with contracts and no externalities and [5] for matching with contracts and externalities).

3.6.2 Irrelevance of Rejected Contracts

The first condition that we define on choice functions is the *Irrelevance of Rejected Contracts* (IRC). A choice function satisfies IRC if some contracts are not chosen by a player, then the player's choice in a subset containing the choice is the same whatever the rejected contracts are in the set or not . The definition below is used in models without externalities,

Definition 67 ([4]). *A choice function c^i is said to satisfy the irrelevance of rejected contracts if for all* $X, X' \subseteq \mathscr{X}$*, we have*

$$c_i(X') \subseteq X \subseteq X' \Rightarrow c_i(X') = c_i(X) \tag{3.42}$$

While the following one is used in matching with contracts and externalities

Definition 68 ([5]). *A choice function C^θ is said to satisfy the irrelevance of rejected contracts if for all* $X, X', \mu \subseteq \mathscr{X}$*, we have*

$$C^\theta(X'|\mu) \subseteq X \subseteq X' \Rightarrow C^\theta(X'|\mu) = C^\theta(X|\mu) \tag{3.43}$$

3.6.3 Subtitutabilities

Now, we consider the substitutability property that asks for the complementarities among contracts to be weak enough not to turn a rejected contract into a selected one by introducing new opportunities. In matching games without externalities, we have the following definition,

Definition 69 ([4]). *Elements of $\mathscr{X}$ are substitute for firm f if for all subsets* $X \subset X' \subset \mathscr{X}$ *we have* $r_f(X) \subset r_f(X')$.

In terms of the lattice theory, the elements of $\mathscr{X}$ are substitutes for firm f if the reject function r_f is isotone (see [4]). In the presence of externalities, the substitute condition has been generalized in [5] but requires the introduction of a preorder over matchings and a consistency property w.r.t. the choice functions of the players (see [5]).

Definition 70 ([5]). *A binary relation $\succsim^\theta$ on a domain $\mathscr{A}^\theta$ is a set of ordered pairs. If it is reflexive and transitive it is a preorder.*

Definition 71 ([5]). *A preorder $\succsim^\theta$ is consistent with the choice function C^θ if for any* $X, X', \mu, \mu' \subseteq \mathscr{X}$*,*

$$X' \supseteq X \;\&\; \mu' \succsim^\theta \mu \Rightarrow C^\theta(X'|\mu') \succsim^\theta C^\theta(X|\mu) \tag{3.44}$$

We now give the recent more general definition of substitutability,

Definition 72 ([5]). *Choice function C^θ satisfies substitutability's if for any* $X, X', \mu, \mu' \subseteq \mathscr{X}$*,*

$$X' \supseteq X \;\&\; \mu' \succsim^\theta \mu \Rightarrow R^\theta(X'|\mu') \supseteq R^\theta(X|\mu) \tag{3.45}$$

Let us insist on the difference in the definitions between the model without and with externalities. Matching with contracts and externalities subsume previous models developed in matching games with contracts and without externalities. The difference lies in the introduction of the conditioning in the definition of the choice functions. Furthermore, preorders are introduced as a form of *common ranking* that is interpreted and reflecting market conditions. If $\mu \tilde{\succeq}^{\theta} \mu'$, then it is said that μ reflects better market conditions than μ'. Without externalities, the appropriate simplified framework is the one defined by Hatfield and Milgrom in [4].

3.6.4 Modified Deferred Acceptance and Existence of Stable Matchings

Finally, to conclude this introduction chapter to stable matchings and the game-theoretic analysis of two-sided markets, we give an existence result for the general setting of matching with contracts and externalities.

Consider the following modified deferred acceptance algorithm,

Algorithm 5: The modified deferred acceptance algorithm

Data: $\Gamma = (\mathcal{F}, \mathcal{W}, \{c_i\}, \mathcal{X})$
Result: μ

Phase 1: Construction of an auxiliary matching μ^* such that $\mu^* \succeq^{F} C^{F}(\mathcal{X}|\mu)$
-Set $\mu_0 = \emptyset$;
while $\nexists l < k$ *such that* $\mu_l = \mu_k$ **do**
  -k = k+1;
  -$\mu_k = C^{F}(\mathcal{X}|\mu_{k-1})$
-Set $\mu_* = \mu_k$;
Phase 2: Construction of a stable matching
-Set $A^{F}(1) = \mathcal{X}$, $A^{W}(1) = \emptyset$, $\mu^{F}(1) = \mu^*$ and $\mu^{W}(1) = \emptyset$;
while $A^{F}(k) \neq A^{F}(k-1), A^{W}(k) \neq A^{W}(k-1), \mu^{F}(k) = \mu^{F}(k-1), \mu^{W}(k) = \mu^{W}(k-1)$ **do**
  - k = k+1;

$$A^{F}(k) = \mathcal{X} \setminus R^{W}(A^{W}(k-1)|\mu^{W}(k-1)) \tag{3.46}$$
$$A^{W}(k) = \mathcal{X} \setminus R^{F}(A^{F}(k-1)|\mu^{F}(k-1)) \tag{3.47}$$
$$\mu^{F}(k) = C^{F}(A^{F}(k-1)|\mu^{F}(k-1)) \tag{3.48}$$
$$\mu^{W}(k) = C^{W}(A^{W}(k-1)|\mu^{W}(k-1)) \tag{3.49}$$

We have the following theorem,

Theorem 73 ([5])**.** *Suppose that the choice functions satisfy substitutability and the irrelevance of rejected contracts. Then, the algorithm terminates, its outcome is stable and*

$$\mu^{F}(T) = \mu^{W}(T) = A^{F}(T) \cap A^{W}(T) \tag{3.50}$$

This result will be extensively studied in Chapter 8, where we study a crowdsourcing marketplace with scheduling contracts using matching games with contracts and externalities. We particularly propose a weaker definition of substitutability, called constrained substitutability, and show a similar theorem proving the existence of stable matchings under new particular conditions (inducing substitutability on the firms' side).

An interesting observation is that to show Theorem 73, one cannot apply the commonly used Tarsi's fixed point theorem (see [5], pp.22 for more details).

3.7 References

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Chapter 4

Nash Bargaining for Resource Allocation in WiFi

In this chapter, we show that the expected performance of the well-known IEEE 802.11 WiFi resource allocation scheme can be modeled as a Nash bargaining. Particularly, the individual throughputs are obtained as Nash bargaining solutions corresponding to Nash bargaining points in a well-defined utility space. This result will be used in Chapter 5 to define a controlled stable matching mechanism for the association problem in WiFi.

4.1 Competition for Resource Allocation in Networks

Most of the resource allocation problems in networks belong to the class bargaining problems. Typically, many players (agents, devices, servers) compete over a resource to be shared. The set of jointly achievable utilities may have various interpretations. Depending on the problem, it may be a set of rate vectors called the rate region of the system, the resource itself (space, time, bandwidth, money) or any other.

4.2 WiFi

Any communication among two or more participants is fundamentally based on a *medium access protocol* which regulates the access to the medium and thus the transmission of the signals. Such protocol defines the rules regulating the transmission events. It does not concern the definition of the transmitted signal such as its content or conveyed message, its amplitude, sampling, power or modulation. The prescribed rules should answer at least the following questions: Is there a centralized controller or arbitrator in the transmission decision-taking process? If yes, what is the arbitration rule? If not, when should an agent decide to talk? What about if two or more agents transmit their signals simultaneously? How long should a transmission last? What about an incomplete message? Observe that these concepts and questions are not restricted to telecommunication networks and extend to any other setting where information has to be transmitted among agents, whatever hardware humans, animals or devices.

More formally, we have the following definition,

Definition 74 ([1])**.** *A Medium Access Control (MAC) protocol regulates the use of the medium by prescribing the rules to initiate a transmission and continue with it. In random access networks, collisions may occur and the MAC protocol has to resolve collisions; i.e. arbitrate among the nodes contending the use of the medium.*

In this thesis, we focus on a specific form of wireless telecommunication networks called *Wireless Local Area Networks* (WLANs). Particularly, on the WiFi (or, Wi-Fi) technology based on the IEEE 802.11 standards. Several MAC protocols have been defined in the framework of the 802.11 standards. We specifically consider the *Carrier Sense Multiple Access with Collision Avoidance* (CSMA/CA) random access based MAC protocol. Basically, it is a distributed resource sharing mechanism without central coordination based on random access techniques.

4.2.1 Channel Usage Model

Before entering the description of the IEEE 802.11 CSMA/CA DCF MAC protocol, we describe the *channel usage model*. This is the model abstracting the interactions between the transmitted signals at the physical layer. The model we consider hereinafter is the one shown by Kumar et al. in [1] (pp.188).

By definition of random access protocols, there is a strictly positive probability that two or more transmitters emit simultaneously (for more details see Section 4.2.2). In WLANs, the emitting nodes transmit over the whole dedicated bandwidth, the consequence is an unavoidable superposition of the signals over the medium, called *interference*. If the interference level is too high at every receiver, then none of the receivers can successfully decode any message. This is called a *collision*. If the interference level is not too high then at least one receiver may be able to decode a message. This is called a *capture*.

The main difference between the capture and the collision is that, in the first case at least one receiver has decoded a signal while in the second none of the transmitted units of information can be taken as a useful unit. If the capture is done by the right receiver (the node the emitter was transmitting to) the corresponding units of information can be taken into account in the computation of the throughput. Observed that assuming no capture gives a lower bound on the achievable throughputs.

Is is clear from the description of the collision and capture events that the issue of a simultaneous transmission strongly depends on the level of the interference signals at the receiver. To take this into account, this channel usage model assumes a spatial discretization around each node in three regions (two boundaries). The discretization is different depending on the role of the node, emitter or receiver. The inner region is called the *decode region*. For a receiver (transmitter), it is the set of locations in space where the Signal-to-Noise Ratio (SNR) is such that the received (transmitted) signal can be decoded and thus interpreted if not collided. The intermediate region is called *interference region* (or *carrier sense region*). For a receiver, it is the set of locations in space such that an interfering signal from these locations induces a collision. For a transmitter, it is the set of locations in space such that if transmitting, the induced interference at any of these locations creates a collision. It is also the set of points in space where any node can sense the channel and detect a channel activity due to the transmitter. In general, the boundaries of the interference and decode regions depend on the interfering nodes. As in [1], this is not taken into account in this description.

We assume the set of nodes of a cell C are within such a distance from each other that only one transmission can occur at any point in time. This assumption is known in the name of *single-cell networks*, see [2] for more details.

We now enter the details of the CSMA/CA protocol, its implementation in the IEEE 802.11 (Wi-Fi standard) Distributed Coordination Function (DCF) protocol and the performance analysis of a cell implementing this random access technique.

4.2.2 CSMA and CA

The *Carrier Sensing Multiple Access* (CSMA) protocol belongs to the class of random access sensing protocols that combine a random access decentralized resource sharing mechanism and sensing techniques. The first has been shown to reduce both the transmission times and the access delay w.r.t *Time Division Multiplexing* (TDM) schemes. The second avoids myopic transmission over the shared medium and thus reduce collisions w.r.t. pure random access schemes such as Aloha. In CSMA, once a node has started transmitting a packet, the rule is to complete the transmission of this packet. Even though completed by additional sensing and avoidance mechanisms, by definition of random access protocols there is a positive probability for two or more transmissions to collide. A collision may be characterized by a time interval called *collision window*. We give below the formal definition,

Definition 75 ([1]). *The collision windows for CSMA is the time interval since the beginning of a transmission during which another node (not having heard the ongoing transmission) can begin its own transmission, and hence collide with the first transmission.*

Another characteristic time is the *duration of a collision* in a network,

Definition 76 ([1]). *The duration of a collision in a network is the time from the beginning of the first transmission in the collision until the earliest time at which a fresh transmission can begin.*

We now turn to the *Collision Avoidance* (CA) mechanism introduced as a complement to the CSMA protocol in WLANs to avoid collision due to *hidden nodes*. As already observed, one of the main problem of random access with channel sensing is the collision. It may happen that a node may starts emitting while another transmission is already going on. There are two reasons for such event to happen. The first is that every signal has a strictly positive propagation delay. Due to such delays, the transmitted signal does instantaneously reach the nodes in the interference region. In the intermediate time, some nodes in the network may have sensed the busy channel as free. The second reason is due to the finiteness of the sensing regions. Some nodes, called *hidden nodes*, may not detect the channel as busy and transmit, thus inducing a collision if the receiver is in their interference region.

The collision avoidance mechanism of the CSMA/CA protocol is based on the principle of *public handshake* which consists in a bilateral agreement on the channel acquisition (reservation) for the transmission of a packet between the source and the receiver. The acceptance of the agreement by the nodes is conditioned on the reception of notification packets called *Request To Send* (RTS) and *Clear To Send* (CTS). The process is the following: the emitter sends an RTS to the receiver who acknowledges a good reception (decoding of the RTS) by sending a CTS packet to the receiver. If the CTS is not received by the source in a certain amount of time, then it is assumed that the RTS has been collided. This RTS and CTS packets can be decoded by any node in the decode region of the packets' transmitters. Thus, any node in the decode region of the transmitter of the RTS is informed of a oncoming transmission of a packet and any node in the decode region of the transmitter of the CTS is informed of the oncoming reception of a packet. This allows for hidden nodes to be aware of the oncoming channel activity and thus defers eventual transmission to a subsequent time-slot. An additional development of the CSMA/CA protocol has introduced an *acknowledgement* packet from the receiver after the successful transmission of a packet.

4.2.3 IEEE 802.11 standards

The CSMA/CA protocol has been normalized by the IEEE in the IEEE 802.11 (b,a,g,n,e) WLAN standards. We focus on the one called IEEE 802.11 *Distributed Coordination Function* (DCF) protocol which relies on individual timers called backoffs. Each node has a backoff that is synchronously decreased among the nodes. The value of the backoff results, at each node, from a sampling over sets of indivisible and standardized time intervals called *slots*. As an example, in the IEEE 802.11b version of the protocol, each (time-)slot is of length 20µs. The backoff of a node is the number of slots the node has to wait in unfrozen backoff intervals before attempting to access the channel. When the backoff of a player is elapsed, the player attempts a reservation of the medium by sending an RTS packet.

In the general case, called *non-homogeneous*, the backoff parameters may vary over the nodes. If these are the same for all nodes, then this is the *homogeneous case*.

If no other player emits an RTS in the same time interval separating its backoff end event and reception of the emitted RTS event, then all backoff timers are frozen. The first emitter exclusively accesses the channel, waits for a CTS from recipient of RTS. The recipient is ready for reception. Packets are transmitted and the reservation ends with an ACK. It is clear from this description that the time taken for a transmission depends on the *physical transmission rate* of the transmitting node. Thus, a node with a low *physical transmission rate* will hold the channel for a longer time (per bit) than a node with a high *physical transmission rate*.

If one or many other players emit an RTS packet in the time interval separating their backoff end and their reception of the emitted RTS event, then each reservation attempting player waits for a time interval SIFS + T_{CTS} + DIFS. Other players have to wait for SIFS + T_{ACK} + DIFS. Sub-waiting times T_{CTS}, T_{ACK} are equal[1]. Let T_o denote the fixed overhead with a packet transmission in slots (52 slots in IEEE 802.11b) and T_c denote the fixed overhead with an RTS transmission (20 slots in IEEE 802.11b). Is is clear from the description of this mechanism that if two or more backoffs end simultaneously, then RTS collide. If only a single backoff ends, then the corresponding node successfully access the channel for transmission. As observed by Kumar et a.l. in [2], it is sufficient to analyze the backoff process to analyze the channel allocation process.

Example 77. *In Figure 4.1, Figure 4.2 and Figure 4.3, we show an example of application of the CSMA/CA mechanism among five nodes. This example shows how each node's data rate has an impact on the others' throughputs. Consider Figure 4.2 (left) where node* 1 *emits a packet and Figure 4.3 (right) where node 3 emits a packet. The data rate of node 1 is higher than the data rate of node 3. The emission of a packet by node 3 is longer than the emission by node 1. During this additional amount of time, no backoff can be decreased and other nodes have to wait for the end of the transmission and the acknowledgement of the recipient (observe that acknowledgements are not shown in the figures). Thus, low data rate nodes have an impact on high data rate ones: the*

[1] due to an equal number of bits of ACK an CTS and fixed control rate of 2 Mb/s

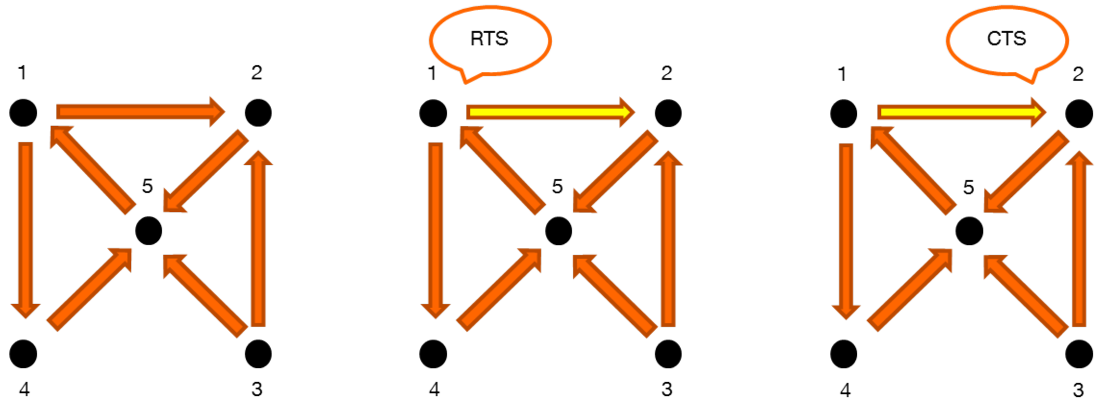

Figure 4.1: **Left:** A set of nodes (black) and their flows (arrows). As an example, node 1 has a flow for node 2. **Middle:** The backoff of node 1 has elapsed. The node sends an RTS to node 2. **Right:** Node 2 receives and decodes the RTS. It sends a CTS.

reduce the expected amount of time they can transmit, thus reducing their expected throughputs. This effect is called WiFi anomaly.

In this chapter, as in [2], we assume that all nodes use the RTS/CTS mechanism for distributed coordination. We furthermore assume that each node always has packets to send. This is known in the name of *saturation assumption.*

4.2.4 Single-Cell Performances

Even though there exists an exact Markov chain based model for the joint backoff process, Kumar et al. in [2] show its intractability and derive a fixed-point formulation of the long run average backoff rate β using the decoupling approximation. The attempts rate is then defined by the following fixed point equation,

$$\beta = G(\Gamma(\beta)) \tag{4.1}$$

where,

$$\Gamma(x) = 1 - e^{-(|C|-1)x} \tag{4.2}$$

where C is the set of nodes in the cell and,

$$G(x) = \frac{1 + x + \ldots + x^K}{b_0 + x b_1 + \ldots + x^K b_K} \tag{4.3}$$

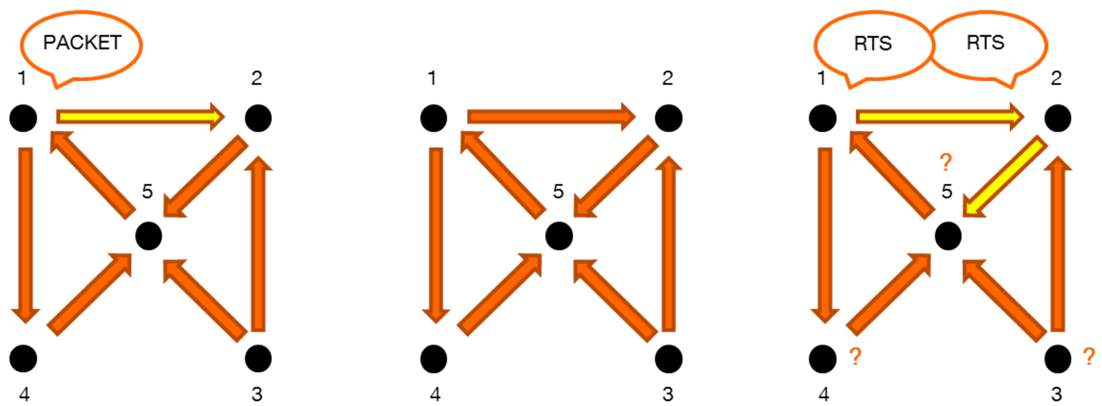

Figure 4.2: **Left:** Node 1 sends a packet at its data rate. **Middle:** Node 1's packet has been transmitted. Backoff timers are decreasing in time. **Right:** Both node 1's backoff and node 2's backoff have elapsed. The two nodes send an RTS and there is a collision. No node can decode the emitted RTS. No CTS packets are emitted.

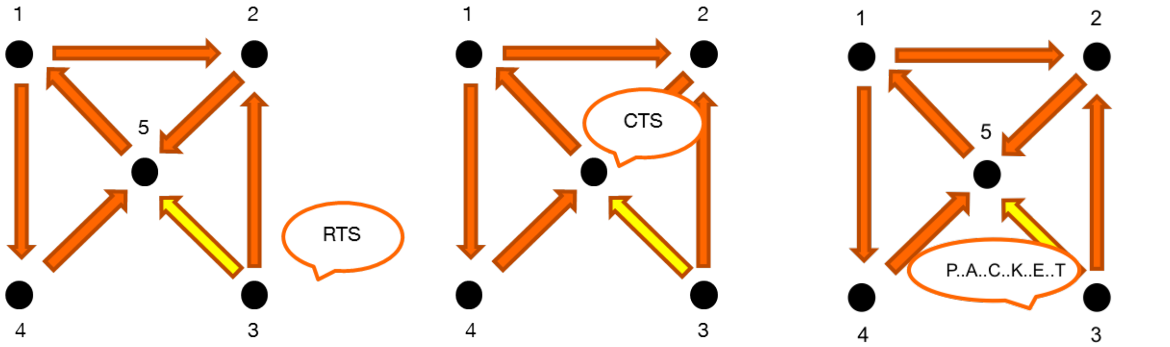

Figure 4.3: **Left:** Nodes 3's backoff has elapsed. The node sends an RTS. **Middle:** Node 5 decodes and answers with a CTS. **Right:** Node 3 transmits a packet at its data rate. This data rate is lower than node 1 data rate. The emission of node 3's packet takes a longer time than node 1's emission. In this additional amount of time, no node can emit.

where K is the limit in the number of attempts before discarding the packet to be transmitted and b_k is the mean backoff duration at the k^{th} attempt for a packet to be transmitted. Define for each player $i \in \mathcal{N}$, the set $\mathcal{B}_i$ of flows $L_{ij} \in \mathcal{B}_i$ generated by i and to be transmitted to a recipient $j \in \mathcal{N}$. If the player i is a user from $\mathcal{W}$, then the only recipient is the access point $\mu(i) \in \mathcal{F}$ it is associated to at association (or matching) μ. We will formally define a model linking the WiFi association and matching games in chapter 5. If the player i is an access point from $\mathcal{F}$, then the only recipients of its flows are the users in its cell $\mu(i) = J \subset \mathcal{W}$. Packet length of a flow is L_{ij}. The fraction $p_{i,j}$ of packets of player i belongs to stream $j \in \mathcal{B}_i$. This fraction can be interpreted as the probability of a packet of player i being from flow j. Each pair (i, j) of players has a data rate θ_{ij} (bits/slot) from the physical transmission rates space $\Theta = \{\theta^1, \ldots, \theta^m\}$. The 802.11b standard defines four rates, 1, 2, 5.5 and 11 Mbps. The physical transmission rate of a pair depends on the control scheme implemented in the sending node and are obtained as functions of the Modulation and Coding Schemes (MCS), the number of spatial streams, the bandwidth, and the Guard Interval (GI).

The saturation throughput of node i for its flow j when in cell C is denoted $r_{ij,C}$ (bits per slot). We have:

$$r_{ij,C} = \frac{p_{ij} L_{ij} \beta(1-\beta)^{|C|}}{1 + \sum_{i=1}^{|C|} (\beta(1-\beta)^{|C|-1}((\sum_{k=1}^{m_i} p_{ik}\frac{L_{ik}}{\theta_{ik}}) + T_0)) + ((1-(1-\beta)^{|C|} - |C|\beta(1-\beta)^{|C|-1})T_C)} \tag{4.4}$$

Thus, we have the saturation throughput of node i in cell C, denoted $r_{i,C}$, as the sum over its flow's saturation throughput,

$$r_{i,C} = \frac{\beta(1-\beta)^{|C|}}{1 + \sum_{i=1}^{|C|} (\beta(1-\beta)^{|C|-1}((\sum_{k=1}^{m_i} p_{ik}\frac{L_{ik}}{\theta_{ik}}) + T_0)) + ((1-(1-\beta)^{|C|} - |C|\beta(1-\beta)^{|C|-1})T_C)} \times \sum_{j=1}^{|B_i|} p_{ij} L_{ij} \tag{4.5}$$

where T_o is the fixed overhead with a packet transmission in slots, T_c is the fixed overhead for an RTS collision in slots and β is the attempt rate of any player. The total cell throughput R_C, where C denotes the cell or coalition and $|C|$ the number of players it contains,

$$R_C = \sum_{i=1}^{|C|} \sum_{j=1}^{|B_i|} r_{ij,C} \tag{4.6}$$

We define the function $\kappa : \mathcal{C} \to \mathbb{R}$ from the set of cells on $\mathbb{R}$ as,

$$\kappa(C) = \frac{\beta(1-\beta)^{|C|}}{1 + \sum_{i=1}^{|C|} (\beta(1-\beta)^{|C|-1}((\sum_{k=1}^{m_i} p_{ik}\frac{L_{ik}}{\theta_{ik}}) + T_0)) + ((1-(1-\beta)^{|C|} - |C|\beta(1-\beta)^{|C|-1})T_C)} \tag{4.7}$$

4.3 WiFi as a Nash Bargaining

We show that the IEEE 802.11 protocol induces an allocation of the resource (and thus saturation throughputs) that can be modeled as a Nash bargaining.

For any node i in a cell C, node i's saturation throughput given by equation (4.5) can be re-written as functions of $\kappa(\mathrm{C})$,

$$r_{i,\mathrm{C}} = \sum_{j=1}^{|\mathscr{B}_i|} p_{ij} \mathrm{L}_{ij} \kappa(\mathrm{C}) \tag{4.8}$$

Equivalently,

$$r_{i,\mathrm{C}} = \frac{\sum_{j=1}^{|\mathscr{B}_i|} p_{ij}\mathrm{L}_{ij}}{\sum_{i\in\mathrm{C}} \sum_{j=1}^{|\mathscr{B}_i|} p_{ij}\mathrm{L}_{ij}} \times \left[\sum_{i\in\mathrm{C}} \sum_{j=1}^{|\mathscr{B}_i|} p_{ij}\mathrm{L}_{ij} \right] \kappa(\mathrm{C}) \tag{4.9}$$

Which can be written as,

$$r_{i,\mathrm{C}} = \frac{\left(\frac{\sum_{j=1}^{|\mathscr{B}_i|} p_{ij}\mathrm{L}_{ij}}{\sum_{i\in\mathscr{N}} \sum_{j=1}^{|\mathscr{B}_i|} p_{ij}\mathrm{L}_{ij}} \right)}{\left(\frac{\sum_{i\in\mathrm{C}} \sum_{j=1}^{|\mathscr{B}_i|} p_{ij}\mathrm{L}_{ij}}{\sum_{i\in\mathscr{N}} \sum_{j=1}^{|\mathscr{B}_i|} p_{ij}\mathrm{L}_{ij}} \right)} \times \left[\sum_{i\in\mathrm{C}} \sum_{j=1}^{|\mathscr{B}_i|} p_{ij}\mathrm{L}_{ij} \right] \kappa(\mathrm{C}) \tag{4.10}$$

or,

$$r_{i,\mathrm{C}} = \frac{\left(\frac{\sum_{j=1}^{|\mathscr{B}_i|} p_{ij}\mathrm{L}_{ij}}{\sum_{i\in\mathscr{N}} \sum_{j=1}^{|\mathscr{B}_i|} p_{ij}\mathrm{L}_{ij}} \right)}{\left(\sum_{i\in\mathrm{C}} \frac{\sum_{j=1}^{|\mathscr{B}_i|} p_{ij}\mathrm{L}_{ij}}{\sum_{i\in\mathscr{N}} \sum_{j=1}^{|\mathscr{B}_i|} p_{ij}\mathrm{L}_{ij}} \right)} \times \left[\sum_{i\in\mathrm{C}} \sum_{j=1}^{|\mathscr{B}_i|} p_{ij}\mathrm{L}_{ij} \right] \kappa(\mathrm{C}) \tag{4.11}$$

Using the results of the Dividing the Dollars game (see Chapter 2), we obtain that the nodes' saturation throughputs (4.17) solve the following optimization program,

$$\begin{aligned} \underset{\mathbf{x}}{\text{maximize}} \quad & \prod s_{i,\mathrm{C}}^{\sum_{j=1}^{|\mathscr{B}_i|} p_{ij}\mathrm{L}_{ij} / \sum_{i\in\mathscr{N}} \sum_{j=1}^{|\mathscr{B}_i|} p_{ij}\mathrm{L}_{ij}} \\ \text{subject to} \quad & \sum_{i=1}^{d} s_{i,\mathrm{C}} = \left[\sum_{i\in\mathrm{C}} \sum_{j=1}^{|\mathscr{B}_i|} p_{ij}\mathrm{L}_{ij} \right] \kappa(\mathrm{C}) \\ & 0 \le s_{i,\mathrm{C}},\ i = 1, \ldots, d. \end{aligned} \tag{4.12}$$

The solution to this problem is the Nash bargaining solution to the bargaining problem modeling the competition between the players of a set C with utility functions such that any node i in C,

$$u_i(s_{i,\mathrm{C}}) = s_{i,\mathrm{C}}^{\alpha_i} \tag{4.13}$$

with $\alpha_i \in [0;1]$ defined by,

$$\alpha_i = \frac{\sum_{j=1}^{|\mathscr{B}_i|} p_{ij}\mathrm{L}_{ij}}{\sum_{i\in\mathscr{N}} \sum_{j=1}^{|\mathscr{B}_i|} p_{ij}\mathrm{L}_{ij}} \tag{4.14}$$

and over the worth $v(\mathrm{C})$ such that,

$$v(\mathrm{C}) = \left[\sum_{i\in \mathrm{C}}\sum_{j=1}^{|\mathscr{B}_i|} p_{ij}\mathrm{L}_{ij}\right]\kappa(\mathrm{C}) \tag{4.15}$$

The set of feasible allocations $\mathscr{S}$ is thus defined as the following,

$$\mathscr{S} = \left\{ s \in \mathbb{R}^{|\mathrm{C}|} | \sum_{i\in \mathrm{C}} s_{i,\mathrm{C}} = \left[\sum_{i\in \mathrm{C}}\sum_{j=1}^{|\mathscr{B}_i|} p_{ij}\mathrm{L}_{ij}\right]\kappa(\mathrm{C}) \;\&\; s_{i,\mathrm{C}} \geq 0 \;\forall i \in \mathrm{C} \right\} \tag{4.16}$$

Finally, observe that one may also have written mobile device i's throughput as,

$$r_{i,\mathrm{C}} = \frac{\left(\frac{\sum_{j=1}^{|\mathscr{B}_i|} p_{ij}\mathrm{L}_{ij}}{\mathrm{L}_{max}}\right)}{\left(\sum_{i\in \mathrm{C}} \frac{\sum_{j=1}^{|\mathscr{B}_i|} p_{ij}\mathrm{L}_{ij}}{\mathrm{L}_{max}}\right)} \times \left[\sum_{i\in \mathrm{C}}\sum_{j=1}^{|\mathscr{B}_i|} p_{ij}\mathrm{L}_{ij}\right]\kappa(\mathrm{C}) \tag{4.17}$$

where L_{max} is the maximum packet length over all flows in the game. In such case, we define the bargaining power α_i of mobile device i's throughput as,

$$\alpha_i = \frac{\sum_{j=1}^{|\mathscr{B}_i|} p_{ij}\mathrm{L}_{ij}}{\mathrm{L}_{max}} \tag{4.18}$$

Observe that α_i equals one if all packet sizes are equal to L_{max}. This leads to a symmetric set of jointly achievable utility points **B** and an equal sharing of the total throughput.

4.4 Conclusion

In this chapter, we have shown that under classical assumptions, the IEEE 802.11 protocol (which basically consists in a competition between heterogeneous nodes for an access to the medium) can be modeled as a Nash bargaining among the nodes (or players from a game-theoretic formalism). This is a surprisingly interesting result that will allows us to use recent results from the theory of stable matchings. Particularly, in this system there exists a core stable partitioning of the nodes in cells. We will particularly focus on the case of a bipartite structure where the players can be partitioned into two disjoint sets, namely the set of mobile users and the set of access points. We will also be able to derive some results on the controllability of such game. Particularly, we will show that one can modify the characteristic function v, defining the total saturation throughput of a cell, to provide the players the incentives for some subset of coalitions and transfer the core from some subset of matchings to another. This will allow to provide priorities in the connectivity management, even though decentralized with selfish players aiming to maximize their individual payoff.

4.5 References

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[2] A. Kumar, E. Altman, D. Morandi, and M. Goyal, "New insights from a fixed-point analysis of single-cell ieee 802.11 wlans," *IEEE/ACM Transactions on Networking*, vol. 13, pp. 588–601, June 2007.

Chapter 5

A Cooperative Game Theoretic Analysis of WiFi

In multi-rate IEEE 802.11 WLANs, the traditional user association based on the strongest received signal and the well known anomaly of the MAC protocol can lead to overloaded Access Points (APs), and poor or heterogeneous performance. Our goal is to propose an alternative game-theoretic approach for association. We model the joint resource allocation and user association as a two-sided matching game with rational players maximizing their individual throughputs. Using the results of chapter 3 and chapter 4, we first show that the IEEE 802.11 WiFi protocol in its Distributed Coordination Function (DCF) implementation and other resource sharing protocols fall in the scope of the set of core stability-inducing resource allocation schemes. The game makes an extensive use of the Nash bargaining and some of its related properties that allow to control the incentives of the players. We show that the proposed mechanism can greatly improve the efficiency of 802.11 with heterogeneous nodes and reduce the negative impact of peer effects such as its MAC anomaly. The mechanism can be implemented as a virtual connectivity management layer to achieve efficient APs-user associations without modification of the MAC layer.

5.1 Introduction

The IEEE 802.11 based wireless local area networks (WLANs) have attained a huge popularity in dense areas as public places, universities and city centers. In such environments, devices have the possibility to use many Access Points (APs) and usually a device selects an AP with the highest received Radio Signal Strength Indicator (best-RSSI association scheme). In this context, the performance of IEEE 802.11 may be penalized by the so called 802.11 *anomaly* and by an imbalance in AP loads (congestion). Moreover, some APs may be overloaded while others are underutilized because of the association rule.

In this chapter, we consider a fully distributed IEEE 802.11 network, in which selfish mobile users and APs look for the associations maximizing their individual throughputs. We analyze this scenario using matching game theory and develop a unified analysis of the joint mobile user association and resource allocation problem for the reduction of the anomaly and for load balancing in IEEE 802.11 WLANs. In a network characterized by a *state of nature* (user locations, channel conditions, physical data rates), composed of a set $\mathscr{W}$ of mobile users and a set $\mathscr{F}$ of APs, the user association problem consists in finding a mapping μ that associates every mobile user to an AP. We call the set formed by an AP and its associated mobile users a *cell*, or a *coalition* in the game framework. The set of coalitions induced by μ is called a matching or a *structure* (partition of the players in coalitions). Once mobile user association has been performed, a resource allocation scheme (also called a *sharing rule* in chapter 3) allocates radio resources of a cell to the associated mobile users.

This matching game is characterized by *complementarities* in the sense that APs have preferences over groups of mobile users and *peer effects* in the sense that mobile users care who their peers are in a cell and thus emit preferences also over groups of mobiles users. Indeed, by definition of DCF implementation of the IEEE 802.11 protocol, a users' throughput does not only depend on its physical data rate but also on the coalition size and composition. We are thus facing the classical association problem with the additional property that the players (mobile users and APs) are selfish and solely interested in the association maximizing their own throughput. The following questions are raised: does there exist associations (or matchings) in which no subset of players prefer deviating and associate with each others, i.e., are there stable associations? Do these associations always exist? Is there unicity? How to reach these equilibria in a decentralized way? Finally, how to provide the players the incentive to make the system converge to another association point with interesting properties in terms of load balancing?

Assuming that players associate solely w.r.t their individual throughput many mobile users may remain unassociated since every AP has the incentive to associate with a single mobile user having the best data rate. We call this problem the *unemployment problem.* To counter this side effect and provide the nodes the incentives to associate with each others, we design a decentralized three steps mechanism to control the set of the stable matchings. In other words, we will manipulate the set of core stable matching so as to make the core (see chapter 3) fall in a set of matchings with interesting properties in terms of load balancing and number of matched users. In the first step, the APs share the load. In the second step, the coalition game is controlled to provide the incentives to enforce the load balancing. In the third step, players play the *controlled* coalition game with individual payoffs obtained from a Nash Bargaining (NB)-based sharing rule. This sharing rule is interesting because it generalizes equal sharing, but also other proposals in the literature such as proportional fairness. Under some assumptions, the NB-based sharing rule guarantees that the set of stable structures is non empty in all states of nature. The control of the game is designed so as to provide the players the incentives to respect the objective of the load balancing (during the first step). The control is based on the notion of Fear of Ruin (FoR) introduced in [1]. The equilibrium point of the third step is obtained by a decentralized algorithm that results in a core stable matching (or structure). We propose here a modified version of the Deferred Acceptance Algorithm (DAA), called Backward Deferred Acceptance Algorithm (BDAA), for matching games with complementarities and peer effects. Similarly to the DAA, the complexity of the BDAA is polynomial. We show through numerical simulations that our mechanism not only ensures that a stable matching will form but is also a way to reduce the impact of the WiFi anomaly. In fact, the equilibrium association relies relies on the agents' incentives to counter the side effects induced by the protocol. Moreover, this mechanism allows us to exploit the overlapping of APs as an opportunity to reduce the anomaly of 802.11 rather than an obstacle.

5.1.1 Related Work

IEEE 802.11 (WiFi) anomaly is a well documented issue in the literature, see e.g. [3; 4; 7]. The first idea to improve the overall performance of a single cell system is to modify the MAC so as to achieve a *time-based fairness* [3; 4]. Authors of [3] propose a leaky-bucket like approach. Banchs et al. [7] achieve *proportional fairness* by adjusting the transmission length or the contention window parameters of the stations depending on their physical data rate. Throughput based fairness, time based fairness and proportional fairness resource allocation schemes are sharing rules that can be obtained a Nash bargaining points of solutions (see chapter 2).

In a multiple cell WLAN network, mobile user-AP association plays a crucial role for improving the network performance and can be seen as a mean to mitigate the WiFi anomaly without modifying the MAC layer. The maximum RSSI association approach, though very simple, may cause an imbalanced traffic load among APs, so that many devices can connect to few APs and obtain low throughput, while few of them benefit from the remaining radio resource. Kumar et al. [6] investigate the problem of maximizing the sum of logarithms of the throughputs. Bejerano et al. [10] formulate a mobile user-AP association problem guaranteeing a max-min fair bandwidth allocation for mobile user. This problem is shown to be NP-hard and constant-factor approximation algorithms are proposed. Li et al. [17].

Arguing for ease of implementation, scalability and robustness, several papers have proposed decentralized heuristics to solve this issue, see e.g. [5; 12; 14]. Reference [5] proposes to enhance the basic RSSI scheme by an estimation of the Signal to Interference plus Noise Ratio (SINR) on both the uplink and the downlink. Bonald et al. in [14] show how performance strongly depends on the frequency assignment to APs and propose to use both data rate and MAC throughput in a combined metric to select the AP. Several papers have approached the problem using game theory based on individual MAC throughput. Due to the WiFi anomaly, this is not a classical *crowding game* in the sense that the mobile user achieved throughput is not necessary a monotonically decreasing function of the number of attached devices, as it can be the case in cellular networks [11; 15]. Compared to proposed decentralized approaches, we do not intend to optimize some network wide objective function, but rather to study the equilibria resulting from selfish behaviors. Compared to other game-theoretic approaches, we consider a fully distributed scenario, in which APs are also players able to accept or reject mobile users. This requires the study of the core stability, a notion stronger than the classical Nash Equilibrium. Moreover, there is a need in understanding the fundamental interactions between mobile user association and resource allocation in the presence of complementarities and peer effects. More generally, the association problem has been studied using congestion and static non-cooperative games in [26–28]. In this chapter, we tackle the mobile user-AP association problem as a two-sided market using matching games.

Some very recent papers in the field of wireless networks have exploited the theoretical results and practical methods of matching games [19–21; 29–32], although none has considered the WLAN association problem and its related WiFi anomaly. Authors of [19] address the problem of downlink association in wireless small-cell networks with device context awareness. The relationship between resource allocation and stability is not investigated and APs are not allowed to reject users. Hamidouche et al. in [20] tackle the problem of video caching in small-cell networks. They propose an algorithm that results in a many-to-many pairwise stable matching. Preferences emitted by servers exhibit complementarities between videos and vice versa. Nevertheless, the model doesn't take into account peer effects within each group. Reference [21] addresses the problem of uplink user association in heterogeneous wireless networks. Invoking a high complexity, complementarities are taken into account by a transfer mechanism that results in a Nash-stable matching, a concept weaker than pairwise stability or core stability. Authors of [31], consider a many-to-one matching between Secondary Users (SU) and Primary Users (PU) of a cognitive radio network. The utility of a SU k on channel l (see equation (4) in the reference) does not depend on the other PU it is connected to. The sum utility does not exhibit complementarities or group effects: each term in the sum is independent of the others because of the orthogonal channels assigned to PUs. The utility of every PU (in equation (5) of the reference) only depends on the interference of the SU on the same channel (there is at most one SU per PU) and not on the other PUs this SU is connected to. There are no complementarities or peer effects. In [32], Mochaourab et a.l. consider the problem of joint user association and beamforming in multi-cell multiple-input single-output systems. The preferences of the users do not exhibit complementarities or peer effects as encountered in WiFi and they propose a proposal budget based control of the users utility to guarantee the convergence of a DAA-like algorithm. The pairwise stability is considered, a solution concept weaker than the core stability considered in this chapter.

5.1.2 Contributions

The contributions of the chapter can be summarized as follows:

• We provide a matching game-theoretic unified approach of mobile user association and resource allocation in IEEE 802.11 WLANs in the presence of complementarities and peer effects. The results of the chapter highlight the importance of the Nash bargaining in wireless networks as a stability inducer but also as a convenient and easy-to-use tool at several levels of network resource management. To the best of our knowledge, this is the first game-theoretic modeling of the IEEE 802.11 protocol covering such a number of resource allocation mechanisms proposed in the literature.

• We use existing theoretical results to show that if the scheduling and/or the MAC protocol result from a Nash bargaining then there exist stable mobile user associations, whatever the user data rates or locations.

• In order to *control* the core (equilibrium solution concept) of matching game, we design a three steps mechanism, which includes 1) a generic load balancing, 2) a control step i.e. the modifications to be applied to the characteristic function of the game (worths of the coalitions) in order to provide the agents the incentives to enforce the result of the load balancing, 3) a coalition game with resource allocation defined as a Nash bargaining over the resource to be allocated and a stable matching algorithm with players' preferences induced by the resource allocation. This three steps mechanism tackles the so called unemployment problem, that would have left mobile users aside from the association otherwise. We show through numerical examples that our mechanism achieves good performance compared to the global optimum solution. We also show how the mechanism can be used to efficiently share the load between APs.
To the best of our knowledge, such a mechanism is absent from both the game theoretic and wireless networks based on matching games literature.

• We show that our BDAA can be efficiently used to find a stable many-to-one matching in a coalition game with complementarities and peer-effects. The algorithm has a polynomial complexity in number of rounds, as the original DAA.

The mechanism has been originally proposed in an extended abstract [24]. The particular case of equal sharing has already been assessed in [23]. BDAA has been originally proposed in a short paper [22] and we provide proofs of convergence to a core stable structure and of polynomial complexity in [25] . In this chapter, we provide a complete description and show the mathematical results. We furthermore generalize the mechanism to a generic load balancing scheme and to the Nash bargaining-based sharing rules (resource allocation schemes). Finally, we show new numerical results.

The rest of the chapter is organized as follows. In Section 5.2, we define the system model. In Section 5.3, we formulate the IEEE 802.11 WLANs resource allocation and decentralized association problem. In Section 5.4, we show that there exist stable coalition structures under certain conditions whatever the individual data rates. Section 5.5 presents our three steps mechanism.

Section 5.6 shows numerical results. Section 5.7 concludes the chapter and provides perspectives.

5.2 System model

We summarize in Table 5.1 the notations used in this chapter. We use both game-theoretic definitions and their networking interpretation. Throughout the chapter, they are used indifferently. Let define the set of players (nodes) $\mathcal{N}$ of cardinality N as the union of the disjoint sets of mobile users $\mathcal{W}$ of cardinality W and APs $\mathcal{F}$ of cardinality F. As in [6], we assume an interference-free model. It is assumed that the AP placement and channel allocation are such that the interference between co-channel APs can be ignored. In game-theoretic terms, there are no externalities on the firms' side. The mobile user association is a mapping μ that associates every mobile user to an AP and every AP to a subset of mobile users.

The IEEE 802.11 standard MAC protocol has been set up to enable any node in $\mathcal{N}$ to access a common medium in order to transmit its packets. The physical data rate between a transmitter and a receiver depends on their respective locations and on the channel conditions. For each mobile user $i \in \mathcal{W}$, let θ_{if} be the (physical) data rate with an AP f where $\theta_{if} \in \Theta = \{\theta^1, \ldots, \theta^m\}$, a finite set of finite rates resulting from the finite set of Modulation and Coding Schemes. If i is not within the coverage of f then $\theta_{if} = 0$. Given an association μ, let $\boldsymbol{\theta}_C = (\theta_{wf})_{(f,w)\in(C\cap\mathcal{F})\times(C\cap\mathcal{W})}$ denote the data rate vector of mobiles users in cell C served by AP f. Let $\mathbf{n_C}$ be the normalized composition vector of C, whose k-th component is the proportion of users in C with data rate $\theta_k \in \Theta$. Observe that in this model, the APs are players and assume that the APs have maximum data rate on the downlink. Within each cell, a resource allocation scheme (e.g. induced by the CSMA/CA MAC protocol) may be formalized as a sharing rule over the resource to be shared in the cell. This resource may be the total cell throughput (as considered in the saturated regime) or the amount of radio resources in time or frequency in the general case. More precisely, a sharing rule is a set of functions $D = (D_{i,C})_{C\in\mathcal{C}, i\in C}$, where $D_{i,C}$ allocates a part of the resource of C to user $i \in C$. Equal sharing, proportional fairness, α-fairness are examples of sharing rules.

Assuming the IEEE 802.11 MAC protocol and the saturated regime, the overall cell resource of cell C is defined as the total throughput. It is a function of the composition vector $\mathbf{n_C}$ and of the cardinality $|C|$. We denote $r_{i,C}$ the throughput obtained by user i in cell C. From the game theoretic point of view, $r_{i,C}$ is understood as i's share of the worth of coalition C denoted $v(C)$. The function $v : \mathcal{C} \to \mathbb{R}$ is called the characteristic function of the coalition game and maps any coalition $C \in \mathcal{C}$ to its worth $v(C)$. Other MAC protocols and regime can however be modeled by this approach. For example time-based fairness proposed in the literature to solve the WiFi anomaly results from the sharing of the time resource. In this case, a user i gets a proportion $\alpha_{i,C}$ of the time resource, which induces a throughout of $\alpha_{i,C}\theta_{if}$, where f is the AP of C. It can be shown that time-based fairness results in a proportional fairness in throughputs.

$\|set\|$	cardinality of the set *set*	$\mathcal{N}$	set of players (mobile users and APs)
$\mathcal{W}$	set of mobile users	$\mathcal{F}$	set of Access Points (APs)
$\mathcal{C}$	set of coalitions (cells)	$\mathcal{C}_f$	set of coalitions containing AP $f \in \mathcal{F}$
C	coalition (cell)	μ	matching (AP-mobile user association)
Θ	set of feasible data rates	θ_{wf}	data rate between w and f
$r_{i,C}$	throughput of node (user or AP) i in cell C	$\alpha_{i,C}$	fraction of resources of i in cell C
D	sharing rule (resource allocation scheme)	$v(C)$	worth of coalition C
$s_{i,C}$	payoff of player i in coalition C	$u_i(.)$	utility function of player i
q_i	quota of player i	χ_C	fear-of-ruin of coalition C
$P(i)$	preferences list of player i over individuals	$P^\#(i)$	preferences list of player i over groups

Table 5.1: Notations

5.3 Matching Games Formulation

5.3.1 Matching Games for Mobile User Association

In this chapter, the mobile user association is modeled as a matching game (in the class of cooperative games as already shown in Chapter 3). The matching theory relies on the existence of individual order relations $\{\succeq_i\}_{i\in\mathcal{N}}$, called preferences, giving the player's ordinal ranking [1] of alternative choices. Here, $w_1 \preceq_{f_1} [w_2, w_3] \preceq_{f_1} w_4$ indicates that the AP f_1 prefers to be associated to mobile user w_4 to any other mobile user, is indifferent between w_2 and w_3, and prefers to be associated to mobile user w_2 or w_3 rather than to be associated to w_1. Following the classical notations, let us denote **P** the set of preference lists $\mathbf{P} = (P_{w_1}, \ldots, P_{w_W}, P_{f_1}, \ldots, P_{f_F})$.

Definition 78 (Many-to-one bi-partite matching [2])**.** *A matching* μ *is a function from the set* $\mathcal{W} \cup \mathcal{F}$ *into the set of all subsets of* $\mathcal{W} \cup \mathcal{F}$ *such that:*
(i) $|\mu(w)| = 1$ *for every mobile user* $w \in \mathcal{W}$ *and* $\mu(w) = w$ *if* $\mu(w) \notin \mathcal{F}$*;*
(ii) $|\mu(f)| \le q_f$ *for every AP* $f \in \mathcal{F}$ ($\mu(f) = \emptyset$ *if* f *isn't matched to any mobile user in* $\mathcal{W}$*);*
(iii) $\mu(w) = f$ *if and only if* w *is in* $\mu(f)$*.*

Condition (i) of the above definition means that a mobile user can be associated to at most one AP and that it is by convention associated to itself if it is not associated to any AP. Condition (ii) states that an AP f cannot be associated to more than q_f mobile users. Condition (iii) means that if a mobile user w is associated to an AP f then the reverse is also true. In this definition, $q_f \in \mathbb{N}^*$ is called the *quota* of AP f and it gives the maximum number of mobile users the AP f can be associated to.

From now on, we focus on many-to-one matchings. In this setting, stability plays the role of equilibrium solution. In this chapter, we particularly have an interest in the pairwise and core stabilities. When the game does not exhibit complementarities or peer effects, it is sufficient for its description that the preferences are emitted over individuals only. In the presence of complementarities or peer effects (particular cases of externalities), players in the same coalition (i.e. the set of mobile users matched to the same AP) have an influence on each others. In such a case, the preferences need to be emitted over subsets of players and are denoted $P^\#$.

In the classical case of matchings with complementarities, the preference lists are of the form $\mathbf{P} = (P_{w_1}, \ldots, P_{w_W}, P^\#_{f_1}, \ldots, P^\#_{f_F})$, i.e., preferences over groups are emitted only by the APs (see the firms and workers problem in [2]). Moreover, it may happen that the preferences over groups may be *responsive* to the individual preferences in the sense that they are aligned with the individual preferences in the preferences over groups differing from at most one player. The preferences over groups may also satisfy the substitutability property. The substitutability of the preferences of a player rules out the possibility that this player considers others as complements.

[1] In this chapter, we use the Individually Rational Coalition Lists (IRCLs) to represent preferences. It can indeed easily be shown that other representations (additively separable preferences, B-preferences, W-preferences) are not adapted to our problem, see [16] for more details.

If the preferences are neither responsive nor substitutable, the equality $S(\mathbf{P}) = C_W(\mathbf{P})$ does not hold in general and the sets of pairwise, weak core and core stable matchings may be empty. An additional difficulty appears if the preferences over groups have to be considered on the mobile users side, i.e., if we have preference lists of the form $\mathbf{P} = (P^{\#}_{w_1},\ldots,P^{\#}_{w_W},P^{\#}_{f_1},\ldots,P^{\#}_{f_F})$. Complementarities and peer effect may arise in both sides of the matching. The user association problem in IEEE 802.11 WLANs falls in this category because the performance of any mobile user in a coalition may depend on the other mobiles in the coalition. To break the indifference, we use the following rule: a mobile user prefers a coalition with AP with the lowest index and an AP prefers coalitions in lexicographic order of users indices.

To see that preferences may not be responsive, consider an example with only uplink communications, two APs f_1 and f_2 and three mobile users w_1, w_2, w_3 such that $\theta_{11} = 300$ Mbps, $\theta_{12} = \theta_{22} = 54$ Mbps, $\theta_{21} = \theta_{32} = 1$ Mbps. Assuming saturated regime and equal packet size, we can show that $P^{\#}(w_1) = f_1 \succ f_2 \succ \{w_3; f_1\} \succ \{w_2; f_2\} \succ \{w_2; f_1\} \succ \{w_3; f_2\}$, which is not responsive. In this example, we also see that substitutability is not even defined since every choice set is reduced to a singleton. After the game has been controlled according the proposed mechanism, preferences of w_1 can be modified as follows: $P^{\#}(w_1) = \{w_3; f_1\} \succ \{w_2; f_2\} \succ \{w_2; f_1\} \succ \{w_3; f_2\} \succ f_1 \succ f_2$. Considering $S = \{w_2, w_3; f_1, f_2\}$, we have $Ch_{w_1}(S) = \{w_3; f_1\}$, while $Ch_{w_1}(S \setminus w_3) = \{w_2; f_2\}$. Preferences are thus not substitutable.

This general many-to-one matching problem has algorithmically been assed by Echenique and Yenmez in [8] who propose a fixed-point formulation and an algorithm to enumerate the set of stable matchings. It is known, that there is no guarantee that this set is non empty if the individual preferences over groups are not of a particular form. The problem of complementarities and peer effects in matchings has been analytically tackled by Pycia in [18]. Nevertheless, no result have been derived concerning the decentralized control of core stable structures and no decentralized algorithm with a limited amount of information and reduced lists of preferences for the mobiles have been derived.

5.3.2 Formulation as a Matching Game

We now assume that a player i in a given coalition C obtains a *payoff* $s_{i,C}$, which is evaluated (or perceived) by it through a *utility function* $u_i : \mathbb{R} \to \mathbb{R}$. In this chapter, we assume that functions u_i are positive, concave (thus log-concave), increasing and differentiable. In such a case, the individual preferences are induced by the player's utilities of these payoffs. We extend our model to the framework of finite coalition games in characteristic form $\Gamma = (\mathcal{N}; v)$, where v is a function mapping any coalition to its worth in $\mathbb{R}^+$. By definition of the characteristic function $v(\emptyset) = 0$. In this chapter we do not assume a particular form of the characteristic function v (e.g. superadditivity[2]). An even particular case of coalition games in characteristic form concerns games with an exogenous sharing rule $\Gamma = (\mathcal{N}; v; E^N; D)$, where E^N is the set of all payoff vectors and D is a sharing rule.

From this definition, the payoff of user i in coalition C is given by $s_{i,C} = D_{i,C} \circ v(C)$ and his utility of this payoff is given by $u_i(s_{i,C})$. We can now formulate the IEEE 802.11 joint user association and resource allocation problem as a matching game.

Definition 79 (Resource Allocation and User Association Game)**.** *Using the above notations, the resource allocation and users association game is defined as a* N*-player many-to-one matching game in characteristic form with sharing rule* D *and rates* $\boldsymbol{\theta} = \{\theta_{wf}\}_{(w,f)\in\mathcal{W}\times\mathcal{F}}$*:* $\Gamma = (\mathcal{W} \cup \mathcal{F}, v, \mathbb{R}^{+N}, D, \boldsymbol{\theta})$. *Each pair of players of the form* $(w, f) \in \mathcal{W} \cup \mathcal{F}$ *is endowed with a rate* θ_{wf} *from the rates space* $\Theta = \{\theta^1,\ldots,\theta^m\}$*. For this game, we define the set of possible coalitions* $\mathcal{C}$*:*

$$\mathcal{C} = \{\{f\} \cup J,\ f \in \mathcal{F},\ J \subseteq \mathcal{W}, |J| \le q_f\} \cup \{\{w\},\ w \in \mathcal{W}\}. \tag{5.1}$$

Note that for IEEE 802.11 MAC protocol and for the saturated regime, $s_{i,C} \triangleq r_{i,C}$. For other time sharing MAC approaches, $s_{i,C} \triangleq \alpha_{i,C}$.

[2] $\forall C, C,\ v(C \cup C') \ge v(C) + v(C')$ if $C \cap C' = \emptyset$

5.4 Existence of Core Stable Matchings in the Game and Unemployment

In this section, we show the the existence of core stable users-AP associations.

Using Pycia's results on matching with complementarities and peer effects with pairwise alignment of the preferences ([18], see chapter 3 for an introduction), we have the existence of stable structure of coalitions whatever the state of nature $\boldsymbol{\theta}$ if and only if in the resource allocation and user association game $F \geq 2$, the firms' quotas are such that $q_f \in \{2,\ldots,W-1\}$ and the sharing rules may be formulated as arising from the maximization of the product of increasing, differentiable and strictly log-concave individual utility functions in all coalitions.

To apply this results, we consider scenarios with at least two APs (which is reasonable when talking about load balancing) and (ii) every AP is supposed to be able to serve at least two users and should not be able to serve the whole set of users. Furthermore, in Chapter 4, we have shown that the expected throughput of a node in a cell C can be modeled as a Nash bargaining solution over a simplex. Thus, there exists a core stable matching in the defined resource allocation and user association problem. The equal sharing resulting from CSMA/CA MAC protocol in saturated regime, single-flow per device and equal packet length is obtained by considering $s_{i,C} = r_{i,C}$ and the identity function for u_i. The players' throughputs in the general saturated regime with multiple flows and heterogenous packet length is obtained by taking $s_{i,C} = r_{i,C}$ and utility functions as shown in chapter 4. Time-based fairness is obtained by setting $s_{i,C} = \alpha_{i,C}$ and the identity function for u_i. It results in turn in proportional fairness in terms of individual throughputs.

Without controlling the coalition game, the core stable matching may not have good properties in terms of load balancing and number of connected users. For example, in CSMA/CA under saturated regime, the cell throughput is increasing with the individual physical data rates and individual throughputs $r_{i,C}$ are sub-additive, i.e., decreasing with the addition of users. Assuming that the payoff is the individual throughput, i.e., $s_{i,C} = r_{i,C}$, then each player has the incentive to form the lowest cardinality coalition with highest composition vector. In this case, the unique stable structure is a one-to-one matching, in which APs are associated to their best mobile user. This will further be mentioned in the name of the *unemployment problem* since it leaves some mobiles users unassociated (unmatched). There is the need for a control of the players incentives for some equilibrium points with satisfying properties, in terms of unemployment in the present case. In other words, since the players have the incentive to match in a one-to-one form, one needs to control the underlying cooperative game so as to provide new incentives for a suitable many-to-one form as an equilibrium.

5.5 Mechanism for controlled matching game

In order to tackle the *unemployment problem*, we propose in this section a mechanism to control the players incentives for coalitions (see Figure 5.1). This mechanism is made of three steps. We start by considering for every AP the set of acceptable mobile users, i.e., the mobile users with non zero data rate with this AP. In the first step (block **LB**), APs share the load defined in number of users. This results in objective quotas that should be enforced by the mechanism. The second step (blocks Ω and Φ) is a controlled coalition game designed so as to provide the players the incentives to form coalitions with cardinalities given by the quotas and reducing heterogeneity (and thus reducing the anomaly in the IEEE 802.11). The third step (block μ) is a decentralized coalition formation (or matching) algorithm which results in a stable structure induced by the individual preferences induced by the controlled coalition game.

Our mechanism can be implemented as a virtual connectivity management layer on top of the IEEE 802.11 MAC protocol. Mobile users and APs form coalitions based on the "virtual rates" provided by this virtual layer. Once associated, users access the channel using the unmodified 802.11 MAC protocol.

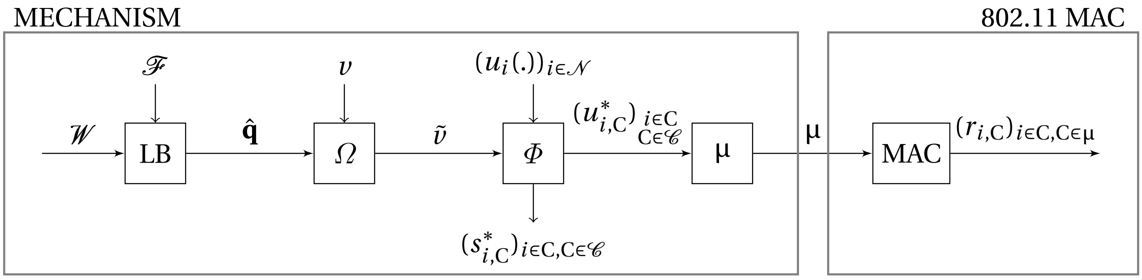


Figure 5.1: Block diagram of the mechanism in the most general form. The APs share the load in the block **LB** which gives the APs' objectives $\hat{\mathbf{q}}$. The characteristic function v of the original coalition game is controlled in Ω and gives the modified characteristic function $\tilde{v}$. The Nash bargaining Φ is played in each coalition for the allocation of the worth of the coalition among its members. The players then emit their preferences over the coalitions on the basis of their shares and enter a stable matching mechanism in block μ. This block outputs an AP-user association μ. Finally, in the block **MAC** the nodes transmit their packets according to the unmodified IEEE 802.11 MAC protocol.

5.5.1 Load Balancing

The first step of the mechanism is a load balancing. This step outputs a quota vector of the form $\hat{\mathbf{q}} = (\hat{q}_1, \ldots, \hat{q}_F)$ that defines the size of the coalitions the players should be incentivized to form with each AP. In other words, $\hat{\mathbf{q}}$ gives the number of connections the players should be incentivized to create with each AP. As in [23], in numerical implementation (see Section 5.6), we take a Nash bargaining based decentralized load balancing scheme between the APs to share the users covered by several APs. This scheme achieves the proportional fair allocation in the utility space. Nevertheless, any load balancing scheme can be used in this mechanism.

5.5.2 Controlling the Coalition Game

The second step of the mechanism is the control of the coalition game. The control step of the mechanism tackles the problem of the control of the set of stable matchings. We observed that when a coalition game is defined by a characteristic function and a sharing rule inducing subadditive strictly positive individual payoffs (except for coalitions of size one or those containing players with zero data rates), the stable structures to be formed are made of coalitions of size two. This step of the mechanism develops an analytical framework and methodology for the control of the equilibria by the way of a control over the players' incitations for individual strategies.

Definition 80 (Controller)**.** *The controller is any entity (player or other) having the legitimacy and ability to change the definition of the game (players, payoffs, worths, information, coalitions).*

The *controller* may not be taking part in the game (e.g. the network operator in a wireless network, the government for a firms and workers association problem) or any player of the game with some kind of additional decisional abilities. In other words, it may be any entity having the ability to create or modify the individual incitations of the players for some strategy and thus the ability to change the definition of the game. These changes in the definition of the game in view of manipulating the players' equilibria strategies are called control transformation,

Definition 81 (Control transformation)**.** *A **control transformation** Ω is a mapping from the set of coalition games in characteristic form in itself.*

In the purpose of this chapter it is sufficient to restrict the definition of the control transformations to the domain of coalition games in characteristic form. In fact, we further assume that the controller cannot arbitrarily move from one game to another without constraints. We assume that he or she can influence the equilibria by partial changes in the definition of the game (characteristic function, individual payoffs, ...) but can neither change the fundamental rules of the game (e.g. the rules of matching games) nor some essential elements such as the players taking part in the game or their strategy spaces. If Γ is a coalition game in characteristic form, then $\Omega(\Gamma)$ is a coalition game in characteristic form modified by the controller according to its (constrained) abilities. The limits of the abilities of such a controller are to be chosen by the game theorist or the designer of the mechanism so as to satisfy the fundamental hypothesis and description of the system he is looking at. The controller and the control transformation may be defined as the result of another game at a higher level (see the application with bargaining APs for quotas). As an example of work on the design of an incitations operator, Auman and Kurz [1] assess the problem of designing the joint taxation and redistribution scheme in the framework of a political majority-minority game. The majority is the controller and the incitations are induced by a multiplicative tax over the worths of the coalitions.

In Appendix C in [25] , we give two simple example of the mechanism we propose to control the player's individual incentives.

We now search for operators modifying the characteristic function v so as to provide players the incentives to form stable structures with coalitions of sizes $\hat{\mathbf{q}}$.

An important lever for controlling our matching game and designing operator Ω is the fear-of-ruin (FoR). Formally, the FoR of user i in coalition C is defined as:

$$\chi_i(s_{i,C}) \triangleq \frac{u_i(s_{i,C})}{u_i^{'}(s_{i,C})}. \tag{5.2}$$

The FoR of coalition C is obtained as the inverse of the Lagrange multiplier associated to the constraint $\sum_{i\in C} s_{i,C} \leq v(C)$ at the optimum of the Nash bargaining optimization problem. Two interesting characteristics of the FoR are that (i) in a coalitional game with Nash bargaining as sharing rule, the FoR is constant over the players in a coalition, i.e., $\chi_i(s_{i,C}) = \chi_C \ \forall i \in C$ at the bargaining solution point $s_{i,C}$ and (ii) with concave increasing utility functions, the individual payoffs increase in the common FoR [18]. Thus, the players have the incentives to form coalitions maximizing their FoR. In terms of control opportunities, this introduces the FoR as a lever to control the set of individual payoff-based incentives for coalitions. As an example, assume two coalitions C and C′ and their FoRs: $\chi_C < \chi_{C'}$. Players in $C \cap C'$ prefer C′ to C. Changing the values of the FoRs to obtain $\chi_C > \chi_{C'}$ changes the individual incentives of these players so that they now prefer C to C′.

Proposition 82. *Assume a coalition game $\Gamma = (\mathscr{F} \cup \mathscr{W}, v, \{u_i\}_{i\in N})$ in characteristic form with the Nash bargaining sharing rule over $v(C)$ for every coalition C in $\mathscr{C}$. Furthermore assume strictly increasing and concave utility functions*[3] *$u_i : \mathbb{R}^{\cdot +} \to \mathbb{R}^+, i \in \mathscr{N}$. The set of transformations Ω from the set of characteristic functions in itself that provide the players the incentive for some subset $\mathscr{C}'$ of coalitions in $\mathscr{C}$ must satisfy:*

$$F_{C'} \circ \Omega(v)(C') < F_C \circ \Omega(v)(C) \quad \forall C' \in \mathscr{C}', \forall C \in \mathscr{C} \backslash \mathscr{C}' \tag{5.3}$$

s.t. $C' \cap C \neq \emptyset$ *and where* $F_C = \left(\sum_{i\in C}\left(\frac{u_i^{'}}{u_i}\right)^{-1}\right)^{-1}$ *and* $\circ$ *is the composition function.*

Proof. See Appendix D in [25] . □

In order to derive our last result, we need to define the concept of single-peaked preferences. Let $X = \{x_1, \dots, x_n\}$ denote a finite set of alternatives, with $n \geq 3$.

Definition 83 (Peak of preferences, [13])**.** *A preference relation $\succ$ on X is a linear order on X. The peak of a preference relation $\succ$ is the alternative $x^* = peak(\succ)$ such that $x^* \succ x$ for all $x \in X \backslash \{x^*\}$.*

Definition 84 (Single-Peaked preferences, [13])**.** *An axis O (noted by $>$) is a linear order on X. Given two alternatives $x_i, x_j \in X$, a preference relation $\succ$ on X whose peak is x^*, and an axis O, we say that x_i and x_j are on the same side of the peak of $\succ$ iff one of the following two condition is satisfied: (i) $x_i > x^*$ and $x_j > x^*$; (ii) $x^* > x_i$ and $x^* > x_j$.*

[3] Such utility functions are bijective and thus injective. Theorem 10 applies.

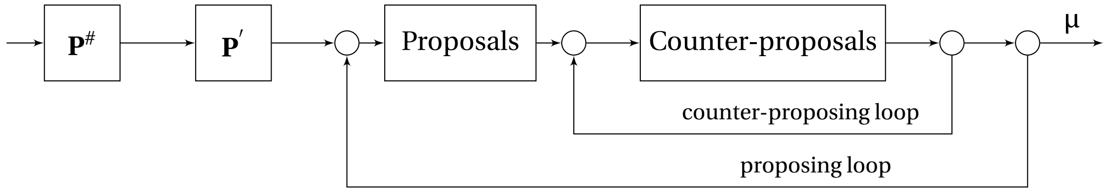


Figure 5.2: Block diagram of the BDAA.

A preference relation $\succ$ is single-peaked with respect to an axis O *if and only if for all* $x_i, x_j \in X$ *such that* x_i *and* x_j *are on the same side of the peak* x^* *of* $\succ$*, one has* $x_i \succ x_j$ *if and only if* x_i *is closer to the peak than* x_j*, that is, if* $x^* > x_i > x_j$ *or* $x_j > x_i > x^*$.

We use the discrete version of this definition over $\mathbb{N}^+$. We immediately obtain the following corollary,

Corollary 85. *Assume a coalition game* $\Gamma = (\mathscr{F} \cup \mathscr{W}, v, \{u_i\}_{i\in N})$ *in characteristic form with the Nash bargaining sharing rule over the* $v(C)$ *in every coalition* $C \in \mathscr{C}$. *Furthermore assume strictly increasing and concave utility functions* $u_i : \mathbb{R}^+ \to \mathbb{R}^+, i \in \mathscr{N}$. *The set of transformations* Ω *from the set of characteristic functions in itself that induce single-peaked preferences (peak at* $\hat{q}_f$*) in cardinalities over the coalitions with an AP* $f \in \mathscr{F}$ *must satisfy:*

$$\max_{\substack{C\in\mathscr{C}_f \\ s.t. |C|=q}} F_C \circ \Omega(v)(C) < \min_{\substack{C\in\mathscr{C}_f \\ s.t. |C|=q+1}} F_C \circ \Omega(v)(C), \quad \forall q \geq \hat{q}_f \tag{5.4}$$

and

$$\max_{\substack{C\in\mathscr{C}_f \\ s.t. |C|=q}} F_C \circ \Omega(v)(C) < \min_{\substack{C\in\mathscr{C}_f \\ s.t. |C|=q-1}} F_C \circ \Omega(v)(C), \quad \forall q \leq \hat{q}_f \tag{5.5}$$

where $F_C = \left(\sum_{i\in C}\left(\frac{u_i'}{u_i}\right)^{-1}\right)^{-1}$.

Proof. See Appendix D in [25] . ☐

5.5.3 Access Point Association

The third step of the mechanism is the joint resource allocation and users association (matching) game where the players (APs and mobile users) share the resource in the coalitions according to a Nash bargaining and then match with each others. The coalition game played has been described in Section 5.3 and Section 5.4. This step corresponds to the blocks Φ and μ of the block diagram in Figure 5.1.

Stable Matching Mechanism

We now show that a modified version of the Gale and Shapley's deferred acceptance algorithm in its college-admission form with APs preferences over groups of users and users preferences over individual APs is a stable matching mechanism for the many-to-one matching games with complementarities, peer effects considered in this chapter (see Algorithm 1: Backward Deferred Acceptance).

BDAA is similar to the DAA in many aspects. It involves two sets of players that have to be matched. Every player from one side has a set of unacceptable players from the other side. In our case, an AP and a mobile user are acceptable to each others if the user is under the AP coverage. As in DAA, the algorithm proceeds by proposals and corresponding acceptances or rejections. The main difference resides in the notion of counter-proposals, introduced to tackle the problem of complementarities.

Algorithm 6: Backward Deferred Acceptance

Data: For each AP: The set of acceptable (covered) users and AP-user data rates.
For each user: The set of acceptable (covering) APs.
Result: A core stable structure $\mathscr{S}$

begin
- *Step 1: Initialization*;
  - **Step 1.a:** All APs and users are marked *unengaged*. $L(f) = L^*(f) = \emptyset, \forall f$;
  - **Step 1.b:** Every AP f computes possible coalitions with its acceptable users, the respective users payoffs and emits its preference list $P^{\#}(f)$;
  - **Step 1.c:** Every AP f transmits to its acceptable users the highest payoff they can achieve in coalitions involving f;
  - **Step 1.d:** Every user w emits its reduced list of preference $P'(w)$;
- *Step 2 (BDAA)*;
  - **Step 2.a, Mobiles proposals:** According to $P'(w)$, every unengaged user w proposes to its most preferred acceptable AP for which it has not yet proposed. If this AP was engaged in a coalition, all players of this coalition are marked *unengaged*;
  - **Step 2.b, Lists update:** Every AP f updates its list with the set of its proposers: $L(f) \longleftarrow L(f) \cup \{\text{proposers}\}$ and $L^*(f) \longleftarrow L(f)$;
  - **Step 2.c, Counter-proposals:** Every AP f computes the set of coalitions with users in the dynamic list $L^*(f)$ and counter-proposes to the users of their most preferred coalition according to $P^{\#}(f)$;
  - **Step 2.d, Acceptance/Rejections:** Based on these counter-proposals and the best achievable payoffs offered by APs in Step 1.c to which they have not yet proposed, users accept or reject the counter-proposals;
    - **Step 2.e:** If all users of the most preferred coalition accept the counter-proposal of an AP f, all these users and f defect from their previous coalitions;
    - all players of these coalitions are marked *unengaged*;
    - users that have accepted the counter-proposal and f are marked *engaged in this new coalition*;
    - **Step 2.f:** Every unengaged AP f updates its dynamic list by removing users both having rejected the counter-proposal and being engaged to another AP:
    - $L^*(f) \longleftarrow L^*(f) \setminus \{\text{engaged rejecters}\}$;
  - **Step 2.g:** Go to Step 2.c while the dynamic list L^* of at least one AP has been strictly decreased (in the sense of inclusion) in Step 2.f;
  - **Step 2.h:** Go to Step 2.a while there are unengaged users that can propose;
  - **Step 2.i:** All players engaged in some coalition are matched.

The block diagram representation of the algorithm is shown in Figure (5.2). In block $\mathbf{P}^{\#}$ the APs emit their preferences over the coalitions. In block $\mathbf{P}^{'}$ the mobiles emit their preferences over the APs. In block *Proposals* the mobiles propose to the APs. In block *counter-proposals* the APs counter-propose. The counter-proposing round continues up to convergence. The next proposing round starts.

We enter the details of the algorithm. Having the information of the data rates with users under their coverage, APs are able to compute all the possible coalitions they can form and the corresponding allocation vectors (throughputs). They can thus build their preference lists (Steps 1b). Then, every AP f transmits to each of its acceptable users the maximum achievable throughput (based on MAC layer and virtual mechanism) it can achieve in the coalitions it can form with f (Step 1.c). Every user w can thus build its reduced list of preferences over individual APs: w prefers f_i to f_j if the maximum achievable throughput with f_i is strictly greater than its maximum achievable throughput with f_j (Step 1.d). BDAA then proceeds by rounds during which users make proposals, AP make counter-proposals and users accept or reject (from Step 2.a to Step 2.h). Every AP that receives a new proposal shall reconsider the set of its opportunities and is thus marked unengaged (Step 2.a). $L(f)$ is the list of all users that have proposed at least once to AP f. $L^*(f)$ is a dynamic list that is reinitialized to $L(f)$ before every AP counter-proposal (Step 2.b). In each round of the algorithm, every unengaged user proposes to its most preferred AP for which it has not yet proposed (Step 2.a). Every AP receiving proposals adds the proposing players to its cumulated list of proposers and reinitializes its dynamic list (Step 2.b). Using $P^{\#}(f)$ it then searches for its most preferred coalition involving only users from the dynamic list and emits a counter-proposal to these users. This counter-proposal contains the throughput every user can achieve in this coalition (Step 2.c). Each user compares the counter-proposals it just received with the best achievable payoffs obtained with the APs it has not proposed to yet (Step 2.d). If one of these best achievable payoffs is strictly greater than the best counter-proposal, the users rejects the counter-proposals and continues proposing (Step 2.d, Step 2.h). Otherwise, the user accepts its most preferred counter-proposal (Step 2.d). Given a counter-proposal, if all users accept it, then they are engaged to the AP. All coalitions in which these users and the AP were engaged are broken and their players are marked unengaged (Step 2.e). If at least one user does not agree, then the AP is unengaged (Step 2.e), it updates its dynamic list by removing the mobiles having rejected its counter-proposal and being engaged to another AP (Step 2.f). The counter-proposals continues up to the point when no AP can emit any new counter-proposal (Step 2.g). The current round ends and the algorithm enters a new round (Step 2.h). The algorithm stops when no more users are rejected (Step 2.h). A stable matching is obtained (Step 2.i).

Proposition 86. *Given a many-to-one matching game, BDAA converges, i.e., outputs a matching in a finite number of steps.*

Proof. See Appendix D in [25]. □

Proposition 87. *Suppose the family of coalitions $\mathscr{C}$ as defined in* (5.1)*, and a sharing rule as defined in proposition 63 (see Chapter 3, Section 3.5.1). Furthermore assume a tie-breaking rule such that there is no indifference (strict preferences). BDAA converges to the unique core stable matching.*

Proof. See Appendix D in [25]. □

Proposition 88. *The complexity of BDAA is* $\mathrm{O}(n^5)$ *in the number of proposals of the players, where* $n = \max(\mathrm{F}, \mathrm{W})$.

Proof. See Appendix D in [25]. □

In Appendix E in [25], we provide an interpretation of BDAA in the economic framework. In Appendix [25], we give an example of application of the BDAA.

5.6 Numerical Results

5.6.1 Simulations Parameters and Scenarios

The numerical computations are performed under the assumption of equal packet sizes and saturated queues (each node has always packets to transmit). Under this assumption the sharing rule is equal sharing. Analytical expressions of the throughputs (individual and total throughputs) are taken from [9] with the parameters of Table 5.2. We further assume that a node compliant with a IEEE 802.11 standard (in chronological order: b, g, n) is compliant with earliest ones. By convention, if all nodes of a cell have the same data rate, we use the MAC parameters of the standard whose maximum physical data rate is the common data rate. Otherwise, we use the MAC parameters of the standard whose maximum physical data rate is the lowest data rate in the cell.

Assume the spatial distributions of nodes of Figure 5.3. The first scenario (a) shows the case of 5 APs with a uniform spatial distribution of 20 mobile users. The second scenario (right) has non-uniform distribution of 10 mobile users in the plane. The green (inner), red (intermediate) and black (outer) circles show the spatial region where the mobiles achieve a data rate of 300 Mbits/s, 54 Mbits/s and 11 Mbits/s respectively. Scenario 2 exhibits a high overlap between AP coverages.

5.6.2 Numerical Work

No mechanism

We show in Figure 5.4 a stable matching. No associated player has an incentive to deviate and form a coalition of size superior to two. The figure shows the natural incentives of the system in forming low cardinalities coalitions with good compositions. This can also be observed on Figure 5.5 which shows the individual throughputs obtained in the coalitions. The coalitions are sorted by cardinalities from low to high. In plot (a) no mechanism is used. In plot (b) a gaussian tax rate is applied. See Section 5.6.2.

Figure 5.4 and Figure 5.5 (a) show the natural incentives of the system in forming low cardinalities coalitions with good compositions. As a result, a one-to-one matching is obtained. Using our mechanism, this structure of throughputs will be changed (as in Figure 5.5 (b)) to move the incentives according to $\hat{\mathbf{q}}$ and thus provide the players the incentives to associate according to a many-to-one matching rather than a one-to-one.

Gaussian Tax Rate in Cardinalities

As an example of family of cost functions, we can use multiplicative symmetric unimodal cost functions. The multiplicative cost functions are commonly called tax rates and are defined such that for any AP $f \in \mathscr{F}$ and any coalition C containing f, we must have:

$$\tilde{v}(\mathrm{C}) = \Omega(v(\mathrm{C})) \triangleq c_f(|\mathrm{C}|)\, v(\mathrm{C}) \tag{5.6}$$

We particularly consider Gaussian tax rates such that:

$$\tilde{v}(\mathrm{C}) = e^{-\frac{(|\mathrm{C}|-\hat{q}_f)^2}{2\sigma_f^2}} v(\mathrm{C}) \tag{5.7}$$

where σ_f is the variance of the function c_f. The Gaussian cost function is convenient in the sense that it does not penalize the mean-sized coalitions and it provides a great amount of flexibility by the way of its variance. Decreasing or increasing the variance σ_f indeed allows for a strict or relaxed control of the incentives for the objective quotas.

Focusing on the first scenario (Figure 5.3 (a)), we consider the three matchings shown in Figure 5.6. The first one (a) is the stable matching resulting from the mechanism (including BDAA and Gaussian costs); The second matching (b) maximizes the sum of modified throughputs (i.e. including Gaussian costs); The third matching (c) maximizes the sum of unmodified throughputs (i.e. without costs).

We first observe that the proposed mechanism induces a stable matching with a drastic reduction of the unemployment problem w.r.t. the result of Figure 5.4. The natural incentives of the system resulting in a one-to-one matching have been countered and a many-to-one matching is obtained. The unemployment has been reduced from 73% to 5% in this particular scenario.

	802.11n	802.11g	802.11b	
Parameter	value			unit
Θ	{300, 54, 11}	{54, 11}	{11}	Mbits/s
slot duration	9	9	20	μs
T_0	3	5	50	slots
T_C	2	10	20	slots
L	8192	8192	8192	bits
K	2	2	2	
b_0	16	16	16	
p	2	2	2	

Table 5.2: Simulation Parameters.

The second point to be raised is that the proposed mechanism allows to obtain (with a polynomial complexity) a stable matching with a high modified total throughput, close to the optimal modified total throughput that is however not stable. For this scenario, we achieve through our mechanism 99% of the total modified maximum throughput (see Figure 5.6 (b)). This means that the cost for stability is very small in this particular scenario. Furthermore, the total throughput performance of the system at the MAC layer (i.e. unmodified throughputs obtained in block MAC of the block diagram representation of the mechanism, see Figure 5.1) is 97% the total unmodified maximum throughput (see Figure 5.6 (b)) and 47% of the total maximum throughput of the uncontrolled system (see Figure 5.6 (c)). This quantifies the cost for control, stability and low unemployment in this scenario. The third point is that the quotas have been enforced by the mechanism (via the cost function) since the quotas vector from the load balancing is $\hat{\mathbf{q}} = (8.0, 4.5, 3.33, 3.83, 4.33)$ (obtained by Nash bargaining[4] over the share $[0,1]$ of the players at the intersection of the coverages of the APs) and the formed coalitions are of sizes 8, 4, 3, 4 and 4.

We go into more details on the difference between the quotas vector and the integer-sized coalitions in the stable matching. Focusing on AP3 with the quota $q_3 = 3.33$, one may observe that in case of a Gaussian cost function with unit variance, the condition for an integer quotas 3 is only satisfied for sizes of coalitions superior or equal to 4. This meaning that the use of a gaussian cost function centered on 3.33 and unit variance even though increasing the penalty with the distance in sizes to $\hat{q}_f$ cannot guarantee the systematic incentive to form coalitions of size 3 with AP3. There exists some coalitions of size 2 giving the players more individual throughputs than the worst coalition of size 3. In such case, the players will have the incentive to form the coalition with the highest individual value among those of cardinalities 2 and 3. In Figure 5.7, we show the performance of the mechanism over a set of 50 scenarios generated by spatial random uniform distribution of the mobile devices. The APs are spatially distributed as in Scenario 1 (see Figure 5.8 for a random distribution of both the mobile devices and APs). The red line shows the empirical mean of the sample and the green dotted lines show the interval $[\hat{m}-\sigma, \hat{m}+\sigma]$ where $\hat{m}$ is the empirical mean of sample and σ is the standard deviation. The empirical mean of the unemployment rate is 6%, the mean modified social welfare is 61Mbits/s and the mean computation time of BDAA is 0.45s. Observe that in 22% of the realizations the unemployment is null and that in 70% of the realizations it is below the mean. In terms of computation times of BDAA, the mean performance is reasonably low (0.45s to match 20 mobiles to 5 APs) and the algorithm performs even better in 62% of the scenarios.

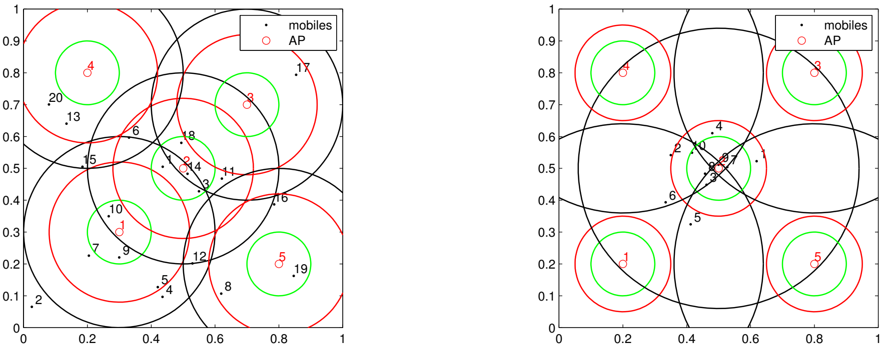
In Figure 5.8, we show the performance of the mechanism over a set of 50 scenarios generated by spatial random uniform distribution of the mobile devices and APs. The red lines show the empirical mean of the sample and the green dotted lines show the interval $[\hat{m}-\sigma, \hat{m}+\sigma]$ where $\hat{m}$ is the empirical mean of sample and σ is the standard deviation. The empirical mean of the unemployment rate is 8%, the mean modified social welfare is 60Mbits/s and the mean computation time of BDAA is 0.51s. Observe that in 22% of the realizations the unemployment is null and that in 56% of the realizations it is below the mean. In terms of computation times of BDAA, the mean is higher than in the previous case but the algorithm performs better than the mean in 68% of the scenarios.

In Figure 5.9, Plot (a), we show the ratios of the modified (mechanism level) social welfare (by definition, the total throughput resulting from BDAA) to the maximum total modified throughput. The mean performance of BDAA achieves 96% of this maximum. Furthermore, observe that the global maximum is achieved by BDAA in 46% of the random networks. The ratio is below $\hat{m}-\sigma$ in only 10% of the cases. In Figure 5.9, Plot (b), we show the ratios of the unmodified (MAC level) social welfare to the unmodified total throughput induced at the matching maximizing the total modified throughput. The mean performance of BDAA achieves 97% of this unmodified total throughput. Finally, observe that in some cases, the ratio is even higher than one. This means that BDAA gives a total throughput at the MAC level that is superior to the (unmodified) total through-

[4]Achieves a proportional fair allocation in the utility space of the APs. Induces the number of players to be connected to each AP.

put resulting from the maximization at the mechanism level (modified values) while the ratio was inferior to one in the modified case. This may come from the fact that in some cases, the equilibrium point (stable matching resulting from BDAA) may contain coalitions with lower modified worths (because of the penalization) w.r.t. those in the global maximum but higher worths at MAC level (real unpenalized setting).

To conclude, we compare our approach to the best-RSSI scheme in Scenario 2. The two matchings are compared Figure 5.10. We observe that the load is effectively shared among the APs and that the individual throughputs are greatly increased from 527 kbits/s when using best-RSSI to 1.64 Mbits/s for the coalition with AP1, 1.93 Mbits/s for the coalition with AP2, 2.59 Mbits/s for the coalition with AP3, 1.64 Mbits/s for the coalition with AP4 and 2.59 Mbits/s for the coalition with AP5. The individual performances are multiplied by a factor 3 to 5.



(a) Scenario 1 (b) Scenario 2

Figure 5.3: Scenario 1 (left): A spatial distribution of APs (smallest red circles) $\mathscr{F} = \{f_1, \ldots, f_5\}$ and devices (black points) $\mathscr{W} = \{w_1, \ldots, w_{20}\}$. Scenario 2 (right): A spatial distribution of APs (smallest red circles) $\mathscr{F} = \{f_1, \ldots, f_5\}$ and devices (black points) $\mathscr{W} = \{w_1, \ldots, w_{10}\}$. Circles show the coverage areas corresponding to different data rates.

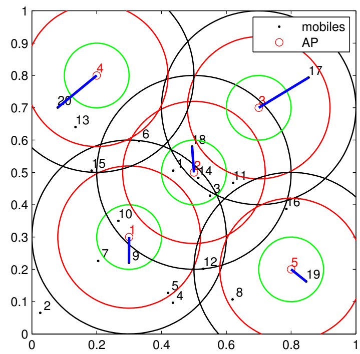
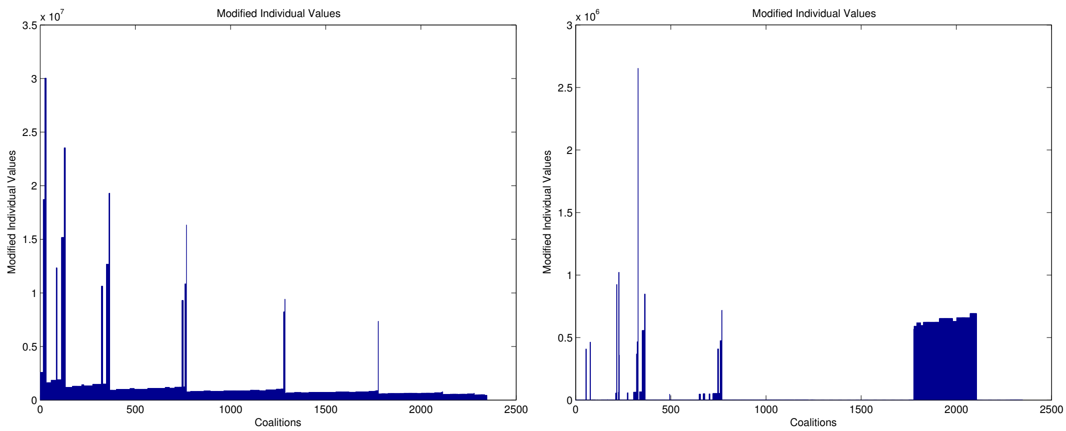
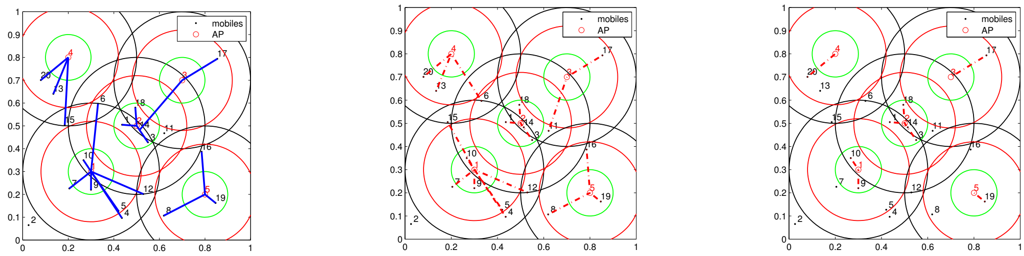


Figure 5.4: A stable matching in the uncontrolled case.



(a) Individual throuputs vs. coalition index. (b) Modified Individual throuputs vs. coalition index.

Figure 5.5: (a)Scenario 1. Structure of the payoffs in the uncontrolled matching game. (b)Scenario 1. Structure of the payoffs in the controlled matching game with a multiplicative tax rate of variance $\sigma_f = 0.3, \forall f \in \mathscr{F}$.

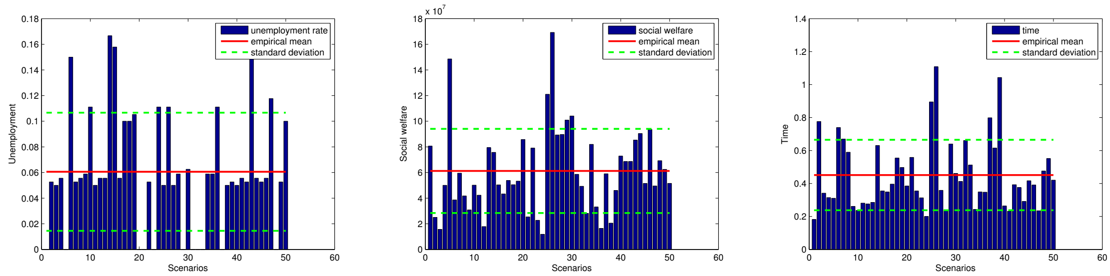


(a) Stable matching resulting from Gaussian costs and BDAA.

(b) A global optimum association with Gaussian costs.

(c) A global optimum association without costs.

Figure 5.6: Controlled matching game in scenario 1. Comparison of the association obtained from (a) BDAA, (b) a global optimum for Gaussian costs with variance σ = 0.2, (c) a global optimum without costs.

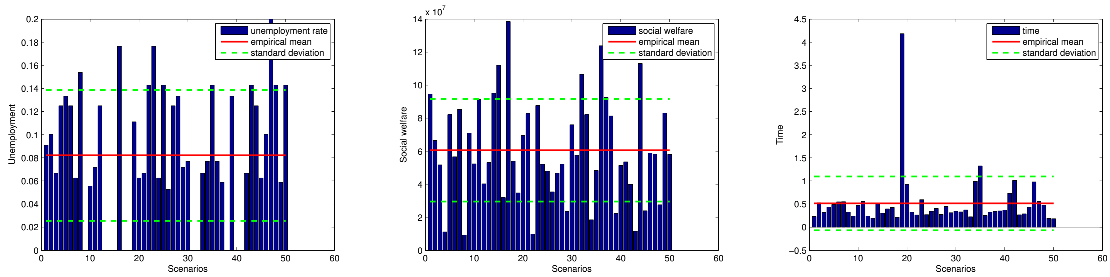


(a) Unemployment rates

(b) Modified social welfares

(c) Computation time of BDAA

Figure 5.7: (a)Unemployment rates, (b)social welfares (the social welfare of a matching is measured as the total throughput of the system at equilibrium) and (c)computation times of BDAA over a sample of 50 scenarios obtained by spatial random uniform distribution of the mobile devices. APs are spatially distributed as in Scenario 1. For each plot, the red line gives the empirical mean $\hat{m}$ of the sample and the green dotted lines the interval $[\hat{m} - \sigma, \hat{m} + \sigma]$ where $\hat{m}$ is the empirical mean of sample and σ is the standard deviation.

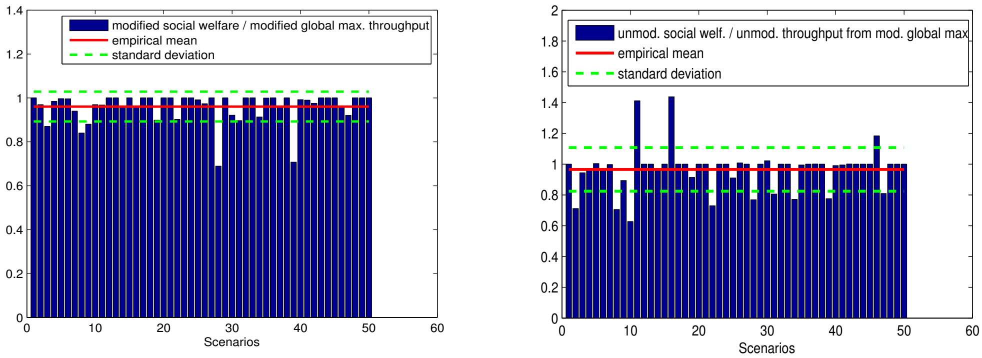


(a) Unemployment rates

(b) Modified social welfares

(c) Computation times of BDAA

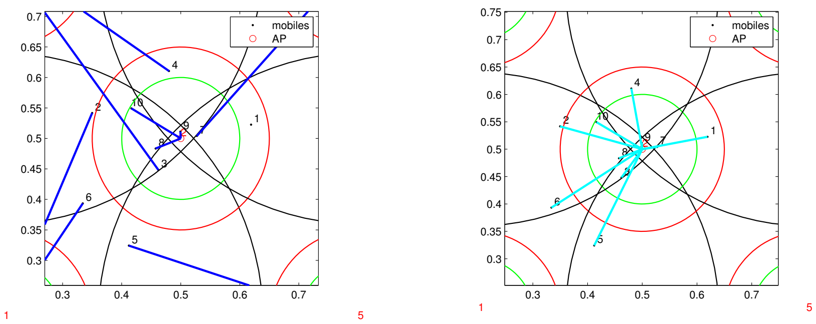
Figure 5.8: (a)Unemployment rates, (b)social welfares and (c)computation times of BDAA over a sample of 50 random networks obtained by spatial random uniform distribution of the mobile devices and APs. For each plot, the red line gives the empirical mean $\hat{m}$ of the sample and the green dotted lines the interval $[\hat{m} - \sigma, \hat{m} + \sigma]$ where $\hat{m}$ is the empirical mean of sample and σ is the standard deviation.



(a) Ratios of modified performances

(b) Ratios of unmodified performances

Figure 5.9: (a)Ratios of the modified social welfares to the maximum modified (mechanism level) total throughput, (b)Ratios of the unmodified (MAC level) social welfares to the unmodified total throughputs corresponding to the matching with maximum modified total throughput. Sample of 50 random networks obtained by spatial random uniform distribution of the mobile devices and APs. For each plot, the red line gives the empirical mean $\hat{m}$ of the sample and the green dotted lines the interval $[\hat{m}-\sigma, \hat{m}+\sigma]$ where $\hat{m}$ is the empirical mean of sample and σ is the standard deviation.



(a) Stable matching resulting from Gaussian cost and BDAA.

(b) Matching resulting from the best-RSSI scheme.

Figure 5.10: Comparison of the association obtained from (a) BDAA and (b) the best-RSSI scheme in scenario 2. These two figures show the AP at center (zoom).

5.7 Conclusion

In this chapter, we have presented a novel AP association mechanism in multi-rate IEEE 802.11 WLANs. We have formulated the problem as a coalition matching game with complementarities and peer effects and we have provided a new practical control mechanism that provides nodes the incentive to form coalitions both solving the unemployment problem and reducing the impact of the anomaly in IEEE 802.11. Simulation results have shown that the proposed mechanism can provide significant gains in terms of increased throughput by minimizing the impact of the anomaly through the overlapping between APs. We have also proposed a polynomial complexity algorithm for computing a stable structure in many-to-one matching games with complementarities and peer effects. This work is a first step in the field of controlled coalition games for achieving core stable associations in distributed wireless networks. Further works includes for example the study of a dynamic number of users or the impact of interference.

5.8 References

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5.9 Appendix: Two examples of control of incentives in games

We give two simple examples of the mechanism we propose to control the players' individual incentives. These examples do not give technical details but illustrate the underlying motivations of the work developed in the framework of the wireless technologies.

Example 89. *Consider the case of a many-to-one matching game with ordinal preferences* $\Gamma = (\mathscr{W}, \mathscr{F}, \mathbf{P})$, *where* $\mathbf{P}$ *is the list of stated preferences. This description of matching games does not allow for the game theorists to catch the strategic incentives the player is facing when emitting preferences since it only shows the emitted preferences. It does not even allow to exhibit the set of players' feasible strategies. As a consequence of this incompleteness and in addition to the standard form, matching games have been defined in a strategic form* $\Gamma = (\mathscr{W}, \mathscr{F}, \{Q_i\}, h, \mathbf{P})$, *where* $\{Q_i\}$ *is the set of players' individual feasible strategies, h is the matching mechanism and* $\mathbf{P}$ *is the set of true preferences. For the ease of understanding, Roth et a.l. assume in [2] that the set of feasible individual strategies is reduced to the set of preferences lists the players' may state. This description allows for each player to state a list of preferences different from his true preferences in view of manipulating the association mechanism (or matching mechanism). In the framework of matching games in wireless networks, we do not allow for the players (devices) to manipulate their preferences or behave strategically by misstating since, we assume that they would intrinsically be built and programmed so as to respect a given protocol asking them for truthfulness. Nevertheless, this framework can be used to allow the controller to modify the game so as to change the true (stated preferences by the previous assumptions) preferences. Thus, the incitation operator* Ω *operates on the matching game in strategic form* $\Gamma = (\mathscr{W}, \mathscr{F}, \{Q_i\}, h, \mathbf{P})$, *in a way that changes the true and stated preferences such that,*

$$\Gamma' = (\mathscr{W}, \mathscr{F}, \{Q_i\}, h, \mathbf{P}') = \Omega(\Gamma) \tag{5.8}$$

The second example is given in terms of the firms and workers model commonly used in stable matchings [2].

Example 90. *Assume a set of firms and a set of workers. Each firm can hire workers. We call a coalition a subset of players containing a single firm and one or more workers. The characteristic function assigns each coalition the worth it produces and this worth is shared via a Nash bargaining among the players in each coalition. The players have the incentive to form coalitions maximizing their own payoffs. In an employment market with performances similar to the IEEE 802.11 standard, the characteristic function would be increasing in the productivity types and the individual payoffs would be sub-additive. In this case, firms and workers would have the incentive to group by pairs of highest productivity types. Nevertheless, these stable structures of coalitions leave some workers unemployed which is not satisfying from the point of view of the unemployment market. Facing the problem, a government solely interested in reducing the number of unemployed workers would thus seek for (well-designed) tax rates as levers to provide the players the incentives to form coalitions reducing the employment. Assuming Nash bargaining for the resource allocation, we propose to manipulate the payoffs (incentives for players via their individual increasing concave utilities) by the way of such tax rates applied to the gross income(s). In other words, we manipulate the individual payoffs so as to change the equilibrium point of the coalition formation process.*

5.10 Appendix: Proofs

5.10.1 Proof of Proposition 82

Proof. Assume a coalition game $\Gamma = (\mathscr{F} \cup \mathscr{W}, v, \{u_i\}_{i\in N})$ in characteristic form with the Nash bargaining sharing rule over the $v(C)$-simplex in each coalition $\mathscr{C}$. Furthermore assume increasing, three-times differentiable, and concave utility functions $u_i : \mathbb{R}^+ \to \mathbb{R}^+, i \in \mathscr{N}$.
As shown in Chapter 2, Nash solution to the bargaining problem in any coalition $C \in \mathscr{C}$ solves,

$$\begin{aligned} \underset{\mathbf{x}}{\text{maximize}} \quad & \prod_{i\in C} u_i(x_i) \\ \text{subject to} \quad & \sum_{i\in C} x_{i,C} \le v(C) \\ & x_i \ge 0 \; \forall i \in C \end{aligned}$$

Which can be equivalently written in the following form,

$$\begin{aligned} \underset{\mathbf{x}}{\text{minimize}} \quad & -\sum_{i\in C} \log(u_i(x_i)) \\ \text{subject to} \quad & \sum_{i\in C} x_{i,C} - v(C) \le 0 \\ & -x_i \le 0 \; \forall i \in C \end{aligned}$$

This optimization problem is convex by convexity of the $v(C)$-simplex and of the objective function.
For any coalition $C \in \mathscr{C}$ such that $v(C) = 0$, solving the allocation problem is irrelevant. The player receives a null payoff. For any other coalition (with strictly positive worth $v(C)$), the interior of the $v(C)$-simplex ($B_C = \{\mathbf{s}_C = (s_{i,C})_{i\in C} | \sum_{i\in C} s_{i,C} \le v(C)\}$) is non-empty. Using Slater's constraint qualification[5], strong duality holds for this convex optimization problem. There is an optimal solution to the optimization problem iff the Karush-Kuhn-Tucker conditions can be satisfied.
We have the Lagrangian $\mathscr{L}(\mathbf{x}, \lambda_0, \{\lambda_i\}_{i\in C})$ such that,

$$\mathscr{L}(\mathbf{x}, \lambda_0, \{\lambda_i\}_{i\in C}) = -\sum_{i\in C} \log(u_i(x_i)) + \lambda_0 \left(\sum_{i\in C} x_i - v(C) \right) - \sum_{i\in C} \lambda_i x_i \quad (5.9)$$

We have the K.K.T. conditions such that:

- Primal constraints: (i) $\sum_{i\in C} x_i - v(C) \le 0$, (ii) $-x_i \le 0 \; \forall i \in C$
- Dual constraints: $\lambda_0 \ge 0$, $\lambda_i \ge 0 \; \forall i \in C$
- Complementary slackness: (i) $\lambda_0 \left(\sum_{i\in C} x_i - v(C) \right) = 0$, (ii)$\lambda_i x_i = 0 \; \forall i \in C$
- Vanishing gradient of the Lagrangian at the solution point: $\frac{u_i^{'}(x_i)}{u_i(x_i)} = \lambda_0 - \lambda_i = \frac{1}{\chi_i(x_i)} \; \forall i \in C$

Case : $\lambda_0 > 0$ and $\lambda_i = 0 \; \forall i \in C$
Due to the complementary slackness conditions, we must have $\sum_{i\in C} x_i = v(C)$ and $x_i \ge 0 \; \forall i \in C$.
The vanishing gradient of the Lagrangian condition gives, $\forall i \in C$:

$$\frac{u_i^{'}(x_i)}{u_i(x_i)} = \lambda_0 = \frac{1}{\chi_i(x_i)} = \frac{1}{\chi_C} \quad (5.10)$$

[5] Strong duality holds for a convex optimization problem if it is strictly feasible.

Knowing that u_i is concave and strictly increasing, we have $\left(\frac{u_i'(x_i)}{u_i}\right)' = \frac{u_i'' u_i - (u_i')^2}{u_i^2} < 0$. Thus, it is a strictly monotonic function and it admits an inverse. We denote $\left(\frac{u_i'}{u_i}\right)^{-1}$ this inverse.
We have:

$$x_i = \left(\frac{u_i'}{u_i}\right)^{-1} (\lambda_0) \quad \forall i \in C \tag{5.11}$$

Due to the complementary slackness conditions, we must have:

$$\sum_{i \in C} \left(\frac{u_i'}{u_i}\right)^{-1} (\lambda_0) = v(C) \tag{5.12}$$

The function $\sum_{i \in C} \left(\frac{u_i'}{u_i}\right)^{-1}$ is also strictly monotonic and has an inverse on $\mathbb{R}^{+*}$. We denote it $\left(\sum_{i \in C} \left(\frac{u_i'}{u_i}\right)^{-1}\right)^{-1}$.
The optimal Lagrange multiplier is obtained as:

$$\lambda_0 = \left(\sum_{i \in C} \left(\frac{u_i'}{u_i}\right)^{-1}\right)^{-1} (v(C)) \tag{5.13}$$

We thus have the optimal solution of the Nash bargaining problem by solving (5.11) and (5.13). In terms of the fear-of-ruin χ_C:

$$x_i = \left(\frac{u_i'}{u_i}\right)^{-1} \left(\frac{1}{\chi_C}\right) \tag{5.14}$$

$$\chi_C = \frac{1}{\left(\sum_{i \in C} \left(\frac{u_i'}{u_i}\right)^{-1}\right)^{-1} (v(C))} \tag{5.15}$$

We now turn to the analysis of the function $b_i \triangleq \frac{u_i'}{u_i}$ called boldness of player i. We have:

$$b_i'(x_i) = \frac{u_i''(x_i) u_i(x_i) - \left(u_i'(x_i)\right)^2}{(u_i(x_i))^2} \tag{5.16}$$

By assumption, u_i is strictly increasing and concave for any player i. For any player i, we obtain that $b_i'(x_i)$ is strictly negative for any x_i and thus that the boldness b_i is a decreasing function of x_i.
Thus, its inverse $\left(\frac{u_i'}{u_i}\right)^{-1}$ is also a decreasing function and the fear-of-ruin of player i, $\frac{u_i}{u_i'}$ is an increasing function of x_i.
The sum of decreasing functions, $\sum_{i \in C} \left(\frac{u_i'}{u_i}\right)^{-1}$ is a decreasing function. So is its inverse $\left(\sum_{i \in C} \left(\frac{u_i'}{u_i}\right)^{-1}\right)^{-1}$.
As a consequence, we obtain that the common boldness λ_0 (solving the Nash bargaining optimization problem) is a decreasing function of the common wealth $v(C)$ and thus (using $\chi_C = \frac{1}{\lambda_0}$) that the common fear-of-ruin is an increasing function of the common wealth $v(C)$.
Finally, using (5.11), we obtain that x_i is decreasing in λ_0 but increasing in $v(C)$ for each player i in C. It is an increasing function of the fear-of-ruin χ_C (by (5.14)).
Now, assume two coalitions C and C′ and their respective Nash solutions to the bargaining problem $\mathbf{x_C}$ (where the bold notation $\mathbf{x_C}$ denotes the vector of individual allocations solving the Nash bargaining optimization program in coalition C) and $\mathbf{x_{C'}}$.
If we want all the players in C and C′ (i.e. in $C \cap C'$) to prefer C to C′, then we must have:

$$x_{i,C} > x_{i,C'} \quad \forall i \in C \cap C' \tag{5.17}$$

which can equivalently be written:

$$(x_{i,\mathrm{C}})_{i\in\mathrm{C}\cap\mathrm{C}'} \succ (x_{i,\mathrm{C}'})_{i\in\mathrm{C}\cap\mathrm{C}'} \tag{5.18}$$

where $\succ$ denotes the component wise strict inequality in $\mathbb{R}^{|\mathrm{C}\cap\mathrm{C}'|}$.
The number of players in $\mathrm{C}\cap\mathrm{C}'$ can be arbitrary large (depending on C and C′). So is the number of inequalities of the form of (5.17) that must be satisfied. This number is of order O(N). In order to reduce the complexity of the control of the incentives of an order N, we use the fact that the Nash solution to the bargaining problem is component-wisely increasing in a quantity that is constant over the players in the coalition, namely the fear-of-ruin.
As a consequence, the set of inequalities (5.17) is induced by the following scalar inequality:

$$\chi_{\mathrm{C}} > \chi_{\mathrm{C}'} \tag{5.19}$$

which can be written as:

$$\frac{1}{\left(\sum_{i\in\mathrm{C}}\left(\frac{u_i'}{u_i}\right)^{-1}\right)^{-1}(\tilde{v}(\mathrm{C}))} > \frac{1}{\left(\sum_{i\in\mathrm{C}'}\left(\frac{u_i'}{u_i}\right)^{-1}\right)^{-1}(\tilde{v}(\mathrm{C}'))} \tag{5.20}$$

where $\tilde{v}$ is a characteristic function.
Taking the inverse, we obtain:

$$\left(\sum_{i\in\mathrm{C}}\left(\frac{u_i'}{u_i}\right)^{-1}\right)^{-1}(\tilde{v}(\mathrm{C})) < \left(\sum_{i\in\mathrm{C}'}\left(\frac{u_i'}{u_i}\right)^{-1}\right)^{-1}(\tilde{v}(\mathrm{C}')) \tag{5.21}$$

Denoting $\mathrm{F}_{\mathrm{C}} = \left(\sum_{i\in\mathrm{C}'}\left(\frac{u_i'}{u_i}\right)^{-1}\right)^{-1}$, we can write:

$$\mathrm{F}_{\mathrm{C}}\circ\tilde{v}(\mathrm{C}) < \mathrm{F}_{\mathrm{C}'}\circ\tilde{v}(\mathrm{C}') \tag{5.22}$$

We thus obtain the set of transformations Ω from the set of characteristic functions in itself that provide the players the incentive for some subset of coalition $\mathscr{C}'\subset\mathscr{C}$ must satisfy the following scalar inequalities, $\forall \mathrm{C}'\in\mathscr{C}', \forall \mathrm{C}\in\mathscr{C}\backslash\mathscr{C}'$ s.t. $\mathrm{C}'\cap\mathrm{C}\neq\emptyset$:

$$\mathrm{F}_{\mathrm{C}'}\circ\Omega(v)(\mathrm{C}') < \mathrm{F}_{\mathrm{C}}\circ\Omega(v)(\mathrm{C}) \tag{5.23}$$

where $\mathrm{F}_{\mathrm{C}} = \left(\sum_{i\in\mathrm{C}}\left(\frac{u_i'}{u_i}\right)^{-1}\right)^{-1}$ and v is the characteristic function of the original coalition game.
This concludes the proof. □

5.10.2 Proof of Corollary 85

Proof. Let $\mathscr{C}_f$ denote the set of coalitions containing the AP $f\in\mathscr{F}$. For every AP $f\in\mathscr{F}$, we want the vector of individual payoffs to be decreasing with the distance to the objective $\hat{q}_f$ where the distance function $d:\mathscr{C}\times\mathscr{C}\to\mathbb{N}$ is defined such that $d(\mathrm{C},\mathrm{C}') = ||\mathrm{C}|-|\mathrm{C}'||$.
In other words, we want any coalition of size q to be strictly preferred to any coalition of size $q+1$ for any size q superior or equal to the objective $\hat{q}_f$. We furthermore want any coalition of size q to be strictly preferred to any coalition of size $q-1$ for any size q inferior or equal to the objective $\hat{q}_f$.
Denoting $\mathbf{u}(\mathbf{x}_{\mathrm{C}})$ the vector of utilities of the allocation $\mathbf{x}_{\mathrm{C}}$ solving the Nash bargaining optimization program in coalition C), we want:

$$\begin{cases} \min\limits_{\substack{\mathrm{C}\in\mathscr{C}_f\\ s.t.|\mathrm{C}|=q}} \mathbf{u}(\mathbf{x}_{\mathrm{C}}) \succ \max\limits_{\substack{\mathrm{C}\in\mathscr{C}_f\\ s.t.|\mathrm{C}|=q+1}} \mathbf{u}(\mathbf{x}_{\mathrm{C}}), & \forall q\geq\hat{q}_f\\ \min\limits_{\substack{\mathrm{C}\in\mathscr{C}_f\\ s.t.|\mathrm{C}|=q}} \mathbf{u}(\mathbf{x}_{\mathrm{C}}) \succ \max\limits_{\substack{\mathrm{C}\in\mathscr{C}_f\\ s.t.|\mathrm{C}|=q-1}} \mathbf{u}(\mathbf{x}_{\mathrm{C}}), & \forall q\leq\hat{q}_f \end{cases} \tag{5.24}$$

Using the fact that the utilities are increasing functions of the payoffs, we obtain the following equivalent condition,

$$\begin{cases} \min\limits_{\substack{C\in\mathscr{C}_f \\ s.t.|C|=q}} \mathbf{x}_C \succ \max\limits_{\substack{C\in\mathscr{C}_f \\ s.t.|C|=q+1}} \mathbf{x}_C, & \forall q \geq \hat{q}_f \\ \min\limits_{\substack{C\in\mathscr{C}_f \\ s.t.|C|=q}} \mathbf{x}_C \succ \max\limits_{\substack{C\in\mathscr{C}_f \\ s.t.|C|=q-1}} \mathbf{x}_C, & \forall q \leq \hat{q}_f \end{cases} \tag{5.25}$$

Using the results if Proposition 82, we immediately obtain the transformation to be applied to the characteristic function to provide the required incentives,

$$\max_{\substack{C\in\mathscr{C}_f \\ s.t.|C|=q}} F_C \circ \Omega(v)(C) < \min_{\substack{C\in\mathscr{C}_f \\ s.t.|C|=q+1}} F_C \circ \Omega(v)(C), \quad \forall q \geq \hat{q}_f \tag{5.26}$$

and

$$\max_{\substack{C\in\mathscr{C}_f \\ s.t.|C|=q}} F_C \circ \Omega(v)(C) < \min_{\substack{C\in\mathscr{C}_f \\ s.t.|C|=q-1}} F_C \circ \Omega(v)(C), \quad \forall q \leq \hat{q}_f \tag{5.27}$$

where $F_C = \left(\sum_{i\in C} \left(\frac{u_i'}{u_i}\right)^{-1}\right)^{-1}$.

This concludes the proof. □

5.10.3 Proof of Proposition 86

Proof. After initialization, BDAA is made of two loops. The first one is a loop of proposals from users. At each iteration of this outer loop, there is an inner loop of counter-proposals from the APs to the users. We show that these two loops stop after a finite number of iterations. Let's first consider the inner loop. At each iteration of the inner loop, the following events can occur:

- An engaged AP remains engaged. Its dynamic list is left unchanged (in Step 2.f, only the dynamic lists of unengaged APs are updated).
- An unengaged AP is now engaged. Its dynamic list is left unchanged (in Step 2.f, only the dynamic lists of unengaged APs are updated).
- An unengaged AP remains unengaged. This is the case when some of the users it counter-proposed in Step 2.d have rejected its counter-proposal and either (a) none of them is engaged with another AP, or (b) some of them are engaged. In (a) the dynamic list remains unchanged. In (b) it is strictly decreasing.
- An engaged AP becomes unengaged. This means that some users in the coalition it was engaged to defected. This is only possible if they are engaged in a new coalition with another AP (Step 2.e). These defecting users are thus removed from the AP dynamic list, which is strictly decreasing.

In all cases, all the dynamic lists are weakly decreasing in the sense of inclusion. The inner loop thus converges in a finite number of steps.

We now consider the outer loop. We immediately have the convergence by finiteness of the number of APs each mobile can propose to and the fact that no mobile can propose more than once to any AP. The algorithm converges in a finite number of steps. □

5.10.4 Proof of Proposition 87

Proof. In [18], it is shown that the stability inducing sharing rules as given in Proposition 63 (see Chapter 3, Section 3.5.1) induce pairwise aligned preferences profiles satisfying the richness condition R1 ([18], pp. 334) of the domain of preferences **R**. It is shown ([18], Lemma 3, pp. 349) that if the family of coalitions $\mathscr{C}$, the preferences domain **R** satisfies R1 and all preference profiles are pairwise aligned, then no profile **R** admits an n-cycle, $n \in \{3, 4, \ldots\}$. As a consequence, there exists a core stable structure. It is unique if the preferences are strict. Furthermore, in the proof of Proposition 5 ([18], Proposition 5, pp. 359) it is shown that without n-cycles, the coalitions $\{C_1, \ldots, C_k\}$ in the stable structure can be re-indexed as $\{C_{i_1}, \ldots, C_{i_k}\}$ so that C_{i_j} is weakly preferred by its members to any coalition of agents in $\mathcal{N} \setminus \{C_{i_1}, \ldots, C_{i_{j-1}}\}$. If the preferences are strict (using a tie-breaking rule in case of indifference), then the members of C_{i_j} strictly prefer it.

We show by induction that these coalitions are formed in BDAA and never blocked by any other coalition once formed.
• The mobiles in C_{i_1} propose to the AP in C_{i_1} in the first proposal round since it is the most preferred coalition of any agent forming it. The AP in C_{i_1} counter-proposes to these mobiles who all accept. The coalition C_{i_1} is formed. No player of this coalition has any incentive to leave this coalition in a subsequent round. It cannot be blocked.

• Assume that the coalitions $\{C_{i_1}, \ldots, C_{i_l}\}$, $l < k$ are formed and are not blocked. We show that $C_{i_{l+1}}$ will be formed and cannot be blocked. Using the previous results, we have that the players in $C_{i_{l+1}}$ prefer it to any other coalition that can be formed with agents in $\mathcal{N} \setminus \{C_{i_1}, \ldots, C_{i_l}\}$. The payoff they receive from any of these coalitions is lower that their payoff in $C_{i_{l+1}}$, thus inferior to the maximum achievable payoff with the AP in $C_{i_{l+1}}$. As a consequence, all the players in $C_{i_{l+1}}$ must have ultimately proposed to the AP in this coalition.
At this point, the cumulated and dynamic lists of this AP contain the players in $C_{i_{l+1}}$. It counter-proposes to the coalitions it prefers to $C_{i_{l+1}}$ which contain players in the coalitions $\{C_{i_1}, \ldots, C_{i_l}\}$. These players reject the counter-proposals and are removed from the AP's dynamic list which ultimately counter-proposes to $C_{i_{l+1}}$. Any mobile in $C_{i_{l+1}}$ rejects this counter-proposal to propose to the other APs with maximum achievable payoff higher than its payoff in $C_{i_{l+1}}$. This continues up to the point where no maximum achievable payoff is higher than the payoff in $C_{i_{l+1}}$. At this point, all the players in this coalition accept forming $C_{i_{l+1}}$. The coalition $C_{i_{l+1}}$ is formed.
No player of this coalition has any incentive to leave this coalition in a subsequent round. This concludes the induction proof.
The unique stable structure $\{C_{i_1}, \ldots, C_{i_k}\}$ is the output of BDAA. □

5.10.5 Proof of Proposition 88

Proof. First we give an upper bound on the number of proposals emitted by the mobile users, then we give an upper bound on the number of proposals emitted by the APs. Finally, we conclude.
In at most F proposals, every mobile user has proposed to all the APs. Thus, in at most $F \times W$ proposals, the mobile users have proposed to all the APs.
In at most W counter-proposals, every AP has proposed to all the mobiles. Furthermore, every AP counter-proposes at each counter-proposing round. Thus, in at most $F \times W \times F$ the APs have emitted all their counter-proposals. We obtain that the total number of proposals (both mobile users proposals and APs counter-proposals) cannot exceed $F^3 \times W^2$. The complexity of BDAA is $O(n^5)$ where $n = \max(F, W)$. □

5.11 Appendix: Interpretation of BDAA in the economic framework

We show that the proposed matching mechanism (backward deferred acceptance) is particularly instinctive and natural in representing a competitive labor market process. We particularly focus on a competitive labor market interpretation.

Consider a competitive labor market made of a set $\mathscr{W}$ of workers and a set $\mathscr{F}$ of firms. Firms and workers can form coalitions as in the considered association and resource allocation game studied in the paper. The workers are looking for some jobs and the firms are looking for hiring some job seekers. We assume backward deferred acceptance mechanism as hiring process.

In Step 1.c, the firms communicate their top perspectives (assumed measured in payoffs) to the job seekers. Such communication can be understood as taking part in some recruiting campaign. In Step 2.a, the workers apply to the firms on the basis of the top perspectives and the promises of the recruiting campaigns. In Step 2.c, the firms consider the set of received applications and propose jobs.The emitted job offers may not necessarily correspond to the top perspectives of the recruiting campaign. In fact some conditions may not have been fulfilled (e.g. the set of workers for optimal production). Each job seeker receives the job proposals and accepts or rejects. In Step 2.d a job seeker may either accept or reject a counter-proposal. The acceptance and rejections are received by the firms. If all the proposals of a firm have been accepted, the workers are engaged (Step 2.e). Hired workers are declared as is. The counter-proposals go on up to the last firm.Each firm in the market having proposed jobs removes from its list of candidates those workers having rejected its counter-proposal and been engaged (Step 2.f). The candidature of the other job seekers are kept into consideration (Step 2.f). The firms may tell the job-seekers still in their lists that they are under consideration and that they may receive new offers. The firms not having recruited yet go on emitting job offers (Step 2.g). This process goes on up to the point where no job seeker receives new proposals (Step 2.g). At this point, the firms may notify the workers that they have received all the proposals. It is now upon the job seekers to create new opportunities. The job seekers attempt to send new applications (to their next most preferred firm in terms of top perspectives). The mechanism goes on up to the point where no job seeker can propose.

5.12 Appendix: Example

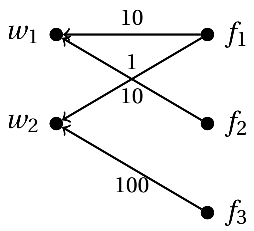

Figure 5.11: The APs send the best achievable payoff to the mobiles.

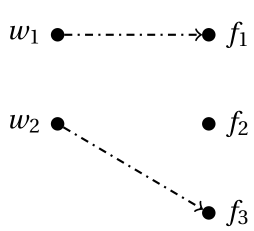

Figure 5.12: The mobiles propose to the APs.

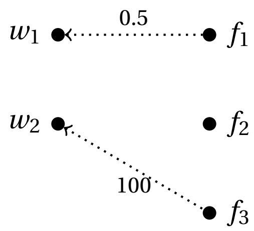

Figure 5.13: On the basis of the proposals, the APs emit counter-proposals.

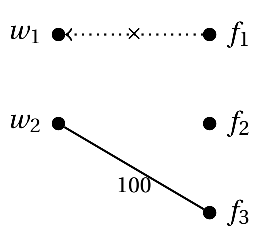

Figure 5.14: The mobiles accept or reject the counter-proposals.

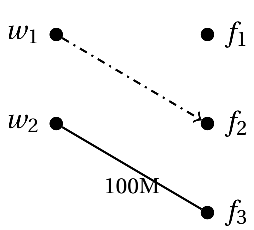

Figure 5.15: Mobile w_1 proposes to AP f_2.

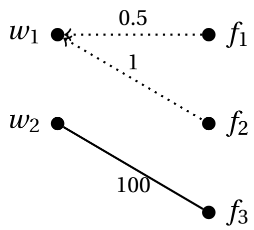

Figure 5.16: APs f_1 and f_2 counter-propose to w_1.

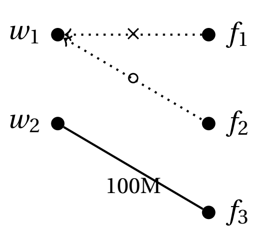

Figure 5.17: Mobile w_1 accept f_2 counter-proposal and rejects f_1' one.

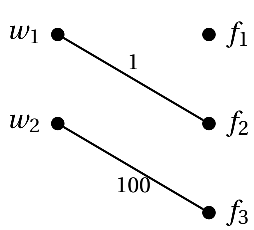

Figure 5.18: The stable matching.

In Figure (5.11) we show of a first example of application of the BDAA. The solid arrows show the best achievable payoffs that the mobiles can obtain with each AP. In dash-dotted we show the proposals of the mobiles to the APs, in dotted the counter-proposals of the APs to the mobiles and in plain the engagement. The cross show a reject and the circles an acceptance.

The APs send the maximum achievable payoff to the players (see Figure 5.11). AP f_1 sends 10 to w_1 and w_2 for the coalition $\{f_1; w_1, w_2\}$. For $\{f_1; w_1\}$ or $\{f_1; w_2\}$ the payoff is 0.5 due to the control of the incentives[6]. AP f_2 sends 1 for $\{f_2; w_1\}$. AP f_3 sends 100 for $\{f_3; w_2\}$.

The induced mobiles' preferences are thus, $f_1 \succ_{w_1} f_2$ and $f_3 \succ_{w_2} f_1$. In the first proposing round, mobile w_1 propose to the AP f_1 and mobile w_2 propose to the AP f_3 (see Figure 5.12). The APs update their cumulated list $\mathrm{L}(f_1) = \{w_1\}$, $\mathrm{L}(f_2) = \emptyset$, $\mathrm{L}(f_3) = \{w_3\}$ and their dynamic list $\mathrm{L}^*(f_1) = \{w_1\}$, $\mathrm{L}^*(f_2) = \emptyset$, $\mathrm{L}^*(f_3) = \{w_3\}$. The APs emit the following counter proposals: $(f_1, w_1, 0.5)$, $(f_3, w_2, 100)$ (see Figure 5.13). The mobiles receive the counter-proposals. Mobile w_2 accepts the counter-proposal of f_3 since none of its maximum achievable payoff gives more than the counter-proposal (shown in solid line in Figure 5.14). Mobile w_1 does not accept the counter-proposal of f_2 since the maximum achievable payoff received from f_2 is higher than the current counter-proposal from f_1 (shown by the cross in Figure 5.14).

The dynamic list of the APs are updated, $\mathrm{L}^*(f_1) = \{w_1\}$ and $\mathrm{L}^*(f_2) = \emptyset$. The algorithm enters the second counter-proposing round. AP f_1 emit the counter-proposal $(f_1, w_1, 0.5)$ which is the same as in the previous counter-proposing round. Mobile w_1 still rejects. The dynamic list of the unengaged APs are updated, $\mathrm{L}^*(f_1) = \{w_1\}$ and $\mathrm{L}^*(f_2) = \emptyset$.

The dynamic lists of the unengaged APs have not changed. The counter-proposing loop stops and the algorithm enters the outer loop for proposals. The second proposing round starts. The

[6] In this example the control provides the mobiles w_1 and w_2 the incentives for the set of coalitions of size 3 w.r.t. those of size 2 in the set of coalitions that the players can form with AP f_1

mobile w_1 is the only unengaged mobile. Its next most preferred AP is f_2 (with best achievable payoff 1). The mobile w_1 propose to f_2 (see Figure 5.15). The APs' lists are updated, $L(f_1) = L^*(f_1) = \{w_1\}$, $L(f_2) = L^*(f_2) = \{w_1\}$ and $L(f_3) = L^*(f_3) = \{w_2\}$. The two unengaged APs f_1 and f_2 counter-propose (see Figure 5.16). They emit the following counter proposals: $(f_1, w_1, 0.5)$, $(f_2, w_1, 1)$. Mobile m_1 receives the two counter-proposals, accepts the one of f_2 and rejects the one of f_1. It is engaged to f_2 (see Figure 5.17). The dynamic list of the unengaged AP f_1 is updated, $L^*(f_1) = \emptyset$. The only unengaged AP cannot propose. There are no more unengaged mobiles.
The algorithm stops. The final stable matching is shown in Figure 5.18.

Chapter 6

Video Caching and an Enumerative Cliques-Based Algorithm

In this chapter, we analyze a video caching problem from a content creator's servers to a service provider's servers using matching games. We show a new algorithm enumerating the set of core stable structures in ordinal coalition potential games. This algorithm is anytime, enumerative and performs on the intersection graphs of coalitions.

6.1 Introduction

Game theorists have been interested in looking at the stability of the structures resulting from the coalition formation process and have shown the importance of the concept in real-life applications. The stability is important since it exhibits the respect of the individual preferences and incentives. An unstable structure would result in deviations [3]. Coalition and matching games open the way through an important number of interesting applications, notably in wireless networks where much work remain to be done in the decentralized decision taking paradigm. Nevertheless, some of the commonly used assumptions (substitutability, responsive preferences, see [3]) limit the modeling to the cases without complementarities or peer effects. Recent theoretical works [9] (and references therein) overcame these difficulties and have paved the way through a suitable modeling of systems with complementarities and peer effects. As examples of such systems in wireless networks we have WiFi and its related anomaly, Device-2-Device with multi-hop relaying, virtual MIMO exploiting multi-users diversity, etc.

6.2 Related Works

Much work has been devoted to super-additive coalition games (where the grand-coalition forms [2]) and to the stabilization of structures by payoff distribution. Nevertheless, real-word problems may not verify the super-additivity assumption. More recently, researchers have been interested in the set of structures maximizing the social welfare in case of non-super-additive coalition games. This problem is called optimal coalition structure generation and is known to be an NP-hard problem with exponential complexity even for sub-optimal solution. Three classes are to be distinguished: dynamic programming, anytime property and heuristics. We have an interest in the first two classes. Dynamic programming has been used in [4; 7]. The worst case complexity is $O(3^n)$. On the opposite, any-time algorithms can be stopped at any-time to provide a sub-optimal result. The coalition structure graph representation[1] is used in [4] for coalition formation. The results are guaranteed to be within a bound from the optimum. Despite of improvements (see [8] and references therein), the complexity remains $O(n^n)$. Our compatibility graph is an alternative graph representation with coalitions as vertices and structures as maximal cliques.

In [1], Gale and Shapley show the non-emptiness of the core of the stable marriage and the college admission problems with preferences over individuals. The Deferred Acceptance Algorithm (DAA) is introduced. In [3], Roth et a.l. survey the existing results related to matching games and develop the theoretical results to provide insights in the understanding of the matching markets. The core stable structures is shown to be non-empty under some simplifying assumptions (e.g. responsive preferences). In [5], Cechlarova et a.l. propose two extensions of the preferences over individuals to sets and propose an algorithm close to Gale's top-trading cycles to find a strict core partition. In [6], Echenique et a.l. show a centralized fixed-point-based algorithm to compute the set of core stable (if non-empty) many-to-one matchings in the case of preferences over colleagues.

6.3 Contributions

In this chapter, we show a new centralized algorithm to enumerate core stable matchings when the players' preferences are emitted over the payoffs they obtain when playing a coalition potential game.

[1] The coalition structure graph is defined as the graph $\mathscr{G} = (\mathscr{S}, \mathscr{E})$, where the set of vertices is the set of coalition structures and the set of edges is defined such that it exists an edge $e = (S, S')$ between vertices S and S′ if S′ can be obtained from S by merging two coalitions of S or by splitting a coalition of S into two disjoint ones. By definition, the graph is of size B(N) (Bell number).

6.4 Potential Coalition Games and Video Caching

6.4.1 Potentials Coalition Games

Let $\Gamma = (\mathcal{N}, v, \{u_i\}_{i\in\mathcal{N}}\}$ define a coalition game in characteristic form. The set $\mathcal{N}$ denotes the set of players of cardinality N, $\{u_i\}_{i\in\mathcal{N}}$ denotes the set of their individual utilities and $v : \mathcal{N} \to \mathbb{R}$ is the characteristic function of the game. We define the set of coalitions $\mathscr{C}$. The potential function Φ is an ordinal coalition potential for the game Γ if for every player $i \in \mathcal{N}$,

$$u_i(\mathrm{C}) > u_i(\mathrm{C}') \text{ iff } \Phi(\mathrm{C}) > \Phi(\mathrm{C}'), \text{ for every } \mathrm{C}, \mathrm{C}' \in \mathscr{C} \tag{6.1}$$

where $u_i(\mathrm{C})$ is player i's utility of the payoff it receives when taking part in the coalition C.
A coalition game in characteristic form with set of players $\mathcal{N}$ admitting an ordinal potential Φ is called an ordinal coalition potential game in characteristic form and denoted,

$$\Gamma = (\mathcal{N}, v, \{u_i\}_{i\in\mathcal{N}}, \Phi) \tag{6.2}$$

6.4.2 The Video Caching problem

As an example consider a video caching system between a content creator $\mathscr{P}$ and a service provider $\mathscr{S}$. The content creator $\mathscr{P}$ has a set $\mathscr{L}$ of L videos. The service provider $\mathscr{S}$ has a set $\mathscr{R}$ of R caching servers. The caching servers may differ in the Quality of Service (QoS) for the cached content. This differentiation is taken to be due to the hierarchical location of the servers in the caching tree. We define $\mathscr{T}$ the set of T QoS of the servers. The content creator's videos are stored in its own servers. They can also be duplicated in a subset of the service provider's servers.
We define the set of coalitions as,

$$\mathscr{C} = \{\{r\} \cup \mathrm{J},\ r \in \mathscr{R}, \mathrm{J} \subseteq \mathscr{L}, |\mathrm{J}| \leq q_r\} \cup \{l \in \mathscr{L}\}$$

i.e. the set of subsets of videos and a single server. We furthermore require the quotas q_r of any server $r \in \mathscr{R}$ to be valued in $\{2, \ldots, \mathrm{L}-1\}$[2]. These quotas give for each server the maximum number of videos that can be cached.

Given a coalition C and the set of videos $\mathscr{L} \cap \mathrm{C}$, the content creator knows about some of the complementarities of the videos (complementarities induced by recommendation lists, management of the content, etc.). We assume that this partial information over the complementarities can be computed in the form of the following matrix of mutual impact factors $\mathbf{A} = (a_{ll'})_{l,l'\in\mathscr{V}}$ where each component $a_{ll'}$ is the impact factor of a video v over another video v'. The content creator does not know more than these informations and cannot estimate the impact of caching in the service provider's servers. Nevertheless, given a coalition C and knowing the natural views and the impact factors of the videos in the coalition, we assume that the service provider can compute the impact of caching over the number of views of the videos in the coalition as the following fixed point equations,

$$n_l(\mathrm{C}) = n_l(1)\gamma_r + \sum_{l'\in\mathscr{V}\cap\mathrm{C}} a_{ll'} n_l(\mathrm{C}) \tag{6.3}$$

where $a_{ll'}$ is an impact factor of l over l' and $\gamma_r \in \mathscr{T}$ is QoS gain obtained by caching in the server r. Thus, the number of views of a video in a coalition C depends on the players in the coalition in three ways,

- (i) the video itself via $n_l(1)$,
- (ii) the other videos in the coalition (cached in the same server) via $\mathbf{A}$,
- (iii) the factor of quality of the caching server via γ_r.

[2] See the regularity conditions over the set of coalitions for the non-emptiness of the set of core stable structures in all states of nature in Chapter 3.

As an example of interpretation, the impact factor a_{ij} may be interpreted as the probability that a user having viewed the video v_i watch video v_j. This shows the importance of the complementarities and peer effects in the modeling. We define the characteristic function $v : \mathscr{C} \to \mathbb{R}$ of a coalition as,

$$v(\mathrm{C}) = \sum_{l \in \mathrm{C} \cap \mathscr{V}} n_l(\mathrm{C})\alpha$$

where α is the constant monetary income generated by a view of a video. We now assume $v(\mathrm{C})$ is shared among the players in C via a Nash bargaining with null threats with concave individual utilities $u_l(x_l) = x_l^{\alpha_{\mathscr{P}}}$ for the content creator's videos and $u_r(x_r) = x_r^{\alpha_r}$ for the service provider's server r. The servers' bargaining powers increase in their factor of quality. Both bargaining powers $\alpha_{\mathscr{P}}$ and $\alpha_{\mathscr{S}}$ are assumed to be valued in $]0;1[$ so as to make the utilities strictly concave. The servers' bargaining powers increase in their factor of quality. We obtain that the individual payoffs increase in $\chi_{\mathrm{C}} = \frac{v(\mathrm{C})}{|\mathrm{C}|\alpha_{\mathscr{P}}+\alpha_r}$ which is called the fear-of-ruin. If the fear of ruin increases then all the payoffs increase. Thus, the players (the content creator via its videos and the service provider via its server) have the incentive to form groups maximizing the fear-of-ruin. In this case the fear-of-ruin is the potential Φ of the matching game between $\mathscr{P}$'s videos and $\mathscr{S}$'s servers. The game $\Gamma = (\mathscr{N} = \mathscr{L} \cup \mathscr{R}, v, \{u_i\})$ is an ordinal coalition potential game with potential function χ.

6.5 An Anytime Centralized Enumerative Algorithm

In the this section we show a new anytime cliques-based algorithm for enumerating the core stable coalition structures in the case of an ordinal coalition potential game Γ with a set of coalitions $\mathscr{C}$.The algorithm has been constructed over the game-theoritic sequential description of convergence to a stable structure provided in [9].

Define the coalitions weighted-vertices undirected graph $(\mathscr{G}, \Phi)$ such that the graph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ is the complement of the intersection graph of the coalitions in $\mathscr{C}$. The set of vertices $\mathscr{V}$ is in bijection α with the set of coalitions $\mathscr{C}$ but for the sake of simplicity and clarity we will further identify a coalition $\mathrm{C}_i \in \mathscr{C}$ with the vertex $v_i \in \mathscr{V}$ of the graph. Let $\mathscr{V}' \subset \mathscr{V}$ be a subset of the nodes, we denote $\mathscr{G}' = (\mathscr{V}', \mathscr{E}(\mathscr{V}'))$ the subgraph induced by $\mathscr{V}'$. Each vertex $v_i \in \mathscr{V}$ is weighted by the value $\Phi(\mathrm{C}_i)$ and the set of edges $\mathscr{E}$ is defined by the adjacency matrix $\mathbf{A} = (a_{ij})_{(i,j)\in\mathscr{C}^2}$ such that:

$$a_{ij} = \begin{cases} 1 & \text{if } \mathrm{C}_i \neq \mathrm{C}_j \text{ and } \mathrm{C}_i \cap \mathrm{C}_j = \{\emptyset\} \\ 0 & \text{if } \mathrm{C}_i = \mathrm{C}_j \text{ or } \mathrm{C}_i \cap \mathrm{C}_j \neq \{\emptyset\} \end{cases}$$

A clique σ of the graph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ is a complete subgraph of $\mathscr{G}$. In case the clique σ may not be included in a superior sized clique σ' without loosing the completeness property, σ is said to be maximal. If σ is the clique with the highest cardinality (in terms of number of vertices $|\sigma|$) the clique is maximum. For the sake of clarity and simplicity, we will further identify a clique σ with the set of its coalitions given by $\mathscr{V}(\sigma)$. Any two coalitions C_1 and C_2 will be called *compatible* if they have no common players, i.e. $\mathrm{C}_1 \cap \mathrm{C}_2 = \emptyset$. In other words, if the nodes v_1 and v_2 are adjacent. More generally, the coalitions of $\mathscr{C}' \subset \mathscr{C}$ are compatible if the set of their nodes form a clique in the graph $\mathscr{G}$. Denote $\mathscr{G}_k$ the subgraph induced by the set of nodes of k^{th} highest weight. As an example, $\mathscr{G}_1$ is the subgraph induced by the nodes with maximum weights, i.e. the subgraph such that any node in $\mathscr{G}_1$ is a coalition giving each player its most preferred achievable share in the game.

The algorithm iteratively builds a forest (i.e. a set of disconnected trees) such that each tree is rooted by one of the maximal clique obtained at initialization and each vertex in each tree is a stable structure. A vertex that is not a leaf is a subgame stable structure (this is the anytime property of the algorithm) and a leaf is a stable structure for the game. The set of leaves is the output of the algorithm. To go into further details, the algorithm starts by defining an initial set of structures, obtained by looking for the set Σ_1 of maximal cliques in the subgraph $\mathscr{G}_1$ (line 3). Each clique $\mathrm{S} \in \Sigma_1$ is the root of a tree of stable structures (line 3). Starting from S, we look in $\mathscr{G} \backslash \mathscr{G}_1$ for

Algorithm 7: Cliques-based algorithm for finding the set of stable structures

Data: Coalitions compatibility weighted graph $(\mathscr{G},\Phi)$
Result: The set of stable structures $\mathscr{S}$

1 **begin**
2   *Step 1 (Initialize)*;
3     $\mathscr{S} := \Sigma_1$ (Σ_1 set of maximal cliques in $\mathscr{G}_1$);
4     $p(\mathscr{S}) = 0$ (vector of size $|\mathscr{S}|$, the structures in $\mathscr{S}$ have not been visited yet);
5   *Step 2 (Form the forest)*;
6   **while** $\exists \mathrm{S} \in \mathscr{S}$ *s.t.* $p(\mathrm{S}) = 0$ **do**
7   take S s.t. $p(\mathrm{S}) = 0$;
8         $p(\mathrm{S}) = 1$;
9   **if** $\mathscr{W} \subset \mathrm{S}$ *or* $\mathscr{F} \subset \mathrm{S}$ *(if the structure includes one or both of the set of players, no coalition can be added)* **then**
10   break;
11   **if** $\mathscr{C}_\mathrm{S}^{\max} = \emptyset$ *($\mathscr{C}_\mathrm{S}^{\max}$ set of maximum valued coalitions compatible with* S*)* **then**
12   break;
13         $\mathscr{S} := \mathscr{S} \backslash \mathrm{S}$;
14   **for** $\mathrm{S}' \in \Sigma'_\mathrm{S}$ *(Σ'_S set of maximal cliques in $\mathscr{C}_\mathrm{S}^{\max}$)* **do**
15             $\mathscr{S} := \mathscr{S} \cup \{\mathrm{S}' \cup \mathrm{S}\}$ (complete the structures by the child $\mathrm{S}' \cup \mathrm{S}$);
16             $p(\mathrm{S}' \cup \mathrm{S}) = 0$;

the set $\mathscr{C}_\mathrm{S}^{\max}$ of highest valued coalitions that are compatible with S (line 12). If $\mathscr{C}_\mathrm{S}^{\max}$ is empty, the remaining players are left unmatched. Otherwise let Σ'_S be the set of maximal cliques in $\mathscr{C}_\mathrm{S}^{\max}$ (line 14). A child of S in the structure tree is a structure $\mathrm{S} \cup \mathrm{S}'$ where $\mathrm{S}' \in \Sigma'_\mathrm{S}$ (line 15). Every time a structure is visited, it is tagged by a binary variable such that it will not be visited again (line 8). Furthermore, in case this structure becomes a parent, then the algorithm gets rid of it (line 13) to only hold the children (tagged as non-visited, line 16). The algorithm goes on iterating as long as it exists non-visited structures (line 6).

Proposition 91. *The algorithm converges in a finite number of iterations.*

The proof uses the finiteness of the set of structures and the non-recurrence of the walk of the algorithm in the space of structures of coalitions. It shows that it exists a finite upper bound in the number of iterations of the algorithm. The existence and finiteness of this bound being sufficient to show the convergence of the algorithm in a finite number of iterations.

Proof. By construction, the algorithm stops if there are no more non-visited structures in the set of built ones, i.e. $p(\mathrm{S}) = 0$ for all $\mathrm{S} \in \mathscr{S}$. At every iteration, the considered S is tagged as visited, i.e. $p(\mathrm{S}) = 1$. Since there is a finite number of players, the set of subsets of coalitions (union of complete and incomplete structures) is also finite. As a consequence, the algorithm will iterate at most as many times as the number of these subsets, thus for a finite number of iterations. □

Proposition 92. *The algorithm outputs stable structures.*

The proof is by contradiction and induction. It assumes that it exists an unstable structure which is an output of the algorithm and shows by recurrence that no subset of players of the structure can deviate.

Proof. Assume it exists an unstable structure S_u which is an output of the algorithm. Then, by definition of stability, it must exist a subset B of player having the incentive to deviate from the current structure and the power to enforce the deviation.

Initial step:
No subset of player in coalitions from the subgraph $\mathscr{G}_1$ has the incentive to deviate since it achieves the maximum payoff, i.e. $\mathrm{B} \cap \mathscr{V}(\mathscr{G}_1) = \emptyset$. Assume the blocking coalition B contains a subset B′ of players from coalitions in $\mathrm{S}_u \cap \mathscr{V}(\mathscr{G}_2)$. As players of B′ are in $\mathscr{V}(\mathscr{G}_2)$, they obtain a payoff of w_2. As they are also in B, they have the incentive to obtain a payoff strictly greater than w_2. By defintion of $\mathscr{G}_1$ and $\mathscr{G}_2$, they have thus the incentive to deviate to $\mathscr{G}_1$ and obtain $w_1 > w_2$. As players of $\mathscr{G}_1$ have no incentive to deviate, players of B′ must extend the clique formed in $\mathscr{G}_1$. This is in contradiction with the fact the algorithm has formed a maximal clique in $\mathscr{G}_1$. Thus, no subset of players in $\mathscr{G}_2$ can deviate.

Induction step:
Assume that no subset of players in $\mathscr{G}_1, \ldots, \mathscr{G}_n$ can deviate and that the blocking coalition B contains a subset B′ of players from coalitions in $\mathrm{S}_u \cap \mathscr{V}(\mathscr{G}_{n+1})$. The players in B′ have the incentive to form a coalition in at least one subgraph in $\mathscr{G}_1, \ldots, \mathscr{G}_n$. By construction, the set of coalitions in $\mathrm{S}_u \cap \{\mathscr{G}_1, \ldots, \mathscr{G}_n\}$ forms maximal cliques in the subgraphs $\mathscr{G}_1, \ldots, \mathscr{G}_n$. Thus, in order that players of B′ form a coalition in any subgraph $\mathscr{G}_l \in \{\mathscr{G}_1, \ldots, \mathscr{G}_n\}$, some players in $\mathrm{S}_u \cap \mathscr{G}_l$ would have to deviate from their coalitions to form another maximal clique in $\mathscr{G}_l$ with players in B′. But neither a player in $\mathrm{S}_u \cap \mathscr{G}_l$ has the incentive to deviate from a clique in $\mathscr{G}_l$ nor can deviate to a clique in $\mathscr{G}_k$, $k < l$. So, by the induction assumption players in B′ cannot deviate.

As a conclusion, S_u cannot contain a blocking coalition, Su is stable. □

Proposition 93. *The algorithm outputs the set of stable structures.*

The proof is by contradiction. It assumes that it exists a stable structure which is not an output of the algorithm and shows that it is not possible that it differs from those built by the algorithm.

Proof. Assume it exits a stable structure S_u which is not an output of the algorithm.

Let $\sigma_1 = \mathrm{S}_u \cap \mathscr{G}_1$ be the set of coalitions both in the structure S_u and in the subgraph $\mathscr{G}_1$. The set σ_1 is non empty because otherwise, there would be a blocking subset of coalition in $\mathscr{G}_k$, $k > 1$ having the incentive to deviate to $\mathscr{G}1$ and S_u wouldn't be stable. Assume now that σ_1 does not form a maximal clique in $\mathscr{G}_1$. Then the clique σ_1 can be augmented by at least one coalition. This latter coalition is necessarily formed by players from $\mathrm{S}_u \cap \mathscr{G}_k$, $k > 1$ (since Su is a partition of the set of players) and is thus blocking for S_u. This is in contradiction with the fact that S_u is stable.
Thus, S_u is rooted by a maximal clique in $\mathscr{G}_1$. As the algorithm (line 3) roots the forest with the set of all maximal cliques Σ_1 in $\mathscr{G}_1$, $\mathrm{S}_u \cap \mathscr{G}_1 \in \Sigma_1$ must root a tree of the algorithm.
Let us now proceed by induction.
Initial step: Let σ_2 be the set of maximum valued coalitions in $\mathrm{S}_u \backslash \sigma_1$. Let show that σ_2 is a maximal clique in $\mathscr{C}_{\sigma_1}^{\max}$. Assume that σ_2 does not form a maximal clique in $\mathscr{C}_{\sigma_1}^{\max}$. Then the clique σ_2 can be augmented by at least one coalition. This latter coalition is necessarily formed by players from $\mathrm{S}_u \cap \mathscr{G}_k$, $k > l$ where $\mathscr{C}_{\sigma_1}^{\max} \subset \mathscr{G}_l$. This coalition blocks S_u. This is in contradiction with the fact that S_u is stable.

Induction step: The induction step is straightforward and similar to the initial step. □

6.6 Some elements about the complexity

Finding stable structures of coalitions and enumerating the set of theses structures are challenging combinatorial tasks. This is mainly due to the exponential grows of the set of coalitions and structures in the number of players in the game. The induced computational difficulty is thus to design practically efficient algorithms with low complexity and bounded execution times.

The number of coalitions in a game with N players is $2^{\mathrm{N}} - 1$ (the empty coalition being irrelevant since it contains no players) and the total number of structures that can be obtained by partitioning the set of players N into k non-empty subsets (where $k \leq \mathrm{N}$) is given by the Stirling number of the second kind $S(N,k)$

$$S(N,k) = \frac{1}{k!}\sum_{l=0}^{k}(-1)^{k-l}\binom{k}{l}l^{N} \tag{6.4}$$

This number may also be recursively defined such as

$$S(N,k) = kS(N-1,k) + Z(N-1,k-1) \tag{6.5}$$

As examples of Stirling numbers of the second kind, we have $S(20,3) = 5.8060e+8$ and $S(20,7) = 1.1143e+12$ and $S(90,10) = 2.7536e+83$ which is more than the upper bound of the estimated number of atoms in the known universe (10^{82}).
Finally, the total number of coalition structures that can be obtained by partitioning the set of N players into non-empty subsets if given by the Bell number $B(N)$

$$B(N) = \sum_{k=1}^{N} S(N,k) \tag{6.6}$$

As examples of Bell numbers and thus number of structures of coalitions (candidate solutions in the coalition formation game), we have $B(10) = 1.1597e+5$, $B(20) = 5.1724e+13$, $B(50) = 1.8572e+47$ and $B(90) = 1.4158e+101$. Sandholm et a.l. have shown in [4] that the number of coalition structures is $O(N^N)$ and $\omega(N^{\frac{N}{2}})$. In the particular case of many-to-one matchings, Echenique et a.l. give in [6] the total number of many-to-one matching in a problem with n colleges and m students ,

$$\sum_{k=1}^{n}\binom{n}{k}S(m,k)k! \tag{6.7}$$

By definition, the set of coalitions is exponential in the number of agents involved in the game. Even though, we consider a restricted set of coalitions, there still is an exponential number of subset of videos. Thus, the input of the algorithm (number of nodes in the intersection graph) is exponential in the number of videos to be cached. Particularly, with L videos and R servers, this graph has a maximum of $(2^L - 1)F$ nodes since, in the worst case, any server $r \in \mathscr{R}$ can form a coalition with any non-empty subset of videos from $\mathscr{L}$ but no subset of player from $\mathscr{R}$ can form a coalition without a player from $\mathscr{F}$. For the rest of this section we assume this worst case scenario. The size of the graph is thus linear in the cardinality of the set of players $\mathscr{F}$ and exponential in the cardinality of the set of players $\mathscr{W}$. Yet, we have not succeeded in obtaining more result on the complexity of the proposed cliques-based algorithm and leave this as an open-question. With respect to the coalitions structure graph defined in [4], our compatibility graph is an alternative graph representation with coalitions as vertices and structures as maximal cliques.

An example of compatibility graph is shown in Figure 6.1.

(F,W)	$\lvert\mathscr{V}\rvert$	Time (s)
(2,4)	$2*2^4$	0.280791
(2,5)	$2*2^5$	0.336115
(2,6)	$2*2^6$	0.358524
(2,7)	$2*2^7$	0.524736
(2,8)	$2*2^8$	0.890548
(2,9)	$2*2^9$	2.941256
(2,10)	$2*2^{10}$	13.896822
(2,11)	$2*2^{11}$	228.499408

Table 6.1: Some running times of the cliques-based algorithm. Intel Core i5vPro @ 1.90GHz, 4Go ram

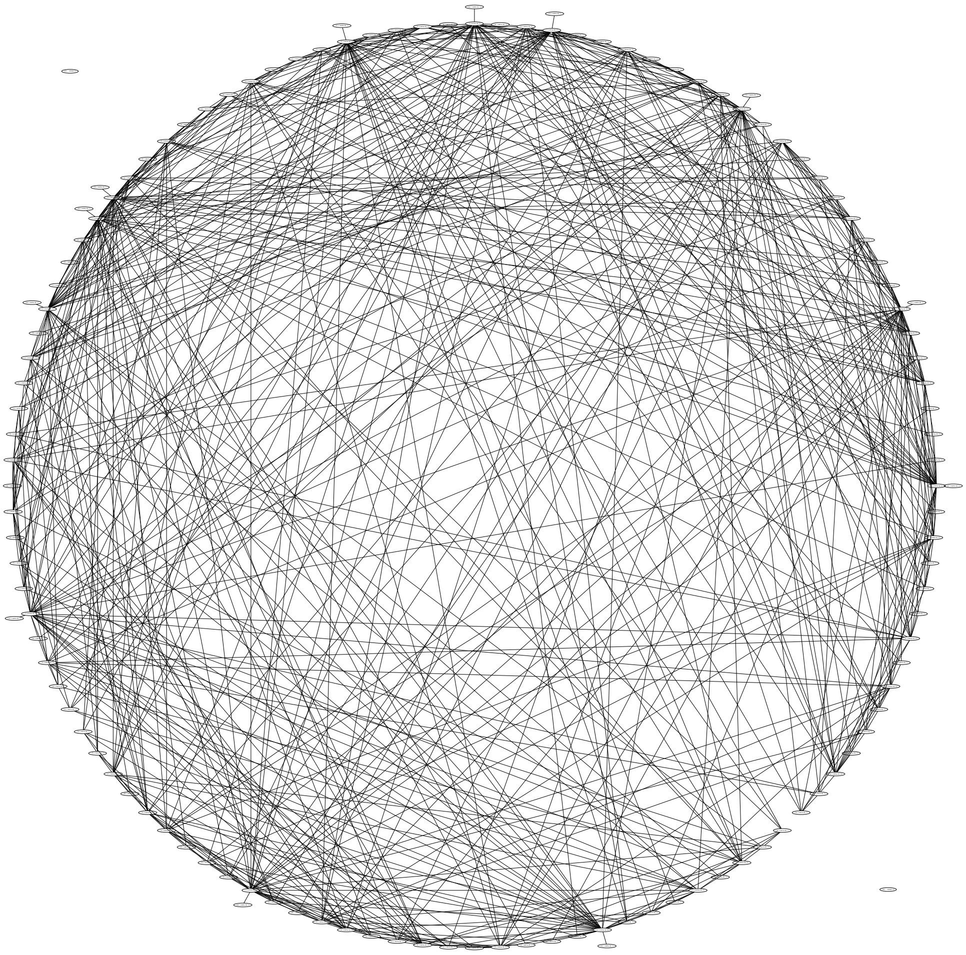

Figure 6.1: Compatibility graph of the coalitions generated by six videos and two servers.

6.7 Example

As an example of application of the algorithm and interpretation of the intermediate and final results, consider the compatibility graph given in the first plot Table 6.2. The other plots of Table 6.2 show the iterative construction of stable structures induced by the algorithm. The plots of Table 6.3 show the resulting stable structures and the plots in Table 6.4 provide the tree-building interpretation of the iterative search.

Let us get into more details about this example. Consider the first graph of Table 6.2 as the weighted compatibility graph such that the nodes are vertically arranged according to their weights in the sense that higher weights give upper nodes. This graph shows that the coalitions are such that they may be grouped in three levels of weights (ie individual value of users in the coalition), i.e. $\mathscr{G}_1, \mathscr{G}_2$ and $\mathscr{G}_3$ where $\mathscr{G}_1$ is the subgraph induced by the coalitions C_1 and C_2, $\mathscr{G}_2$ is the subgraph induced by the coalitions C_3, C_4, C_5, C_6 and $\mathscr{G}_3$ is the subgraph induced by the coalitions C_7, C_8 and C_9. In its first step, the algorithm works in the subgraph $\mathscr{G}_1$, looking for the set Σ_1 of maximal cliques (line 3). In this example, there is a single clique in $\mathscr{G}_1$, which is $\mathscr{G}_1$ itself. The algorithm returns $\Sigma_1 = \{\mathscr{G}_1\}$. This is shown in the second plot of Table 6.2 where the single maximal clique is highlightenned in green. Thus, $\mathscr{S} = \{S_0\}$ where $S_0 = \{C_1, C_2\}$ (line 3). In terms of trees, there will be a unique tree of stable structures with S_0 as the root. In this particular case, the forest is reduced to a single tree.

Then, the algorithm enters the second step dedicated to iteratively (line 6) completing the previously initialized stable structures. This step of the algorithm results in the set of stable structure shown in red in the Table 6.3. In the first iteration, the algorithm creates three structures $S_1 = S_0 \cup S'_1$, $S_2 = S_0 \cup S'_2$ and $S_3 = S_0 \cup S'_3$ where S'_1, S'_2 and S'_3 are the sets of coalitions from the

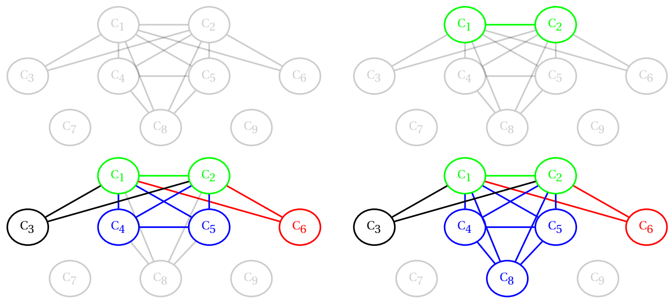

Table 6.2: The maximal cliques algorithm. The set of coalitions Σ is represented by the nodes of the graph. The vertical placement of a coalition associated node in the graph is in agreement with their weight ω of this node i.e. $[\omega_{C_1}, \omega_{C_2}] > [\omega_{C_3}, \omega_{C_4}, \omega_{C_5}, \omega_{C_6}] > [\omega_{C_7}, \omega_{C_8}, \omega_{C_9}]$. The algorithm iteratively builds the maximal cliques from higher weighs to lower ones.

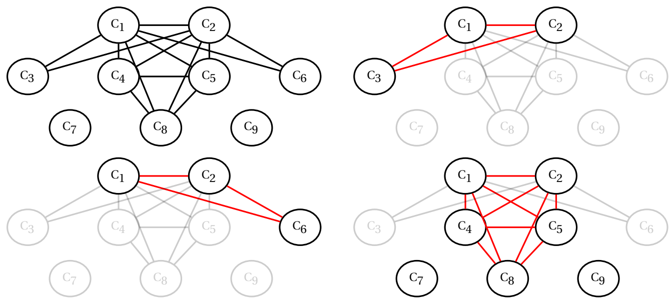

Table 6.3: The maximal cliques algorithm outputs, i.e. the stable structures of coalition. Each graph exhibits a structures as a red maximal clique.

three maximal cliques in $\mathscr{C}^{\max}_{S_0}$. The third plot of Table 6.2 shows these as the union of the green subgraph and each of the colored one. The completion phase and associated trees may be observed on the second plot of Table 6.4.

From now on, the algorithm performs identically by iterating over the built structures as if they were from the initialization step (line 6). For the second and third iterations, $\mathscr{C}^{max}_{S_1}$ and $\mathscr{C}^{max}_{S_3}$ are empty. No more coalitions are compatible either with S_1 or S_3 (line 12). The structures are tagged as visited $p(S1) = p(S3) = 1$ and S_1 and S_3 are leaves of the tree. The fourth iteration visits S_2, and we have $\mathscr{C}^{max}_{S_2} = \{C_8\}$. The search for maximal cliques in it trivially gives $S"_2 = \{C_8\}$ and we have the ultimate leaf of the tree $S_4 = S"_2 \cup S_2$. The third plot of Table 6.2 shows this as the union of the green subgraph and the blue one and the completion is also shown by the blue branch in the fifth plot of Table 6.4. This ends the algorithm which results in the following three stable structures: $\mathscr{S}_1 = \{C_1, C_2, C_3\}$, $\mathscr{S}_2 = \{C_1, C_2, C_4, C_5, C_8\}$ and $\mathscr{S}_3 = \{C_1, C_2, C_6\}$. As already pointed out at the beginning of this paragraph, these three stable structures are shown in Table 6.3. Note that C_7 and C_9 are not compatible with others.

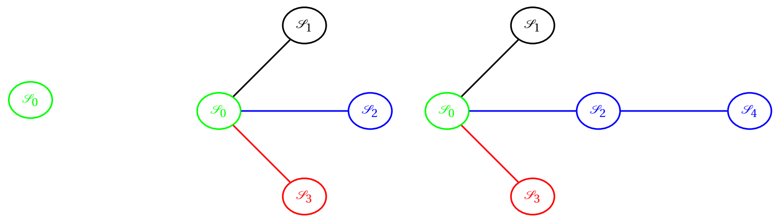

Table 6.4: The iterative construction of the tree of stable structure. Each node is a subgame perfect equilibrium, i.e. no player in each node has the incentive to deviate from the coalition he belongs to on the structure associated to the node.

6.8 Conclusion

We have shown a new algorithm to enumerate the set of core stable structure in coalition potential games. The algorithm is centralized, anytime and enumerative. We need to go further in the analysis of the complexity of the enumerative algorithm and search for the requirements over the coalitions compatibility graph to make it fall in classes of graphs with low complexity in the maximal cliques enumeration. Another decentralized algorithm, called BDAA, was shown in chapter 5 in the framework of the analysis of the WiFi association problem.

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Chapter 7

Fear of Ruin and Concavity Conditions in Coalition Games with Generalized α-Fair Resource Allocation

In this chapter, we study a general multiagent system where agents can form groups, or coalitions, and share a resource. In particular, we derive some of the conditions for the generalized α-fair scheme to guarantee the existence of a stable partitioning of the players in groups. We give a game-theoretic interpretation of these conditions using measures of risk aversion and fear of ruin. The results are both theoretically and practically useful in networks with applications to the decentralized assignment and migration of tasks in cloud computing or computer architecture, the user association problem in wireless networks (including Device-to-Device) and the post of messages on the timelines of social networks. This chapter aims at strengthening the links existing between some well-known and used fairness schemes in networks, particularly the generalized α-fair allocation, and the game-theoretic analysis.

7.1 Introduction

Multiagent systems are based on a decentralized interaction of the decision takers. Multiple reasons have led to the design of such systems. Decentralization may be an exogenous constraint. As an example, consider a social network, where distributed decision takers create flows and links and influence the structure of the graph underlying the system operations. In this case, the customers, users or devices are distributed by nature. Another reason may be, broadly speaking, related to the cost. Whereas in some systems decision can be taken by a central entity, distributed implementations may be preferred for cost, simplicity, or robustness reasons. As an example, we can think about the optimal user association to access points in a wireless network. In this chapter, we focus on a generic model coalition formation problem with exogenous sharing rule where the decision makers can form groups, each producing a worth shared among the players of the group according to the sharing rule. This model covers a wide range of resource allocation situations, where multiple agents can distributedly take decisions and form coalitions to share a given resource. Given any agent, her incentive for the groups he can participate to are defined by the shares he would receive if the group were formed. In team problems [7], the fundamental assumption is that the decision taking process must maximize a common objective function. The underlying behavioral interpretation is that the agents' preferences are all aligned toward the optimum of the objective. The decision taking process is not centralized and the agents follow a behavior that is altruistically guided toward optimality of the global objective function.

In many games, players are assumed to be selfish rational payoff maximizers. In this case, the interests may not be aligned by mean of a common objective function. This differs from the team problem in the sense that a player takes the decisions with the aim of improving its own welfare despite the fact that some other players may be negatively affected. The cooperative game theory studies the games where the players can cooperate. The cooperative transformation of a game proposed by Myerson [5] as a mapping from a non-cooperative game to another one with additional cooperative strategies highlights the reasons for cooperative behaviors to emerge: new opportunities. The equilibrium of this transformed game results from a process of bargaining among the players. Solution concepts to the bargaining problem consist in deriving equilibrium cooperative strategies and resource allocations in such a way that individual incentives and some fairness properties are respected. As examples, consider the Nash's solution to the two-person bargaining problem [1][2] or the *core* allocation [5][6] defined as the set of allocations of the worth $v(\mathcal{N})$ providing the players the incentive to form the grand coalitions. As other examples consider the matching [4] and coalition formation problems [17]. The solution concepts of these cooperative association games may be defined in terms of stability such as the pairwise stability, the core stability, or the group stability.

It is well-known that the existence of equilibria (in non-cooperative and cooperative games) and their characteristics strongly depend on the properties of the players' utilities or preferences. Some of these properties have interpretations in terms of behaviors. As an example, a concave increasing utility function induces a risk averse behavior. Several measures of aversion have been defined to quantify the players behavior in their decision making process in presence of uncertainty. As examples where the risk averse behavior of the player has been shown to have an impact on the solution of the game, consider the bargaining problem [1][2], the taxation mechanisms [10], or more recently routing games in networks [20]. In these papers, some measures such as risk aversion, boldness, fear of ruin, and pure fear of ruin are used. We will particularly show that such indicators may be useful in the understanding of well-known fairness schemes (we focus on the generalized α-fair allocation) in game-theoretic applications to networks. We will see that these indicators appear naturally when attempting to derive some of the conditions for the generalized α-fair allocation to belong to the class of resource allocation schemes that guarantee the existence of a stable partitioning of the players in a coalition game.

Game-theory and networks have somewhat progressed in parallel on the definition of fairness schemes. They have converged to common solutions. As an example, the proportional fairness introduced by Kelly in [12] is a particular case of the generalized proportional fairness achieved by the Nash bargaining. The proportional fair allocation is now one of most popular and used resource allocation scheme in networks. The interesting properties of this scheme has led researchers to generalizations and unifications such as the α-fairness defined by Mo and Walrand in [16]. The sum throughput optimization can be obtained with $\alpha = 0$, the proportional fairness with $\alpha \to 1$, the average delay minimization with $\alpha = 2$ and the max-min fairness with $\alpha \to \infty$. Finally, in [15], Altman et a.l. defined the generalized α-fair allocation which is, to the best of our knowledge, the most general version of α-fair allocation to date.

In this chapter, we go further in the understanding of some game-theoretic properties of the well-known generalized α-fair allocation. Assuming decentralized decision-taking systems, we derive some of the conditions for the generalized α-fair scheme to guarantee the existence of a stable partitioning of the players in groups. Particularly, we focus on the concavity constraints over the utilities. We give the non-trivial game-theoretic interpretation of these conditions in terms of measures of risks. The results are both theoretically and practically useful in networks in their decentralized decision-taking paradigm (particularly coalitions formation problem and the stable matching problem) with applications to the problems of decentralized tasks assignment (e.g. in cloud computing or computer multi-core architectures) or connectivity management (D2D, classical mobile users association). This chapter aims at strengthening the links existing between some well-known and used resource allocation schemes in networks, particularly the generalized α-fair allocation, and the game-theoretic analysis to provide new insights for applications in multiagent systems.

7.2 Contributions

Our contributions can be summarized as follows:

- We show the importance of the game-theoretic analysis of risks and behaviors of agents under uncertainty in networks. To the best of our knowledge, this is the first joint analysis of multiagents system in networks, coalition games, resource allocation and risks.

- We derive some of the conditions on the utility functions used in the generalized α-fair allocation required for the existence of a core stable partitioning of the players in a multiagents system. We particularly focus on the concavity constraints over the utilities. We give these conditions in terms of pure fear of ruin. To the best of our knowledge, this is the first link between the generalized α-fair scheme, the existence of game-theoretic stable matchings or structures of players and the measures of risk aversion.

- We show that the fear of ruin of a decision maker can be obtained as the limit of a sequence of expected ratios of probabilities in even money gambles with decreasing gains. To the best of our knowledge, this is a new interpretation of the fear of ruin.

In Section 7.5, we derive our main results in term of measures of risks aversion: the fear of ruin and the pure fear of ruin. In Section 7.4, we give three examples of multiagent systems that fall in the scope of the proposed modeling and analysis. In Section 7.6, we introduce the risk and measures of aversion of a player. We show that the results of the previous sections can be interpreted in terms gambles. Finally, we conclude.

7.3 Model

Let $\mathcal{N}$ be a set of players. The players can form groups, called coalitions. We denote $\mathcal{C}$ the set of coalitions. Define a function $v : \mathcal{C} \to \mathcal{R}^+$, called the characteristic function. The worth of a coalition may be produced by the coalition (e.g. C contains some workers and a firm producing goods having a total worth) or may be exogenously fixed (e.g. a fixed amount of resource allocated to any group such as a time-interval or a bandwidth). The worth $v(C)$ of $C \in \mathscr{C}$ is shared among the players in the group using a resource allocation scheme or fairness scheme. The game-theoretical term is sharing rule. We denote $(\mathscr{N}, \mathscr{C}, v, D)$ this coalition game.

As a payoff maximizer[1], any player is thus seeking for forming the group maximizing its own payoff. Figure 7.1 shows a structure in the coalition formation problem and Figure 7.2 a structure in the matching problem, or simply a matching.

Figure 7.1: Examples of structures in the coalition formation problem. In (a), the structure is a single coalition made of all the players and called the grand coalition. In (b), the structure is composed of three coalitions.

The core stability is a cooperative solution concept that is robust to strong group deviations (all the players in the group strictly prefer deviating). As already defined in chapter 3, no player wants to leave her coalition except if there is a strictly positive gain for this player. The existence of core stable structures is not guaranteed for all coalition games (multiagent systems) of the form $(\mathscr{N}, \mathscr{C}, v, D)$ and may exist in some states of the system but not others.

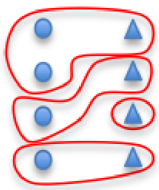

Figure 7.2: Example of a matching with players belonging to two disjoint sets: the set of circles and the set of triangles.

Observe that the cardinal nature of the coalition game (characteristic function, sharing rules and players' utilities) is not mandatory. The ordinal-based theory has preferences as primitives (see chapter 3). In this chapter, we focus on the cardinal theory where the players have utility functions defined over the agents' a set of payoffs.

In Figure 7.3 we show an example of a structure resulting from a coalition formation game with equal sharing as resource allocation scheme. In a coalition all the players receive the same payoff. The players have partitioned in three coalitions. Each player receives a slice of the worth of the coalition displayed as a circle. The size of the circle shows the worth of the coalition.

[1] By assumption, all the players are rational, thus utility and payoff maximizers (increasing utilities).

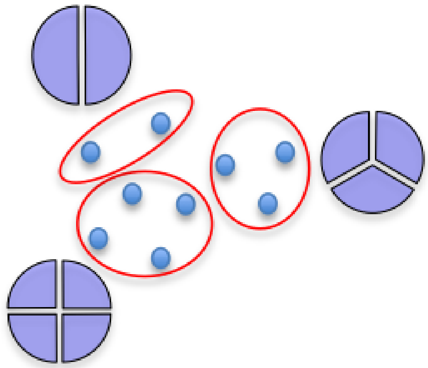

Figure 7.3: Example of a structure and corresponding shares to the players. The worth of the coalition is shown as a circle and each payoff by a slice.

7.4 Motivating Examples

We give three examples of game-theoretic multiagent systems where a resource is to be shared among the players forming groups.

7.4.1 Competition Over Visibility on Social Networks

Assume a set of content creators and social networks such that the expected payoff results from the individual posting strategies of the content creators over the social networks. It is surprisingly interesting to observe how such a scheme can result from a mechanism not originally designed for stability.

Figure 7.4 shows an example of matching between content creators and social networks. Content

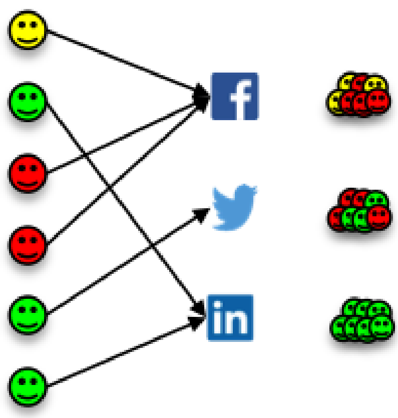

Figure 7.4: An example of a matching with peer effects between content creators and social networks.

creator i sends messages to only one social network according to a Poisson process of intensity λ_i. It is shown in [19] that the expected number of views of a content provider with respect to the social network k is a Tullock rent seeking allocation of the form:

$$\frac{\lambda_i}{\sum_{j\in\mu(k)}\lambda_k} n_k, \tag{7.1}$$

where $\mu(k)$ is the set of content creators posting on the social network k and n_k is the number of viewers of the social network k. In some cases, the number of users of a social network may not be constant and may depend on the content creators sending messages to it. In this case, n_k is a function of the coalition made of the content creators and the social network k and we can define $v(C) = n_k(C)$ where k is the social network in C. Using the previous result, one can show that this is a stability inducing sharing rule and that it always exists a core stable matching of the content creators to the social networks.

7.4.2 Example: D2D Connectivity Management

Consider a set of devices and a set of access points (APs). These two sets are disjoints. The devices and APs are players. Devices can connect to APs or connect with each others (for a D2D communication) to form connectivity groups. The set of players is naturally partitioned in two as the set of APs $\mathscr{F}$ and the set of devices $\mathscr{W}$. The family of coalitions (see chapter 3) $\mathscr{C}$ considered here is,

$$\mathscr{C} = \left\{\{f\} \cup \mathrm{S} : f \in \mathscr{F}, \mathrm{S} \subseteq \mathscr{W}, |\mathrm{S}| \leq \mathrm{M}_f\right\} \cup \{\mathrm{Q} \subset \mathscr{W}\}\} \tag{7.2}$$

By definition of $\mathscr{C}$, a coalition is either an AP and a set of devices (at most that total number of devices minus one) connected to it or a subset of devices forming a connectivity group (at most the total number of devices minus one). The characteristic function v of the coalition game may both take into account the productivity types[2] of the devices (i.e. their limitations in transmission, e.g. physical data rates) and the connectivity graph (a node is a device and an edge is a flow to be transmitted between the nodes). In this case, we have $v(\mathrm{C}, \mathrm{G}_\mathrm{C})$ the worth of the coalition C with connectivity graph G_C.

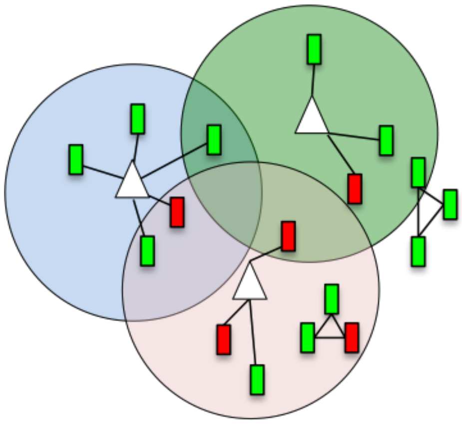

Figure 7.5: An example of a mixed decentralized connectivity setting. The devices (squared) can either connect to the access points (triangles) or connect with each others. All the nodes are players (devices or access points).

7.4.3 Example: Assignment of Tasks to Machines

Assume a set of finite-size queues $\mathscr{Q} = \{q_1, \ldots, q_\mathrm{Q}\}$. The size of the buffer of q_i is l_{q_i}. Each queue has a server and these servers may differ in their processing properties. Furthermore assume that each server can be controlled. We denote e_i the control parameter of the server of queue q_i. As examples, consider a processing power or an amount of energy. Such controls can be found in computer architecture (e.g. the Dynamic Voltage Scaling[3] (DVS) and the Dynamic Frequency Scaling[4] (DFS, or CPU throttling)) or cloud computing.
We assume an ideal fluid model such that the service rate is shared among the tasks according to a generalized α-fair allocation. We furthermore assume that the allocation satisfies the conditions (98) and (99) so that it exists a core stable matching of tasks to queues. We furthermore assume

[2]Observe that the productivity types may be encoded in the form of a weighted graph such that the players are the nodes and weighted edges between them give the productivity of the link.
[3]The DVS technique is a power management technique which consist in a dynamically changing the voltage of some components such as the CPU or others.
[4]The DFS technique is another technique which consists in dynamically changing the frequency of a CPU.

that the machines and the tasks are selfish players[5]. The tasks and the servers can match (group in coalitions) to maximize their individual payoffs. The set of coalitions they can form is,

$$\mathscr{C} = \left\{\{q\} \cup \mathrm{S} : q \in \mathscr{Q}, \mathrm{S} \subseteq \mathscr{W}, |\mathrm{S}| \leq l_q\right\} \cup \{\{w\} : w \in \mathscr{W}\} \tag{7.3}$$

Each coalition has a worth $v(\mathrm{C})$ that depends on the tasks, the server in the coalition and the control parameter. As examples, the worth may be the service rate of the server or a unit time interval. The dependance of the worth $v(\mathrm{C})$ on the players can be illustrated by assuming a server optimized for processing some kind of tasks (e.g. video) but being able to process any other tasks at a cost in the resource to be shared. In Figure 7.6, we show example of a structure of coalitions. The tasks and the servers are players. The servers have quotas and cannot accept more tasks than their quotas. The top server can only perform green tasks, the middle one green and yellow asks and the bottom one yellow and red tasks. By definition of the coalitions and sharing rules, in any

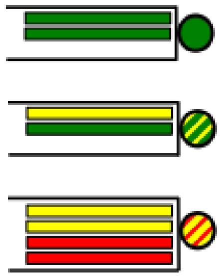

Figure 7.6: An example of a matching between machines (and their queues) and tasks. The top machine can only process green tasks, the middle one can process both green and yellow tasks, the bottom one can process red and yellow ones.

coalition the worth is shared among the set of players, including the machine. This means that each machine is assigned a positive payoff, as any task in its queue. This payoff can be interpreted in at least two ways. On the first hand, it may be the amount of worth the machine will spend in processing its own tasks. As a simple example, take the machine to be an operating system with maintenance routines such as memory cleaning (or any other). On the other hand, it may be the worth or power the machine saves up in an idle mode. Consider an example from computer architecture. If the payoffs are measured in time, then it is the amount of time the machine spends in an idle mode, thus saving the energy required for processing. If the payoffs are measured in clock rate (or equivalently, percentage of the CPU's clock rate), then the CPU's payoff is the amount of clock rate it does not use to process instructions. Whatever the one or the other metric, this clock cycles saving can be turned into a reduction of the dynamic consumption of the circuit by switching off the clock signal of some part of the circuit (unused active devices such as flip-flops switching states at clock signal) when needed. This technique is known in the name of *clock gating*. Some famous example of application of the clock gating is the IBM PowerPC-based Xenon used in the Microsoft Xbox 360 or the more recent Xbox One. Because it is not the purpose of the chapter, we do not enter into more details the implementation of the clock gating in a system or the clock gating policies. It is sufficient here to observe that the share of a machine may be interpreted as the target of the clock gating system control policy when the machine is a CPU. Furthermore observe that such a service discipline is not work conserving[6] if we assume that the machine uses its payoff in an idle form.

7.5 The Generalized α-fair Allocation: Stability, Risks and Behaviors

Consider a coalition game $(\mathscr{N}, \mathscr{C}, v, \mathrm{D})$ such that the allocation scheme D is the generalized α-fair scheme in every coalition. In this section, we focus on the strict log-concavity condition of

[5]Equivalently, we assume that players control machines and tasks. Each player controlling a single-task or a single-machine.

[6]The work conservation is the fact that a machine must be busy as long as there are packets in the system.

Corollary 63 (see Chapter 3, Section 3.5) and on the utility functions used in the generalized α-fair to guarantee the existence of core stable structures. The conditions are given in terms of an indicator of behavior under uncertainty, the Pure Fear of Ruin (PFoR) (defined by Aumann and Kurz in [10]). To the best of our knowledge this is the first connection between the generalized α-fair allocation, the stability inducing sharing rules and the game theoretic notions of fear of ruin and pure fear-of-ruin. In [15], Altman et a.l. defined the generalized α-fairness as resulting from the following optimization problem:

$$\max_{x\in X} \nu(x) = \begin{cases} \frac{1}{1-\alpha}\sum_{i=1}^{n} \pi_i (f_i(x_i))^{1-\alpha}, \alpha \neq 1 \\ \sum_{i=1}^{n} \pi_i \log(f_i(x_i)) \qquad , \alpha = 1 \end{cases}$$

where $\alpha \in [0,+\infty]$, f_i are increasing and concave functions valued in $[0,+\infty)$ and $0 \le \pi_i \le 1$. This maximization problem can equivalently by formulated in the following product form:

$$\max_{x\in X} \tilde{\nu}(x) = \begin{cases} \prod_{i=1}^{n} u_i(x_i) = \prod_{i=1}^{n} e^{\frac{\pi_i}{1-\alpha} f_i(x_i)^{1-\alpha}}, \alpha \neq 1 \\ \prod_{i=1}^{n} u_i(x_i) = \prod_{i=1}^{n} f_i(x_i)^{\pi_i} \qquad , \alpha = 1 \end{cases}$$

We call this problem the equivalent product form (or bargaining form) of the generalized α-fair optimization problem. For the rest of this chapter, we relax the concavity assumption on the utility functions f_i, $i \in \{1,\dots,n\}$ and seek for the constraints on these functions induced by the concavity conditions for the existence of core stable structures. Before showing the results, we need to define the fear of ruin and the pure fear of ruin of a player i with utility function f_i.

Definition 94 (Fear of Ruin, [10]). *The fear of ruin of player i with utility function f_i at allocation x is defined as:*

$$\text{FoR}_{f_i}(x) = \frac{f_i(x)}{f_i'(x)} \tag{7.4}$$

Remark 95. *The risk aversion, boldness, fear of ruin and pure fear of ruin are functions of the utility function of the player. It is important to be precise on the considered utility function, particularly in this chapter where any player has two utility functions: the utility f_i in the generalized α-fair program or the utility u_i in the equivalent product form.*

Definition 96 (Pure Fear of Ruin, [10]). *The pure fear of ruin of player i with utility function f_i at allocation x is defined as:*

$$\text{PFoR}_{f_i}(x) = -\frac{f_i"(x) f_i(x)}{(f_i'(x))^2} \tag{7.5}$$

Particularly, observe that $\text{FoR}'_{f_i}(x) = 1 + \text{PFoR}_{f_i}(x)$ and thus:

$$\text{FoR}_{f_i}(x) = \int_0^x \text{PFoR}_{f_i}(s)ds + \text{FoR}_{f_i}(0) + x \tag{7.6}$$

In Section 7.6, we give more results, details and interpretations of these quantities. In Section 7.6.4, we give a new interpretation on the fear of ruin that can be used to interpret the following results. When it is clear from the context which utility function is used, we denote PFoR_i the pure fear of ruin function of player i. Otherwise we use the notation in Definition 96.

We have the following results.

Proposition 97. *The utility function u_i of any player $i \in \mathcal{N}$ is strictly log-concave in the equivalent product form of the generalized α-fair optimization problem if:*

$$\text{PFoR}_{f_i} > -\alpha \tag{7.7}$$

Proof. For $\alpha \neq 1$. We have the second order derivative of $\frac{\pi_i}{1-\alpha} f_i(x_i)^{1-\alpha}$ such that:

$$\frac{\partial^2}{\partial x_i^2}\left(\frac{\pi_i}{1-\alpha} f_i(x_i)^{1-\alpha}\right) = \pi_i\left[\frac{\partial^2 f_i}{\partial x_i^2} f_i(x_i)^{-\alpha} - \alpha\left(\frac{\partial f_i}{\partial x_i}\right)^2 f_i(x_i)^{-\alpha-1}\right] \tag{7.8}$$

For $\frac{\pi_i}{1-\alpha} f_i(x_i)^{1-\alpha}$ to be strictly concave, the second-order derivative must be strictly negative. This gives:

$$\frac{\partial^2 f_i}{\partial x_i^2} f_i(x_i)^{-\alpha} < \alpha\left(\frac{\partial f_i}{\partial x_i}\right)^2 f_i(x_i)^{-\alpha-1} \tag{7.9}$$

multiplying both sides by $f_i(x_i)^{\alpha+1}$ gives the following second-order nonlinear ordinary differential inequation:

$$\frac{\partial^2 f_i}{\partial x_i^2} f_i(x_i) < \alpha\left(\frac{\partial f_i}{\partial x_i}\right)^2 \tag{7.10}$$

which gives:

$$\frac{\frac{\partial^2 f_i}{\partial x_i^2} f_i(x_i)}{\left(\frac{\partial f_i}{\partial x_i}\right)^2} < \alpha \tag{7.11}$$

By definition of the pure fear-or-ruin, multiplying both sides by minus one gives the result:

$$-\frac{\frac{\partial^2 f_i}{\partial x_i^2} f_i(x_i)}{\left(\frac{\partial f_i}{\partial x_i}\right)^2} > -\alpha \tag{7.12}$$

The case $\alpha = 1$ is obtained the same way. □

Proposition 98. *The individual utility function u_i of any player $i \in \mathcal{N}$ is increasing in the equivalent product form of the generalized α-fair optimization problem for any α.*

Proof. We have,

$$\frac{\partial u_i}{\partial x_i} = \frac{\partial}{\partial x_i}\left(e^{\frac{\pi_i}{1-\alpha} f_i(x_i)^{1-\alpha}}\right) = \pi_i \frac{\partial f_i}{\partial x_i} f_i(x_i)^{-\alpha} e^{\frac{\pi_i}{1-\alpha} f(x_i)^{1-\alpha}} \tag{7.13}$$

By positivity of the exponential, of f_i and the fact that f_i is increasing, we have that the first order derivative of u_i is always positive. □

Proposition 99. *The individual utility function u_i of any player $i \in \mathcal{N}$ is concave in the bargaining problem equivalent to the generalized α-fair optimization problem if:*

$$\mathrm{PFoR}_{f_i}(x_i) \geq \pi_i f_i(x_i)^{1-\alpha} - \alpha \tag{7.14}$$

Proof. For $\alpha \neq 1$. In the equivalent product form of the generalized α-fair allocation, we have shown that the individual utilities must be of the form $u_i(x_i) = e^{\frac{\pi_i}{1-\alpha} f_i(x_i)^{1-\alpha}}$. We have:

$$\frac{\partial^2 u_i}{\partial x_i^2} = \pi_i\left[\frac{\partial^2 f_i}{\partial x_i^2} f_i(x_i)^{-\alpha} - \alpha\left(\frac{\partial f_i}{\partial x_i}\right)^2 f_i(x_i)^{-\alpha-1} + \pi_i \frac{\partial^2 f_i}{\partial x_i^2} f_i(x_i)^{-2\alpha}\right] u_i(x_i) \tag{7.15}$$

For u_i to be concave, we must have $\frac{\partial^2 u_i}{\partial x_i^2} \leq 0$. We thus have:

$$\frac{\partial^2 f_i}{\partial x_i^2} f_i(x_i)^{-\alpha} - \alpha\left(\frac{\partial f_i}{\partial x_i}\right)^2 f_i(x_i)^{-\alpha-1} + \pi_i \frac{\partial^2 f_i}{\partial x_i^2} f_i(x_i)^{-2\alpha} \leq 0 \tag{7.16}$$

Multiplying on both sides by $f_i(x_i)^{1+\alpha}$, we obtain:

$$\frac{\partial^2 f_i}{\partial x_i^2} f_i(x_i) + \left(\frac{\partial f_i}{\partial x_i}\right)^2 (\pi_i f_i(x_i)^{1-\alpha} - \alpha) \leq 0 \tag{7.17}$$

which can be written on the following form:

$$\frac{\frac{\partial^2 f_i}{\partial x_i^2} f_i(x_i)}{(\frac{\partial f_i}{\partial x_i})^2} \leq \alpha - \pi_i f_i(x_i)^{1-\alpha} \tag{7.18}$$

we thus obtain the result:

$$-\frac{\frac{\partial^2 f_i}{\partial x_i^2} f_i(x_i)}{(\frac{\partial f_i}{\partial x_i})^2} \geq \pi_i f_i(x_i)^{1-\alpha} - \alpha \tag{7.19}$$

The case $\alpha = 1$ is obtained the same way. □

We now give two examples of the constraint induced by Proposition 97 on the design of a multiagents system with generalized α-fair allocation scheme.

Example 100. *For any player i in the set of players $\mathcal{N}$, take $f_i(x) = \ln(1 + k_i x)$, $k_i \in \mathbb{R}^{+*}$. We have:*

$$\text{PF}o\text{R}_i = \ln(1 + k_i x) \tag{7.20}$$

Proposition 97 requires $\text{PF}o\text{R}_i > -\alpha$. We conclude that in a system with logarithmic utility functions of the form $f_i(x) = \ln(1 + k_i x)$, $k_i \in \mathbb{R}^{+*}$, for any player i in $\mathcal{N}$, condition 97 is satisfied if α is strictly positive.

Example 101. *For any player i in the set of players $\mathcal{N}$, take $f_i(x) = k_i x$, $k_i \in \mathbb{R}^{+*}$. We have:*

$$\text{PF}o\text{R}_i = 0 \tag{7.21}$$

Condition 97 is satisfied if α is strictly positive.

7.6 Risks and Measures of Aversion

The aim of this section is to give an interpretation of the results of Section 7.5. These results have shown that some conditions for existence of core stable structures in multiagent systems with generalized α-fair allocation of the resource can be formulated as constraint on the pure fear of ruin. In the game-theoretic literature, the pure fear of ruin is defined as one of the measure of risk aversion of a player when facing uncertainty. These measures quantify the (un)willingness of the player in entering a situation where there is a positive probability to loose some resource. In this section, we define the risk, its related concepts and the measures of risk aversion, boldness, fear-of-ruin and pure fear-of-ruin. We provide a new interpretation of the fear of ruin.

7.6.1 Risks and Gambles

Assume a player i with resource x (also called risk-free resource, measured in units of incomes[7]) and its increasing utility function u_i.

Definition 102. *The risk is a real-value random variable $\tilde{z}$ with probability distribution $\mathbb{P}$. The expected value of the risk $\mathbb{E}[\tilde{z}]$ is called the actuarial value. If it is is null the risk is said to be actuarially neutral (or fair). When receiving a risk $\tilde{z}$ in resource, the original resource x of the player is turned into random variable, $x + \tilde{z}$. If the support of the risk is a closed interval of the form $[z_{\min}, z_{\max}]$, the minimum realization gives resource $x + z_{\min}$ and the maximum realization gives resource $x + z_{\max}$.*

Two risky situations are of particular importance: the even money gamble and the asymmetric ruin gamble.

[7] e.g. dollars, bandwidth, throughput, number of views.

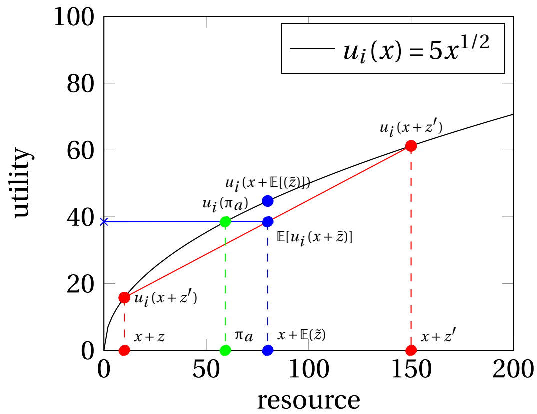

Figure 7.7: An example of risk aversion in an actuarially neutral ($\mathbb{E}(\tilde{z}) = 0$) even-money gamble where the (random) risk $\tilde{z}$ is valued in $\{+70, -70\}$. The fortune x is 80 units of resource.

Definition 103. *A player entering an even money gamble with amount x has a probability $\mathbb{P}(\tilde{z} = +h) = 1 - p$ to win an amount $+h$ and a probability $\mathbb{P}(\tilde{z} = -h) = p$ to loose an amount $-h$.*

Definition 104. *A player entering an asymmetric gamble with amount x has a probability $\mathbb{P}(\tilde{z} = +h) = 1 - q$ to win an amount $+h$ and a probability $\mathbb{P}(\tilde{z} = -x) = q$ to loose x.*

In figure 7.7, we show an example of an actuarially neutral even-money bet. There is probability p for the player to win z and $1 - p$ for her to win z'. Here $x = 80$, $z = +70$, $z' = -70$ and $p = 0.5$.

The expected resource is $\mathbb{E}[x + \tilde{z}] = x + \mathbb{E}[\tilde{z}]$ (since x is the original deterministic resource). By definition of the utility function which maps an absolute worth in an subjective one, the player perceives any resource x as $u_i(x)$. Thus, the actuarial value $\mathbb{E}[\tilde{z}]$ gives the player a utility $u(\mathbb{E}[x + \tilde{z}]) = u(x + \mathbb{E}[\tilde{z}])$ of the expected resource. The expected utility of the risky resource is given by $\mathbb{E}[u(x + \tilde{z})]$. The expected utility is the one to be considered by the player when receiving the risk. Observe that if u_i is convex $\mathbb{E}[u(x + \tilde{z})] \geq u(\mathbb{E}[x + \tilde{z}])$ and if it is concave $\mathbb{E}[u(x + \tilde{z})] \leq u(\mathbb{E}[x + \tilde{z}])$.

Definition 105 ([8])**.** *The cash equivalent (also called certainty equivalent or value of the risk), denoted $\pi_a(x, \tilde{z})$ is defined as the amount of resource to be given to the player to make her indifferent between receiving the risk $\tilde{z}$ or receiving $x + \pi_a(x, \tilde{z})$ without risk (i.e. with certainty), i.e.,*

$$u_i(x + \pi_a(x, \tilde{z})) = \mathbb{E}[u_i(x + \tilde{z})] \tag{7.22}$$

Definition 106 ([8])**.** *The insurance premium $\pi_I(x, \tilde{z})$ is defined as the opposite of the cash equivalent.*

As explained in [8], $\pi_a(x, \tilde{z})$ can be interpreted as the smallest amount (in units of resource) for which the decision maker would willingly sell the expected allocation if he had it. The most natural interpretation of the insurance premium can be given when considering unfavorable risks. In this context, the insurance premium is the maximum amount of resource that the player is willing to pay for a third party (called insurance) to lever the risk and guarantee her $x - \pi_I(x, \tilde{z})$ with certainty.

In real-life applications, this does not necessarily mean that the risk is actually transferred to a third party and that the player is free of risk. Usually, the third party restores the resource to the amount x (for an example, see Example 107). The insurance premium is equal to the cash equivalent in absolute value. Remark that if the cash equivalent is negative and thus is a loss, then the insurance premium is a positive amount to pay.

Example 107. *Assume that player i owns a housed valued $x = 10$ units and assume that there is a material risk of damage (fire, flood, etc.) measured in units of resource (e.g. a fire can decrease the worth by 8 units). One can consider that the material risk is mapped to a risk in resource. Assume that the player pays an insurance premium $\pi_I = 2$ to the insurance (third party). Naturally, the* *damage can still occur: the material risk still exists. In case it occurs, the insurance restores the house (or pays back the player for her to restore it) at the original worth* 10. *In the end, from the point of view of the player, there is no risk in resource. This is the reason why it is said that the risk is transferred to the insurance and that the player receives* $x - \pi_I = 8$ *with certainty.*

Definition 108 ([8])**.** *The risk premium is defined as the difference between the actuarial value and the value of the risk:*

$$\pi(x, \tilde{z}) = \mathbb{E}[\tilde{z}] - \pi_a(x, \tilde{z}) \tag{7.23}$$

It gives the amount of resource to be added to (or picked from, depending on the sign of the difference) the actuarial value $\mathbb{E}[\tilde{z}]$ *to obtain the cash equivalent* $\pi_a(x, \tilde{z})$.

In other words, it is the gain (or loss) in units of resource from accepting the cash equivalent and not receiving the risks (hence the name risk premium). The value of the risk can be written:

$$\pi_a(x, \tilde{z}) = \mathbb{E}[\tilde{z}] - \pi(x, \tilde{z}) \tag{7.24}$$

Example 109. *Assume that a decision maker i owns a house that is assessed* $x = 10$ *units of resource. The utility function* u_i *of the decision maker is given by* $u_i(x) = x^{\frac{1}{2}}$. *Assume the risk* $\tilde{z}$ *valued in* $\{-5, 0\}$ *such that* $P(\tilde{z} = -5) = 0.01$ *and* $P(\tilde{z} = 0) = 0.99$. *The expected utility of receiving the risk is* $\mathbb{E}(u(x + \tilde{z})) = 0.01 * 5^{1/2} + 0.99 * 10^{1/2} = 3.15$. *The cash equivalent of the risk is obtained by looking for the amount of resource y such that,*

$$u(x + y) = \mathbb{E}(u(x + \tilde{z})) = 3.15 \tag{7.25}$$

We obtain,

$$(10 + y)^{1/2} = 3.15 \tag{7.26}$$

which gives $y = -0.0775$. *The player negatively values the risk. By definition, he is willing to pay a maximum of* 0.0775 *units of resource to the insurance to not take the risk.*

7.6.2 Risk Aversion

One of the most important measure of the decision-taking analysis in a risky settings is called the *Absolute Risk Aversion* (ARA) or absolute measure of risk aversion. It is used in game theory and economy (e.g. finance, insurance).

Definition 110 ([8])**.** *The absolute risk aversion r is a local measure that quantifies the attitude of this player toward risk and can be interpreted as a measure of the curvature of the utility function of the player.*

$$r(x) = -\frac{u_i''}{u_i'}(x) \tag{7.27}$$

Assume a risk $\tilde{z}$ of mean $\mathbb{E}[\tilde{z}]$ and small variance [8], $\sigma_{\tilde{z}}^2 \to 0$. We have (see [8] for the details) the second order approximation of the risk premium $\pi(x, \tilde{z})$:

$$\pi(x, \tilde{z}) = \frac{1}{2}\sigma_z^2 r(x + \mathbb{E}[\tilde{z}]) + o(\sigma_z^2) \tag{7.28}$$

If the risk aversion increases then the risk premium $\pi(x, \tilde{z})$ and the insurance premium $\pi_I(x, \tilde{z})$ increase. The player wants to pay more to secure her resource and is said to be more risk averse.

If $u''(x)$ is negative then $r(x)$, the risk premium $\pi(x, \tilde{z})$ and the insurance premium $\pi_I(x, \tilde{z})$ are positive and the player is willing to pay $\pi_I(x, \tilde{z})$ not to receive the risk but $x - \pi_I(x, \tilde{z})$ with certainty. The player is risk averse. The higher the absolute risk aversion, the higher he is willing to pay. If the risk aversion increases the player is more risk averse, if it decreases the player is less risk averse. In

[8] The risk is a random variable.

other words, a player is risk averse when her marginal utility of the resource is decreasing. If the utility is linear the player is said to be risk neutral and if it is strictly convex he is said to be risk lover.

The risk aversion can also be shown to be an approximation in the small loss regime of the bias to be introduced in the probability distribution of an even money gamble for the player to be indifferent between entering the gamble or not. This interpretation has initially been proposed by Pratt in [8] and is the starting point of Aumann and Kurz in [10] who consider an asymmetric gamble with risk of ruin to define the fear of ruin. We will use it later in the chapter to interpret some of our results.

Definition 111 ([8])**.** *The probability premium is defined as the difference $p(x,\tilde{z}) = \mathbb{P}(\tilde{z} = +h) - \mathbb{P}(\tilde{z} = -h)$ between the probability of the events of the gamble that makes the player indifferent between the initial resource x and receiving the risk $\tilde{z}$:*

$$u_i(x) = \frac{1}{2}\left[1 + p(x,\tilde{z})\right] u_i(x+h) + \frac{1}{2}[1 - p(x,\tilde{z})] u_i(x-h) \tag{7.29}$$

The probability premium can be understood as the bias to be added in the probability distribution in favor of the winning event for the player to be indifferent between entering the gamble or not. In the limit case of infinitely small risks ($h \to 0$), we obtain the following second-order development:

$$p(x,\tilde{z}) = \frac{h}{2} r(x) + \mathrm{O}(h^2) \tag{7.30}$$

The risk aversion is thus defined as twice the difference per unit risked between the probability distribution of the actuarially neutral risk ($\mathbb{P}(\tilde{z} = +h) = \mathbb{P}(\tilde{z} = h) = \frac{1}{2}$) and the probability distribution (of the risk) that makes the player indifferent in expectation between receiving x resource and the risk $\tilde{z}$. In the small loss regime, if $r(x)$ increases, then the player requires a higher probability to win to be indifferent.

Remark 112. *The absolute risk aversion is invariant under positive and linear affine transformations of the utility. As a consequence, it is constant over any set of fully equivalent utility functions*[9]*.*

7.6.3 Boldness, Fear of Ruin and Pure Fear of Ruin

Consider the asymmetric ruin gamble. For the player to be indifferent between not entering the bet and risking her allocation x, the following must be satisfied:

$$u_i(x) = (1-q) u_i(x+h) + q u_i(0) = (1-q) u_i(x+h) \tag{7.31}$$

where $u_i(0) = 0$. If equality (7.31) holds at higher q then the player accepts a higher risk of ruin. If it holds at lower q then he accepts a lower risk of ruin. The probability q is thus a measure of boldness.

Definition 113 ([10])**.** *The boldness b_i of player i is the limit for small bets regime ($h \to 0$) of the probability of ruin per unit of bets:*

$$b_i(x) = \frac{u_i'}{u_i}(x) = \lim_{h\to 0} \frac{q}{h} = \frac{(u_i(x+h) - u_i(x))/h}{u_i(x+h)} \tag{7.32}$$

The fear of ruin is the inverse of the boldness:

$$\mathrm{FoR}_i(x) = \frac{u_i(x)}{u_i'(x)} \tag{7.33}$$

[9] Equivalent utility functions induce the same preferences over strategy profiles or outcomes of a game. See [5] and the utility theory.

In [14], Foncel and Treich have shown that when the probability of ruin q is small, the insurance premium $\pi_I(x,\tilde{z})$ can be approximated by:

$$\pi_I(x,\tilde{z}) \sim q\frac{u_i(x)-u_i(0)}{u^{'}(x)} \tag{7.34}$$

which shows that if u_i is an increasing function and has a negative second-order derivative (risk-averse player) then the more resource x the player has, the higher is the amount he is willing to pay to secure her resource. In fact, as observed in [14], there are two reasons. First, there is more to loose ($u_i(x') > u_i(x)$ for $x' > x$). Second her marginal utility for the resource is lower (decreasing first-order derivative, i.e. the richer the player the lower the utility of a unit of resource), i.e. he perceives the resource as having less worth in utility as before so he can spend more in insurance. The fear of ruin captures the risk aversion towards large risks. It has received a particular interest in the value of life literature in the purpose of the study of willingness to pay for small reductions in mortality risks (the ruin of the player is her death of utility $u_i(0)$), see [14] and references therein for more details.

The definition of the pure fear of ruin of a player was given in Definition 96. It is the product of the fear or ruin and the risk aversion (the utility function of the player is u_i):

$$\mathrm{PF}o\mathrm{R}_i(x) = \mathrm{FoR}_i(x)\times r_i(x) = -\frac{u_i"(x)u_i(x)}{(u_i'(x))^2} \tag{7.35}$$

7.6.4 Result

We now show the main result of this section.

Proposition 114. *Assume a player i with utility function u_i and resource x.*

$$\mathrm{FoR}_i(x) = \lim_{n\to+\infty} x\left(1+2\sum_{k=1}^{n}\frac{1}{n}\frac{p_{k,n}}{q_{k,n}}\right)+\mathrm{FoR}_i(0) \tag{7.36}$$

where $p_{k,n}$ is the probability premium in the even money gamble $\{-g(x,k,n),+g(x,k,n)\}$ at $k\frac{x}{n}$, $q_{k,n}$ is the probability of ruin that makes the player indifferent in the ruin gamble with winning amount $+g(x,k,n)$ at $k\frac{x}{n}$ and $g:\mathbb{R}^+\times\mathbb{N}\times\mathbb{N}\to\mathbb{R}^+$ is any function such that, $\forall n\in\mathbb{N}^*, \forall k\in\{1,\ldots,n\}$:

$$g(x,n,k) \le (k-1)\frac{x}{n} \text{ and } \lim_{n\to+\infty} g(x,n,k)=0 \tag{7.37}$$

The proof of this proposition is given in Appendix 7.9.

This result can be interpreted by considering a sequence of even money gamble on the path of ruin from x to 0.

Assume an amount x of resource and a sequence of n even money gambles at fixed resource level x_k. Gamble k, $k\in\{1,\ldots,n\}$ is at resource allocation $x_k=(k-1)\frac{x}{n}$ with $\tilde{z}=-g(x,n,k)$ and $\tilde{z}=+g(x,n,k)$. Furthermore assume the function g such that $g(x,n,k)=+\frac{x}{n}$. When winning gamble k, the player's enters the next gamble $k+1$. Starting at x, loosing all the gambles from $k=n$ to $k=1$ leads the player to the gamble at 0. Consider gamble k. The probability premium (that makes the decision maker indifferent) is denoted $p_{k,n}$ (see equation (7.29) for the definition of this premium).

The expected resource in the actuarially fair even money gamble at x_k is $\mathbb{E}[x_k+\tilde{z}]=x_k$. In the biased one that makes the player indifferent, we have $\mathbb{E}[x_k+\tilde{z}]=x_k+p_{k,n}g(x,n,k)$. The difference between these two expected amounts is $\Delta\mathbb{E}_{k,n}=p_{k,n}g(x,n,k)$. We call this amount the *money equivalent of the probability premium*. It is an equivalent expected amount of resources to be introduced in the gamble to make the player indifferent between staying out or taking on the gamble. This amount is not given to the player but is introduced in the form of a probability premium or bias in probability. It is a constraint imposed by the player on the probability distribution of the events of the gamble for her to be indifferent in entering the gamble and not. This may be understood as a requirement or a cost emitted by the player to a third party controlling the probability distribution. We obtain that $p_{k,n} = \frac{\Delta\mathbb{E}_{k,n}}{g(x,n,k)}$ is the expected amount to be invested in the gamble per unit risked (by the player) to make her indifferent between staying out or taking on the gamble.

The fear of ruin can be written as:

$$\mathrm{FoR}_i(x) = \lim_{n\to+\infty} x\left(1+2\sum_{k=1}^{n}\frac{1}{n}\frac{1}{q_{k,n}}\frac{\Delta\mathbb{E}_{k,n}}{g(x,n,k)}\right) + \mathrm{FoR}_i(0) \tag{7.38}$$

By assumption, $g(x,n,k) = \frac{x}{n}$ which gives:

$$\mathrm{FoR}_i(x) = \lim_{n\to+\infty} x + 2\sum_{k=1}^{n}\frac{1}{q_{k,n}}\Delta\mathbb{E}_{k,n} + \mathrm{FoR}_i(0) \tag{7.39}$$

By definition of $q_{k,n}$, we have:

$$\mathrm{FoR}_i(x) = \lim_{n\to+\infty} x + 2\sum_{k=1}^{n}\frac{u_i((k-1)\frac{x}{n}+g(x,n,k))}{u_i((k-1)\frac{x}{n}+g(x,n,k)) - u_i((k-1)\frac{x}{n})} \times \Delta\mathbb{E}_{k,n} + \mathrm{FoR}_i(0) \tag{7.40}$$

Observe that the terms in the sum in (7.40) can be written in the following form:

$$100 \times \frac{\text{expected investment in resources}}{\frac{\text{loss in utility}}{\text{total utility}} \times 100} \tag{7.41}$$

where the expected investment is $\Delta\mathbb{E}_{k,n}$, the total utility is $u_i((k-1)\frac{x}{n}+g(x,n,k))$ and the loss is $u_i((k-1)\frac{x}{n}+g(x,n,k)) - u_i((k-1)\frac{x}{n})$.
The denominator ($\frac{\text{loss in utility}}{\text{total utility}} \times 100$) in (7.41) is the rate of loss in total utility (or owned resources). It gives the fraction (in percentage) of the total utility (in utility) $u_i((k-1)\frac{x}{n}+g(x,n,k)$ that the loss $g(x,n,k) = \frac{x}{n}$ represents. The numerator in (7.41) is the equivalent expected amount of money to be introduced in the gamble to make the player indifferent at $(k-1)\frac{x}{n}$. We interpret it as the minimum cost for the player to enter the gamble and take the risk to loose again the loss or get it back.
Thus, the ratio,

$$\tau_{k,n} = \frac{\text{expected investment in resources}}{\frac{\text{loss in utility}}{\text{total utility}} \times 100} \tag{7.42}$$

is the equivalent expected amount of resource per percentage of loss in total utility to be introduced in the gamble for the player to be indifferent between staying out of the gamble and entering in the gamble and take the risk to take back her loss or loose it again.

We conclude that the fear of ruin of player i at x,

$$\mathrm{FoR}_i(x) = \lim_{n\to+\infty} x + 200\sum_{k=1}^{n}\frac{\Delta\mathbb{E}_{k,n}}{\tau_{k,n}} + \mathrm{FoR}_i(0) \tag{7.43}$$

measures the cumulated equivalent expected investment per percentage of loss in total utility along the pass from x to the ruin. According to this interpretation, at resource allocation x, the player knows the cumulated cost to give her the opportunity to get back the lost resource x. The limit of this cost is the fear of ruin. The higher this cumulated cost, the higher the fear of ruin.

7.7 Conclusion

In this chapter we have analyzed some of the conditions required for the generalized α-fair allocation of a resource to fall in the class of game-theoretic (core) stability inducing sharing rules under certain conditions on the utility functions of the players. Particularly, we have focused on the strict log-concavity of the utility function of any player in the equivalent product form of the generalized α-fair program. We have given these conditions in terms of pure fear of ruin, a game-theoretic measure of the non-willingness of a player in entering a risky situation where there is a non-zero probability for ruin. To the best of our knowledge, this is the first use of this measure since Aumann and Kurz's [10]. This chapter shows some new constraints on the use of fairness schemes in the design of multiagents system in networks. Possible future works include the analysis of the other conditions required for the guaranteed existence of core stables structures.One may also go further in the understanding of applications of the pure fear of ruin in networks and in the analysis of the risk in matching games and the coalition formation problem.

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$$\tilde{S}_n = -2\frac{x}{n}\sum_{k=1}^{n}\frac{u_i((k-1)\frac{x}{n}+g(x,n,k))}{u_i((k-1)\frac{x}{n}+g(x,n,k))-u_i((k-1)\frac{x}{n})} \times \frac{u_i((k-1)\frac{x}{n}+g(x,n,k))-2u((k-1)\frac{x}{n})+u((k-1)\frac{x}{n}-g(x,n,k))}{u_i((k-1)\frac{x}{n}+g(x,n,k))-u_i((k-1)\frac{x}{n}-g(x,n,k))} \quad (7.54)$$

7.9 Appendix: Proof of Proposition 114

Proof. Taking the first derivative of the fear of ruin, we have:

$$\frac{d\mathrm{FoR}_i}{dx} = \frac{d}{dx}\left(\frac{u_i}{u_i^{'}}\right) = \frac{(u_i^{'})^2 - u_i u_i^{''}}{(u_i^{'})^2} \quad (7.44)$$

$$= 1 + \mathrm{FoR}_i(x) r_i(x) \quad (7.45)$$

$$= 1 + \mathrm{PFoR}_i(x) \quad (7.46)$$

where:

$$\mathrm{PFoR}_i(x) = \mathrm{FoR}_i(x) r_i(x) = -\frac{u_i u_i^{''}}{(u_i^{'})^2} \quad (7.47)$$

is called the pure fear of ruin of player i. We thus have the following:

$$\mathrm{FoR}_i(x) - \mathrm{FoR}_i(0) = \int_0^x 1 + \mathrm{FoR}_i(s) \times r_i(s) ds \quad (7.48)$$

$$= x + \int_0^x \mathrm{FoR}_i(s) \times r_i(s) ds \quad (7.49)$$

$$= x + \int_0^x \mathrm{PFoR}_i(s) ds \quad (7.50)$$

which gives:

$$\mathrm{FoR}_i(x) = x + \int_0^x \mathrm{PFoR}_i(s) ds + \mathrm{FoR}_i(0) \quad (7.51)$$

We show that the fear of ruin FoR_i at x can be approximated as x plus the product of x and the limit of a sequence of expected pure fear of ruins.

By Riemann integrability of function PFoR_i, we can construct a Riemann sum S_n that converges to $\int_0^x \mathrm{PFoR}_i(s)ds$. As an example, we have the left Riemann sum[10]:

$$S_n = \frac{x}{n}\sum_{k=1}^{n} \mathrm{PFoR}_i((k-1)\frac{x}{n}) \quad (7.52)$$

such that:

$$\lim_{n\to+\infty} S_n = \int_0^x \mathrm{PFoR}_i(s) ds \quad (7.53)$$

For the rest of this section, we denote $I(x)$ the integral $\int_0^x \mathrm{PFoR}_i(s)ds$. Consider the sum $\tilde{S}_n$ given in equation (7.54) where $g : \mathbb{R}^+ \times \mathbb{N} \times \mathbb{N} \to \mathbb{R}^+$ is any function such that, $\forall n \in \mathbb{N}^*, \forall k \in \{1,\dots,n\}$,

$$g(x,n,k) \le (k-1)\frac{x}{n} \text{ and } \lim_{n\to+\infty} g(x,n,k) = 0 \quad (7.55)$$

We show that one can use $\tilde{S}_n$ as an approximation of S_n in (7.53), i.e.,

$$\lim_{n\to+\infty} |\tilde{S}_n - \int_0^x \mathrm{PFoR}_i(s) ds| \to 0 \quad (7.56)$$

[10] Using $\lim_{n\to+\infty} S_n = \lim_{n\to+\infty} \frac{\beta-\alpha}{n}\sum_{k=1}^{n} f(\alpha + (k-1)\frac{\beta-\alpha}{n}) = \int_\alpha^\beta f(s)ds$

Using Taylor's formula,

$$f(x+h)=\sum_{k=0}^{n}\frac{f^{(k)}(x)}{k!}h^k+\mathrm{R}_n(x) \tag{7.57}$$

where $f^{(k)}(x)$ is the k-th derivative of f at x and $\mathrm{R}_n(x)$ is the $o(h^n)$ remainder term.

One can write,

$$u_i((k-1)\frac{x}{n}+g(x,n,k))=\sum_{k=0}^{n}\frac{u_i^{(k)}((k-1)\frac{x}{n})}{k!}(g(x,n,k))^k+\mathrm{R}_n((k-1)\frac{x}{n}) \tag{7.58}$$

and,

$$u_i((k-1)\frac{x}{n}-g(x,n,k))=\sum_{k=0}^{n}\frac{u_i^{(k)}((k-1)\frac{x}{n})}{k!}(-g(x,n,k))^k+\mathrm{Q}_n((k-1)\frac{x}{n}) \tag{7.59}$$

Introducing this in $\tilde{\mathrm{S}}_n$, we obtain:

$$\tilde{\mathrm{S}}_n=-2\frac{x}{n}\sum_{k=1}^{n}\frac{u_i((k-1)\frac{x}{n}+g(x,n,k))}{u_i'((k-1)\frac{x}{n})+\frac{\mathrm{R}_1((k-1)\frac{x}{n})}{g(x,n,k)}}\times\frac{u_i"((k-1)\frac{x}{n})+\frac{\mathrm{R}_2((k-1)\frac{x}{n})+\mathrm{Q}_2((k-1)\frac{x}{n})}{g(x,n,k)^2}}{2u_i'((k-1)\frac{x}{n})+\frac{\mathrm{R}_1((k-1)\frac{x}{n})-\mathrm{Q}_1((k-1)\frac{x}{n})}{g(x,n,k)}} \tag{7.60}$$

Since S_n converges to $\mathrm{I}(x)$ as n goes to infinity, by definition we have:

$$\forall\epsilon\in\mathbb{R}^{+*},\exists\mathrm{N}\,\mathrm{s.t.}\forall n>\mathrm{N},|\mathrm{S}_n-\mathrm{I}(x)|<\epsilon \tag{7.61}$$

and, since the remainders R_n and Q_n are $o(h^n)$ for all n, we have:

$$\forall\epsilon'\in\mathbb{R}^{+*},\exists\mathrm{N}'\mathrm{s.t.}\forall n>\mathrm{N}',|\tilde{\mathrm{S}}_n-\mathrm{S}_n)|<\epsilon \tag{7.62}$$

Using $\tilde{\mathrm{S}}_n-\mathrm{I}(x)=\tilde{\mathrm{S}}_n-\mathrm{S}_n+\mathrm{S}_n-\mathrm{I}(x)$ and the triangular inequality we have,

$$|\tilde{\mathrm{S}}_n-\mathrm{I}(x)|\le|\tilde{\mathrm{S}}_n-\mathrm{S}_n|+|\mathrm{S}_n-\mathrm{I}(x)| \tag{7.63}$$

We conclude,

$$\forall\epsilon\in\mathbb{R}^{+*},\exists\mathrm{M}=\max(\mathrm{N},\mathrm{N}^{'})\text{ s.t. }\forall n>\mathrm{M},|\tilde{\mathrm{S}}_n-\mathrm{I}(x))|<2\epsilon \tag{7.64}$$

The difference can be made arbitrary small. The sum $\tilde{\mathrm{S}}_n$ converges to $\mathrm{I}(x)$ as n goes to infinity. Using the previous definitions, $\tilde{\mathrm{S}}_n$ can be written as:

$$\tilde{\mathrm{S}}_n=2\frac{x}{n}\sum_{k=1}^{n}\frac{p_{k,n}}{q_{k,n}} \tag{7.65}$$

where $q_{k,n}$ is the probability making the player indifferent in the gamble $\tilde{z}\in\{-(k-1)\frac{x}{n},g(x,n,k)\}$ and $p_{k,n}$ is the probability premium making the player indifferent in the even money gamble $\tilde{z}\in\{-g(x,n,k),g(x,n,k)\}$.
Introducing the term $\frac{1}{n}$ in the sum, we obtain the following:

$$\tilde{\mathrm{S}}_n=2x\sum_{k=1}^{n}\frac{1}{n}\frac{p_{k,n}}{q_{k,n}} \tag{7.66}$$

We finally obtain:

$$\mathrm{FoR}_i(x)=\lim_{n\to+\infty}x\left(1+2\sum_{k=1}^{n}\frac{1}{n}\frac{p_{k,n}}{q_{k,n}}\right)+\mathrm{FoR}_i(0) \tag{7.67}$$

□

Chapter 8

Matching Games and Crowdsourcing

In this chapter, we analyze a two-sided crowdsourcing marketplace with externalities and scheduling constraints on the firms' side using game theory, particularly matching games. We consider the introduction of the stability property in such markets, a property that has been shown by A.E. Roth and others as essential for the long-term participation of the agents in the market. This problem we consider allows for crowdsourcing platforms with more rich and complete opportunities both for workers and firms because we allow for intra and inter-firms scheduling constraints. As examples of such settings, consider supply chains or subcontracting with outsourcing. We show that the problem does not fall in the scope of existing work in the stable matching theory and extends the theory of stable matching with contracts and externalities. We particularly introduce constrained substitutability condition to deal with constraints inducing the non-substitutability. We show the conditions for the existence of pairwise stable matching, obtained as the fixed point of a modified deferred acceptance algorithm proposed by Pycia and Yenmez in [1]. Furthermore, we define new stabilities adapted to the problem and propose an approach to bridge the gap between the game-theoretic analysis of the considered two-sided markets and non-cooperative game, as proposed in [2]. These results rely on a transformation of the crowdsourcing problem in a non-cooperative problem in normal and extensive form.

8.1 Introduction

Crowdsourcing is an effective paradigm for human-powered problem in which many companies propose to a group of individual users varying knowledge, heterogeneity, via a flexible open call, the voluntary undertaking of tasks.

There exist several commercial systems used in crowdsourcing [3–7]. The most popular is Amazon Mechanical Turk (MT) [3] so that there exists a wide body of works reporting on usage of MT to complete tasks ranging from natural language annotation to opinion evaluation and even spam identification. As another example, TaskCN is the largest platform covering the Chinese market [8]. Furthermore, specialized platforms also exist, such as [5] or [6] for software development and [4] providing the support of small labour service in neighborhoods and [7] provides crowdsourced services for data labeling and semantic data analysis.

Crowdsourcing systems fall in the scope of two-sided labor markets with a tripartite structure composed of workers, firms and a platform (in our framework: the matching mechanism or engine), as identified in [9]. As known in the game-theoretic analysis of two-sided markets, in such markets there are some fundamental properties (such as stability or truthfulness) to be verified by the matching mechanism for the agents to sustainably take part in the market and avoid its unravelling (agents turn to alternative matching solutions that benefit them). In this chapter, we focus on the stability property, as defined in matching games, that has experimentally been shown to be necessary for the participation of agents in markets such as college admission or hiring markets such as interns to hospitals. The theoretical tools develop by game theorists and economists to study such markets and design their matching mechanism have evolved along with the markets and their increasing complexity in terms of number of participants and specificities (contracts, externalities, etc.). In this chapter, we study a two-sided crowdsourcing problem with contracts, externalities and scheduling constraints from the game-theoretic point of view, particularly focusing on the stability property.

Based on recent work from the matching theory [1], we show the conditions for the existence of a pairwise stable matching and give an algorithm converging to such equilibrium. This requires the definition of a new substitutability, called *constrained substitutability* and related conditions exhibiting new specific constraints on choice functions. Furthermore, following [2] in the game-theoretic unification of two-sided markets with non-cooperative game theory, we transform our original constrained crowdsourcing problem in a non-cooperative game in normal form and give the first analysis of the matching obtained as a mixed Nash equilibrium. We show that, in the general case of our problem, there may not exist Nash equilibrium in pure strategies. Furthermore, we also transform the problem in a non-cooperative game in extensive form and give the first analysis of the matching obtained as a Subgame Pure Nash Equilibrium (SPNE).

A survey covering a classification and several technical aspects of crowdsourcing is found in [10]. Issues requiring theoretical modeling are identified in [9]: in our chapter we address specifically the strategic interactions which lead to match workers and tasks.

Several works have proposed mechanism design for crowdsourcing. E.g., reverse auction algorithms have been recently proposed in [11]: the aim is to implement a truthful mechanism able to maximize correct binary labeling under budget constraints. The authors of [12] address the problem of heterogeneity in crowdsourcing markets on the Internet. They propose a truthful budget-feasible mechanism which is incentive compatible and applies to the one-to-one matching scenario for splittable tasks. Budget feasible mechanisms have been proposed first by Singer and specialized for one-to-one matching crowdsourcing [13].

Mobile crowdsensing is a recent paradigm studied in a number of works, e.g., [14; 15]. In [14] a mechanism with budget constraint is applied. The authors of [15] provide a game theoretical framework for users' path selection when they perform geographically referenced mobile crowdsourcing tasks; a distributed asyncronous solution based on potential games is devised. This work does not look at the stability of the marketplace as studied in this paper.

In the crowdsourcing problem, we consider a constrained two-sided market with contracts and externalities in the classical framework of firms and workers where the firms want their tasks to be performed by workers who get paid for the execution. Nevertheless, there are other applications to the considered setting.

As a first alternative application, consider a reversed market obtained by permuting firms and workers. The worker have tasks to be performed according to scheduling constraints. Then, turn the tasks into places and interpret the directed scheduling graph as a set of physical paths between locations. This is a problem where mobile users move in space from place to place and can execute tasks along their path.

As a second application, consider a matching mechanism for cloud services. Define the workers as cloud servers of companies, the firms as software applications seeking for a decentralized computing of their tasks. The scheduling constraints defines the constraints among the tasks to be executed for the applications. The contracts define the terms of execution of the tasks in the companies' servers. As basic examples of such terms one can consider the amount of storage or the processing power dedicated by the server to the task involved in the contract. One can also consider the definition of payments from the application (thus its owner) to the companies for the execution of the tasks by the servers. A stable matching in this case would be a set of triplets (application, task, execution terms) defining not only which servers perform the tasks of the clients' applications (given the constraints) but also the terms of the execution and such that no single server, application or pair (server,application) would prefer deviating for a contract (not in the matching) defining the execution of a task.

In [16], Crawford and Knoer analyze a competitive labor market with heterogeneous firms and workers and perfect information (i.e. the agents know the market). They study the core stable allocations of the market which is shown to be an appropriate equilibrium concept for such market. An allocation is an assignment between firms and workers and the salaries for the corresponding jobs. In the paper, their focus on one-to-one matchings but the results extends up to the many-to-many case. The characteristics of any pair such as the satisfaction, the productivity and the salaries for the jobs are assumed to be integers. As a consequence, the model falls in the class of discrete ones. The satisfactions and productivities are assumed separable across firm-worker pairs, i.e. are independent from the assignment in the market. They furthermore assume continuous downward-sloping utility frontier for each pair (worker,firm) and transferable utility (salaries paid in a good that can be transferred between the players of a pair). A dynamic salary adjustment process (also called competitive adjustment process) is proposed and shown to converge to an allocation in the core. This guarantees its existence (non-emptiness) and the strict Pareto-efficiency of the assignment. The authors show that this process can be viewed as a computational alternative to solve the optimal assignment and transportation problem. The process allows for the agents in the market to fix the salary characteristic endogenously. They particularly show that the strict core of the continuous market is non-empty (emptiness of it would contradict the non-emptiness of the core of the discrete market) and that the assignment at the convergence point of the process is associated to an allocation of the strict core of this market. Observe that a discrete core allocation may not be in the continuous cores. They also show that, as in [17], that there exists a polarization of the interests of the agents over the core and that the salary adjustment process converges to the proposers optimal allocation under the assumption of strict preferences. This work generalizes Gale and Shapley's work in [17] the preference vary in time due to the dynamic adjustment of the salaries. It also generalizes Shapley and Shubik's one in [18] since the results are still valid for arbitrary utility functions as long as the separability and continuous downward-sloping frontier assumptions hold and with an arbitrary number of endogenous jobs characteristics (salaries and more).

In [19], Roth studies a many-to-many model of job-matching between firms and workers with contracts. A contract is defined by a set of elements such as the salary, the working conditions or others. Firms and workers have preferences over sets of contracts. The preferences of a worker over sets of jobs do not depend on the co-workers and the preferences of a firm over sets of jobs do not depend on the other firms the workers are working for. Each worker can be hired by a firm for at most one job. The preferences of the agents are strict (no indifference) and the formalism of choice functions is used. A matching, also called outcome, is a set of contracts. It is more complete than an association since it gives the working conditions. Coalitions are made of firms and workers. A matching is stable if there does not exist a coalition of players that would all prefer re-contracting while potentially maintaining other contracts. The set of stable outcomes is a subset of the core defined by strict domination. Roth assumes the Pareto separability which mainly requires that, given any pair (firm,worker), the set of Pareto optimal contracts between the two is independent of the matching. He furthermore assumes the substitutability of the contracts, which says that any contract chosen among a group will be chosen in a less preferred group. Roth shows that the blocking coalitions can be reduced to pairs, that the set of stable outcomes is always nonempty and that there exists firm-optimal and worker-optimal outcomes. He furthermore gives a finite stable algorithm and shows at the side-optimal outcomes no player on the optimal side would choose contracts from any other stable matching.

In [20], Sasaki and Toda study the two-sided matching with externalities. They show that one-to-one matchings may not exist. As a solution to the problem they proposed a new (weaker) stability: a pair of agent can block a matching if they prefer forming the pair under all possible rematching of the other players. The set of possible rematchings being predicted by the players of the pair by an estimation function. Such a solution is shown to always exist. They showed that the set of weakly stable matchings is always non-empty if and only if each agent has a universal estimation function which considers all matchings possible. Their idea to define a new stability in settings where classical conditions cannot be satisfied departed from the usual approach assuming pairwise stability as a solution concept (see [1] for more details).

In [21], Echenique and Yenmez study the many-to-one matchings with preferences over colleagues (or co-matched agents). The paper is formulated in terms of the academic labor market where workers are students and firms are colleges. They analyze a fixed point formulation of the problem and show an algorithm converging to the core stable matching if the core is not empty.

In [22], Ostrovsky studies the stability in supply chains. In a supply chain, the agents can be distinguished in three classes: (i) the suppliers, (ii) the buyers and sellers or intermediaries, and (iii) the consumers. These agents are partially ordered along the supply chain and the corresponding contract network is acyclic. In this paper, Ostrovsky focuses on another stability concept called chain-stability. A set of contracts or matching in a supply chain is called chain-stable if there are no blocking downstream chains of contracts. Assuming full substitutability[1], he has shown that there exists a chain stable matching for any supply chain satisfying full-substitutability. It was further shown by Hatfield and Kominers in [23] that if the contract network is acyclic, then chain-stable outcomes are equivalent to set-stable outcomes[2].

In [24], Fleiner et a.l. study a general networks of bilateral contracts which generalizes the previous works on matching with contracts, particularly Ostrovsky's work on supply chains [22]. By definition, a contract network is a multi-sided matching market in which firms form downstream contracts to sell outputs and upstream contracts to buy inputs. Fleiner et a.l. define the trail stability as a new stability subsuming chain stability but weaker than set stability. They also define the weaker notion of full trail-stability. They show that the trail-stability is equivalent to the chain stability in acyclic contract networks and to pairwise stability in two-sided many-to-many matching markets with contracts. Trail-stable and full trail-stable outcomes are shown to exist in network contracts under full substitutability and irrelevance of rejected contracts. Furthermore, if the preferences also satisfy the separability condition, the set of trail-stable outcomes contains buyer-optimal and seller-optimal outcomes. Finally, in the setting where contract specify prices, if in addition the complete prices and price separability assumptions are satisfied then competitive equilibrium exists and it is trail stable. A generalized salary dynamic adjustment process is proposed, as in [19][16].

As observed in [1], only few papers look at standard stability in the general matching problem

[1] Same-side substitutability and cross-side complementarity.

[2] Robust to deviations by sets of firms.

with externalities. This work developed in this chapter falls in this class. In [25], Bando studies a many-to-one matching market (without contracts) with externalities only on the firm's side and due to hired workers only. The model is formulated in terms of choice functions. As in [20], a new stability concept called weak stability is defined. They define the notion of *incredible deviation* and *strongly blocking pair* such that weakly stable matching that cannot be strongly blocked by any pair. Such weak stable matching is show to exist under the assumptions of extended substitutability, increasing choice [3] and no external effect by unchosen worker. If such conditions are not satisfied, then a weak stable matching may not exist. These are thus necessary conditions for the existence of weak stable matchings in this setting. In [26], a modified deferred acceptance algorithm is proposed and analyzed. This algorithm works the following: the workers (simultaneously) propose to their most preferred firm (not having rejected them yet) and the firm (after receiving the proposals) chose some workers in the cumulative set (set of workers having proposed to it) assuming that the workers proposing to other firms are hired. The author shows that this algorithm converges (when there are no more rejections) to a worker-optimal quasi stable matching. He also shows that this algorithm can be generalized into a fixed point algorithm.

8.2 Crowdsourcing System with Scheduling Constraints

8.2.1 System

Symbol	*Meaning*
$\mathscr{F}$	Set of firms f_k, $k=1,\dots,l$
$\mathscr{W}$	Set of workers w_i, $i=1,\dots,m$
$\mathscr{T}$	Set of tasks τ_j, $j=1,\dots,n$
$\mathscr{T}_f$	Set of tasks of the firm f
$\mathscr{X}$	Set of contracts
μ	Matching (subset of contracts)
X	Subset of contracts
$\mathscr{T}(\mathrm{X})$	Tasks of contracts in X
$\mathscr{F}(\mathrm{X})$	Firm of contracts in X
$\mathscr{W}(\mathrm{X})$	Workers of contracts in X
$\mathrm{X}(\mu)$	Subset of feasible contracts in X at μ
$\mathbf{A}$	Adjacency matrix of the (directed) scheduling graph $\mathscr{G}$
$\mathscr{N}^-(\tau)$	Set of predecessors of task τ in the scheduling graph $\mathscr{G}=(\mathscr{T},\mathbf{A})$
$\mathscr{N}^-(\mathscr{T}_f)$	Set of predecessors of f's tasks (not including those in $\mathscr{T}_f$) in the scheduling graph $\mathscr{G}=(\mathscr{T},\mathbf{A})$
$\succeq_i$	Order relation of i
$c_i(.\vert.)$	Choice function of i
$\mathrm{C}^{\mathrm{F}}(.\vert.)$	Choice function of the firms
$\mathrm{C}^{\mathrm{W}}(.\vert.)$	Choice function of the workers

Table 8.1: Main notations

In Figure 8.1 we show a crowdsourcing system (top) and its related matching market (bottom). In this chapter, the system operates in the following way. Each firms in $\mathscr{F}$ (splitted squares, right) submit the tasks in $\mathscr{T}_f$ and their corresponding rewards to the platform (center) and workers in $\mathscr{W}$ (left) connect to it so as to choose (or apply for) the tasks they can perform. In this chapter we do not investigate or discuss the motivations of the firms or the workers in using the crowdsourcing system. We assume the business model well-defined and do not take into account an eventual unraveling of the market in favor of an alternative solution. We also assume that each firm has a

[3] By definition: (i) the choice set of a firm depends only on the set of workers hired by its rival firms, and (ii) the choice set of a firm expands when the set of workers hired by the rival firms expands.

private scheduling (interdependence constraints between the tasks) over the tasks in the purpose of its activity. For more information about the scheduling and its modeling in this chapter, see Section 8.2.2. The workers are not required to provide an exhaustive list of their abilities w.r.t. all the proposed tasks. We assume that the given informations are sufficient to deduce these abilities.

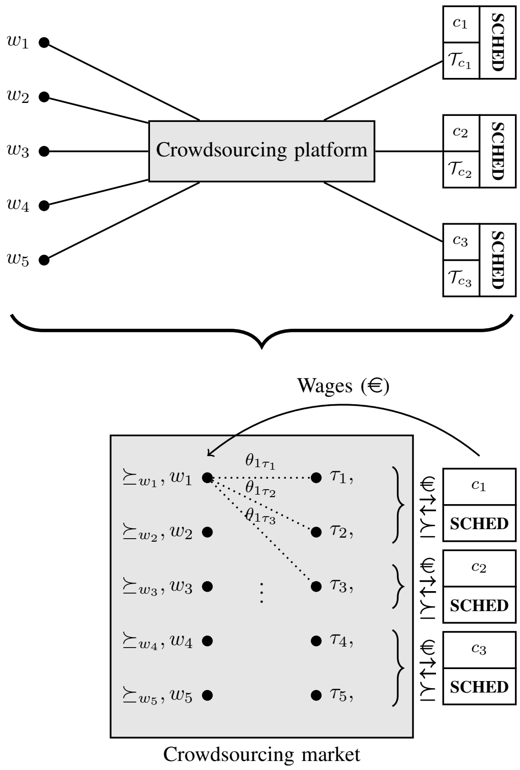

Figure 8.1: The crowsourcing platform and its associated crowdsourcing matching matching market. The productivity of worker i w.r.t. task τ_j is denoted $\theta_{i\tau_j}$. Such productivity is an example of measure of efficiency of the execution of a task by a worker.

8.2.2 Representation

We give two examples of practically well-known modeling of scheduling constraints in project management. These techniques have been used in the industry for a long time. In this chapter, we use a graph-theoretic representation of the precedence constraints that is an abstract version of the practical methods. It allows for a wide range of externalities (not necessarily temporal).

The Project Evaluation and Review Technique (PERT)

The Project Evaluation and Review Technique (PERT) chart is a commonly used graph-theoretic representation in operation research and scheduling or project management. This method has been developed by the U.S. Navy in the 1950s and first used in the development of their Polaris missile program. This program had to be achieved in fixed delays with 250 suppliers and 9000 subcontractors. The PERT is used to show the logic (in terms or precedence) and temporal dependencies between the tasks that have to be performed. It is a systematic method for scheduling, control and correction. The PERT chart is a graph representation where vertices are milestones and directed edges are objects called activities or tasks which show the execution of the related task. In the representation, the length of the arcs are not necessarily proportional to the execution time The arcs are weighted by the execution time before completion of their generating task (vertex). It exists various execution times, namely the optimistic time, the pessimistic time, the normal time and the expected time. An example of a PERT chart is given in Figure 8.2. The tasks with direct edges to a given task are called the predecessors of this task. Reciprocally, this task is a successor of its predecessors. In Figure 8.2, τ_1 and τ_2 are the predecessors of τ_3. Reciprocally, the task τ_3 is a successor of τ_1 and τ_2. Using graph theory and dedicated algorithms, the task manager can compute some characteristic quantities that allow a better planning and scheduling of the project. As examples of such quantities, we have the expected start time of a task (expected completion time of all prerequisites), the slackness of the start time (distance to expected start time the earliest start time of any successor of a task), the maximum delay over a start before delaying the successors and the critical path (longest path from start to stop of the PERT). By definition, the graph must contain two extreme vertices, start and stop.

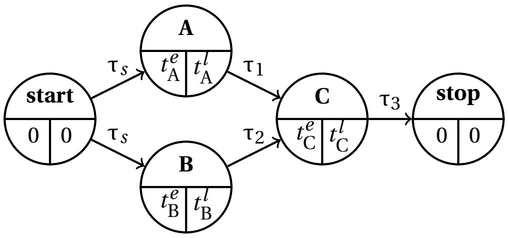

Figure 8.2: Pert chart of a set of tasks $\mathscr{T} = \{\tau_1, \tau_2, \tau_3\}$.

The Metra Potential Method (MPM)

The Metra Potential Method has been proposed by Bernard Roy in 1958 in the purpose of the construction of the paquebot France and the first French EDF's nuclear power plant. The MPM representation is equivalent to the PERT one, there are no extra information provided neither by the one nor the other. The MPM only differs from the PERT in the definition of the nodes. A vertex of the MPM graph is no more a step or a fictive task but the task itself. Except for this difference, the two representations are the same. Thus, there is need to go further in the description of the MPM method. Similarly to the PERT, the graph must contain only two extreme vertices, start and stop.

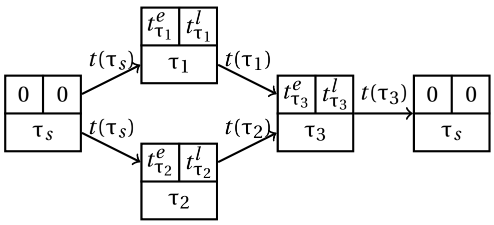

Figure 8.3: MPM chart of a set of tasks $\mathscr{T} = \{\tau_1, \tau_2, \tau_3\}$.

Simplified Representation

In this chapter, we use a simplified representation of the scheduling constraints. Consider a directed graph $\mathscr{G} = (\mathscr{T}, \mathbf{A})$ where $\mathscr{T}$ denotes the set of nodes in the graphs and $\mathbf{A}$ the set of edges (or arcs), or equivalently, the adjacency matrix with coefficients in $\{0, +1\}$. In fact, we assume a one-to-one mapping between the tasks and the nodes of the graph. Given any pair (τ, τ') of nodes of $\mathscr{G}$, if τ dominates[4] τ' then it is required that a worker performs (is assigned) the task τ for the task τ' to be performed (assigned).

In this chapter we allow the graph to span over the set the tasks in the game, whatever the firm they correspond to. This allows, not only for an intra-firms scheduling but also for inter-firms scheduling in the sense that the tasks of a firm may be constrained by the tasks of another firm. As an example, there is subcontracting. Such constraints are sufficient to make the problem fall

[4] A node τ dominates another node τ' is there is an edge from τ to τ'.

in the scope of matching with *complementarities* (intra-firms scheduling) and *externalities* (inter-firms scheduling). Because the set of nodes $\mathscr{T}$ of scheduling graph contains all the tasks we will use a global index for the tasks. Given the graph $\mathscr{G} = (\mathscr{T}, \mathbf{A})$, we denote $\mathscr{G}_i = (\mathscr{T}_i, \mathbf{A}_i)$, the subgraph induced by the set of nodes corresponding to the tasks in $\mathscr{T}_i$ and their dominating nodes (i.e. the set of nodes in $\mathscr{T} - \mathscr{T}_i$ that are dominating nodes in $\mathscr{T}_i$) also called set of direct predecessors. We denote $\mathscr{N}^-(\tau)$ the set of predecessors[5] of τ.

In Figure 8.4, we show an example of a simplified representation. Tasks τ_1 and τ_2 have no predecessors. Task τ_3 has τ_1 and τ_2 as predecessors ($\mathscr{N}^-(\tau_3) = \{\tau_1, \tau_2\}$).

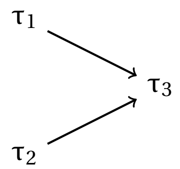

Figure 8.4: Graph of scheduling constraints $\mathscr{T} = \{\tau_1, \tau_2, \tau_3\}$.

8.2.3 Motivating Examples

In this section we give small examples showing that the existing results are not sufficient to solve the many-to-many matching problems with scheduling constraints. The first example is a many-to-many matching problem with contracts and individual scheduling constraints (intra-firms scheduling). We turn this problem into a many-to-many matching with inter-firms scheduling by using a *task-agent representation* (transforming the original matching problem in a new one by turning the tasks into players) and show that in this case it does not satisfy the generalized substitutability proposed by Pycia in [1] and there are no stable matchings.

Matching with Contracts and Intra-firms Scheduling

In this example, consider a many-to-many crowdsourcing marketplace with contracts and individual scheduling with individual scheduling on the firms' side. The example is shown in Figure 8.5.

There are two firms $\mathscr{F} = \{f_1, f_2\}$, two workers $\mathscr{W} = \{w_1, w_2\}$ and three tasks $\mathscr{T} = \{\tau_1, \tau_2, \tau_3\}$. Each firm can sign contracts with many workers and each worker can sign many contracts with any given firm, and sign contracts with many different firms. We assume that only one contract can be signed for any given task. In other words, no firm can sign two contracts for any given task. This assumption will be used throughout the chapter. We assume the set of contracts $\mathscr{X} = \{x^1_{11}, x^2_{22}, x^3_{22}\}$, where x^m_{kl} the contract (w_k, f_l, τ_m).

Assume the preferences of the firm f_2 given by the choice function in Table 8.2. Firm f_2 is the only player exhibiting complementarities in its choice function (or preferences, see [1] and references therein for a discussion on the link between choice functions and preferences). Particularly, it prefers x^3_{22} to the null contract if x^2_{22} can be signed and prefers the null contract to x^3_{22} if x^2_{22} cannot not signed: $\{x^2_{22}, x^3_{22}\} \succ_{f_2} \{x^2_{22}\} \succ_{f_2} \emptyset \succ_{f_2} \{x^3_{22}\}$ Thus, the firm prefers having its two tasks performed rather a single one but in this latter case the only acceptable contract is x^2_{22} due to the scheduling constraints. Firm f_1 prefers x^1_{11} to the null contract: $x^1_{11} \succ_{f_1} \emptyset$. On the workers' side, worker w_1 prefers working at x^1_{11} rather than being unemployed: $x^1_{11} \succ_{w_1} \emptyset$. Worker w_2 wants to perform only one task and prefers working at x^3_{22} rather than x^2_{22} but prefers any contract to being unemployed: $x^3_{22} \succ_{w_2} x^2_{22} \succ_{w_2} \emptyset \succ_{w_2} \{x^2_{22}, x^3_{22}\}$. The preferences of worker w_2 imply that any stable matching (if there exists any) would be of the *one-to-one* form since any worker in the example would be matched to only one firm to perform a single task.

[5] A node τ is a predecessor of another node τ' if τ' is reachable from τ, i.e. if there is a directed path in the graph from τ to τ'

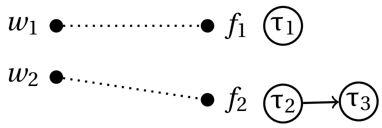

Figure 8.5: A matching market with contracts and intra-firms scheduling. Dotted lines between two agents show that there exists a contract among them in the set of contracts.

	$\{x_{22}^2\}$	$\{x_{22}^3\}$	$\{x_{22}^2, x_{22}^3\}$	$\emptyset$
$c_{f_2}(.\|\emptyset)$	$\{x_{22}^2\}$	$\emptyset$	$\{x_{22}^2, x_{22}^3\}$	$\emptyset$
$c_{f_2}(.\|\{x_{11}^1\})$	$\{x_{22}^2\}$	$\emptyset$	$\{x_{22}^2, x_{22}^3\}$	$\emptyset$
$c_{f_2}(.\|\{x_{22}^2\})$	$\{x_{22}^2\}$	$\emptyset$	$\{x_{22}^2, x_{22}^3\}$	$\emptyset$
$c_{f_2}(.\|\{x_{22}^3\})$	$\{x_{22}^2\}$	$\emptyset$	$\{x_{22}^2, x_{22}^3\}$	$\emptyset$
$c_{f_2}(.\|\{x_{11}^1, x_{22}^2\})$	$\{x_{22}^2\}$	$\emptyset$	$\{x_{22}^2, x_{22}^3\}$	$\emptyset$
$c_{f_2}(.\|\{x_{11}^1, x_{22}^3\})$	$\{x_{22}^2\}$	$\emptyset$	$\{x_{22}^2, x_{22}^3\}$	$\emptyset$
$c_{f_2}(.\|\{x_{22}^2, x_{22}^3\})$	$\{x_{22}^2\}$	$\emptyset$	$\{x_{22}^2, x_{22}^3\}$	$\emptyset$
$c_{f_2}(.\|\{x_{11}^1, x_{22}^2, x_{22}^3\})$	$\{x_{22}^2\}$	$\emptyset$	$\{x_{22}^2, x_{22}^3\}$	$\emptyset$

Table 8.2: Choice function c_{f_2} of firm f_2.

There is no stable matching in this case. In fact, assume the matching $\mu = \{x_{11}^1\}$. This matching is blocked by w_2 and f_2 who both prefer x_{22}^2 to the unemployment. Alternatively, consider the matching $\mu = \{x_{11}^1, x_{22}^2\}$. This matching is blocked by w_2 and f_2 who both prefer x_{22}^3 (knowing that x_{22}^2 is signed). Finally, consider the matching $\mu = \{x_{11}^1, x_{22}^3\}$. This matching is blocked by f_2 which prefers the null contract knowing that x_{22}^3 only is signed. Any other matching such that x_{11}^1 is not signed is blocked by the pair w_1 and f_1 for the contract x_{11}^1 or such that both x_{22}^2 and x_{22}^3 are signed is blocked by w_2 who rejects x_{22}^2.

Nevertheless, one may expect that the equilibrium of this small market would be $\{x_{11}^1, x_{22}^2\}$ because of the scheduling constraints that give τ_2 a priority over τ_3 in the execution and the fact that deviating for x_{22}^3 induces a self-penalization for f_2. There lies the problem we want to assess in this chapter.

For now, we conclude that in this case either there is no stable matching and we have to weaken our objectives and make further assumptions on the choice functions or preferences of the agents (such as the well-known substitutability, Irrelevance of Rejected Contracts (IRC, see Chapter 3, Section 3.6), separability, quasi-linearity of the firm's profit functions or other). Observe that no assumption on the preferences of the agents could remove the scheduling constraints and thus the complementarities of contracts. An alternative solution would be to define a new adapted stability that would take into account the scheduling constraints.

Remark 115. *If the preferences of worker w_2 were $\{x_{22}^2, x_{22}^3\} \succ_{w_2} x_{22}^3 \succ_{w_2} x_{22}^2 \succ_{w_2} \emptyset$. The only stable matching would be $\{x_{11}^1, x_{22}^2, x_{22}^3\}$.*

To provide more intuition on this problem we show that the choice functions defined in this example do not satisfy one of the fundamental sufficient condition for the existence of stable matching (in many-to-many matching problems with contracts), namely substitutability. Particularly, we show that the choice function c_{f_2} of firm f_2 does not satisfy substitutability. Qualitatively, a firm's choice function satisfies substitutability if the firms rejects in a set of contracts at a given matching are included in its rejects in a superset at a matching with better market conditions (see definition 69 for the formal definition). By definition, given a matching, a choice function satisfies substitutability if the rejects from a set of contracts are included from those in a superset.

Take $X = \{x_{22}^3\} \subseteq X' = \{x_{22}^3, x_{22}^2\}$. We have, $\{x_{22}^3\} = r_{f_2}(\{x_{22}^3\}) \not\subset \emptyset = r_{f_2}(\{x_{22}^3, x_{22}^2\})$.

The non-substitutability in this example comes from the complementarities between the contracts x^2_{22} and x^3_{22}. Firm f_2 signs x^3_{22} only if x^2_{22} is also signed. Intuitively, on can conclude that the non-substitutability can be solved by either changing the preferences of the firm to make $\{x^3_{22}\}$ acceptable or remove the complementarities between x^3_{22} and x^2_{22}.

This basic example shows that matching with contracts and scheduling constraints may not have pairwise stable matchings and that the choice functions of players submitted to scheduling constraints (namely, the firms in this problem) may not satisfy substitutability.

Matching with Contracts and Inter-firms Scheduling

Consider a second example, as shown in Figure 8.6. This example can be considered as a self-sufficient matching problem with externalities with choice functions as defined in Table 8.3, Table 8.4 and Table 8.5. It can also be considered as the result of a transformation of the example of Section 8.2.3 (see Figure 8.5), called *task-agent representation*. In the task-agent representation, the firms' side is turned into a tasks' side with tasks as players. This transformation is called task-agent representation of the original problem because it consists in replacing any firm by the set of task-agents playing in the name of the tasks they represent. The motivation for this interpretation is to asses wether a solution to the previous example could be obtained using such simple transformation of the problem. We show that no stable matching exists in this case. The conclusion is that this simple example with inter-firms scheduling has no stable matching and the considered transformation can not successfully solve the instability of Example 8.2.3.

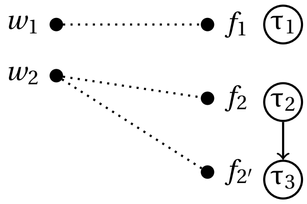

Figure 8.6: Example: A many-to-one matching game with contracts and scheduling constraints in the task-agent representation. Dotted lines between two agents show that there exists a contract among them in the set of contracts.

Remark 116. *In the task-agent representation, the players are the nodes $\mathcal{T}$ of the graph $\mathcal{G}$. The scheduling constraints of the example shown in Section 8.2.3 now consist in externalities. The set of firms is increased to a superset in bijection with the set of tasks. Firm f_1 is left as is since it has only τ_1 to be performed. Firm f_2 is transformed in f_2 with task τ_2 and $f_{2'}$ with task τ_3. The first difficulty encountered when using the task-agent representation is the definition of the choice functions. A naive approach consists in assigning each sub-firm the choice function of the corresponding original firm restricted to the contracts of the corresponding task.*

$$\forall i \in \mathcal{F}, \quad c_i : 2^{\mathcal{X}_i} \times 2^{\mathcal{X}} \to 2^{\mathcal{X}_i} \Rightarrow \{c_j : \mathcal{X}(j) \times 2^{\mathcal{X}} \to \mathcal{X}(j)\}_{j \in \mathcal{T}_i} \tag{8.1}$$

where $\mathcal{X}(j)$ denotes the set of contracts with task j in $\mathcal{T}_i$.

We show the choice functions in Table 8.3, Table 8.4 and Table 8.5. Table 8.6 shows the corresponding rejects of $f_{2'}$. In Table 8.7, we show the firms' choices as union of individual choices. The corresponding firms' choice function C^F will be used throughout the chapter (see Chapter 3, Section 3.6), as in [27] in the analysis of matching with contracts and no externalities and [1] in the analysis of matching with contracts and externalities.

In this example, there are three firms $\mathcal{F} = \{f_1, f_2, f_{2'}\}$, two workers $\mathcal{W} = \{w_1, w_2\}$ and three contracts $\mathcal{X} = \{x^1_{11}, x^2_{22}, x^3_{22'}\}$ such that $x^1_{11} = (f_1, w_1, \tau_1)$, $x^2_{22} = (f_2, w_2, \tau_2)$ and $x^3_{22'} = (f_{2'}, w_2, \tau_3)$. On the firms' side, firm f_1 prefers x^1_{11} to the null contract: $x^1_{11} \succ_{f_1} \emptyset$. Firm f_2 prefers x^2_{22} to the null contract: $x^2_{22} \succ_{f_2} \emptyset$. Firm $f_{2'}$ is the only player exhibiting externalities in its preference relation: it prefers $x^3_{22'}$ to the null contract if x^2_{22} is signed and prefers the null contract to $x^3_{22'}$ if x^2_{22} is not

	$\{x^1_{11}\}$	$\emptyset$
$c_{f_1}(.\vert\{x^2_{22}, x^3_{22'}\})$	$\{x^1_{11}\}$	$\emptyset$
$c_{f_1}(.\vert\{x^2_{22}\})$	$\{x^1_{11}\}$	$\emptyset$
$c_{f_1}(.\vert\{x^3_{22'}\})$	$\{x^1_{11}\}$	$\emptyset$
$c_{f_1}(.\vert\emptyset)$	$\{x^1_{11}\}$	$\emptyset$

Table 8.3: Choice function c_{f_1}.

	$\{x^2_{22}\}$	$\emptyset$
$c_{f_2}(.\vert\{x^1_{11}, x^3_{22'}\})$	$\{x^2_{22}\}$	$\emptyset$
$c_{f_2}(.\vert\{x^1_{11}\})$	$\{x^2_{22}\}$	$\emptyset$
$c_{f_2}(.\vert\{x^3_{22'}\})$	$\{x^2_{22}\}$	$\emptyset$
$c_{f_2}(.\vert\emptyset)$	$\{x^2_{22}\}$	$\emptyset$

Table 8.4: Choice function c_{f_2}.

	$\{x^3_{22'}\}$	$\emptyset$
$c_{f'_2}(.\vert\{x^1_{11}, x^2_{22}\})$	$\{x^3_{22'}\}$	$\emptyset$
$c_{f'_2}(.\vert\{x^1_{11}\})$	$\emptyset$	$\emptyset$
$c_{f'_2}(.\vert\{x^2_{22}\})$	$\{x^3_{22'}\}$	$\emptyset$
$c_{f'_2}(.\vert\emptyset)$	$\emptyset$	$\emptyset$

Table 8.5: Choice function $c_{f'_2}$.

	$\{x^3_{22'}\}$	$\emptyset$
$r_{f'_2}(.\vert\{x^1_{11}, x^2_{22}\})$	$\emptyset$	$\emptyset$
$r_{f'_2}(.\vert\{x^1_{11}\})$	$\{x^3_{22'}\}$	$\emptyset$
$r_{f'_2}(.\vert\{x^2_{22}\})$	$\emptyset$	$\emptyset$
$r_{f'_2}(.\vert\emptyset)$	$\{x^3_{22'}\}$	$\emptyset$

Table 8.6: Reject function $r_{f'_2}$.

	$\{x^1_{11}, x^2_{22}, x^3_{22'}\}$	$\{x^1_{11}, x^2_{22}\}$	$\{x^1_{11}, x^3_{22'}\}$	$\{x^2_{22}, x^3_{22'}\}$	$\{x^1_{11}\}$	$\{x^2_{22}\}$	$\{x^3_{22'}\}$	$\emptyset$
$C^F(.\vert\{x^1_{11}, x^2_{22}, x^3_{22'}\})$	$\{x^1_{11}, x^2_{22}, x^3_{22'}\}$	$\{x^1_{11}, x^2_{22}\}$	$\{x^1_{11}, x^3_{22'}\}$	$\{x^2_{22}, x^3_{22'}\}$	$\{x^1_{11}\}$	$\{x^2_{22}\}$	$\{x^3_{22'}\}$	$\emptyset$
$C^F(.\vert\{x^1_{11}, x^2_{22}\})$	$\{x^1_{11}, x^2_{22}, x^3_{22'}\}$	$\{x^1_{11}, x^2_{22}\}$	$\{x^2_{22}, x^3_{22'}\}$	$\{x^2_{22}, x^3_{22'}\}$	$\{x^1_{11}\}$	$\{x^2_{22}\}$	$\{x^3_{22'}\}$	$\emptyset$
$C^F(.\vert\{x^1_{11}, x^3_{22'}\})$	$\{x^1_{11}, x^2_{22}\}$	$\{x^1_{11}, x^2_{22}\}$	$\{x^1_{11}\}$	$\{x^2_{22}\}$	$\{x^1_{11}\}$	$\{x^2_{22}\}$	$\emptyset$	$\emptyset$
$C^F(.\vert\{x^2_{22}, x^3_{22'}\})$	$\{x^1_{11}, x^2_{22}, x^3_{22'}\}$	$\{x^1_{11}, x^2_{22}\}$	$\{x^2_{22}, x^3_{22'}\}$	$\{x^2_{22}, x^3_{22'}\}$	$x^1_{11}\}$	$\{x^2_{22}\}$	$\{x^3_{22'}\}$	$\emptyset$
$C^F(.\vert\{x^1_{11}\})$	$\{x^1_{11}, x^2_{22}\}$	$\{x^1_{11}, x^2_{22}\}$	$\{x^1_{11}\}$	$\{x^2_{22}\}$	$\{x^1_{11}\}$	$\{x^2_{22}\}$	$\emptyset$	$\emptyset$
$C^F(.\vert\{x^2_{22}\})$	$\{x^1_{11}, x^2_{22}, x^3_{22'}\}$	$\{x^1_{11}, x^2_{22}\}$	$\{x^2_{22}, x^3_{22'}\}$	$\{x^2_{22}, x^3_{22'}\}$	$\{x^1_{11}\}$	$\{x^2_{22}\}$	$\{x^3_{22'}\}$	$\emptyset$
$C^F(.\vert\{x^3_{22'}\})$	$\{x^1_{11}, x^2_{22}\}$	$\{x^1_{11}, x^2_{22}\}$	$\{x^1_{11}\}$	x^2_{22}	x^1_{11}	$\{x^2_{22}\}$	$\emptyset$	$\emptyset$
$C^F(.\vert\emptyset)$	$\{x^1_{11}, x^2_{22}\}$	$\{x^1_{11}, x^2_{22}\}$	$\{x^1_{11}\}$	$\{x^2_{22}\}$	$\{x^1_{11}\}$	$x^2_{22}\}$	$\emptyset$	$\emptyset$

Table 8.7: Firms' choice function C^F.

signed: $x^3_{22'} \succ_{f'_2} \emptyset$ if x^2_{22} is signed and $\emptyset \succ_{f'_2} x^3_{22'}$ if x^2_{22} is not signed. On the workers' side, worker w_1 prefers working at x^1_{11} to the unemployement: $x^1_{11} \succ_{w_1} \emptyset$. Worker w_2 prefers working at $x^3_{22'}$ rather than x^2_{22} but prefers any contract to being unemployed: $x^2_{22'} \succ_{w_2} x^2_{22} \succ_{w_2} \emptyset$.

There is no stable matching in this case, as in the previous one. In fact, assume matching $\mu = \{x^1_{11}\}$. This matching is blocked by w_2 and f_2 who both prefer x^2_{22} to the null contract. Alternatively, consider the matching $\mu = \{x^1_{11}, x^2_{22}\}$. This matching is blocked by w_2 and $f_{2'}$ who both prefer $x^3_{22'}$ (knowing that x^2_{22} is signed). Finally, consider the matching $\mu = \{x^1_{11}, x^3_{22'}\}$. This matching is blocked by $f_{2'}$ which prefers the null contract knowing that x^3_{22} only is signed. Any other matching such that x^1_{11} is not signed is blocked by the pair w_1 and f_1 for the contract x^1_{11} or such that both x^2_{22} and $x^3_{22'}$ are signed is blocked by w_2 who rejects x^2_{22}. As in the previous example, there is no stable matching in this case. This conclusion also shows that the task-agent representation does lead to a solution to the problem shown in Section 8.2.3 as a fairly simple transformation.

At this point, we consider three ways to solve the problem: (i) make some assumptions on the agents' choice functions[6], (ii) consider alternative stabilities as solution concepts or (iii) solve the problem using another formulation (such as non-cooperative strategic or extensive forms).

[6] Or preferences, see Chapter 3, Section 3.6 for more details.

These ways are well-known in matching games. Using the classical stabilities and assuming that the agents' preferences satisfy some conditions or considering new stabilities are two well-known and studied options to solve matching problems (see [28] for a recent survey on matching with externalities and existing methods).

The third approach is less common. An interesting work in this direction was proposed in [2]. The authors consider a specific congestion game and show that the solutions are stable marriages. Generalizations to many-to-one settings are proposed. The interesting point is that this bridges the gap between some non-cooperative games (such as congestion ones) and cooperative problems (such as stable matchings) and their solution concepts (such as the Nash equilibrium and stability). As other examples, consider [29] where core stable matchings can be obtained from some non-cooperative games or the bargaining problem (see [30] and references therein) where cooperative solutions to the bargaining problem can be obtained as non-cooperative equilibria.

In this chapter, we consider these three ways to solve the problem. We particularly focus on the first (see Section 8.3) and third ones (see Section 8.4 and Section 8.5). Furthermore, we provide new appropriate definitions of stability in Appendix 8.10. In the next section and in Appendix 8.8, we give further details on the insufficiency of these solutions.

8.2.4 Insufficiency of Existing Models and Solutions

In this section, we compare the problem of many-to-many matching game with contracts and scheduling constraints to those assessed in existing works and give the reasons for their insufficiency. A more detailed analysis (with applications of the modified deferred acceptance from [1] or transformation of the problem as a contract network) is given in Appendix 8.8.

- Sasaki and Toda's [20], *Matching with Externalities*:

  The results are limited to one-to-one matchings whereas the problem we consider is a many-to-many matching problem.

- Hatfield and Milgrom's [27], *Matching with Contracts*:

  This model does not take into account externalities and many-to-one matching with contracts. This is insufficient.

- Bando's [25][26], *Matching with Externalities*:

  This work studies a many-to-one matching market with externalities only on the firm's side and due to hired workers only. The model is formalized in terms of choice functions on sets of players and not contracts. This model cannot be used despite of interesting elements such as the weak stability and an adaptation of the assumption on externalities only due to hired workers.

- Fleiner et a.l. [24], *Trading Networks with Bilateral Contracts*:

  The model proposed by Fleiner et a.l. in [24] considers a contract network modeled by a directed graph such that the nodes are firms and the directed edges are the contracts. In this setting, the firms trade with each others over the contract network. In our setting, firms do not trade with each others but with workers. Nevertheless, it is shown that many-to-many markets are special cases of such trading networks. Both our model and their model consider a directed graph but the motivations differ since in our case, the graph describes scheduling constraints and not trading opportunities between adjacent nodes.

- Pycia and Yenmez's [1], *Matching with Externalities*:

  The studied model is compatible with the particular many-to-many structure of our problem. The authors use the substitutability and irrelevance of rejected contracts as sufficient conditions for the convergence of their modified deferred acceptance algorithm to a stable matching and thus to show the existence of stable matchings. As already observed, one can find counter-examples of matching problems with scheduling constraints that have no pairwise stable matchings and thus do not satisfy their sufficient conditions.

8.3 Matching with Contracts, Externalities and Scheduling Constraints

8.3.1 Model

We model the crowdsourcing market as a many-to-many matching game with contracts and externalities. The model of matching with contracts generalizes the classical formulation [17][31] of the stable matching theory and incorporates the well-known college admission problem, the Kelso-Crawford labor market matching model and some ascending packet auctions[7]. Futhermore, it has been extended to allow for externalities in many-to-many settings. In this model, the agents interact with each others through bilateral contracts. For the rest of this section we define the model and give some of the definitions, properties and results that have been used in the literature. More particularly, most of the definitions are given in terms of the recent general model of matching with contracts and externalities defined by Pycia and Yenmez in [1]. For the sake of clarity and completeness we also give some definitions in terms of the models of matching with contracts without externalities as defined by Hatfield and Milgrom in [27]. These last notations are naturally embedded in those used in matching games with externalities and can immediately be derived by removing the conditioning from the expression.

Consider a finite sets of firms $\mathscr{F}$, workers $\mathscr{W}$, tasks $\mathscr{T}$ and scheduling constraints $\mathscr{G} = (\mathscr{T}, \mathbf{A})$ as defined in Section 8.2.2. Let define the finite set of *contracts* $\mathscr{X}$ between workers in $\mathscr{W}$ and firms in $\mathscr{F}$ as the set of bilateral pairwise agreements that the workers and the firms can sign with each others. A contract $x \in \mathscr{X}$ specifies a worker, a firm and additional terms such as the wage, the execution constraints or the penalties to the signatories in case of improper execution. For the sake of simplicity, we assume that any contract is defined as four-tuple of the form (f, w, τ, s) where f is a firm, w is a worker, τ is a task in $\mathscr{T}$ and s is the salary (wage) in the finite set $\mathscr{S} = \{\underline{s}, \dots, \overline{s}\}$ (see [19] and [27]) paid to the worker for the execution of the task.

Given any subset of contracts X, $\mathscr{T}(X)$ is the set of tasks corresponding to the contracts in X. For any subset of contracts $X \subseteq \mathscr{X}$, X_i denotes the maximal set of contracts in X involving i,

$$X_i = \{x \in X | i \in \{f(x), w(x)\}\} \tag{8.2}$$

A matching μ between workers and tasks is defined as a set of contracts $\mu \subseteq \mathscr{X}$. This definition used in matching with contracts generalizes the one used in the stable matching theory without contracts in the sense that the association binary relation between the players of a pair (*matched* or *unmatched*) is completed by the terms of the signed contract (among those that can be signed between the two) and that more than one contract can be signed simultaneously between any two players. We define player i's choice function,

$$c_i : 2^{\mathscr{X}} \times 2^{\mathscr{X}} \to 2^{\mathscr{X}} \tag{8.3}$$

such that $c_i(X|\mu) = c_i(X_i|\mu)$ is the choice set of i in X_i knowing $\mu = (\mu_i, \mu_{-i})$. It is the set of contracts that i chooses from X_i given the set μ of contracts signed. In this chapter, we consider choice functions without prediction as in definition 64, chapter 3. Similarly, the workers' choice function C^W and the firms' one C^F are defined as in equation 3.40, chapter 3. We now give the definition of a set of feasible contracts.

Definition 117. *Given a firm $f \in \mathscr{F}$, a set of contracts $X \subseteq \mathscr{X}_f$ and a matching μ, the set X is called feasible for f at μ if for any contract $x \in X$, there exists a subset $X' \subseteq X$ with $x \in X'$ and one contract per task in $\mathscr{T}(X')$ such that the predecessors of any task in X' are in $\mathscr{T}(X' \cup \mu_{-f})$.*

Furthermore, let us define the maximal subset of feasible contracts in X at μ.

[7] For more information on the link between stable matchings and auctions, see [27].

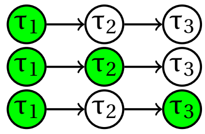

Figure 8.7: Three sets of contracts. Each set contains contracts for the tasks in green. The two first sets are feasible, the third one is not feasible.

Definition 118. *Given a firm $f \in \mathscr{F}$, a set of contracts $\mathrm{X} \subseteq \mathscr{X}_f$ and a matching μ, the set X is called maximal set of feasible contracts for f at μ if it is feasible and of maximal cardinality. The maximal set of feasible contracts for f in $\mathrm{X} \subseteq \mathscr{X}_f$ at μ is denoted $\mathrm{X}(\mu)$.*

A task $\tau \in \mathscr{T}$ is called *feasible* in X at μ if there is a feasible contract $x \in \mathrm{X}$ at μ such that $\mathscr{T}(x) = \tau$.

Example 119. *In Figure 8.7, we show two sets of feasible contracts and an unfeasible one for a simple scheduling. The contracts in the set are shown in green. In the third case, the set of contracts is not feasible since task τ_3 cannot be executed because there is no contract with τ_2 in the set.*

Example 120. *Consider the example shown Figure 8.5. The sets $\{x_{22}^2\}$ and $\{x_{22}^2, x_{22}^3\}$ are feasible for f_2 in $\{x_{22}^2, x_{22}^3\}$ at $\mu = \emptyset$. Furthermore, the set $\{x_{22}^2, x_{22}^3\}$ is the maximal set of feasible contracts for f_2 at $\mu = \emptyset$ and $\{x_{22}^3\}$ is not feasible because f_2 needs a contract to be assigned to task τ_2 to execute τ_3.*

The set of feasible tasks is $\mathscr{T}(\mathrm{X}(\mu))$. It is the set of tasks corresponding to contracts in the maximal set of feasible ones $\mathrm{X}(\mu)$.

Definition 121. *We define $\underline{\mathrm{X}}_f(\mu)$, the set of min-salary contracts in $\mathrm{X}_f(\mu)$ for each feasible task in $\mathscr{T}(\mathrm{X}_f(\mu))$.*

We assume profit-maximizing firms (see [27], pp.918 and references therein for more elements on profit maximization).

Definition 122. *The choice function of any firm $f \in \mathscr{F}$ is defined by,*

$$
\begin{aligned}
c_f(\mathrm{X}|\mu) &= \bigcup_{\tau \in \mathscr{T}(\mathrm{X}_f(\mu))} \operatorname*{arg\,min}_{\substack{x \in \mathrm{X}_f(\mu) \\ s.t. \tau(x) = \tau}} s(x) && (8.4) \\
&= \underline{\mathrm{X}}_f(\mu) && (8.5)
\end{aligned}
$$

As defined in Chapter 3 (Section 3.6), we have the firms' choice function C^{F} such that,

$$
\mathrm{C}^{\mathrm{F}}(\mathrm{X}|\mu) = \bigcup_{f \in \mathscr{F}} \underline{\mathrm{X}}_f(\mu) \tag{8.6}
$$

In this chapter, we consider for simplicity that firms prefer contracts with min-salary for a given task. The interpretation is that in the set of contracts X_f, firm f chooses a min-salary contract for each feasible task (i.e. each task in $\mathscr{T}(\mathrm{X}_f(\mu))$) at μ. However any strict preference order over contracts for a task can be used. Observe that we have,

$$
c_f(\mathrm{X}|\mu) = c_f(\mathrm{X}(\mu)|\mu) = c_f(\mathrm{X}(\mu_{-f})|\mu_{-f}) = c_f(\mathrm{X}(\mu_{-f})) \tag{8.7}
$$

The first equality comes from the fact that the firms only considers the maximal subset of feasible contracts in any set X at μ. The second equality comes from the fact that for each firm the feasible sets are defined as functions of the contracts signed by the other firms. The third equality comes from the fact that by definition of the choice functions, externalities in this problem are limited to their impact on the maximal set of feasible contracts $\mathrm{X}(\mu_{-f})$. Given the firms' choice functions, we define the firms' preorder $\succeq^{\mathrm{F}}$ (see Chapter 3, Section 3.6) as,

Definition 123. *The preorder $\succeq^{\mathrm{F}}$ is defined such that, for any $\mu, \mu' \subseteq \mathscr{X}$, $\mu' \succeq^{\mathrm{F}} \mu$ iff for any $f \in \mathscr{F}$,*

$$
\mathscr{T}(\mu'_{-f}) \cap \mathscr{N}^{-}(\mathscr{T}_f) \supseteq \mathscr{T}(\mu_{-f}) \cap \mathscr{N}^{-}(\mathscr{T}_f) \tag{8.8}
$$

In words, $\mu' \succeq^F \mu$ if for any firm $f \in \mathscr{F}$, the set of predecessors of f's tasks in $\mathscr{T}_{-f}$ at μ, denoted $\mathscr{T}(\mu_{-f}) \cap \mathscr{N}^-(\mathscr{T}_f)$, is included in the set of predecessors of f's tasks in $\mathscr{T}_{-f}$ at μ', denoted $\mathscr{T}(\mu'_{-f}) \cap \mathscr{N}^-(\mathscr{T}_f)$. In other words, $\mu' \succeq^F \mu$ if for any firm $f \in \mathscr{F}$ there are weakly more feasible tasks in $\mathscr{T}_f$ at μ' than at μ. Another interpretation is that for any firm the feasibility of its tasks is improved w.r.t. the firms' external constraints. We have the following result giving the consistency (see Chapter 3, Section 3.6) of $\succeq^F$ with C^F.

Proposition 124. *The preorder* $\succeq^F$ *is consistent with the choice function* C^F.

The proof is given in Appendix 8.9.1.

On the workers' side, we keep on assuming generic choice functions $\{c_w\}_{w \in \mathscr{W}}$ without particular form. Further properties such as substitutability and irrelevance of rejected contracts (see Chapter 3, Section 3.6 for the definitions) will be assumed later on. Let $\succeq^W$ denote the workers' preorder consistent with C^W.

8.3.2 Constrained Substitutability

In this section, we show that there exist pairwise stable matchings in the matching problem with contracts, externalities and scheduling constraints if the choice functions of the agents satisfy some conditions.

We now turn to substitutability and define the *constrained substitutability* property that, in addition to better market conditions, asks for a specific structure in the feasible sets w.r.t. feasible tasks and feasible min-salary contracts. The intuition is that a contract rejected by a firm in a set X at μ keep on being rejected in a superset X' at matching μ' (with more feasibility for f) if either the corresponding task is still not feasible or becomes feasible but the contract is not of min-salary w.r.t. this task (thus is not chosen).

Definition 125. *Choice function* C^F *satisfies constrained substitutability if for any* $X, X', \mu, \mu' \subseteq \mathscr{X}$, *such that*
C1. $X \subseteq X'$
C2. $\mu' \succeq^F \mu$
C3. $\forall f \in \mathscr{F}$, $\mathscr{T}^f_{\substack{X \to X' \\ \mu \to \mu'}} = \emptyset$ *or* $X_f(\mathscr{T}^f_{\substack{X \to X' \\ \mu \to \mu'}}) \not\subseteq \underline{X}'_f(\mu')$
then,

$$R^F(X'|\mu') \supseteq R^F(X|\mu) \tag{8.9}$$

where, $\mathscr{T}^f_{\substack{X \to X' \\ \mu \to \mu'}} = \left[\mathscr{T}(X_f) \setminus \mathscr{T}(X_f(\mu))\right] \cap \mathscr{T}(X'_f(\mu'))$.

The first condition is the well-known condition of inclusion of sets used to define classical substitutability (see [27]). The second condition asks for better market conditions as defined by Pycia and Yenmez in [1]. Condition C3 is an additional constraint introduced so as to guarantee that the sets X and X' and matchings μ and μ' do not exhibit complementarities from the set of feasible contracts $X_f(\mu)$ to $X'_f(\mu')$ that would make some contracts rejected in X at μ and chosen in X' at μ. The first part of condition C3, $\mathscr{T}^f_{\substack{X \to X' \\ \mu \to \mu'}} = \emptyset$, requires that no non-feasible tasks become feasible. Thus, no unfeasible contract (in X at μ) can be chosen in X' at μ' because it is maintained non-feasible in X' at μ'. The second part of condition C3, $X_f(\mathscr{T}^f_{\substack{X \to X' \\ \mu \to \mu'}}) \not\subseteq \underline{X}'_f(\mu')$, requires that if a task becomes feasible, any of the corresponding contract rejected in X at μ is not of min-salary in X' at μ'. This prevent from the rejected contracts to be chosen if they become feasible.

Given firms' choice functions as defined in (8.4), we have the following result,

Proposition 126. *The choice function* C^F *defined in 8.4 satisfies the constrained substitutability.*

The proof of the proposition in given in Appendix 8.9.2.

We now show that the firms' choice function C^F satisfies the Irrelevance of Rejected Contracts (IRC).

Proposition 127. *The firms' choice function* C^F *defined in 8.6 satisfies the irrelevance of rejected contracts.*

The proof of the proposition is given in Appendix 8.9.3.

Finally, we define function f, that abstracts the set-wise operations of an iteration of the modified deferred acceptance algorithm defined in [1] (see Chapter 3, Section 3.6). This function (and related properties) is the main component of the proof of existence of a pairwise stable matching in [1] and in this chapter. As in [1], consider function $f : 2^{\mathscr{X}} \times 2^{\mathscr{X}} \times 2^{\mathscr{X}} \times 2^{\mathscr{X}} \to 2^{\mathscr{X}} \times 2^{\mathscr{X}} \times 2^{\mathscr{X}} \times 2^{\mathscr{X}}$ such that for any $A^F, A^W, \mu^F, \mu^W \subseteq \mathscr{X}$

$$f(A^F, A^W, \mu^F, \mu^W) = (\mathscr{X} \setminus R^W(A^W|\mu^W), \mathscr{X} \setminus R^F(A^F|\mu^F), C^F(A^F|\mu^F), C^W(A^W|\mu^W)) \tag{8.10}$$

Let us define a constraint on function f requiring that the inputs and outputs of function f are constrained such that condition C3 of definition 125 is satisfied with X' and μ' as inputs of f on the firms side and X and μ as outputs of f on the firms' side. In other words, this condition requires that the inputs and outputs of function have a structure.

C4. Given any sets $A^F, A^W, \mu^F, \mu^W \subseteq \mathscr{X}$, the image $f(A^F, A^W, \mu^F, \mu^W) = (\tilde{A}^F, \tilde{A}^W, \tilde{\mu}^F, \tilde{\mu}^W)$ is such that $A^F, \tilde{A}^F, \mu^F, \tilde{\mu}^F$ satisfies

$$\mathscr{T}^{f}_{\substack{\tilde{A}^F \to A^F \\ \tilde{\mu}^F \to \mu^F}} = \emptyset \text{ or } \tilde{A}^F_f(\mathscr{T}^{f}_{\substack{\tilde{A}^F \to A^F \\ \tilde{\mu}^F \to \mu^F}}) \not\subseteq \underline{A}^F_f(\mu^F) \tag{8.11}$$

By definition of f, this condition implies that the choice agents' choice function must be such that the generated sets satisfy the condition.

8.3.3 Existence of Stable Matchings

Now we turn to the main results of this section and follow the same approach as Pycia and Yenmez in [1]. We show that their results hold in our problem as long as conditions C3 (see Definition 125) and C4 hold. First, let us define the preorder $\sqsubseteq$.

Definition 128. *The preorder* $\sqsubseteq$ *is defined as follows,*

$$(A^F, A^W, \mu^F, \mu^W) \sqsubseteq (\tilde{A}^F, \tilde{A}^W, \tilde{\mu}^F, \tilde{\mu}^W) \Leftrightarrow A^F \subseteq \tilde{A}^F, A^W \supseteq \tilde{A}^W, \mu^F \preceq^F \tilde{\mu}^F, \mu^W \succeq^W \tilde{\mu}^W \tag{8.12}$$

We have the following lemma,

Lemma 129. *Suppose that* C^W *satisfies substitutability. The function* f *is monotone increasing w.r.t. the preorder* $\sqsubseteq$ *if* $A^F, \mu^F, \tilde{A}^F, \tilde{\mu}^F$ *satisfy condition C3 of Definition 125 with* $X = A^F$, $X' = \tilde{A}^F$, $\mu = \mu^F$ *and* $\mu' = \tilde{\mu}^F$.

The proof is given in Appendix 8.9.4.

The following lemma (see [1]) shows that the reference matchings at fixed point of function f are the same for both sides and can be obtained as the intersection of proposals.

Lemma 130. *Let* (A^F, A^W, μ^F, μ^W) *be a fixed point of function* f. *Then* $A^F \cup A^W = \mathscr{X}$ *and*

$$\mu^F = \mu^W = A^F \cap A^W = C^F(A^F|\mu^F) = C^W(A^W|\mu^W) \tag{8.13}$$

Proof. The proof is the same as in [1], Lemma 3, pp.20. □

We now state one of the two main theorems of this section. Our proof is closed to the one given in [1] except that, because of the constrained substitutability, we need to take care about the fact that at any point the conditions for substitutability are satisfied, including C3 of Definition 125.

Theorem 131. *Suppose that the choice function* C^W *satisfies substitutability and the irrelevance of rejected contracts. Suppose that the firms choice functions* $\{c_f\}_{f\in\mathcal{F}}$ *are defined as in Definition 122.*

Then a matching μ *is stable if and only if there exists sets of contracts* $A^F, A^W \subseteq \mathcal{X}$ *s.t.* (A^F, A^W, μ, μ) *is a fixed point of function* f.

The proof is given in Appendix 8.9.5.

We now show the existence theorem stating that if C^W satisfies substitutability, irrelevance of rejected contracts and f satisfies the conditions for constrained substitutability at any point, then the algorithm (see Chapter 3, Section 3.6) defined by f converges to a stable matching.

Theorem 132. *Suppose that the choice function* C^W *satisfies substitutability and the irrelevance of rejected contracts. Suppose that the firms choice functions* $\{c_f\}_{f\in\mathcal{F}}$ *are defined as in Definition 122. Finally, suppose that* f *satisfies condition C4 at every iteration of the algorithm. Then, the algorithm terminates, its outcome is stable and*

$$\mu^F(T) = \mu^W(T) = A^F(T) \cap A^W(T) \tag{8.14}$$

The proof is given in Appendix 8.9.6

This result concludes this section. We have shown that by defining a new substitutability that asks for the choice functions of the agents to be such that at any point in the algorithm the conditions for substitutability on the firms' side to be satisfied, the algorithm converges to a pairwise stable matching. This solution induces conditions on the agent's choice functions indirectly given by requiring for condition C4 on the function f. In Appendix 8.10, we consider alternative stabilities that extend some of the existing ones. Nevertheless, this solution leads to a setting that looks hardly tractable from an analytic point of view. The analysis of the proposed stabilities in this problem is left as an open question.

In the next sections, we focus on alternative formulations of the crowdsourcing problem. Particularly, we use games in normal and extensive forms. Even though there is in this case, a priori, no equivalence of the stability and the equilibria of this non-cooperative games, these formulations show interesting properties both in interpretations and in connecting cooperative solutions used for two-sided markets to non-cooperative ones.

8.4 The Crowdsourcing Problem in Normal form

In this section, we consider an alternative formulation of the problem in the non-cooperative normal form. The aim is to explore the link between the classical non-cooperative game theory and the game-theoretic analysis of two-sided markets and see how the stabilities and conditions transform from one branch to the other. This unification was already engaged by Ackermann et a.l. in [2] in a similar way and by Roth in [31] in a somewhat different one where the non-cooperative strategic aspect are formalized in a game of stated preferences in order to manipulate the matching mechanism. In this case, the non-cooperative game combines with the matching mechanism, while in our case we model the matching mechanism as the result of the non-cooperative game and thus define a stability of the matching game as a non-cooperative equilibrium solution concept. Particularly, our transformation turns the two-sided market problem in a one-shot contract proposing game with firms as players and workers as receivers of these proposals, implicitly modeled as a response function Ψ of the market.

Let define a crowdsourcing problem in normal (or strategic form) $\Gamma = (\mathcal{N}, (\mathcal{S}_i)_{i\in\mathcal{N}}, (u_i)_{i\in\mathcal{N}})$, where $\mathcal{N} = \mathcal{F}$ is the set of players, $u_i : \mathcal{X} \to \mathbb{R}$ is the utility function of i from the set of matchings in $\mathbb{R}$ and $\mathcal{S}_i \subseteq 2^{\mathcal{X}_i}$ is i's set of pure strategies defined as the set of subsets of contracts with exactly one contract (including the null contract) per task in $\mathcal{T}_i$,

$$\mathcal{S}_i = \{X \subseteq \mathcal{X}_i | \forall \tau \in \mathcal{T}_i, \exists! x \in X \text{ s.t. } \mathcal{T}(x) = \tau\} \tag{8.15}$$

equivalently,

$$\mathcal{S}_i = \{X \subseteq \mathcal{X}_i | \forall \tau \in \mathcal{T}_i, |X(\tau)| = 1\} \tag{8.16}$$

We now define the worker' response function Ψ as the output of the market when being proposed a set of contracts by the firms.

Definition 133. *The workers' response function (or, mechanism), is the function* $\Psi : \underset{i\in\mathcal{N}}{\times} \mathcal{S}_i \to 2^{\mathcal{X}}$ *from the set of firms' strategy profiles (contract proposals) to the set of subset of contracts (set of matchings) induced by the choice function of the workers such that, given strategy profile* $s \in \mathcal{S}$, $\psi(s \in \mathcal{S})$ *is the limit of the sequence,*

$$\mu^{(0)} = \emptyset \tag{8.17}$$

$$\mu^{(1)} = \mathrm{C^W}(s|\mu^{(0)}) = \bigcup_{w\in\mathcal{W}} c_w(s|\mu^{(0)}) \tag{8.18}$$

$$\mu^{(2)} = \mathrm{C^W}(s|\mu^{(1)}) \bigcup_{w\in\mathcal{W}} c_w(s|\mu^{(1)}) \tag{8.19}$$

$$\vdots \tag{8.20}$$

$$\Psi(s) = \mu^{(k)} = \mathrm{C^W}(s|\mu^{(k-1)}) = \bigcup_{w\in\mathcal{W}} c_w(s|\mu^{(k-1)}) \tag{8.21}$$

For the rest of this section, we assume that the sequence converges in a finite number of steps. Thus, there exists a well-defined response of the workers to the emitted proposals. If not, then an appropriate solution may be to assume convergence to a subset of subsets of contracts or to the maximum w.r.t. an order relation such as the preorder $\succeq^{\mathrm{W}}$ considered in Section 8.3. As shown in [1], if the choice function $\mathrm{C^W}$ satisfies substitutability, then the sequence converges to a most preferred matching w.r.t. $\succeq^{\mathrm{W}}$.

Remark 134. *If the workers where the players of the game (i.e.* $\mathcal{N} = \mathcal{W}$*), we would similarly define the firms' response function* $\Psi : \underset{i\in\mathcal{N}}{\times} \mathcal{S}_i \to 2^{\mathcal{X}}$ *taking as input the set of contracts proposed by the workers and giving as output the firms' choices in this set.*

In this non-cooperative setting, we consider the well-known Nash equilibrium as solution concept

Definition 135. *Given a strategy profile* $s \in \mathcal{S}$*, the utility profile is* $(u_i(\Psi(s))_{i\in\mathcal{F}}$*. The strategy profile* s^* *is a Nash equilibrium if and only if,*

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*), \quad \forall s_i \in \mathcal{S}_i \tag{8.22}$$

In terms of the response function Ψ*, the strategy profile* s^* *is a Nash equilibrium if and only if for any firm* $i \in \mathcal{F}$,

$$u_i(\Psi(s_i^*, s_{-i}^*)) \geq u_i(\Psi(s_i, s_{-i}^*)), \quad \forall s_i \in \mathcal{S}_i \tag{8.23}$$

Thus, at a Nash equilibrium, no firm prefers unilaterally deviating by proposing an alternative set of contracts, given the (converging) response mechanism Ψ *of the workers.*

Remark 136. *Alternatively, if there are no utility functions in the model but only preferences* $(\succeq_f)_{f\in\mathcal{F}}$*, we have the following definition.*

Definition 137. *The strategy profile* s^* *is a Nash equilibrium if and only if for any firm* $i \in \mathcal{F}$,

$$\Psi(s_i^*, s_{-i}^*) \succeq_i \Psi(s_i, s_{-i}^*), \quad \forall s_i \in \mathcal{S}_i \tag{8.24}$$

8.4.1 The Marriage Problem and Player-specific Singleton Congestion Game with Priorities

We now show that the non-cooperative transformation of the crowdsourcing problem generalizes some interesting congestion games defined by Ackermann et a.l. in [2] to unify the non-cooperative theory and the two-sided market problems.

Consider the particular case of the marriage problem obtained from the crowdsourcing problem by removing scheduling constraints, constraining to one-to-one matchings (equivalently, restricting each firm to have a single task and a single contract with each worker) and removing externalities on both sides. Taking $\mathcal{M} = \mathcal{F}$ and $\mathcal{S}_m = \mathcal{W} \cup \{m\}$ for any $m \in \mathcal{M}$, a strategy profile s^* (consisting of a set of proposals to the women, one proposal per man) is a Nash equilibrium if and only if for any firm $m \in \mathcal{M}$,

$$\Psi(s^*_m, s^*_{-m}) \succeq_f \Psi(s_m, s^*_{-m}), \quad \forall s_m \in \mathcal{S}_m \tag{8.25}$$

where $\Psi(s)$ is defined such that $\Psi(s) = \bigcup_{w \in \mathcal{W}} c_w(s)$ and $c_w(s)$ is defined such that, for any man m such that $s(m) = w$, $c_w(s) \succ_w m$. As shown in [2], this game falls in the class of player-specific singleton congestion games with priorities (by taking strict priorities). In such game, as in any congestion game, we define a set of players $\mathcal{M}$ and a set of resources $\mathcal{W}$. Each player has a specific cost function such that when m is matched to w, he incurs a cost $c_m(w)$. Furthermore, resources emit priorities over the players. These are given by the strict preferences $(\succ_w)_{w \in \mathcal{W}}$ or weak preferences with eventual ties $(\succeq_w)_{w \in \mathcal{W}}$.

Definition 138. *[2] A player-specific singleton congestion game with priorities is a congestion game (with m players and n resources) where the players' strategies are reduced to singletons (they can choose a single resource), have individual resource-specific cost function (e.g. $c_{i,r}$ for player i with resource r) depending only in the number of other players associated to the resource and resources rank the players, so that given a strategy, a player choosing a resource is associated to i if it has maximal rank among those having chosen this resource.*

In [2], if the preferences are strict, we have the following definition of stability,

Definition 139. *[2] A strategy profile s^* is a stable matching if none of the players can unilaterally increase her payoff by changing her proposal given the proposals of the other players.*
That is, for each player i who is assigned to a resource r_i, each resource r from which she receives a higher payoff than from r_i is matched to a player whom r prefers over i.

Observe that, even though s^* in the previous definition is called *matching*, the effective matching is obtained by acceptance or rejection of the proposals s by the resources.

In such game, we have the following existence theorem,

Theorem 140 ([2])**.** *Every player-specific singleton congestion game with priorities possesses a pure Nash equilibrium that can be computed in polynomial time by* $O(m^2.n^3)$ *strategy changes.*

The result is valid for both strict preferences for the resources (which leads to the previously described marriage problem) or non-strict allowing for ties (which leads to many-to-one matchings because any man with maximum rank in the set of men proposing to a given woman is accepted).

In the player-specific singleton congestion game with priorities as proposed in [2], particularly in, a resource (here, a woman) accepts all the most-preferred men among proposers (the resource must be indifferent among them, i.e. they all have the same priority).

Our crowdsourcing setting is more general, in fact the firms can be matched to many workers for many tasks, the workers can have preferences over groups more complex than those induced by the preferences individuals. Particularly, we allow for complementarities and externalities. The firms' utilities (if not implicitly given by preferences and choice functions) may not be the sum of specific congestion costs (functions of the numbers of players associated to the resource) over the matched resources. Finally, we allow for scheduling constraints and externalities on the firms' side.

In Appendix 8.11, we consider a more general class of congestion game with priorities called player-specific matroid congestion games with priorities. These games are interesting because they generalize the previous player-specific congestion game with priorities that have been shown to be linked to the marriage problem and allow to asses some of the many-to-one problems. Furthermore, by definition of our non-cooperative transformation and the firms' strategy spaces, our non-cooperative game is linked to such congestion games by some matroidal properties (that are not studied in this document, but are left as open questions).

8.4.2 Results

As a first solution, we use Nash's well-known theorem on the existence of mixed Nash equilibriums in finite games.

Theorem 141 (Nash, [30]). *Every finite game (finitely many players, each with a finite set of strategies) has a mixed strategy Nash equilibrium.*

Thus there exists a mixed Nash equilibrium in the crowdsourcing problem in normal form. We have defined $\mathscr{S}_i$ as i's set of pure strategies and each strategy $s_i \in \mathscr{S}_i$ is a set of contracts in $\mathscr{X}_i$, one for each task. A randomized strategy $\delta_i \in \Delta(\mathscr{S}_i)$ assigns a probability distribution to each pure strategy. We consider two interpretations of randomized strategies:

- **Probabilistic interpretation:** each subset of contracts in $s_i \in \mathscr{S}_i$ has a probability $\delta_i(s_i)$ to be played and each contract $x \in \mathscr{X}_i$ is executed with probability $\sum_{s_i \in \mathscr{S}_i | x \in s_i} \delta_i(s_i)$.
- **Deterministic interpretation:** each subset of contracts in $\mathscr{S}_i$ is played for a fraction $\delta_i(s_i)$ of time and each contract $x \in \mathscr{X}_i$ is executed for a fraction $\sum_{s_i \in \mathscr{S}_i | x \in s_i} \delta_i(s_i)$ of time.

By definition, at mixed Nash equilibrium δ^*, we have,

$$u_i(\Psi(\delta_i^*, \delta_{-i}^*)) \geq u_i(\Psi(\delta_i, \delta_{-i}^*)), \quad \forall \delta_i \in \Delta(\mathscr{S}_i) \tag{8.26}$$

where,

$$u_i(\Psi(\delta_i, \delta_{-i})) = \sum_{s \in \mathscr{S}} \delta(s) u_i(\Psi(s)) = \sum_{s \in \mathscr{S}} \prod_{i \in \mathscr{N}} \delta_i(s_i) u_i(\Psi(s)) \tag{8.27}$$

According to the deterministic interpretation, for any task τ with a contract in the support of $\Psi(\delta^*)$ (response of the workers to the mixed strategy obtained as the convex combination of the response to the pure strategies) each contract x such that $\mathscr{T}(x) = \tau$ is executed for a fraction of time by $\mathscr{W}(x)$ according to the terms specified in x. We call fractional matching such distribution over contracts. The notion of fractional matching is already known in matching games and has been particularly studied in the framework of the analysis of the stable matching polytope (see [31] and references therein). A fractionally stable matching is obtained as a convex combination of stable matchings (whatever probablilistic or deterministic). In our setting, the distribution over strategy profiles induces a distribution over workers' response and thus over matchings as sets of contracts chosen by the workers in the set of those proposed by the firms. To the best of our knowledge, this is the first joint consideration of the notion of fractional matching and matching with contracts (a fortiori with externalities).

If the contracts are uniquely identified by a firm, a worker, a task and a salary, then the mixed distribution gives for τ the profile of payments to be distributed among the workers with contracts and obtained by giving, for each contract x w.r.t. τ, the worker $\mathscr{W}(x)$ a salary $s(x) \times \sum_{s_i \in \mathscr{S}_i | x \in s_i} \delta_i(s_i)$. If the contracts contain more than a salary, then there is an equivalent time-sharing over the conditions of the contracts and the mixed distribution gives the profile of conditions for the execution of τ.

At equilibrium, no firm has the incentive to chose an alternative mixed distribution to change the workers' response (expected terms of the executed contracts, or profile of executions of contracts) given other firms' profiles and workers response function Ψ. In other words, by choosing another mixed profile, no firm $f \in \mathscr{F}$ can generate sequences of choices on workers' side converging to a response $\Psi(\delta)$ that would strictly improve the expected utility of the resulting fractional matching for f.

Furthermore, observe that the existence of mixed equilibrium in the non-cooperative transformation of the two-sided crowdsourcing problem does not rely on a particular assumption on the firms' utilities (or preferences in case they are the primitives of the model). We have only assumed convergence of the workers' choice function C^W.

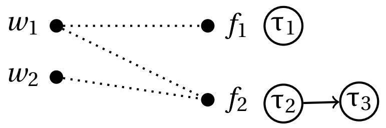

Figure 8.8: A matching market with contracts and non-cross-constrained scheduling.

We now turn to pure strategies and show that there may not exists a Nash equilibrium in pure strategies $\mathscr{S} = \prod_{i \in \mathscr{N}} \mathscr{S}_i$. To show this, simply consider the example shown in Figure 8.8.

Example 142. *In this example, consider that the set of contracts* $\mathscr{X} = \{x_{11}^{1,2\$}, x_{11}^{1,4\$}, x_{21}^{3,1\$}, x_{21}^{3,3\$}, x_{22}^{2,1\$}, x_{22}^{2,2\$}\}$ *where any contract is of the form* (w_k, f_l, τ_m, s)*:* $x_{kl}^{m,s\$}$ *and* s *is the salary. Furthermore, assume the following preferences*

- $P_{w_1}: x_{11}^{1,4\$} \succ_{w_1} x_{21}^{3,3\$} \succ_{w_1} x_{21}^{3,1\$} \succ_{w_1} \emptyset$
- $P_{w_2}: x_{22}^{2,2\$} \succ_{w_2} x_{22}^{2,1\$} \succ_{w_2} \emptyset$
- $P_{f_1}: x_{11}^{1,2\$} \succ_{f_1} x_{11}^{1,4\$} \succ_{f_2} \emptyset$
- $P_{f_2}: \{x_{21}^{1,1\$}, x_{22}^{2,2\$}\} \succ_{f_2} \{x_{21}^{1,3\$}, x_{22}^{2,1\$}\} \succ_{f_2} x_{22}^{2,2\$} \succ_{f_2} x_{22}^{2,1\$} \succ_{f_2} \emptyset$

To justify $\{x_{22}^{2,2\$}\} \succ_{f_2} \{x_{22}^{2,1\$}\}$*, assume that the productivity of* w_2 *when being paid* 2\$ *is higher than at* 1\$*. The firm prefers, because the tradeoff salary-productivity is good. As another example, at* 4\$*, the tradeoff is bad for* f_1*, thus* $x_{11}^{1,2\$} \succ_{f_1} x_{11}^{1,4\$}$

In the Table 8.8, we give the matching resulting from the workers' choice at any pure strategy profile (firm f_1 *in columns and* f_2 *in rows). In Table 8.9, we show the corresponding utilities. Observe that given any pure strategy (i.e. pair row-column), a player has an incentive to deviate to strictly increase her payoff, thus there is no Nash equilibrium.*

	$x_{11}^{1,2\$}$	$x_{11}^{1,4\$}$
$x_{21}^{3,1\$}, x_{22}^{2,2\$}$	$\mu = \{x_{22}^{2,2\$}, x_{11}^{1,2\$}\}$, (A)	$\mu = \{x_{22}^{2,2\$}, x_{11}^{1,4\$}\}$, (B)
$x_{21}^{3,3\$}, x_{22}^{2,1\$}$	$\mu = \{x_{21}^{3,3\$}, x_{22}^{2,2\$}\}$, (C)	$\mu = \{x_{22}^{2,1\$}, x_{11}^{1,4\$}\}$, (D)
$x_{21}^{3,1\$}$, null	$\mu = \{x_{11}^{1,2\$}\}$, (E)	$\mu = \{x_{11}^{1,4\$}\}$, (F)
$x_{21}^{3,3\$}$, null	$\mu = \{x_{21}^{3,3\$}\}$, (G)	$\mu = \{x_{11}^{1,4\$}\}$, (H)
null, $x_{22}^{2,1\$}$	$\mu = \{x_{22}^{2,1\$}, x_{11}^{1,2\$}\}$, (I)	$\mu = \{x_{22}^{2,1\$}, x_{11}^{1,4\$}\}$, (J)
null, $x_{22}^{2,2\$}$	$\mu = \{x_{22}^{2,2\$}, x_{11}^{1,2\$}\}$, (K)	$\mu = \{x_{22}^{2,2\$}, x_{11}^{1,4\$}\}$, (L)

Table 8.8: Matchings in the normal form representation with workers' response function Ψ.

	$x_{11}^{1,2\$}$	$x_{11}^{1,4\$}$
$x_{21}^{3,1\$}, x_{22}^{2,2\$}$	(2,2), (A)	(2,1), (B)
$x_{21}^{3,3\$}, x_{22}^{2,1\$}$	(3,0), (C)	(1,1), (D)
$x_{21}^{3,1\$}$, null	(0,2), (E)	(0,1), (F)
$x_{21}^{3,3\$}$, null	$(-\infty, 0)$, (G)	(0,1), (H)
null, $x_{22}^{2,1\$}$	(1,2), (I)	(1,1), (J)
null, $x_{22}^{2,2\$}$	(2,2), (K)	(2,1), (L)

Table 8.9: Utilities of the matchings (see Table 8.8) in the normal form representation.

In the next section, we consider non-cooperative games in extensive form. First we introduce the formalism and give a well-known existence result in perfect information. Then we study the transformation of the original crowdsourcing problem (and the non-cooperative transformation studied in this section) in a game in extensive form with perfect information.

8.5 The Crowdsourcing Problem in Extensive Form

In this section, we define the crowdsourcing problem with contracts and externalities as a non-cooperative game in extensive form. Basically, the idea is to introduce a sequential decision taking in the previous normal form. The motivation for such transformation is that the unstability in the motivating examples (see Section 8.2.3 and Section 8.2.3) arise from self-penalizing actions for the decision-takers. Particularly, consider matching $\{x_{11}^1, x_{22}^2\}$. In such case, f_2 proposes the contract x_{22}^3 to w_2. Such proposal leads to a self-penalization of f_2 because w_2 accepts x_{22}^3 but rejects x_{22}^2 giving a non-feasible (unrational) matching $\{x_{11}^1, x_{22}^3\}$. Avoiding such self-penalizations have motivated the definition of alternative stabilities such as considered in [28] and in Appendix 8.10. On the non-cooperative side, the concept of *Subgame Perfect Nash Equilibrium* has been considered to get rid off non-credible threats and actions on the path of play. We aim at exploiting this in the crowdsourcing problem. The order of play depends on priorities based on the scheduling graph. First, we give a few basic notions of games in extensive form. Then, we define the crowdsourcing matching problem with externalities and scheduling constraints in extensive form and with perfect information. This game is not equivalent to the original one but exhibit interesting properties in terms of existence of equilibrium.

8.5.1 Games in Extensive Form

Defining a game in extensive form requires a more complete description than in normal or strategic form. In fact, the extensive form introduces the notion of sequential decision taking, the agents play in a given order. Depending on their level of information, they may or not be aware of the previous choices (or path of play). Formally, a game in extensive form is defined as follows

Definition 143 (Game in Extensive Form)**.** $\Gamma^e = (\mathcal{N}, \mathcal{V}, \nu_{root}, \alpha, \{\mathcal{I}_i\}_{i\in\mathcal{N}}, \{u_i\}_{i\in\mathcal{N}})$*, where:*

- $\mathcal{N}$*: set of players*
- $(\mathcal{V}, \nu_{root}, \alpha)$ *is a tree*
  - $\mathcal{V}$*: the set of nodes*
  - $\nu_{root} \in \mathcal{V}$*: the root*
  - α*: the predecessor function mapping each node to its predecessor from* $\mathcal{V} \setminus \{\nu_{root}\}$
- $\{\mathcal{I}_i\}_{i\in\mathcal{N}}$ *is a partitioning of the nodes of the tree into subsets called information sets. We denote* $\mathcal{I}_i$ *the set of information sets of player* i*. Given an information set* $v \in \mathcal{I}_i$*, if the path of play reaches this set then player* i *is unable to distinguish which of the nodes in* v *has been reached*
- $\mathcal{Y}_s$*: set of nodes belonging to player* i *with information state* s

In Figure 8.9, we show the basic setting of a game in extensive form. The players play in a given order and choose an action per decision node. Depending on the game, they may or not observe past moves. When a player does not observe some past moves, then he cannot uniquely identify the decision node. A set of nodes that cannot be distinguished from each others is called information sets. In Figure 8.9, each information set is shown as a yellow square, the predecessor function is shown by the red arrow (maps any node to its predecessor) and the terminal nodes are the leaves of the tree. They map each path of play to an outcome (whatever a matching or a utility profile).

When using the extensive form, one must also appropriately re-define the strategy that must now embed the fact that the path of play may lead to distinct nodes and that the players may behave differently depending on the node reached by the sequential decision-taking process. A strategy must explicitly give the set of decisions that would be taken by any player in the game.

Definition 144. *Denote* $\mathcal{I}_i$ *the set of information sets of* i*.*

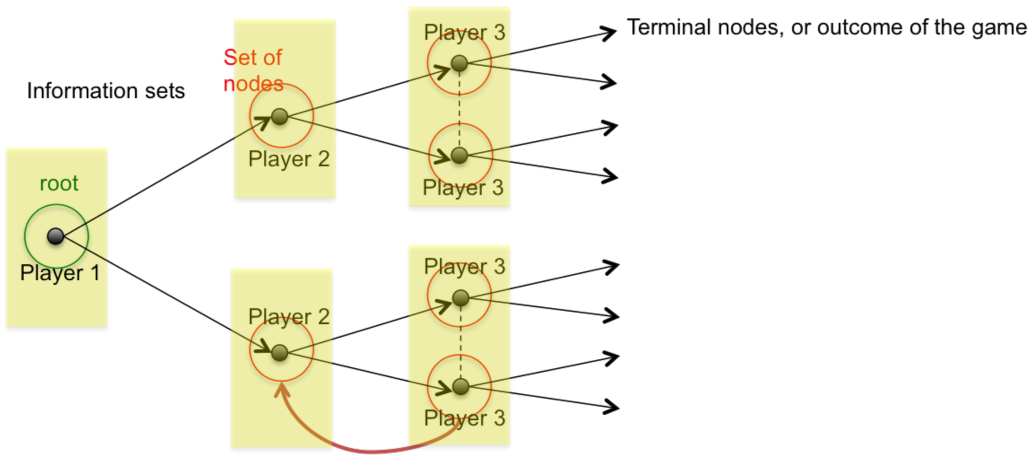

Figure 8.9: Game in extensive form. The green circle shows the root node, the red ones other decision-nodes in the tree. The red arrow shows the application of the predecessor function on one of player 3's node. Finally, yellow rectangles show information sets.

- *For each $v \in \mathcal{I}_i$, we define $\mathcal{D}_v$ the set of moves that i could play if the game entered v.*
- *A strategy for a player is a mapping from the information sets into moves. A pure strategy maps every information state to a move.*
- *Set of pure strategies of player i: $\mathcal{S}_i = \times_{v \in \mathcal{I}_i} \mathcal{D}_v$.*
- *Set of mixed strategies of player i: $\Delta(\mathcal{S}_i)$.*
- *Set of mixed strategy profile: $\times_{i \in \mathcal{N}} \Delta(\mathcal{S}_i)$.*
- *Set of behavioral strategies of player i: $\times_{v \in \mathcal{I}_i} .\Delta(\mathcal{D}_v)$*
  - *A behavioral strategy: $\sigma_i \in \times_{v \in \mathcal{I}_i} .\Delta(\mathcal{D}_v)$.*
- *Set of behavioral strategy profiles: $\times_{i \in \mathcal{N}} \times_{v \in \mathcal{I}_i} \Delta(\mathcal{D}_v)$.*

In Figure 8.10, we show the set of moves of player 2. The set D_{v1} (respectively D_{v2}) is the set of moves of player 2 at node $v1$ (respectively $v2$). In Figure 8.11, we show a pure strategy profile among any node to an action. In Figure 8.12, we show a mixed strategy profile for player 2. Each pure strategy (blue or red) is defined by a action at any decision node of player 2. The mixed strategy is defined by the probability distribution $(p, 1-p)$ over the pure strategies. The blue strategies played with probability p, the red one with probability $1-p$. In Figure 8.13, we show an example of a mixed behavioral strategy for player 2. At any decision node, there is a probability distribution over actions. At node $v1$, the blue action is played with probability p, the red one with probability $1-p$. At node $v2$, the black action is played with probability p', the green one with probability $1-p'$.

As already stated, the amount of information about past moves in a game in extensive form may vary from one game to another. As an example, in chess as in tic-tac-toe the players play sequentially and observe all past moves (history) so that they know the history of the game at any time. In such a case, the game is said to be in perfect information and the information sets are reduced to singletons. No player can be misleading about past moves and a strategy σ_i of player i is defined over the set of i's nodes and maps any of these node to a successor. A strategy profile $\sigma = (\sigma_1, \ldots, \sigma_N)$ inductively induces a path of play, thus an outcome (final node). For the rest of this section, we focus on perfect information (see [30] or [32] for further details on perfect and imperfect information).

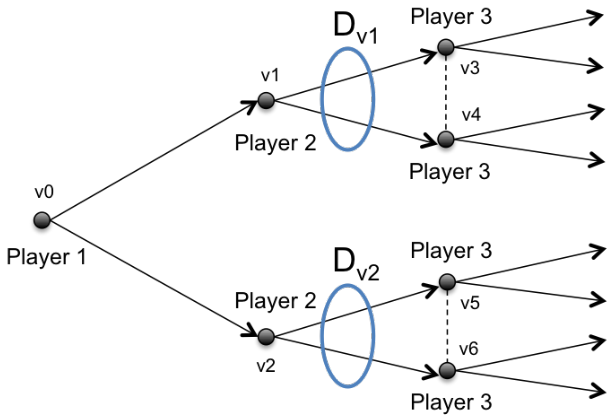

Figure 8.10: Game in extensive form. Blue circles show the set of moves at player 2's decision nodes.

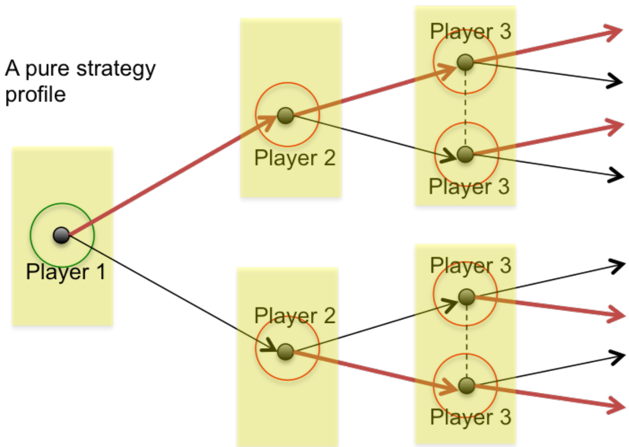

Figure 8.11: Game in extensive form, a pure strategy profile. Each node is mapped to an action.

The link between the extensive form and the normal form is defined by an application mapping the complete strategy profile of the tree to the path-of-play, thus to the termination node and thus the outcome of the game. Formally,

Definition 145 ([32], pp.67)**.** *The application* F *mapping any strategy profile* σ *to an outcome in the set of final nodes is called reduction in normal form or strategic form.*

Definition 146 ([33])**.** *Given a node* v *in the game tree and fixing* $s(v')$ *for all* v' *below*[8] v*, we can define an induced normal-form game in node* v *by* s *as the game with strategy space* $\times S_i(v)$ *such that the utility for player* i *by playing* $\tilde{s}(v), \tilde{s}_i(v) \in S_i(v)$ *is* $u_i^v(s_i, s_{-i})$ *where player* i *plays* $\tilde{s}_i(v)$ *in node* v *and according to* $s_i(v')$ *in all nodes* v' *below* v.

Assuming that the sequential decision-taking has reached a node in the tree, the subtree induced by this node defines a smaller game in extensive form, called subgame. Formally,

Definition 147 ([33])**.** *A subgame of sequential game is the game resulting after fixing some initial history of play, (i.e., starting the game from a node* v *of the game tree). Let* $u_i^v(s)$ *denote the utility that* i *gets from playing* s *starting from node* v *in the tree.*

Based on this notion of subgames, game-theorists have refined the Nash equilibrium to another solution concept, called Subgame Perfect Nash Equilibrium, asking for the strategies to form an equilibrium in any subgame, thus guaranteeing the credibility of any threat in the strategy profile.

[8] v' is a successor of v in the extensive form tree.

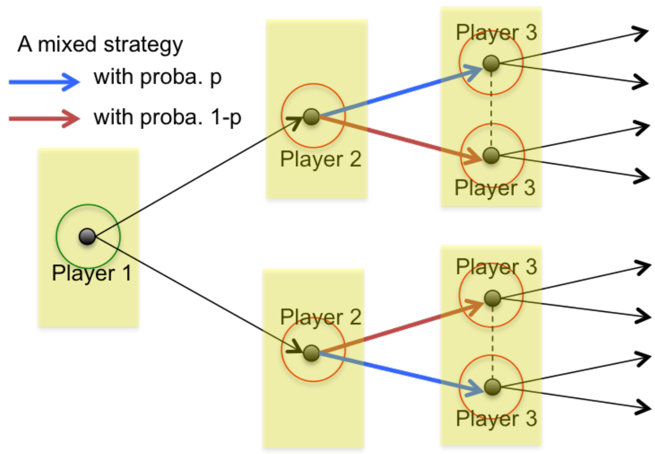

Figure 8.12: Game in extensive form, a mixed strategy for player 2. Each pure strategy is assigned a probability.

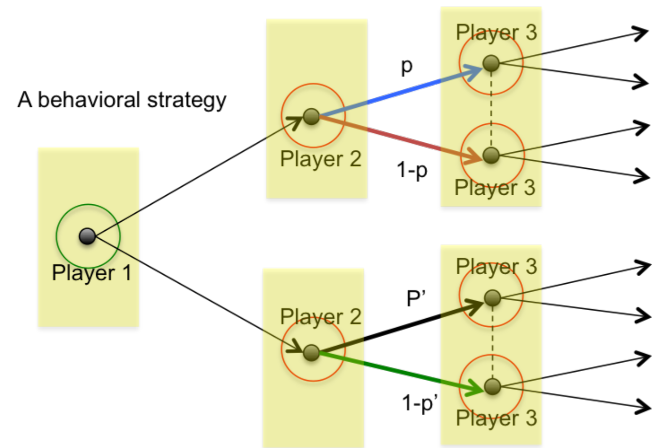

Figure 8.13: Game in extensive form, a mixed behavioral strategy for player 2. Each node is assigned a probability distribution over its set of moves.

Definition 148 ([32], pp.71)**.** *A strategy profile* σ *is an S-perfect equilibrium, or Subgame Perfect Nash Equilibrium (SPNE), if for any non-terminal node* v*,* $\sigma(p)$ *induced by* σ *in the subgame* $G(p)$ *is a Nash equilibrium in this subgame*

In other words ([33]), we say that a profile s is a SPNE if it is a Nash equilibrium for each subgame of the game, that is, for all nodes v

$$\forall s'_i : u_i^v(s_i, s_{-i}) \geq u_i^v(s'_i, s_{-i}) \tag{8.28}$$

It is known that set of subgame perfect Nash equilibriums is subset of the set of Nash equilibriums. We have the following important existence theorem guaranteeing the non emptiness of the set of SPNEs in any finite game in extensive form with perfect information.

Theorem 149 ([32])**.** *Any finite game in extensive form admits an S-perfect equilibrium in pure strategies.*

Furthermore, such equilibriums can be obtained by backward induction in finite games and Kuhn's Theorem states that s is a subgame perfect equilibrium iff $s(v)$ is a Nash equilibrium on the induced normal-form game in node v for all v.

8.5.2 Formulation as a Game Extensive Form

We now show that a solution to the crowdsourcing problem with contracts and scheduling can be found by using an extensive form with perfect information and the subgame perfect Nash equilib-

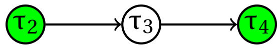

Figure 8.14: Non-feasible matching.

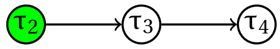

Figure 8.15: Feasible matching.

rium solution concept (see the previous section).

Consider the original crowdsourcing matching problem. If a matching μ is not feasible (i.e. violates the scheduling constraints of at least one firm), then the firm with violated constraints strictly prefers rejecting the contracts in μ with non-feasible tasks. These tasks may be considered as individually discarded from the left to the right of the scheduling graph of the firm.

We define the task-agent representation as a simple transformation of the matching game such that each task is a player $\mathcal{N} = \mathcal{T}$ and the action space of task i is $\mathcal{A}_i = \mathcal{X}_{\tau_i} \cup \emptyset$, where $\emptyset$ is the null contract. In the task-agent representation, the chosen contracts are proposed by the tasks to the workers. This is similar to the the previously studied normal form where each firm proposed a set of contracts to the workers.

We define the crowdsourcing problem in extensive form with perfect information as,

- $\Gamma = (\mathcal{N}, \mathcal{V}, \nu_{\text{root}}, \alpha, \{\mathcal{I}_i\}_{i\in\mathcal{N}}, \{u_i\}_{i\in\mathcal{N}})$
- $\mathcal{N} = \mathcal{T}$
- Tree $(\mathcal{V}, \nu_{\text{root}}, \alpha)$
- Information sets are reduced to the singletons $\mathcal{V}$
- Outcomes: at a termination node, each task has the utility of its firm at the resulting matching

Observe that all the tasks $\mathcal{T}(\mathcal{X}_f)$ of any firm $f \in \mathcal{F}$ have identical utilities and will play to maximize the utility of the firm.

To make the definition complete, we define the predecessor function of the game. Consider the graph of scheduling constraints $\mathcal{G} = (\mathcal{T}, \mathbf{A})$. Let $\Pi = \{\pi(1), \ldots, \pi(m)\}$ denote a partition of the players (tasks or nodes in $\mathcal{G}$) into subsets. The subset $\pi(j)$ contains the set of nodes of $\mathcal{G}$ that are at distance j from the root. The distance of a node to the virtual root is defined as the length of the longest path from the virtual root to the node. Let $\beta(\pi(i))$ be a permutation of the nodes in $\pi(i)$. The players play in the following order:

- From $\pi(1)$ to $\pi(m)$
- In any $\pi(j) \in \Pi$, according to the order defined by $\beta(\pi(j))$

Observe that the game is such that each player plays only once.

By Theorem 149, there exists a SPNE (thus a NE) in the defined extensive form game (that is also a NE of its normal form representation). At equilibrium, no task has the incentive to propose another contract at any node given the strategy profile and the workers response function. By definition, given the strategy profile and on the path of play, the deviation of a task to another contract would generate a sequence of acceptance and rejections (by the workers) of the contracts defined by the new path of play that would converge to the workers' response defined by Ψ. In this setting, each node (actually the corresponding task) anticipates the deviations of the other nodes prescribed by the strategy (at any node the playing task anticipates the deviations of its successors). At SPNE, no decision-node in the tree (i.e. task $\tau \in \mathcal{T}$) can strictly increase its utility (thus firm $\mathcal{F}(\tau)$'s utility) by proposing the workers another contract than the one prescribed by the strategy, given the strategy at other nodes. In other words, at any node v there is no deviation (i.e. alternative contract proposed by the task at v, particularly on the path of play) that would benefit the node given the strategy of the successors. This anticipation looks similar to the far-sighted behaviors considered in classical matching games (see [1] and references therein). The particularity is that when proposing a contract at a node, a task anticipates only the impact on the decisions in the induced subgame, i.e. of its successors in the induced subtree.

Example 150. *We now consider the example of Section 8.2.3. In Figure 8.16, we show a transformation of the original problem as a non-cooperative game in extensive form. To obtain this transformation, we use the task-agent representation introduced in Section 8.2.3. By definition, the transformation is such that each task is a player. At any node, the task proposes a contract in the set of contracts related to it. Given a path, the outcome of the game is obtained as the result of the contracts along the path to the workers. The corresponding matching is given by the workers' the response function Ψ with input the set of contracts defined by the path. The utility of each task at any outcome (leaf of the tree) is the utility of its firm of the matching resulting from the proposal of the contracts on the path from the root to the leaf. In Figure 8.17, we show an example of a pure strategy. Each branch of the tree is defined by a contract. In Figure 8.19, we show (in red) the SPNE of the game. This SPNE corresponds to the expected outcome $\{x_{11}^1, x_{22}^2\}$ because the non-credible action x_{22}^3 cannot be played by rationality and polarization of the interests of τ_2 and τ_3. If player τ_3 had play x_{22}^3 instead of the null contract on the path of play, the result would have been a matching $\mu = \{x_{11}^1, x_{22}^3\}$ which is not feasible ($-\infty$ to both τ_2 and τ_3). This is a self-penalization, similar to non-credible threat because at this point the agent's best response to the others' strategy is not this action. In this case*

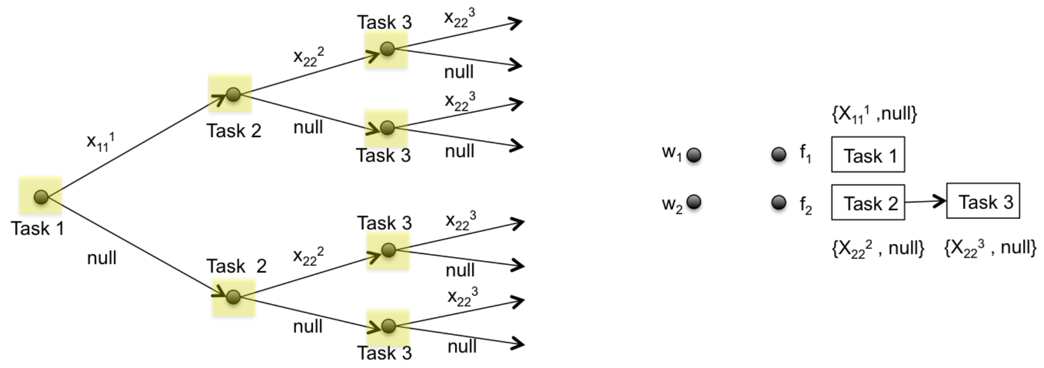

Figure 8.16: Crowdsourcing market in extensive form.

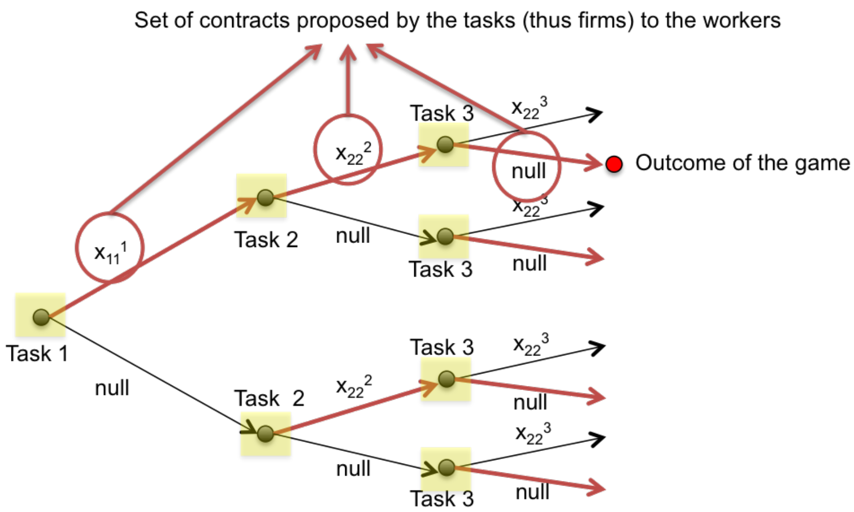

Figure 8.17: Extensive Crowdsourcing: contracts. Each branch of the tree is assigned a contract. At every node, the set of branches corresponds to the set of contracts for the task corresponding to the node.

As shown in Example 150, the proposed formulation of the crowdsourcing problem as a non-cooperative game in extensive form leads to interesting equilibria, both in terms of existence and decision-taking properties. In fact, SPNEs are guaranteed to exist in our setting and naturally avoid non-rational decisions (or non-credible choices) that would lead the decision-taker to a self-penalization in the outcome of the game. This is aligned with the stabilities proposed in Appendix

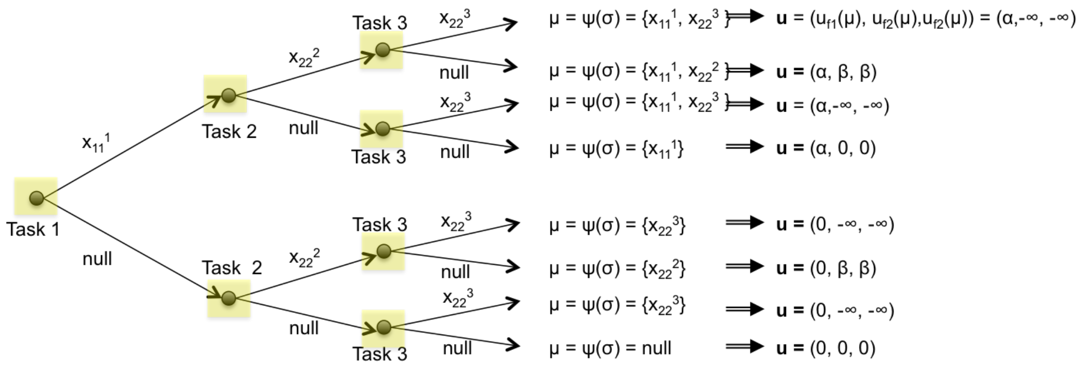

Figure 8.18: Extensive Crowdsourcing: utilities. Every outcome is defined by matching (thus utilities) resulting from the proposals of contracts defined by the path from the root to the leaf and from the workers' response function Ψ.

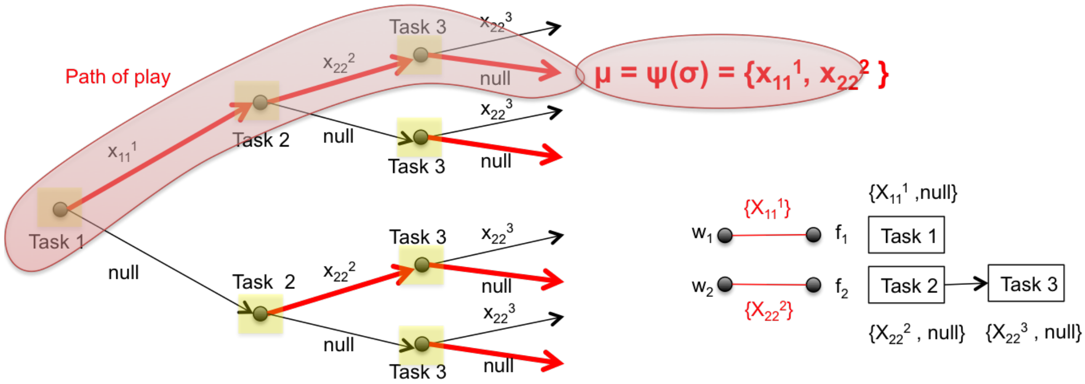

Figure 8.19: Extensive Crowdsourcing: SPNE. In red, we show the SPNE strategy. The path of play gives the set of proposed contracts and the outcome the matching.

8.10 that have been designed so as to avoid self-penalizing choices and incredible deviations that would lead to non-persistent matchings (see [28] for more details on the account of non-credible deviations in stability solution concepts). Nevertheless, by definition an SPNE is defined as a strategy profile mapping the nodes of the tree to actions. When deviating, the players anticipate the reaction of others. Such anticipation gets rid of the *myopic* nature of the agents as assumed by *short-sighted* choice functions as considered in Section 8.3 and defined in Chapter 3, Section 3.6 (see [1], [28] and references therein). As already observed, in the definitions of alternative stabilities (see 8.10) we have taken this into account. There is a priori no equivalence between the stabilities considered in this chapter and SPNEs of the crowdsourcing problem in extensive form. Nevertheless, it would be interesting to see in further details how such solution concepts interconnect with each others and wether they should be considered more as complements or substitutes.

8.6 Conclusion

In this chapter, we have analyzed a crowdsourcing system allowing for firms' intra and inter-scheduling constraints using the theory of stable matchings, particularly matching with contracts and externalities. Nevertheless, as shown there is no guarantee of existence of a stable matching defined as a set of contracts between the firms and the workers. In fact, the scheduling constraints induce that the firms choice functions do not satisfy the substitutability condition. We have defined a constrained substitutability, generalizing the well-known substitutability condition to constrained settings, and shown sufficient conditions for the existence of stable matchings. Finally, we have considered two transformations of the original problem (without equivalence) to the non-cooperative formulation: the normal and extensive forms. This allows to benefit from some fundamental and well-known existence theorems in non-cooperative games and not consider the substitutability condition. The link between cooperative stable matchings and non-cooperative formulations and solution concepts is to be explored in further details and remains as an open question.

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8.8 Appendix: Insufficiency of Existing Solutions

In this appendix, we compare the problem of many-to-many matching game with contracts and scheduling constraints to those assessed in existing works and give the reasons for their insufficiency.

- Sasaki and Toda's [20], *Matching with Externalities*:

  This work solves the problem of stability in the marriage problem with externalities. The general existence of stable matchings is guaranteed only for a new stability. The results are limited to the one-to-one matchings whereas the problem we consider is a many-to-many matching problem.

- Hatfield and Milgrom's [27], *Matching with Contracts*:

  This model does not take into account externalities, i.e. the current matching does not affect the choice of the players. Furthermore, they consider many-to-one matching with contracts. A doctor can be hired by a single firm. By definition, the problem we want to assess deals with many-to-many matchings (particularly: any worker can both work for a firm on many tasks and for many firms) with contracts and externalities. This model of matching with contracts cannot be used.

- Bando's [25][26], *Matching with Externalities*:

  This work studies a many-to-one matching market with externalities only on the firm's side and due to hired workers only. The model is formalized in terms of choice functions on sets of players and not contracts. A new stability concept called weak stability is defined and the set of weak stable matchings is shown to be non-empty under the assumptions of extended substitutability, increasing choice and no external effect by unchosen worker. By definition, the problem we want to assess is of the many-to-many form (particularly: any worker can both work for a firm on many tasks and for many firms) with contracts and externalities due to an exogenous scheduling graph. This model of matching cannot be used directly despite of the fact that it may be interesting to consider the weak stability and an adaptation of the assumption on externalities only due to hired workers.

- Fleiner et a.l. [24], *Trading Networks with Bilateral Contracts*:

  The model proposed by Fleiner et a.l. in [24] considers a contract network modeled by a directed graph such that the nodes are firms and the directed edges are the contracts. In this setting, the firms trade with each others over the contract network. Contract networks subsume the bi-partite structure of the graph in matching games. For any firm (node in the graph) and set of contracts, the externalities in the choice of upstream contracts are due to the set of available downstream contracts and reciprocally [9]. The authors consider the trail and full trail stabilities, new solution concepts for settings contract networks allowing cycles. In our setting, firms do not trade with each others but with workers. Nevertheless, it is shown that many-to-many markets are special cases of such trading networks. Both our model and their model consider a directed graph but the motivations differ since in our case, the graph describes scheduling constraints and not trading opportunities between adjacent nodes. Nevertheless, it may be interesting to see how the trail and full trail stabilities could be adapted to our setting. See 8.8.3 for further details on the transformation of the crowdsourcing problem in the framework of trading networks.

- Pycia and Yenmez's [1], *Matching with Externalities*:

  The studied model is compatible with the particular many-to-many structure of our problem: any worker can both work for a firm on many tasks and for many firms. The authors use

[9] The externalities in the choice of downstream contracts are due to the set of available upstream contracts

	$A^f(t)$	$A^w(t)$	$\mu^f(t)$	$\mu^w(t)$	$C^f(A^f(t)\|\mu^f(t))$	$C^w(A^w(t)\|\mu^w(t))$
$t=1$	$\{x_{11}^1, x_{22}^2, x_{22}^3\}$	$\emptyset$	$\{x_{11}^1, x_{22}^2, x_{22}^3\}$	$\emptyset$	$\{x_{11}^1, x_{22}^2, x_{22}^3\}$	$\emptyset$
$t=2$	$\{x_{11}^1, x_{22}^2, x_{22}^3\}$	$\{x_{11}^1, x_{22}^2, x_{22}^3\}$	$\{x_{11}^1, x_{22}^2, x_{22}^3\}$	$\emptyset$	$\{x_{11}^1, x_{22}^2, x_{22}^3\}$	$\{x_{11}^1, x_{22}^3\}$
$t=3$	$\{x_{11}^1, x_{22}^3\}$	$\{x_{11}^1, x_{22}^2, x_{22}^3\}$	$\{x_{11}^1, x_{22}^2, x_{22}^3\}$	$\{x_{11}^1, x_{22}^3\}$	$\{x_{11}^1\}$	$\{x_{11}^1, x_{22}^3\}$
$t=4$	$\{x_{11}^1, x_{22}^3\}$	$\{x_{11}^1, x_{22}^2\}$	$\{x_{11}^1\}$	$\{x_{11}^1, x_{22}^3\}$	$\{x_{11}^1\}$	$\{x_{11}^1, x_{22}^2\}$
$t=5$	$\{x_{11}^1, x_{22}^2, x_{22}^3\}$	$\{x_{11}^1, x_{22}^2\}$	$\{x_{11}^1\}$	$\{x_{11}^1, x_{22}^2\}$	$\{x_{11}^1, x_{22}^2, x_{22}^3\}$	$\{x_{11}^1, x_{22}^2\}$
$t=6$	$\{x_{11}^1, x_{22}^2, x_{22}^3\}$	$\{x_{11}^1, x_{22}^2, x_{22}^3\}$	$\{x_{11}^1\}$	$\{x_{11}^1, x_{22}^2, x_{22}^3\}$	$\{x_{11}^1, x_{22}^2, x_{22}^3\}$	$\{x_{11}^1, x_{22}^3\}$
$t=7$	$\{x_{11}^1, x_{22}^3\}$	$\{x_{11}^1, x_{22}^2, x_{22}^3\}$	$\{x_{11}^1, x_{22}^2, x_{22}^3\}$	$\{x_{11}^1, x_{22}^3\}$	$\{x_{11}^1\}$	$\{x_{11}^1, x_{22}^3\}$

Table 8.10: Modified Deferred Acceptance with firms proposing and non-substitutability of the choice function C^F.

the substitutability and irrelevance of rejected contracts as sufficient conditions for the convergence of their modified deferred acceptance algorithm to a stable matching and thus to show the existence of stable matchings. As already observed, one can find counter-examples of matching problems with scheduling constraints that have no pairwise stable matchings and thus do not satisfy Pycia and Yenmez's sufficient conditions. See 8.8.1 for further details on the insufficiency of the model and solution proposed in [1] to solve the crowdsourcing problem.

8.8.1 Non-Substitutability and the Modified Deferred Acceptance with Intra-firms Scheduling

Consider the first motivating example shown in Section 8.2.3. We show that the firms' choice function C^F do not satisfy the substitutability condition because c_{f_2} does not. Thus, one cannot use the results in [1] to show the existence of pairwise stable matchings (that actually does not exist as shown in Section 8.2.3).

Consider $X = \{x_{22}^3\} \subseteq X' = \{x_{22}^2, x_{22}^3\}$ and $\mu = \emptyset$. We have,

$$r_{f_2}(X|\mu) = x_{22}^3 \not\subseteq r_{f_2}(X'|\mu) = \emptyset \tag{8.29}$$

The choice function c_{f_2} does not satisfy the substitutability condition. Thus C^F does not and the existence of stable matchings is not guaranteed. Nevertheless, let us apply the modified deferred acceptance algorithm defined in [1] (see Chapter 3, Section 3.6) and show the cyclic behavior.

The first phase of the algorithm converges to the matching $\mu^* = \{x_{11}^1, x_{22}^2, x_{22}^3\}$. We show in Table 8.10 how the second phase of the algorithm performs in it firms proposing version with non-substitutability of the choice function C^F. Observe that the algorithm cycles.. The condition for convergence [10] is not satisfied in any intermediate round.

The first time the algorithm goes through a choice that does not satisfies the substitutability condition and thus induces non-monotonicity of the function f is at round four ($t = 3$). In fact, at $t = 3$ we have $R^F(\{x_{11}^1, x_{22}^3\}|\{x_{11}^1, x_{22}^2, x_{22}^3\}) = \{x_{22}^3\}$ which is not included in the set contracts $R^f(\{x_{11}^1, x_{22}^2, x_{22}^3\}|\{x_{11}^1, x_{22}^2, x_{22}^3\}) = \emptyset$ rejected by the firms at $t = 2$. This induces that the set of offers $A^w(4)$ at $t = 4$ is strictly included in the set of offers $A^w(3)$ at $t = 3$ and that the set of contracts $A^f(5)$ at $t = 5$ is a superset $A^f(4)$ at $t = 4$. As a consequence of the non-substitutability, f is no more a monotone function. Furthermore, $A^f(t)$ and $A^w(t)$ does not define anymore respectively the set of not yet rejected contracts and the cumulated set of offered contracts.

[10] $A^f(t) = A^f(t+1)$, $A^w(t) = A^w(t+1)$, $\mu^f(t) = \mu^f(t+1)$, $\mu^w(t) = \mu^w(t+1)$

8.8.2 Non-Substitutability and the Modified Deferred Acceptance with Inter-firms Scheduling

Consider the second motivating example 8.2.3 with firms' choice functions C^{F} as given in Table 8.7. Consider the preorder $\overset{f}{\succeq}$ defined as,

$$\forall \mu, \mu' \subseteq \mathscr{X}, \quad \mu' \succeq^{\mathrm{F}} \mu \Leftrightarrow \mu'(i) \underset{i}{\succeq} \mu(i) \quad \forall i \in \mathscr{F} \tag{8.30}$$

We show that this preorder is consistent with $\mathscr{C}^{\mathrm{F}}$. Since the choice function of firms f_1 and f_2 do not exhibit externalities, as there are more contracts available, the two firms are better off (each chooses a preferred subset of contract). In fact, for any $i \in \{f_1, f_2\}$ and any $\mathrm{X} \subseteq \mathrm{X}' \subseteq \mathscr{X}$, $\mathrm{C}_i(\mathrm{X}') \subseteq_i \mathrm{C}_i(\mathrm{X})$. By definition of $c_{f_{2'}}$ (see Table 8.5), we can easily show that $\overset{f}{\succeq}$ is consistent with $c_{f_{2'}}$. In fact, we have,

$$c_{f_{2'}}(\{x^3_{22'}\}|\{x^1_{11}, x^2_{22}\}) \succeq_{f_{2'}} c_{f_{2'}}(\{x^3_{22'}\}|\{x^1_{11}\}), c_{f_{2'}}(\{x^3_{22'}\}|\{x^2_{22}\}) \succeq_{f_{2'}} \emptyset \tag{8.31}$$

The preorder $\succeq^{\mathrm{F}}$ is consistent with C^{F}.

We show that the contracts are not substitutes for the firms (see (see Chapter 3, Section 3.6 for the definition of the substitutes condition in matchings). For any $\mathrm{X}, \mathrm{X}', \mu, \mu' \subseteq \mathscr{X}$ such that $\mathrm{X} \subseteq \mathrm{X}'$ and $\mu' \succeq^{\mathrm{F}} \mu$, we have $r_{f_1}(\mathrm{X}|\mu) \subseteq r_{f_1}(\mathrm{X}'|\mu')$ and $r_{f_2}(\mathrm{X}|\mu) \subseteq r_{f_2}(\mathrm{X}'|\mu')$. Considering the reject function of $f_{2'}$ (see Table 8.6), we have,

$$r_{f_{2'}}(.|\{x^1_{11}\}) = r_{f_{2'}}(.|\emptyset) = \{x^3_{22'}\} \supseteq r_{f_{2'}}(\{x^3_{22'}\}|\{x^1_{11}, x^2_{22}\}) = \emptyset \tag{8.32}$$

Using this result, for all $\mathrm{X}, \mathrm{X}', \mu, \mu' \subseteq \mathscr{X}$ such that $x^3_{22'} \in \mathrm{X}, x^2_{22} \not\in \mu, x^2_{22} \in \mu', \mu' \overset{f}{\succeq} \mu$, we have,

$$\mathrm{R}^{\mathrm{F}}(\mathrm{X}'|\mu') \subseteq \mathrm{R}^{\mathrm{F}}(\mathrm{X}|\mu) \tag{8.33}$$

where R^{F} is the firms' reject function. Thus, the firms' choice function C^{F} does not satisfy the substitutes condition. There is no guarantee of convergence to a solution (whatever stable or not) of the transformation function f (i.e. of the algorithm).

The first phase of the algorithm converges to the matching $\mu^* = \{x^1_{11}, x^2_{22}, x^3_{22'}\}$. We show in Table 8.11 how the algorithm performs in it firms proposing version with non-substitutability of the choice function C^{F} (see Table 8.7) and the consistent preorder $\overset{\mathrm{F}}{\succeq}$ defined in (8.30). Observe that the algorithm cycles except for μ^{W}. In fact, the sets obtained at round eight ($t = 8$) are the same as those obtained at round three ($t = 2$) except for μ^{W}. The first time the algorithm goes through a choice that does not satisfies the substitutability condition and thus induces non-monotonicity of function f w.r.t. the order $\sqsupseteq$ is at round four ($t = 4$). In fact, at $t = 4$ we have $\mathrm{R}^{\mathrm{F}}(\{x^1_{11}, x^3_{22'}\}|\{x^1_{11}, x^3_{22'}\}) = \{x^3_{22'}\}$ which is not included in the set contracts $\mathrm{R}^{\mathrm{F}}(\{x^1_{11}, x^3_{22'}\}|\{x^1_{11}, x^2_{22}, x^3_{22'}\}) = \{x^1_{11}, x^3_{22'}\}$ rejected by the firms at $t = 3$. This induces that the set of offers $\mathrm{A}^{\mathrm{W}}(4)$ at $t = 4$ is not included in the set of offers $\mathrm{A}^{w}(5)$ at $t = 5$ and that the set of not rejected contracts at $t = 6$, $\mathrm{A}^{\mathrm{F}}(6)$ is not included in the set of not rejected contracts at $t = 5$ $\mathrm{A}^{\mathrm{F}}(5)$. As a consequence of the non-substitutability, $\mathrm{A}^{\mathrm{F}}(t)$ and $\mathrm{A}^{\mathrm{W}}(t)$ does not define anymore, respectively the set of not yet rejected contracts and the cumulated set of offered contracts.

8.8.3 Matching with Contracts and Inter-firms Scheduling via Contract Networks

To show that graph-based existing solutions in matching games do not solve the problem, we formulate the previous example using trading networks as developed by Fleinet et al. in [24]. Figure 8.20 shows the corresponding setting. In the model, each contract is mapped to a directed edge of a graph with nodes as players of the game. Given a node and the set of contracts pointing from this node to others, it is said that the node is the seller of this output stream. Similarly, given a node and the set of contracts pointing to this node from others, it is said that the node is the

	$A^F(t)$	$A^W(t)$	$\mu^F(t)$	$\mu^W(t)$	$C^F(A^F(t)\vert\mu^f(t))$	$R^F(A^F(t)\vert\mu^f(t))$	$C^W(A^W(t)\vert\mu^w(t))$	$R^W(A^W(t)\vert\mu^w(t))$
$t=1$	$\{x^1_{11}, x^2_{22}, x^3_{22'}\}$	$\emptyset$	$\{x^1_{11}, x^2_{22}, x^3_{22'}\}$	$\emptyset$	$\{x^1_{11}, x^2_{22}, x^3_{22'}\}$	$\emptyset$	$\emptyset$	$\emptyset$
$t=2$	$\{x^1_{11}, x^2_{22}, x^3_{22'}\}$	$\{x^1_{11}, x^2_{22}, x^3_{22'}\}$	$\{x^1_{11}, x^2_{22}, x^3_{22'}\}$	$\emptyset$	$\{x^1_{11}, x^2_{22}, x^3_{22'}\}$	$\emptyset$	$\{x^1_{11}, x^3_{22'}\}$	$\{x^2_{22}\}$
$t=3$	$\{x^1_{11}, x^3_{22'}\}$	$\{x^1_{11}, x^2_{22}, x^3_{22'}\}$	$\{x^1_{11}, x^2_{22}, x^3_{22'}\}$	$\{x^1_{11}, x^3_{22'}\}$	$\{x^1_{11}, x^3_{22'}\}$	$\emptyset$	$\{x^1_{11}, x^3_{22'}\}$	$\{x^2_{22}\}$
$t=4$	$\{x^1_{11}, x^3_{22'}\}$	$\{x^1_{11}, x^2_{22}, x^3_{22'}\}$	$\{x^1_{11}, x^3_{22'}\}$	$\{x^1_{11}, x^3_{22'}\}$	$\{x^1_{11}\}$	$\{x^3_{22'}\}$	$\{x^1_{11}, x^3_{22'}\}$	$\{x^2_{22}\}$
$t=5$	$\{x^1_{11}, x^3_{22'}\}$	$\{x^1_{11}, x^2_{22}\}$	$\{x^1_{11}\}$	$\{x^1_{11}, x^3_{22'}\}$	$\{x^1_{11}\}$	$\{x^3_{22'}\}$	$\{x^1_{11}, x^2_{22}\}$	$\emptyset$
$t=6$	$\{x^1_{11}, x^2_{22}, x^3_{22'}\}$	$\{x^1_{11}, x^2_{22}\}$	$\{x^1_{11}\}$	$\{x^1_{11}, x^2_{22}\}$	$\{x^1_{11}, x^2_{22}\}$	$\{x^3_{22'}\}$	$\{x^1_{11}, x^2_{22}\}$	$\emptyset$
$t=7$	$\{x^1_{11}, x^2_{22}, x^3_{22'}\}$	$\{x^1_{11}, x^2_{22}\}$	$\{x^1_{11}, x^2_{22}\}$	$\{x^1_{11}, x^2_{22}\}$	$\{x^1_{11}, x^2_{22}, x^3_{22'}\}$	$\emptyset$	$\{x^1_{11}, x^2_{22}\}$	$\emptyset$
$t=8$	$\{x^1_{11}, x^2_{22}, x^3_{22'}\}$	$\{x^1_{11}, x^2_{22}, x^3_{22'}\}$	$\{x^1_{11}, x^2_{22}, x^3_{22'}\}$	$\{x^1_{11}, x^2_{22}\}$	$\{x^1_{11}, x^2_{22}, x^3_{22'}\}$	$\emptyset$	$\{x^1_{11}, x^3_{22'}\}$	$\{x^2_{22}\}$

Table 8.11: Modified Deferred Acceptance with firms proposing and non-substitutability of the choice function C^F (see Table 8.7) when using the preorder $\succeq^F$ defined in (8.30).

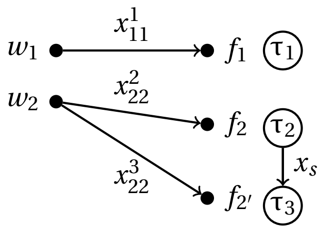

Figure 8.20: Example: A many-to-one matching game with contracts and scheduling constraints as a contract network.

buyer of this input stream. In this model, the scheduling constraint is introduced as an additional contract x_s from τ_2 to τ_3 and selected in the output stream of τ_2 when x^2_{22} is signed. We have the following choice functions:

- τ_2 as a buyer (input stream):

$$c^B_{\tau_2}(x^2_{22}|\emptyset) = x^2_{22} \text{ and } c^B_{\tau_2}(x^2_{22}|x_s) = x_2 \tag{8.34}$$

- τ_2 as a seller (output stream):

$$c^S_{\tau_2}(x_s|\emptyset) = \emptyset \text{ and } c^S_{\tau_2}(x_s|x^2_{22}) = x_s \tag{8.35}$$

- w_2 as a seller (output stream):

$$c^S_{w_2}(x^2_{22}|\emptyset) = x^2_{22} \text{ and } c^S_{w_2}(x^3_{22}|\emptyset) = x^3_{22} \text{ and } c^S_{w_2}(\{x^2_{22}, x^3_{22}\}|\emptyset) = x^3_{22} \tag{8.36}$$

- τ_3 as a buyer (input stream)

$$c^B_{\tau_3}(x^3_{22}|\emptyset) = \emptyset \text{ and } c^B_{\tau_3}(x_s|\emptyset) = x_s \text{ and } c^B_{\tau_3}(\{x^3_{22}, x_s\}|\emptyset) = x^3_{22}, x_s \tag{8.37}$$

Since, $R^B_{\tau_3}(x^3_{22}|\emptyset) \supset R^B_{\tau_3}(\{x^3_{22}, x_s\}|\emptyset)$, the *same-side substitutability (SSS)* condition for input streams (τ_3 as a buyer) from the *full substitutability* condition is not satisfied (see [24] for a complete analysis).

Observe that by reversing the directed edges between the firms and the workers (but not of x_s to maintain the scheduling constraint, see Figure 8.21) one would have obtained complementary input and output streams for $f_{2'}$ (x_s and x^3_{22}) and complementary output streams since

$$r_{f_2}(\{x_s\}|\emptyset) = \{x_s\} \not\subseteq r_{f_2}(\{x_s, x^2_{22}\}|\emptyset) = \emptyset \tag{8.38}$$

Based on this transformation, on can see that there are complementarities in the output streams of the firms so the conditions (full substitutability) for the results to hold are not satisfied but we leave as an open question the analysis of the crowdsourcing problem by trading networks with bilateral contracts either in the original formulation with the following configuration of the trading network:

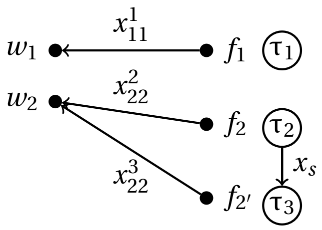

Figure 8.21: Example: A many-to-one matching game with contracts and scheduling constraints as a contract network.

- The nodes of the network are the firms and the workers.
- The contracts of the original problem are directed edges from the firms to the workers.
- Any oriented edge from τ to τ' not belonging to the same firm in the scheduling graph is transformed into a contract from $\mathscr{F}(\tau)$ to $\mathscr{F}(\tau')$ (i.e. on the output stream of $\mathscr{F}(\tau)$ and in the input stream of $\mathscr{F}(\tau')$). This contract would be chosen by $\mathscr{F}(\tau)$ as soon as the the scheduling constraints for the execution of τ are satisfied.

or in the task-agent representation of the original problem with the following configuration of the trading network:

- The nodes of the network are the tasks and workers.
- The contracts of the original problem are directed edges from the firms to the workers.
- Any oriented edge from τ to τ' in the scheduling graph is transformed into a contract from τ to τ' (i.e. in the output stream of τ and in the input stream of τ'). This contract would be chosen by τ as soon as a contract from τ to the workers is signed.

8.9 Appendix: Proofs

8.9.1 Appendix: Proof of Proposition 124

Proof. Consider any $X, X', \mu, \mu' \subseteq \mathscr{X}$ such that $X' \supseteq X$ and $\mu' \succeq^F \mu$. Using the fact that any feasible contract in X at μ is also feasible at μ' because weakly more feasibility conditions are satisfied, for any $f \in \mathscr{F}$, we have

$$X_f(\mu') \supseteq X_f(\mu) \tag{8.39}$$

Furthermore, since any feasible contract in X at μ' is feasible in any superset and $X'_f \supseteq X_f$, then

$$X'_f(\mu') \supseteq X_f(\mu') \supseteq X_f(\mu) \tag{8.40}$$

By definition of c_f, we have,

$$c_f(X|\mu) = \bigcup_{\tau \in \mathscr{T}(X_f(\mu))} \operatorname{arg\,min}_{\substack{x \in X_f(\mu) \\ s.t. \tau(x)=\tau}} s(x) \tag{8.41}$$

Each firm, chooses weakly more contracts in X' at μ' than in X at μ. Thus, the set of tasks assigned in $\bigcup_{f' \in \mathscr{F} \setminus f} c_{f'}(X'_{f'}(\mu'))$ contains $\bigcup_{f' \in \mathscr{F} \setminus f} c_{f'}(X_{f'}(\mu))$,

$$\mathscr{T}\left(\bigcup_{f' \in \mathscr{F} \setminus f} c_{f'}(X'_{f'}(\mu'))\right) \supseteq \mathscr{T}\left(\bigcup_{f' \in \mathscr{F} \setminus f} c_{f'}(X_{f'}(\mu))\right) \tag{8.42}$$

Then,

$$\mathscr{T}\left(\bigcup_{f' \in \mathscr{F} \setminus f} c_{f'}(X'_{f'}(\mu'))\right) \cap \mathscr{N}^-(\mathscr{T}_f) \supseteq \mathscr{T}\left(\bigcup_{f' \in \mathscr{F} \setminus f} c_{f'}(X_{f'}(\mu))\right) \cap \mathscr{N}^-(\mathscr{T}_f) \tag{8.43}$$

which implies consistency,

$$C^F(X'|\mu') \succeq^F C^F(X|\mu) \tag{8.44}$$

□

8.9.2 Appendix: Proof of Proposition 126

Proof. Assume $X, X', \mu, \mu' \subseteq \mathscr{X}$, such that $X \subseteq X'$, $\mu' \succeq^F \mu$ and for any $f \in \mathscr{F}$,

$$\mathscr{T}^f_{\substack{X\to X'\\ \mu\to\mu'}} = \emptyset \text{ or } X_f(\mathscr{T}^f_{\substack{X\to X'\\ \mu\to\mu'}}) \not\subseteq \underline{X}'_f(\mu') \tag{8.45}$$

By definition, the firms' choice function C^F satisfies constrained substitutability iff for any firm $f \in \mathscr{F}$, the firm's choice function c_f satisfy constrained substitutability. Consider any firm $f \in \mathscr{F}$ with choice function c_f. By definition of c_f, a contract $x \in X_f$ is rejected by f in X_f at μ (i.e. $x \not\in c_f(X_f|\mu)$), if,

- $x \in X_f \backslash X_f(\mu)$ or,
- $x \in X_f(\mu)$ and $\exists x' \in X_f(\mu)$ such that $s(x) > s(x')$ and $\tau(x) > \tau(x')$

First, if the condition $\mathscr{T}^f_{\substack{X\to X'\\ \mu\to\mu'}} = \emptyset$ is satisfied, then no non-feasible task τ in X_f at μ is feasible in X'_f at μ'. In such a case, if a contract $x \in X_f$ is rejected at μ because of non-feasibility then it is also rejected in X' at μ'.

Else if the feasible contract x is rejected because it is not of min-salary, i.e. $x \not\in \underline{X}_f(\mu)$, by inclusion $X' \supseteq X$, x is still rejected from X'_f at μ' because it is still not of min-salary w.r.t. $\tau(x)$.

Second, if $\mathscr{T}^f_{\substack{X\to X'\\ \mu\to\mu'}} \neq \emptyset$ and $X_f(\mathscr{T}^f_{\substack{X\to X'\\ \mu\to\mu'}}) \not\subseteq \underline{X}'_f(\mu')$, then there exists a non-feasible task τ in X at μ that is feasible in X' at μ' with min-salary contract x' for τ in X'_f not in X_f. As previously, in such a case, if a contract $x \in X_f$ is non-feasible in X_f at μ and in X'_f at μ', i.e. $x \not\in X'_f \backslash X_f(\mu)$ and $x \not\in X'_f \backslash X'_f(\mu)$, then it is rejected.
Else the non-feasible contract x in X_f at μ, even though feasible in X'_f at μ' keep on being rejected by f because x' has lower salary.

Finally, we need to check that any contract $x \in X_f(\mu) \backslash \underline{X}_f(\mu)$ (i.e. feasible but not of min-salary in X at μ) is rejected by f in X'_f at μ'. Since $\underline{X}_f(\mu) \subseteq \underline{X'}_f(\mu')$ (using $X \subseteq X'$ and $\mu' \succeq^F \mu$) then we obtain that x is still not of min-salary in X' at μ' and is thus rejected.

Thus, the choice function of any firm $f \in \mathscr{F}$ satisfies constrained substitutability and the firms' choice functions satisfy constrained substitutability by equivalence.
This concludes the proof. □

8.9.3 Appendix: Proof of Proposition 127

Proof. Assume $X', X, \mu \subseteq \mathscr{X}$ and the firms' choice function C^F such that $C^F(X'|\mu) \subseteq X \subseteq X'$.
By definition of C^F (as the union of individual choices), for any firm $f' \in \mathscr{F}$ we have $c_{f'}(X'_{f'}|\mu) = \underline{X}'_{f'}(\mu) \subseteq X_{f'} \subseteq X'_{f'}$. The set of min-salary feasible contracts in $X'_{f'}$ is in $X_{f'}$.
Furthermore, assume $C^F(X'|\mu) \neq C^F(X|\mu)$.
Then, there exists a firm $f \in \mathscr{F}$ such that $c_f(X'_f|\mu) = \underline{X}'_f(\mu) \neq c_f(X_f|\mu) = \underline{X}_f(\mu)$.

By definition of feasibility and inclusion $X_f \subseteq X'_f$, we have $X_f(\mu) \subseteq X'_f(\mu)$ (since any feasible contract in a set is feasible in a superset). Thus, $\underline{X}_f(\mu) \subseteq X_f(\mu) \subseteq X'_f(\mu) \subseteq X'_f$ and there must exists a contract $x \in X_f$ such that $x \in \underline{X}'_f(\mu) \subseteq X_f$ but $x \notin \underline{X}_f(\mu)$. Either $x \notin X_f(\mu)$, or there exists another contract $x' \in X_f(\mu)$ such that $\tau(x) = \tau(x')$ and $s(x') < s(x)$.

In the first case, $x \notin X_f(\mu)$. But, since $\underline{X}'_f(\mu) \subseteq X_f$, $\underline{X}'_f(\mu)$ is feasible in X_f at μ there is a contradiction with $x \notin X_f(\mu)$.
The second case is in contradiction with $x \in \underline{X}'_f(\mu)$ since $X_f(\mu) \subseteq X'_f(\mu)$ implies $x' \in \underline{X}'_f(\mu)$ and $x \notin \underline{X}'_f(\mu)$.
This shows by contradiction that $c_f(X'_f|\mu) = c_f(X_f|\mu)$, for any firm $f \in \mathscr{F}$. Thus, $C^F(X'|\mu) = C^F(X|\mu)$.
This concludes the proof. □

8.9.4 Appendix: Proof of Lemma 129

Proof. The proof is based on the same arguments than the one provided in [1] (Lemma 2, pp. 20), except that we require for condition C3 of Definition 125 to be satisfied for the substitutability to be used on the firms' side. Function f is monotonic in $\sqsubseteq$ because for any $A^F \subseteq \tilde{A}^F, A^W \supseteq \tilde{A}^W, \mu^F \preceq \tilde{\mu}^F, \mu^W \succeq \tilde{\mu}^W$, such that condition C3 is satisfied, substitutability implies that

$$R^W(\tilde{A}^W|\tilde{\mu}^W) \subseteq R^W(A^W|\mu^W) \tag{8.46}$$

thus,

$$\mathscr{X}\backslash R^W(A^W|\mu^W) \subseteq \mathscr{X}\backslash R^W(\tilde{A}^W|\tilde{\mu}^W) \tag{8.47}$$

and constrained substitutability implies that

$$R^F(A^F|\mu^F) \subseteq R^F(\tilde{A}^F|\tilde{\mu}^F) \tag{8.48}$$

thus,

$$\mathscr{X}\backslash R^F(A^F|\mu^F) \supseteq \mathscr{X}\backslash R^F(\tilde{A}^F|\tilde{\mu}^F) \tag{8.49}$$

Furthermore, consistency implies that

$$C^F(A^F|\mu^F) \preceq^F C^F(\tilde{A}^F|\tilde{\mu}^F) \tag{8.50}$$

$$C^W(A^W|\mu^W) \preceq^W C^W(\tilde{A}^W|\tilde{\mu}^W) \tag{8.51}$$

Therefore, if $(A^F, A^W, \mu^F, \mu^W) \sqsubseteq (\tilde{A}^F, \tilde{A}^W, \tilde{\mu}^F, \tilde{\mu}^W)$ and condition C3 is satisfied, then we have,

$$f(A^F, A^W, \mu^F, \mu^W) \sqsubseteq f(\tilde{A}^F, \tilde{A}^W, \tilde{\mu}^F, \tilde{\mu}^W) \tag{8.52}$$

This concludes the proof. □

8.9.5 Appendix: Proof of Theorem 131

Proof. The proof is in four claims and is quite similar original to the one in [1]. Suppose that (A^F, A^W, μ, μ) is a fixed point of f.

Claim 151. *Suppose that the choice function* C^W *satisfies substitutability and the irrelevance of rejected contracts. Then matching* μ *is stable.*

Proof. As a preliminary result, we show that if (A^F, A^W, μ, μ) is a fixed point of f, then condition C3 of definition 125 is satisfied such that for any $f \in \mathscr{F}$,

$$\mathscr{T}^{f}_{\substack{\mu \to A^F \\ \mu \to \mu}} = \emptyset \text{ or } \mu\left(\mathscr{T}^{f}_{\substack{\mu \to A^F \\ \mu \to \mu}}\right) \not\subseteq \underline{A}^F_f(\mu) \tag{8.53}$$

By assumption (fixed point), $\mu = C^F(A^F|\mu)$ and by definition of c_f, we have $\mu_f = c_f(A^F_f|\mu)$ and $\mu_f(\mu) = \mu_f$. Thus, $\mathscr{T}(\mu_f(\mu)) = \mathscr{T}(\mu_f)$ and $\mathscr{T}(\mu_f) \backslash \mathscr{T}(\mu_f(\mu)) = \mathscr{T}(\mu_f) \backslash \mathscr{T}(\mu_f) = \emptyset$. This implies

$$\mathscr{T}^{f}_{\substack{\mu \to A^F \\ \mu \to \mu}} = \emptyset \tag{8.54}$$

which shows the preliminary result.

Now, suppose for contradiction that μ is not stable. Then, there are three possibilities, all of which we proceed to rule out.

- First, matching μ is not individually rational for some firm j, that is $c_j(\mu|\mu) \subsetneq \mu_j$. Since (A^F, A^W, μ, μ) is a fixed point of f, $C^F(A^W|\mu) = \mu$ and $A^F \supseteq \mu$. But constrained substitutability (condition C3 of Definition 125 verified by the preliminary result) and $c_j(\mu|\mu) \subsetneq \mu_j$ imply that there is a contract $x \in \mu_j$ rejected out of A^F by agent j, that is $x \not\subseteq C^F$.

- Second, matching μ is not individually rational for some worker i, that is $c_i(\mu|\mu) \subsetneq \mu_i$.
  The proof is the same as the original one provided in [1]. We provide it here for the sake of completeness and to show the asymmetry w.r.t. substitutability between firms and workers. Since (A^F, A^W, μ, μ) is a fixed point of f, $C^W(A^W|\mu) = \mu$ and $A^W \supseteq \mu$. But substitutability and $c_i(\mu|\mu) \subsetneq \mu_i$ imply that there is a contract $x \in \mu_i$ rejected out of A^W by agent i, that is $x \not\subseteq C^W$.

- Then, there exists a blocking pair with contract $x \in \mathscr{X} \backslash \mu$.
  Since (A^F, A^W, μ, μ) is a fixed point of f, by Lemma 130 $A^F \cup A^W = \mathscr{X}$.
  Assume $x \in A^W$. Since $\{x\}$ is a blocking set, there exists worker i s.t. $x \in c_i(\mu \cup \{x\}|\mu) \backslash \mu$. Again, since (A^F, A^W, μ, μ) is a fixed point of f, by Lemma 130

  $$C^W(A^W|\mu) = \mu \tag{8.55}$$

  which implies $c_i(A^W|\mu) = \mu_i$.

  By irrelevance of rejected contracts, for any set Y s.t. $A^W \supseteq Y \supseteq \mu$, $c_i(Y|\mu) = \mu_i$. In particular for $Y = \mu \cup \{x\}$, $c_i(\mu \cup \{x\}) = \mu_i$, which is a contradiction because $x \in c_i(\mu \cup \{x\}|\mu) \backslash \mu$.

  It remains to show that there is a contradiction if $x \in A^F$. The proof is the same as in the previous case since IRC also holds for the firms.

This concludes the proof of the claim.

□

Claim 152. *Suppose that the choice function* C^W *satisfies substitutability and the irrelevance of rejected contracts. Then the function* $M^W(\mu) = \max\{X \subseteq \mathscr{X} | C^W(X|\mu) = \mu\}$, *where the maximum is w.r.t. set inclusion, is well-defined. Moreover, for any contract* $z \not\subseteq M^W(\mu)$, $z \in C^W(M^W(\mu) \cup z|\mu)$.

Proof. The proof is the same as the one given for Claim 2 for $\theta = W$, in [1], pp.40. □

Claim 153. *The function* $M^F(\mu) = \max\{X \subseteq \mathscr{X} | C^F(X|\mu) = \mu\}$, *where the maximum is w.r.t. set inclusion, is well-defined. Moreover, for any contract* $z \notin M^F(\mu)$, $z \in C^F(M^F(\mu) \cup z|\mu)$.

Proof. By definition of C^F, we have

$$M^F(\mu) = \bigcup_{f \in \mathscr{F}} \left[\left[\mathscr{X}_f \backslash \mathscr{X}_f(\mu) \right] \cup Y_f \cup \mu \right] \tag{8.56}$$

where $Y_f = \left\{ x \in \mathscr{X}_f | \forall \tau \in \mathscr{T}(\mu_f), s(x) > s(\mu(\tau)) \right\}$, i.e. $M^F(\mu)$ contains for any firm $f \in \mathscr{F}$, all contracts in $\mathscr{X}_f$ with non-feasible tasks and for every feasible tasks in $\mathscr{X}_f$ at μ (i.e. $\mathscr{X}_f(\mu)$) all contracts with salary superior or equal to those in μ (knowing that by definition, only the contracts in μ have the corresponding salaries). By definition, if a contract $z \notin M^F(\mu)$ and $x \in \mu$ s.t. $\tau(z) = \tau(x)$ and $s(z) < s(x)$, then $z \in C^F(M \cup z|\mu)$. □

Claim 154. *Suppose that the matching* μ *is stable, the choice function* C^W *satisfies substitutability. Then, there exist sets of contracts* A^F *and* A^W *s.t.* (A^F, A^W, μ, μ) *is a fixed point of* f.

Proof. By claim 152 and claim 153, there exists the largest set

$$M^\theta(\mu) = \max\{X \subseteq \mathscr{X} | C^\theta(X|\mu) = \mu\} \tag{8.57}$$

for $\theta = W$ and $\theta = F$. Let $A^F = M^F(\mu)$ and $A^W = \mathscr{X} \backslash R^F(A^F|\mu)$. By definition of A^F and $\mu = C^F(A^F|\mu)$.

Thus, we get $A^F \cap A^W = A^F \cap \left(\mathscr{X} \backslash R^F(A^F|\mu)\right) = C^F(A^F|\mu) = \mu$. To finish the proof, we need to show $\mu = C^W(A^W|\mu)$ and $A^F = \mathscr{X} \backslash R^W(A^W|\mu)$. Note that

$$A^W = \mathscr{X} \backslash R^F(A^F|\mu) \tag{8.58}$$
$$= \left[\mathscr{X} \backslash A^F\right] \cup C^F(A^F|\mu) \tag{8.59}$$
$$= [\mathscr{X} \backslash] A^F \cup \mu \tag{8.60}$$

In particular, $A^W \supseteq \mu$. If $C^W(A^W|\mu) = Y \neq \mu$, there are two cases, both of which contradict stability of μ.

- First, if $Y \subsetneq \mu$, then the irrelevance of rejected contracts implies $C^W(\mu|\mu) = Y$, implying that μ is not individually rational for some workers, contradicting stability.
- Second, if $Y \not\subseteq \mu$, then there exists a $y \in Y \backslash \mu$ and $y \in C^W(\mu \cup \{y\}|\mu)$ by substitutability since $y \in C^W(A^W|\mu)$ and $A^W \supseteq \mu \cup \{y\}$. But, we also have that $y \in C^F(A^F \cup \{y\}|\mu)$ by claim 153. The set $\{y\}$ blocks μ, contradicting stability. Thus, the only case consistent with stability is $C^W(A^W|\mu) = \mu$.
- Finally we show that

  $$A^F = \mathscr{X} \backslash R^W(A^W|\mu) = \mathscr{X} \backslash R^W\left(\mathscr{X} \backslash R^F(A^F|\mu)|\mu\right) \tag{8.61}$$

  Since $C^W(A^W|\mu) = \mu$, then

  $$\mathscr{X} \backslash R^W(A^W|\mu) = \mathscr{X} \backslash \left(A^W \backslash \mu\right) \tag{8.62}$$
  $$= \mathscr{X} \backslash \left(\left(\left(\mathscr{X} \backslash A^F\right) \cup \mu\right) \backslash \mu\right) \tag{8.63}$$
  $$= \mathscr{X} \backslash \left(\mathscr{X} \backslash A^F\right) \tag{8.64}$$
  $$= A^F \tag{8.65}$$

This concludes the proof of the claim. □

The theorem is proved. □

8.9.6 Appendix: Proof of Theorem 132

Proof. First, let us consider the first phase of the algorithm and check out that $\mu^* \succeq^F C^F(\mathscr{X}|\mu^*)$. By the irrelevance of rejected contracts, we get $C^F(\mu_k|\mu_{k-1}) = \mu_k$ for every $k \geq 1$. We show that $\mu_k \succeq^F \mu_{k-1}$ for every $k \geq 1$. The proof is by mathematical induction on k. For the base case when $k = 1$, note that $\mathscr{X} \supseteq \emptyset$ and consistency imply that

$$\mu_1 = C^F(\mathscr{X}|\emptyset) \succeq^F C^F(\emptyset|\emptyset) = \emptyset = \mu_0 \tag{8.66}$$

For the general case, $\mu_k \succeq^F \mu_{k-1}$ and $\mathscr{X} \supseteq \mu_k$ imply that (by consistency)

$$\mu_{k+1} = C^F(\mathscr{X}|\mu_k) \succeq^F C^F(\mu_k|\mu_{k-1}) = \mu_k \tag{8.67}$$

Then, for $\{\mu_k\}_{k\geq 1}$ is a monotone sequence w.r.t. the preorder $\succeq^F$.

By definition of C^F and $\{c_f\}_{f\in\mathscr{F}}$, one can show that the sequence converges to the matching μ^* such that for any task $\tau \in \mathscr{T}$,

$$\mu^*(\tau) = \arg\min_{\substack{x\in\mathscr{X} \\ \tau(x)=\tau}} s(x) \tag{8.68}$$

Thus, we have $\mu^* \succeq^F C^F(\mathscr{X}|\mu^*)$ and $\mu^* = C^F(\mathscr{X}|\mu^*)$.

It remains to show that the second phase converges and that the resulting matching is stable. It is easy to see that

$$f(\mathscr{X},\emptyset,\mu^*,\emptyset) \sqsubseteq (\mathscr{X},\emptyset,\mu^*,\emptyset) \tag{8.69}$$

Since $C^F(\mathscr{X}|\mu^*) = \mu^* \preceq^F \mu^*$ and $C^W(\emptyset|\emptyset) = \emptyset \succeq^W \emptyset$. By reflexivity of $\succeq^F$ and $\succeq^W$ respectively, we have

$$\begin{aligned} f(\mathscr{X},\emptyset,\mu^*,\emptyset) &= (\mathscr{X},\mathscr{X}\setminus(\mathscr{X}\setminus\mu^*),\mu^*,\emptyset) & (8.70)\\ &= (\mathscr{X},\mu^*,\mu^*,\emptyset) & (8.71)\end{aligned}$$

Thus, for any $f \in \mathscr{F}$

$$\mathscr{T}_{\substack{\mathscr{X}\to\mathscr{X} \\ \mu^*\to\mu^*}} = \left[\mathscr{T}(\mathscr{X}_f)\backslash\mathscr{T}(\mathscr{X}_f(\mu^*))\right] \cap \mathscr{T}(\mathscr{X}_f(\mu^*)) = \emptyset \tag{8.72}$$

The conditions for constrained substitutability are satisfied for $k = 1$.

By assumption, f satisfies C3 (i.e. (A^F, A^W, μ^F, μ^W) and (A^F, A^W, μ^F, μ^W) satisfy C3), thus by Lemma 129, f is monotone increasing, so we can repeatedly apply it to the last inequality to get

$$f^k(\mathscr{X},\emptyset,\mu^*,\emptyset) \sqsubseteq f^{k-1}(\mathscr{X},\emptyset,\mu^*,\emptyset) \tag{8.73}$$

for every k.

We consider two separate cases.

Suppose first that this sequence converges. Therefore there exists k such that

$$f^{k-1}(\mathscr{X},\emptyset,\mu^*,\emptyset) = f^k(\mathscr{X},\emptyset,\mu^*,\emptyset) \tag{8.74}$$

As a result, $f^{k-1}(\mathscr{X},\emptyset,\mu^*,\emptyset)$ is a fixed point of f. Let $(A^{*F}, A^{*W}, \mu^{*F}, \mu^{*W}) = f^{k-1}(\mathscr{X},\emptyset,\mu^*,\emptyset)$. By Lemma 130, $\mu^{*F} = \mu^{*W} = A^{*F} \cap A^{*W}$ and μ^{*F} is a sable matching by Theorem 131.

Otherwise, if sequence does not converge, there exists a subsequence

$$f^n(\mathscr{X},\emptyset,\mu^*,\emptyset) \sqsupseteq f^{n+1}(\mathscr{X},\emptyset,\mu^*,\emptyset) \sqsupseteq \ldots \sqsupseteq f^m(\mathscr{X},\emptyset,\mu^*,\emptyset) \sqsupseteq f^{m+1}(\mathscr{X},\emptyset,\mu^*,\emptyset) = f^n(\mathscr{X},\emptyset,\mu^*,\emptyset) \tag{8.75}$$

because the number of contracts is finite.

By transitivity of the preorder $\sqsupseteq$ and the previous inequality, we get

$$f^n(\mathscr{X},\emptyset,\mu^*,\emptyset) \quad = \quad f^{m+1}(\mathscr{X},\emptyset,\mu^*,\emptyset) \quad \sqsupseteq \quad f^m(\mathscr{X},\emptyset,\mu^*,\emptyset) \quad \sqsupseteq \quad f^n(\mathscr{X},\emptyset,\mu^*,\emptyset) \tag{8.76}$$

Let $f^n(\mathscr{X}, \emptyset, \mu^*, \emptyset) = (A_1^F, A_1^W, \mu_1^F, \mu_1^W)$ and $f^m(\mathscr{X}, \emptyset, \mu^*, \emptyset) = (A_2^F, A_2^W, \mu_2^F, \mu_2^W)$. By definition of $\sqsupseteq$, we get that

$$A_1^F = A_2^F \tag{8.77}$$
$$A_1^W = A_2^W \tag{8.78}$$
$$\mu_1^F \sim^F \mu_2^F \tag{8.79}$$
$$\mu_1^W \sim^W \mu_2^W \tag{8.80}$$
$$\tag{8.81}$$

Now, by construction $C^F(A_2^F|\mu_2^F) = \mu_1^F$ (because $f^n(\mathscr{X}, \emptyset, \mu^*, \emptyset) = f^{m+1}(\mathscr{X}, \emptyset, \mu^*, \emptyset)$) and by definition of C^F, $C^F(A_2^F|\mu_2^F) = C^F(A_1^F|\mu_1^F)$, which imply that $C^F(A_1^F|\mu_1^F) = \mu_1^F$. Similarly, by construction $C^W(A_2^W|\mu_2^W) = \mu_1^W$ (because $f^n(\mathscr{X}, \emptyset, \mu^*, \emptyset) = f^{m+1}(\mathscr{X}, \emptyset, \mu^*, \emptyset)$) and by substitutability of C^W, $C^W(A_2^W|\mu_2^W) = C^W(A_1^W|\mu_1^W)$, which imply that $C^W(A_1^W|\mu_1^W) = \mu_1^W$. Furthermore, by substitutability, $\mathscr{X} \setminus R^W(A_2^W|\mu_2^W) = \mathscr{X} \setminus R^W(A_1^W|\mu_1^W)$, and by construction $\mathscr{X} \setminus R^W(A_2^W|\mu_2^W) = A_1^F$, which imply $\mathscr{X} \setminus R^W(A_1^W|\mu_1^W) = A_1^F$. Similarly, we get $\mathscr{X} \setminus R^F(A_2^F|\mu_2^F) = A_1^W$. Therefore, $(A_1^F, A_1^W, \mu_1^F, \mu_1^W)$ is a fixed point of f. This shows, that the sequence converges as in the previous paragraph, so there exists a stable matching.

□

8.10 Appendix: Alternative Stabilities

In this appendix, we define new stabilities that, to our point of view, would correspond to appropriate equilibrium concepts for the matching problem with contracts, externalities and scheduling constraints. To develop these new stabilities, we have used existing (see [28] for a recent survey on matching with externalities and alternative stabilities). In developing these definition, we have particularly taken into account non-credible deviations as already done for the *weak-stability*, see [28]. The proposed stabilities exhibit similarities w.r.t. existing ones such as the optimistic stability for the marriage problem where a man and a woman deviate to engage with each others if there is a matching marrying each others and they prefer to the current one. In this appendix, we do not compare our stabilities to those already proposed in the literature even though these have been used to develop ours.

By definition of the problem, a matching not satisfying the scheduling constraints (i.e. with contracts corresponding to tasks without assigned predecessors) is not feasible. In Section 8.3, this notion was naturally embedded in the firms' choice functions by considering feasible sets. In this appendix, this also holds true and requiring individually rationality implies feasibility for any firm.

As in the well-known definitions of substitutability's given by Sasaki and Toda in [20], we define the matching μ_x induced by the deviation of a pair $(\mathcal{F}(x), \mathcal{W}'x))$ for contract x as,

Definition 155. *The matching* μ_x *induced by the deviation of the pair* (w, f) *with contract* $x \notin \mu$ *is such that:*

- $\mu^x(f) = c_f(\mu \cup \{x\}|\mu) \setminus r_w(\mu \cup \{x\}|\mu)$
- $\mu^x(w) = c_w(\mu \cup \{x\}|\mu) \setminus r_f(\mu \cup \{x\}|\mu)$
- $\mu^x(i) = \mu(i) \setminus r_f(\mu \cup \{x\}|\mu) \setminus r_w(\mu \cup \{x\}|\mu) \quad \forall i \notin \{w, f\}.$

8.10.1 Optimistic Pairwise Stability with Scheduling Externalities

Let us define the notion of *optimistic schedule compatible blocking pair* that embeds the fact that the deviation of a pair (w, f) with contract x cannot induce a matching μ_x that is not feasible and no feasible matching containing x can be enforced by a subsequent multi-contract deviation with set of contracts Y ($x \in$ Y). Such deviation would be inconsistent and would lead to a non-persistent matching because it is actually not a "good choice" for the firm to deviate.

Definition 156 (Optimistic Schedule Compatible Blocking Pair)**.** *Given a matching* μ*, a pair* (i, j) *with contract x optimistically schedule-compatibly blocks* μ *if it blocks* μ *and there exists a subset of contracts* $\mathrm{Y} \subseteq \mathcal{X}$ *such that,*

- $x \in \mathrm{Y}$
- $\mathrm{Y}_i \subseteq c_i(\mu^x \cup \mathrm{Y}|\mu^x)$ *for every agent i in* $\mathcal{F}(\mathrm{Y}) \cup \mathcal{W}(\mathrm{Y})$
- $\mathcal{T}(\mathrm{Y}) \supseteq \mathcal{N}^-(\mathcal{T}(x))$.

An interpretation is that if the path of play leads to the matching μ and the pair (f, w) deviates for x at μ there is no "best-response" (even though considering simultaneous group deviations for set of contracts Y) to the resulting matching that would make the realization of x feasible.

The corresponding *optimistic pairwise stability with scheduling externalities* is defined as,

Definition 157 (Optimistic Pairwise Stability with Scheduling Externalities)**.** *A matching* μ *is optimistically pairwise stable with scheduling externalities if,*

- *It is individually rational for all agents,*
- *There are no optimistic schedule compatible blocking pair.*

Thus, the only admitted blocking pairs must be schedule incompatible. An interpretation is that when deviating from μ to μ_x by choosing $c_w(\mu \cup \{x\}|\mu)$ and $c_f(\mu \cup \{x\}|\mu)$, the agents in $\mathscr{F}(Y) \cup \mathscr{W}(Y)$ must have the incentive to cooperatively maintain at least the feasibility of $\mathscr{T}(x)$ w.r.t. f's constraints. Furthermore, observe that no scheduling constraints implies that any blocking pair (w, f) with x is *schedule-compatible*. Thus, in this setting the optimistic pairwise stability with scheduling externalities is pairwise stable (there is no blocking pair). In such case, simply take $Y = \{x\}$.

Example 158. *Consider the motivating example shown in Figure 8.5 (see section 8.2.3).*

As already shown, there is no pairwise stable matching. Nevertheless, there exists an optimistic pairwise stable matching: $\mu = \{x^1_{11}, x^2_{22}\}$. *Matching* $\mu = \{x^1_{11}\}$ *is blocked by* x^2_{22} *and* (f_2, w_2) *deviates for* $\mu' = \{x^1_{11}, x^2_{22}\}$. *Matching* $\mu = \{x^1_{11}, x^2_{22}\}$ *is blocked by* x^3_{22} *and* (f_2, w_2) *have the incentive to deviate for* $\mu' = \{x^1_{11}, x^3_{22}\}$*, but by taking* $Y = \{x^2_{22}, x^3_{22}\}$ *(since it is the only subset of contracts that can make* x^3_{22} *feasible) we obtain* $x^2_{22} \not\in c_{w_2}(Y)$. *Thus* (w_2, f_2) *with* x^3_{22} *is schedule incompatible and* $\mu = \{x^1_{11}, x^2_{22}\}$ *is optimistically pairwise stable with scheduling externalities.*

8.10.2 Weak Pairwise Stability with Scheduling Externalities

We now consider a weaker stability that allows only for a single pairwise deviation to make the matching induced by the deviation of the blocking pair feasible. The idea is that if a blocking pair leads to a matching μ_x that is not feasible (otherwise, taking $x' = x$ is sufficient) and there exists another other contract that both w' and f want to enforce at μ_x that gives a feasible matching w.r.t. $\mathscr{T}(x)$, then the pair $(\mathscr{F}(x), \mathscr{W}(x))$ blocks μ.

Definition 159 (Pairwise Schedule Compatible Blocking)**.** *Given a matching* μ*, a pair* (m, w) *with contract* x *weakly pairwise schedule-compatibly blocks* μ *if it blocks* μ *and there exists a contract* $x' \in \mathscr{X}(f)$ *between a worker* w' *and* f *such that,*

- $x' \not\in \mu^x$
- $x' \in c_{w'}(\mu^x \cup \{x'\}|\mu^x)$
- $x' \in c_f(\mu^x \cup \{x'\}|\mu^x)$
- $\mathscr{N}^-(\mathscr{T}(x)) \subseteq \mathscr{T}(\mu^x)$.

The path of play leads to the matching μ and the pair (i, j) deviates for x at μ there is no "best-response" to the resulting matching that would make the realization of x feasible As previously, the deviation (w, f) with x is inconsistent because it is actually not a "good choice" for the firm. An interpretation is that when deviating from μ to μ_x by choosing $c_w(\mu \cup \{x\}|\mu)$ and $c_f(\mu \cup \{x\}|\mu)$, w' and f must have the incentive to cooperatively maintain at least the feasibility of $\mathscr{T}(x)$ w.r.t. f's constraints.

Definition 160 (Weak Pairwise Stability with Scheduling Externalities)**.** *A matching* μ *is weakly optimistically pairwise stable with scheduling externalities if,*

- *It is individually rational for all agents,*
- *There are no pairwise schedule-compatible blocking pair.*

Example 161. *As previously, consider the first example from Section 8.2.3. There exists a weakly pairwise stable matching* $\mu = \{x^1_{11}, x^2_{22}\}$. *Assume the matching* $\mu = \{x^1_{11}, x^2_{22}\}$ *that blocked by* x^3_{22} *and the pair* (f_2, w_2) *have the incentive to deviate for* $\mu' = \{x^1_{11}, x^3_{22}\}$*, but by taking* $x' = x^2_{22}$ *(since it is the only contract that can make* x^3_{22} *feasible), we have,* $x^2_{22} \not\in c_{w_2}(\{x^2_{22}, x^3_{22}\})$. *Thus* (w_2, f_2) *with* x^3_{22} *is not pairwise schedule compatible blocking* $\mu = \{x^1_{11}, x^2_{22}\}$. *The matching* $\mu = \{x^1_{11}, x^2_{22}\}$ *is weakly pairwise stable with scheduling externalities.*

8.10.3 Pairwise stability with scheduling externalities

Finally, we define the *pairwise stability with scheduling externalities*, that simply asks for no blocking pair such that the resulting matching is feasible. This stability does not allow for two-steps sighted agents that would deviate and expect a further deviation leading to feasibility of the first one.

Definition 162 (Pairwise Stability with Scheduling Externalities)**.** *A matching* μ *is pairwise stable with scheduling externalities if,*

- *It is individually rational for all agents,*
- *There is no blocking pair* (w, f) *with contract* x *such that* μ_x *is feasible.*

Example 163. *As in the previous two examples, consider the first example from Section 8.2.3. We show that the matching* $\mu = \{x_{11}^1, x_{22}^2\}$ *is a pairwise stability matching with scheduling externalities. Assume* $\mu = \{x_{11}^1, x_{22}^2\}$*. This matching is blocked by* x_{22}^3 *and the pair* (f_2, w_2) *have the incentive to deviate for* $\mu' = \{x_{11}^1, x_{22}^3\}$ *that is not feasible. Thus, the matching* $\mu = \{x_{11}^1, x_{22}^2\}$ *is pairwise stable with scheduling externalities.*

In this section we have considered the introduction of three new stabilities in the crowdsourcing problem. Nevertheless, even though such concepts look relevant and appropriate (as observed in our basic example where (w_2, f_2, x_{22}^3) would no more be a blocking pair), they have not been proved (compared to pairwise stability) to be those required to avoid an unravelling of the marketplace. Furthermore, these concept (at least the first two ones) look hardly tractable from an analytical standpoint. Thus, we leave as open further analysis and questions about the relevance and existence of such equilibriums.

8.11 Appendix: Player-Specific Matroid Congestion Games with Priorities

A player-specific matroid congestion games with priorities is defined as,

Definition 164. *Player-Specific Matroid Congestion Games with Priorities, [2] In a player-specific matroid congestion game with priorities, each strategy space $\mathcal{S}_i$ must be the set of bases of a matroid over the set of resources.*

As examples of such games, consider

- Singleton games and games in which the resources are the edges of a graph and every player has to allocate a spanning tree.
- Extension of two-sided markets in which each player can propose to a subset of resources instead of only one, so-called many-to-one markets, and in which the preference lists of the resources can have ties.

For the sake of completeness an in view of the next result (the firms' strategy spaces of our non-cooperative game are basis of firms-specific matroids), we give the definition of a matroid

Definition 165 (Matroid, [2]).

1. *A set system $(\mathcal{R},\mathcal{I})$ with $\mathcal{I} \subseteq 2^{\mathcal{R}}$ [11] is said to be a matroid if $X \in \mathcal{I}$ implies $Y \in \mathcal{I}$ for all $Y \subseteq X$ and if for every $X, Y \in \mathcal{I}$ with $|Y| < |X|$ there exists an $x \in X$ with $Y \cup \{x\} \in \mathcal{I}$.*
2. *A basis of a matroid $(\mathcal{R},\mathcal{I})$ is a set $X \in \mathcal{I}$ with maximum cardinality.*
3. *Every basis of a matroid has the same cardinality which is called the rank of the matroid.*
4. *For a matroid congestion game Γ, we denote by $rk(\Gamma)$ the maximal rank of one of the strategy spaces of the players.*

We have the following results,

Theorem 166 ([2]). *In matroid congestion games with consistent priorities[12], the best response dynamics reaches a Nash equilibrium after a polynomial number of rounds.*

Theorem 167 ([2]). *Matroid congestion games with priorities are potential games with respect to lazy better responses.*

Theorem 168 ([2]). *Every player-specific matroid congestion game Γ with priorities possesses a pure Nash equilibrium that can be computed in polynomial time by $O(m^2.n^3.rk(\Gamma))$ strategy changes.*

As an example of matroid, consider $(\mathcal{X}_i,\mathcal{I}_i)$, where $\mathcal{I}_i$ is the set of contracts of firm i such that,

$$\mathcal{I}_i = \{X \subseteq \mathcal{X}_i | \forall \tau \in \mathcal{T}_i, |X(\tau)| \leq 1\} \tag{8.82}$$

where $X(\tau)$ is the set of contracts in X involving task τ, $\mathcal{I}_i$ is the set of subsets of contracts in $\mathcal{X}_i$ such that in each set, each task of i is assigned at most one contract and the strategy space $\mathcal{S}_i$ is the set of basis of the matroid $(\mathcal{X}_i,\mathcal{I}_i)$, where (recall),

$$\mathcal{S}_i = \{X \subseteq \mathcal{X}_i | \forall \tau \in \mathcal{T}_i, \exists! x \in X \text{ s.t. } \mathcal{T}(X) = \tau\} \tag{8.83}$$

equivalently,

$$\mathcal{S}_i = \{X \subseteq \mathcal{X}_i | \forall \tau \in \mathcal{T}_i, |X(\tau)| = 1\} \tag{8.84}$$

The matroid $(\mathcal{X}_i,\mathcal{I}_i)$ has rank T_i.

[11] $\mathcal{I}$ is thus a set of subsets of elements in the set $\mathcal{R}$.

[12] The priorities assigned to the players by different resources coincide.

Proposition 169.
For any firm $i \in \mathscr{F}$, $(\mathscr{X}_i, \mathscr{I}_i)$ is a matroid of rank T_i with set of basis $\mathscr{S}_i$.

The proof is in four steps,

1. First, show that $\mathscr{I}_i \subseteq 2^{\mathscr{R}}$.
2. Second, show that if $X \in \mathscr{I}$ implies $Y \in \mathscr{I}$ for all $Y \subseteq X$.
3. Third, show that for every $X, Y \in \mathscr{I}$ with $|Y| < |X|$ there exists an $x \in X$ with $Y \cup \{x\} \in \mathscr{I}$.
4. Fourth, show that $\mathscr{S}_i$ is the set of basis of the matroid of rank T_i.

Proof. First, we show that $\mathscr{I}_i \subseteq 2^{\mathscr{R}}$. By definition $\mathscr{I}_i \subseteq \mathscr{X}_i$. Let denote, $\mathscr{I}_i = \{X_k\}_{k \in K}$.

Second, we show that if $X \in \mathscr{I}$ then $Y \in \mathscr{I}$ for all $Y \subseteq X$. Take any $X \in \mathscr{I}$. By definition X is a subset of contracts with at most one contract per task. Take any $Y \subseteq X$, i.e. any subset of the subset of contracts X. Y is also a subset of contracts with at most one contract per task. Thus, $Y \in \mathscr{I}_i$.

Third, we show that for every $X, Y \in \mathscr{I}$ with $|Y| < |X|$ there exists an $x \in X$ with $Y \cup \{x\} \in \mathscr{I}$. Take any pair X and Y in $\mathscr{I}_i$ (i.e. any pair of subsets of contracts with at most one contracts per task), such that $|Y| < |X|$. By definition, there are strictly more contracts in X than in Y. Because there is at most one contract per task, $|Y| < |X|$ means that strictly more tasks are assigned a contract in X than in Y. In other words, there exists a task $\tau \in \mathscr{T}_i$ and a contract $x \in X$ such that $\tau = \mathscr{T}_i(x)$ but, there exists no contract $y \in Y$ such that $\tau = \mathscr{T}_i(y)$. We obtain that the set $Y \cup \{x\}$ is a subset of contracts with at most one contract by task. Thus, there exists $x \in X$ with $Y \cup \{x\} \in \mathscr{I}_i$.

This concludes the proof that $(\mathscr{X}_i, \mathscr{I}_i)$ is a matroid.

Fourth, we show that $\mathscr{S}_i$ is the set of basis of the matroid of rank T_i. By definition, we have

$$\mathscr{I}_i = \{X \subseteq \mathscr{X}_i | \forall \tau \in \mathscr{T}_i, |X(\tau)| \leq 1\} \tag{8.85}$$

and,

$$\mathscr{S}_i = \{X \subseteq \mathscr{X}_i | \forall \tau \in \mathscr{T}_i, |X(\tau)| = 1\} \tag{8.86}$$

For any set X in $\mathscr{S}_i$, every task in $\mathscr{T}_i$ has a contract in X. By definition of $\mathscr{I}_i$, no contract in $x \in \mathscr{X}_i \backslash X$ can be added to X such that $X \cup \{x\} \in \mathscr{I}_i$. In fact, in $X \cup \{x\}$ there is a task with two contracts. The set X has maximum cardinality. Thus, X is a basis of the matroid $(\mathscr{X}_i, \mathscr{I}_i)$.

By definition, $\mathscr{S}_i$ contains the set of such basis and the matroid has rank T_i. □

There are interesting similarities between matroid congestion games with priorities and the crowdsourcing problem in normal form, particularly,

- In player-specific matroid congestion games with priorities: each strategy space consists of the bases of a matroid over the resources.
- In the crowdsourcing problem in normal form: each strategy space consists of the basis of a player-specific matroid (based on the player's set of contracts).

Nevertheless, our crowdsourcing setting is more general than the player-specific matroid congestion games with priorities, in fact:

- Workers can have preferences over groups more complex than those induced by the preferences individuals (priorities of resources over players). Particularly, we allow for complementarities and externalities.
- Firms' utilities (if not implicitly given by preferences and choice functions) may be more general than the sum of specific congestion costs (functions of the numbers of players associated to the resource) over the chosen resources
- We allow for scheduling constraints and externalities on the firms' side that are more general than the congestion-like externalities.

Chapter 9

Open Questions

9.1 Nash Bargaining, Markov Chains and Relative Entropy

In this section, we show that there exists a link between Nash's solution to the bargaining problem, the second law of thermodynamics, information theory and stationary distributions of some Markov chains. Particularly, we show that the minimization of the relative entropy, a well-known measure of divergence between probability distributions, may be the fundamental reason for the existence of such connection. First, we give a brief background of Markov chains, then we give some motivating examples where the generalized Nash's solution is the stationary distribution of the chains, finally we give our result. Further developments and analysis are left as open question.

9.1.1 Background on Markov Chains

Let $(\Omega, \mathscr{F}, \mu)$ be a probability space and let $\{X_t\}_{t\geq 0}$ be a sequence of random variables. The sequence $\{X_t\}_{t\geq 0}$ is a Markov chain with state space D if, for all $t \geq 1$ and all$x_0, \ldots, x_t \in D$,

$$\mathbb{P}(X_{t+1} = x_{n+1} | X_1 = x_1, \ldots, X_t = x_t) = \mathbb{P}(X_{t+1} = x_{t+1} | X_t = x_t) \tag{9.1}$$

If the probability transitions do not depend on t, the chain is called *time-homogeneous* and can be described by a $|D| \times |D|$ stochastic matrix[1] $\mathbf{P}$.

A Markov chain is thus a stochastic process such that, at any time $t \geq 1$, the probability that the process enters a given state (in the state space) does only depend on the time and the state visited by the process at the previous instant. If the chain is homogeneous, the transition probabilities do not depend on time. For the rest of this chapter, we assume the time-homogeneity property.

We give some of the important properties characterizing the states and the chain itself (see [5] for more details). A state i in D is recurrent if the chain returns to the state with probability one, and is transient if not recurrent. A state j in D is *accessible* from another state i in D, if $P^n_{ij} > 0$ for some $n \geq 1$. A state i in D is called *essential*, for any state j accessible from i, i is also accessible from j. A state is called *inessential* is it is not essential. States i and j *communicate* with each other if j is accessible from i and reciprocally. A set of states C in D is *closed* if no state outside of C is accessible from any state in C ($P^n_{ij} = 0, \forall i \in C, j \in D - C$). If a closed set is reduced to a singleton, the state is called *absorbing*. The set C is irreducible if the states of any pair in C communicate. A Markov chain is irreducible if its state space is an irreducible set. The period r_i of a state is the GCD[2] of all n that satisfy $P^n_{ii} > 0$. If the period r_i of i is equal to one, the state is called *aperiodic*.

[1] A stochastic matrix has positive components with each row summing to one.
[2] Greatest Common Divisor.

We now turn to the characterization of the occupation of the states. A probability measure π on the state space D is called *stationary distribution* for the chains if,

$$\pi = \pi \mathbf{P} \tag{9.2}$$

which can equivalently be written,

$$\pi(j) = \sum_{i \in D} \pi(i) P_{ij}, \ \forall j \in D \tag{9.3}$$

Such distribution is not guaranteed to exist for all Markov chains, and in case it exists, it is not guaranteed to be unique. The following result shows the equivalence between the unicity of the stationary distribution and the existence of a unique essential communicating class.

Proposition 170 ([5], pp.17)**.** *The stationary distribution π for a transition matrix $\mathbf{P}$ is unique iff there is a unique essential communicating class.*

We have the following sufficiency result,

Proposition 171 ([5], pp.14)**.** *Let $\mathbf{P}$ be the transition matrix of a Markov chain with state space* D*. Any distribution π satisfying the detail balance equations,*

$$\pi(i) P_{i,j} = \pi(j) P(j,i), \ \forall i, j \in D \tag{9.4}$$

is stationary for $\mathbf{P}$.

A chain satisfying the detail balance equations is called *reversible*. For irreducible Markov chains, we have the following existence and unicity results,

Proposition 172 ([5], pp.14)**.** *Let $\mathbf{P}$ be the transition matrix of an irreducible Markov chain. There exists a unique probability distribution π on* D *satisfying $\pi = \pi \mathbf{P}$.*

9.1.2 The Kullback-Leibler Divergence

The Kullback-Leibler divergence (also known in the name of Kullback-Leibler information or relative entropy) between two probability distributions p and q, is defined as,

$$D_{KL}(p||q) = \sum_x p(x) \log\left(\frac{p(x)}{q(x)}\right) \tag{9.5}$$

We also have the following well-known theorem,

Theorem 173. $D_{KL}(p||q) \geq 0$ *with equality if and only if $p(x) = q(x)$ for all x.*

Consider two joint probability mass functions $p(x,y)$ and $q(x,y)$. The conditional relative entropy $D_{KL}(p(y|x)||q(y|x))$ between $p(y|x)$ and $q(y|x)$ is defined as,

$$D\left(p(y|x)||q(y|x)\right) = \sum_x p(x) \sum_y p(y|x) \log\left(\frac{p(y|x)}{q(y|x)}\right) \tag{9.6}$$

We have the chain rule for the relative entropy,

Theorem 174 ([4], pp.24)**.** $D_{KL}(p(x,y)||q(x,y)) = D(p(x)||q(x)) + D(p(y|x)||q(y|x))$

As shown in [2] the Kullback-Leibler divergence can be interpreted in many ways. We give two of these interpretations, (i) the relative entropy is the expected log likelihood ratio between distributions p and q, (ii) the relative entropy is the degree of difficulty in distinguishing two distributions. One of the main result shown in [2] is the following,

Theorem 175 ([2])**.** *Let π_n and π'_n be two probability distributions on the state space of a finite state Markov chain at time n. Then $D(\pi_n||\pi'_n)$ is monotonically decreasing. In particular, if π is the unique stationary distribution,*

$$D_{KL}(\pi_n||\pi) \searrow 0 \tag{9.7}$$

9.1.3 Motivating Examples

The examples used in this section come from [5].

Example 176. *Consider a* 2*-state Markov chain with transition matrix* P *such that,*

$$\mathrm{P} = \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix} \tag{9.8}$$

for some $p, q \in (0,1)$*. The stationary distribution* π *of this chains is,*

$$\pi_1 = \frac{q}{p+q} \quad \pi_2 = \frac{p}{p+q} \tag{9.9}$$

Now, consider a 2*-players bargaining problem over the probability simplex* $\Delta^1 = \{\mathbf{x} \in [0,1]^d : x_1 + x_2 = 1\}$*. The utility function player* 1 *is* $u_1(x_1) = x_1^{1/p}$ *and the utility function of player* 2 *is* $u_2(x_2) = x_2^{1/q}$*. Assume a null threat vector* $\mathbf{t}$*, where* t_i *is the threat of player* i*.*
The generalized (asymmetric) Nash solution to this bargaining problem solves,

$$\begin{aligned} \underset{\mathbf{x}}{\text{maximize}} \quad & x_1^{1/p} x_2^{1/q} \\ \text{subject to} \quad & x_1 + x_2 = 1 \\ & 0 \le x_i \le 1, \; i = 1,2. \end{aligned} \tag{9.10}$$

We have the solution such that,

$$x_1 = \frac{1/p}{1/p + 1/q} = \frac{q}{p+q} \tag{9.11}$$

and

$$x_2 = \frac{1/q}{1/p + 1/q} = \frac{p}{p+q} \tag{9.12}$$

Example 177. *Consider a generalization of the previous example to* d*-state Markov chains of the following form:*

$$\mathrm{P}_{i,j} = \begin{cases} 1-\epsilon_i & \text{if } i = j; \\ \frac{\epsilon_i}{d-1} & \text{if } i \neq j; \end{cases} \tag{9.13}$$

for some $\epsilon_1, \epsilon_2, \ldots, \epsilon_d \in (0,1)$*. This chain is ergodic and reversible and has a unique stationary distribution* π *such that,*

$$\pi_i = \frac{1/\epsilon_i}{\sum_{j=1}^d 1/\epsilon_j}, \quad \forall i \in \{1, \ldots, d\} \tag{9.14}$$

Now, consider a d*-players bargaining problem over the probability simplex* $\Delta^{d-1} = \{\mathbf{x} \in [0,1]^d : \sum_{i=1}^d x_i = 1\}$*. The utility function any player* i *in* $\{1, \ldots, d\}$ *is,*

$$u_i(x_i) = x_i \tag{9.15}$$

Assume a null threat vector $\mathbf{t}$*, where* t_i *is the threat of player* i*.*
The generalized Nash solution with individual bargaining powers solves,

$$\begin{aligned} \underset{\mathbf{p}}{\text{maximize}} \quad & \prod u_i(x_i)^{\alpha_i} \\ \text{subject to} \quad & \sum_{i=1}^d x_i = 1 \\ & 0 \le x_i \le 1, \; i = 1, \ldots, d. \end{aligned} \tag{9.16}$$

where α_i *is called the bargaining power of player* i*.*
Taking $\alpha_i = \frac{1}{\epsilon_i}$*, we obtain,*

$$\begin{aligned} \underset{\mathbf{p}}{\text{minimize}} \quad & -\prod x_i^{1/\epsilon_i} \\ \text{subject to} \quad & \sum_{i=1}^d x_i = 1 \\ & 0 \le x_i \le 1, \; i = 1, \ldots, d. \end{aligned} \tag{9.17}$$

The unique solution to this problem is the allocation vector $\mathbf{p}$ *such that,*

$$x_i = \frac{1/\epsilon_i}{\sum_{j=1}^{d} 1/\epsilon_j} \tag{9.18}$$

Thus, the generalized solution to the d*-players bargaining is the unique stationary distribution of the Markov chain.*

Example 178. *Consider a simple random walk on a graph* $G = (V, E)$*,*

$$P(x, y) = \begin{cases} \frac{1}{deg(x)} & \text{if } y \sim x, \\ 0 & \text{otherwise.} \end{cases} \tag{9.19}$$

where $x \sim y$ *denotes that* y *is a neighbor of* x *(and reciprocally).*
For any vertex $y \in V$*,*

$$\sum_{x \in V} deg(x) P(x, y) = \sum_{x \sim y} \frac{deg(x)}{deg(x)} = deg(y) \tag{9.20}$$

Normalize by $\sum_{y \in V} deg(y)$ *to obtain a probability. It is shown that a stationary distribution for the walk is always given by the probability measure,*

$$\pi(y) = \frac{deg(y)}{\sum_{v \in V} deg(v)}, \quad \forall y \in \Omega \tag{9.21}$$

As in the previous example, consider a $|V|$*-players bargaining problem over the probability simplex* $\Delta^{d-1} = \{\mathbf{x} \in [0,1]^d : \sum_{i=1}^{d} x_i = 1\}$*. Any player* i *is mapped to a node* v_i *in* V *and no two players can be mapped to the same node. The utility function of any player* i *in* $\{1, \ldots, d\}$ *is,*

$$u_i(x_i) = x_i^{deg(v_i)} \tag{9.22}$$

where v_i *is the degree of the node* $v_i \in V$ *corresponding to player* i*. Assume a null threat vector* $\mathbf{t}$*, where* t_i *is the threat of player* i*.*

Using the results of the previous example, we have the solution to the allocation problem as,

$$x_i = \frac{deg(v_i)}{\sum_j deg(v_j)}, \quad \forall i \in \{1, \ldots, d\} \tag{9.23}$$

Example 179. *A spin system is a probability distribution on* $D = \{-1, 1\}^V$*, where* V *is the vertex set of a graph* $G = (V, E)$*. Each state* σ *in* D *characterizes the configuration of the set of vertices by mapping each vertex* v *to its corresponding state* $\sigma(v)$ *in* $\{-1, +1\}$*. The state* $\sigma(v)$ *of a vertex* v *is called the spin at* v*. Consider the nearest-neighbor Ising model. The energy* H *of a configuration* σ *is,*

$$H(\sigma) = -\sum_{v, w \in V v \sim w} \sigma(v)\sigma(w) \tag{9.24}$$

The Gibbs distribution corresponding to H *is the probability distribution* μ *on* D *such that,*

$$\mu(\sigma) = \frac{\exp^{-\beta H(\sigma)}}{Z(\beta)} \tag{9.25}$$

where $\beta \geq 0$ *and* $Z(\beta)$ *(called the partition function) is defined by,*

$$Z(\beta) = \sum_{\sigma \in D} e^{-\beta H(\sigma)} \tag{9.26}$$

Assuming the Glauber dynamic, it is known that the stationary distribution of the system is given by the Gibbs distribution μ*. Observe, the Glauber dynamics for* π *is a reversible Markov chain with stationary distribution* π*.*

According to the previous results, it is clear that the D*-players generalized Nash bargaining over* Δ^{D-1} *with bargaining powers* $\{\alpha_\sigma = \exp^{-\beta} H(\sigma)\}_{\sigma \in D}$ *gives the stationary distribution* μ*.*

9.1.4 Result

As a first result toward the understanding of the link between Nash solution to the bargaining problem and information theory, we have shown the following result.

Proposition 180. *Consider a d-person Nash bargaining* $(\Delta^{d-1}, \{x_i\}, \mathbf{t} = \mathbf{0})$. *Furthermore, consider the probability distribution* $\mathbb{P}$ *over* $\{1, \ldots, d\}$ *such that,*

$$p(i) = \mathbb{P}(X = i) = \frac{\alpha_i}{\sum_j \alpha_j}, \; \forall i \in \{1, \ldots, d\} \tag{9.27}$$

The generalized solution to the bargaining problem with utility functions $u_i(x_i) = x_i$ *and bargaining powers* $\alpha_i > 0$, *for any* i, *minimizes the Kullback-Leibler divergence* $D_{KL}(p||\mathbf{x})$ *to the probability distribution* p.

Proof. Consider the bargaining problem $\Gamma = (\Delta^{d-1}, \{x_i^{\alpha_i}\}, \mathbf{t} = \mathbf{0})$ with positive bargaining powers α_i. The generalized Nash solution to this problem solves the following optimization problem,

$$\begin{aligned} \underset{\mathbf{x}}{\text{maximize}} \quad & \prod x_i^{\alpha_i} \\ \text{subject to} \quad & \sum_{i=1}^{d} x_i = 1 \\ & 0 \le x_i \le 1, \; i = 1, \ldots, d. \end{aligned} \tag{9.28}$$

which can equivalently be written,

$$\begin{aligned} \underset{\mathbf{x}}{\text{minimize}} \quad & -\sum_i \alpha_i \log(x_i) \\ \text{subject to} \quad & \sum_{i=1}^{d} x_i = 1 \\ & 0 \le x_i \le 1, \; i = 1, \ldots, d. \end{aligned} \tag{9.29}$$

or,

$$\begin{aligned} \underset{\mathbf{x}}{\text{minimize}} \quad & \sum_i \alpha_i \log(\frac{1}{x_i}) \\ \text{subject to} \quad & \sum_{i=1}^{d} x_i = 1 \\ & 0 \le x_i \le 1, \; i = 1, \ldots, d. \end{aligned} \tag{9.30}$$

Any linear transformation $af_0 + b$ with $a > 0$ of the objective function $f_o(\mathbf{x}) = \sum_i \alpha_i \log(\frac{1}{x_i})$ gives an equivalent problem. Taking,

$$a = \frac{1}{\sum_j \alpha_j}, \quad b = \sum_i \frac{\alpha_i}{\sum_j \alpha_j} \log\left(\frac{\alpha_i}{\sum_j \alpha_j}\right) \tag{9.31}$$

where a is introduced as a normalization factor so that $\frac{\alpha_i}{\sum_j \alpha_j}$ is a probability for any i in $\{1, \ldots, d\}$. We obtain the following equivalent problem,

$$\begin{aligned} \underset{\mathbf{x}}{\text{minimize}} \quad & \sum_i \frac{\alpha_i}{\sum_j \alpha_j} \log(\frac{\alpha_i / \sum_j \alpha_j}{x_i}) \\ \text{subject to} \quad & \sum_{i=1}^{d} x_i = 1 \\ & 0 \le x_i \le 1, \; i = 1, \ldots, d. \end{aligned} \tag{9.32}$$

which, by definition of the KL-divergence, can be written in the following form,

$$\begin{aligned} \underset{\mathbf{q}}{\text{minimize}} \quad & \mathrm{D}_{\mathrm{KL}}(p||q) \\ \text{subject to} \quad & \sum_{i=1}^{d} q_i = 1 \\ & 0 \le q_i \le 1,\ i = 1, \dots, d. \end{aligned} \tag{9.33}$$

where p is the probability distribution over $\{1, \dots, d\}$ such that,

$$p(i) = \mathbb{P}(X = i) = \frac{\alpha_i}{\sum_j \alpha_j} \tag{9.34}$$

We conclude that the generalized Nash solution to the bargaining problem minimizes the Kullback-Leibler divergence to the probability distribution p. □

It is interesting to observe that Nash's solution to some bargaining problem minimizes an entropy. This shows the eventual link between Nash's solution and information theoretic properties. Did Nash embedded in his solution an uncertainty principle? What about the others (Kalai, Smorodinsky, Tijs, etc.)?

More generally, this shows that there may be a link between game-theoretic solutions to the bargaining problem as cooperative resource allocation problems and well-known information theoretic results. This may lead to both new interpretations and stochastic processes (leading to protocols) seen as bargaining mechanisms for resource allocation as done in statistics with probability distributions sampling based Markov Chain Monte Carlo (MCMC) algorithms.

9.2 Strategic Information Transmission and Recommendations Systems

As shown in this thesis, the theory of stable matchings is one of the most successful branch of game-theory. The set of applications to the sector of Information Technology (IT) is important and remains unexploited. Here, we focus a problem related to the control of information in matching problems. Consider a repeated association or matching game with incomplete information where buyers and sellers repeatedly use an online platforms and match with each others using the marketplace's matching mechanism. By buying and selling players acquire information on their own as a result of their decisions. Nevertheless, in the internet, the platform also provides recommendations to the players (basically recommending goods or sellers to the buyers and recommending buyers to the sellers). In the most basic example of the marriage problem (see chapter 3), buyers are men and sellers are women. In such case, the platform would recommend women to the men and men to the women, as commonly done in online dating services. Furthermore, assume that the players receive a feedback from others on the quality of the advices provided by the recommendation system. In such case, how should this recommendation system control the information to be sent to the players to turn some points into equilibrium or to speed up the convergence to some equilibrium?

Intuitively, because of the incomplete information setting, the players acquire information and transform it into decisions through their belief and learning process. Using its ability to control some information the players do not have, the recommendation system may have the incentive to manipulate this to change the decision-taking process and make it converge to a preferred matching compared to the one obtained without control. Nevertheless, because the player acquire information on their own there must be a tradeoff between non-truthful recommendations w.r.t. the players' preferences (namely, lying to the agents on what they should do) and truthful ones. Furthermore, assuming that there is a feedback from the players to the players introduces an additional correlation across them, namely the reputation of the platform. Recommending wrong items to an agent may have an impact on the decision-taking of another agent w.r.t. its recommendations, even though truthful.

Let formalize the concepts of recommendations system in an entity called the *sender*. At this point, using the term *sender* is abusive because this term is usually devoted to a specific player in the sender-receiver game (see [3]). A rigorous analysis should clearly determine wether or not our sender is actually one as defined sender-receiver games and why it is not if so. In our setting, the sender of the game abstractly models the information conveying property of the environment of the game. In other words, the sender models the fact that the game happens in a wider system that measures (observes) and, naturally or in a controlled way, disseminates information to the players of the game.

From the point of view of the ordinal theory, one may reformulate the previous model and consider that the players repeatedly emit preferences as a function of the acquired information (on their own, with feedbacks and through recommendations). In such case, the sender or information sender may want to manipulate the information so as to induce a appropriate permutations in the players preferences. Formally, the permutations in the players preferences may be defined by a transition kernel of the form

$$\mathbb{P}(\text{preferences of } i \text{ at epoch } t | \text{history at } t-1, \text{recommendations at } t_-) \tag{9.35}$$

where the history may be composed of the set of past matchings, feedbacks, recommendations and preferences to the user i, recommendations at t_- is the information transmitted by the system to i. The adapted approach to tackle this problem may be a combination of the analysis developed in [6] and [7].

9.2.1 Motivating Example

Example 181. *Consider a dynamic house allocation problem where a set of buyers have preferences over (sellers') goods to be allocated. There is one good per seller and as many sellers as buyers. We show an example of the setting in Figure 9.1. We assume that the sellers delegate the sale to a rep-*

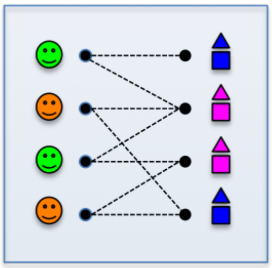

Figure 9.1: The house allocation problem.

resentative called real estate agent (the sender) and that there is a unique real estate agent on the market. The buyers can visit the houses for a finite number of rounds according to a visit mechanism and then emit their final preferences over the goods that will be allocated to them according to a matching mechanism. In Figure 9.2 we show the visit mechanism and the discovery of the first player of the state of the houses.

The preferences of the buyers evolve over time as functions of their visits and their knowledge about the goods. The real estate agent can send informations to the players (mails, phone, etc.) in order to influence the players' true preferences and make the decision process converge faster or to manipulate the equilibria. The set of information provided to the buyers by the real estate agent may contain information about a product that is assumed known to the buyer (e.g. already visited) or about an assumed unknown product. It may also be that the buyers having already visited the house can give a feedback on it and on the recommendations given by the real estate agent.

Example 182. *Consider a bipartite social network or online dating services. This setting is close to the the historical Gale and Shapley's marriage problem [1]. We assume that the players cannot see each others perfectly. Each player has an individual private state that the other players do not know.*

Figure 9.2: Example: Visiting houses. On the left, the green agent knows that there exists two houses on the market but has no information about these. These houses are shown in white. Uncolored ones are unknown to the green agent (who does not know that they exist). On the right, the green agent has visited the houses and know their true state as shown in Figure 9.1.

In terms of preferences, the preferences emitted by a player are based on a partial or incomplete information. Furthermore, let define the sender as the online dating service's recommendation engine. The engine can selectively send information to the players. As an example, the sender provides the players recommendations over strangers (either show their existence or give more information about them). For a player to know the state of another player, a date must be organized. The dates allow for the players to partially observe their respective states. As long as a date has not occurred between the players of a pair the two players are strangers and use the prior information they have over each others. If the states of the players change [3] *then a single date between any pair of players is only sufficient for them to mutually observe their instantaneous states. Due to the limited validity of the acquired information, subsequent attempts are necessary for them to learn from each others about their dynamics or expected properties.*

Figure 9.3: Example: Dating with strangers.

In Figure (9.4), the sender recommends the top-left player called Bob (green state) two unknown women. Before the recommendation, Bob did not know about the existence of these two players exist.

In Figure (9.5), the sender recommends the second top-right player called Alice (green state) to Bob and provides some information about Alice's state. It may be that the sender does not know, partially knows or fully knows the state of a player. The state of knowledge of the sender may even vary among the players. Before the recommendation, Bob knew that Alice existed but did not know about her.

The information given by the sender may or not be true w.r.t. the agents' preferences. We mainly identify three reasons for the sender to provide false informations. The first reason is the imperfect observation or estimation of the game (e.g. of the agents' preferences). The second reason is that it is imperfect in itself. We assume that the sender may be misleading (due to an incorrect analysis of the environment or estimation process) even though receiving right signals from the game. The third reason is that it is a strategic entity that may attempt manipulating the matching mechanism to achieve his or her own objectives.

[3] Either deterministically of stochastically.

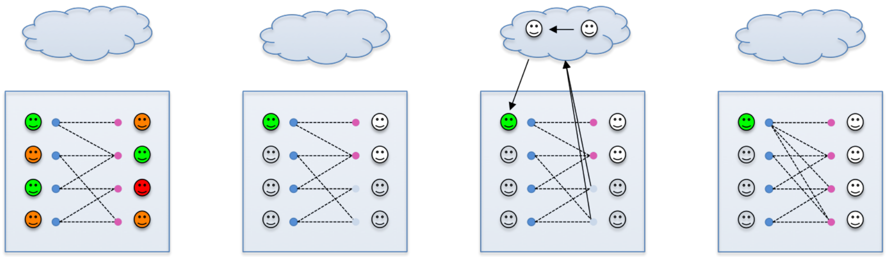

Figure 9.4: Example: Recommending outsiders. The arrows shows the recommendation (by the sender) of outsiders to Bob. The first figure shows the true state of the market (the agents properties). The second one shows that Bob originally knows that he can have a date with two women, but doesn't know them. The third figure shows a recommendation to Bob: there are two women involved that Bob doesn't know. The fourth figure shows that Bob has received the information and now considers the four women (he doesn't know) instead of the two original ones.

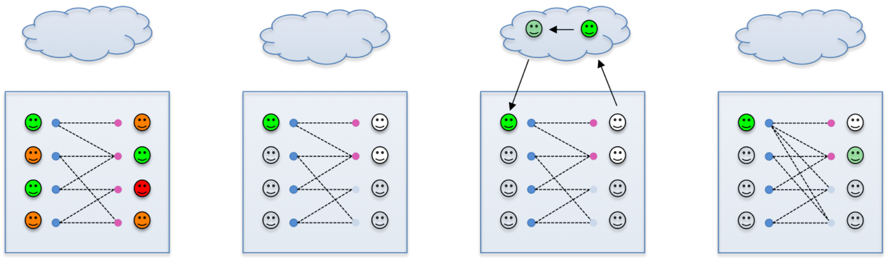

Figure 9.5: Example: Recommending strangers. The first figure show the true state of the market (interns of the agents properties). The second one shows that Bob originally knows that he can have a date with two women, but doesn't know them. The third figure shows a recommendation to Bob: some (but not all) information about a woman (let us call her Alice) are given to Bob. The fourth figure shows that Bob has received the information and integrates it to his knowledge. Bob partially knows Alice's properties. The sender has given Bob some true but partial information on the state of Alice.

One can intuitively conclude that there should be a tradeoff somewhere between truthful and non-truthful recommendations in the repeated framework for the sender to be able to control the state of the market and maintain its impact on the agents' preferences (they should keep on believing that the recommendations are valuable to them).

Such tradeoff shows the problem of control of information in marketplaces. There is a non-negligible potential impact on the design of matching and recommendation mechanisms, advertising mechanisms and on the design of social networks. Among many others, we raise the following questions: Which and how information should be sent to the players so as to make the agents' learning process and the repeated matching mechanism converge to a specific point? Can the sender strategically manipulate the information to control the equilibrium of such game knowing that the agents rate the recommendations?

9.3 References

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Chapter 10

Conclusion

In this thesis, we have used cooperative game theory to solve network problems. We have modeled the systems as two-sided markets and analyzed them using the theory of stable matchings. This branch of game theory in the class of cooperative games has been successfully used by theorists and economists to design matching mechanisms for two-sided marketplaces such as college admissions, the association of interns to hospitals or kidney exchange. As a proof of the impact of matching games, A.E. Roth and L. Shapley received the Nobel prize in 2012 for both the development of a theory of stable allocations and their impact on real-life applications through the practice of market design. The theory of stable matchings has historically been build for the analysis of economic two-sided marketplaces but is also adapted to network problems. In fact, the fundamental motivation of the theory is to assess stability problems in allocation systems and provide tools to design matching mechanisms. These topics are also at the basis of network problems that have been assessed using other efficient tools. Recent generalizations to matching with contracts and externalities or to trading networks strengthen this proximity. As an example, externalities may be interpreted as the influence in social networks, interferences in cellular networks, or as a peer effect such as those observed in WiFi. As another example, trading networks embed supply chain problems that may also be interpreted as scheduling constraints. The literature shows us that there has always been strong links between some network analysis and game-theoretic ones, historically paired to economy. As an example among many others, consider the well-known and celebrated proportional fair allocation that actually is a particular case of Nash's solution to the bargaining problems or routing and congestion management problems that have attracted interests from both sides. Nevertheless, even though there is a densification of the game theoretic analysis in networks, this does not mean that there is convergence between economic and network problems.

At application levels, we observe an increasing concentration of online marketplaces in the internet for business-to-business (B2B, e.g. online advertising market), business-to-consumer (B2C, e.g.) and customer-to-customer (C2C, e.g. airbnb, eBay, Le Bon Coin) applications. Such marketplaces have emerged as new business opportunities. In some cases, the demand and supply of the market is parametrized by a networking activity such as the clicks and views of some users in webpages, videos, etc. As an example, the online advertising business is based on the sell of advertising slots on an online platform (using auctions as a matching mechanism between buyers and the goods sold by the sellers). Sold slots are then used buy the buyers to display their advertisements on the webpages, or contents viewed by users. Furthermore, oncoming technological developments such as online systems for spectrum renting among operators are considered as promising improvements with a strong potential impacts on the existing economics models. Thus, networks and economy have already started to merge in new (potentially complex) business models, systems and applications requiring in their design and conception a unified approach often including game theory (as an example, consider the success of auctions in online applications or the natural propensity of matching with contracts among firms and workers to develop matching mechanisms for crowdsourcing systems). Based on this trend, it looks rather logical to predict that the oncoming and future systems will also heavily rely on such unified approach.

Our analysis falls in the scope of this observation. In this thesis, we have studied problems at the frontier of networks, game theory and economy, either because we have pushed them from pure allocation problems in networks toward game theory and made them fall in the scope of some economic works or because they naturally fall in the scope of both worlds. As an example consider the WiFi association problem. We have shown that the WiFi protocol induces a medium allocation and individual throughputs that can be modeled as resulting from a Nash bargaining, a solution proposed by Nash to the cooperative bargaining problem and shown to belong to the set of core stability inducing sharing rules in coalition formation problems and matching games. We have formulated the WiFi association problem as a two-sided matching game and proposed a controlled matching mechanism, using in the numerical implementations a tax rate over the gross incomes of the coalitions, namely their total throughputs. Then, we have shown that when considering the generalized α-fair allocation, the stability inducing concavity conditions can be easily formulated in terms of risk aversion indicators, usually used in economy or finance to study the behaviors of agents w.r.t. risks. As another example, we have considered an online crowdsourcing platform with scheduling constraints on the firms' side and shown that this unsolved problem falls in the scope of analysis of the theory of stable matchings. Using some of the most recent works in the domain, we analyzed this two-sided marketplace and derived new conditions for the existing mechanisms to converge to stable matchings. Furthermore, based on previous works partially unifying two-sided markets and congestion games, we have proposed an approach to solve the original problem using a non-cooperative formulation (in normal and extensive form).

New applications at the intersection of networks and economy on one hand and game theory and artificial intelligence or machine learning on the other hand give rise to fascinating and challenging problems. As an example, consider social networks, recommendations systems or marketplaces where the existence, structure, volume and propagation of data and information may have a huge impact on the individual and overall performances. This is not only about signal processing but also on the modeling of the information itself and the identification of its role in the system to control it in an efficient way. How to influence the graph properties of a social network by controlling the information? How to influence the amount of goods sold to customers on a marketplace by recommendations and the introduction of feedbacks and rates from the customers to the customers? How much should customers know about the others' experience?

Chapter 11

Publications

11.1 Journals

- M. Touati and R. El-Azouzi and M. Coupechoux and E. Altman and J.M. Kelif, Controlled Matching Game for User Association and Resource Allocation in Multi-Rate WLANs, *Under major review, IEEE Journal on Selected Areas in Communications Special Issue on Game Theory for Networks*, 2016.
- M. Touati and R. El-Azouzi and M. Coupechoux and E. Altman and J.M. Kelif, Core Stable Algorithms for Coalition Games with Complementarities and Peer Effects, *ACM SIGMETRICS Performance Evaluation Review*, vol. 43, no.3, pp. 72–75, December, 2015.
- M. Touati and R. El-Azouzi and M. Coupechoux and E. Altman and J.M. Kelif, About Joint Stable User Association and Resource Allocation in Multi-Rate IEEE 802.11 WLANs, *ACM SIGMETRICS Performance Evaluation Review*, vol. 43, no.2, pp. 30–31, September, 2015.

11.2 Conferences with review committee

- M. Touati and R. El-Azouzi and M. Coupechoux and E. Altman and J.M. Kelif, About Joint Stable User Association and Resource Allocation in Multi-Rate IEEE 802.11 WLANs, *The 33rd International Symposium on Computer Performance, Modeling, Measurements and Evaluation (IFIP WG 7.3 Performance)*, Sydney, Australia, October, 2015.
- M. Touati and R. El-Azouzi and M. Coupechoux and E. Altman and J.M. Kelif, Controlled matching game for user association and resource allocation in multi-rate WLANs, *The 53rd Annual Allerton Conference on Communication, Control, and Computing (Allerton)*, Monticello, USA, September, 2015.
- M. Touati and R. El-Azouzi and M. Coupechoux and E. Altman and J.M. Kelif, Core Stable Algorithms for Coalition Games with Complementarities and Peer Effects, *The 10th Workshop on the Economics of Networks, Systems and Computation (NETECON), in Conjunction with ACM Sigmectrics 2015 and ACM Conference on Economy and Computations (EC)*, Portland, USA, June, 2015.
- M. Touati and M. Coupechoux and J.M. Kelif, Impact of mobility on QoS in heterogeneous wireless networks, *IEEE International Conference on Communication (ICC)*, London, United Kingdom, June, 2015.

11.3 Talks

- M. Touati, A Cooperative Game Theoretic Analysis of WiFi, GdR ISIS Learning in Networks and Beyond, Paris, France, June, 2016.

- M. Touati, A Brief Introduction to Game Theory and Applications to Networks, Orange Labs, *Lundis de la radio* (internal communication event), Châtillon, France, March, 2016.
- M. Touati, Controlled Matching Game for Connectivity Management with Nash Bargaining Resource Allocation, Poster presented at the Seminar on Modeling, Optimization and Control in Wireless Networks, Paris, France, September, 2015.
- M. Touati, A Cooperative Game Theoretic Analysis of WiFi, Coordinated Science Laboratory, University of Illinois, Urbana-Champaign, USA, October, 2015.
- M. Touati, A Tutorial on Matching Games, Laboratoire Informatique d'Avignon, University of Avignon, June, 2014.

Jeux Coopératifs et d'Appariements Stables dans les Réseaux

Mikaël TOUATI

RESUME : Dans cette thèse, nous proposons des solutions à plusieurs problèmes d' allocation de ressources et d'associations dans les réseaux. Pour cela, nous employons les jeux coopératifs, particulièrement les jeux d'appariements stables, classiquement utilisés en économie pour l'analyse de marchés bifaces et la conception de leurs mécanismes d'allocations. Dans une première partie, nous introduisons les jeux de négociation et d'appariements stables. Dans une seconde partie, nous proposons un nouveau mécanisme stable d'association des utilisateurs en WiFi réduisant l'impact de l'anomalie du protocole. Nous présentons également une analyse d'un problème de stockage de videos et un nouvel algorithme d'énumération de structures stables. Dans une troisième partie, nous analysons des conditions pour la stabilité de certains schémas d'équité connus en termes de mesures d'aversion au risque. Dans une quatrième partie, nous analysons la stabilité d'une place de marché biface de crowdsourcing avec contraintes d'ordonnancement de tâches. La classique propriété de substitution des biens n'étant pas satisfaite, nous introduisons des nouvelles conditions et montrons l'existence d'appariements stables. Nous proposons également une résolution du problème par une formulation non-coopérative en forme extensive.

ABSTRACT : In this thesis, we propose new solutions to matching problems in networks. We use cooperative games, particularly stable matchings, classically used in economy to analyze two-sided markets and design matching mechanisms. In the first part, we introduce bargaining and stable matching games. In the second part, we propose a new stable matching mechanism for user association in WiFi reducing the impact of the anomaly in the protocol. Furthermore, we analyze a video caching problem and show a new algorithm enumerating stable structures. In the third part, we analyze conditions for the stability of some fairness schemes in terms of risk aversion indicators. In the fourth part, we analyze the stability of a two-sided crowdsourcing marketplace with scheduling constraints on the tasks. The classical substitutability condition does not hold in this case. We introduce new conditions and show the existence of stable matchings. We also solve the crowdsourcing problem as a non-cooperative game in extensive form.

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