Theoretical and numerical analysis of invariant measures of viscous stochastic scalar conservation laws

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Après les rapports de MM. Gabriel Lord et Julien Vovelle

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Préambule

Cette thèse se consacre à une analyse théorique puis numérique d'une certaine classe d'équations aux dérivées partielles stochastiques (EDPS) : les lois de conservation scalaires avec viscosité et avec un forçage aléatoire de type additif et bruit blanc en temps. Un exemple typique est l'équation de Burgers stochastique, motivée par la théorie de la turbulence. On s'intéresse particulièrement au comportement en temps long des solutions de ces équations à travers une étude des mesures invariantes.

La partie théorique de la thèse constitue le chapitre 2. Dans ce chapitre, on prouve l'existence et l'unicité d'une solution au sens fort. Pour cela, des estimations sur les normes de Sobolev jusqu'à l'ordre 2 sont établies. Dans la seconde partie du chapitre 2, on montre que la solution de l'EDPS admet une unique mesure invariante.

On se propose dans le chapitre 3 d'approcher numériquement cette mesure invariante. À cette fin, on introduit un schéma numérique dont la discrétisation spatiale est de type Volumes Finis et dont la discrétisation temporelle est une méthode d'Euler à pas fractionnaire. Il est montré que ce type de schéma respecte certaines propriétés fondamentales de l'EDPS telles que la dissipation d'énergie et la contraction L^1. Ces propriétés assurent l'existence et l'unicité d'une mesure invariante pour le schéma. À l'aide d'un certain nombre d'estimations de régularité, on montre ensuite que cette mesure invariante discrète converge, lorsque le pas de temps et le pas d'espace tendent vers zéro, vers l'unique mesure invariante pour l'EDPS au sens de la distance de Wasserstein d'ordre 2.

Dans le chapitre 4, des expériences numériques sont effectuées sur l'équation de Burgers pour illustrer cette convergence ainsi que des propriétés à petites échelles spatiales relatives à la turbulence.

Un appendice est consacré à l'étude de la stationnarité des schémas numériques introduits au chapitre 3 dans le cas où la viscosité est nulle et où le domaine spatial est de dimension quelconque.

Preamble

This thesis is devoted to the theoretical and numerical analysis of a certain class of stochastic partial differential equations (SPDEs), namely scalar conservation laws with viscosity and with a stochastic forcing which is an additive white noise in time. A particular case of interest is the stochastic Burgers equation, which is motivated by turbulence theory. We focus on the long time behaviour of the solutions of these equations through a study of the invariant measures.

The theoretical part of the thesis constitutes the second chapter. In this chapter, we prove the existence and uniqueness of a solution in a strong sense. To this end, estimates on Sobolev norms up to the second order are established. In the second part of Chapter 2, we show that the solution of the SPDE admits a unique invariant measure.

In the third chapter, we aim to approximate numerically this invariant measure. For this purpose, we introduce a numerical scheme whose spatial discretisation is of the finite volume type and whose temporal discretisation is a split-step backward Euler method. It is shown that this kind of scheme preserves some fundamental properties of the SPDE such as energy dissipation and L^1-contraction. Those properties ensure the existence and uniqueness of an invariant measure for the numerical scheme. Thanks to a few regularity estimates, we show that this discrete invariant measure converges, as the space and time steps tend to zero, towards the unique invariant measure for the SPDE in the sense of the second order Wasserstein distance.

In Chapter 4, numerical experiments are performed on the Burgers equation in order to illustrate this convergence as well as some small-scale properties related to turbulence.

An appendix is devoted to the study of the stationarity of the numerical schemes introduced in Chapter 3, in the case where the viscosity coefficient is zero and the spatial domain is of arbitrary dimension.

List of publications

Here is a list of articles (accepted or submitted) that were written during this thesis:

- [58] Numerical schemes for the aggregation equation with pointy potentials.
  With Benoît Fabrèges, Hélène Hivert, Kevin Le Balc'h, François Delarue, Frédéric Lagoutière and Nicolas Vauchelet.
  Published in *ESAIM: Proceedings and Surveys (CEMRACS 2017 - Numerical methods for stochastic models: control, uncertainty quantification, mean-field)*.

- [92] Viscous scalar conservation law: strong solution and invariant measure.
  With Julien Reygner.
  Preprint (submitted), 2019.

- [19] Finite-volume approximation of the invariant measure of a viscous stochastic scalar conservation law.
  With Sébastien Boyaval and Julien Reygner.
  Preprint (submitted), 2019.

Contents

Chapter 1

Introduction

1.1 Viscous scalar conservation laws

Conservation laws are among the most fundamental principles describing a physical process. They state that a particular measurable quantity of an isolated physical system does not evolve over time. This quantity may be a mass, an energy, a momentum, an electric charge, etc... When the physical system consists of a continuous material, this conservation principle is appropriately complemented with the statement that the conserved quantity is also *locally* conserved: the amount of the quantity in a given volume of space varies only by the amount flowing in or out of the volume through its boundaries. In more general words, the quantity moves in space as a continuous flow. This assumption, called the *continuum hypothesis*, constitutes the core of continuum mechanics [37]. The physical system may also create and destroy some of the conserved quantity by itself. We say that it contains a *source* at the time and place where the conserved quantity is created, and a *sink* where it is destroyed. The conservation law still holds as long as the sources and the sinks compensate each other.

Let $u(t, \mathbf{x})$ denote the density per unit volume of the conserved quantity at a time t and at a point $\mathbf{x}$ of the physical space. Similarly, let $\mathbf{F}(t, \mathbf{x})$ denote the flux of the conserved quantity, and $S(t, \mathbf{x})$ the generation of the conserved quantity per unit volume per unit time (it corresponds to a source when $S > 0$ and a sink when $S < 0$). Let V be an arbitrary volume of space. The conservation law in the volume V has the expression

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V u\mathrm{d}\mathbf{x} = -\int_{\partial V} \mathbf{F}(t, \mathbf{x}) \cdot \mathbf{n}\mathrm{d}\mathbf{x} + \int_V S(t, \mathbf{x})\mathrm{d}\mathbf{x}.$$

Indeed, the left-hand side expresses the variation per unit time of the amount of the conserved quantity inside V. As $\mathbf{n}$ denotes the outward unit normal to V, the first term of the right-hand side expresses the amount of the conserved quantity flowing into V through its boundary at time t, and the second term expresses the amount of the conserved quantity generated inside V at time t. Using Green's formula, we can rewrite this equation in the following way:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V u\mathrm{d}\mathbf{x} = -\int_V \mathrm{div}(\mathbf{F}(t, \mathbf{x}))\mathrm{d}\mathbf{x} + \int_V S(t, \mathbf{x})\mathrm{d}\mathbf{x}.$$

Since this equation must hold for any volume of space V, the integrals may vanish and we shall write

$$\partial_t u = -\mathrm{div}(\mathbf{F}(t, \mathbf{x})) + S(t, \mathbf{x}). \tag{1.1}$$

In most physical applications, the function $\mathbf{F}$ depends on t and $\mathbf{x}$ through the density u. When it depends on u up to the first-order space differential, *i.e.* when

$$\mathbf{F}(t, \mathbf{x}) = \widetilde{\mathbf{F}}(t, \mathbf{x}, u(t, \mathbf{x}), \nabla u(t, \mathbf{x})),$$

then Equation (1.1) defines a partial differential equation that we may call a *second-order scalar conservation law*.

In the present work, we will restrict ourselves to second-order scalar conservation laws where the flux is homogeneous in time and space and takes the form:

$$\widetilde{\mathbf{F}}\left(t, \mathbf{x}, u(t, \mathbf{x}), \nabla u(t, \mathbf{x})\right)=\mathbf{A}(u(t, \mathbf{x}))-\nu \nabla u(t, \mathbf{x}),$$

where $\mathbf{A}$ is a continuous function and ν is a positive constant. Furthermore, the time interval considered will always be $[0,+\infty)$. Taking these assumptions into account, Equation (1.1) now writes

$$\partial_t u=-\operatorname{div}(\mathbf{A}(u))+\nu \Delta u+S(t, \mathbf{x}), \quad t \geq 0, \quad \mathbf{x} \in D, \tag{1.2}$$

where D is the space domain. It could be either $\mathbb{R}^d$, a subdomain[1] of $\mathbb{R}^d$, or even the d-dimensional torus $\mathbb{T}^d$, where $\mathbb{T}:=\mathbb{R}/\mathbb{Z}$. Observe that the expression of the flux of the conserved quantity has been split in two parts:

- The term $-\operatorname{div}(\mathbf{A}(u))$ accounts for the transport of the conserved quantity. Indeed, the equation $\partial_t u=-\operatorname{div}(\mathbf{A}(u))$ which by differentiation can also be written $\partial_t u=-\mathbf{A}'(u)\cdot\nabla u$, is a transport equation where the solution u is transported with a velocity $\mathbf{A}'(u)$. As it relates to the non-diffusive part of the transport, we call $\mathbf{A}$ the *flux function*, and by extension, we call $-\operatorname{div}(\mathbf{A}(u))$ the *flux term*.

- The term $\nu\Delta u$ accounts for the spatial diffusion of the conserved quantity. Observe that when $\mathbf{A}\equiv 0$ and $S\equiv 0$, Equation (1.2) is merely the heat equation. By analogy with the hydrodynamical context, the term $\nu\Delta u$ is called the *viscous term* and the constant ν is the *viscosity coefficient*.

In this regard, Equation (1.2) is often called a *viscous scalar conservation law*. A notable non-linear viscous conservation law is the *viscous Burgers equation*, namely Equation (1.2) in dimension $d=1$ and with the flux function $A(u)=u^2/2$. Particular physical motivations behind this equation will be discussed in Section 1.4.

In the recent years, a great interest has been brought to *hyperbolic* (or *inviscid*, or *first-order*) scalar conservation laws, *i.e.* to equations of the type (1.2) with $\nu=0$:

$$\partial_t u=-\operatorname{div}(\mathbf{A}(u))+S(t, \mathbf{x}), \quad t \geq 0, \quad \mathbf{x} \in D. \tag{1.3}$$

When $\mathbf{A}$ is non-linear, solutions of (1.3) usually develop shocks in finite time, even when the initial condition u_0 is smooth. In general, formulating Equation (1.3) in the weak sense is not sufficient to establish well-posedness as several weak solutions may co-exist after a shock has occured. The usual approach thus consists in establishing a formulation of (1.3) that contains a criterion discriminating the weak solution that has a proper physical meaning. In Kruzkov's seminal paper [84], such a formulation was introduced with the notion of *entropic* solution. Another notable formulation of (1.3) is the notion of *kinetic* solution introduced by Lions, Perthame and Tadmor [89] as a generalisation of the entropic one.

To establish a proper notion of solution for Equation (1.3), one usually first considers the solution to Equation (1.2). Indeed, (1.2) is a second-order parabolic equation for which it is often possible (depending on the choice of the different parameters) to find a unique solution in the classical sense [85]. In this regard, (1.2) is often presented as the parabolic approximation of (1.3). For instance, the entropic (or the kinetic) solution of (1.3) is found by the so-called *vanishing viscosity method*: it is sought as the limit of the solutions of (1.2) as ν tends to 0.

[1]in which case Equation (1.2) is usually complemented with boundary conditions, *e.g.* the Dirichlet condition $u_{|\partial D}=0$.

When the physical system described by the conservation law is subject to unpredictable external disturbances, it can be relevant to consider the source term S to be random. According to the nature of these disturbances, the term S may be chosen among a wide variety of random forcing terms. In this respect, white noises are almost always successful candidates. Indeed, they express the chaotic nature of the physical system at all scales of time and/or space and they offer mathematical convenience by the use of familiar probabilistic tools provided by Gaussian processes. The noise is said to be

- white in time and in space when $\mathbb{E}[S(t,\mathbf{x})S(s,\mathbf{y})] = \delta_{t=s}\delta_{\mathbf{x}=\mathbf{y}}$;
- white in time and spatially correlated when $\mathbb{E}[S(t,\mathbf{x})S(s,\mathbf{y})] = \delta_{t=s}c(\mathbf{x},\mathbf{y})$, where c is a function on D^2 expressing the spatial correlations.

We will be interested in cases where the random source is written $S(t,\mathbf{x}) = \Phi(t,\mathbf{x},u(t,\mathbf{x}))\dot{W}(t)$, where $(W(t))_{t\geq 0}$ is a cylindrical Brownian motion, which formally writes

$$W(t) = \sum_{k\geq 1} W^k(t)e_k, \quad t \geq 0,$$

where $(e_k)_{k\geq 1}$ is a complete orthonormal basis of a Hilbert space H and $(W^k)_{k\geq 1}$ is a family of independent real Brownian motions. The mapping Φ is the *noise operator*. For any $(t,\mathbf{x},u) \in [0,+\infty) \times D \times \mathbb{R}$, $\Phi(t,\mathbf{x},u)$ is a linear form on H. When Φ does not depend on u, the noise is said to be *additive*, otherwise it is *multiplicative*.

With such a source term, (1.2) falls into the realm of infinite dimensional stochastic differential equations.

1.2 Viscous scalar conservation laws with stochastic forcing

1.2.1 Well-posedness

Brief survey

Stochastic calculus is certainly the most appropriate framework to study Equation (1.2) forced with a white noise. Starting from the 1960's, efforts have been made to interpret evolutionary PDEs with random noise as generalised stochastic differential equations (SDEs) where the state space is Hilbert or Banach. Extensive results have been obtained since then and the essence of the theory is contained in Da Prato and Zabczyk's reference book [35] in the domain of evolutionary stochastic partial differential equations (SPDEs). In [35], the notion of infinite dimensional Wiener process is introduced together with the associated Itô integral. The Itô formula (the most important tool) is established for Hilbert-valued stochastic processes. Furthermore, well-posedness results are given for linear and semi-linear SPDEs.

In this framework, Equation (1.2) shall be expressed under the formulation

$$\mathrm{d}u = (-\mathrm{div}(\mathbf{A}(u)) + \nu\Delta u)\,\mathrm{d}t + \Phi(t,\cdot,u)\mathrm{d}W(t), \quad t \geq 0. \tag{1.4}$$

The sought solution is a stochastic process $(u(t))_{t\geq 0}$ taking values in a functional state space E, where E is a Hilbert or a Banach space. Observe that the mapping $u \mapsto -\mathrm{div}(\mathbf{A}(u)) + \nu\Delta u$ constitutes the drift of the (generalised) SDE. There are several possible definitions for the solution of Equation (1.4). Let us mention the three main ones.

- The most basic one is the **strong** solution, that is, a solution $(u(t))_{t\geq 0}$ with an initial condition $u(0) = u_0$ and which satisfies the strong formulation

$$u(t) = u_0 + \int_0^t (-\mathrm{div}(\mathbf{A}(u)) + \nu\Delta u)\mathrm{d}s + \int_0^t \Phi(s,\mathbf{x},u)\mathrm{d}W(s), \quad t \geq 0, \quad \text{almost surely},$$

where the second term of the right-hand side is a Bochner integral and the third term is an infinite dimensional Itô integral. Notice in particular that the solution u must have enough spatial regularity for the Laplacian Δu to have a classical meaning. Suitable state spaces for the stochastic process $(u(t))_{t\geq 0}$ are for instance the Sobolev space $W^{m,p}(D)$, with $m \geq 2$, $p \geq 1$, or the space $C^m(D)$, $m \geq 2$.

- A weaker definition of solution is given by the **mild** formulation

$$u(t) = S_t u_0 - \int_0^t S_{t-s}\mathrm{div}(\mathbf{A}(u))\mathrm{d}s + \int_0^t S_{t-s}\Phi(s,\mathbf{x},u)\mathrm{d}W(s), \quad t \geq 0, \quad \text{almost surely,}$$

where $(S_t)_{t\geq 0}$ is the semigroup generated by the operator $\nu\Delta$ (*i.e.* the heat semigroup). It can be observed that in this formulation, the Laplacian need not exist as the spatial diffusion acts on each term of the dynamics through the semigroup $(S_t)_{t\geq 0}$. Therefore, a mild solution might be defined on a larger class of state spaces than a strong solution, *e.g.* in $W^{m,p}(D)$, $m \geq 1$.

- Finally, the **weak** formulation generally writes: for all $\varphi \in C_c^2(D)$, for all $t \geq 0$, almost surely,

$$\begin{aligned}\int_D u(t,\mathbf{x})\varphi(\mathbf{x})\mathrm{d}\mathbf{x} &= \int_D u_0(\mathbf{x})\varphi(\mathbf{x})\mathrm{d}\mathbf{x} + \nu\int_0^t\int_D u(s,\mathbf{x})\Delta\varphi(\mathbf{x})\mathrm{d}\mathbf{x}\mathrm{d}s \\ &\quad + \int_0^t\int_D \mathbf{A}(u(s,\mathbf{x}))\cdot\nabla\varphi(\mathbf{x})\mathrm{d}\mathbf{x}\mathrm{d}s + \int_0^t\int_D \varphi(x)\Phi(s,\mathbf{x},u(s,\mathbf{x}))\mathrm{d}\mathbf{x}\mathrm{d}W(s).\end{aligned}$$

In the case where D has boundaries, the test function φ is usually needed to satisfy some Dirichlet boundary conditions. Notice that the weak formulation asks no spatial regularity on u. The state spaces for such solutions are often the spaces $L^p(D)$, $p \geq 1$.

A particular observation arising from the mild formulation is that it looks like a mollified version of the hyperbolic counterpart. As in the deterministic case, weak, mild or strong solutions to Equation (1.4) are used to approximate solutions of

$$\mathrm{d}u = -\mathrm{div}(\mathbf{A}(u))\mathrm{d}t + \Phi(t,\mathbf{x},u)\mathrm{d}W(t), \quad t \geq 0, \quad \mathbf{x} \in D, \tag{1.5}$$

and the vanishing viscosity method is still a standard tool. Two main notions of solution are used in practice to establish the well-posedness of (1.5). Those are the generalisations to the stochastic case of the solutions in the entropic and kinetic sense. Let us give the formal definitions.

- To establish an **entropic** formulation of (1.5), one has first to define a set of entropy-entropy flux functions. That is a family of couples $(\eta,\mathbf{q})$ where η is a real convex function at least twice differentiable, called *entropy*, and $\mathbf{q}$ is a continuously differentiable function from $\mathbb{R}$ to $\mathbb{R}^d$ satisfying $\mathbf{q}'(v) = \eta'(v)\mathbf{A}'(v)$, for all $v \in \mathbb{R}$, and which represents the flux of the entropy η. An entropic solution of (1.5) satisfies: for all entropy-entropy flux $(\eta,\mathbf{q})$, and all non-negative test functions $\varphi \in C_c^2(D)$, for all $t \geq 0$, almost surely,

$$\begin{aligned}0 \leq &\int_D \eta(u_0(\mathbf{x}))\varphi(\mathbf{x})\mathrm{d}\mathbf{x} - \int_D \eta(u(t,\mathbf{x}))\varphi(\mathbf{x})\mathrm{d}\mathbf{x} + \int_0^t\int_D \mathbf{q}(u(s,\mathbf{x}))\cdot\nabla\varphi(\mathbf{x})\mathrm{d}\mathbf{x}\mathrm{d}s \\ &+ \int_0^t\left(\int_D \eta'(u(s,\mathbf{x}))\Phi(s,\mathbf{x},u(s,\mathbf{x}))\varphi\mathrm{d}\mathbf{x}\right)\mathrm{d}W(s) + \frac{1}{2}\int_0^t\int_D |\Phi(s,\mathbf{x},u(s,\mathbf{x}))|^2\,\eta''(u(s,\mathbf{x}))\varphi\mathrm{d}\mathbf{x}\mathrm{d}s.\end{aligned}$$

- The **kinetic** formulation is expressed through the characteristic function $f(\mathbf{x},t,\xi) := \mathbf{1}_{u(\mathbf{x},t)>\xi}$. More precisely, u is said to be a kinetic solution of (1.5) if for all $T > 0$, there exists a random positive measure m on $D \times [0,T] \times \mathbb{R}$, such that for all test functions $\varphi \in C_c^\infty([0,T] \times D \times \mathbb{R})$, the function f satisfies almost surely

$$\begin{aligned}\int_0^T \langle f(t), \partial_t \varphi(t) \rangle \mathrm{d}t + \langle f_0, \varphi(0) \rangle + \int_0^t \langle f(t), \mathbf{A}' \cdot \nabla \varphi(t) \rangle \mathrm{d}t \\ = - \int_0^T \int_D \varphi(t, \mathbf{x}, u(t,\mathbf{x})) \Phi(t, \mathbf{x}, u(t,\mathbf{x})) \mathrm{d}\mathbf{x} \mathrm{d}W(t) \\ - \frac{1}{2} \int_0^T \int_D |\Phi(t, \mathbf{x}, u(t,\mathbf{x}))|^2 \partial_\xi \varphi(t, \mathbf{x}, u(t,\mathbf{x})) \mathrm{d}x \mathrm{d}t + m(\partial_\xi \varphi).\end{aligned}$$

where $\langle \cdot, \cdot \rangle$ denotes the $L^2(D \times \mathbb{R})$ scalar product.

In Figure 1.1, we recorded, in the chronological order, a list of well-posedness results for scalar conservation laws with stochastic forcing where we give some details on the setting. Some precautions must however be taken while examining this table:

- In some works, the equation considered is more general than a scalar conservation law. In these cases, the assumptions we mention in Figure 1.1 are those that remain after restricting the general equation to the particular case of a scalar conservation law. For instance, in [67, 68, 69], a non-linear deterministic source term is added in the equation; in [69, 73], the viscous term consists of a uniformly elliptic diffusion operator instead of a Laplacian; in [73], the non-linear term is defined in a more general way than the ordinary flux term; in [74, 39, 63], a positive semi-definite diffusion operator holds in place of the viscous term and thus, depending on the nature of this operator, the equation may be of the first or of the second order (such equations are usually called *quasilinear degenerate parabolic-hyperbolic SPDEs*).
- As much as it aims to be extensive, the list is still non-exhaustive.
- Only some particular assumptions are specified. For instance, we do not give details on the noise coefficients. Furthermore, assumptions of polynomial growth on the flux usually concern the derivatives of the flux function, we do not specify up to which order these derivatives are subject to growth constraints.
- In the cited articles, the aspects of the results that may go beyond well-posedness are not mentioned (such aspects concern for instance the regularity of the solutions, the dependence on initial conditions, the existence and uniqueness of an invariant measure, the Markov and/or Feller properties satisfied by the solution, etc...)

Another result of well-posedness

The work contained in [92] (see the last line of Table 1.1) consitutes the second chapter of this manuscript. In the first part of this chapter, we are concerned with the well-posedness of Equation (1.4) in the following setting: the domain D is the one-dimensional torus $\mathbb{T}$ and the noise is additive and white in time. More precisely, given a cylindrical Brownian motion $(W(t))_{t \geq 0}$ in a Hilbert space H with the expression $W(t) = \sum_{k \geq 1} W^k(t) e_k$, our noise operator Φ is defined by the coefficients

$$g_k(x) := \Phi(x) e_k, \quad x \in \mathbb{T}, \quad k \geq 1.$$

In short, the equation becomes

$$\mathrm{d}u = (-\partial_x A(u) + \nu \partial_{xx} u) \, \mathrm{d}t + \sum_{k \geq 1} g_k \mathrm{d}W^k(t), \quad t \geq 0, \quad x \in \mathbb{T}. \tag{1.6}$$

[2] depending on the chosen assumptions.

Authors and reference	**Flux**	**Viscosity**	**Noise**	**Notion of solution**	**Space domain, boundary conditions**
Bertini, Cancrini, Jona-Lasinio [9]	Burgers'	$\nu > 0$	Space-time white noise	Mild	$\mathbb{R}$
Da Prato, Debussche, Temam [34]	Burgers'	$\nu > 0$	Space-time white noise	Mild	$[0,1]$ Dirichlet b.c.
Da Prato, Gatarek [96]	Burgers'	$\nu > 0$	Multiplicative space-time white noise	Mild	$[0,1]$ Dirichlet b.c.
Da Prato, Zabczyk [36, Chapter 14]	Burgers'	$\nu > 0$	Space-time white noise	Mild	$[0,1]$ Dirichlet b.c.
Gyöngy [67]	Non-homogeneous, Locally Lipschitz, quadratic growth	$\nu > 0$	Multiplicative space-time white noise	Weak	$[0,1]$ Dirichlet b.c.
Gyöngy, Nualart [68]	Burgers'	$\nu > 0$	Multiplicative space-time white noise	Weak	$\mathbb{R}$
Gyöngy, Rovira [69]	Locally Lipschitz, polynomial growth	$\nu > 0$	White-in-time noise	Weak	Smooth bounded convex in $\mathbb{R}^d$, Dirichlet b.c.
E, Khanin, Mazel, Sinai [52]	Burgers'	$\nu = 0$	White-in-time noise	Entropic	$\mathbb{T}$
Dong, Xu [44]	Burgers'	$\nu > 0$	Lévy process	Mild, weak, strong[2]	$[0,1]$ Dirichlet b.c.
Feng, Nualart [60]	C^2, polynomial growth	$\nu = 0$	Space-time white noise	Entropic	$\mathbb{R}$
Vallet, Wittbold [100]	Lipschitz continuous	$\nu = 0$	Space-time white noise	Entropic	Lipschitz, bounded domain of $\mathbb{R}^d$, Dirichlet b.c.
Chen, Ding Karlsen [27]	C^2, polynomial growth	$\nu = 0$	Multiplicative white-in-time noise	Entropic	$\mathbb{R}^d$
Boritchev [15, Appendice A]	Strongly convex, C^∞, polynomial growth	$\nu > 0$	White-in-time noise	Strong	$\mathbb{T}$
Bauzet, Vallet, Wittbold [7]	Lipschitz continuous	$\nu = 0$	Multiplicative white-in-time noise	Entropic	$\mathbb{R}^d$
Hausenblas, Giri [72]	$A(u) = u^p/p$ $p \in [2, +\infty)$	$\nu > 0$	Lévy process	Mild	$[0,1]$ Dirichlet b.c.
Hofmanová [73]	Lipschitz continuous	$\nu > 0$	White-in-time noise	Strong	$\mathbb{T}^d$
Hofmanová [74]	C^1, polynomial growth	$\nu \geq 0$	Multiplicative white-in-time noise	Kinetic	$\mathbb{T}^d$
Debussche, Vovelle [40]	C^2, polynomial growth	$\nu = 0$	White-in-time noise	Kinetic	$\mathbb{T}^d$
Debussche, Hofmanová, Vovelle [39]	C^2, polynomial growth	$\nu \geq 0$	Multiplicative white-in-time noise	Kinetic	$\mathbb{T}^d$
Lewis, Nualart [87]	Burgers'	$\nu > 0$	Multiplicative space-time white noise	Mild	$\mathbb{R}$
Gess, Hofmanová [63]	Non-degenerate	$\nu = 0$	Multiplicative white-in-time noise	Kinetic	$\mathbb{T}^d$
	C^2	$\nu > 0$			
Martel, Reygner [92]	C^2, polynomial growth	$\nu > 0$	White-in-time noise	Strong	$\mathbb{T}$

Figure 1.1: Well-posedness results for scalar conservation laws

For each $k \geq 1$, g_k is a continuous function from $\mathbb{T}$ to $\mathbb{R}$ such that $\int_{\mathbb{T}} g_k(x)\mathrm{d}x = 0$. This condition implies in particular that any solution of (1.6) satisfies $\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}} u(t,x)\mathrm{d}x = 0$. Thus, we may restrict our attention to solutions whose spatial average is zero. To this end, we will denote by $W_0^{m,p}(\mathbb{T})$ the space of functions $v \in W^{m,p}(\mathbb{T})$ satisfying $\int_{\mathbb{T}} v(x)\mathrm{d}x = 0$, and we endow it with the norm $\|\cdot\|_{W_0^{m,p}(\mathbb{T})} = \|\partial_x^m \cdot\|_{L^p(\mathbb{T})}$. Incidentally, we will write $H_0^m(\mathbb{T}) := W_0^{m,2}(\mathbb{T})$. Note that the Poincaré inequality implies that $\|\cdot\|_{L^2(\mathbb{T})} \leq \|\cdot\|_{H_0^1(\mathbb{T})} \leq \|\cdot\|_{H_0^2(\mathbb{T})} \leq \cdots$. Given this setting, we assume that

$$\sum_{k\geq 1} \|g_k\|^2_{H_0^2(\mathbb{T})} =: D_0 < +\infty.$$

Moreover, the function A is assumed to be of class C^2, its first derivative A' to have at most polynomial growth, and its second derivative A'' to be locally Lipschitz continuous. Under these assumptions, we show the following

Theorem (Theorem 2.4)**.** *There exists a unique $H_0^2(\mathbb{T})$-valued stochastic process $(u(t))_{t\geq 0}$ satisfying Equation (1.4) in the strong sense.*

The proof follows a standard approach (used for instance in [34] or [36, Section 14.2]) but is driven by regularity issues required for establishing a strong solution (such issues are also addressed for instance in [15, Appendice A] or [73]). Let us formally explain the overall approach of the proof. As the non-linear term is locally Lipschitz and thanks to the regularising properties of the heat semigroup, it is possible to establish, via a fixed point argument applied for each trajectory, the existence and uniqueness of a local-in-time mild solution $(u(t))_{t\in[0,\tau]} \in C([0,\tau], H_0^1(\mathbb{T}))$ to (1.6), where τ is an almost surely positive stopping time. The mild formulation is best suited to apply the fixed point theorem, but is actually sufficient even if we ultimately seek well-posedness in the strong sense, as the following property holds:

Proposition (Proposition 2.12)**.** *Any mild solution to (1.6) (local or global in time) with a regular enough initial condition is a strong solution.*

To extend the local solution to a global solution, one usually needs a priori estimates that prevent any blow-up in finite time. Since the non-linearity is not globally Lipschitz, such estimates are not forthright. However, the viscosity endows Equation (1.6) with a decisive property for the stability of the solution.

Proposition (Dissipativity)**.** *The drift function in the SPDE (1.6) satisfies:*

$$\langle -\partial_x A(v) + \nu\partial_{xx} v, v\rangle_{L^2(\mathbb{T})} = -\nu\|v\|^2_{H_0^1(\mathbb{T})} \leq 0, \quad v \in H_0^2(\mathbb{T}).$$

This property is actually quite standard for semi-linear equations and the proof follows from simple computations. To see its significance, let us apply the Itô formula to (1.6) with the squared L^2-norm. We get

$$\mathrm{d}\,\|u\|^2_{L^2(\mathbb{T})} = 2\,\langle -\partial_x A(u) + \nu\partial_{xx} u, u\rangle_{L^2(\mathbb{T})}\,\mathrm{d}t + 2\sum_{k\geq 1}\langle g_k, u\rangle_{L^2(\mathbb{T})}\,\mathrm{d}W^k(t) + \sum_{k\geq 1}\|g_k\|^2_{L^2(\mathbb{T})}\,\mathrm{d}t.$$

Taking the expectation and applying the dissipativity property, it appears that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\left[\|u(t)\|^2_{L^2(\mathbb{T})}\right] \leq -2\nu\mathbb{E}\left[\|u(t)\|^2_{H_0^1(\mathbb{T})}\right] + D_0, \tag{1.7}$$

meaning that the growth of the L^2-norm of the solution is controlled. We should notice that, at this point, we formally have the existence and uniqueness of an $L_0^2(\mathbb{T})$-valued weak solution to (1.6). Indeed, our local mild solution $(u(t))_{t\in[0,\tau]}$ with values in $H_0^1(\mathbb{T})$ is even more so a local weak solution with values in $L_0^2(\mathbb{T})$. The L^2 estimate arising from the growth control (1.7) ensures that this weak local solution does not blow-up and is actually global. However, such estimates do not give information on the behaviour of the H_0^1-norm, whose growth needs to be controlled in order to consider a global mild solution. In this regard, the dissipativity property is not sufficient but it can be strengthened to the following

Proposition (Generalised dissipativity). *The drift function in the SPDE (1.6) satisfies for all positive even number p:*

$$\langle -\partial_x A(v) + \nu\partial_{xx}v, v^{p-1}\rangle_{L^2(\mathbb{T})} = -\frac{4\nu(p-1)}{p^2}\left\|v^{p/2}\right\|_{H_0^1(\mathbb{T})} \leq 0, \quad v \in H_0^2(\mathbb{T}).$$

When $p = 2$, this corresponds to the usual dissipativity property. This proposition is not plainly stated in the content of Chapter 2 but appears covertly in the step 3 of the proof of Lemma 2.15. It yields the a priori L^p bounds given in the statement of Lemma 2.15, which eventually leads, using the polynomial growth of A', to have a control over the H_0^1-norm, extending thereby the local solution to a global one.

1.2.2 Invariant measure

After well-posedness, the most addressed topic regarding stochastic conservation laws is the existence and uniqueness of an invariant measure.

Definition (Invariant measure). *Let $(X_t)_{t\geq 0}$ be a Markov process on a state space E. A probability measure μ on E is said to be an* invariant measure *of $(X_t)_{t\geq 0}$ if*

$$X_0 \sim \mu \quad \Longrightarrow \quad \forall t \geq 0, \quad X_t \sim \mu.$$

The analysis of invariant measures sheds light on the long time behaviour of solutions, the ergodic properties, and the statistical properties that are inherent to the system and do not depend on the initial conditions. More details on this last point will be discussed in Subsection 1.4. We mention here some of the previous works done in this direction but we refer the reader to [93, 28] for more detailed reviews on the subject.

Some of the works mentioned in Table 1.1 contain results regarding the invariant measure. For the stochastic Burgers equation, existence is proved by Da Prato, Debussche and Temam in [34], by Dong and Xu in [44], by Hausenblas and Giri in [72]. Existence and uniqueness are proved by Da Prato and Gatarek in [96], by Da Prato and Zabczyk in [36, Chapter 14], by E, Khanin, Mazel and Sinai in [52]. For more general conservation laws, existence and uniqueness of an invariant measure have been proved by Boritchev in [15, Chapter 4], by Debussche and Vovelle in [41] in a setting very close to [40] but where the flux function satisfies a non-degeneracy condition. For stochastic anisotropic parabolic-hyperbolic equations, existence and uniqueness of an invariant measure have been proved by Chen and Pang in [28].

An invariant measure for (1.6)

The second main result of the second chapter is

Theorem (Theorem 2.7). *The strong solution $(u(t))_{t\geq 0}$ of Equation (1.6) admits a unique invariant measure.*

As for the well-posedness result presented in Subsection 1.2.1, the proof follows a standard approach but has to deal with regularity concerns. The usual method to address existence of an invariant measure is to use the Krylov-Bogoliubov theorem (see *e.g.* [36, Theorem 3.1.1]):

Theorem (Krylov-Bogoliubov). *A Feller process $(X_t)_{t\geq 0}$ on a Polish space E admits at least one invariant measure if for some initial condition X_0, there exists an increasing sequence of positive numbers $(T_n)_{n\geq 1}$ tending to $+\infty$ such that the family of probability measures $\{\mu_{T_n} : n \geq 1\}$ defined by*

$$\mu_{T_n} : \Gamma \in \mathcal{B}(E) \longmapsto \frac{1}{T_n}\int_0^{T_n} \mathbb{P}\left(X_t \in \Gamma\right) \mathrm{d}t \in [0,1]$$

(where $\mathcal{B}(E)$ denotes the Borel σ-algebra on E) is tight.

The proof of the Krylov-Bogoliubov theorem is done in two steps. The first step is merely an application of the Prokhorov theorem stating that the sequence $(\mu_{T_n})_{n\geq 1}$ admits a limit μ in the weak sense. The second step consists in showing that the limit μ is invariant for the process $(X_t)_{t\geq 0}$.

In our case, if we consider the strong solution $(u(t))_{t\geq 0}$ as an $L^2_0(\mathbb{T})$-valued process, the conditions of the Krylov-Bogoliubov theorem are easily satisfied as the viscosity induces regularity in space and thus compactness. Indeed, from Equation (1.7), taking the average in time, we get

$$\frac{1}{T}\int_0^T \mathbb{E}\left[\|u(t)\|^2_{H^1_0(\mathbb{T})}\right] \mathrm{d}t \leq \frac{1}{2\nu T}\mathbb{E}\left[\|u_0\|^2_{L^2(\mathbb{T})}\right] + D_0.$$

It follows from the Markov inequality that for all $\varepsilon > 0$,

$$\frac{1}{T}\int_0^T \mathbb{P}\left(\|u(t)\|^2_{H^1_0(\mathbb{T})} > \varepsilon\right) \mathrm{d}t \leq \varepsilon\left(\frac{1}{2\nu T}\mathbb{E}\left[\|u_0\|^2_{L^2(\mathbb{T})}\right] + D_0\right).$$

Hence, since the space $H^1_0(\mathbb{T})$ is compactly embedded in $L^2_0(\mathbb{T})$, the requirements of the Krylov-Bogoliubov theorem are met. However, $(u(t))_{t\geq 0}$ is not stricto sensu a Markov process on $L^2_0(\mathbb{T})$ but on $H^2_0(\mathbb{T})$. The weak limit of the sequence $(\mu_{T_n})_{n\in\mathbb{N}}$ we obtain by this method do not qualify to be an invariant measure as we do not know if it gives full weight to $H^2_0(\mathbb{T})$. That is where higher regularity estimates need to intervene and where the original proof of the Krylov-Bogoliubov theorem needs to be adapted in order to conclude for the existence part of our proof (Lemma 2.23).

The proof of uniqueness relies on the following standard property for scalar conservation laws:

Proposition (L^1-contraction, Proposition 2.21). *Let $(u(t))_{t\geq 0}$ and $(v(t))_{t\geq 0}$ be two strong solutions of Equation (1.6). Then, for all $0 \leq s \leq t$, almost surely,*

$$\|u(t) - v(t)\|_{L^1(\mathbb{T})} \leq \|u(s) - v(s)\|_{L^1(\mathbb{T})}.$$

That is to say, two solutions of (1.6) with possibly different initial conditions, but driven by the same noise, get closer (or at least do not move away from each other) with respect to the L^1 distance. Recall however that the dissipativity property induces that each of these solutions drifts towards the center of the space $L^2_0(\mathbb{T})$. This type of behaviour is the starting point for the following *coupling* argument: the two solutions are attracted to the center of $L^2_0(\mathbb{T})$ and thus, they get closer to one another for the L^2 distance and even more so for the L^1 distance, and each time they get closer, they stay closer as the L^1-contraction prevents them to move apart. Applying this argument leads to the reinforced

Proposition (L^1-confluence, Equation (2.56)). *Let $(u(t))_{t\geq 0}$ and $(v(t))_{t\geq 0}$ be two strong solutions of Equation (1.6). Then, almost surely,*

$$\lim_{t\to\infty} \|u(t) - v(t)\|_{L^1(\mathbb{T})} = 0.$$

Assuming that there exist two different invariant measures μ_1 and μ_2 for (1.6), and choosing two solutions $(u(t))_{t\geq 0}$ and $(v(t))_{t\geq 0}$ such that $u_0 \sim \mu_1$ and $v_0 \sim \mu_2$, the above proposition yields promptly that $\mu_1 = \mu_2$.

Authors and reference	Equation	Space discretisation	Time discretisation
Bréhier [20]	Semi-linear parabolic SPDE	Galerkin method	Semi-implicit[3] Euler scheme
Bréhier, Kopec [21]	Semi-linear parabolic SPDE	Spectral Galerkin method	Semi-implicit Euler scheme
Chen, Hong, Wang [26]	Damped stochastic non-linear Schrödinger equation	Spectral Galerkin method	Modified implicit Euler scheme
Chen, Gan, Wang [29]	Semi-linear parabolic SPDE	Spectral Galerkin method	Exponential Euler scheme
Cui, Hong, Sun [33]	Semi-linear parabolic SPDE with cubic polynomial non-linearity	Spectral Galerkin method	Implicit Euler scheme
Boyaval, Martel, Reygner [19]	Stochastic viscous scalar conservation law	Finite Volume method	Split-step Euler scheme

Figure 1.2: Numerical approximation results for invariant measures of SPDEs

1.3 Numerical approximation

Most of the numerical schemes developed to approximate solutions of SPDEs in finite time are adapted from the usual deterministic methods. For semi-linear SPDEs such as Equation (1.4), a common approach is the finite element approximation and in particular the Spectral Galerkin method (see for instance [90, 104, 4, 32, 3, 83]). This method is not suited for the inviscid equation (1.5). In this hyperbolic case, the discontinuous Galerkin method may be considered [88], but the most common approach is the finite volume method [5, 6, 47, 45]. A notable advantage of finite volumes is that they are well suited for conservation laws in general, should they be parabolic or hyperbolic [57].

The numerical approximation results for the invariant measure of an SPDE are quite recent. In Figure 1.2, we recorded, in the chronological order, a list of these results. As far as we are aware, the list is exhaustive.

The last line of Figure 1.2 corresponds to the content of Chapter 3, which is devoted to the approximation of the invariant measure of the solution of Equation (1.6) with a finite volume method for the space discretisation and a split-step backward Euler method for the time discretisation. We should point out that in all the other mentioned references of Figure 1.2, convergence rates are established for the weak error of discretisation. In our case, the lack of such results is essentially due the non-globally Lipschitz nature of the flux term, but we will get back to this point after introducing our numerical scheme.

In order to discretise (1.6) with respect to the space variable, we first define the following regular mesh on the torus:

$$\left\{\left(\frac{i-1}{N}, \frac{i}{N}\right], i \in \mathbb{Z}/N\mathbb{Z}\right\}.$$

Averaging in (1.6) over each cell of the mesh, we get

$$\begin{aligned}\mathrm{d}\left(N\int_{\frac{i-1}{N}}^{\frac{i}{N}} u(t,x)\mathrm{d}x\right) &= -N\left(A\left(u\left(t,\frac{i}{N}\right)\right) - A\left(u\left(t,\frac{i-1}{N}\right)\right)\right)\mathrm{d}t \\ &\quad + \nu N\left(\partial_x u\left(t,\frac{i}{N}\right) - \partial_x u\left(t,\frac{i-1}{N}\right)\right)\mathrm{d}t + \sum_{k\geq 1} N\int_{\frac{i-1}{N}}^{\frac{i}{N}} g_k(x)\mathrm{d}x\mathrm{d}W^k(t), \quad i \in \mathbb{Z}/N\mathbb{Z}. \qquad (1.8)\end{aligned}$$

[3]For semi-linear equations, the semi-implicit scheme is implicit with respect to the linear term and explicit with respect to the non-linear term.

Finite-volume schemes aim to approximate the dynamics of the average value of the solution over each cell of the mesh. This leads to the introduction of a *numerical flux* function $\overline{A}(u,v)$ approximating the flux of the conserved quantity at the interface between two adjacent cells. As regards the viscous term in (1.8), we replace the space derivatives by their finite difference approximations. As for the noise coefficients, we introduce the shorthand notation

$$\sigma_i^k := N\int_{\frac{i-1}{N}}^{\frac{i}{N}} g_k(x)\mathrm{d}x, \quad k\geq 1, \quad i\in\mathbb{Z}/N\mathbb{Z},$$

and we write $\boldsymbol{\sigma}^k := (\sigma_1^k,\dots,\sigma_N^k)$. In particular, $\boldsymbol{\sigma}^k$ belongs to the space

$$\mathbb{R}_0^N := \{\mathbf{u}=(u_1,\dots,u_N)\in\mathbb{R}^N : u_1+\cdots+u_N=0\}.$$

These operations result in the following stochastic differential equation

$$\begin{aligned}\mathrm{d}U_i(t) = &-N\left(\overline{A}\left(U_i(t),U_{i+1}(t)\right)-\overline{A}\left(U_{i-1}(t),U_i(t)\right)\right)\mathrm{d}t\\ &+\nu N^2\left(U_{i+1}(t)-2U_i(t)+U_{i-1}(t)\right)\mathrm{d}t+\sum_{k\geq 1}\sigma_i^k\mathrm{d}W^k(t), \quad i\in\mathbb{Z}/N\mathbb{Z}, \quad t\geq 0, \end{aligned}\tag{1.9}$$

as a semi-discrete finite-volume approximation of (1.6) in the sense that $U_i(t)$ is meant to be an approximation of the spatial average $N\int_{\frac{i-1}{N}}^{\frac{i}{N}} u(t,x)\mathrm{d}x$. In order to be consistent with Equation (1.6), the numerical flux function has to satisfy $\overline{A}(v,v) = A(v)$ for all $v\in\mathbb{R}$. Moreover, we make the assumption of a *monotone* numerical flux, *i.e.* we assume that $\overline{A}$ is non-decreasing with respect to the first variable and non-increasing with respect to the second.

Denoting by $\mathbf{b}=(b_1,\dots,b_N)$ the function defined from $\mathbb{R}_0^N$ to $\mathbb{R}_0^N$ by

$$b_i(\mathbf{v}) := -N\left(\overline{A}(v_i,v_{i+1})-\overline{A}(v_{i-1},v_i)\right)+\nu N^2(v_{i+1}-2v_i+v_{i-1}), \quad i\in\mathbb{Z}/N\mathbb{Z}, \quad \mathbf{v}\in\mathbb{R}_0^N,$$

we can express (1.9) in the vectorised form

$$\mathrm{d}\mathbf{U}(t) = \mathbf{b}(\mathbf{U}(t))\mathrm{d}t+\sum_{k\geq 1}\boldsymbol{\sigma}^k\mathrm{d}W^k(t), \quad t\geq 0. \tag{1.10}$$

Let us now discretise (1.10) with respect to the time variable. For this, we introduce a time step $\Delta t>0$, and we write $\Delta W_n^k := W^k(n\Delta t)-W^k((n-1)\Delta t)$ for all $k,n\geq 1$. As it was already noticed in [94], explicit numerical schemes for SDEs with non-globally Lipschitz continuous coefficients do not preserve in general the large time stability, whereas implicit schemes are more robust. Therefore, since our main focus in this thesis is to approximate invariant measures, we propose the following *split-step backward Euler method*:

$$\begin{cases}\mathbf{U}_{n+\frac{1}{2}} = \mathbf{U}_n+\Delta t\mathbf{b}\left(\mathbf{U}_{n+\frac{1}{2}}\right),\\ \mathbf{U}_{n+1} = \mathbf{U}_{n+\frac{1}{2}}+\sum_{k\geq 1}\boldsymbol{\sigma}^k\Delta W_{n+1}^k.\end{cases}\tag{1.11}$$

The first result of Chapter 3 is the following

Theorem (Proposition 3.15, Proposition 3.23 and Theorem 3.5)**.** *Let* $\mathbf{U}_0$ *be an* $\mathbb{R}_0^N$*-valued random variable. Then,*

(i) there exists a unique $\mathbb{R}_0^N$*-valued stochastic process* $(\mathbf{U}(t))_{t\geq 0}$ *solution to (1.10) with initial condition* $\mathbf{U}_0$*;*

(ii) there exists a unique $\mathbb{R}_0^N$-valued Markov chain $(\mathbf{U}_n)_{n\in\mathbb{N}}$ solution to (1.11) with initial condition $\mathbf{U}_0$.

Furthermore,

(iii) $(\mathbf{U}(t))_{t\geq 0}$ admits a unique invariant measure ν_N;

(iv) $(\mathbf{U}_n)_{n\in\mathbb{N}}$ admits a unique invariant measure $\nu_{N,\Delta t}$.

The proof of this theorem relies on properties analogous to the SPDE. In particular, the monotonicity of the numerical flux ensures the following

Proposition (Lemma 3.13)**.** *For any $\mathbf{u},\mathbf{v}\in\mathbb{R}_0^N$, the function $\mathbf{b}$ satisfies*

$$\sum_{i\in\mathbb{Z}/N\mathbb{Z}} u_i b_i(\mathbf{u}) \leq -\nu N^2 \sum_{i\in\mathbb{Z}/N\mathbb{Z}} (u_{i+1}-u_i)^2 \quad \textit{(dissipativity)} \tag{1.12}$$

and

$$\sum_{i\in\mathbb{Z}/N\mathbb{Z}} \operatorname{sign}(u_i-v_i)\left(b_i(\mathbf{u})-b_i(\mathbf{v})\right) \leq 0. \tag{1.13}$$

The well-posedness of (1.10) is proved by use of Inequality (1.12) in a similar way as for the SPDE, while (1.12) and (1.13) allow to prove respectively the existence and uniqueness of a solution $(\mathbf{U}_n)_{n\in\mathbb{N}}$ to (1.11) and more than that, to prove respectively the existence and uniqueness of an invariant measure for both the processes $(\mathbf{U}(t))_{t\geq 0}$ and $(\mathbf{U}_n)_{n\in\mathbb{N}}$. In particular, thanks to (1.13), the L^1-contraction has been preserved by the dicretisation:

Proposition (Discrete L^1-contraction, Proposition 3.17 and Lemma 3.24)**.** *Let $(\mathbf{U}(t))_{t\geq 0}$ and $(\mathbf{V}(t))_{t\geq 0}$ be two solutions of (1.10), with possibly different initial conditions, but perturbed by the same Wiener process. Then, for all $0\leq s\leq t$, almost surely*

$$\|\mathbf{U}(t)-\mathbf{V}(t)\|_1 \leq \|\mathbf{U}(s)-\mathbf{V}(s)\|_1 .$$

Similarly, two solutions $(\mathbf{U}_n)_{n\in\mathbb{N}}$ and $(\mathbf{V}_n)_{n\in\mathbb{N}}$ to (1.11) satisfy for all $n\in\mathbb{N}$, almost surely,

$$\|\mathbf{U}_{n+1}-\mathbf{V}_{n+1}\|_1 \leq \|\mathbf{U}_n-\mathbf{V}_n\|_1 .$$

From the numerical scheme (1.11), we now reconstruct a piecewise constant approximation $(u_N(t))_{t\geq 0}$ to the solution $(u(t))_{t\geq 0}$ of (1.6) by setting

$$u_N(t,x) := \sum_{i\in\mathbb{Z}/N\mathbb{Z}} U_i(t)\mathbf{1}_{\left(\frac{i-1}{N},\frac{i}{N}\right]}(x). \tag{1.14}$$

This way $(u_N(t))_{t\geq 0}$ is an $L_0^2(\mathbb{T})$-valued process with an invariant probability measure μ_N on $L_0^2(\mathbb{T})$. This measure μ_N is actually the embedding of ν_N in $L_0^2(\mathbb{T})$ via the reconstruction given by (1.14), and in the same way, we can define the measure $\mu_{N,\Delta t}$ on $L_0^2(\mathbb{T})$ as the pushforward of the measure $\nu_{N,\Delta t}$ via the same reconstruction.

It is now possible to compare $\mu_{N,\Delta t}$, μ_N and μ as all of these measures belong to the space $\mathcal{P}(L_0^2(\mathbb{T}))$ of probability measures over $L_0^2(\mathbb{T})$. Note that $\mathcal{P}(L_0^2(\mathbb{T}))$ (resp. $\mathcal{P}(\mathbb{R}_0^N)$) is naturally endowed with the topology associated to the weak convergence. The subset of $\mathcal{P}(L_0^2(\mathbb{T}))$ (resp. $\mathcal{P}(\mathbb{R}_0^N)$) containing all the measures with finite second order moment is denoted $\mathcal{P}_2(L_0^2(\mathbb{T}))$ (resp. $\mathcal{P}_2(\mathbb{R}_0^N)$). We may endow this subspace with the topology associated to the following metric:

$$
\begin{tikzcd}
\mathbf{U}_n \arrow[r, "\text{Invariant measure}"] & \nu_{N,\Delta t} \arrow[d, "\Delta t \to 0"] \\
\mathbf{U}(t) \arrow[r, "\text{Invariant measure}"] & \nu_N \arrow[d, "N \to +\infty"] \\
u(t) \arrow[r, "\text{Invariant measure}"] & \mu
\end{tikzcd}
$$

Figure 1.3: Approximation of the invariant measure μ

Definition (Wasserstein distance, Definition 3.6)**.** *Let $(E, \|\cdot\|_E)$ be a normed vector space. Let α and β be two probability measures on E with finite second order moment. The* second order Wasserstein distance *between α and β is defined by*

$$W_2(\alpha, \beta) := \inf_{\substack{u\sim\alpha \\ v\sim\beta}} \mathbb{E}\left[\|u-v\|_E^2\right]^{1/2}.$$

Let us now state the second result of Chapter 3, and main result of this thesis:

Theorem (Convergence of invariant measures, Theorem 3.7)**.** *The following equality holds:*

$$\lim_{N\to\infty} \lim_{\Delta t\to 0} W_2(\mu_{N,\Delta t}, \mu) = 0.$$

As illustrated by Figure 1.3, a specific order for the discretisation leads to the proof of the above theorem. In particular, we show that:

A) μ_N converges towards μ as $N \to +\infty$;

B) for any integer $N \geq 1$, $\nu_{N,\Delta t}$ (or $\mu_{N,\Delta t}$) converges towards ν_N (or μ_N) as $\Delta t \to 0$.

The step A is the subject of Section 3.3 while the step B is addressed in Section 3.4. Each of these steps is itself divided in two substeps as we will show that

A.1) the family $(\mu_N)_{N\geq 1}$ is relatively compact in $\mathcal{P}_2(L_0^2(\mathbb{T}))$;

A.2) any subsequential μ^* is invariant for the solution $(u(t))_{t\geq 0}$ of (1.6), and thus $\mu^* = \mu$;

B.1) the family $(\nu_{N,\Delta t})_{\Delta t>0}$ is relatively compact in $\mathcal{P}_2(\mathbb{R}_0^N)$;

B.2) any subsequential limit ν_N^* (as $\Delta t \to 0$) is invariant for the solution $(\mathbf{U}(t))_{t\geq 0}$ of (1.10), and thus $\nu_N^* = \nu_N$.

As the proofs of A.1 and A.2 are very much alike those of B.1 and B.2 respectively, let us give only the intuition regarding the step A. The proof of relative compactness of the family $(\mu_N)_{N\geq 1}$ in $\mathcal{P}_2(L_0^2(\mathbb{T}))$ involves the Prokhorov theorem and the following uniform discrete $H_0^1(\mathbb{T})$ estimate (Lemma 3.31):

$$\mathbb{E}\left[N \sum_{i\in\mathbb{Z}/N\mathbb{Z}} (U_{i+1}-U_i)^2\right] \leq \frac{D_0}{2\nu}, \qquad \mathbf{U} \sim \nu_N.$$

Now, let μ^* be the limit in $\mathcal{P}_2(L^2_0(\mathbb{T}))$ of some subsequence $(\mu_{N_j})_{j\in\mathbb{N}}$. By virtue of the Skorokhod representation theorem, we choose random variables $u(0) \sim \mu^*$ and $u_{N_j}(0) \sim \mu_{N_j}$ such that $u_{N_j}(0)$ converges almost surely in $L^2_0(\mathbb{T})$ towards $u(0)$ as $j \to +\infty$. Consequently, we can define on the same probability space the processes $(u(t))_{t\geq 0}$ and $(u_{N_j}(t))_{t\geq 0}$ respectively as the solution of (1.6) with initial condition $u(0)$ and the reconstructed solution of (1.10) via (1.14) with initial condition $u_{N_j}(0)$. In this setting, we show the following finite time convergence result (Proposition 3.36):

$$\forall t \geq 0, \quad \lim_{j\to\infty} \mathbb{E}\left[\left\|u_{N_j}(t) - u(t)\right\|^2_{L^2_0(\mathbb{T})}\right] = 0. \tag{1.15}$$

Since for all $t \geq 0$, the law of $u_{N_j}(t)$ is μ_{N_j}, it follows immediately that $W_2(\mu_{N_j}, \mathcal{L}(u(t)))$ converges to 0 as $j \to +\infty$. Thus, $W_2(\mu^*, \mathcal{L}(u(t))) = 0$ for all $t \geq 0$, which means that μ^* is actually invariant for the process $(u(t))_{t\geq 0}$.

In establishing the finite time approximation (1.15), we had to deal with a non-Lipschitz flux term. Our control of the strong error relied on moment estimates, which themselves relied on the assumption of polynomial growth of the flux function. The approximation in time is treated in the same way. A drawback of this approach is that no convergence rate of the discretisation error can be derived. A possible way to establish orders of convergence of the invariant measure for the W_2-distance would be to make the stronger assumption that A is globally Lipschitz and start from the convergence rates for the strong error in finite time that would come up. One could also follow a more standard procedure, and analyse the weak error, in particular when $\Delta t \to 0$, with the techniques that were pioneered by Talay [98, 99].

1.4 Application to turbulence

The rigorous understanding of the turbulence phenomenon is one of the most important open problems in mathematical physics. Among the extensive research efforts that have been made in this field, stochastic scalar conservation laws, and more specifically the stochastic Burgers equation, were found to play some role. To explain how these SPDEs were thrown under the light of such physical motivations, some context ought to be introduced.

The most common model for describing the motion of an incompressible viscous fluid flow is given by the *Navier-Stokes equations*. Derived from the fundamental principles of physics that are the continuum hypothesis (see the beginning of Section 1.1) and Newton's second law, and from taking into account the internal friction (*i.e.* the viscosity), this system of equations is given by:

$$\begin{cases} \partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \nu\Delta\mathbf{u} + \mathbf{f} \\ \mathrm{div}(\mathbf{u}) = 0, \end{cases} \tag{1.16}$$

where the unkowns are the velocity field $\mathbf{u}(t,\mathbf{x})$ and the pressure $p(t,\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^3$ (see for instance [86] for the complete derivation of the system). The parameters include the kinematic viscosity coefficient $\nu > 0$ and the external forces $\mathbf{f}(t,\mathbf{x})$ acting on the fluid. To study the general features of the flow that do not depend on the time and space scales, one may non-dimensionalise (1.16). In this regard, a most notable indicator of the behaviour of the flow is the dimensionless *Reynolds number* $\mathrm{Re} = \frac{UL}{\nu}$, where U and L are respectively the typical velocity and length scales. When the Reynolds number is low enough, the flow is characterised by an orderly and predictable motion. The fluid particles follow smooth paths that do not cross each other. In this case, the flow is said to be *laminar*. Such a stable behaviour contrasts drastically with high Reynolds number flows. Indeed, when Re is high enough (usually around the order $\mathrm{Re} \approx 10^3$ and beyond), the flow follows a quite unsteady motion. Vortices of many sizes appear in a disorganised manner. In particular, fluctuations in the fluid occur at a wide range of space and time scales. A flow displaying such a chaotic behaviour is said to be *turbulent*.

Motivated by industrial and theoretical issues, hydrodynamic turbulence constitutes an active field of research. Numerous works testing numerical simulations against experimental results have attested the relevance of the Navier-Stokes model (for turbulent flows as much as for laminar ones).

Groundbreaking advances in the understanding of turbulence came forth in 1941 with the works of Kolmogorov [81, 80, 79] (see also [61]). Using scaling arguments, Kolmogorov postulated that the (seemingly chaotic) turbulent flows actually display some *universal* behaviour, *i.e.* they show particular features that depend only on Re[4]. His framework relies on several circumstantial assumptions. In particular, the velocity $\mathbf{u}(t, \mathbf{x})$ is assumed to be a random field whose space and time increments are pairwise independent and invariant with respect to rotations and reflections in some domain (the turbulence is said locally homogeneous and isotropic). Let us give some physical insight as regards these universality predictions. The external forces applied on the flow supply an amount of kinetic energy which is described by large space scales (of an order L). Due to the convection happening within the fluid, the energy is transported to smaller and smaller scales, down to a scale l, under which the energy is dissipated by the effect of viscosity. Between the energy scale L and the dissipation scale l, the transfer of kinetic energy does not depend on the forcing nor on the viscosity. Thus, in this interpretation of turbulence as an *energy cascade*, the emergence of universal properties in a particular scale range seems conceivable. The universality prediction is asserted via quantitative estimates such as the following:

$$\left\langle \left| \frac{(\mathbf{u}(t, \mathbf{x}+\mathbf{r}) - \mathbf{u}(t, \mathbf{x})) \cdot \mathbf{r}}{\|\mathbf{r}\|_2} \right|^n \right\rangle \sim C_n \, \|\mathbf{r}\|_2^{\frac{n}{3}}, \quad l \ll \|\mathbf{r}\|_2 \ll L, \quad n \geq 0, \tag{1.17}$$

where $\langle \cdot \rangle$ denote the expectation of the underlying random variable, and where C_n is a universal constant. When $n = 2$, (1.17) may be expressed in the Fourier space, which gives the following relation for the space Fourier tranform $\hat{\mathbf{u}}$ of $\mathbf{u}$:

$$\left\langle \sum_{k < \|\mathbf{k}\|_2 \leq k+1} \|\hat{\mathbf{u}}(\mathbf{k})\|_2^2 \right\rangle \sim k^{-\frac{5}{3}}, \quad L^{-1} \ll k \ll l^{-1}, \tag{1.18}$$

which is the well-known *Kolmogorov spectrum*.

It should be noticed that (1.17), and thus (1.18), have been derived solely from scaling analysis and physical arguments. In particular, they do not rely on the Navier-Stokes equations. By the way, their mathematical status remains at the conjecture level. A rigorous analysis of the universal properties of turbulence ought to be achieved through (1.16). For instance, one could consider the stochastic process $(\mathbf{u}(t, \mathbf{x}))_{t \geq 0}$ solution to (1.16) with a random initial condition. In this perspective, one could furthermore study the invariant measures for (1.16) and compute the energy spectrum of stationary solutions. However, the well-posedness of (1.16) remains a notorious open problem and despite many efforts to overcome it, there are only very limited partial results on this topic [59].

Nonetheless, the concept of energy cascade may be addressed via simpler models. In this perspective, if we bring Equation (1.16) to one space dimension, thereby dropping the pressure term and incompressibility condition which do not have a meaning in 1D, it boils down to

$$\partial_t u + u \partial_x u = \nu \partial_{xx} u + f,$$

that is, the viscous Burgers equation. This equation was actually studied by J. M. Burgers as a toy model for hydrodynamic turbulence [24]. Many works followed in this direction and some adjustments were made to this initial equation, by considering for instance a random initial condition or a random forcing. Indeed, the restriction from three to one space dimension induces the loss of the chaotic behaviour in the model. The stochastic forcing allows to maintain a steady yet unpredictable supply of energy in the system and thus to keep a cascade-like energy transfer.

[4] These properties constitute what is now called the *K41 theory*.

More than that, what was originally meant to be a testing ground for the Navier-Stokes equations (for theoretical as much as for numerical concerns) turned out to be a prototype for a wide range of physical systems where the non-linear effects induce a non-trivial flux of energy across scales. Applications were found for instance in cosmology [105, 66, 101, 2], vehicle traffic models [31], vortices in superconductors [13]... The stochastic Burgers equation has also close ties with the KPZ equation [10, 65] and in this respect, happened to be an appropriate model for growing interfaces and directed polymers [77, 17, 78].

The mathematical questions raised by the randomly forced Burgers equation are the well-posedness, the invariant measures and the regularity of solutions. These were discussed in Section 1.2. Questions of a more physical nature concern the energy spectrum and the small-scale characteristics [25, 30, 76, 64] as well as the asymptotic behaviour of probability distribution functions [53, 51, 55, 50]. For general surveys, see [62, 8, 48, 49, 54].

Let us mention here an analogous result to the Kolmogorov spectrum (1.18) in the framework of Equation (1.6):

Theorem (Boritchev, [15, Theorem 4.7.3])**.** *Let μ be the invariant measure for (1.6) where the flux function A is C^∞ and strictly convex, and let u be a random variable with distribution μ. Then the Fourier transform $\hat{u}$ of u satisfies for some $M > 0$*

$$\mathbb{E}\left[\sum_{kM^{-1}\leq n\leq kM} |\hat{u}_n|\right] \sim k^{-2}.$$

In particular, the Burgers energy spectrum has a decay rate of order -2. In Chapter 4, numerical experiments are realised on Burgers' equation with the finite volume scheme introduced in Section 1.3. After testing the stationarity of the scheme and establishing empirical convergence rates in space and in time, we compute some small-scale characteristics regarding turbulence. Notably, the slope of the energy spectrum is plotted and the result of the above theorem is recovered numerically.

Chapter 2

Viscous scalar conservation law with stochastic forcing: strong solution and invariant measure

Résumé. *Ce chapitre correspond à la pré-publication [92], écrite en collaboration avec J. Reygner. On s'intéresse aux lois de conservation scalaires avec de la viscosité et un bruit blanc en temps mais spatialement correlé. Le domaine spatial considéré est de dimension* 1 *et périodique. La fonction de flux est supposée localement lipschitzienne et à croissance polynomiale. Aucune hypothèse de non-dégénerescence n'est imposée au flux ou au bruit. Dans un premier temps, on prouve l'existence et l'unicité d'une solution globale au sens fort. Dans un second temps, on établit l'existence et l'unicité d'une mesure invariante pour cette solution forte.*

Abstract. This chapter corresponds to the preprint [92], written in collaboration with J. Reygner. We are interested in viscous scalar conservation laws with a white-in-time but spatially correlated stochastic forcing. The equation is assumed to be one-dimensional and periodic in the space variable, and its flux function to be locally Lipschitz continuous and have at most polynomial growth. Neither the flux nor the noise need to be non-degenerate. In a first part, we show the existence and uniqueness of a global solution in a strong sense. In a second part, we establish the existence and uniqueness of an invariant measure for this strong solution.

2.1 Introduction

2.1.1 Stochastic viscous scalar conservation law

We are interested in the existence, uniqueness, regularity and large time behaviour of solutions of the following viscous scalar conservation law with additive and time-independent stochastic forcing

$$\mathrm{d}u = -\partial_x A(u)\mathrm{d}t + \nu\partial_{xx}u\mathrm{d}t + \sum_{k\geq 1} g_k \mathrm{d}W^k(t), \quad x \in \mathbb{T}, \quad t \geq 0, \tag{2.1}$$

where $(W^k(t))_{t\geq 0}$, $k \geq 1$, is a family of independent Brownian motions. Here, $\mathbb{T}$ denotes the one-dimensional torus $\mathbb{R}/\mathbb{Z}$, meaning that the sought solution is periodic in space. The *flux function* A is assumed to satisfy the following set of conditions.

Assumption 2.1 (on the flux function)**.** *The function $A : \mathbb{R} \to \mathbb{R}$ is C^2 on $\mathbb{R}$, its first derivative has at most polynomial growth:*

$$\exists C_1 > 0, \quad \exists p_A \in \mathbb{N}^*, \quad \forall v \in \mathbb{R}, \qquad |A'(v)| \leq C_1 \left(1 + |v|^{p_A}\right), \tag{2.2}$$

and its second derivative A'' is locally Lipschitz continuous on $\mathbb{R}$.

The parameter $\nu > 0$ is the *viscosity* coefficient. In order to present our assumptions on the family of functions $g_k : \mathbb{T} \to \mathbb{R}$, $k \geq 1$, which describe the spatial correlation of the stochastic forcing of (2.1), we first introduce some notation. For any $p \in [1, +\infty]$, we denote by $L^p_0(\mathbb{T})$ the subset of functions $v \in L^p(\mathbb{T})$ such that

$$\int_{\mathbb{T}} v \mathrm{d}x = 0.$$

The L^p norm induced on $L^p_0(\mathbb{T})$ is denoted by $\|\cdot\|_{L^p_0(\mathbb{T})}$. For any integer $m \geq 0$, we denote by $H^m_0(\mathbb{T})$ the intersection of the Sobolev space $H^m(\mathbb{T})$ with $L^2_0(\mathbb{T})$. Equipped with the norm

$$\|v\|_{H^m_0(\mathbb{T})} := \left(\int_{\mathbb{T}} |\partial^m_x v|^2 \mathrm{d}x\right)^{1/2},$$

and the associated scalar product $\langle \cdot, \cdot \rangle_{H^m_0(\mathbb{T})}$, it is a separable Hilbert space. On the one-dimensional torus, the Poincaré inequality implies that $H^{m+1}_0(\mathbb{T}) \subset H^m_0(\mathbb{T})$ and $\|\cdot\|_{H^m_0(\mathbb{T})} \leq \|\cdot\|_{H^{m+1}_0(\mathbb{T})}$. Actually, the following stronger inequality holds: if $v \in H^1_0(\mathbb{T})$, then $v \in L^\infty_0(\mathbb{T})$ and for all $p \in [1, +\infty)$,

$$\|v\|_{L^p_0(\mathbb{T})} \leq \|v\|_{L^\infty_0(\mathbb{T})} \leq \|v\|_{H^1_0(\mathbb{T})}. \tag{2.3}$$

The spaces $H^m_0(\mathbb{T}), m \geq 0$, generalise to the class of fractional Sobolev spaces $H^s_0(\mathbb{T})$, where $s \in [0, +\infty)$, which will be defined in Section 2.2.1. We may now state:

Assumption 2.2 (on the noise functions)**.** *For all $k \geq 1$, $g_k \in H^2_0(\mathbb{T})$ and*

$$D_0 := \sum_{k \geq 1} \|g_k\|^2_{H^2_0(\mathbb{T})} < +\infty. \tag{2.4}$$

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, equipped with a normal filtration $(\mathcal{F}_t)_{t \geq 0}$ in the sense of [35, Section 3.3], on which $(W^k)_{k \geq 1}$ is a family of independent Brownian motions. Under Assumption 2.2, the series $\sum_k g_k W^k$ converges in $L^2(\Omega, C([0,T], H^2_0(\mathbb{T})))$, for any $T > 0$, towards an $H^2_0(\mathbb{T})$-valued Wiener process $(W^Q(t))_{t \in [0,T]}$ with respect to the filtration $(\mathcal{F}_t)_{t \geq 0}$, defined in the sense of [35, Section 4.2], with the trace class covariance operator $Q : H^2_0(\mathbb{T}) \to H^2_0(\mathbb{T})$ given by

$$\forall u, v \in H^2_0(\mathbb{T}), \qquad \langle u, Qv \rangle_{H^2_0(\mathbb{T})} = \sum_{k \geq 1} \langle u, g_k \rangle_{H^2_0(\mathbb{T})} \langle v, g_k \rangle_{H^2_0(\mathbb{T})}. \tag{2.5}$$

Thus, almost surely, $t \mapsto W^Q(t)$ is continuous in $H^2_0(\mathbb{T})$ and for all $u \in H^2_0(\mathbb{T})$, the process $(\langle W^Q(t), u \rangle_{H^2_0(\mathbb{T})})_{t \geq 0}$ is a real-valued Wiener process with variance

$$\mathbb{E}\left[\langle W^Q(t), u \rangle^2_{H^2_0(\mathbb{T})}\right] = t \sum_{k \geq 1} \langle g_k, u \rangle^2_{H^2_0(\mathbb{T})}. \tag{2.6}$$

2.1.2 Main results and previous works

First, we are interested in the well-posedness in the strong sense of Equation (2.1). In particular, we look for solutions that admit at least a second spatial derivative in order to give a classical meaning to the viscous term, in the sense of the following definition:

Definition 2.3 (Strong solution to (2.1))**.** *Let $u_0 \in H^2_0(\mathbb{T})$. Under Assumptions 2.1 and 2.2, a strong solution to Equation (2.1) with initial condition u_0 is an $(\mathcal{F}_t)_{t \geq 0}$-adapted process $(u(t))_{t \geq 0}$ with values in $H^2_0(\mathbb{T})$ such that, almost surely:*

1. *the mapping $t \mapsto u(t)$ is continuous from $[0, +\infty)$ to $H_0^2(\mathbb{T})$;*

2. *for all $t \geq 0$, the following equality holds:*

$$u(t) = u_0 + \int_0^t \left(-\partial_x A\left(u(s)\right) + \nu \partial_{xx} u(s)\right) \mathrm{d}s + W^Q(t). \tag{2.7}$$

In the above definition, the first condition ensures that the time integral in Equation (2.7) is a well-defined Bochner integral in $L_0^2(\mathbb{T})$. For a careful introduction of the general concepts of random variables and stochastic processes in Hilbert spaces, the reader is referred to the third and fourth chapters of the reference book [35].

Our first result is the following:

Theorem 2.4 (Well-posedness)**.** *Let $u_0 \in H_0^2(\mathbb{T})$. Under Assumptions 2.1 and 2.2, there exists a unique strong solution $(u(t))_{t\geq 0}$ to Equation (2.1) with initial condition u_0. Moreover, the solution depends continuously on initial data in the following sense: if $(u_0^{(j)})_{j\geq 1}$ is a sequence of $H_0^2(\mathbb{T})$ satisfying*

$$\lim_{j\to\infty} \left\| u_0 - u_0^{(j)} \right\|_{H_0^2(\mathbb{T})} = 0,$$

then, denoting by $(u^{(j)}(t))_{t\geq 0, j\geq 1}$ the family of associated solutions, for any $T \geq 0$, we have almost surely

$$\lim_{j\to\infty} \sup_{t\in[0,T]} \left\| u(t) - u^{(j)}(t) \right\|_{H_0^2(\mathbb{T})} = 0.$$

Similar results have already been established: the case where the flux A is strictly convex is treated in [15, Appendix A], and the case where A is globally Lipschitz continuous is treated in [73]. Furthermore, the case of *mild* solutions (in L^p spaces) has been looked at in [69]. Here, no global Lipschitz continuity assumption nor restrictions on the convexity of the flux function are made. We can also point out that the well-posedness of stochastically forced conservations laws in the inviscid case (*i.e.* when $\nu = 0$) has been under a great deal of investigation in the recent years. In this "hyperbolic" framework, the appearance of shocks prevents the solutions to be smooth enough to be considered in a strong sense as in our present work. Therefore, the study of *entropic* solutions [60] or *kinetic* solutions [40] to the SPDE have been the two main approaches, both of which rely on a *vanishing viscosity* argument: the entropic or kinetic solution is sought as the limit of its viscous approximation as the viscosity coefficient tends to 0.

Let $C_b(H_0^2(\mathbb{T}))$ denote the set of continuous and bounded functions from $H_0^2(\mathbb{T})$ to $\mathbb{R}$. As a consequence of Theorem 2.4, we can define a family of functionals $(P_t)_{t\geq 0}$ on $C_b(H_0^2(\mathbb{T}))$ by writing

$$P_t\varphi(u_0) := \mathbb{E}_{u_0}\left[\varphi(u(t))\right], \qquad t \geq 0, \quad u_0 \in H_0^2(\mathbb{T}),$$

where the notation $\mathbb{E}_{u_0}$ indicates that the random variable $u(t)$ is the solution to (2.1) at time t starting from the initial condition u_0.

Corollary 2.5. *Under Assumptions 2.1 and 2.2, the family $(P_t)_{t\geq 0}$ is a Feller semigroup and the process $(u(t))_{t\geq 0}$ is a strong Markov process in $H_0^2(\mathbb{T})$ with semigroup $(P_t)_{t\geq 0}$.*

Proof. The uniqueness of a strong solution and the fact that, for all $t \geq 0$, the processes $(W^Q(t + s) - W^Q(t))_{s\geq 0}$ and $(W^Q(s))_{s\geq 0}$ have the same distribution, ensure that $(P_t)_{t\geq 0}$ is a semigroup, and therefore that $(u(t))_{t\geq 0}$ is a Markov process. The Feller property is a straightforward consequence of the result of continuous dependence on initial conditions given in Theorem 2.4, whereas it is a classical result that the strong Markov property of $(u(t))_{t\geq 0}$ follows from the Feller property of $(P_t)_{t\geq 0}$ (see for instance the proof of [23, Theorem 16.21]). □

Let $\mathcal{B}(H_0^2(\mathbb{T}))$ denote the Borel σ-algebra of the metric space $H_0^2(\mathbb{T})$, and $\mathcal{P}(H_0^2(\mathbb{T}))$ refer to the set of Borel probability measures on $H_0^2(\mathbb{T})$. The Markov property allows us to extend the notion of strong solution to (2.1) by considering not only a deterministic initial condition but any $\mathcal{F}_0$-measurable random variable u_0 on $H_0^2(\mathbb{T})$. In this perspective, we define the dual semigroup $(P_t^*)_{t\geq 0}$ of $(P_t)_{t\geq 0}$ by

$$P_t^*\alpha(\Gamma) := \int_{H_0^2(\mathbb{T})} \mathbb{P}_{u_0}\left(u(t)\in\Gamma\right) \mathrm{d}\alpha(u_0), \qquad t\geq 0, \quad \alpha\in\mathcal{P}\left(H_0^2(\mathbb{T})\right), \quad \Gamma\in\mathcal{B}\left(H_0^2(\mathbb{T})\right).$$

In particular, $P_t^*\alpha$ is the law of $u(t)$ when u_0 is distributed according to α.

Definition 2.6 (Invariant measure)**.** *We say that a probability measure $\mu\in\mathcal{P}(H_0^2(\mathbb{T}))$ is an invariant measure for the semigroup $(P_t)_{t\geq 0}$ (or equivalently for the process $(u(t))_{t\geq 0}$) if and only if*

$$\forall t\geq 0, \quad P_t^*\mu = \mu.$$

Theorem 2.7 (Existence, uniqueness and estimates on the invariant measure)**.** *Under Assumptions 2.1 and 2.2, the process $(u(t))_{t\geq 0}$ solution to the SPDE (2.1) admits a unique invariant measure μ. Besides, if $u\in H_0^2(\mathbb{T})$ is distributed according to μ, then $\mathbb{E}[\|u\|^2_{H_0^2(\mathbb{T})}] < +\infty$ and, for all $p\in[1,+\infty)$, $\mathbb{E}[\|u\|^p_{L_0^p(\mathbb{T})}] < +\infty$.*

A few similar results exist in the literature. Da Prato, Debussche and Temam [34] have studied the viscous Burgers equation (which corresponds to the flux function $A(u) = u^2/2$) perturbed by an additive space-time white noise whereas Da Prato and Gatarek [96] studied the same equation but with a multiplicative white noise. Both showed the well-posedness of the equation as well as the existence of an invariant measure. These results are moreover put in a much detailed context in the two reference books [35, 36]. Boritchev [14, 15, 16] showed the existence and uniqueness of an invariant measure for the viscous *generalised* Burgers equation (which corresponds to the case of strictly convex flux function) perturbed by a white-in-time and spatially correlated noise. E, Khanin, Mazel and Sinai [52] showed the existence and uniqueness of an invariant measure for the inviscid Burgers equation with a white-in-time and spatially correlated noise. Debussche and Vovelle [41] generalised this last result by extending it to *non-degenerate* flux functions (roughly speaking, there is no non-negligible subset of $\mathbb{R}$ on which A is linear). Besides, the fact that these results from [52, 41] also hold when $\nu = 0$ makes them quite powerful: it shows indeed that the presence of a viscous term is not a necessary condition for the solution to be stationary.

The stochastic Burgers equation is mainly studied as a one-dimensional model for *turbulence*. By showing a stable behaviour at large times, this model manages, to some extent, to fit the predicitions of Kolmogorov's "K41" theory about the *universal* properties of a turbulent flow [81, 80]. Whether it is modelled by the Burgers equation or a by more general process such as Equation (2.1), turbulence is then described through the statistics of some particular small-scale quantities in the stationary state [48, 49]. Sharp estimates were given by Boritchev for these small-scale quantities [15], which were furthermore shown to be independent of the viscosity coefficient. One of the purposes of this chapter is to lay the groundwork for the numerical analysis of Equation (2.1). In Chapter 3, we introduce a finite-volume approximation of (2.1) which allows to approximate the invariant measure μ. Generating random variables with distribution μ shall eventually lead us to compute said small-scale quantities and analyse the development of turbulence in the model established by Equation (2.1).

2.1.3 Outline of the chapter

The proofs of Theorems 2.4 and 2.7 are respectively detailed in Sections 2.2 and 2.3.

2.2 Well-posedness and regularity

This section is dedicated to the proof of Theorem 2.4. This proof is decomposed as follows. In Subsection 2.2.1, we introduce a weaker formulation of Equation (2.1), the so-called *mild formulation*. In Subsection 2.2.2, we show that Equation (2.1) is well-posed locally in time both in the mild and in the strong sense. In Subsection 2.2.3, we give higher bounds for the Lebesgue and Sobolev norms of this local solution. Eventually, these estimates allow us to extend the local solution to a global-in-time solution, and thus to prove Theorem 2.4 in Subsection 2.2.4.

2.2.1 Mild formulation of (2.1)

In this subsection, we collect preliminary results which shall enable us to provide a *mild* formulation of Equation (2.1), for which we prove the existence and uniqueness of a solution on a small interval. The proofs of several results are postponed to Subsection 2.2.5.

Fractional Sobolev spaces

For all $m' \geq 1$, let us define $\lambda_{2m'-1} = \lambda_{2m'} = -(2\pi m')^2$, and $e_{2m'-1}(x) = \sqrt{2}\sin(2\pi m' x)$, $e_{2m'}(x) = \sqrt{2}\cos(2\pi m' x)$. The family $(e_m)_{m\geq 1}$ is a complete orthogonal basis of $L^2_0(\mathbb{T})$ such that, for all $m \geq 1$, e_m is C^∞ on $\mathbb{T}$ and $\partial_{xx} e_m = \lambda_m e_m$. With respect to this basis, we define the fractional Sobolev space $H^s_0(\mathbb{T})$, for any $s \in [0, +\infty)$, as the space of functions $v \in L^2_0(\mathbb{T})$ such that

$$\|v\|_{H^s_0(\mathbb{T})} := \left(\sum_{m \geq 1} (-\lambda_m)^s \langle v, e_m \rangle^2_{L^2_0(\mathbb{T})} \right)^{1/2} < +\infty. \tag{2.8}$$

We take from [15, Appendice A] the following proposition and adapt it to our case of a flux function satisfying Assumption 2.1:

Proposition 2.8. *Under Assumption 2.1, for any $s \in [1, 2]$, the mapping*

$$v \in H^s_0(\mathbb{T}) \longmapsto \partial_x A(v) \in H^{s-1}_0(\mathbb{T})$$

is bounded on bounded subsets of $H^s_0(\mathbb{T})$. Moreover, when $s = 1$ or $s = 2$, it is Lipschitz continuous on bounded subsets of $H^s_0(\mathbb{T})$.

The proof of Proposition 2.8 is postponed to Subsection 2.2.5.

By virtue of Proposition 2.8, for all $m \geq 1$, we denote by $C_2^{(m)}$ and $C_3^{(m)}$ two finite constants such that:

- for all $v \in H^1_0(\mathbb{T})$ such that $\|v\|_{H^1_0(\mathbb{T})} \leq m$, $\|\partial_x A(v)\|_{L^2_0(\mathbb{T})} \leq C_2^{(m)}$;
- for all $v_1, v_2 \in H^1_0(\mathbb{T})$ such that $\|v_1\|_{H^1_0(\mathbb{T})} \vee \|v_2\|_{H^1_0(\mathbb{T})} \leq m$, $\|\partial_x A(v_1) - \partial_x A(v_2)\|_{L^2_0(\mathbb{T})} \leq C_3^{(m)} \|v_1 - v_2\|_{H^1_0(\mathbb{T})}$.

Heat kernel

Let us denote by $(S_t)_{t\geq 0}$ the semigroup generated by the operator $\nu\partial_{xx}$:

$$S_t v := \sum_{m \geq 1} \mathrm{e}^{\nu \lambda_m t} \langle v, e_m \rangle_{L^2_0(\mathbb{T})} e_m, \quad v \in L^2_0(\mathbb{T}), \quad t \geq 0. \tag{2.9}$$

Some of its properties are gathered in the following proposition.

Proposition 2.9 (Properties of the heat kernel). *The semigroup $(S_t)_{t\geq 0}$ satisfies the following properties.*

1. *For any $s \geq 0$, for any $v \in H^s_0(\mathbb{T})$, for any $t \geq 0$, $S_t v \in H^s_0(\mathbb{T})$ and $\|S_t v\|_{H^s_0(\mathbb{T})} \leq \|v\|_{H^s_0(\mathbb{T})}$; besides, the mapping $t \mapsto S_t v \in H^s_0(\mathbb{T})$ is continuous on $[0, +\infty)$.*

2. *For all $0 \leq s_1 \leq s_2$, there exists a constant $C_4 = C_4(s_1, s_2) > 0$ such that*
$$\forall v \in H^{s_1}_0(\mathbb{T}), \quad \forall t \geq 0, \qquad \|S_t v\|_{H^{s_2}_0(\mathbb{T})} \leq C_4 t^{\frac{s_1 - s_2}{2}} \|v\|_{H^{s_1}_0(\mathbb{T})}.$$

3. *For any $s \in [0, +\infty)$, $T > 0$ and $(v(t))_{t\in[0,T]} \in C([0,T], H^s_0(\mathbb{T}))$, the process $(\int_0^t S_{t-r} v(r)\mathrm{d}r)_{t\in[0,T]}$ belongs to $C([0,T], H^{s+3/2}_0(\mathbb{T}))$.*

The proof of Proposition 2.9 is postponed to Subsection 2.2.5.

Stochastic convolution and mild formulation of (2.1)

Let $(\overline{\mathcal{F}}_t)_{t\geq 0}$ be a normal filtration on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $(\overline{W}^Q(t))_{t\geq 0}$ be a Q-Wiener process in $H^2_0(\mathbb{T})$ with respect to this filtration. Given that the orthonormal basis $(e_m)_{m\geq 1}$ of the space $L^2_0(\mathbb{T})$ satisfies $\partial_{xx} e_m = \lambda_m e_m$, the family $(e_m/\lambda_m)_{m\geq 1}$ is an orthonormal basis of $H^2_0(\mathbb{T})$. We set
$$\overline{W}_m(t) := \left\langle \overline{W}^Q(t), \frac{e_m}{\lambda_m} \right\rangle_{H^2_0(\mathbb{T})}, \qquad m \geq 1, \quad t \geq 0,$$
so that by (2.6), $(\overline{W}_m(t))_{t\geq 0}$ is a real-valued Brownian motion with variance $\sum_{k\geq 1} \langle g_k, e_m/\lambda_m\rangle^2_{H^2_0(\mathbb{T})}$. Next, we write
$$\overline{w}_m(t) := \int_0^t \mathrm{e}^{\nu\lambda_m(t-s)} \mathrm{d}\overline{W}_m(s), \qquad m \geq 1, \quad t \geq 0.$$

Proposition 2.10. *Under Assumption 2.2, for all $T > 0$, the series*
$$\sum_{m\geq 1} \frac{e_m}{\lambda_m} (\overline{w}_m(t))_{t\in[0,T]}$$
converges in $L^2(\Omega, C([0,T], H^2_0(\mathbb{T})))$, and its sum defines an $(\overline{\mathcal{F}}_t)_{t\geq 0}$-adapted, $H^2_0(\mathbb{T})$-valued process $(\overline{w}(t))_{t\geq 0}$ almost surely continuous.

The proof of Proposition 2.10 is postponed to Subsection 2.2.5. The process $(\overline{w}(t))_{t\geq 0}$ is called the *stochastic convolution* associated to the Q-Wiener process $(\overline{W}^Q(t))_{t\geq 0}$.

In the sequel, we let $\overline{\tau}$ be a $(\overline{\mathcal{F}}_t)_{t\geq 0}$-stopping time, almost surely finite. We shall say that a process $(\overline{u}(t))_{t\in[0,\overline{\tau}]}$ is $(\overline{\mathcal{F}}_t)_{t\geq 0}$-adapted if for all $t \geq 0$, the random variable $\overline{u}(t)\mathbf{1}_{t\leq\overline{\tau}}$ is $\overline{\mathcal{F}}_t$-measurable.

Definition 2.11 (Local mild solution). *Let $\overline{u}_0$ be an $\overline{\mathcal{F}}_0$-measurable, $H^1_0(\mathbb{T})$-valued random variable. Under Assumptions 2.1 and 2.2, a (local) mild solution to the SPDE*
$$\mathrm{d}\overline{u}(t) = -\partial_x A(\overline{u}(t))\mathrm{d}t + \nu\partial_{xx}\overline{u}(t)\mathrm{d}t + \mathrm{d}\overline{W}^Q(t) \tag{2.10}$$
on $[0, \overline{\tau}]$ is an $H^1_0(\mathbb{T})$-valued, $(\overline{\mathcal{F}}_t)_{t\geq 0}$-adapted process $(\overline{u}(t))_{t\in[0,\overline{\tau}]}$ such that, almost surely:

1. *the mapping $t \mapsto \overline{u}(t) \in H^1_0(\mathbb{T})$ is continuous on $[0, \overline{\tau}]$;*

2. *for all $t \in [0, \overline{\tau}]$,*
$$\overline{u}(t) = S_t\overline{u}_0 - \int_0^t S_{t-s}\partial_x A(\overline{u}(s))\mathrm{d}s + \overline{w}(t). \tag{2.11}$$

The combination of Propositions 2.8 and 2.9 ensures that all terms of the identity (2.11) are well-defined.

We now clarify the relationship between the notions of mild and strong solutions.

Proposition 2.12 (Mild and strong solutions). *Under the assumptions of Definition 2.11, let* $(\overline{u}(t))_{t\in[0,\overline{\tau}]}$ *be a mild solution to (2.10) on* $[0,\overline{\tau}]$. *If* $\overline{u}_0 \in H_0^2(\mathbb{T})$, *then:*

1. *for all* $t \in [0,\overline{\tau}]$, $\overline{u}(t) \in H_0^2(\mathbb{T})$ *and the mapping* $t \mapsto \overline{u}(t) \in H_0^2(\mathbb{T})$ *is continuous on* $[0,\overline{\tau}]$*;*
2. *for all* $t \in [0,\overline{\tau}]$,
$$\overline{u}(t) = \overline{u}_0 + \int_0^t \left(-\partial_x A\left(\overline{u}(s)\right) + \nu \partial_{xx}\overline{u}(s)\right) \mathrm{d}s + \overline{W}^Q(t).$$

Conversely, any $H_0^2(\mathbb{T})$*-valued,* $(\overline{\mathcal{F}}_t)_{t\geq 0}$*-adapted process* $(\overline{u}(t))_{t\in[0,\overline{\tau}]}$ *satisfying these two conditions almost surely is a mild solution to (2.10) on* $[0,\overline{\tau}]$.

The proof of Proposition 2.12 is postponed to Subsection 2.2.5.

Existence and uniqueness of a mild solution on a small interval

For any integer $\overline{m}_0 \geq 0$, let us define

$$\tau_{\overline{m}_0}\left(\overline{W}^Q\right) = \frac{1}{8\left(C_1 C_3^{(\overline{m}_0+1)}\right)^2} \wedge \inf\left\{t \geq 0 : 2C_4 C_2^{(\overline{m}_0+1)}\sqrt{t} + \|\overline{w}(t)\|_{H_0^1(\mathbb{T})} \geq 1\right\},$$

where we recall that the constant C_4 is defined in Proposition 2.9, the constants $C_2^{(m)}$ and $C_3^{(m)}$ are defined after Proposition 2.8, and the constant C_1 is defined in (2.2).

Notice that $\tau_{\overline{m}_0}(\overline{W}^Q) \in (0,+\infty)$, almost surely.

In the spirit of [34, 15], we obtain the existence and uniqueness of a mild solution to (2.10) on the "small" interval $[0,\tau_{\overline{m}_0}(\overline{W}^Q)]$ by a fixed-point argument.

Lemma 2.13 (Local existence and uniqueness). *Let* $\overline{u}_0$ *and* $\overline{m}_0$ *be two* $\overline{\mathcal{F}}_0$*-measurable random variables taking values respectively in* $H_0^1(\mathbb{T})$ *and* $\mathbb{N}$ *such that* $\|\overline{u}_0\|_{H_0^1(\mathbb{T})} \leq \overline{m}_0$. *Furthermore, let us set* $\overline{\tau} := \tau_{\overline{m}_0}(\overline{W}^Q)$. *Then, under Assumptions 2.1 and 2.2, there is a unique mild solution* $(\overline{u}(t))_{t\in[0,\overline{\tau}]}$ *to (2.10) on* $[0,\overline{\tau}]$.

Proof. Let us introduce the random set

$$\Sigma := \left\{(v(t))_{t\in[0,\overline{\tau}]} \in C\left([0,\overline{\tau}], H_0^1(\mathbb{T})\right) : \forall t \in [0,\overline{\tau}], \|v(t)\|_{H_0^1(\mathbb{T})} \leq \overline{m}_0 + 1\right\}.$$

Thanks to Propositions 2.9 and 2.10, we may define the random operator $G : C([0,\overline{\tau}], H_0^1(\mathbb{T})) \to C([0,\overline{\tau}], H_0^1(\mathbb{T}))$ by

$$(Gv)(t) = S_t\overline{u}_0 - \int_0^t S_{t-s}\partial_x A(v(s))\mathrm{d}s + \overline{w}(t), \qquad t \in [0,\overline{\tau}],$$

and notice that any $v \in C([0,\overline{\tau}], H_0^1(\mathbb{T}))$ satisfies Equation (2.11) if and only if $Gv = v$.

We first write, for some $v \in C([0,\overline{\tau}], H_0^1(\mathbb{T}))$ and for any $t \in [0,\overline{\tau}]$,

$$\|(Gv)(t)\|_{H_0^1(\mathbb{T})} \leq \|S_t\overline{u}_0\|_{H_0^1(\mathbb{T})} + \int_0^t \|S_{t-s}\partial_x A(v(s))\|_{H_0^1(\mathbb{T})} \mathrm{d}s + \|\overline{w}(t)\|_{H_0^1(\mathbb{T})}. \tag{2.12}$$

On the one hand, by the first assertion of Proposition 2.9, $\|S_t\overline{u}_0\|_{H^1_0(\mathbb{T})} \leq \|\overline{u}_0\|_{H^1_0(\mathbb{T})} \leq \overline{m}_0$; on the other hand, we know thanks to the second assertion of Proposition 2.9 that

$$\|S_{t-s}\partial_x A(v(s))\|_{H^1_0(\mathbb{T})} \leq \frac{C_4}{\sqrt{t-s}}\|\partial_x A(v(s))\|_{L^2_0(\mathbb{T})}, \tag{2.13}$$

furthermore, thanks to Proposition 2.8, if $v \in \Sigma$, then $\partial_x A(v)$ is bounded in $L^2_0(\mathbb{T})$ uniformly in time, *i.e.* for all $s \in [0,\overline{\tau}]$, $\|\partial_x A(v(s))\|_{L^2_0(\mathbb{T})} \leq C_2^{(\overline{m}_0+1)}$. Thus,

$$\|(Gv)(t)\|_{H^1_0(\mathbb{T})} \leq \overline{m}_0 + 2C_4 C_2^{(\overline{m}_0+1)}\sqrt{t} + \|\overline{w}(t)\|_{H^1_0(\mathbb{T})}, \qquad t \in [0,\overline{\tau}]. \tag{2.14}$$

By definition of $\overline{\tau}$, it follows that $Gv \in \Sigma$ whenever $v \in \Sigma$.

We now take $(v_1(t))_{t\in[0,\overline{\tau}]}, (v_2(t))_{t\in[0,\overline{\tau}]} \in \Sigma$. Then, for any $t \in [0,\overline{\tau}]$,

$$\begin{aligned}\|(Gv_1)(t) - (Gv_2)(t)\|_{H^1_0(\mathbb{T})} &= \left\|\int_0^t S_{t-s}\left(\partial_x A(v_1(s)) - \partial_x A(v_2(s))\right)\mathrm{d}s\right\|_{H^1_0(\mathbb{T})} \\ &\leq \int_0^t \frac{C_4}{\sqrt{t-s}}\|\partial_x A(v_1(s)) - \partial_x A(v_2(s))\|_{L^2_0(\mathbb{T})}\mathrm{d}s,\end{aligned} \tag{2.15}$$

where we have used the same arguments as above. Using now the Lipschitz continuity result in Proposition 2.8 and the definition of $\overline{\tau}$, we get for all $t \in [0,\overline{\tau}]$,

$$\begin{aligned}\|(Gv_1)(t) - (Gv_2)(t)\|_{H^1_0(\mathbb{T})} &\leq 2C_1 C_3^{(\overline{m}_0+1)}\sqrt{t}\sup_{s\in[0,t]}\|v_1(s) - v_2(s)\|_{H^1_0(\mathbb{T})} \\ &\leq \frac{1}{2}\sup_{s\in[0,\overline{\tau}]}\|v_1(s) - v_2(s)\|_{H^1_0(\mathbb{T})},\end{aligned}$$

meaning that G is a contraction mapping on Σ, which is complete. Then, by the Banach fixed-point theorem, G admits a unique fixed point $(\overline{u}(t))_{t\in[0,\overline{\tau}]}$ in Σ. To show that this solution to Equation (2.11) is unique among all the $H^1_0(\mathbb{T})$-valued continuous processes, let us first notice that our choice of $\overline{\tau}$ implies

$$\forall t < \overline{\tau}, \qquad \|\overline{u}(t)\|_{H^1_0(\mathbb{T})} < \overline{m}_0 + 1.$$

Assume that there is another solution $(\widetilde{u}(t))_{t\in[0,\overline{\tau}]}$ of (2.11) not belonging almost surely to Σ. Then we have with positive probability

$$\exists \widetilde{\tau} < \overline{\tau}, \qquad \|\widetilde{u}(\widetilde{\tau})\|_{H^1_0(\mathbb{T})} \geq \overline{m}_0 + 1.$$

This means that the double inequality $\|\overline{u}\left(\widetilde{\tau}\right)\|_{H^1_0(\mathbb{T})} < \overline{m}_0 + 1 \leq \|\widetilde{u}\left(\widetilde{\tau}\right)\|_{H^1_0(\mathbb{T})}$ holds on some non-negligible event. On this event, the fixed-point argument also holds in the set

$$\widetilde{\Sigma} := \left\{(v(t))_{t\in[0,\widetilde{\tau}]} : \forall t \in [0,\widetilde{\tau}], \|v(t)\|_{H^1_0(\mathbb{T})} \leq \overline{m}_0 + 1\right\}$$

which is formally a subset of Σ. Thus, by uniqueness of the fixed point, we have $\overline{u}_{|[0,\widetilde{\tau}]} = \widetilde{u}_{|[0,\widetilde{\tau}]}$ and in particular $\overline{u}\left(\widetilde{\tau}\right) = \widetilde{u}\left(\widetilde{\tau}\right)$, which is absurd. As a consequence, $(\overline{u}(t))_{t\in[0,\overline{\tau}]}$ is the only $H^1_0(\mathbb{T})$-valued process with continuous trajectories satisfying Equation (2.11) on $[0,\overline{\tau}]$.

Finally, let $v^{(0)} = 0$ and define the sequence of processes $v^{(j)} \in C([0,\overline{\tau}], H^1_0(\mathbb{T}))$, $j \geq 1$ by $v^{(j)} = Gv^{(j-1)}$. It is clear from the definition of the operator G and from Proposition 2.10 that each process $(v^{(j)}(t)\mathbf{1}_{t\leq\overline{\tau}})_{t\geq 0}$ is $(\overline{\mathcal{F}}_t)_{t\geq 0}$-adapted. On the other hand, the Banach fixed-point theorem asserts that almost surely, the sequence $(v^{(j)}(t))_{t\in[0,\overline{\tau}]}$ converges to $(\overline{u}(t))_{t\in[0,\overline{\tau}]}$ in $C([0,\overline{\tau}], H^1_0(\mathbb{T}))$. As a consequence, for any $t \geq 0$, the sequence of $\overline{\mathcal{F}}_t$-measurable random variables $\mathbf{1}_{t\leq\overline{\tau}}v^{(j)}(t)$ converges almost surely to $\mathbf{1}_{t\leq\overline{\tau}}\overline{u}(t)$, which makes this limit also $\overline{\mathcal{F}}_t$-measurable. Thus, the process $(\mathbf{1}_{t\leq\overline{\tau}}\overline{u}(t))_{t\geq 0}$ is $(\overline{\mathcal{F}}_t)_{t\geq 0}$-adapted. ∎

2.2.2 Construction of a maximal solution to (2.1)

In this subsection, we use the notions introduced in Subsection 2.2.1 to prove the following existence and uniqueness result for (2.1).

Lemma 2.14 (Existence and uniqueness result of a maximal solution to (2.1))**.** *Under Assumptions 2.1 and 2.2, for any $u_0 \in H_0^1(\mathbb{T})$, there exists a pair $(T^*, (u(t))_{t\in[0,T^*)})$ such that:*

1. *for any $(\mathcal{F}_t)_{t\geq 0}$-stopping time T such that almost surely, $T < +\infty$ and $T \leq T^*$, $(u(t))_{t\in[0,T]}$ is the unique mild solution to (2.1) on $[0,T]$;*
2. *almost surely, $T^* = +\infty$ or $\limsup_{t\to T^*} \|u(t)\|_{H_0^1(\mathbb{T})} = +\infty$.*

The random time T^* is called the *explosion time* and the process $(u(t))_{t\in[0,T^*)}$ is called the *maximal solution* to (2.1).

Proof. Let $u_0 \in H_0^1(\mathbb{T})$. Let $m_0^{(0)} = \lceil \|u_0\|_{H_0^1(\mathbb{T})} \rceil$. By Lemma 2.13, Equation (2.1) possesses a unique mild solution $(u(t))_{t\in[0,\tau^{(0)}]}$ on $[0,\tau^{(0)}]$, where $\tau^{(0)} = \tau_{m_0^{(0)}}$. We now define the filtration $(\mathcal{F}_t^{(1)})_{t\geq 0}$ by

$$\mathcal{F}_t^{(1)} = \mathcal{F}_{\tau^{(0)}+t} = \left\{ B \in \mathcal{F} : \forall s \geq 0, B \cap \{\tau^{(0)} + t \leq s\} \in \mathcal{F}_s \right\},$$

and recall that the process $W^{Q,(1)}$ defined by $W^{Q,(1)}(t) = W^Q(\tau^{(0)} + t) - W^Q(t)$ is a Q-Wiener process with respect to $(\mathcal{F}_t^{(1)})_{t\geq 0}$. Therefore, applying Lemma 2.13 again with this Q-Wiener process, and initial condition $u_0^{(1)} = u(\tau^{(0)})$ and $m_0^{(1)} = \lceil \|u(\tau^{(0)})\|_{H_0^1(\mathbb{T})} \rceil \vee m_0^{(0)}$, we obtain a mild solution $(u^{(1)}(t))_{t\in[0,\tau^{(1)}]}$ of $\mathrm{d}u = -\partial_x A(u)\mathrm{d}t + \nu\partial_{xx}u\mathrm{d}t + \mathrm{d}W^{Q,(1)}$ on $[0,\tau^{(1)}]$, where $\tau^{(1)} = \tau_{m_0^{(1)}}(W^{Q,(1)})$. It is then easily checked that defining $T^{(1)} = \tau^{(0)} + \tau^{(1)}$ and $u(t+\tau^{(0)}) = u^{(1)}(t)$ for any $t \in (0,\tau^{(1)}]$, we obtain a unique mild solution $(u(t))_{t\in[0,T^{(1)}]}$ to Equation (2.1) on $[0,T^{(1)}]$.

We now proceed by induction and set for all $n \geq 1$,

$$\begin{aligned}
T^{(n)} &:= \sum_{i=0}^{n} \tau^{(i)}, \\
m_0^{(n+1)} &:= \left\lceil \left\| u\left(T^{(n)}\right) \right\|_{H_0^1(\mathbb{T})} \right\rceil \vee m_0^{(n)}, \\
\tau^{(n+1)} &:= \tau_{m_0^{(n+1)}} \left(W^Q\left(T^{(n)} + \cdot\right) - W^Q\left(T^{(n)}\right) \right), \\
T^* &:= \sup_{n\geq 1} T^{(n)},
\end{aligned}$$

where at each iteration we use Lemma 2.13 to extend the process $(u(t))_{t\in[0,T^{(n)}]}$ to the unique mild solution of Equation (2.1) on $[0,T^{(n)}]$. It is then clear that $(u(t))_{t\in[0,T^*)}$ satisfies the first assertion of Lemma 2.14.

Since the sequence of integers $(m_0^{(n)})_{n\geq 0}$ is nondecreasing, $\sup_{n\geq 0} m_0^{(n)} < +\infty$ if and only if there exists $n_0 \geq 0$ and $m \geq 0$ such that, for all $n \geq n_0$, $m_0^{(n)} = m$. Hence, we can write

$$\begin{aligned}
\left\{ T^* < +\infty, \sup_{n\geq 0} m_0^{(n)} < +\infty \right\} &= \bigcup_{n_0\geq 0, m\geq 0} \left\{ \sum_{n=0}^{\infty} \tau^{(n)} < +\infty, \forall n \geq n_0, m_0^{(n)} = m \right\} \\
&= \bigcup_{n_0\geq 0, m\geq 0} \left\{ \sum_{n=n_0+1}^{\infty} \tau^{(n)} < +\infty, \forall n \geq n_0, m_0^{(n)} = m \right\} \\
&\subset \bigcup_{n_0\geq 0, m\geq 0} \left\{ \sum_{n=n_0+1}^{\infty} \tau_m \left(W^Q\left(T^{(n)} + \cdot\right) - W^Q\left(T^{(n)}\right) \right) < +\infty \right\}.
\end{aligned}$$

However, by the strong Markov property, for any $m \geq 0$, the random variables $\tau_m(W^Q(T^{(n)}+\cdot) - W^Q(T^{(n)}))$, $n \geq 1$, are independent and identically distributed, and by the definition of $\tau_m(\cdot)$, they are almost surely positive. As a consequence, by Borel's 0-1 law,

$$\forall n_0, m \geq 0, \quad \mathbb{P}\left(\sum_{n=n_0+1}^{\infty} \tau_m\left(W^Q\left(T^{(n)}+\cdot\right) - W^Q\left(T^{(n)}\right)\right) < +\infty\right) = 0.$$

As the countable union of negligible events is still negligible, we get

$$\mathbb{P}\left(T^* < +\infty, \sup_{n\geq 0} m_0^{(n)} < +\infty\right) = 0.$$

This implies that almost surely, if $T^* < +\infty$ then $\sup_{n\geq 0} m_0^{(n)} = +\infty$, so that $\limsup_{n\to\infty} \|u(T^{(n)})\|_{H_0^1(\mathbb{T})} = +\infty$, which is the wanted result. $\square$

2.2.3 Estimates on the maximal solution

Let $u_0 \in H_0^2(\mathbb{T})$. Let $(T^*, (u(t))_{t\in[0,T^*)})$ be the maximal solution to Equation (2.1) given by Lemma 2.14. By Proposition 2.12, $(u(t))_{t\in[0,T^*)}$ is a continuous $H_0^2(\mathbb{T})$-valued process. Besides, Lemma 2.14 allows us to define, for any $r \geq 0$, the stopping time

$$T_r := \inf\left\{t \in [0, T^*) : \|u(t)\|^2_{H_0^1(\mathbb{T})} \geq r\right\}, \tag{2.16}$$

which always satisfies $T_r \leq T^*$. In the sequel, we shall prove that $\lim_{r\to\infty} T_r = +\infty$, which shall imply that $T^* = +\infty$, almost surely.

Lemma 2.15. *Under Assumptions 2.1 and 2.2, for any $p \in 2\mathbb{N}^*$ and for all $t \geq 0$, we have:*

$$\frac{4\nu}{p}(p-1)\mathbb{E}\left[\int_0^{t\wedge T_r}\int_{\mathbb{T}}\left(\partial_x u(s)^{p/2}\right)^2 \mathrm{d}x\mathrm{d}s\right] \leq \|u_0\|^p_{L_0^p(\mathbb{T})} + \frac{p(p-1)}{2}D_0\mathbb{E}\left[\int_0^{t\wedge T_r}\|u(s)\|^{p-2}_{L_0^{p-2}(\mathbb{T})}\mathrm{d}s\right]. \tag{2.17}$$

Moreover, there exist two constants $C_5^{(p)}, C_6^{(p)} > 0$ depending only on ν, p and D_0 such that

$$\mathbb{E}\left[\int_0^{t\wedge T_r}\|u(s)\|^p_{L_0^p(\mathbb{T})}\mathrm{d}s\right] \leq C_5^{(p)}\left(1 + \|u_0\|^p_{L_0^p(\mathbb{T})}\right) + C_6^{(p)}t. \tag{2.18}$$

Proof. Let $p \in 2\mathbb{N}^*$. We want to apply Itô's formula on $[0, t\wedge T_r]$ to the $H_0^2(\mathbb{T})$-valued process $(u(t))_{t\in[0,T^*)}$ with the function $F_p : u \mapsto \|u\|^p_{L_0^p(\mathbb{T})}$. Since this process writes

$$u(t) = u_0 + \int_0^t \varphi(s)\mathrm{d}s + W^Q(t)$$

with $\varphi(t) = -\partial_x A(u(t)) + \nu\partial_{xx}u(t) \in L_0^2(\mathbb{T})$, the standard formulation of Itô's formula in Hilbert spaces [35, Theorem 4.32] requires at least F_p to be continuous on $L_0^2(\mathbb{T})$, which is not the case for $p > 2$ here. Hence, we shall proceed to approximate F_p with a sequence of smooth functions $F_{M,p}$, $M \geq 1$, apply Itô's formula to the functions $F_{M,p}$ and then take the limit $M \to +\infty$.

Step 1. Approximation of the $L_0^p(\mathbb{T})$-norm. Let ρ be a C^∞ function from $\mathbb{R}$ to $\mathbb{R}_+$ such that $\int_{\mathbb{R}} \rho(u)\mathrm{d}u = 1$ and whose support is contained in the interval $(-\frac{1}{2}, \frac{1}{2})$. For any $M \geq 1$, we set the regularised Heaviside function $\psi_M := \mathbf{1}_{(-\infty, M+\frac{1}{2}]} * \rho$ and its antiderivative

$$\phi_M : u \in \mathbb{R}_+ \longmapsto \int_0^u \psi_M(v)\mathrm{d}v \in \mathbb{R}_+.$$

We now define a truncated $L^p_0(\mathbb{T})$-norm by setting

$$F_{M,p}:\begin{cases} L^2_0(\mathbb{T}) & \longrightarrow \mathbb{R}_+ \\ v & \longmapsto \int_{\mathbb{T}} \phi_M\left(v(x)^p\right) \mathrm{d}x. \end{cases}$$

The first differential $\mathrm{D}F_{M,p}$ and the second differential $\mathrm{D}^2F_{M,p}$ have the following expressions: $\forall v,h\in L^2_0(\mathbb{T})$,

$$\langle \mathrm{D}F_{M,p}(v),h\rangle_{L^2_0(\mathbb{T})} = p\int_{\mathbb{T}} h(x)v(x)^{p-1}\phi'_M\left(v(x)^p\right)\mathrm{d}x,$$

$$\langle \mathrm{D}^2F_{M,p}(v)\cdot h,h\rangle_{L^2_0(\mathbb{T})} = p(p-1)\int_{\mathbb{T}} h(x)^2v(x)^{p-2}\phi'_M\left(v(x)^p\right)\mathrm{d}x + p^2\int_{\mathbb{T}} h(x)^2v(x)^{2(p-1)}\phi''_M\left(v(x)^p\right)\mathrm{d}x.$$

Step 2. Itô's formula. First, let us notice that the process $(W^Q(t))_{t\geq 0}$ can be seen as an $L^2_0(\mathbb{T})$-valued Q'-Wiener process where the operator $Q' : L^2_0(\mathbb{T}) \to L^2_0(\mathbb{T})$ has covariance

$$\langle u, Q'v\rangle_{L^2_0(\mathbb{T})} = \sum_{k\geq 1}\langle g_k,u\rangle_{L^2_0(\mathbb{T})}\langle g_k,v\rangle_{L^2_0(\mathbb{T})}.$$

Indeed, Assumption 2.2 ensures that $Q'(L^2_0(\mathbb{T}))\subset H^2_0(\mathbb{T})$ and $Q'_{|H^2_0(\mathbb{T})} = Q$. We now have

$$\mathrm{Tr}\left(\mathrm{D}^2F_{M,p}(v)Q'\right) = \sum_{k\geq 1}\langle \mathrm{D}^2F_{M,p}(v)g_k,g_k\rangle_{L^2_0(\mathbb{T})},$$

so that we can apply Itô's formula [35, Theorem 4.32] for the real-valued process $(F_{M,p}(u(t)))_{t\in[0,T^*)}$, which leads to

$$\begin{aligned} F_{M,p}\left(u(t)\right) =& F_{M,p}(u_0) + p\int_0^t\int_{\mathbb{T}}\left(-\partial_x A(u(s)) + \nu\partial_{xx}u(s)\right)u(s)^{p-1}\phi'_M\left(u(s)^p\right)\mathrm{d}x\mathrm{d}s \\ &+ \int_0^t \langle \mathrm{D}F_{M,p}(u(s)), \mathrm{d}W^Q(s)\rangle_{L^2_0(\mathbb{T})} \\ &+ \frac{1}{2}p(p-1)\sum_{k\geq 1}\int_0^t\int_{\mathbb{T}} g_k^2 u(s)^{p-2}\phi'_M(u(s)^p)\mathrm{d}x\mathrm{d}s \\ &+ \frac{1}{2}p^2\sum_{k\geq 1}\int_0^t\int_{\mathbb{T}} g_k^2 u(s)^{2(p-1)}\phi''_M(u(s)^p)\mathrm{d}x\mathrm{d}s. \end{aligned}$$

Since the $L^2_0(\mathbb{T})$-norm of $\mathrm{D}F_{M,p}(u(s))$ is bounded uniformly in time, the third term of the right-hand side is a square integrable martingale [35, Theorem 4.27]. Thus, for $t\geq 0$, integrating in time up to $t\wedge T_r$ and taking the expectation, we get

$$\begin{aligned} \mathbb{E}\left[F_{M,p}\left(u(t\wedge T_r)\right)\right] =& F_{M,p}(u_0) - p\mathbb{E}\left[\int_0^{t\wedge T_r}\int_{\mathbb{T}}\partial_x A(u(s))u(s)^{p-1}\phi'_M\left(u(s)^p\right)\mathrm{d}x\mathrm{d}s\right] && (2.19)\\ &+ p\mathbb{E}\left[\int_0^{t\wedge T_r}\int_{\mathbb{T}}\nu\partial_{xx}u(s)u(s)^{p-1}\phi'_M\left(u(s)^p\right)\mathrm{d}x\mathrm{d}s\right] && (2.20)\\ &+ \frac{1}{2}p(p-1)\mathbb{E}\left[\sum_{k\geq 1}\int_0^{t\wedge T_r}\int_{\mathbb{T}} g_k^2u(s)^{p-2}\phi'_M(u(s)^p)\mathrm{d}x\mathrm{d}s\right] && (2.21)\\ &+ \frac{1}{2}p^2\mathbb{E}\left[\sum_{k\geq 1}\int_0^{t\wedge T_r}\int_{\mathbb{T}} g_k^2u(s)^{2(p-1)}\phi''_M(u(s)^p)\mathrm{d}x\mathrm{d}s\right]. && (2.22) \end{aligned}$$

Step 3. Passing $M \to +\infty$. We want now to pass to the limit $M \to +\infty$. Regarding the left-hand side in the above equation, the family of functions ϕ_M is non-decreasing with respect to M, so that the monotone convergence theorem yields

$$\lim_{M\to\infty} \mathbb{E}\left[F_{M,p}\left(u(t\wedge T_r)\right)\right] = \mathbb{E}\left[\int_{\mathbb{T}} \lim_{M\to\infty} \phi_M\left(u(t\wedge T_r)^p\right) \mathrm{d}x\right] = \mathbb{E}\left[\|u(t\wedge T_r)\|^p_{L^p_0(\mathbb{T})}\right].$$

For the flux term, we have almost surely, for all $s \in [0, t\wedge T_r]$ and for all $M \geq 0$, $\partial_x A(u(s))u(s)^{p-1}\phi'_M(u(s)^p) \leq |\partial_x A(u(s))||u(s)|^{p-1}$. Furthermore,

$$\begin{aligned}
\mathbb{E}\left[\int_0^{t\wedge T_r}\int_{\mathbb{T}} |\partial_x A(u(s))|\,|u(s)|^{p-1}\mathrm{d}x\mathrm{d}s\right] &\leq \mathbb{E}\left[\sup_{s\in[0,t\wedge T_r]} \|u(s)\|^{p-1}_{L^\infty_0(\mathbb{T})} \int_0^{t\wedge T_r}\int_{\mathbb{T}} |\partial_x A(u(s))|\,\mathrm{d}x\mathrm{d}s\right] \\
&\leq r^{\frac{p-1}{2}} \mathbb{E}\left[\int_0^{t\wedge T_r} \|\partial_x A(u(s))\|_{L^2_0(\mathbb{T})}\,\mathrm{d}s\right] \quad \text{(from (2.3) and (2.16))} \\
&\leq r^{\frac{p-1}{2}} L_r \mathbb{E}\left[\int_0^{t\wedge T_r} \|u(s)\|_{L^2_0(\mathbb{T})}\,\mathrm{d}s\right] \quad \text{(from Proposition 2.8)} \\
&\leq L_r r^{\frac{p}{2}} t < +\infty.
\end{aligned}$$

Thus, the dominated convergence theorem applies and yields

$$\lim_{M\to\infty} p\mathbb{E}\left[\int_0^{t\wedge T_r}\int_{\mathbb{T}} \partial_x A(u(s))u(s)^{p-1}\phi'_M\left(u(s)^p\right)\mathrm{d}x\mathrm{d}s\right] = p\mathbb{E}\left[\int_0^{t\wedge T_r}\int_{\mathbb{T}} \partial_x A(u(s))u(s)^{p-1}\mathrm{d}x\mathrm{d}s\right].$$

We now integrate by parts the viscous term:

$$\begin{aligned}
&p\nu\mathbb{E}\left[\int_0^{t\wedge T_r}\int_{\mathbb{T}} \partial_{xx}u(s)u(s)^{p-1}\phi'_M\left(u(s)^p\right)\mathrm{d}x\mathrm{d}s\right] \\
&= -p\nu\mathbb{E}\left[\int_0^{t\wedge T_r}\int_{\mathbb{T}} \partial_x u(s)\left(\partial_x\left(u(s)^{p-1}\right)\phi'_M\left(u(s)^p\right) + u(s)^{p-1}\partial_x\left(\phi'_M\left(u(s)^p\right)\right)\right)\mathrm{d}x\mathrm{d}s\right] \\
&= -p\nu\mathbb{E}\left[\int_0^{t\wedge T_r}\int_{\mathbb{T}} (\partial_x u(s))^2\left((p-1)u(s)^{p-2}\phi'_M\left(u(s)^p\right) + pu(s)^{2(p-1)}\phi''_M\left(u(s)^p\right)\right)\mathrm{d}x\mathrm{d}s\right],
\end{aligned}$$

and this last integrand is dominated uniformly in M by $(\partial_x u(s))^2\left((p-1)u(s)^{p-2} + \kappa p u(s)^{2(p-1)}\right)$, where $\kappa = \sup_{\mathbb{R}} |\rho|$. Furthermore, thanks to (2.16), we have

$$\begin{aligned}
&\mathbb{E}\left[\int_0^{t\wedge T_r}\int_{\mathbb{T}} (\partial_x u(s))^2\left((p-1)u(s)^{p-2} + \kappa p u(s)^{2(p-1)}\right)\mathrm{d}x\mathrm{d}s\right] \\
&\leq \mathbb{E}\left[\left((p-1)\sup_{s\in[0,t\wedge T_r]}\|u(s)\|^{p-2}_{L^\infty_0(\mathbb{T})} + \kappa p \sup_{s\in[0,t\wedge T_r]}\|u(s)\|^{2(p-1)}_{L^\infty_0(\mathbb{T})}\right)\int_0^{t\wedge T_r}\|u(s)\|^2_{H^1_0(\mathbb{T})}\mathrm{d}s\right] \\
&\leq \left((p-1)r^{\frac{p-2}{2}} + \kappa p r^{p-1}\right) rt < +\infty.
\end{aligned}$$

Thus, we get from the dominated convergence theorem,

$$\lim_{M\to\infty} p\mathbb{E}\left[\int_0^{t\wedge T_r}\int_{\mathbb{T}} \nu\partial_{xx}u(s)u(s)^{p-1}\phi'_M\left(u(s)^p\right)\mathrm{d}x\mathrm{d}s\right] = -\nu p(p-1)\mathbb{E}\left[\int_0^{t\wedge T_r}\int_{\mathbb{T}} (\partial_x u(s))^2 u(s)^{p-2}\mathrm{d}x\mathrm{d}s\right].$$

With similar computations, for the noise term, we have

$$\lim_{M\to\infty} p(p-1)\mathbb{E}\left[\sum_{k\geq 1}\int_0^{t\wedge T_r}\int_{\mathbb{T}} g_k^2 u(s)^{p-2}\phi'_M(u(s)^p)\mathrm{d}x\mathrm{d}s\right] = p(p-1)\mathbb{E}\left[\sum_{k\geq 1}\int_0^{t\wedge T_r}\int_{\mathbb{T}} g_k^2 u(s)^{p-2}\mathrm{d}x\mathrm{d}s\right],$$

and

$$\lim_{M\to\infty} p^2\mathbb{E}\left[\sum_{k\geq 1}\int_0^{t\wedge T_r}\int_{\mathbb{T}} g_k^2 u(s)^{2(p-1)}\phi_M''(u(s)^p)\mathrm{d}x\mathrm{d}s\right] = 0.$$

Letting M go to $+\infty$ in (2.19), (2.20), (2.21) and (2.22), we get

$$\begin{aligned}\mathbb{E}\left[\|u(t\wedge T_r)\|^p_{L^p_0(\mathbb{T})}\right] &= \|u_0\|^p_{L^p_0(\mathbb{T})} - p\mathbb{E}\left[\int_0^{t\wedge T_r}\int_{\mathbb{T}}\partial_x A(u(s))u(s)^{p-1}\mathrm{d}x\mathrm{d}s\right]\\ - \nu p(p-1)\mathbb{E}\left[\int_0^{t\wedge T_r}\int_{\mathbb{T}}(\partial_x u(s))^2 u(s)^{p-2}\mathrm{d}x\mathrm{d}s\right] &+ \frac{1}{2}p(p-1)\sum_{k\geq 1}\mathbb{E}\left[\int_0^{t\wedge T_r}\int_{\mathbb{T}} u(s)^{p-2}g_k^2\mathrm{d}x\mathrm{d}s\right].\end{aligned} \tag{2.23}$$

It turns out that the flux term disappears:

$$\int_{\mathbb{T}} u(s)^{p-1}\partial_x A(u(s))\mathrm{d}x = \int_{\mathbb{T}} u(s)^{p-1}A'(u(s))\partial_x u(s)\mathrm{d}x = \int_{\mathbb{T}}\partial_x\left(\mathcal{A}_p(u(s))\right)\mathrm{d}x = 0, \tag{2.24}$$

where $\mathcal{A}_p$ is an antiderivative of $v\mapsto v^{p-1}A'(v)$. As regards the noise coefficients, we have

$$\sum_{k\geq 1} g_k(x)^2 \leq \sum_{k\geq 1}\|g_k\|^2_{L^\infty_0(\mathbb{T})} \leq \sum_{k\geq 1}\|g_k\|^2_{H^1_0(\mathbb{T})} \leq D_0,$$

thanks to (2.3) and (2.4). As a consequence, we get from (2.23) the inequality

$$\nu p(p-1)\mathbb{E}\left[\int_0^{t\wedge T_r}\int_{\mathbb{T}}\left(u(s)^{\frac{p}{2}-1}\partial_x u(s)\right)^2\mathrm{d}x\mathrm{d}s\right] \leq \|u_0\|^p_{L^p_0(\mathbb{T})} + \frac{1}{2}p(p-1)D_0\mathbb{E}\left[\int_0^{t\wedge T_r}\|u(s)\|^{p-2}_{L^{p-2}_0(\mathbb{T})}\mathrm{d}s\right]. \tag{2.25}$$

Rewriting the integrand in the left-hand side, we get

$$\frac{4\nu}{p}(p-1)\mathbb{E}\left[\int_0^{t\wedge T_r}\int_{\mathbb{T}}\left(\partial_x\left(u(s)^{p/2}\right)\right)^2\mathrm{d}x\mathrm{d}s\right] \leq \|u_0\|^p_{L^p_0(\mathbb{T})} + \frac{p(p-1)}{2}D_0\mathbb{E}\left[\int_0^{t\wedge T_r}\|u(s)\|^{p-2}_{L^{p-2}_0(\mathbb{T})}\mathrm{d}s\right]. \tag{2.26}$$

Since $u(s)$ has a zero space average and is continuous in space (because it belongs to $H^1_0(\mathbb{T})$), almost surely the function $u(s)^{p/2}$ vanishes somewhere on the torus. Thus, we can apply the Poincaré inequality on the left-hand side which leads, after multiplying by $p/(4\nu(p-1))$ on both sides, to the inequality

$$\mathbb{E}\left[\int_0^{t\wedge T_r}\|u(s)\|^p_{L^p_0(\mathbb{T})}\mathrm{d}s\right] \leq \frac{p}{4\nu(p-1)}\|u_0\|^p_{L^p_0(\mathbb{T})} + \frac{p^2 D_0}{8\nu}\mathbb{E}\left[\int_0^{t\wedge T_r}\|u(s)\|^{p-2}_{L^{p-2}_0(\mathbb{T})}\mathrm{d}s\right]. \tag{2.27}$$

For $p=2$, we get

$$\mathbb{E}\left[\int_0^{t\wedge T_r}\|u(s)\|^2_{L^2_0(\mathbb{T})}\mathrm{d}s\right] \leq \frac{1}{2\nu}\|u_0\|^2_{L^2_0(\mathbb{T})} + \frac{D_0 t}{2\nu},$$

and the claimed result for arbitrary $p\in 2\mathbb{N}^*$ follows by induction and from the inequalities $\|u_0\|^{p-2r}_{L^{p-2r}_0(\mathbb{T})} \leq 1 + \|u_0\|^p_{L^p_0(\mathbb{T})}$ and $\mathbb{E}[t\wedge T_r]\leq t$. □

Remark 2.16. *By Jensen's inequality, the bound (2.18) also holds for any real number $p\geq 2$.*

Lemma 2.17. *Under Assumptions 2.1 and 2.2, there exist two constants $C_7, C_8 > 0$ depending only on ν, p_A, C_1 and D_0, such that for all $t\geq 0$ and all $r\geq 0$,*

$$\mathbb{E}\left[\|u(t\wedge T_r)\|^2_{H^1_0(\mathbb{T})}\right] + \nu\mathbb{E}\left[\int_0^{t\wedge T_r}\|u(s)\|^2_{H^2_0(\mathbb{T})}\mathrm{d}s\right] \leq \|u_0\|^2_{H^1_0(\mathbb{T})} + C_7\left(1+\|u_0\|^{2p_A+2}_{L^{2p_A+2}_0(\mathbb{T})}\right) + C_8 t.$$

Proof. We want to apply Itô's formula to the squared $H_0^1(\mathbb{T})$-norm of the process $(u(t))_{t\in[0,T^*)}$. As for the proof of Lemma 2.15, we proceed by truncation of this function.

Step 1. Approximation of the $H_0^1(\mathbb{T})$-norm. We set

$$G_M : \begin{cases} L_0^2(\mathbb{T}) & \longrightarrow \mathbb{R}_+ \\ v & \longmapsto \sum_{m=1}^M (-\lambda_m)\langle v, e_m\rangle^2_{L_0^2(\mathbb{T})} \end{cases}$$

The first differential $\mathrm{D}G_M$ and the second differential $\mathrm{D}^2 G_M$ have the following expressions: $\forall h \in L_0^2(\mathbb{T})$,

$$\langle \mathrm{D}G_M(v), h\rangle_{L_0^2(\mathbb{T})} = -2\sum_{m=1}^M \lambda_m \langle v, e_m\rangle_{L_0^2(\mathbb{T})}\langle h, e_m\rangle_{L_0^2(\mathbb{T})},$$

$$\langle \mathrm{D}^2 G_M(v)\cdot h, h\rangle_{L_0^2(\mathbb{T})} = -2\sum_{m=1}^M \lambda_m \langle h, e_m\rangle^2_{L_0^2(\mathbb{T})}.$$

Step 2. Itô's formula. Itô's formula applied to G_M yields almost surely and for all $r \geq 0$,

$$\begin{aligned} G_M(u(t\wedge T_r)) = G_M(u_0) - 2\int_0^{t\wedge T_r}\sum_{m=1}^M \lambda_m \langle u(s), e_m\rangle_{L_0^2(\mathbb{T})}\langle -\partial_x A(u(s)) + \nu\partial_{xx}u(s), e_m\rangle_{L_0^2(\mathbb{T})}\mathrm{d}s \\ - 2\int_0^{t\wedge T_r}\langle \mathrm{D}G_M(u(s)), \mathrm{d}W^Q(s)\rangle_{L_0^2(\mathbb{T})} - 2\sum_{k\geq 1}\int_0^{t\wedge T_r}\sum_{m=1}^M \lambda_m\langle g_k, e_m\rangle^2_{L_0^2(\mathbb{T})}\mathrm{d}s. \end{aligned} \tag{2.28}$$

We first check that the third term of the right-hand side is a square-integrable martingale:

$$\begin{aligned} \mathbb{E}\left[\int_0^{t\wedge T_r}\|DG_M(u(s))\|^2_{L_0^2(\mathbb{T})}\,\mathrm{d}s\right] &= 4\sum_{m=1}^M \lambda_m^2 \mathbb{E}\left[\int_0^{t\wedge T_r}\langle u(s), e_m\rangle^2_{L_0^2(\mathbb{T})}\mathrm{d}s\right] \\ &\leq 4\left(\sum_{m=1}^M \lambda_m^2\right)\mathbb{E}\left[\int_0^{t\wedge T_r}\|u(s)\|^2_{L_0^2(\mathbb{T})}\mathrm{d}s\right] \leq 4\left(\sum_{m=1}^M \lambda_m^2\right) tr < +\infty. \end{aligned}$$

Thus, taking the expectation, the stochastic integral disappears and we get

$$\begin{aligned} \mathbb{E}\left[G_M(u(t\wedge T_r))\right] = G_M(u_0) + 2\mathbb{E}\left[\int_0^{t\wedge T_r}\sum_{m=1}^M \lambda_m\langle u(s), e_m\rangle_{L_0^2(\mathbb{T})}\langle \partial_x A(u(s)), e_m\rangle_{L_0^2(\mathbb{T})}\mathrm{d}s\right] \\ - 2\mathbb{E}\left[\int_0^{t\wedge T_r}\sum_{m=1}^M \lambda_m\langle u(s), e_m\rangle_{L_0^2(\mathbb{T})}\langle \nu\partial_{xx}u(s), e_m\rangle_{L_0^2(\mathbb{T})}\mathrm{d}s\right] - \mathbb{E}\left[\sum_{k\geq 1}\int_0^{t\wedge T_r}\sum_{m=1}^M \lambda_m\langle g_k, e_m\rangle^2_{L_0^2(\mathbb{T})}\mathrm{d}s\right]. \end{aligned} \tag{2.29}$$

On one hand, we can rewrite the viscous term as follows:

$$\begin{aligned} \sum_{m=1}^M \lambda_m\langle u(s), e_m\rangle_{L_0^2(\mathbb{T})}\langle \nu\partial_{xx}u(s), e_m\rangle_{L_0^2(\mathbb{T})} &= \sum_{m=1}^M \lambda_m\langle u(s), e_m\rangle_{L_0^2(\mathbb{T})}\langle \nu u(s), \partial_{xx}e_m\rangle_{L_0^2(\mathbb{T})} \\ &= \sum_{m=1}^M \lambda_m\langle u(s), e_m\rangle_{L_0^2(\mathbb{T})}\langle \nu u(s), \lambda_m e_m\rangle_{L_0^2(\mathbb{T})} \\ &= \nu\sum_{m=1}^M \lambda_m^2\langle u(s), e_m\rangle^2_{L_0^2(\mathbb{T})}. \end{aligned} \tag{2.30}$$

On the other hand, applying Young's inequality on the flux term, we get

$$
\begin{aligned}
2\mathbb{E}&\left[\int_0^{t\wedge T_r}\sum_{m=1}^{M}\lambda_m\langle u(s),e_m\rangle_{L_0^2(\mathbb{T})}\langle\partial_x A(u(s)),e_m\rangle_{L_0^2(\mathbb{T})}\mathrm{d}s\right]\\
&\le 2\nu\mathbb{E}\left[\int_0^{t\wedge T_r}\sum_{m=1}^{M}\lambda_m^2\langle u(s),e_m\rangle_{L_0^2(\mathbb{T})}^2\mathrm{d}s\right]+\frac{1}{2\nu}\mathbb{E}\left[\int_0^{t\wedge T_r}\sum_{m=1}^{M}\langle\partial_x A(u(s)),e_m\rangle_{L_0^2(\mathbb{T})}^2\mathrm{d}s\right]. \quad (2.31)
\end{aligned}
$$

Injecting (2.30) and (2.31) into (2.29), we get the inequality

$$
\mathbb{E}\left[G_M(u(t\wedge T_r))\right]\le G_M(u_0)+\frac{1}{2\nu}\mathbb{E}\left[\int_0^{t\wedge T_r}\sum_{m=1}^{M}\langle\partial_x A(u(s)),e_m\rangle_{L_0^2(\mathbb{T})}^2\mathrm{d}s\right]-\mathbb{E}[t\wedge T_r]\sum_{k\ge1}\sum_{m=1}^{M}\lambda_m\langle g_k,e_m\rangle_{L_0^2(\mathbb{T})}^2. \quad (2.32)
$$

Step 3. Passing $M\to+\infty$. From Proposition 2.8, for any $r\ge0$, there is a constant L_r such that for all $M\ge1$, we have

$$
\sum_{m=1}^{M}\langle\partial_x A(u(s)),e_m\rangle_{L_0^2(\mathbb{T})}^2\le\|\partial_x A(u(s))\|_{L_0^2(\mathbb{T})}^2\le L_r\|u(s)\|_{H_0^1(\mathbb{T})}^2\le rL_r.
$$

Thus, we can use the dominated convergence theorem to let M go to infinity in (2.32) and we get

$$
\mathbb{E}\left[\|u(t\wedge T_r)\|_{H_0^1(\mathbb{T})}^2\right]\le\|u_0\|_{H_0^1(\mathbb{T})}^2+\frac{1}{2\nu}\mathbb{E}\left[\int_0^{t\wedge T_r}\|\partial_x A(u(s))\|_{L_0^2(\mathbb{T})}^2\mathrm{d}s\right]+\mathbb{E}[t\wedge T_r]\sum_{k\ge1}\|g_k\|_{H_0^1(\mathbb{T})}^2. \quad (2.33)
$$

Since from Assumption 2.1, A' has polynomial growth, we can bound the second term of the right-hand side: using (2.2) and (2.17) with $p=2$ and $p=2p_A+2$, we get

$$
\begin{aligned}
\mathbb{E}\left[\int_0^{t\wedge T_r}\|\partial_x A(u(s))\|_{L_0^2(\mathbb{T})}^2\mathrm{d}s\right]&=\mathbb{E}\left[\int_0^{t\wedge T_r}\int_{\mathbb{T}}(\partial_x u(s))^2A'(u(s))^2\mathrm{d}x\mathrm{d}s\right]\\
&\le 2C_1^2\mathbb{E}\left[\int_0^{t\wedge T_r}\int_{\mathbb{T}}(\partial_x u(s))^2\left(1+|u(s)|^{2p_A}\right)\mathrm{d}x\mathrm{d}s\right]\\
&=2C_1^2\left(\mathbb{E}\left[\int_0^{t\wedge T_r}\|u(s)\|_{H_0^1(\mathbb{T})}^2\mathrm{d}s\right]+\mathbb{E}\left[\int_0^{t\wedge T_r}\int_{\mathbb{T}}(\partial_x u(s))^2u(s)^{2p_A}\mathrm{d}x\mathrm{d}s\right]\right)\\
&\le\frac{C_1^2}{\nu}\Bigg(\|u_0\|_{L_0^2(\mathbb{T})}^2+D_0\mathbb{E}[t\wedge T_r]\\
&\quad+\frac{2}{(2p_A+2)(2p_A+1)}\|u_0\|_{L_0^{2p_A+2}(\mathbb{T})}^{2p_A+2}+D_0\mathbb{E}\left[\int_0^{t\wedge T_r}\|u(s)\|_{L_0^{2p_A}(\mathbb{T})}^{2p_A}\mathrm{d}s\right]\Bigg).
\end{aligned}
$$

Applying now Lemma 2.15, we get

$$
\begin{aligned}
\mathbb{E}\left[\int_0^{t\wedge T_r}\|\partial_x A(u(s))\|_{L_0^2(\mathbb{T})}^2\mathrm{d}s\right]\le\frac{C_1^2}{\nu}\Bigg(2\left(1+\|u_0\|_{L_0^{2p_A+2}(\mathbb{T})}^{2p_A+2}\right)+D_0t\\
+D_0C_5^{(2p_A)}\left(1+\|u_0\|_{L_0^{2p_A}(\mathbb{T})}^{2p_A}\right)+C_6^{(2p_A)}t\Bigg).
\end{aligned}
$$

Injecting this last bound in (2.33), we get the wanted result. □

Corollary 2.18 (Limit of T_r)**.** *Under Assumptions 2.1 and 2.2, $T_r\to+\infty$ almost surely, and thus $T^*=+\infty$ almost surely.*

Proof. Let $t \geq 0$. Writing

$$\mathbb{P}\left(T_r < t\right) = \mathbb{P}\left(\|u(t \wedge T_r)\|^2_{H^1_0(\mathbb{T})} \geq r\right),$$

we get from Markov's inequality,

$$\mathbb{P}\left(T_r < t\right) \leq \frac{1}{r}\mathbb{E}\left[\|u(t \wedge T_r)\|^2_{H^1_0(\mathbb{T})}\right].$$

We apply now Lemma 2.17 to get

$$\mathbb{P}\left(T_r < t\right) \leq \frac{1}{r}\left(\|u_0\|^2_{H^1_0(\mathbb{T})} + C_7\left(1 + \|u_0\|^{2p_A+2}_{L^{2p_A+2}_0(\mathbb{T})}\right) + C_8 t\right) \underset{r\to\infty}{\longrightarrow} 0.$$

Since t has been chosen arbitrarily, it follows that almost surely, T_r tends to $+\infty$ as $r \to +\infty$. Then, since $T_r \leq T^*$, we have $T^* = +\infty$ almost surely. □

2.2.4 Proof of Theorem 2.4

Under Assumptions 2.1 and 2.2, let $u_0 \in H^2_0(\mathbb{T})$, and $(T^*, (u(t))_{t\in[0,T^*)})$ be the maximal solution to Equation (2.1) given by Lemma 2.14. By Corollary 2.18, $T^* = +\infty$ almost surely. Therefore, $(u(t))_{t\geq 0}$ is the unique (global) mild solution to Equation (2.1), and by Proposition 2.12, it is also the unique (global) strong solution to this equation. It remains to check that this solution depends continuously on u_0.

Lemma 2.19 (Continuous dependence on initial conditions). *If $(u_0^{(j)})_{j\geq 1}$ is a sequence of $H^2_0(\mathbb{T})$ satisfying*

$$\lim_{j\to\infty}\left\|u_0 - u_0^{(j)}\right\|_{H^2_0(\mathbb{T})} = 0,$$

then, denoting by $(u^{(j)}(t))_{t\geq 0, j\geq 1}$ the family of associated solutions, for any $T \geq 0$, we have almost surely

$$\lim_{j\to\infty}\sup_{t\in[0,T]}\left\|u(t) - u^{(j)}(t)\right\|_{H^2_0(\mathbb{T})} = 0.$$

Proof. Let us fix a time horizon $T > 0$. Subtracting the mild formulations of $(u(t))_{t\geq 0}$ and $(u^{(j)}(t))_{t\geq 0}$ given by Proposition 2.12 and taking the $H^2_0(\mathbb{T})$-norm, we get by the triangle inequality and Proposition 2.9, for all $t \in [0, T]$,

$$\begin{aligned}\left\|u(t) - u^{(j)}(t)\right\|_{H^2_0(\mathbb{T})} &\leq \left\|S_t\left(u_0 - u_0^{(j)}\right)\right\|_{H^2_0(\mathbb{T})} + \int_0^t \left\|S_{t-s}\partial_x\left(A(u(s)) - A\left(u^{(j)}(s)\right)\right)\right\|_{H^2_0(\mathbb{T})} \\ &\leq \left\|u_0 - u_0^{(j)}\right\|_{H^2_0(\mathbb{T})} + \int_0^t \frac{C_4}{\sqrt{t-s}}\left\|\partial_x A(u(s)) - \partial_x A\left(u^{(j)}(s)\right)\right\|_{H^1_0(\mathbb{T})} \mathrm{d}s. \quad (2.34)\end{aligned}$$

Now, for any $M > 0$, we define the stopping times

$$\tau_M := \inf\left\{t \geq 0 : \|u(t)\|_{H^2_0(\mathbb{T})} \geq M\right\}, \quad \tau_M^{(j)} := \inf\left\{t \geq 0 : \|u^{(j)}(t)\|_{H^2_0(\mathbb{T})} \geq M\right\}, \quad j \in \mathbb{N},$$

and we denote by L_M, according to Proposition 2.8, the Lipschitz constant of the mapping $v \in H^2_0(\mathbb{T}) \mapsto \partial_x A(v) \in H^1_0(\mathbb{T})$ over the centered ball in $H^2_0(\mathbb{T})$ of radius M. For an arbitrarily fixed $t \in [0, T]$, the inequality (2.34) implies

$$\begin{aligned}\left\|u\left(t \wedge \tau_M \wedge \tau_M^{(j)}\right) - u^{(j)}\left(t \wedge \tau_M \wedge \tau_M^{(j)}\right)\right\|_{H^2_0(\mathbb{T})} &\leq \left\|u_0 - u_0^{(j)}\right\|_{H^2_0(\mathbb{T})} \\ &+ \int_0^{t\wedge\tau_M\wedge\tau_M^{(j)}} \frac{C_4 L_M}{\sqrt{t \wedge \tau_M \wedge \tau_M^{(j)} - s}}\left\|u(s) - u^{(j)}(s)\right\|_{H^2_0(\mathbb{T})} \mathrm{d}s.\end{aligned}$$

In the next step, we iterate this last inequality and apply the Fubini theorem on the double time integral:

$$\begin{aligned}
&\left\| u\left(t\wedge\tau_M\wedge\tau_M^{(j)}\right) - u^{(j)}\left(t\wedge\tau_M\wedge\tau_M^{(j)}\right)\right\|_{H_0^2(\mathbb{T})} \\
&\leq \left\| u_0 - u_0^{(j)}\right\|_{H_0^2(\mathbb{T})}\left(1+2\sqrt{t\wedge\tau_M\wedge\tau_M^{(j)}}C_4L_M\right) \\
&\quad + C_4^2L_M^2\int_0^{t\wedge\tau_M\wedge\tau_M^{(j)}}\int_0^s \frac{1}{\sqrt{(t\wedge\tau_M\wedge\tau_M^{(j)}-s)(s-r)}}\left\| u(r)-u^{(j)}(r)\right\|_{H_0^2(\mathbb{T})}\mathrm{d}r\mathrm{d}s \\
&\leq \left\| u_0 - u_0^{(j)}\right\|_{H_0^2(\mathbb{T})}\left(1+2\sqrt{T}C_4L_M\right) \\
&\quad + C_4^2L_M^2\int_0^{t\wedge\tau_M\wedge\tau_M^{(j)}}\left(\int_r^{t\wedge\tau_M\wedge\tau_M^{(j)}}\frac{1}{\sqrt{(t\wedge\tau_M\wedge\tau_M^{(j)}-s)(s-r)}}\mathrm{d}s\right)\left\| u(r)-u^{(j)}(r)\right\|_{H_0^2(\mathbb{T})}\mathrm{d}r.
\end{aligned}$$

However, by a change of variable, we have

$$\int_s^{t\wedge\tau_M\wedge\tau_M^{(j)}}\frac{1}{\sqrt{(t\wedge\tau_M\wedge\tau_M^{(j)}-r)(r-s)}}\mathrm{d}r = \int_{-1}^1\frac{1}{\sqrt{1-y^2}}\mathrm{d}y = \pi.$$

Hence, Grönwall's lemma yields the following control

$$\left\| u\left(t\wedge\tau_M\wedge\tau_M^{(j)}\right) - u^{(j)}\left(t\wedge\tau_M\wedge\tau_M^{(j)}\right)\right\|_{H_0^2(\mathbb{T})} \leq \left\| u_0 - u_0^{(j)}\right\|_{H_0^2(\mathbb{T})}\left(1+2\sqrt{T}C_4L_M\right)\mathrm{e}^{C_4^2L_M^2\pi t\wedge\tau_M\wedge\tau_M^{(j)}}.$$

It follows from this inequality that $\liminf_{j\to\infty}\tau_M^{(j)}\geq\tau_M\wedge T$. Indeed, assuming the opposite, we would have (along a subsequence)

$$\left\| u\left(\tau_M^{(j)}\right) - u^{(j)}\left(\tau_M^{(j)}\right)\right\|_{H_0^2(\mathbb{T})} \leq \left\| u_0 - u_0^{(j)}\right\|_{H_0^2(\mathbb{T})}\left(1+2\sqrt{T}C_4L_M\right)\mathrm{e}^{C_4^2L_M^2T}\underset{j\to\infty}{\longrightarrow}0,$$

which would imply

$$M\leq\lim_{j\to\infty}\left\| u^{(j)}\left(\tau_M^{(j)}\right)\right\|_{H_0^2(\mathbb{T})} = \lim_{j\to\infty}\left\| u\left(\tau_M^{(j)}\right)\right\|_{H_0^2(\mathbb{T})} < M.$$

Hence, necessarily, beyond a certain rank j, we have

$$\left\| u(t\wedge\tau_M) - u^{(j)}(t\wedge\tau_M)\right\|_{H_0^2(\mathbb{T})} \leq \left\| u_0 - u_0^{(j)}\right\|_{H_0^2(\mathbb{T})}\left(1+2\sqrt{T}C_4L_M\right)\mathrm{e}^{C_4^2L_M^2t\wedge\tau_M}.$$

Since the solutions of (2.7) do not explode, the stopping time τ_M tends almost surely to $+\infty$ as M tends to $+\infty$. As a consequence, there exists $M_T>0$ such that $T<\tau_{M_T}$ almost surely, so that for all $t\in[0,T]$,

$$\left\| u(t) - u^{(j)}(t)\right\|_{H_0^2(\mathbb{T})} \leq \left\| u_0 - u_0^{(j)}\right\|_{H_0^2(\mathbb{T})}\left(1+2\sqrt{T}C_4L_{M_T}\right)\mathrm{e}^{C_4^2L_{M_T}^2T}.$$

Hence the result. □

2.2.5 Proofs of preliminary results

In this subsection, we detail the proofs of the preliminary results from Subsection 2.2.1, namely Propositions 2.8, 2.9, 2.10 and 2.12.

Proof of Proposition 2.8. We shall prove only the second claim when $s = 2$. For the first claim, we refer the reader to [15, Lemma A.0.5]. The second claim with $s = 1$ is proved in the same way as the case $s = 2$, but more easily.

Let $M > 0$ and let $u, v \in H_0^2(\mathbb{T})$ such that $\|u\|_{H_0^2(\mathbb{T})} \leq M$ and $\|v\|_{H_0^2(\mathbb{T})} \leq M$. Note in particular that using (2.3), all quantities $\|u\|_{L_0^\infty(\mathbb{T})}$, $\|v\|_{L_0^\infty(\mathbb{T})}$, $\|\partial_x u\|_{L_0^\infty(\mathbb{T})}$, $\|\partial_x v\|_{L_0^\infty(\mathbb{T})}$ are bounded from above by M. We write

$$\begin{aligned}
\|\partial_x A(u) - \partial_x A(v)\|^2_{H_0^1(\mathbb{T})} &= \int_{\mathbb{T}} \left(\partial_x \left(\partial_x u A'(u) - \partial_x v A'(v)\right)\right)^2 \mathrm{d}x \\
&= \int_{\mathbb{T}} \left(\partial_{xx} u A'(u) + (\partial_x u)^2 A''(u) - \partial_{xx} v A'(v) - (\partial_x v)^2 A''(v)\right)^2 \mathrm{d}x \\
&\leq 2\int_{\mathbb{T}} \left(\partial_{xx} u A'(u) - \partial_{xx} v A'(v)\right)^2 \mathrm{d}x + 2\int_{\mathbb{T}} \left((\partial_x u)^2 A''(u) - (\partial_x v)^2 A''(v)\right)^2 \mathrm{d}x \\
&\leq 4\int_{\mathbb{T}} (\partial_{xx} u)^2 \left(A'(u) - A'(v)\right)^2 \mathrm{d}x + 4\int_{\mathbb{T}} (\partial_{xx} u - \partial_{xx} v)^2 A'(v)^2 \mathrm{d}x \\
&\quad + 4\int_{\mathbb{T}} (\partial_x u)^4 \left(A''(u) - A''(v)\right)^2 \mathrm{d}x + 4\int_{\mathbb{T}} (\partial_x u - \partial_x v)^2 (\partial_x u + \partial_x v)^2 A''(v)^2 \mathrm{d}x \\
&\leq 4\|u\|^2_{H_0^2(\mathbb{T})} \operatorname*{ess\,sup}_{\mathbb{T}} \left|A'(u) - A'(v)\right|^2 + 4\|u - v\|^2_{H_0^2(\mathbb{T})} \sup_{[-M,M]} \left|A'\right|^2 \\
&\quad + 4\left(\int_{\mathbb{T}} (\partial_x u)^4 \mathrm{d}x\right) \operatorname*{ess\,sup}_{\mathbb{T}} \left|A''(u) - A''(v)\right|^2 \\
&\quad + 4 \operatorname*{ess\,sup}_{\mathbb{T}} |\partial_x u + \partial_x v|^2 \sup_{[-M,M]} |A''|^2 \|u - v\|^2_{H_0^1(\mathbb{T})}
\end{aligned}$$

By Assumption 2.1, A' and A'' are locally Lipschitz continuous on $\mathbb{R}$ (and thus locally bounded). Hence, there exist constants C_M and L_M such that

$$\begin{aligned}
\|\partial_x A(u) - \partial_x A(v)\|^2_{H_0^1(\mathbb{T})} &\leq 4M^2 L_M \|u - v\|^2_{L_0^\infty(\mathbb{T})} + 4C_M \|u - v\|^2_{H_0^2(\mathbb{T})} \\
&\quad + 4M^4 L_M \|u - v\|^2_{L_0^\infty(\mathbb{T})} + 16 C_M M^4 \|u - v\|^2_{H_0^1(\mathbb{T})} \\
&\leq \left(4M^2 L_M + 4C_M + 4M^4 L_M + 16 C_M M^4\right) \|u - v\|^2_{H_0^2(\mathbb{T})},
\end{aligned}$$

where we used (2.3) thrice in the last line. □

Proof of Proposition 2.9. The equations (2.8) and (2.9) yield the immediate estimate

$$\|S_t v\|^2_{H_0^s(\mathbb{T})} = \sum_{m \geq 1} (-\lambda_m)^s \mathrm{e}^{2\nu\lambda_m t} \langle v, e_m \rangle^2_{L_0^2(\mathbb{T})} \leq \sum_{m \geq 1} (-\lambda_m)^s \langle v, e_m \rangle^2_{L_0^2(\mathbb{T})} = \|v\|^2_{H_0^s(\mathbb{T})},$$

which ensures that $S_t v \in H_0^s(\mathbb{T})$ and then implies the first assertion of Proposition 2.9 thanks to the dominated convergence theorem.

The second assertion is proved in [15, 97, 34].

We now detail the proof of the third assertion, part of which can also be found in [15, Lemma A.0.6]. Let $s \in [0, +\infty)$, $T > 0$ and $(v(t))_{t \in [0,T]} \in C\left([0,T], H_0^s(\mathbb{T})\right)$. For any $0 \leq t_1 \leq t_2 \leq T$, we have

$$\begin{aligned}
&\left\| \int_0^{t_1} S_{t_1 - r} v(r) \mathrm{d}r - \int_0^{t_2} S_{t_2 - r} v(r) \mathrm{d}r \right\|_{H_0^{s+3/2}(\mathbb{T})} \\
&\qquad \leq \int_0^{t_1} \|S_{t_1 - r} v(r) - S_{t_2 - r} v(r)\|_{H_0^{s+3/2}(\mathbb{T})} \mathrm{d}r + \int_{t_1}^{t_2} \|S_{t_2 - r} v(r)\|_{H_0^{s+3/2}(\mathbb{T})} \mathrm{d}r. \qquad (2.35)
\end{aligned}$$

Thanks to the second assertion of Proposition 2.9, we get a bound over the second term of the right-hand side:

$$\int_{t_1}^{t_2} \|S_{t_2-r}v(r)\|_{H_0^{s+3/2}(\mathbb{T})} \mathrm{d}r \leq \int_{t_1}^{t_2} C_4(t_2-r)^{-3/4}\|v(r)\|_{H_0^s(\mathbb{T})}\mathrm{d}r \leq 4C_4(t_2-t_1)^{1/4} \sup_{r\in[0,T]} \|v(r)\|_{H_0^s(\mathbb{T})}, \tag{2.36}$$

as well as for the first term:

$$\begin{aligned}\int_0^{t_1} \|S_{t_1-r}v(r) - S_{t_2-r}v(r)\mathrm{d}r\|_{H_0^{s+3/2}(\mathbb{T})} &= \int_0^{t_1} \|S_{t_1-r}\left(\mathrm{Id} - S_{t_2-t_1}\right)v(r)\|_{H_0^{s+3/2}(\mathbb{T})} \mathrm{d}r \\ &\leq \int_0^{t_1} C_4(t_1-r)^{-3/4} \|(\mathrm{Id} - S_{t_2-t_1})v(r)\|_{H_0^s(\mathbb{T})} \mathrm{d}r.\end{aligned} \tag{2.37}$$

Using (2.8) and (2.9), we write for all $r \in [0,T]$,

$$\begin{aligned}\|(\mathrm{Id} - S_{t_2-t_1})v(r)\|_{H_0^s(\mathbb{T})} &= \left(\sum_{m\geq 1}(-\lambda_m)^s\left(1-\mathrm{e}^{\nu\lambda_m(t_2-t_1)}\right)^2\langle v(r), e_m\rangle^2_{L_0^2(\mathbb{T})}\right)^{1/2}. \\ &\leq \left(\sum_{m\geq 1}(-\lambda_m)^s\langle v(r), e_m\rangle^2_{L_0^2(\mathbb{T})}\right)^{1/2} \\ &= \|v(r)\|_{H_0^s(\mathbb{T})} < +\infty,\end{aligned}$$

and thus, we can apply the dominated convergence theorem which yields

$$\begin{aligned}\lim_{t_2-t_1\to 0}\|(\mathrm{Id} - S_{t_2-t_1})v(r)\|_{H_0^s(\mathbb{T})} &\leq \left(\sum_{m\geq 1}(-\lambda_m)^s\lim_{t_2-t_1\to 0}\left(1-\mathrm{e}^{\nu\lambda_m(t_2-t_1)}\right)^2\langle v(r), e_m\rangle^2_{L_0^2(\mathbb{T})}\right)^{1/2} \\ &= 0.\end{aligned}$$

To pass to the limit $t_2 - t_1 \to 0$ in (2.35), we use the dominated convergence theorem once again: injecting (2.36) and (2.37) into (2.35), we get

$$\begin{aligned}&\lim_{t_2-t_1\to 0}\left\|\int_0^{t_1} S_{t_1-r}v(r)\mathrm{d}r - \int_0^{t_2} S_{t_2-r}v(r)\mathrm{d}r\right\|_{H_0^{s+3/2}(\mathbb{T})} \\ &\leq \int_0^{t_1} C_4(t_1-r)^{-3/4}\lim_{t_2-t_1\to 0}\|(\mathrm{Id} - S_{t_2-t_1})v(r)\|_{H_0^s(\mathbb{T})}\mathrm{d}r + 4C_4\sup_{r\in[0,T]}\|v(r)\|_{H_0^s(\mathbb{T})}\lim_{t_2-t_1\to 0}(t_2-t_1)^{1/4} = 0,\end{aligned}$$

from which we derive the wanted result. □

Proof of Proposition 2.10. For any $m \geq 1$, the process $(\overline{w}_m(t))_{t\geq 0}$ is an Ornstein-Uhlenbeck process. In particular, it is the solution of the stochastic differential equation:

$$\overline{w}_m(t) = \nu\lambda_m\int_0^t \overline{w}_m(s)\mathrm{d}s + \overline{W}_m(t), \qquad t \geq 0. \tag{2.38}$$

As such, it satisfies the inequality

$$\overline{w}_m(t) - \nu\lambda_m\int_0^t \overline{w}_m(s)\mathrm{d}s \leq \sup_{s\in[0,T]} \overline{W}_m(s), \qquad t \in [0,T],$$

where we fixed a time horizon $T > 0$. Thus, for all $t \in [0, T]$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{-\nu\lambda_m t}\int_0^t \overline{w}_m(s)\mathrm{d}s\right) = \mathrm{e}^{-\nu\lambda_m t}\left(-\nu\lambda_m\int_0^t \overline{w}_m(s)\mathrm{d}s + \overline{w}_m(t)\right) \le \mathrm{e}^{-\nu\lambda_m t}\sup_{s\in[0,T]}\overline{W}_m(s),$$

which leads, after integrating in time and dividing by $\mathrm{e}^{-\nu\lambda_m t}$ on each side, to the inequality

$$\int_0^t \overline{w}_m(s)\mathrm{d}s \le \frac{1-\mathrm{e}^{\nu\lambda_m t}}{-\nu\lambda_m}\sup_{s\in[0,T]}\overline{W}_m(s), \qquad t\in[0,T]. \tag{2.39}$$

In a similar way, from the inequality

$$\overline{w}_m(t) - \nu\lambda_m\int_0^t \overline{w}_m(s)\mathrm{d}s \ge \inf_{s\in[0,T]}\overline{W}_m(s), \qquad t\in[0,T],$$

we deduce that

$$\int_0^t \overline{w}_m(s)\mathrm{d}s \ge \frac{1-\mathrm{e}^{\nu\lambda_m t}}{-\nu\lambda_m}\inf_{s\in[0,T]}\overline{W}_m(s), \qquad t\in[0,T]. \tag{2.40}$$

Combining (2.39) and (2.40), we get

$$\left|\int_0^t \overline{w}_m(s)\mathrm{d}s\right| \le \frac{1}{-\nu\lambda_m}\sup_{s\in[0,T]}|\overline{W}_m(s)|, \qquad t\in[0,T].$$

Taking the supremum in time, the expectation and applying Doob's inequality, recalling (2.6), we have

$$\begin{aligned}
\mathbb{E}\left[\sup_{t\in[0,T]}\left|\int_0^t \overline{w}_m(s)\mathrm{d}s\right|^2\right] &\le \frac{1}{(\nu\lambda_m)^2}\mathbb{E}\left[\sup_{t\in[0,T]}|\overline{W}_m(t)|^2\right] &(2.41)\\
&\le \frac{4}{(\nu\lambda_m)^2}\mathbb{E}\left[\left\langle \overline{W}^Q(T), \frac{e_m}{\lambda_m}\right\rangle_{H_0^2(\mathbb{T})}^2\right] &(2.42)\\
&= \frac{4T}{(\nu\lambda_m)^2}\sum_{k\ge1}\left\langle g_k, \frac{e_m}{\lambda_m}\right\rangle_{H_0^2(\mathbb{T})}^2. &(2.43)
\end{aligned}$$

We deduce from (2.38) that

$$\sup_{t\in[0,T]}\overline{w}_m(t)^2 \le 2\left((\nu\lambda_m)^2\sup_{t\in[0,T]}\left|\int_0^t \overline{w}_m(s)\mathrm{d}s\right|^2 + \sup_{t\in[0,T]}|\overline{W}_m(t)|^2\right),$$

and therefore it follows from (2.41), (2.42), (2.43) that

$$\mathbb{E}\left[\sup_{t\in[0,T]}\overline{w}_m(t)^2\right] \le 16T\sum_{k\ge1}\left\langle g_k, \frac{e_m}{\lambda_m}\right\rangle_{H_0^2(\mathbb{T})}^2 < +\infty. \tag{2.44}$$

As a consequence, for any $M \ge 1$, we have

$$\sum_{m=1}^M \overline{w}_m\frac{e_m}{\lambda_m} \in L^2\left(\Omega, C\left([0,T], H_0^2(\mathbb{T})\right)\right).$$

By completeness of $L^2(\Omega, C([0,T], H_0^2(\mathbb{T})))$, to prove the statement, it suffices to show that the sequence $(\sum_{m=1}^M \overline{w}_m e_m/\lambda_m)_{M\ge1}$ is Cauchy in this space. That is,

$$\lim_{M,N\to\infty}\mathbb{E}\left[\sup_{t\in[0,T]}\left\|\sum_{m=M}^{M+N}\overline{w}_m(t)\frac{e_m}{\lambda_m}\right\|_{H_0^2(\mathbb{T})}^2\right] = 0. \tag{2.45}$$

We prove this last equality using (2.44):

$$\begin{aligned}\mathbb{E}\left[\sup_{t\in[0,T]}\left\|\sum_{m=M}^{M+N}\overline{w}_m(t)\frac{e_m}{\lambda_m}\right\|^2_{H^2_0(\mathbb{T})}\right] &= \mathbb{E}\left[\sup_{t\in[0,T]}\sum_{m=M}^{M+N}\overline{w}_m(t)^2\right]\\ &\leq \sum_{m=M}^{M+N}\mathbb{E}\left[\sup_{t\in[0,T]}\overline{w}_m(t)^2\right]\\ &\leq 16T\sum_{m=M}^{M+N}\sum_{k\geq 1}\left\langle g_k,\frac{e_m}{\lambda_m}\right\rangle^2_{H^2_0(\mathbb{T})}.\end{aligned}$$

Recall that thanks to Assumption 2.2 combined with the fact that the family $(e_m/\lambda_m)_{m\geq 1}$ is an orthonormal basis of $H^2_0(\mathbb{T})$,

$$\sum_{k\geq 1}\sum_{m\geq 1}\left\langle g_k,\frac{e_m}{\lambda_m}\right\rangle^2_{H^2_0(\mathbb{T})} = \sum_{k\geq 1}\|g_k\|^2_{H^2_0(\mathbb{T})} < +\infty.$$

Hence, (2.45) is proved. Moreover, $(\overline{w}(t))_{t\geq 0}$ is $(\mathcal{F}_t)_{t\geq 0}$-adapted as the limit of the sequence of $(\mathcal{F}_t)_{t\geq 0}$-adapted processes $\sum_{m\geq 1}\overline{w}_m e_m/\lambda_m$. □

Proof of Proposition 2.12. We first show that the assumption that $\overline{u}_0 \in H^2_0(\mathbb{T})$ ensures that $t \mapsto \overline{u}(t)$ is a continuous, $H^2_0(\mathbb{T})$-valued mapping. The proof consists in the two first iterations of the bootstrap argument that was used in [15, Theorem A.0.7]: since the mild solution $(\overline{u}(t))_{t\in[0,\overline{\tau}]}$ belongs to $C([0,\overline{\tau}],H^1_0(\mathbb{T}))$ almost surely, then from Proposition 2.8, for all $s\in[0,\tau]$, we have $\partial_x A(\overline{u}(s))\in L^2_0(\mathbb{T})$. As a consequence of the third assertion of Proposition 2.9, the mapping $t\mapsto\int_0^t S_{t-s}\partial_x A(\overline{u}(s))\mathrm{d}s$ is continuous from $[0,\overline{\tau}]$ to $H^{3/2}_0(\mathbb{T})$, and by the first assertion of Proposition 2.9 as well as Proposition 2.10, so are the mappings $t\mapsto S_t\overline{u}_0$ and $t\mapsto\overline{w}(t)$. Therefore, (2.11) shows that $(\overline{u}(t))_{t\in[0,\overline{\tau}]}\in C([0,\overline{\tau}],H^{3/2}_0(\mathbb{T}))$. Iterating this argument, we get $(\overline{u}(t))_{t\in[0,\overline{\tau}]}\in C([0,\overline{\tau}],H^2_0(\mathbb{T}))$.

We now show that $(\overline{u}(t))_{t\in[0,\overline{\tau}]}$ satisfies the strong formulation of (2.10) on $[0,\overline{\tau}]$. Along each coordinate of the basis $(e_m)_{m\geq 1}$, the mild formulation (2.11) writes for all $t\in[0,\overline{\tau}]$

$$\langle\overline{u}(t),e_m\rangle_{L^2_0(\mathbb{T})} = \mathrm{e}^{\nu\lambda_m t}\langle\overline{u}_0,e_m\rangle_{L^2_0(\mathbb{T})} - \int_0^t \mathrm{e}^{\nu\lambda_m(t-s)}\langle\partial_x A(\overline{u}(s)),e_m\rangle_{L^2_0(\mathbb{T})}\mathrm{d}s + \frac{1}{\lambda_m}\overline{w}_m(t).$$

Multiplying on each side by $\mathrm{e}^{-\nu\lambda_m t}$, we get the decomposition:

$$\mathrm{e}^{-\nu\lambda_m t}\langle\overline{u}(t),e_m\rangle_{L^2_0(\mathbb{T})} = \langle\overline{u}_0,e_m\rangle_{L^2_0(\mathbb{T})} - \int_0^t \mathrm{e}^{-\nu\lambda_m s}\langle\partial_x A(\overline{u}(s)),e_m\rangle_{L^2_0(\mathbb{T})}\mathrm{d}s + \frac{1}{\lambda_m}\int_0^t \mathrm{e}^{-\nu\lambda_m s}\mathrm{d}\overline{W}_m(s).$$

Then, the Itô formula yields

$$\langle\overline{u}(t),e_m\rangle_{L^2_0(\mathbb{T})} = \langle\overline{u}_0,e_m\rangle_{L^2_0(\mathbb{T})} + \nu\lambda_m\int_0^t\langle\overline{u}(s),e_m\rangle_{L^2_0(\mathbb{T})}\mathrm{d}s - \int_0^t\langle\partial_x A(\overline{u}(s)),e_m\rangle_{L^2_0(\mathbb{T})}\mathrm{d}s + \frac{1}{\lambda_m}\overline{W}_m(t). \tag{2.46}$$

Since for all $s\in[0,t]$, $\overline{u}(s)$ belongs to $H^2_0(\mathbb{T})$, it is possible to perform an integration by parts on the viscous term in the following way:

$$\nu\lambda_m\int_0^t\langle\overline{u}(s),e_m\rangle_{L^2_0(\mathbb{T})}\mathrm{d}s = \nu\int_0^t\langle\overline{u}(s),\partial_{xx}e_m\rangle_{L^2_0(\mathbb{T})}\mathrm{d}s = \nu\int_0^t\langle\partial_{xx}\overline{u}(s),e_m\rangle_{L^2_0(\mathbb{T})}\mathrm{d}s.$$

Equation (2.46), after being injected with the above equality, multiplied by e_m and summed over m, becomes the strong formulation of (2.10).

The converse statement follows from the same computations. □

2.3 Invariant measure

This section is dedicated to the proof of Theorem 2.7. The existence of an invariant measure is proved in Subsection 2.3.2 using the Krylov-Bogoliubov theorem, whereas the uniqueness is addressed through a coupling argument relying on the $L^1_0(\mathbb{T})$-contraction property established in Proposition 2.21.

The proof of existence of an invariant measure we provide in the next subsection relies plainly on the presence of viscosity. Indeed, the viscous term provides the process $u(t)$ with a dissipative – and thus a more stable – behaviour. Still, it has to be borne in mind that when the flux term is nonlinear enough, the presence of a viscous term is not a necessary condition for the stability of the underlying stochastic process. On the physical side, in his theory of turbulent flows [81, 80], Kolmogorov already predicted this idea: the statistical distribution of scales of intermediate size in turbulence are not determined by the viscosity coefficient. On the theoretical side, the same idea was validated theoretically by powerful results on the invariant measure for the inviscid stochastic Burgers' equation [52] and, quite a few years later, for inviscid stochastic conservation laws with "non-degenerate" flux [41]. However, our framework differs substantially from the inviscid case in the sense that our stability results are driven by regularity issues which cannot be tackled without viscosity.

2.3.1 Preliminary results

By Definition 2.6, an invariant measure for Equation (2.1) is a Borel probability measure on $H^2_0(\mathbb{T})$. Our proofs of existence and uniqueness however involve estimates in various spaces, namely $L^1_0(\mathbb{T})$, $L^2_0(\mathbb{T})$ and $H^1_0(\mathbb{T})$. In particular, we shall manipulate and identify Borel probability measures on these spaces. We first clarify the relation between the associated Borel σ-fields thanks to the following result. For any metric space E, we respectively denote by $\mathcal{B}(E)$ and $\mathcal{P}(E)$ the Borel σ-field and the set of Borel probability measures on E.

Lemma 2.20 (Borel probability measures on $L^q_0(\mathbb{T})$ and $H^s_0(\mathbb{T})$)**.** *For all $q \in [1,2]$ and $s \geq 1$, $\mathcal{B}(H^s_0(\mathbb{T})) = \{B \cap H^s_0(\mathbb{T}) : B \in \mathcal{B}(L^q_0(\mathbb{T}))\}$. As a consequence:*

(1) for any $\mu \in \mathcal{P}(H^s_0(\mathbb{T}))$, the mapping $\widetilde{\mu}(\cdot) = \mu(\cdot \cap H^s_0(\mathbb{T}))$ defines a Borel probability measure on $L^q_0(\mathbb{T})$;

(2) conversely, for any $\widetilde{\mu} \in \mathcal{P}(L^q_0(\mathbb{T}))$ which gives full weight to $H^s_0(\mathbb{T})$, there exists a unique $\mu \in \mathcal{P}(H^s_0(\mathbb{T}))$ such that $\widetilde{\mu}(B) = \mu(B \cap H^s_0(\mathbb{T}))$ for any $B \in \mathcal{B}(L^q_0(\mathbb{T}))$.

Proof. Let $q \in [1,2]$ and $s \geq 1$. The set $\mathcal{T}$ defined by

$$\mathcal{T} = \{B \cap H^s_0(\mathbb{T}) : B \in \mathcal{B}\left(L^q_0(\mathbb{T})\right)\}.$$

is a σ-field on $H^s_0(\mathbb{T})$, called the *trace σ-field* of $H^s_0(\mathbb{T})$ in $\mathcal{B}(L^q_0(\mathbb{T}))$.

(1) We denote by I the injection $H^s_0(\mathbb{T}) \to L^q_0(\mathbb{T})$, so that $\mathcal{T} = \{I^{-1}(B) : B \in \mathcal{B}(L^q_0(\mathbb{T}))\}$. Since I is continuous, and therefore Borel measurable, we have $\mathcal{T} \subset \mathcal{B}(H^s_0(\mathbb{T}))$. Thus, for any $\mu \in \mathcal{P}(H^s_0(\mathbb{T}))$, the pushforward measure $\widetilde{\mu}$ defined by

$$\widetilde{\mu}(B) := \mu \circ I^{-1}(B) = \mu\left(B \cap H^s_0(\mathbb{T})\right), \qquad B \in \mathcal{B}\left(L^q_0(\mathbb{T})\right),$$

is a Borel probability measure on $L^q_0(\mathbb{T})$.

(2) Let us first notice that since $H^s_0(\mathbb{T})$ is separable, the Borel σ-field $\mathcal{B}(H^s_0(\mathbb{T}))$ is the smallest σ-field on $H^s_0(\mathbb{T})$ containing all closed balls. Let $A \subset H^s_0(\mathbb{T})$ be such a ball. Since the $H^s_0(\mathbb{T})$-norm is lower semi-continuous on $L^q_0(\mathbb{T})$, then A is closed in $L^q_0(\mathbb{T})$ as a level set of a lower semi-continuous function, and thus $A \in \mathcal{B}(L^q_0(\mathbb{T}))$. It is then clear that $A \in \mathcal{T}$, which by the minimality property of $\mathcal{B}(H^s_0(\mathbb{T}))$ entails $\mathcal{B}(H^s_0(\mathbb{T})) \subset \mathcal{T}$, and thus $\mathcal{B}(H^s_0(\mathbb{T})) = \mathcal{T}$.

Now let $\widetilde{\mu}$ be a Borel probability measure on $L^q_0(\mathbb{T})$ which gives full weight to $H^s_0(\mathbb{T})$, that is to say such that there exists $\widetilde{B} \in \mathcal{B}(L^q_0(\mathbb{T}))$ such that $\widetilde{B} \subset H^s_0(\mathbb{T})$ and $\widetilde{\mu}(\widetilde{B}) = 1$. Let us define the Borel probability measure μ on $H^s_0(\mathbb{T})$ by

$$\mu(B \cap H^s_0(\mathbb{T})) := \widetilde{\mu}(B), \qquad B \in \mathcal{B}\left(L^q_0(\mathbb{T})\right).$$

Notice that this definition is not ambiguous, because the identity $\mathcal{T} = \mathcal{B}(H^s_0(\mathbb{T}))$ ensures that any element of $\mathcal{B}(H^s_0(\mathbb{T}))$ writes under the form $B \cap H^s_0(\mathbb{T})$ for some $B \in \mathcal{B}(L^q_0(\mathbb{T}))$; besides, if $B_1, B_2 \in \mathcal{B}(L^q_0(\mathbb{T}))$ are such that $B_1 \cap H^s_0(\mathbb{T}) = B_2 \cap H^s_0(\mathbb{T})$, then $\widetilde{\mu}(B_1) = \widetilde{\mu}(B_1 \cap \widetilde{B}) = \widetilde{\mu}(B_2 \cap \widetilde{B}) = \widetilde{\mu}(B_2)$ because the identity $B_1 \cap H^s_0(\mathbb{T}) = B_2 \cap H^s_0(\mathbb{T})$ implies that $B_1 \cap \widetilde{B} = B_2 \cap \widetilde{B}$. Finally, the fact that any $\nu \in \mathcal{P}(H^s_0(\mathbb{T}))$ such that $\widetilde{\mu}(B) = \nu(B \cap H^s_0(\mathbb{T}))$ for any $B \in \mathcal{B}(L^q_0(\mathbb{T}))$ needs to coincide with μ follows again from the identity $\mathcal{B}(H^s_0(\mathbb{T})) = \mathcal{T}$. $\square$

To prove Theorem 2.7, we will need a standard property of scalar conservation laws, namely the $L^1_0(\mathbb{T})$-contraction. In the stochastic setting, we mention that a similar proof of the following proposition is done in [16, Theorem 6.1], but in the case where the flux function is C^∞.

Proposition 2.21 ($L^1_0(\mathbb{T})$-contraction)**.** *Under Assumptions 2.1 and 2.2, let $(u(t))_{t\geq 0}$ and $(v(t))_{t\geq 0}$ be two strong solutions of (2.1) starting from different initial conditions u_0 and v_0. Then, almost surely and for every $0 \leq s \leq t$, we have*

$$\|u(t) - v(t)\|_{L^1_0(\mathbb{T})} \leq \|u(s) - v(s)\|_{L^1_0(\mathbb{T})}.$$

Proof. We define a continuous approximation of the sign function by setting for all $\eta > 0$,

$$\operatorname{sign}_\eta(u) := \begin{cases} \frac{u}{\eta}, & u \in [-\eta, \eta], \\ 1, & u \geq \eta, \\ -1, & u \leq \eta, \end{cases}$$

which gives rise to the following continuously differentiable approximation of the absolute value function:

$$|v|_\eta := \int_0^v \operatorname{sign}_\eta(u)\mathrm{d}u, \quad v \in \mathbb{R}.$$

Let $0 \leq s \leq t$. We have

$$\begin{aligned}
&\int_{\mathbb{T}} |u(t) - v(t)|_\eta \,\mathrm{d}x - \int_{\mathbb{T}} |u(s) - v(s)|_\eta \mathrm{d}x = \int_{\mathbb{T}} \int_s^t \frac{\mathrm{d}}{\mathrm{d}r} |u(r) - v(r)|_\eta \,\mathrm{d}r\mathrm{d}x \qquad (2.47)\\
&= \int_{\mathbb{T}} \int_s^t \frac{\mathrm{d}}{\mathrm{d}r} \left(u(r) - v(r)\right) \operatorname{sign}_\eta(u(r) - v(r))\mathrm{d}r\mathrm{d}x \\
&= \int_s^t \int_{\mathbb{T}} \left(A(u(r)) - A(v(r)) - \nu\partial_x \left(u(r) - v(r)\right)\right) \partial_x \left(\operatorname{sign}_\eta(u(r) - v(r))\right) \mathrm{d}x\mathrm{d}r \\
&\qquad\qquad \text{(where we used the Fubini theorem and an integration by parts)} \\
&= \int_s^t \int_{\mathbb{T}} \left(A(u(r)) - A(v(r)) - \nu\partial_x \left(u(r) - v(r)\right)\right) \partial_x \left(u(r) - v(r)\right) \frac{1}{\eta} \mathbf{1}_{|u(r)-v(r)|\leq\eta} \mathrm{d}x\mathrm{d}r \\
&\leq \int_s^t \int_{\mathbb{T}} (A(u(r)) - A(v(r)))\partial_x(u(r) - v(r)) \frac{1}{\eta} \mathbf{1}_{|u(r)-v(r)|\leq\eta} \mathrm{d}x\mathrm{d}r
\end{aligned}$$

We fix

$$M := \sup_{r\in[s,t]} \|u(r)\|_{L^\infty_0(\mathbb{T})} \vee \sup_{r\in[s,t]} \|v(r)\|_{L^\infty_0(\mathbb{T})},$$

and we denote by L_M a Lipschitz constant of A over the interval $[-M, M]$. Since $(u(r))_{r\in[s,t]}$ and $(v(r))_{r\in[s,t]}$ belong to $C([s,t], H_0^2(\mathbb{T}))$ almost surely, then M is finite almost surely and for all $r \in [s,t]$

$$|A(u(r)) - A(v(r))||\partial_x(u(r) - v(r))|\frac{1}{\eta}\mathbf{1}_{|u(r)-v(r)|\leq\eta} \leq L_M \left|\partial_x(u(r) - v(r))\right|,$$

with

$$\int_s^t \int_{\mathbb{T}} L_M \left|\partial_x(u(r) - v(r))\right| \mathrm{d}x\mathrm{d}r < +\infty.$$

Thus, we get from the dominated convergence theorem:

$$\begin{aligned}\lim_{\eta\to 0} \int_s^t \int_{\mathbb{T}} &(A(u(r)) - A(v(r)))\partial_x(u(r) - v(r))\frac{1}{\eta}\mathbf{1}_{|u(r)-v(r)|\leq\eta}\mathrm{d}x\mathrm{d}r \\ &= \int_s^t \int_{\mathbb{T}} \lim_{\eta\to 0}(A(u(r)) - A(v(r)))\partial_x(u(r) - v(r))\frac{1}{\eta}\mathbf{1}_{|u(r)-v(r)|\leq\eta}\mathrm{d}x\mathrm{d}r = 0. \quad (2.48)\end{aligned}$$

As for the left-hand side of (2.47), noticing that $|\cdot|_\eta$ increases to $|\cdot|$ as η decreases, we have from the monotone convergence theorem

$$\lim_{\eta\to 0}\int_{\mathbb{T}} |u(t) - v(t)|_\eta \,\mathrm{d}x = \|u(t) - v(t)\|_{L_0^1(\mathbb{T})}, \qquad \lim_{\eta\to 0}\int_{\mathbb{T}} |u(s) - v(s)|_\eta \mathrm{d}x = \|u(s) - v(s)\|_{L_0^1(\mathbb{T})}.$$

Hence, (2.47) yields the wanted result. □

2.3.2 Existence

From the semigroup $(P_t)_{t\geq 0}$ introduced in Subsection 2.1.2, we define its time-averaged semigroup $(R_T)_{T\geq 0}$ by $R_0 = \mathrm{Id}$, and for all $T > 0$,

$$R_T\varphi(u_0) = \frac{1}{T}\int_0^T P_t\varphi(u_0)\mathrm{d}t, \qquad \varphi \in C_b(H_0^2(\mathbb{T})), \quad u_0 \in H_0^2(\mathbb{T}),$$

$$R_T^*\alpha(\Gamma) = \frac{1}{T}\int_0^T P_t^*\alpha(\Gamma)\mathrm{d}t, \qquad \alpha \in \mathcal{P}(H_0^2(\mathbb{T})), \quad \Gamma \in \mathcal{B}(H_0^2(\mathbb{T})).$$

Following the first part of Lemma 2.20, for any $\alpha \in \mathcal{P}(H_0^2(\mathbb{T}))$ and $T \geq 0$, we denote by $\widetilde{R}_T^*\alpha$ the Borel probability measure on $L_0^1(\mathbb{T})$ defined by $\widetilde{R}_T^*\alpha(\cdot) = R_T^*\alpha(\cdot \cap H_0^2(\mathbb{T}))$.

Lemma 2.22. *Under Assumptions 2.1 and 2.2, for any $u_0 \in H_0^2(\mathbb{T})$, there exists an increasing sequence $T^n \overset{n\to\infty}{\longrightarrow} +\infty$ and a probability measure $\widetilde{\mu} \in \mathcal{P}(L_0^1(\mathbb{T}))$, such that the sequence of measures $(\widetilde{R}_{T^n}^*\delta_{u_0})_{n\geq 1}$ converges weakly to $\widetilde{\mu}$ in $\mathcal{P}(L_0^1(\mathbb{T}))$.*

Proof. Let $u_0 \in H_0^2(\mathbb{T})$. From the inequality (2.17) with $p = 2$, we can pass to the limit $r \to +\infty$ (which we recall implies that $T_r \to +\infty$ almost surely), and we get for all $T \geq 0$,

$$\mathbb{E}\left[\int_0^T \|u(t)\|_{H_0^1(\mathbb{T})}^2 \mathrm{d}t\right] \leq \frac{1}{2\nu}\|u_0\|_{L_0^2(\mathbb{T})}^2 + \frac{D_0 T}{2\nu}.$$

Applying now the Markov inequality when $T \geq 1$, we have for all $\varepsilon > 0$,

$$\frac{1}{T}\int_0^T \mathbb{P}\left(\|u(t)\|_{H_0^1(\mathbb{T})}^2 > \frac{1}{\varepsilon}\right)\mathrm{d}t \leq \frac{\varepsilon}{2\nu}\left(\|u_0\|_{L_0^2(\mathbb{T})}^2 + D_0\right). \quad (2.49)$$

Setting

$$K_\varepsilon := \left\{ v \in H_0^1(\mathbb{T}) : \|v\|_{H_0^1(\mathbb{T})}^2 \leq \frac{1}{\varepsilon} \right\},$$

we know from the compact embedding $H_0^1(\mathbb{T}) \subset\subset L_0^1(\mathbb{T})$ that the set K_ε is compact in $L_0^1(\mathbb{T})$. Thus, rewriting (2.49) as

$$\widetilde{R}_T^* \delta_{u_0} \left(L_0^1(\mathbb{T}) \setminus K_\varepsilon \right) \leq \frac{\varepsilon}{2\nu} \left(\|u_0\|_{L_0^2(\mathbb{T})}^2 + D_0 \right),$$

we deduce that the family of measures $\{\widetilde{R}_T^* \delta_{u_0} : T \geq 1\}$ is tight in the space $\mathcal{P}(L_0^1(\mathbb{T}))$. The result is then a consequence of Prokhorov's theorem [12, Theorem 5.1]. $\square$

Lemma 2.23. *Under the assumptions of Lemma 2.22, for all $p \geq 1$, if v is a random variable in $L_0^1(\mathbb{T})$ distributed according to $\widetilde{\mu}$, then*

$$\mathbb{E}\left[\|v\|_{L_0^p(\mathbb{T})}^p \right] < +\infty \qquad \text{and} \qquad \mathbb{E}\left[\|v\|_{H_0^2(\mathbb{T})}^2 \right] < +\infty.$$

Besides, the probability measure $\mu \in \mathcal{P}(H_0^2(\mathbb{T}))$ associated with $\widetilde{\mu}$ by the second part of Lemma 2.20 is invariant for the semigroup $(P_t)_{t\geq 0}$.

Proof. We start to show that the measure $\widetilde{\mu} \in \mathcal{P}(L_0^1(\mathbb{T}))$ gives full weight to $H_0^2(\mathbb{T})$. Thanks to Lemma 2.17, since $T_r \underset{r\to\infty}{\longrightarrow} +\infty$ almost surely, we have:

$$\forall T > 0, \quad \frac{1}{T} \int_0^T \mathbb{E}\left[\|u(s)\|_{H_0^2(\mathbb{T})}^2 \right] \mathrm{d}s \leq \frac{1}{T\nu} \left(\|u_0\|_{H_0^1(\mathbb{T})}^2 + C_7 \left(1 + \|u_0\|_{L_0^{2p_A+2}(\mathbb{T})}^{2p_A+2} \right) \right) + \frac{C_8}{\nu}. \tag{2.50}$$

Let $(v_n)_{n\geq 1}$ be a sequence of $H_0^2(\mathbb{T})$-valued random variables such that $v_n \sim R_{T_n}^* \delta_{u_0}$ and v_n converges in distribution in $L_0^1(\mathbb{T})$ towards a random variable $v \sim \widetilde{\mu}$. From (2.50) and the definition of $(R_T)_{T\geq 0}$, we have

$$\limsup_{n\to\infty} \mathbb{E}\left[\|v_n\|_{H_0^2(\mathbb{T})}^2 \right] = \limsup_{n\to\infty} \frac{1}{T_n} \int_0^{T_n} \mathbb{E}_{u_0} \left[\|u(s)\|_{H_0^2(\mathbb{T})}^2 \right] \mathrm{d}s \leq \frac{C_8}{\nu}.$$

Now, since $\|\cdot\|_{H_0^2(\mathbb{T})}^2$ is lower semi-continuous on $L_0^1(\mathbb{T})$, we get from Portemanteau's theorem:

$$\mathbb{E}\left[\|v\|_{H_0^2(\mathbb{T})}^2 \right] \leq \liminf_{n\to\infty} \mathbb{E}\left[\|v_n\|_{H_0^2(\mathbb{T})}^2 \right] \leq \frac{C_8}{\nu}.$$

In particular, $v \in H_0^2(\mathbb{T})$ almost surely, and thus $\widetilde{\mu}$ gives full weight to $H_0^2(\mathbb{T})$.

We now show that for any $p \geq 1$, $\mathbb{E}[\|v\|_{L_0^p(\mathbb{T})}^p] < +\infty$. Let $p \geq 1$. From Lemma 2.15, we have for all $T > 0$,

$$\frac{1}{T} \int_0^T \mathbb{E}_{u_0} \left[\|u(s)\|_{L_0^p(\mathbb{T})}^p \right] \mathrm{d}s \leq \frac{C_5^{(p)}}{T} \left(1 + \|u_0\|_{L_0^p(\mathbb{T})}^p \right) + C_6^{(p)}.$$

Once again, we use Portemanteau's theorem and the lower semi-continuity, this time of $\|\cdot\|_{L_0^p(\mathbb{T})}^p$, on $L_0^1(\mathbb{T})$:

$$\mathbb{E}\left[\|v\|_{L_0^p(\mathbb{T})}^p \right] \leq \liminf_{n\to\infty} \mathbb{E}\left[\|v_n\|_{L_0^p(\mathbb{T})}^p \right] = \liminf_{n\to\infty} \frac{1}{T_n} \int_0^{T_n} \mathbb{E}_{u_0} \left[\|u(s)\|_{L_0^p(\mathbb{T})}^p \right] \mathrm{d}s \leq C_6^{(p)},$$

and the wanted result follows.

To prove the invariance of the measure μ with respect to $(P_t)_{t\geq 0}$, we wish to apply the Krylov-Bogoliubov theorem [36, Theorem 3.1.1]. However, $(P_t)_{t\geq 0}$ is a Feller semigroup on the space $H_0^2(\mathbb{T})$ (Corollary 2.5) whereas our tightness result (Lemma 2.22) holds in $\mathcal{P}(L_0^1(\mathbb{T}))$. To overcome this inconvenience, we use Lemma 2.20 and we place ourselves at the level of the embedded probability measures in $\mathcal{P}(L_0^1(\mathbb{T}))$, where we can adapt, thanks to Proposition 2.21, the proof of [36, Theorem 3.1.1].

Let $\mu \in \mathcal{P}(H_0^2(\mathbb{T}))$ be associated with $\widetilde{\mu}$ by the second part of Lemma 2.20, and let $\varphi \in C_b(L_0^1(\mathbb{T}))$. In particular, the restriction $\varphi_{|H_0^2(\mathbb{T})}$ is bounded and continuous on $H_0^2(\mathbb{T})$ and we can write

$$\int_{H_0^2(\mathbb{T})} \varphi \mathrm{d}P_t^*\mu = \int_{H_0^2(\mathbb{T})} P_t\varphi \mathrm{d}\mu. \tag{2.51}$$

It follows from the $L_0^1(\mathbb{T})$-contraction property that the map $P_t\varphi : H_0^2(\mathbb{T}) \to \mathbb{R}$ is continuous with respect to the $L_0^1(\mathbb{T})$-norm. To prove this fact, let $v_0 \in H_0^2(\mathbb{T})$ and let $(v_0^{(j)})_{j\geq 1}$ be a sequence of $H_0^2(\mathbb{T})$ such that $\|v_0^{(j)} - v_0\|_{L_0^1(\mathbb{T})} \to 0$, $j \to +\infty$. Let $(v(t))_{t\geq 0}$ and $(v^{(j)}(t))_{t\geq 0}$, $j \geq 1$, be the strong solutions of (2.1) respectively with initial conditions v_0 and $v_0^{(j)}$, $j \geq 1$. From Proposition 2.21, we get almost surely and for all $t \geq 0$,

$$\lim_{j\to\infty} \left\| v^{(j)}(t) - v(t) \right\|_{L_0^1(\mathbb{T})} = 0.$$

Since φ is bounded and continuous with respect to the $L_0^1(\mathbb{T})$-norm, we have

$$\lim_{j\to\infty} \left| P_t\varphi\left(v_0^{(j)}\right) - P_t\varphi(v_0) \right| \leq \lim_{j\to\infty} \mathbb{E}\left[\left| \varphi\left(v^{(j)}(t)\right) - \varphi(v(t)) \right| \right] = 0,$$

so that $P_t\varphi$ is continuous with respect to the $L_0^1(\mathbb{T})$-norm.

As a consequence, from Lemma 2.22, we have for all $t \geq 0$

$$\begin{aligned}
\int_{H_0^2(\mathbb{T})} P_t\varphi \mathrm{d}\mu &= \int_{L_0^1(\mathbb{T})} P_t\varphi \mathrm{d}\widetilde{\mu} \\
&= \lim_{n\to\infty} \int_{L_0^1(\mathbb{T})} P_t\varphi \mathrm{d}\widetilde{R}_{T^n}^* \delta_{u_0} \\
&= \lim_{n\to\infty} \int_{H_0^2(\mathbb{T})} P_t\varphi \mathrm{d}R_{T^n}^* \delta_{u_0} \\
&= \lim_{n\to\infty} \frac{1}{T^n} \int_0^{T^n} \int_{H_0^2(\mathbb{T})} \varphi \mathrm{d}P_{s+t}^* \delta_{u_0} \mathrm{d}s \\
&= \lim_{n\to\infty} \frac{1}{T^n} \int_t^{T^n+t} \int_{H_0^2(\mathbb{T})} \varphi \mathrm{d}P_s^* \delta_{u_0} \mathrm{d}s \\
&= \lim_{n\to\infty} \left(\frac{1}{T^n} \int_0^{T^n} \int_{H_0^2(\mathbb{T})} \varphi \mathrm{d}P_s^* \delta_{u_0} \mathrm{d}s + \frac{1}{T^n} \int_{T^n}^{T^n+t} \int_{H_0^2(\mathbb{T})} \varphi \mathrm{d}P_s^* \delta_{u_0} \mathrm{d}s - \frac{1}{T^n} \int_0^t \int_{H_0^2(\mathbb{T})} \varphi \mathrm{d}P_s^* \delta_{u_0} \mathrm{d}s \right) \\
&= \lim_{n\to\infty} \int_{H_0^2(\mathbb{T})} \varphi \mathrm{d}R_{T^n}^* \delta_{u_0} \\
&= \lim_{n\to\infty} \int_{L_0^1(\mathbb{T})} \varphi \mathrm{d}\widetilde{R}_{T^n}^* \delta_{u_0} = \int_{L_0^1(\mathbb{T})} \varphi \mathrm{d}\widetilde{\mu}.
\end{aligned}$$

For any $t \geq 0$, $P_t^*\mu$ gives full weight to $H_0^2(\mathbb{T})$ and therefore, following the first part of Lemma 2.20, we can define the associated Borel probability measure on $L_0^1(\mathbb{T})$ by $\widetilde{P}_t^*\mu = P_t^*\mu(\cdot \cap H_0^2(\mathbb{T}))$. From Equation (2.51) and the above sequence of computations, it follows that for all $t \geq 0$,

$$\int_{L_0^1(\mathbb{T})} \varphi \mathrm{d}\widetilde{P}_t^*\mu = \int_{L_0^1(\mathbb{T})} \varphi \mathrm{d}\widetilde{\mu},$$

Given that φ has been chosen arbitrarily in $C_b(L_0^1(\mathbb{T}))$, this last equality says that $\widetilde{P}_t^*\mu = \widetilde{\mu}$. The second part of Lemma 2.20 now ensures that $P_t^*\mu = \mu$. □

2.3.3 Uniqueness

The proof of the uniqueness part of Theorem 2.7 follows the ideas of the "small-noise" coupling argument from Dirr and Souganidis [43]. On one hand, due to the dissipative nature of the drift, two solutions of (2.1) perturbed by the same noise and starting from different initial conditions are driven to balls of $L_0^2(\mathbb{T})$ with small radius whenever this noise is small over sufficiently long time intervals. On the other hand, the $L_0^1(\mathbb{T})$-contraction property ensures that when these two solutions get close to one another they stay close forever. Hence, each time the noise gets small enough, the two solutions get closer and closer and eventually, they show the same asymptotical behaviour. This idea allows to show that the law of two solutions have the same limit as the time goes to infinity. Therefore, starting from two invariant measures leads to the equality of these measures. The same kind of argument was used in [41] for the invariant measure of kinetic solutions of inviscid scalar conservation laws and in [38] for the stochastic Navier-Stokes equations.

Let $(u(t))_{t\geq 0}$ and $(v(t))_{t\geq 0}$ be two solutions of (2.1) driven by the same Q-Wiener process $(W^Q(t))_{t\geq 0}$. For all $R>0$, we define the stopping time:

$$\tau_R := \inf\left\{t\geq 0 : \|u(t)\|^2_{H_0^1(\mathbb{T})} + \|v(t)\|^2_{H_0^1(\mathbb{T})} \leq R\right\}.$$

Lemma 2.24. *Under Assumptions 2.1 and 2.2, there exists $R>0$ such that for any u_0 and v_0 in $H_0^2(\mathbb{T})$, the stopping time τ_R is finite almost surely.*

Proof. We can use here, from the statement of Lemma 2.15, the inequality (2.17) with $p=2$. In this case, we get

$$2\nu\mathbb{E}\left[\int_0^{t\wedge\tau_R}\left(\|u(s)\|^2_{H_0^1(\mathbb{T})} + \|v(s)\|^2_{H_0^1(\mathbb{T})}\right)\mathrm{d}s\right] \leq \|u_0\|^2_{L_0^2(\mathbb{T})} + \|v_0\|^2_{L_0^2(\mathbb{T})} + 2D_0\mathbb{E}[t\wedge\tau_R],$$

from which we deduce, by definition of the stopping time τ_R, that

$$2\nu R\mathbb{E}[t\wedge\tau_R] \leq \|u_0\|^2_{L_0^2(\mathbb{T})} + \|v_0\|^2_{L_0^2(\mathbb{T})} + 2D_0\mathbb{E}[t\wedge\tau_R].$$

Taking $R > D_0/\nu$ yields

$$\mathbb{E}[\tau_R] = \lim_{t\to\infty}\mathbb{E}[\tau_R\wedge t] \leq \frac{\|u_0\|^2_{L_0^2(\mathbb{T})} + \|v_0\|^2_{L_0^2(\mathbb{T})}}{2(\nu R - D_0)} < +\infty,$$

from which we derive the wanted result. □

The following result asserts that when the coupled processes $(u(t))_{t\geq 0}$ and $(v(t))_{t\geq 0}$ start from deterministic initial conditions inside some ball of $L_0^2(\mathbb{T})$, then they both attain in finite time any neighbourhood of 0 with positive probability:

Lemma 2.25. *Under Assumptions 2.1 and 2.2, for any $M>0$ and any $\varepsilon>0$, there exist a time $t_{\varepsilon,M}>0$ and a value $p_{\varepsilon,M}\in(0,1)$ such that for all $u_0, v_0\in H_0^2(\mathbb{T})$ satisfying $\|u_0\|^2_{H_0^1(\mathbb{T})} + \|v_0\|^2_{H_0^1(\mathbb{T})} \leq M$,*

$$\mathbb{P}\left(\|u(t_{\varepsilon,M})\|^2_{L_0^2(\mathbb{T})} + \|v(t_{\varepsilon,M})\|^2_{L_0^2(\mathbb{T})} \leq \varepsilon\right) \geq p_{\varepsilon,M}.$$

Proof. Let u_0, $v_0\in H_0^2(\mathbb{T})$ be such that $\|u_0\|_{H_0^1(\mathbb{T})} + \|v_0\|_{H_0^1(\mathbb{T})} \leq M$, and let us define

$$t_{\varepsilon,M} = -\frac{1}{2\nu}\log\left(\frac{\varepsilon}{4M}\right).$$

To prove the lemma, we are going to compare the trajectories of $(u(t))_{t\geq 0}$ and $(v(t))_{t\geq 0}$ with the trajectories of their noiseless counterparts $(\overline{u}(t))_{t\geq 0}$ and $(\overline{v}(t))_{t\geq 0}$, defined by

$$\begin{cases} \partial_t \overline{u}(t) = -\partial_x A\left(\overline{u}(t)\right) + \nu \partial_{xx} \overline{u}(t) \\ \overline{u}(0) = u_0 \end{cases} \qquad \begin{cases} \partial_t \overline{v}(t) = -\partial_x A\left(\overline{v}(t)\right) + \nu \partial_{xx} \overline{v}(t) \\ \overline{v}(0) = v_0. \end{cases}$$

Recall that the viscosity yields energy dissipation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\overline{u}(t)\|^2_{L^2_0(\mathbb{T})} + \|\overline{v}(t)\|^2_{L^2_0(\mathbb{T})}\right) = -2\nu\left(\|\overline{u}(t)\|^2_{H^1_0(\mathbb{T})} + \|\overline{v}(t)\|^2_{H^1_0(\mathbb{T})}\right).$$

Applying (2.3) on the right-hand side, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\overline{u}(t)\|^2_{L^2_0(\mathbb{T})} + \|\overline{v}(t)\|^2_{L^2_0(\mathbb{T})}\right) \leq -2\nu\left(\|\overline{u}(t)\|^2_{L^2_0(\mathbb{T})} + \|\overline{v}(t)\|^2_{L^2_0(\mathbb{T})}\right),$$

and we can now apply Grönwall's lemma:

$$\|\overline{u}(t)\|^2_{L^2_0(\mathbb{T})} + \|\overline{v}(t)\|^2_{L^2_0(\mathbb{T})} \leq \left(\|u_0\|^2_{L^2_0(\mathbb{T})} + \|v_0\|^2_{L^2_0(\mathbb{T})}\right)\mathrm{e}^{-2\nu t} \leq M\mathrm{e}^{-2\nu t}.$$

With our choice of $t_{\varepsilon,M}$, the above inequality means that as soon as $t \geq t_{\varepsilon,M}$, we have $\|\overline{u}(t)\|^2_{L^2_0(\mathbb{T})} + \|\overline{v}(t)\|^2_{L^2_0(\mathbb{T})} \leq \varepsilon/4$.

Furthermore, it is a consequence of Lemma 2.17 that $(\overline{u}(t))_{t\geq 0}$ satisfies

$$\|\overline{u}(t)\|^2_{H^1_0(\mathbb{T})} \leq \|u_0\|^2_{H^1_0(\mathbb{T})} + C_7\left(1 + \|u_0\|^{2p_A+2}_{L^{2p_A+2}_0(\mathbb{T})}\right), \quad t \geq 0.$$

Indeed, when all the noise coefficients g_k are equal to zero, the constant C_8 in the statement of Lemma 2.17 can also be taken equal to zero. Since the same inequality also applies to $(\overline{v}(t))_{t\geq 0}$, we have

$$\|\overline{u}(t)\|^2_{H^1_0(\mathbb{T})} + \|\overline{v}(t)\|^2_{H^1_0(\mathbb{T})} \leq M + 2C_7\left(1 + M^{p_A+1}\right) =: C_9^{(M)}.$$

We focus now on the trajectories of the random processes $(u(t))_{t\geq 0}$ and $(v(t))_{t\geq 0}$. We introduce the stopping time

$$\widetilde{\tau}_M := \inf\left\{t \geq 0 : \|u(t)\|_{H^1_0(\mathbb{T})} \vee \|v(t)\|_{H^1_0(\mathbb{T})} \geq \frac{1}{2} + \sqrt{C_9^{(M)}}\right\}.$$

Following Proposition 2.12, we may use the expressions of $(u(t))_{t\geq 0}$ and $(\overline{u}(t))_{t\geq 0}$ in the mild sense. From these mild formulations, we write

$$\|u(t) - \overline{u}(t)\|_{H^1_0(\mathbb{T})} \leq \int_0^t \|S_{t-s}\partial_x\left(A(u(s)) - A(\overline{u}(s))\right)\|_{H^1_0(\mathbb{T})}\mathrm{d}s + \|w(t)\|_{H^1_0(\mathbb{T})}, \tag{2.52}$$

where $(w(t))_{t\geq 0}$ is the stochastic convolution associated with the Q-Wiener process $(W^Q(t))_{t\geq 0}$. According to Proposition 2.8, we call L_M a local Lipschitz constant of the map $z \in H^1_0(\mathbb{T}) \mapsto \partial_x A(z) \in L^2_0(\mathbb{T})$ over the ball $\{z \in H^1_0(\mathbb{T}) : \|z\|^2_{H^1_0(\mathbb{T})} \leq \frac{1}{2} + \sqrt{C_9^{(M)}}\}$, and we place ourselves in the event

$$\left\{\sup_{t\in[0,t_{\varepsilon,M}]} \|w(t)\|_{H^1_0(\mathbb{T})} \leq \delta_{\varepsilon,M}\right\}, \qquad \text{where} \quad \delta_{\varepsilon,M} := \frac{\sqrt{\varepsilon}}{2\sqrt{2}}\frac{1}{1 + 2\sqrt{t_{\varepsilon,M}}C_4 L_M}\mathrm{e}^{-C_4^2 L_M^2 t_{\varepsilon,M}},$$

where C_4 has been defined at Proposition 2.8. Taking $t \leq \widetilde{\tau}_M \wedge t_{\varepsilon,M}$, applying the second part of Proposition 2.9 and Proposition 2.8 to (2.52), we get

$$\begin{aligned}\|u(t)-\overline{u}(t)\|_{H_0^1(\mathbb{T})} &\leq \int_0^t \frac{C_4}{\sqrt{t-s}}\,\|\partial_x(A(u(s))-A(\overline{u}(s)))\|_{L_0^2(\mathbb{T})}\,\mathrm{d}s+\delta_{\varepsilon,M}\\ &\leq \int_0^t \frac{C_4L_M}{\sqrt{t-s}}\|u(s)-\overline{u}(s)\|_{H_0^1(\mathbb{T})}\mathrm{d}s+\delta_{\varepsilon,M}.\end{aligned}$$

Iterating this inequality and using the same arguments as in the proof of Lemma 2.19, we get for all $t \leq t_{\varepsilon,M}\wedge\widetilde{\tau}_M$,

$$\|u(t)-\overline{u}(t)\|_{H_0^1(\mathbb{T})} \leq \delta_{\varepsilon,M}\left(1+2\sqrt{t_{\varepsilon,M}\wedge\widetilde{\tau}_M}C_4L_M\right)+C_4^2L_M^2\pi\int_0^t\|u(s)-\overline{u}(s)\|_{H_0^1(\mathbb{T})}\mathrm{d}s.$$

Using now Grönwall's lemma, we deduce

$$\|u(t)-\overline{u}(t)\|_{H_0^1(\mathbb{T})} \leq \delta_{\varepsilon,M}\left(1+2\sqrt{t_{\varepsilon,M}\wedge\widetilde{\tau}_M}C_4L_M\right)\mathrm{e}^{C_4^2L_M^2\pi t} \leq \frac{\sqrt{\varepsilon}}{2\sqrt{2}}.$$

Since the same arguments apply for the processes $(v(t))_{t\geq0}$ and $(\overline{v}(t))_{t\geq0}$, and given Equation (2.3), we have shown that for all $t\leq\widetilde{\tau}_M\wedge t_{\varepsilon,M}$,

$$\begin{aligned}\|u(t)\|^2_{L_0^2(\mathbb{T})}+\|v(t)\|^2_{L_0^2(\mathbb{T})} &\leq 2\left(\|\overline{u}(t)\|^2_{L_0^2(\mathbb{T})}+\|\overline{v}(t)\|^2_{L_0^2(\mathbb{T})}\right)+2\left(\|u(t)-\overline{u}(t)\|^2_{L_0^2(\mathbb{T})}+\|v(t)-\overline{v}(t)\|^2_{L_0^2(\mathbb{T})}\right)\\ &\leq \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.\end{aligned}$$

We shall prove now that the event $\widetilde{\tau}_M < t_{\varepsilon,M}$ is impossible. Indeed, assume for instance that $\|u\left(\widetilde{\tau}_M\right)\|_{H_0^1(\mathbb{T})} \geq \frac{1}{2}+\sqrt{C_9^{(M)}}$, then we would have

$$\|u\left(\widetilde{\tau}_M\right)-\overline{u}\left(\widetilde{\tau}_M\right)\|_{H_0^1(\mathbb{T})} \leq \frac{\sqrt{\varepsilon}}{2} \quad\text{and}\quad \|\overline{u}\left(\widetilde{\tau}_M\right)\|^2_{H_0^1(\mathbb{T})}\leq C_9^{(M)},$$

and thus,

$$\frac{\sqrt{\varepsilon}}{2}\geq\|u\left(\widetilde{\tau}_M\right)-\overline{u}\left(\widetilde{\tau}_M\right)\|_{H_0^1(\mathbb{T})}\geq\left|\|u\left(\widetilde{\tau}_M\right)\|_{H_0^1(\mathbb{T})}-\|\overline{u}\left(\widetilde{\tau}_M\right)\|_{H_0^1(\mathbb{T})}\right|\geq\left(\frac{1}{2}+\sqrt{C_9^{(M)}}\right)-\sqrt{C_9^{(M)}}=\frac{1}{2},$$

which is false for too small values of ε.

We just have proved that for $M>0$ arbitrarily chosen and for all $u_0,v_0\in H_0^2(\mathbb{T})$ such that $\|u_0\|^2_{H_0^1(\mathbb{T})}+\|v_0\|^2_{H_0^1(\mathbb{T})}\leq M$, we have

$$\mathbb{P}\left(\|u(t_{\varepsilon,M})\|^2_{L_0^2(\mathbb{T})}+\|v(t_{\varepsilon,M})\|^2_{L_0^2(\mathbb{T})}\leq\varepsilon\right)\geq\mathbb{P}\left(\sup_{t\in[0,t_{\varepsilon,M}]}\|w(t)\|_{H_0^1(\mathbb{T})}\leq\delta_{\varepsilon,M}\right).$$

To conclude the proof, it remains to check that

$$p_{\varepsilon,M}:=\mathbb{P}\left(\sup_{t\in[0,t_{\varepsilon,M}]}\|w(t)\|_{H_0^1(\mathbb{T})}\leq\delta_{\varepsilon,M}\right)>0. \tag{2.53}$$

We can write $\{\sup_{t\in[0,t_{\varepsilon,M}]}\|w(t)\|_{H_0^1(\mathbb{T})}\leq\delta_{\varepsilon,M}\}=\{(w(t))_{t\in[0,t_{\varepsilon,M}]}\in B\}$ where B is the closed ball of $C([0,t_{\varepsilon,M}],H_0^1(\mathbb{T}))$ with radius $\delta_{\varepsilon,M}$. Since the process $(w(t))_{t\in[0,t_{\varepsilon,M}]}$ is the mild solution to the stochastic heat equation (*i.e.* Equation (2.7) with initial condition $w(0)\equiv0$ and flux $A\equiv0$), we can apply the support theorem from [95, Theorem 1.1] which implies $\mathbb{P}((w(t))_{t\in[0,t_{\varepsilon,M}]}\in B)>0$, so that (2.53) is satisfied. □

Lemma 2.26. *Under Assumptions 2.1 and 2.2, any invariant measure μ for the process $(u(t))_{t\geq 0}$ solution to (2.1) is unique.*

Proof. **Step 1. Almost sure confluence.** We start by fixing $\varepsilon > 0$ small to which we associate the value $t_{\varepsilon,R}$ defined at Lemma 2.25, where R has been defined at Lemma 2.24. We define the increasing stopping time sequence

$$\begin{aligned}
\mathbf{T}_1 &:= \tau_R \\
\mathbf{T}_2 &:= \inf\left\{t \geq \mathbf{T}_1 + t_{\varepsilon,R} : \|u(t)\|^2_{H^1_0(\mathbb{T})} + \|v(t)\|^2_{H^1_0(\mathbb{T})} \leq R\right\} \\
\mathbf{T}_3 &:= \inf\left\{t \geq \mathbf{T}_2 + t_{\varepsilon,R} : \|u(t)\|^2_{H^1_0(\mathbb{T})} + \|v(t)\|^2_{H^1_0(\mathbb{T})} \leq R\right\} \\
&\vdots
\end{aligned}$$

Lemma 2.24 and the strong Markov property (Corollary 2.5) ensure that every $\mathbf{T}_j$ is finite almost surely. We claim that

$$\forall J \in \mathbb{N}^*, \qquad \mathbb{P}\left(\forall j = 1, \dots, J, \quad \|u(\mathbf{T}_j + t_{\varepsilon,R})\|^2_{L^2_0(\mathbb{T})} + \|v(\mathbf{T}_j + t_{\varepsilon,R})\|^2_{L^2_0(\mathbb{T})} > \varepsilon\right) \leq (1 - p_{\varepsilon,R})^J. \tag{2.54}$$

Indeed, it is true for $J = 1$ thanks to the strong Markov property and Lemma 2.25:

$$\begin{aligned}
&\mathbb{P}_{(u_0,v_0)}\left(\|u(\tau_R + t_{\varepsilon,R})\|^2_{L^2_0(\mathbb{T})} + \|v(\tau_R + t_{\varepsilon,R})\|^2_{L^2_0(\mathbb{T})} > \varepsilon\right) \\
&= \mathbb{E}_{(u_0,v_0)}\left[\mathbb{P}_{(u_0,v_0)}\left(\|u(\tau_R + t_{\varepsilon,R})\|^2_{L^2_0(\mathbb{T})} + \|v(\tau_R + t_{\varepsilon,R})\|^2_{L^2_0(\mathbb{T})} > \varepsilon | \mathcal{F}_{\tau_R}\right)\right] \\
&= \mathbb{E}_{(u_0,v_0)}\left[\mathbb{P}_{(u(\tau_R),v(\tau_R))}\left(\|u(t_{\varepsilon,R})\|^2_{L^2_0(\mathbb{T})} + \|v(t_{\varepsilon,R})\|^2_{L^2_0(\mathbb{T})} > \varepsilon\right)\right] \\
&\leq 1 - p_{\varepsilon,R},
\end{aligned}$$

and the general case follows by induction: assuming that inequality (2.54) is true for some $J \in \mathbb{N}^*$, we have

$$\begin{aligned}
&\mathbb{P}_{(u_0,v_0)}\left(\forall j = 1, \dots, J+1, \quad \|u(\mathbf{T}_j + t_{\varepsilon,R})\|^2_{L^2_0(\mathbb{T})} + \|v(\mathbf{T}_j + t_{\varepsilon,R})\|^2_{L^2_0(\mathbb{T})} > \varepsilon\right) \\
&= \mathbb{E}_{(u_0,v_0)}\left[\mathbb{P}_{(u_0,v_0)}\left(\forall j = 1, \dots, J+1, \quad \|u(\mathbf{T}_j + t_{\varepsilon,R})\|^2_{L^2_0(\mathbb{T})} + \|v(\mathbf{T}_j + t_{\varepsilon,R})\|^2_{L^2_0(\mathbb{T})} > \varepsilon | \mathcal{F}_{\mathbf{T}_{J+1}}\right)\right] \\
&= \mathbb{E}_{(u_0,v_0)}\left[\left(\prod_{j=1}^{J} \mathbf{1}_{\|u(\mathbf{T}_j + t_{\varepsilon,R})\|^2_{L^2_0(\mathbb{T})} + \|v(\mathbf{T}_j + t_{\varepsilon,R})\|^2_{L^2_0(\mathbb{T})} > \varepsilon}\right) \mathbb{P}_{(u(\mathbf{T}_{J+1}),v(\mathbf{T}_{J+1}))}\left(\|u(t_{\varepsilon,R})\|^2_{L^2_0(\mathbb{T})} + \|v(t_{\varepsilon,R})\|^2_{L^2_0(\mathbb{T})} > \varepsilon\right)\right] \\
&\leq (1 - p_{\varepsilon,R})^J \times (1 - p_{\varepsilon,R}) = (1 - p_{\varepsilon,R})^{J+1}.
\end{aligned}$$

Taking the limit when J goes to infinity, we get

$$\begin{aligned}
&\mathbb{P}\left(\forall j \in \mathbb{N}^*, \quad \|u(\mathbf{T}_j + t_{\varepsilon,R})\|^2_{L^2_0(\mathbb{T})} + \|v(\mathbf{T}_j + t_{\varepsilon,R})\|^2_{L^2_0(\mathbb{T})} > \varepsilon\right) \\
&= \lim_{J\to\infty} \mathbb{P}\left(\forall j = 1, \dots, J, \quad \|u(\mathbf{T}_j + t_{\varepsilon,R})\|^2_{L^2_0(\mathbb{T})} + \|v(\mathbf{T}_j + t_{\varepsilon,R})\|^2_{L^2_0(\mathbb{T})} > \varepsilon\right) \\
&\leq \lim_{J\to\infty} (1 - p_{\varepsilon,R})^J = 0,
\end{aligned}$$

and consequently,

$$\mathbb{P}\left(\exists t \geq 0, \quad \|u(t)\|^2_{L^2_0(\mathbb{T})} + \|v(t)\|^2_{L^2_0(\mathbb{T})} \leq \varepsilon\right) = 1. \tag{2.55}$$

Since $\|u(t)-v(t)\|_{L_0^1(\mathbb{T})}^2 \leq \|u(t)-v(t)\|_{L_0^2(\mathbb{T})}^2 \leq 2(\|u(t)\|_{L_0^2(\mathbb{T})}^2 + \|v(t)\|_{L_0^2(\mathbb{T})}^2)$ and since the value $\varepsilon > 0$ has been chosen arbitrarily at the beginning of this proof, then Equality (2.55) means that almost surely,

$$\forall \varepsilon > 0, \quad \exists t \geq 0, \quad \|u(t)-v(t)\|_{L_0^1(\mathbb{T})}^2 \leq 2\varepsilon.$$

Recall however that Proposition 2.21 states that almost surely, the mapping $t \mapsto \|u(t)-v(t)\|_{L_0^1(\mathbb{T})}$ is non-decreasing. It follows that almost surely,

$$\lim_{t\to\infty} \|u(t)-v(t)\|_{L_0^1(\mathbb{T})} = 0. \tag{2.56}$$

Step 2. Uniqueness. Let us now assume that there exist two invariant measures μ_1, μ_2 for the solution of (2.1), and let us take initial conditions u_0 and v_0 with distributions μ_1 and μ_2 respectively. For any test function $\phi : L_0^1(\mathbb{T}) \to \mathbb{R}$ bounded and Lipschitz continuous, we have for all $t \geq 0$,

$$|\mathbb{E}[\phi(u_0)] - \mathbb{E}[\phi(v_0)]| = |\mathbb{E}[\phi(u(t))] - \mathbb{E}[\phi(v(t))]| \leq \mathbb{E}[|\phi(u(t)) - \phi(v(t))|].$$

Since ϕ is Lipschitz continuous, from (2.56), we have almost surely

$$\lim_{t\to\infty} |\phi(u(t)) - \phi(v(t))| = 0.$$

Moreover, for any $t \geq 0$, we have almost surely $|\phi(u(t)) - \phi(v(t))| \leq 2\sup|\phi|$. Thus, we may apply the dominated convergence theorem, which yields

$$|\mathbb{E}[\phi(u_0)] - \mathbb{E}[\phi(v_0)]| \leq \lim_{t\to\infty} \mathbb{E}[|\phi(u(t)) - \phi(v(t))|] = 0,$$

so that $\mathbb{E}[\phi(u_0)] = \mathbb{E}[\phi(v_0)]$, or in other words,

$$\int_{H_0^2(\mathbb{T})} \phi \mathrm{d}\mu_1 = \int_{H_0^2(\mathbb{T})} \phi \mathrm{d}\mu_2. \tag{2.57}$$

According to Lemma 2.20, let $\widetilde{\mu}_1$ and $\widetilde{\mu}_2$ be the probability measures on $\mathcal{P}(L_0^1(\mathbb{T}))$ associated to μ_1 and μ_2 respectively. Equation (2.57) rewrites

$$\int_{L_0^1(\mathbb{T})} \phi \mathrm{d}\widetilde{\mu}_1 = \int_{L_0^1(\mathbb{T})} \phi \mathrm{d}\widetilde{\mu}_2, \qquad \forall \phi \in C_b\left(L_0^1(\mathbb{T})\right),$$

so that $\widetilde{\mu}_1 = \widetilde{\mu}_2$ and thus, by Lemma 2.20, $\mu_1 = \mu_2$. □

Proof of Theorem 2.7. It follows from Lemmas 2.23 and 2.26. □

Chapter 3

Finite-volume approximation of the invariant measure of a viscous stochastic scalar conservation law

Résumé. *Ce chapitre correspond aux quatre premières sections de la pré-publication [19], écrite en collaboration avec S. Boyaval et J. Reygner. On se propose d'établir une approximation numérique de la mesure invariante de la solution de l'équation étudiée dans le chapitre 2, c'est-à-dire d'une loi de conservation scalaire avec viscosité, uni-dimensionnelle et périodique en espace, et forcée aléatoirement avec un bruit blanc en temps mais spatialement correlé. La fonction de flux est supposée localement lipschitzienne et à croissance polynomiale. Le schéma numérique utilisé discrétise l'EDPS en espace selon la méthode des volumes finis, et en temps selon une méthode d'Euler à pas fractionné. En premier lieu, pour la semi-discrétisation spatiale puis pour le schéma totalement discrétisé, on prouve l'existence et l'unicité d'une solution puis d'une mesure invariante pour cette solution. Le résultat principal est alors la convergence des mesures invariantes de ces approximations, lorsque les pas de temps et d'espace tendent vers zéro, vers l'unique mesure invariante de l'EDPS par rapport à la distance de Wasserstein d'ordre deux.*

Abstract. This chapter corresponds to the first four sections of the preprint [19], written in collaboration with S. Boyaval and J. Reygner. We aim to give a numerical approximation of the invariant measure of the solution of the equation studied in Chapter 2, that is, a viscous scalar conservation law, one-dimensional and periodic in the space variable, and stochastically forced with a white-in-time but spatially correlated noise. The flux function is assumed to be locally Lipschitz and to have at most polynomial growth. The numerical scheme we employ discretises the SPDE according to a finite volume method in space, and a split-step backward Euler method in time. As a first result, we prove the well-posedness as well as the existence and uniqueness of an invariant measure for both the spatial semi-discretisation and the fully discrete scheme. Our main result is then the convergence of the invariant measures of the discrete approximations, as the space and time steps go to zero, towards the invariant measure of the SPDE, with respect to the second-order Wasserstein distance.

3.1 Introduction

3.1.1 Viscous scalar conservation law with random forcing

We consider the following viscous scalar conservation law with stochastic forcing

$$\mathrm{d}u = -\partial_x A(u)\mathrm{d}t + \nu\partial_{xx}u\mathrm{d}t + \sum_{k\geq 1} g_k \mathrm{d}W^k(t), \quad x \in \mathbb{T}, \quad t \geq 0. \tag{3.1}$$

Periodic boundary conditions are assigned over the space variable x as $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ denotes the one-dimensional torus, and $(W^k)_{k\geq 1}$ is a family of independent real Brownian motions. The viscosity coefficient ν is assumed to be positive. In Chapter 2, we have shown the well-posedness in a strong sense of Equation (3.1), as well as the existence and uniqueness of an invariant measure for its solution. These results are recalled in Proposition 3.2 below. In this work, we aim to provide a numerical scheme, based on the finite-volume method, that allows to approximate this invariant measure. In this perspective, we place ourselves in the setting of Chapter 2 and recall our main notations and assumptions.

Notations. For all $p \in [1, +\infty]$, we denote by $L^p_0(\mathbb{T})$ the set of functions $f \in L^p(\mathbb{T})$ such that $\int_{\mathbb{T}} f(x)\mathrm{d}x = 0$. We write $\|\cdot\|_{L^p_0(\mathbb{T})}$ the L^p-norm induced on $L^p_0(\mathbb{T})$ and $\langle\cdot,\cdot\rangle_{L^2_0(\mathbb{T})}$ the L^2-scalar product induced on $L^2_0(\mathbb{T})$. In a similar manner, for any integer $m \geq 0$ and any $p \in [1, +\infty]$, we introduce the Sobolev space $W^{m,p}_0(\mathbb{T}) := L^p_0(\mathbb{T}) \cap W^{m,p}(\mathbb{T})$ which we equip with the norm $\|\cdot\|_{W^{m,p}_0(\mathbb{T})} := \|\partial^m_x \cdot\|_{L^p_0(\mathbb{T})}$. Incidentally, we will denote by $H^m_0(\mathbb{T})$ the space $W^{m,2}_0(\mathbb{T})$, which we recall is separable and Hilbert when endowed with the norm $\|\cdot\|_{H^m_0(\mathbb{T})} := \|\cdot\|_{W^{m,2}_0(\mathbb{T})}$ and the associated scalar product $\langle\cdot,\cdot\rangle_{H^m_0(\mathbb{T})}$. We recall the following inequalities: for all $1 \leq p \leq q \leq +\infty$,

$$\|u\|_{L^p_0(\mathbb{T})} \leq \|u\|_{L^q_0(\mathbb{T})}, \quad \forall u \in L^q_0(\mathbb{T}), \tag{3.2}$$

and

$$\|u\|_{L^\infty_0(\mathbb{T})} \leq \|u\|_{W^{1,1}_0(\mathbb{T})}, \quad \forall u \in W^{1,1}_0(\mathbb{T}). \tag{3.3}$$

In the sequel, we denote by $\mathbb{N}$ the set of non-negative integers, and by $\mathbb{N}^*$ the set of positive integers.

Assumption 3.1. *The function $A : \mathbb{R} \to \mathbb{R}$ is of class C^2, its first derivative has at most polynomial growth:*

$$\exists C_A > 0, \quad \exists p_A \in \mathbb{N}^*, \quad \forall v \in \mathbb{R}, \qquad |A'(v)| \leq C_A \left(1 + |v|^{p_A}\right), \tag{3.4}$$

and its second derivative A'' is locally Lipschitz continuous on $\mathbb{R}$. Furthermore, for all $k \geq 1$, $g_k \in H^2_0(\mathbb{T})$ and

$$D_0 := \sum_{k\geq 1} \|g_k\|^2_{H^2_0(\mathbb{T})} < +\infty. \tag{3.5}$$

The assumptions (3.4) and (3.5) will be needed in the arguments contained in this chapter while the local Lipschitz continuity of A'' is only necessary for Proposition 3.2.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, equipped with a normal filtration $(\mathcal{F}_t)_{t\geq 0}$ in the sense of [35, Section 3.3], on which $(W^k)_{k\geq 1}$ is a family of independent Brownian motions. Under Assumption 3.1, the series $\sum_k g_k W^k$ converges in $L^2(\Omega, C([0,T], H^2_0(\mathbb{T})))$, for any $T > 0$, towards an $H^2_0(\mathbb{T})$-valued Wiener process $(W^Q(t))_{t\in[0,T]}$ with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$, defined in the sense of [35, Section 4.2], with the trace class covariance operator Q defined by

$$Q : \begin{cases} H^2_0(\mathbb{T}) & \longrightarrow H^2_0(\mathbb{T}) \\ v & \longmapsto \sum_{k\geq 1} g_k \langle v, g_k\rangle_{H^2_0(\mathbb{T})}. \end{cases}$$

Given a normed vector space E, $\mathcal{B}(E)$ denotes the Borel sets of E, $\mathcal{P}(E)$ denotes the set of Borel probability measures over E, and for $p \in [1, +\infty)$, $\mathcal{P}_p(E)$ denotes the subset of $\mathcal{P}(E)$ of probability measures with finite p-th order moment. The well-posedness of (3.1) as well as the existence and uniqueness of an invariant measure for its solution is proved in Chapter 2 (Theorems 2.4 and 2.7):

Proposition 3.2. *Let $u_0 \in H^2_0(\mathbb{T})$. Under Assumption 3.1, there exists a unique strong solution $(u(t))_{t\geq 0}$ to Equation (3.1) with initial condition u_0. That is, an $(\mathcal{F}_t)_{t\geq 0}$-adapted process $(u(t))_{t\geq 0}$ with values in $H^2_0(\mathbb{T})$ such that, almost surely:*

1. *the mapping $t \mapsto u(t)$ is continuous from $[0, +\infty)$ to $H^2_0(\mathbb{T})$;*

2. for all $t \geq 0$, the following equality holds:

$$u(t) = u_0 + \int_0^t \left(-\partial_x A\left(u(s)\right) + \nu \partial_{xx} u(s)\right) \mathrm{d}s + W^Q(t). \tag{3.6}$$

Furthermore, the process $(u(t))_{t\geq 0}$ admits a unique invariant measure $\mu \in \mathcal{P}(H_0^2(\mathbb{T}))$. Besides, if v is a random variable with distribution μ, then $\mathbb{E}[\|v\|^2_{H_0^2(\mathbb{T})}] < +\infty$ and for all $p \in [1, +\infty)$, $\mathbb{E}[\|v\|^p_{L_0^p(\mathbb{T})}] < +\infty$.

Let us precise that for any $t \geq 0$, $u(t)$ will always refer to an element of the space $H_0^2(\mathbb{T})$. The scalar values taken by this function are denoted by $u(t,x)$, for $x \in \mathbb{T}$.

3.1.2 Space discretisation

In order to discretise (3.1) with respect to the space variable, we first define a regular mesh $\mathcal{T}$ on the torus:

$$\mathcal{T} := \left\{ \left(\frac{i-1}{N}, \frac{i}{N} \right], i \in \mathbb{Z}/N\mathbb{Z} \right\}.$$

Averaging in (3.1) over each cell of $\mathcal{T}$, we get

$$\begin{aligned}\mathrm{d}\left(N \int_{\frac{i-1}{N}}^{\frac{i}{N}} u(t,x)\mathrm{d}x\right) &= -N\left(A\left(u\left(t,\frac{i}{N}\right)\right) - A\left(u\left(t,\frac{i-1}{N}\right)\right)\right)\mathrm{d}t \\ &+ \nu N\left(\partial_x u\left(t,\frac{i}{N}\right) - \partial_x u\left(t,\frac{i-1}{N}\right)\right)\mathrm{d}t + \sum_{k\geq 1} N\int_{\frac{i-1}{N}}^{\frac{i}{N}} g_k(x)\mathrm{d}x\mathrm{d}W^k(t), \quad i \in \mathbb{Z}/N\mathbb{Z}. \end{aligned} \tag{3.7}$$

Finite-volume schemes aim to approximate the dynamics of the average value of the solution over each cell of the mesh. This leads to the introduction of a *numerical flux* function $\overline{A}(u,v)$ approximating the flux of the conserved quantity at the interface between two adjacent cells. As regards the viscous term in (3.7), we replace the space derivatives by their finite difference approximations. As for the noise coefficients, we introduce the shorthand notation

$$\sigma_i^k := N \int_{\frac{i-1}{N}}^{\frac{i}{N}} g_k(x)\mathrm{d}x, \quad k \geq 1, \quad i \in \mathbb{Z}/N\mathbb{Z}.$$

These operations result in the following stochastic differential equation

$$\begin{aligned}\mathrm{d}U_i(t) &= -N\left(\overline{A}\left(U_i(t), U_{i+1}(t)\right) - \overline{A}\left(U_{i-1}(t), U_i(t)\right)\right)\mathrm{d}t \\ &+ \nu N^2\left(U_{i+1}(t) - 2U_i(t) + U_{i-1}(t)\right)\mathrm{d}t + \sum_{k\geq 1}\sigma_i^k \mathrm{d}W^k(t), \quad i \in \mathbb{Z}/N\mathbb{Z}, \quad t \geq 0, \end{aligned} \tag{3.8}$$

as a semi-discrete finite-volume approximation of (3.1) in the sense that $U_i(t)$ is meant to be an approximation of the spatial average $N\int_{\frac{i-1}{N}}^{\frac{i}{N}} u(t,x)\mathrm{d}x$. We may interpret the noise term (*i.e.* the last term in (3.8)) as a discrete version of the Q-Wiener process $(W^Q(t))_{t\geq 0}$ introduced in Section 3.1.1. Let us notice that the $\mathbb{R}^N$-valued stochastic process $(\mathbf{W}^{Q,N}(t))_{t\geq 0}$ whose components are defined by

$$W_i^{Q,N}(t) := \sum_{k\geq 1}\sigma_i^k W^k(t), \quad i \in \mathbb{Z}/N\mathbb{Z}, \quad t \geq 0,$$

is a Wiener process with the covariance

$$\mathbb{E}\left[W_i^{Q,N}(t) W_j^{Q,N}(t)\right] = t\sum_{k\geq 1}\sigma_i^k \sigma_j^k,$$

which is finite as the Jensen inequality, Assumption 3.1 and (3.3) ensure that for all $i \in \mathbb{Z}/N\mathbb{Z}$,

$$\sum_{k\geq 1}\left|\sigma_i^k\right|_2^2 = \sum_{k\geq 1}\left|N\int_{\frac{i-1}{N}}^{\frac{i}{N}} g_k(x)\mathrm{d}x\right|^2 \leq \sum_{k\geq 1} N\int_{\frac{i-1}{N}}^{\frac{i}{N}} \|g_k\|^2_{L_0^\infty(\mathbb{T})}\,\mathrm{d}x \leq \sum_{k\geq 1}\|g_k\|^2_{H_0^2(\mathbb{T})} \leq D_0. \tag{3.9}$$

Furthermore, each vector $\boldsymbol{\sigma}^k = (\sigma_1^k, \dots, \sigma_N^k)$ satisfies a discrete cancellation condition:

$$\sum_{i=1}^N \sigma_i^k = N\int_{\mathbb{T}} g_k(x)\mathrm{d}x = 0.$$

Thus, denoting

$$\mathbb{R}_0^N := \{\mathbf{u} = (u_1, \dots, u_N) \in \mathbb{R}^N : u_1 + \cdots + u_N = 0\},$$

we get that $(\mathbf{W}^{Q,N}(t))_{t\geq 0}$ is an $\mathbb{R}_0^N$-valued process. We equip the space $\mathbb{R}_0^N$ with the renormalised L^p norm $\|\cdot\|_p$ and scalar product $\langle\cdot,\cdot\rangle$: for any $\mathbf{u}, \mathbf{v} \in \mathbb{R}_0^N$ and any $p \in [1, +\infty)$,

$$\|\mathbf{u}\|_p^p := \frac{1}{N}\sum_{i=1}^N |u_i|^p, \qquad \langle \mathbf{u}, \mathbf{v}\rangle := \frac{1}{N}\sum_{i=1}^N u_i v_i.$$

Furthermore, for any $\mathbf{u} \in \mathbb{R}_0^N$, we set by convention $\|\mathbf{u}\|_0^0 = 1$. Besides, notice that for any $1 \leq p \leq q < +\infty$, we have

$$\|\mathbf{u}\|_p \leq \|\mathbf{u}\|_q, \quad \forall \mathbf{u} \in \mathbb{R}_0^N. \tag{3.10}$$

The drift function in (3.8) is the function $\mathbf{b}$ defined on $\mathbb{R}_0^N$ by the components

$$b_i(\mathbf{v}) := -N\left(\overline{A}(v_i, v_{i+1}) - \overline{A}(v_{i-1}, v_i)\right) + \nu N^2(v_{i+1} - 2v_i + v_{i-1}), \quad i \in \mathbb{Z}/N\mathbb{Z}.$$

These notations being set, we can write the SDE (3.8) in the vectorised form

$$\mathrm{d}\mathbf{U}(t) = \mathbf{b}(\mathbf{U}(t))\mathrm{d}t + \mathrm{d}\mathbf{W}^{Q,N}(t), \quad t \geq 0. \tag{3.11}$$

It appears that $\mathbf{b}$ takes values in $\mathbb{R}_0^N$. As a consequence, Equation (3.11) is conservative in the following sense: if $\mathbf{U}_0 \in \mathbb{R}_0^N$, then for all $t \geq 0$, $\mathbf{U}(t) \in \mathbb{R}_0^N$.

We may now state our assumptions on the numerical flux:

Assumption 3.3. *The function $\overline{A}$ belongs to $C^1(\mathbb{R}^2, \mathbb{R})$, its first derivatives $\partial_1\overline{A}$ and $\partial_2\overline{A}$ are locally Lipschitz continuous on $\mathbb{R}^2$, and it satisfies the following properties:*

(i) Consistency:

$$\forall u \in \mathbb{R}, \quad \overline{A}(u, u) = A(u); \tag{3.12}$$

(ii) Monotonicity:

$$\forall u, v \in \mathbb{R}, \quad \partial_1\overline{A}(u, v) \geq 0, \quad \partial_2\overline{A}(u, v) \leq 0; \tag{3.13}$$

(iii) Polynomial growth:

$$\exists C_{\overline{A}} > 0, \quad \exists p_{\overline{A}} \in \mathbb{N}^*, \quad \forall u, v \in \mathbb{R}, \quad |\partial_1\overline{A}(u, v)| \leq C_{\overline{A}}(1 + |u|^{p_{\overline{A}}}), \quad |\partial_2\overline{A}(u, v)| \leq C_{\overline{A}}(1 + |v|^{p_{\overline{A}}}). \tag{3.14}$$

Note in particular that the flux function, and therefore the non-linearity of Equation (3.1), is not subject to a global Lipschitz continuity assumption. Nevertheless, we will prove in Proposition 3.15 below that (3.11) is well-posed under Assumption 3.3.

Remark 3.4 (Engquist-Osher numerical flux)**.** *A notable class of numerical fluxes satisfying the monotonicity and polynomial growth conditions (under Assumption 3.1) are the flux-splitting schemes [57, Example 5.2], among which a commonly employed example is the Engquist-Osher flux [56] defined by*

$$\overline{A}_{\mathrm{EO}}(u, v) := \frac{A(u) + A(v)}{2} - \frac{1}{2}\int_u^v |A'(z)|\mathrm{d}z.$$

3.1.3 Space and time discretisation

The second stage in constructing a numerical scheme for (3.1) is the time discretisation of the SDE (3.11). Considering a time step $\Delta t > 0$ and a positive integer n, we introduce the notation $\Delta \mathbf{W}_n^{Q,N} := \mathbf{W}^{Q,N}(n\Delta t) - \mathbf{W}^{Q,N}((n-1)\Delta t)$.

As it was already noticed in [94], explicit numerical schemes for SDEs with non-globally Lipschitz continuous coefficients do not preserve in general the large time stability, whereas implicit schemes are more robust. Therefore, since our main focus in this chapter is to approximate invariant measures, we propose the following *split-step stochastic backward Euler method*:

$$\begin{cases} \mathbf{U}_{n+\frac{1}{2}} = \mathbf{U}_n + \Delta t \mathbf{b}\left(\mathbf{U}_{n+\frac{1}{2}}\right), \\ \mathbf{U}_{n+1} = \mathbf{U}_{n+\frac{1}{2}} + \Delta \mathbf{W}_{n+1}^{Q,N}. \end{cases} \tag{3.15}$$

The well-posedness of the scheme, *i.e.* the existence and uniqueness of the value $\mathbf{U}_{n+\frac{1}{2}}$ in the first line of (3.15), is ensured by Proposition 3.23.

3.1.4 Main results

Our first focus is on the large-time behaviour of the processes $(\mathbf{U}(t))_{t\geq 0}$ and $(\mathbf{U}_n)_{n\in\mathbb{N}}$. In this context, we state our first result:

Theorem 3.5. *Under Assumptions 3.1 and 3.3, the following two statements hold:*

(i) for any $N \geq 1$, the process $(\mathbf{U}(t))_{t\geq 0}$ solution of the SDE (3.11) admits a unique invariant measure $\nu_N \in \mathcal{P}(\mathbb{R}_0^N)$;

(ii) for any $N \geq 1$ and any $\Delta t > 0$, the process $(\mathbf{U}_n)_{n\in\mathbb{N}}$ defined by (3.15) admits a unique invariant measure $\nu_{N,\Delta t} \in \mathcal{P}(\mathbb{R}_0^N)$.

Moreover, for any $N \geq 1$ and any $\Delta t > 0$, the measures ν_N and $\nu_{N,\Delta t}$ belong to $\mathcal{P}_2(\mathbb{R}_0^N)$.

The proofs for these two statements are given separately in Section 3.2. The structure of the proof is the same as for Theorem 2.7 where we derived the existence and uniqueness of an invariant measure for the solution of (3.1) from two important properties: respectively the dissipativity of the solution and an L^1-contraction property. In Lemma 3.13 below, we show that both of these properties are preserved at the discrete level. Therefore, we will address the existence part with a tightness argument (the Krylov-Bogoliubov theorem) and the uniqueness with a coupling argument. To compare two different probability measures, we will make use of the following metric:

Definition 3.6 (Wasserstein distance)**.** *Let $(E, \|\cdot\|_E)$ be a normed vector space and let $\alpha, \beta \in \mathcal{P}_2(E)$. The* second order Wasserstein distance *between α and β is defined by*

$$W_2(\alpha, \beta) := \inf_{\pi\in\Pi(\alpha,\beta)} \left(\int_{E\times E} \|u - v\|_E^2 \, \mathrm{d}\pi(u, v) \right)^{1/2},$$

where $\Pi(\alpha, \beta)$ is the set of probability measures on $E \times E$ with marginals α and β:

$$\Pi(\alpha, \beta) := \{\pi \in \mathcal{P}_2(E \times E) : \forall B \in \mathcal{B}(E), \pi(B \times E) = \alpha(B) \text{ and } \pi(E \times B) = \beta(B)\}.$$

The reader is referred to [102, Chapter 6] for further details on the Wasserstein distance, and in particular for the proof that it actually defines a distance on $\mathcal{P}_2(E)$. From now on, the space $\mathcal{P}_2(E)$ will be endowed with the distance W_2: convergence of elements of $\mathcal{P}_2(E)$ will always be meant in the sense of the Wasserstein distance. The only cases we will address in this chapter are $E = L^2_0(\mathbb{T})$ and $E = \mathbb{R}^N_0$.

As a first step to approximate numerically the measure μ, we start to embed the measures ν_N and $\nu_{N,\Delta t}$ into $\mathcal{P}(L^2_0(\mathbb{T}))$. For $m = 0, 1, 2$, let $\Psi^{(m)}_N : \mathbb{R}^N_0 \to H^m_0(\mathbb{T})$ denote embedding functions in such a way that for any $\mathbf{u} \in \mathbb{R}^N_0$, $\Psi^{(0)}_N \mathbf{u}$, $\Psi^{(1)}_N \mathbf{u}$ and $\Psi^{(2)}_N \mathbf{u}$ correspond respectively to piecewise constant, piecewise affine, and piecewise quadratic interpolations of the vector $\mathbf{u}$ on the mesh $\mathcal{T}$. The functions $\Psi^{(m)}_N$ will be precisely defined at the beginning of Section 3.3.

For $m = 0, 1, 2$, we define the pushforward measures $\mu^{(m)}_N := \nu_N \circ (\Psi^{(m)}_N)^{-1}$, and $\mu^{(m)}_{N,\Delta t} := \nu_{N,\Delta t} \circ (\Psi^{(m)}_N)^{-1}$. In particular, the measures $\mu^{(m)}_N$ and $\mu^{(m)}_{N,\Delta t}$ give full weight to $H^m_0(\mathbb{T})$. Sections 3.3 and 3.4 are devoted to the proof of our main result:

Theorem 3.7. *Under Assumptions 3.1 and 3.3, we have for all $m = 0, 1, 2$,*

$$\lim_{N\to\infty} \mu^{(m)}_N = \mu \quad \text{in} \quad \mathcal{P}_2(L^2_0(\mathbb{T})), \tag{3.16}$$

and moreover, for any $N \geq 1$,

$$\lim_{\Delta t\to 0} \nu_{N,\Delta t} = \nu_N \quad \text{in} \quad \mathcal{P}_2(\mathbb{R}^N_0). \tag{3.17}$$

In short, we have the following approximation result for all $m = 0, 1, 2$:

$$\lim_{N\to\infty} \lim_{\Delta t\to 0} \mu^{(m)}_{N,\Delta t} = \mu \quad \text{in} \quad \mathcal{P}_2(L^2_0(\mathbb{T})).$$

Remark 3.8. *In Theorem 3.7, μ is seen as a probability measure of $\mathcal{P}_2(L^2_0(\mathbb{T}))$ giving full weight to $H^2_0(\mathbb{T})$, as opposed to Proposition 3.2 where μ was directly seen as a measure of the space $\mathcal{P}_2(H^2_0(\mathbb{T}))$.*

Remark 3.9 (Ergodicity)**.** *As the invariant measure μ of the process $(u(t))_{t\geq 0}$ is unique from Proposition 3.2, it is ergodic. In particular, a consequence of Birkhoff's ergodic theorem (see for instance [35, Theorem 1.2.3]) is that for any $\varphi \in L^1(\mu)$ and for μ-almost every initial condition $u_0 \in H^2_0(\mathbb{T})$, almost surely,*

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \varphi(u(s))\mathrm{d}s = \mathbb{E}\left[\varphi(v)\right], \quad \text{where } v \sim \mu.$$

By virtue of Theorem 3.5, this property also holds at the discrete level: the process $(\mathbf{U}_n)_{n\in\mathbb{N}}$ satisfies for any $\varphi \in L^1(\nu_{N,\Delta t})$ and for $\nu_{N,\Delta t}$-almost every initial condition $\mathbf{U}_0 \in \mathbb{R}^N_0$, almost surely,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{l=0}^{n-1} \varphi(\mathbf{U}_l) = \mathbb{E}\left[\varphi(\mathbf{V})\right], \quad \text{where } \mathbf{V} \sim \nu_{N,\Delta t}.$$

Thanks to this property, it is possible to approximate numerically expectations of functionals under the invariant measure by averaging in time the simulated process. We used this method to perform the numerical experiments presented in Chapter 4.

3.1.5 Review of literature

Many results are found concerning the numerical approximation *in finite time* of stochastic conservation laws. A particular case of interest is the stochastic Burgers equation which corresponds to the case of the flux function $A(u) = u^2/2$. Finite difference schemes are presented in [1, 71] to approximate its solution. When the viscosity coefficient is equal to zero, the SPDE falls into a different framework. Convergence of finite-volume schemes in this *hyperbolic* case have been established both under the kinetic [46, 47, 45] and the entropic formulations [5, 6].

As regards the numerical approximation of the invariant measure of an SPDE, we may start by mentioning [26] concerning the damped stochastic non-linear Schrödinger equation, where a spectral Galerkin method is used for the space discretisation and a modified implicit Euler scheme for the temporal discretisation. Several works of Bréhier are devoted to the numerical approximation of the invariant measures of semi-linear SPDEs in Hilbert spaces perturbed with white noise [20, 21, 22], where spectral Galerkin and semi-implicit Euler methods are used. Those results hold under a global Lipschitz assumption on the nonlinearity. In the more recent works [29, 33], non-Lipschitz nonlinearities are considered, but they still need to satisfy a one-sided Lipschitz condition.

In the present work, our assumptions on the flux function do not imply that the non-linear term is globally Lipschitz in $L^2_0(\mathbb{T})$ nor even one-sided Lipschitz. In particular, the case of the Burgers' equation is covered. However, Equation (3.1) satisfies an L^1-contraction property (Proposition 2.21) which may be viewed as a one-sided Lipschitz condition in the Banach space $L^1_0(\mathbb{T})$.

3.1.6 Outline of the chapter

The existence and uniqueness of an invariant measure for the solution of (3.11), *i.e.* the first part of Theorem 3.5, is proved in Section 3.2.2, and for the split-step scheme (3.15), *i.e.* the second part of Theorem 3.5, it is proved in Section 3.2.3.

The proof of Theorem 3.7 is also split in two separate parts. The convergence in space (3.16) is proved in Section 3.3 and then, in Section 3.4, we prove the convergence with respect to the time step, *i.e.* Equation (3.17).

We performed numerical experiments to test the stationarity and convergence results in the Burgers case. The results of these experiments are exposed in the next chapter, where we furthermore illustrate some properties regarding the turbulent behaviour of the process in its stationary regime.

3.2 Semi-discrete and split-step schemes: well-posedness and invariant measure

Preliminary results are given in Subsection 3.2.1. In Subsection 3.2.2, we prove the well-posedness of (3.11), and after establishing some properties for the solution, we prove the existence of an invariant measure at Proposition 3.19. Then, Lemmas 3.21 and 3.22 lead to the proof of uniqueness of this invariant measure, *i.e.* to the proof of the first assertion of Theorem 3.5.

3.2.1 Notations and properties

All the lemmas stated in this section will be proved in the appendix. We define the discrete differential operators $\mathbf{D}_N^{(m)} : \mathbb{R}^N \to \mathbb{R}^N$, $m = 0, 1, 2$, by

$$\left(\mathbf{D}_N^{(0)}\mathbf{u}\right)_i := u_i, \quad \left(\mathbf{D}_N^{(1)}\mathbf{u}\right)_i := N(u_{i+1} - u_i), \quad \left(\mathbf{D}_N^{(2)}\mathbf{u}\right)_i := N^2(u_{i+1} - 2u_i + u_{i-1}), \quad i \in \mathbb{Z}/N\mathbb{Z}.$$

We will often make use, in this chapter, of the summation by parts identity

$$\left\langle \mathbf{D}_N^{(1)}\mathbf{u}, \mathbf{D}_N^{(1)}\mathbf{v} \right\rangle = -\left\langle \mathbf{D}_N^{(2)}\mathbf{u}, \mathbf{v} \right\rangle, \quad \mathbf{u}, \mathbf{v} \in \mathbb{R}_0^N. \tag{3.18}$$

These operators satisfy furthermore the following properties:

Lemma 3.10 (Discrete Poincaré inequality)**.** *Let* $\mathbf{u} \in \mathbb{R}^N$*. If there exist* $i_-, i_+ \in \mathbb{Z}/N\mathbb{Z}$ *such that* $u_{i_-} \leq 0 \leq u_{i_+}$*, then for any* $m = 0, 1$*, any* $p \in [1, +\infty)$*,*

$$\left\| \mathbf{D}_N^{(m)}\mathbf{u} \right\|_p \leq \left\| \mathbf{D}_N^{(m+1)}\mathbf{u} \right\|_p.$$

It should be noted that this discrete Poincaré inequality holds in particular for $\mathbf{u} \in \mathbb{R}_0^N$.

Several times in this chapter, we will establish estimates uniformly in N (resp. in Δt) over the moment of the discrete Sobolev norm $\mathbb{E}[\|\mathbf{D}_N^{(m)}\mathbf{V}\|_p^p]$ where $\mathbf{V}$ is an invariant measure for the semi-discrete scheme (resp. the fully discrete scheme). Whenever this situation appears, we will denote by $C^{m,p}$ (resp. $\overline{C}^{m,p}$) the uniform upper bound.

Lemma 3.11. *For any $\mathbf{u} \in \mathbb{R}_0^N$ and $p \in 2\mathbb{N}^*$, we have*

$$\left\langle \mathbf{D}_N^{(1)}\left(\mathbf{u}^{p-1}\right), \mathbf{D}_N^{(1)}\mathbf{u} \right\rangle \geq \frac{4(p-1)}{p^2}\|\mathbf{u}\|_p^p,$$

where $\mathbf{u}^{p-1} := (u_1^{p-1}, \dots, u_N^{p-1})$.

Lemma 3.12. *Under Assumption 3.3, for any $\mathbf{u} \in \mathbb{R}_0^N$ and any $q \in 2\mathbb{N}^*$, we have*

$$\sum_{i=1}^N u_i^{q-1}\left(\overline{A}(u_i, u_{i+1}) - \overline{A}(u_{i-1}, u_i)\right) \geq 0.$$

For any $z \in \mathbb{R}$, we write $\operatorname{sign}(z) := \mathbf{1}_{z\geq 0} - \mathbf{1}_{z<0}$. By extension, for $\mathbf{u} \in \mathbb{R}_0^N$, $\mathbf{sign}(\mathbf{u})$ denotes the vector of $\{-1,+1\}^N$ defined by $(\mathbf{sign}(\mathbf{u}))_i = \operatorname{sign}(u_i)$.

The discretised drift $\mathbf{b}$ preserves some nice properties of Equation (3.1) that we will use repeatedly throughout this chapter:

Lemma 3.13. *Under Assumption 3.3, for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}_0^N$, the function $\mathbf{b}$ satisfies*

(i) $\langle \mathbf{sign}(\mathbf{u}-\mathbf{v}), \mathbf{b}(\mathbf{u}) - \mathbf{b}(\mathbf{v})\rangle \leq 0$ *(L^1-contraction);*

(ii) $\langle \mathbf{u}, \mathbf{b}(\mathbf{u})\rangle \leq -\nu\|\mathbf{D}_N^{(1)}\mathbf{u}\|_2^2$ *(dissipativity).*

Remark 3.14. *The dissipativity property actually holds for the family of E-fluxes [91], a larger family than the class of monotone numerical fluxes. The monotonicity assumption (3.13) seems however necessary as regards the L^1-contraction property.*

3.2.2 The semi-discrete scheme

Before addressing the invariant measure of the solution of (3.11), we first ensure the existence and uniqueness of this solution:

Proposition 3.15. *Let $\mathbf{U}_0$ be an $\mathbb{R}_0^N$-valued, $\mathcal{F}_0$-measurable random variable. Under Assumptions 3.1 and 3.3, the stochastic differential equation (3.11) admits a unique strong solution $(\mathbf{U}(t))_{t\geq 0}$ taking values in $\mathbb{R}_0^N$ and with initial condition $\mathbf{U}_0$.*

Proof. Let $\mathbf{u}_0 \in \mathbb{R}_0^N$. Since the function $\mathbf{b}$ is locally Lipschitz continuous, there exists a unique strong solution $(\mathbf{U}(t))_{t\in[0,T^*)}$ to Equation (3.11) with initial condition $\mathbf{u}_0$ defined up to an explosion time T^*, *i.e.* a stopping time taking values in $(0,+\infty]$ such that almost surely, if $T^* < +\infty$, then

$$\lim_{t\to T^*}\|\mathbf{U}(t)\|_2 = +\infty$$

(see for instance [75, Theorem 2.3, Lemma 2.1, Theorem 3.1]). In particular, if we define the stopping times

$$\tau_M := \inf\left\{t \geq 0 : \|\mathbf{U}(t)\|_2^2 \geq M\right\}, \tag{3.19}$$

then for all $M \geq 0$, we have $\tau_M \leq T^*$ almost surely.

From Dynkin's formula applied to Equation (3.11), we get for all $t \geq 0$,

$$\mathbb{E}\left[\|\mathbf{U}(t\wedge\tau_M)\|_2^2\right] = \|\mathbf{u}_0\|_2^2 + 2\mathbb{E}\left[\int_0^{t\wedge\tau_M} \langle \mathbf{U}(s), \mathbf{b}(\mathbf{U}(s))\rangle \mathrm{d}s\right] + \mathbb{E}[t\wedge\tau_M]\sum_{k\geq 1}\|\boldsymbol{\sigma}^k\|_2^2$$
$$\leq \|\mathbf{u}_0\|_2^2 - 2\nu\mathbb{E}\left[\int_0^{t\wedge\tau_M} \left\|\mathbf{D}_N^{(1)}\mathbf{U}(s)\right\|_2^2 \mathrm{d}s\right] + \mathbb{E}[t\wedge\tau_M]\sum_{k\geq 1}\|\boldsymbol{\sigma}^k\|_2^2,$$

where the inequality comes from Lemma 3.13.(ii). As a consequence, using (3.9),

$$\mathbb{E}\left[\|\mathbf{U}(t\wedge\tau_M)\|_2^2\right] \leq \|\mathbf{u}_0\|_2^2 + tD_0.$$

From Markov's inequality, we now derive

$$\mathbb{P}(\tau_M \leq t) = \mathbb{P}\left(\|\mathbf{U}(t\wedge\tau_M)\|_2^2 \geq M\right) \leq \frac{\mathbb{E}\left[\|\mathbf{U}(t\wedge\tau_M)\|_2^2\right]}{M} \leq \frac{\|\mathbf{u}_0\|_2^2 + tD_0}{M} \underset{M\to\infty}{\longrightarrow} 0.$$

As the random variable $\mathbf{1}_{\tau_M\leq t}$ is almost surely non-decreasing as M increases, it admits an almost sure limit as $M \to +\infty$. From the dominated convergence theorem, this limit is actually zero:

$$\mathbb{E}\left[\lim_{M\to\infty} \mathbf{1}_{\tau_M\leq t}\right] = \lim_{M\to\infty} \mathbb{P}(\tau_M \leq t) = 0.$$

As a consequence, almost surely, $\lim_{M\to\infty}\tau_M = +\infty$, and then $T^* = +\infty$ almost surely, meaning that $(\mathbf{U}(t))_{t\geq 0}$ is a global solution of (3.11).

Now, if the initial condition of $(\mathbf{U}(t))_{t\geq 0}$ is an $\mathcal{F}_0$-measurable random variable $\mathbf{U}_0$ distributed under some probability measure α on $\mathbb{R}_0^N$, then we have

$$\mathbb{P}(T^* = +\infty) = \int_{\mathbb{R}_0^N} \mathbb{P}_{\mathbf{u}_0}(T^* = +\infty)\mathrm{d}\alpha(\mathbf{u}_0) = 1, \tag{3.20}$$

where $\mathbb{P}_{\mathbf{u}_0}$ is the conditional probability given the event $\mathbf{U}(0) = \mathbf{u}_0$. □

We now turn to the proof of existence of an invariant measure. The following lemma, Proposition 3.17, and Corollary 3.18 are preliminary results.

Lemma 3.16 (Moment estimates on the semi-discrete approximation). *Let $p \in 2\mathbb{N}^*$ and let $\mathbf{U}_0$ be an $\mathcal{F}_0$-measurable random variable such that $\mathbb{E}[\|\mathbf{U}_0\|_p^p] < +\infty$. Then, under Assumptions 3.1 and 3.3, the strong solution $(\mathbf{U}(t))_{t\geq 0}$ of (3.11) with initial condition $\mathbf{U}_0$ satisfies:*

(i) For all $t \geq 0$,

$$\mathbb{E}\left[\|\mathbf{U}(t)\|_p^p\right] + \nu p\mathbb{E}\left[\int_0^t \left\langle \mathbf{D}_N^{(1)}\left(\mathbf{U}(s)^{p-1}\right), \mathbf{D}_N^{(1)}\mathbf{U}(s)\right\rangle \mathrm{d}s\right]$$
$$\leq \mathbb{E}\left[\|\mathbf{U}_0\|_p^p\right] + D_0\frac{p(p-1)}{2}\mathbb{E}\left[\int_0^t \|\mathbf{U}(s)\|_{p-2}^{p-2}\mathrm{d}s\right] \tag{3.21}$$

where $\mathbf{U}(s)^p$ denotes the vector $(U_1(s)^p, \ldots, U_N(s)^p)$ and when $p = 2$, we recall the convention $\|\cdot\|_{p-2}^{p-2} = 1$.

(ii) There exist six positive constants $c_0^{(p)}$, $c_1^{(p)}$, $c_2^{(p)}$, $\theta_0^{(p)}$, $\theta_1^{(p)}$ and $\theta_2^{(p)}$ depending only on D_0, ν and p such that we have

$$\forall t > 0, \quad \mathbb{E}\left[\int_0^t \|\mathbf{U}(s)\|_p^p \mathrm{d}s\right] \leq \theta_0^{(p)} + \theta_1^{(p)}\mathbb{E}\left[\|\mathbf{U}_0\|_p^p\right] + \theta_2^{(p)}t, \tag{3.22}$$

and

$$\forall T > 0, \quad \sup_{t\in[0,T]} \mathbb{E}\left[\|\mathbf{U}(t)\|_p^p\right] \leq c_0^{(p)} + c_1^{(p)}\mathbb{E}\left[\|\mathbf{U}_0\|_p^p\right] + c_2^{(p)}T. \tag{3.23}$$

Proof. Let τ_M be the stopping time defined at (3.19). Applying Dynkin's formula to Equation (3.11), we get the following dynamics for the p-th order moment: for all $t \geq 0$ and all $M \geq 0$,

$$\mathbb{E}\left[\|\mathbf{U}(t\wedge\tau_M)\|_p^p\right] = \mathbb{E}\left[\|\mathbf{U}_0\|_p^p\right] - p\mathbb{E}\left[\int_0^{t\wedge\tau_M}\sum_{i=1}^N U_i(s)^{p-1}\left(\overline{A}\left(U_i(s),U_{i+1}(s)\right)-\overline{A}\left(U_{i-1}(s),U_i(s)\right)\right)\mathrm{d}s\right]$$
$$+\nu Np\mathbb{E}\left[\int_0^{t\wedge\tau_M}\left\langle \mathbf{U}(s)^{p-1},\mathbf{D}_N^{(2)}\mathbf{U}(s)\right\rangle\mathrm{d}s\right]+\frac{p(p-1)}{2N}\mathbb{E}\left[\int_0^{t\wedge\tau_M}\sum_{i=1}^N U_i(s)^{p-2}\sum_{k\geq 1}\left(\sigma_i^k\right)^2\mathrm{d}s\right]. \quad (3.24)$$

From (3.9), we have for all $i=1,\dots,N$,

$$\sum_{k\geq 1}\left(\sigma_i^k\right)^2 \leq D_0.$$

On the other hand, the second term of the right-hand side in (3.24) is non-positive thanks to Lemma 3.12. Hence, using (3.18) in the viscous term, we get

$$\mathbb{E}\left[\|\mathbf{U}(t\wedge\tau_M)\|_p^p\right] \leq \mathbb{E}\left[\|\mathbf{U}_0\|_p^p\right] - \nu Np\mathbb{E}\left[\int_0^{t\wedge\tau_M}\left\langle \mathbf{D}_N^{(1)}\left(\mathbf{U}(s)^{p-1}\right),\mathbf{D}_N^{(1)}\mathbf{U}(s)\right\rangle\mathrm{d}s\right]$$
$$+\frac{p(p-1)}{2}D_0\mathbb{E}\left[\int_0^{t\wedge\tau_M}\|\mathbf{U}(s)\|_{p-2}^{p-2}\mathrm{d}s\right]. \quad (3.25)$$

Letting M go to $+\infty$, applying the monotone convergence theorem on the right-hand side and Fatou's lemma on the left-hand side yields the first assertion of the lemma.

From the first assertion and Lemma 3.11, we have

$$\frac{4\nu(p-1)}{p}\mathbb{E}\left[\int_0^t\|\mathbf{U}(s)\|_p^p\mathrm{d}s\right] \leq \mathbb{E}\left[\|\mathbf{U}_0\|_p^p\right]+\frac{D_0p(p-1)}{2}\mathbb{E}\left[\int_0^t\|\mathbf{U}(s)\|_{p-2}^{p-2}\mathrm{d}s\right]. \quad (3.26)$$

Noticing that $\|\cdot\|_{p-2}^{p-2} \leq 1+\|\cdot\|_p^p$, by (3.26) and an induction argument, we can show that for all $p\in 2\mathbb{N}^*$, (3.22) holds. Now, from the first assertion once again and (3.22), we have for all $p \in 2\mathbb{N}^*$ and all $0 \leq t \leq T$,

$$\begin{aligned}
\mathbb{E}\left[\|\mathbf{U}(t)\|_p^p\right] &\leq \mathbb{E}\left[\|\mathbf{U}_0\|_p^p\right]+\frac{D_0p(p-1)}{2}\mathbb{E}\left[\int_0^T\|\mathbf{U}(s)\|_{p-2}^{p-2}\mathrm{d}s\right]\\
&\leq \mathbb{E}\left[\|\mathbf{U}_0\|_p^p\right]+\frac{D_0p(p-1)}{2}\left(\theta_0^{(p-2)}+\theta_1^{(p-2)}\mathbb{E}\left[\|\mathbf{U}_0\|_{p-2}^{p-2}\right]+\theta_2^{(p-2)}T\right)\\
&\leq \mathbb{E}\left[\|\mathbf{U}_0\|_p^p\right]+\frac{D_0p(p-1)}{2}\left(\theta_0^{(p-2)}+\theta_1^{(p-2)}\left(1+\mathbb{E}\left[\|\mathbf{U}_0\|_p^p\right]\right)+\theta_2^{(p-2)}T\right)\\
&=: c_0^{(p)}+c_1^{(p)}\mathbb{E}\left[\|\mathbf{U}_0\|_p^p\right]+c_2^{(p)}T.
\end{aligned}$$

Since the right-hand side does not depend on t, we get (3.23). □

Proposition 3.17 (L^1-contraction). *Under Assumptions 3.1 and 3.3, two strong solutions $(\mathbf{U}(t))_{t\geq 0}$ and $(\mathbf{V}(t))_{t\geq 0}$ of (3.11) (driven by the same Wiener process $\mathbf{W}^{Q,N}$) with possibly different initial conditions satisfy almost surely:*

$$\|\mathbf{U}(t)-\mathbf{V}(t)\|_1 \leq \|\mathbf{U}(s)-\mathbf{V}(s)\|_1, \qquad 0\leq s\leq t.$$

Proof. Since $(\mathbf{U}(t))_{t\geq 0}$ and $(\mathbf{V}(t))_{t\geq 0}$ are driven by the same Wiener process, then $(\mathbf{U}(t)-\mathbf{V}(t))_{t\geq 0}$ is an absolutely continuous process:

$$\mathrm{d}(\mathbf{U}(t)-\mathbf{V}(t)) = (\mathbf{b}(\mathbf{U}(t))-\mathbf{b}(\mathbf{V}(t)))\,\mathrm{d}t.$$

In particular, we can write for all $t\geq 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{U}(t)-\mathbf{V}(t)\|_1 = \langle \mathbf{sign}\left(\mathbf{U}(t)-\mathbf{V}(t)\right), \mathbf{b}(\mathbf{U}(t))-\mathbf{b}(\mathbf{V}(t))\rangle \leq 0,$$

where the inequality comes from Lemma 3.13.(i), and the result follows by integrating in time. □

This last property ensures the following result which we state without a proof:

Corollary 3.18 (Feller property). *Under Assumption 3.3, the strong solution $(\mathbf{U}(t))_{t\geq 0}$ of Equation (3.11) satisfies the Feller property,* i.e. *for any continuous and bounded function $\varphi : \mathbb{R}_0^N \to \mathbb{R}$ and any $t\geq 0$, the mapping*

$$\mathbf{u}_0 \in \mathbb{R}_0^N \longmapsto \mathbb{E}_{\mathbf{u}_0}\left[\varphi(\mathbf{U}(t))\right] \in \mathbb{R}$$

is continuous and bounded, where $\mathbb{E}_{\mathbf{u}_0}$ is the conditional expectation given the event $\mathbf{U}(0)=\mathbf{u}_0$.

Proposition 3.19 (Existence of an invariant measure for the semi-discrete scheme). *Under Assumptions 3.1 and 3.3, the strong solution $(\mathbf{U}(t))_{t\geq 0}$ of (3.11) admits an invariant measure $\nu_N \in \mathcal{P}(\mathbb{R}_0^N)$. Moreover, for all $p\in[1,+\infty)$, there exists a constant $C^{0,p}$ not depending on N such that if $\mathbf{V}$ is a random variable with distribution ν_N, then*

$$\mathbb{E}\left[\|\mathbf{V}\|_p^p\right] \leq C^{0,p}.$$

Proof. Let $(\mathbf{U}(t))_{t\geq 0}$ be the solution of (3.11) with a deterministic initial condition $\mathbf{u}_0 \in \mathbb{R}_0^N$. From Lemma 3.16.(ii), we have for all $t>0$ and all $p\in 2\mathbb{N}^*$,

$$\frac{1}{t}\int_0^t \mathbb{E}_{\mathbf{u}_0}\left[\|\mathbf{U}(s)\|_p^p\right]\mathrm{d}s \leq \frac{1}{t}\theta_0^{(p)} + \frac{1}{t}\theta_1^{(p)}\|\mathbf{u}_0\|_2^2 + \theta_2^{(p)}. \tag{3.27}$$

Let us take $p=2$. Applying the Markov inequality and taking the limit superior in t, we get for all $\varepsilon>0$,

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t \mathbb{P}_{\mathbf{u}_0}\left(\|\mathbf{U}(s)\|_2^2 > \frac{1}{\varepsilon}\right)\mathrm{d}s \leq \varepsilon\theta_2^{(2)}.$$

Since from Corollary 3.18, $(\mathbf{U}(t))_{t\geq 0}$ is Feller, the existence of an invariant measure $\nu_N\in\mathcal{P}(\mathbb{R}_0^N)$ for $(\mathbf{U}(t))_{t\geq 0}$ is now a consequence of the Krylov-Bogoliubov theorem [36, Corollary 3.1.2].

Let $\mathbf{V}$ be a random variable with distribution ν_N. We will derive now from Equation (3.27) that $\mathbf{V}$ has finite moments. A computation of the same kind as the one below is found for instance in the proof of [70, Proposition 4.24]. For any $M>0$ and any $p\in 2\mathbb{N}^*$,

$$\begin{aligned}
\mathbb{E}\left[\|\mathbf{V}\|_p^p\wedge M\right] &= \frac{1}{t}\int_0^t\int_{\mathbb{R}_0^N}\mathbb{E}_{\mathbf{u}_0}\left[\|\mathbf{U}(s)\|_p^p\wedge M\right]\mathrm{d}\nu_N(\mathbf{u}_0)\mathrm{d}s\\
&= \int_{\mathbb{R}_0^N}\frac{1}{t}\int_0^t\mathbb{E}_{\mathbf{u}_0}\left[\|\mathbf{U}(s)\|_p^p\wedge M\right]\mathrm{d}s\mathrm{d}\nu_N(\mathbf{u}_0)\\
&\leq \int_{\mathbb{R}_0^N}\left(\frac{1}{t}\int_0^t\mathbb{E}_{\mathbf{u}_0}\left[\|\mathbf{U}(s)\|_p^p\right]\mathrm{d}s\right)\wedge M\mathrm{d}\nu_N(\mathbf{u}_0)\\
&\leq \int_{\mathbb{R}_0^N}\left(\frac{1}{t}\theta_0^{(p)} + \theta_1^{(p)}\frac{\|\mathbf{u}_0\|_p^p}{t} + \theta_2^{(p)}\right)\wedge M\mathrm{d}\nu_N(\mathbf{u}_0).
\end{aligned}$$

Now, letting $t \to +\infty$, we get from the dominated convergence theorem,

$$\mathbb{E}\left[\|\mathbf{V}\|_p^p \wedge M\right] \leq \int_{\mathbb{R}_0^N} \lim_{t\to\infty}\left(\frac{1}{t}\theta_0^{(p)} + \theta_1^{(p)}\frac{\|\mathbf{u}_0\|_p^p}{t} + \theta_2^{(p)}\right) \wedge M \mathrm{d}\nu_N(\mathbf{u}_0) = \theta_2^{(p)} \wedge M \leq \theta_2^{(p)},$$

and the result for $p \in 2\mathbb{N}^*$ follows by letting $M \to +\infty$ and using the monotone convergence theorem. This result extends readily to the general case $p \in [1, +\infty)$ by using for instance the Jensen inequality. □

Corollary 3.20. *Under Assumptions 3.1 and 3.3, let ν_N be an invariant measure for the solution $(\mathbf{U}(t))_{t\geq 0}$ of (3.11) and let $\mathbf{V}$ be a random variable with distribution ν_N. Then, for all $p \in 2\mathbb{N}^*$, $\mathbf{V}$ satisfies*

$$\mathbb{E}\left[\left\langle \mathbf{D}_N^{(1)}\left(\mathbf{V}^{p-1}\right), \mathbf{D}_N^{(1)}\mathbf{V}\right\rangle\right] \leq \frac{D_0(p-1)}{2\nu}C^{0,p-2},$$

where we set $C^{0,0} = 1$.

Proof. Let $(\mathbf{U}(t))_{t\geq 0}$ be a solution of (3.11) whose initial condition $\mathbf{U}_0$ has distribution ν_N. According to Proposition 3.19, one has $\mathbb{E}[\|\mathbf{U}_0\|_p^p] < +\infty$. Thus, one can apply Lemma 3.16.(i) which in the stationary case, writes

$$\nu p\mathbb{E}\left[\left\langle \mathbf{D}_N^{(1)}\left(\mathbf{V}^{p-1}\right), \mathbf{D}_N^{(1)}\mathbf{V}\right\rangle\right] \leq D_0\frac{p(p-1)}{2}\mathbb{E}\left[\|\mathbf{V}\|_{p-2}^{p-2}\right],$$

and it remains to apply Proposition 3.19 to conclude. □

We now turn to the proof of uniqueness of the invariant measure ν_N. We will first need some preliminary results:

Lemma 3.21. *Let Assumptions 3.1 and 3.3 hold and let $(\mathbf{U}(t))_{t\geq 0}$ and $(\mathbf{V}(t))_{t\geq 0}$ be two strong solutions of (3.11) driven by the same Wiener process. Then, for all $M > 0$ and all $\varepsilon > 0$, there exists $t_{\varepsilon,M} > 0$ such that*

$$p_{\varepsilon,M} := \inf_{\|\mathbf{u}_0\|_2 \vee \|\mathbf{v}_0\|_2 \leq M} \mathbb{P}_{(\mathbf{u}_0,\mathbf{v}_0)}\left(\|\mathbf{U}(t_{\varepsilon,M})\|_1 + \|\mathbf{V}(t_{\varepsilon,M})\|_1 \leq \varepsilon\right) > 0.$$

Proof. We recall that $\mathbf{b} : \mathbb{R}_0^N \to \mathbb{R}_0^N$ is locally Lipschitz continuous (for every norm over $\mathbb{R}_0^N$). Let $M > 0$ and $\varepsilon > 0$. Let us also fix the deterministic values $\mathbf{u}_0, \mathbf{v}_0 \in \mathbb{R}_0^N$ satisfying $\|\mathbf{u}_0\|_2 \vee \|\mathbf{v}_0\|_2 \leq M$, along with the following constants:

$$t_{\varepsilon,M} := -\frac{1}{2\nu}\log\frac{\varepsilon^2}{16M^2};$$

$$L_{M+\varepsilon} := \text{ Lipschitz constant of } b \text{ over the ball } \left\{\mathbf{u} \in \mathbb{R}_0^N : \|\mathbf{u}\|_1 \leq M + \varepsilon\right\};$$

$$\delta_\varepsilon := \frac{\varepsilon}{4}\mathrm{e}^{-L_{M+\varepsilon}t_{\varepsilon,M}}.$$

Let $(\mathbf{U}(t))_{t\geq 0}$ and $(\mathbf{V}(t))_{t\geq 0}$ denote two solutions of (3.11) with the initial conditions $\mathbf{u}_0$ and $\mathbf{v}_0$. We introduce the stopping times

$$\widetilde{\tau}^{\mathbf{U}} := \inf\left\{t \geq 0 : \|\mathbf{U}(t)\|_1 \geq M + \varepsilon\right\};$$

$$\widetilde{\tau}^{\mathbf{V}} := \inf\left\{t \geq 0 : \|\mathbf{V}(t)\|_1 \geq M + \varepsilon\right\}.$$

Furthermore, we denote by $(\mathbf{u}(t))_{t\geq 0}$ and $(\mathbf{v}(t))_{t\geq 0}$ the noiseless counterparts of $(\mathbf{U}(t))_{t\geq 0}$ and $(\mathbf{V}(t))_{t\geq 0}$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u}(t) = \mathbf{b}\left(\mathbf{u}(t)\right), \qquad \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v}(t) = \mathbf{b}\left(\mathbf{v}(t)\right),$$

with respective initial conditions $\mathbf{u}_0$ and $\mathbf{v}_0$.

By the dissipativity property (Lemma 3.13.(ii)) and Lemma 3.10, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\mathbf{u}(t)\|_2^2+\|\mathbf{v}(t)\|_2^2\right) &= 2\left(\langle \mathbf{u}(t),\mathbf{b}\left(\mathbf{u}(t)\right)\rangle+\langle \mathbf{v}(t),\mathbf{b}\left(\mathbf{v}(t)\right)\rangle\right)\\ &\le -2\nu\left(\left\|\mathbf{D}_N^{(1)}\mathbf{u}(t)\right\|_2^2+\left\|\mathbf{D}_N^{(1)}\mathbf{v}(t)\right\|_2^2\right)\\ &\le -2\nu\left(\|\mathbf{u}(t)\|_2^2+\|\mathbf{v}(t)\|_2^2\right),\end{aligned}$$

so that Grönwall's lemma yields the upper bound

$$\|\mathbf{u}(t)\|_2^2+\|\mathbf{v}(t)\|_2^2\le\left(\|\mathbf{u}_0\|_2^2+\|\mathbf{v}_0\|_2^2\right)\mathrm{e}^{-2\nu t},$$

meaning that for $t\ge t_{\varepsilon,M}$, we have

$$\|\mathbf{u}(t)\|_2^2+\|\mathbf{v}(t)\|_2^2\le\frac{\varepsilon^2}{8},$$

and consequently, by (3.10),

$$\|\mathbf{u}(t)\|_1+\|\mathbf{v}(t)\|_1\le\|\mathbf{u}(t)\|_2+\|\mathbf{v}(t)\|_2\le\frac{\varepsilon}{2}.$$

We now restrict ourselves to the situation where

$$\omega\in\left\{\sup_{t\in[0,t_{\varepsilon,M}]}\left\|\mathbf{W}^{Q,N}(t)\right\|_1\le\delta_\varepsilon\right\}.$$

For any $t\le\widetilde{\tau}^{\mathbf{U}}\wedge\widetilde{\tau}^{\mathbf{V}}\wedge t_{\varepsilon,M}$, the four vectors $\mathbf{U}(t)$, $\mathbf{V}(t)$, $\mathbf{u}(t)$ and $\mathbf{v}(t)$ stay in the ball $\{\|\cdot\|_1\le M+\varepsilon\}$, and thanks to the local Lipschitz continuity assumption on $\mathbf{b}$ we have

$$\begin{aligned}\|\mathbf{U}(t)-\mathbf{u}(t)\|_1+\|\mathbf{V}(t)-\mathbf{v}(t)\|_1 &= \left\|\int_0^t\left(\mathbf{b}\left(\mathbf{U}(s)\right)-\mathbf{b}\left(\mathbf{u}(s)\right)\right)\mathrm{d}s+\mathbf{W}^{Q,N}(t)\right\|_1\\ &\quad+\left\|\int_0^t\left(\mathbf{b}\left(\mathbf{V}(s)\right)-\mathbf{b}\left(\mathbf{v}(s)\right)\right)\mathrm{d}s+\mathbf{W}^{Q,N}(t)\right\|_1\\ &\le\int_0^t\left(\|\mathbf{b}\left(\mathbf{U}(s)\right)-\mathbf{b}\left(\mathbf{u}(s)\right)\|_1+\|\mathbf{b}\left(\mathbf{V}(s)\right)-\mathbf{b}\left(\mathbf{v}(s)\right)\|_1\right)\mathrm{d}s+2\left\|\mathbf{W}^{Q,N}(t)\right\|_1\\ &\le L_{M+\varepsilon}\int_0^t\left(\|\mathbf{U}(s)-\mathbf{u}(s)\|_1+\|\mathbf{V}(s)-\mathbf{v}(s)\|_1\right)\mathrm{d}s+2\delta_\varepsilon,\end{aligned}$$

so by Grönwall's lemma, we have

$$\|\mathbf{U}(t)-\mathbf{u}(t)\|_1+\|\mathbf{V}(t)-\mathbf{v}(t)\|_1\le 2\delta_\varepsilon\mathrm{e}^{L_{M+\varepsilon}t}\le 2\delta_\varepsilon\mathrm{e}^{L_{M+\varepsilon}t_{\varepsilon,M}}=\frac{\varepsilon}{2},\tag{3.28}$$

for every $t\in[0,\widetilde{\tau}^{\mathbf{U}}\wedge\widetilde{\tau}^{\mathbf{V}}\wedge t_{\varepsilon,M}]$. But it appears that the case $\widetilde{\tau}^{\mathbf{U}}\wedge\widetilde{\tau}^{\mathbf{V}}<t_{\varepsilon,M}$ is impossible for small values of ε. Indeed, it would either imply $\|(\mathbf{U}-\mathbf{u})(\widetilde{\tau}^{\mathbf{U}})\|_1\le\varepsilon/2$ or $\|(\mathbf{V}-\mathbf{v})(\widetilde{\tau}^{\mathbf{V}})\|_1\le\varepsilon/2$ which is impossible because we have on the one hand

$$\left\|\mathbf{U}\left(\widetilde{\tau}^{\mathbf{U}}\right)\right\|_1\ge M+\varepsilon\qquad\left(\text{or }\left\|\mathbf{V}\left(\widetilde{\tau}^{\mathbf{V}}\right)\right\|_1\ge M+\varepsilon\right),$$

and on the other hand

$$\left\|\mathbf{u}\left(\widetilde{\tau}^{\mathbf{U}}\right)\right\|_1\le\left\|\mathbf{u}\left(\widetilde{\tau}^{\mathbf{U}}\right)\right\|_2\le\|\mathbf{u}_0\|_2\le M\qquad\left(\text{or }\left\|\mathbf{v}\left(\widetilde{\tau}^{\mathbf{V}}\right)\right\|_1\le M\right).$$

Therefore, Inequality (3.28) holds for all $t \in [0, t_{\varepsilon,M}]$. Thus,

$$\|\mathbf{U}(t_{\varepsilon,M})\|_1 + \|\mathbf{V}(t_{\varepsilon,M})\|_1 \leq \|\mathbf{U}(t_{\varepsilon,M}) - \mathbf{u}(t_{\varepsilon,M})\|_1 + \|\mathbf{V}(t_{\varepsilon,M}) - \mathbf{v}(t_{\varepsilon,M})\|_1 + \|\mathbf{u}(t_{\varepsilon,M})\|_1 + \|\mathbf{v}(t_{\varepsilon,M})\|_1 \leq \varepsilon,$$

and we have just shown that

$$\left\{ \sup_{t\in[0,t_{\varepsilon,M}]} \left\|\mathbf{W}^{Q,N}(t)\right\|_1 \leq \delta_\varepsilon \right\} \subset \left\{ \|\mathbf{U}(t_{\varepsilon,M})\|_1 + \|\mathbf{V}(t_{\varepsilon,M})\|_1 \leq \varepsilon \right\}.$$

and therefore,

$$\mathbb{P}_{(\mathbf{u}_0,\mathbf{v}_0)} \left(\|\mathbf{U}(t_{\varepsilon,M})\|_1 + \|\mathbf{V}(t_{\varepsilon,M})\|_1 \leq \varepsilon \right) \geq \mathbb{P}\left(\sup_{t\in[0,t_{\varepsilon,M}]} \left\|\mathbf{W}^{Q,N}(t)\right\|_1 \leq \delta_\varepsilon \right).$$

Notice that the right-hand side does not depend on $\mathbf{u}_0$ nor $\mathbf{v}_0$. Furthermore, it is positive since $\mathbf{W}^{Q,N}$ is an $\mathbb{R}^N$-valued Wiener process. Hence, taking the infimum over $\mathbf{u}_0$ and $\mathbf{v}_0$ on the left-hand side yields the wanted result. □

For two solutions $(\mathbf{U}(t))_{t\geq 0}$ and $(\mathbf{V}(t))_{t\geq 0}$ of (3.11) driven by the same Wiener process $\mathbf{W}^{Q,N}$, we define the following entrance time for all $M \geq 0$:

$$\tau_M := \inf \left\{ t \geq 0 : \|\mathbf{U}(t)\|_2 \vee \|\mathbf{V}(t)\|_2 \leq M \right\}. \tag{3.29}$$

Lemma 3.22. *Under Assumptions 3.1 and 3.3, there exists $M > 0$ such that for any deterministic initial conditions $\mathbf{u}_0$, $\mathbf{v}_0 \in \mathbb{R}^N_0$, $\tau_M < +\infty$ almost surely.*

Proof. From Itô's formula, we have for all $t \geq 0$,

$$\begin{aligned}\|\mathbf{U}(\tau_M\wedge t)\|_2^2 + \|\mathbf{V}(\tau_M\wedge t)\|_2^2 = \|\mathbf{U}_0\|_2^2 + \|\mathbf{V}_0\|_2^2 + \int_0^{\tau_M\wedge t} \langle \mathbf{b}(\mathbf{U}(s)), \mathbf{U}(s)\rangle \,\mathrm{d}s + \int_0^{\tau_M\wedge t} \langle \mathbf{b}(\mathbf{V}(s)), \mathbf{V}(s)\rangle \,\mathrm{d}s \\ + \sum_{k\geq 1} \int_0^{\tau_M\wedge t} \left\langle \mathbf{U}(s) + \mathbf{V}(s), \boldsymbol{\sigma}^k \right\rangle \mathrm{d}W^k(s) + 2\sum_{k\geq 1} \int_0^{\tau_M\wedge t} \left\|\boldsymbol{\sigma}^k\right\|_2^2 \mathrm{d}s.\end{aligned} \tag{3.30}$$

The fifth term of the right-hand side is a martingale. Indeed, by the Cauchy-Schwarz inequality, Inequality (3.9), and the bound (3.22), we have

$$\begin{aligned}\mathbb{E}\left[\sum_{k\geq 1} \int_0^{\tau_M\wedge t} \left| \left\langle \mathbf{U}(s) + \mathbf{V}(s), \boldsymbol{\sigma}^k \right\rangle \right|^2 \mathrm{d}s \right] &\leq \left(\sum_{k\geq 1} \left\|\boldsymbol{\sigma}^k\right\|_2^2 \right) \mathbb{E}\left[\int_0^t \|\mathbf{U}(s) + \mathbf{V}(s)\|_2^2 \,\mathrm{d}s \right] \\ &\leq 2D_0 \left(\mathbb{E}\left[\int_0^t \|\mathbf{U}(s)\|_2^2 \,\mathrm{d}s \right] + \mathbb{E}\left[\int_0^t \|\mathbf{V}(s)\|_2^2 \,\mathrm{d}s \right] \right) \\ &\leq 2D_0 \left(2\theta_0^{(2)} + \theta_1^{(2)} \left(\|\mathbf{u}_0\|_2^2 + \|\mathbf{v}_0\|_2^2 \right) + 2\theta_2^{(2)} t \right) \\ &< +\infty.\end{aligned}$$

Thus, taking the expectation in (3.30), applying Lemma 3.13.(ii), Inequality (3.9), Lemma 3.10

and (3.29), we get

$$\begin{aligned}
&\mathbb{E}\left[\|\mathbf{U}(\tau_M\wedge t)\|_2^2+\|\mathbf{V}(\tau_M\wedge t)\|_2^2\right]\\
&\quad=\|\mathbf{u}_0\|_2^2+\|\mathbf{v}_0\|_2^2+2\mathbb{E}\left[\int_0^{\tau_M\wedge t}\left(\langle\mathbf{b}\left(\mathbf{U}(s)\right),\mathbf{U}(s)\rangle+\langle\mathbf{b}\left(\mathbf{V}(s)\right),\mathbf{V}(s)\rangle\right)\mathrm{d}s\right]+2\mathbb{E}\left[\int_0^{\tau_M\wedge t}\sum_{k\geq1}\left\|\boldsymbol{\sigma}^k\right\|_2^2\mathrm{d}s\right]\\
&\quad\leq\|\mathbf{u}_0\|_2^2+\|\mathbf{v}_0\|_2^2-2\nu\mathbb{E}\left[\int_0^{\tau_M\wedge t}\left(\left\|\mathbf{D}_N^{(1)}\mathbf{U}(s)\right\|_2^2+\left\|\mathbf{D}_N^{(1)}\mathbf{U}(s)\right\|_2^2\right)\mathrm{d}s\right]+2\mathbb{E}\left[\tau_M\wedge t\right]D_0\\
&\quad\leq\|\mathbf{u}_0\|_2^2+\|\mathbf{v}_0\|_2^2-2\nu\mathbb{E}\left[\int_0^{\tau_M\wedge t}\left(\|\mathbf{U}(s)\|_2^2+\|\mathbf{V}(s)\|_2^2\right)\mathrm{d}s\right]+2\mathbb{E}\left[\tau_M\wedge t\right]D_0\\
&\quad\leq\|\mathbf{u}_0\|_2^2+\|\mathbf{v}_0\|_2^2+2\left(D_0-\nu M^2\right)\mathbb{E}\left[\tau_M\wedge t\right]
\end{aligned}$$

So if we choose $M>\sqrt{D_0/\nu}$, we get

$$\mathbb{E}[\tau_M\wedge t]\leq\frac{\|\mathbf{u}_0\|_2^2+\|\mathbf{v}_0\|_2^2}{2\left(\nu M^2-D_0\right)},$$

so that we can apply the monotone convergence theorem:

$$\mathbb{E}[\tau_M]=\lim_{t\to\infty}\mathbb{E}[\tau_M\wedge t]<+\infty,$$

which concludes the proof. □

Proof of Theorem 3.5, Assertion (i). We start to fix $\varepsilon>0$ to which we associate the quantities $t_{\varepsilon,M}$ and $p_{\varepsilon,M}$ defined at Lemma 3.21, where M has been defined at Lemma 3.22. Let $(\mathbf{U}(t))_{t\geq0}$ and $(\mathbf{V}(t))_{t\geq0}$ start respectively from arbitrary deterministic initial conditions $\mathbf{u}_0$ and $\mathbf{v}_0$ and be driven by the same Wiener process. We define the increasing stopping time sequence

$$\begin{aligned}
T_1&:=\tau_M\\
T_2&:=\inf\left\{t\geq T_1+t_{\varepsilon,M}:\|\mathbf{U}(t)\|_2\vee\|\mathbf{V}(t)\|_2\leq M\right\}\\
T_3&:=\inf\left\{t\geq T_2+t_{\varepsilon,M}:\|\mathbf{U}(t)\|_2\vee\|\mathbf{V}(t)\|_2\leq M\right\}\\
&\vdots
\end{aligned}$$

By the strong Markov property and Lemma 3.22, each term of this sequence is finite almost surely. We claim that

$$\forall J\in\mathbb{N}^*,\qquad\mathbb{P}\left(\forall j=1,\dots,J,\quad\|\mathbf{U}(T_j+t_{\varepsilon,M})\|_1+\|\mathbf{V}(T_j+t_{\varepsilon,M})\|_1>\varepsilon\right)\leq\left(1-p_{\varepsilon,M}\right)^J.\tag{3.31}$$

Indeed, it is true for $J=1$ thanks to the strong Markov property and Lemma 3.21:

$$\mathbb{P}\left(\|\mathbf{U}(\tau_M+t_{\varepsilon,M})\|_1+\|\mathbf{V}(\tau_M+t_{\varepsilon,M})\|_1>\varepsilon\right)=\mathbb{E}\left[\mathbb{P}\left(\|\mathbf{U}(\tau_M+t_{\varepsilon,M})\|_1+\|\mathbf{V}(\tau_M+t_{\varepsilon,M})\|_1>\varepsilon|\mathcal{F}_{\tau_M}\right)\right]\leq1-p_{\varepsilon,M},$$

and the general case follows by induction: assuming that Inequality (3.31) is true for some $J\in\mathbb{N}^*$, we have

$$\begin{aligned}
&\mathbb{P}\left(\forall j=1,\dots,J+1,\|\mathbf{U}(T_j+t_{\varepsilon,M})\|_1+\|\mathbf{V}(T_j+t_{\varepsilon,M})\|_1>\varepsilon\right)\\
&\quad=\mathbb{E}\left[\mathbb{P}\left(\forall j=1,\dots,J+1,\quad\|\mathbf{U}(T_j+t_{\varepsilon,M})\|_1+\|\mathbf{V}(T_j+t_{\varepsilon,M})\|_1>\varepsilon|\mathcal{F}_{T_{J+1}}\right)\right]\\
&\quad=\mathbb{E}\left[\left(\prod_{j=1}^J\mathbf{1}_{\|\mathbf{U}(T_j+t_{\varepsilon,M})\|_1+\|\mathbf{V}(T_j+t_{\varepsilon,M})\|_1>\varepsilon}\right)\mathbb{P}\left(\|\mathbf{U}(T_{J+1}+t_{\varepsilon,M})\|_1+\|\mathbf{V}(T_{J+1}+t_{\varepsilon,M})\|_1>\varepsilon|\mathcal{F}_{T_{J+1}}\right)\right]\\
&\quad\leq\left(1-p_{\varepsilon,M}\right)^J\times\left(1-p_{\varepsilon,M}\right)=\left(1-p_{\varepsilon,M}\right)^{J+1}.
\end{aligned}$$

Letting $J \to +\infty$, we get

$$\begin{aligned}
&\mathbb{P}\left(\forall j \in \mathbb{N}^*, \quad \|\mathbf{U}(T_j + t_{\varepsilon,M})\|_1 + \|\mathbf{V}(T_j + t_{\varepsilon,M})\|_1 > \varepsilon\right) \\
&\quad = \lim_{J\to\infty} \mathbb{P}\left(\forall j = 1, \dots, J, \quad \|\mathbf{U}(T_j + t_{\varepsilon,M})\|_1 + \|\mathbf{V}(T_j + t_{\varepsilon,M})\|_1 > \varepsilon\right) \\
&\quad \le \lim_{J\to\infty} (1 - p_{\varepsilon,M})^J = 0,
\end{aligned}$$

and consequently,

$$\mathbb{P}\left(\exists t \ge 0, \quad \|\mathbf{U}(t)\|_1 + \|\mathbf{V}(t)\|_1 \le \varepsilon\right) = 1,$$

meaning that almost surely,

$$\exists t \ge 0, \quad \|\mathbf{U}(t) - \mathbf{V}(t)\|_1 \le \varepsilon.$$

Now recall that thanks to Proposition 3.17, $\|\mathbf{U}(t) - \mathbf{V}(t)\|_1$ is non-increasing in time almost surely. Since ε has been chosen arbitrarily, the above inequality actually indicates that $\|\mathbf{U}(t) - \mathbf{V}(t)\|_1$ converges almost surely to 0 as $t \to +\infty$ when the initial conditions are deterministic. However, this almost sure consergence extends naturally to random and $\mathcal{F}_0$-measurable initial conditions using the same argument as for (3.20). Let $\phi : \mathbb{R}_0^N \to \mathbb{R}$ be a Lipschitz continuous and bounded test function, with Lipschitz constant L_ϕ. We have in particular, almost surely,

$$\lim_{t\to\infty} |\phi(\mathbf{U}(t)) - \phi(\mathbf{V}(t))| \le L_\phi \lim_{t\to\infty} \|\mathbf{U}(t) - \mathbf{V}(t)\|_1 = 0. \tag{3.32}$$

To conclude the proof, assume that there exist two invariant measures $\nu_N^{(1)}$ and $\nu_N^{(2)}$ for the solution of (3.11), and take random initial conditions $\mathbf{U}_0$ and $\mathbf{V}_0$ with distributions $\nu_N^{(1)}$ and $\nu_N^{(2)}$ respectively. We have for all $t \ge 0$,

$$|\mathbb{E}[\phi(\mathbf{U}_0)] - \mathbb{E}[\phi(\mathbf{V}_0)]| = |\mathbb{E}[\phi(\mathbf{U}(t))] - \mathbb{E}[\phi(\mathbf{V}(t))]| \le \mathbb{E}[|\phi(\mathbf{U}(t)) - \phi(\mathbf{V}(t))|].$$

Letting t go to $+\infty$, by (3.32) and the dominated convergence theorem, we have

$$|\mathbb{E}[\phi(\mathbf{U}_0)] - \mathbb{E}[\phi(\mathbf{V}_0)]| \le \lim_{t\to\infty} \mathbb{E}[|\phi(\mathbf{U}(t)) - \phi(\mathbf{V}(t))|] = 0.$$

As a consequence, $\mathbf{U}_0$ and $\mathbf{V}_0$ have the same distribution, meaning that $\nu_N^{(1)} = \nu_N^{(2)}$. □

3.2.3 Invariant measure for the split-step scheme

In this subsection, we aim to prove the existence and uniqueness of an invariant measure for the discrete time process $(\mathbf{U}_n)_{n\in\mathbb{N}}$ defined by (3.15). The general argument is the same as the one used in Subsection 3.2.2 for the semi-discrete case and the intermediary results are stated in the same order. Therefore, the proofs which are not affected by the time discretisation are omitted.

As the time step Δt is meant to converge towards 0, we may consider that it will always lie in an interval $(0, \Delta t_{\max}]$ for some arbitrarily chosen $\Delta t_{\max} > 0$.

The following preliminary result ensures that the scheme (3.15) is well-posed.

Proposition 3.23. *Under Assumption 3.3, given $\Delta t > 0$ and $\mathbf{v} \in \mathbb{R}_0^N$, there exists a unique $\mathbf{w} \in \mathbb{R}_0^N$ such that $\mathbf{w} = \mathbf{v} + \Delta t\mathbf{b}(\mathbf{w})$.*

Proof. **Uniqueness.** It is a straightforward consequence of Lemma 3.13.(i): if $\mathbf{w}_1$ and $\mathbf{w}_2$ are two solutions, then

$$\begin{aligned}
\|\mathbf{w}_1 - \mathbf{w}_2\|_1 &= \langle \mathbf{sign}(\mathbf{w}_1 - \mathbf{w}_2), \mathbf{w}_1 - \mathbf{w}_2 \rangle \\
&= \Delta t \langle \mathbf{sign}(\mathbf{w}_1 - \mathbf{w}_2), \mathbf{b}(\mathbf{w}_1) - \mathbf{b}(\mathbf{w}_2) \rangle \le 0.
\end{aligned}$$

Existence. The mapping $\mathbf{Id} - \Delta t\mathbf{b} : \mathbb{R}_0^N \to \mathbb{R}_0^N$ is continuous. Furthermore, by Lemmas 3.13.(ii) and 3.10, we have for all $\mathbf{w} \in \mathbb{R}_0^N$,

$$\frac{\langle(\mathbf{Id} - \Delta t\mathbf{b})(\mathbf{w}), \mathbf{w}\rangle}{\|\mathbf{w}\|_2} = \|\mathbf{w}\|_2 - \Delta t\frac{\langle \mathbf{b}(\mathbf{w}), \mathbf{w}\rangle}{\|\mathbf{w}\|_2}$$
$$\geq \|\mathbf{w}\|_2 + \nu\Delta t\frac{\left\|\mathbf{D}_N^{(1)}\mathbf{w}\right\|_2^2}{\|\mathbf{w}\|_2} \geq (1+\nu\Delta t)\|\mathbf{w}\|_2.$$

Thus, as a consequence of [42, Theorem 3.3], $\mathbf{Id} - \Delta t\mathbf{b}$ is surjective in $\mathbb{R}_0^N$ and, for any $\mathbf{v} \in \mathbb{R}_0^N$, there exists $\mathbf{w} \in \mathbb{R}_0^N$ such that $\mathbf{w} = \mathbf{v} + \Delta t\mathbf{b}(\mathbf{w})$. ◻

Lemma 3.24 (L^1-contraction)**.** *Let Assumptions 3.1 and 3.3 hold and let $(\mathbf{U}_n)_{n\in\mathbb{N}}$ and $(\mathbf{V}_n)_{n\in\mathbb{N}}$ be two solutions of (3.15) (driven by the same Wiener process $\mathbf{W}^{Q,N}$). Then, almost surely and for any $n \in \mathbb{N}$,*

$$\|\mathbf{U}_{n+1} - \mathbf{V}_{n+1}\|_1 \leq \|\mathbf{U}_n - \mathbf{V}_n\|_1 .$$

Proof. From Equations (3.15) and Lemma 3.13.(ii), we write

$$\begin{aligned}
\|\mathbf{U}_{n+1} - \mathbf{V}_{n+1}\|_1 &= \left\|\mathbf{U}_{n+\frac{1}{2}} - \mathbf{V}_{n+\frac{1}{2}}\right\|_1 \\
&= \left\langle \mathbf{sign}\left(\mathbf{U}_{n+\frac{1}{2}} - \mathbf{V}_{n+\frac{1}{2}}\right), \mathbf{U}_{n+\frac{1}{2}} - \mathbf{V}_{n+\frac{1}{2}}\right\rangle \\
&= \left\langle \mathbf{sign}\left(\mathbf{U}_{n+\frac{1}{2}} - \mathbf{V}_{n+\frac{1}{2}}\right), \mathbf{U}_n - \mathbf{V}_n\right\rangle \\
&\quad + \Delta t\left\langle \mathbf{sign}\left(\mathbf{U}_{n+\frac{1}{2}} - \mathbf{V}_{n+\frac{1}{2}}\right), \mathbf{b}\left(\mathbf{U}_{n+\frac{1}{2}}\right) - \mathbf{b}\left(\mathbf{V}_{n+\frac{1}{2}}\right)\right\rangle \\
&\leq \left\langle \mathbf{sign}\left(\mathbf{U}_{n+\frac{1}{2}} - \mathbf{V}_{n+\frac{1}{2}}\right), \mathbf{U}_n - \mathbf{V}_n\right\rangle \\
&\leq \|\mathbf{U}_n - \mathbf{V}_n\|_1 .
\end{aligned}$$
◻

Remark 3.25. *Note that the choice of the split-step backward Euler scheme is essential for the L^1-contraction property to hold. Indeed, consider for instance two processes $(\widetilde{\mathbf{U}}_n)_{n\in\mathbb{N}}$ and $(\widetilde{\mathbf{V}}_n)_{n\in\mathbb{N}}$ built via a explicit Euler method, that is,*

$$\widetilde{\mathbf{U}}_{n+1} = \widetilde{\mathbf{U}}_n + \Delta t\mathbf{b}\left(\widetilde{\mathbf{U}}_n\right) + \Delta\mathbf{W}_{n+1}^{Q,N}$$

(and naturally, the same construction for $(\widetilde{\mathbf{V}}_n)_{n\in\mathbb{N}}$), then the expansion of the L^1 distance gives

$$\left\|\widetilde{\mathbf{U}}_{n+1} - \widetilde{\mathbf{V}}_{n+1}\right\|_1 = \mathbf{sign}\left\langle\left(\widetilde{\mathbf{U}}_{n+1} - \widetilde{\mathbf{V}}_{n+1}\right), \widetilde{\mathbf{U}}_n - \widetilde{\mathbf{V}}_n\right\rangle + \Delta t\left\langle\mathbf{sign}\left(\widetilde{\mathbf{U}}_{n+1} - \widetilde{\mathbf{V}}_{n+1}\right), \mathbf{b}\left(\widetilde{\mathbf{U}}_n\right) - \mathbf{b}\left(\widetilde{\mathbf{V}}_n\right)\right\rangle.$$

Thus, we would need to control the second term of the right-hand side in the above equation, which is delicate given that $\mathbf{b}$ is not globally Lipschitz.

As for the semi-discrete scheme, Lemma 3.24 induce the following property:

Corollary 3.26. *Under Assumptions 3.1 and 3.3, the solution $(\mathbf{U}_n)_{n\in\mathbb{N}}$ of (3.15) is a Feller process.*

Proposition 3.27. *Under Assumptions 3.1 and 3.3, for any time step $\Delta t > 0$, the process $(\mathbf{U}_n)_{n\in\mathbb{N}}$ solution of the split-step backward Euler method (3.15) admits an invariant measure $\nu_{N,\Delta t}$. Moreover, if $\mathbf{V}$ is a random variable with distribution $\nu_{N,\Delta t}$, then*

$$\mathbb{E}\left[\left\|\mathbf{D}_N^{(1)}\mathbf{V}\right\|_2^2\right] \leq D_0\left(\frac{1}{2\nu} + \Delta t_{\max}\right) =: \overline{C}^{1,2} \tag{3.33}$$

and

$$\mathbb{E}\left[\left\|\mathbf{D}_N^{(1)}\mathbf{V}_{\frac{1}{2}}\right\|_2^2\right] \leq \frac{D_0}{2\nu}, \tag{3.34}$$

where $\mathbf{V}_{\frac{1}{2}}$ *denotes the solution of* $\mathbf{V}_{\frac{1}{2}} = \mathbf{V} + \Delta t\mathbf{b}(\mathbf{V}_{\frac{1}{2}})$.

Proof. Let $\mathbf{u}_0 \in \mathbb{R}_0^N$ be the deterministic initial condition of the process $(\mathbf{U}_n)_{n\in\mathbb{N}}$. Starting from the first equation in (3.15), we have

$$\left\|\mathbf{U}_{n+\frac{1}{2}} - \Delta t\mathbf{b}\left(\mathbf{U}_{n+\frac{1}{2}}\right)\right\|_2^2 = \|\mathbf{U}_n\|_2^2,$$

by expanding the left-hand side, we derive the inequality

$$\left\|\mathbf{U}_{n+\frac{1}{2}}\right\|_2^2 \leq \|\mathbf{U}_n\|_2^2 + 2\Delta t\left\langle \mathbf{b}\left(\mathbf{U}_{n+\frac{1}{2}}\right), \mathbf{U}_{n+\frac{1}{2}}\right\rangle.$$

Using the dissipativity inequality (Lemma 3.13.(ii)), we get

$$\left\|\mathbf{U}_{n+\frac{1}{2}}\right\|_2^2 \leq \|\mathbf{U}_n\|_2^2 - 2\nu\Delta t\left\|\mathbf{D}_N^{(1)}\mathbf{U}_{n+\frac{1}{2}}\right\|_2^2. \tag{3.35}$$

Now, from the second equation in (3.15), we have

$$\|\mathbf{U}_{n+1}\|_2^2 = \left\|\mathbf{U}_{n+\frac{1}{2}}\right\|_2^2 + 2\left\langle \mathbf{U}_{n+\frac{1}{2}}, \Delta\mathbf{W}_{n+1}^{Q,N}\right\rangle + \left\|\Delta\mathbf{W}_{n+1}^{Q,N}\right\|_2^2. \tag{3.36}$$

Injecting Inequality (3.35) into Equation (3.36), we get

$$\|\mathbf{U}_{n+1}\|_2^2 - \|\mathbf{U}_n\|_2^2 \leq -2\nu\Delta t\left\|\mathbf{D}_N^{(1)}\mathbf{U}_{n+\frac{1}{2}}\right\|_2^2 + 2\left\langle \mathbf{U}_{n+\frac{1}{2}}, \Delta\mathbf{W}_{n+1}^{Q,N}\right\rangle + \left\|\Delta\mathbf{W}_{n+1}^{Q,N}\right\|_2^2. \tag{3.37}$$

By definition of $\mathbf{W}^{Q,N}$ and from (3.9), we have

$$\mathbb{E}\left[\left\|\Delta\mathbf{W}_{n+1}^{Q,N}\right\|_2^2\right] = \frac{1}{N}\Delta t\sum_{i=1}^{N}\sum_{k\geq 1}\left(\sigma_i^k\right)^2 \leq D_0\Delta t. \tag{3.38}$$

On the other hand, the variables $\mathbf{U}_{n+\frac{1}{2}}$ and $\Delta\mathbf{W}_{n+1}^{Q,N}$ are independent, so that taking the expectation in (3.37) yields

$$\mathbb{E}\left[\|\mathbf{U}_{n+1}\|_2^2\right] - \mathbb{E}\left[\|\mathbf{U}_n\|_2^2\right] \leq -2\nu\Delta t\mathbb{E}\left[\left\|\mathbf{D}_N^{(1)}\mathbf{U}_{n+\frac{1}{2}}\right\|_2^2\right] + D_0\Delta t,$$

which is valid for any $n \in \mathbb{N}$, so that we can get a telescopic sum:

$$\begin{aligned}\mathbb{E}\left[\|\mathbf{U}_n\|_2^2\right] - \|\mathbf{u}_0\|_2^2 &= \sum_{l=0}^{n-1}\left(\mathbb{E}\left[\|\mathbf{U}_{l+1}\|_2^2\right] - \mathbb{E}\left[\|\mathbf{U}_l\|_2^2\right]\right)\\ &\leq -2\nu\Delta t\sum_{l=0}^{n-1}\mathbb{E}\left[\left\|\mathbf{D}_N^{(1)}\mathbf{U}_{l+\frac{1}{2}}\right\|_2^2\right] + n\Delta tD_0. \end{aligned}\tag{3.39}$$

Hence,

$$2\nu\Delta t\sum_{l=0}^{n-1}\mathbb{E}\left[\left\|\mathbf{D}_N^{(1)}\mathbf{U}_{l+\frac{1}{2}}\right\|_2^2\right] \leq \|\mathbf{u}_0\|_2^2 + n\Delta tD_0. \tag{3.40}$$

Besides,

$$\mathbb{E}\left[\left\|\mathbf{D}_N^{(1)}\mathbf{U}_{l+1}\right\|_2^2\right] = \mathbb{E}\left[\left\|\mathbf{D}_N^{(1)}\mathbf{U}_{l+\frac{1}{2}}\right\|_2^2\right] + \mathbb{E}\left[\left\|\mathbf{D}_N^{(1)}\Delta\mathbf{W}_{l+1}^{Q,N}\right\|_2^2\right], \tag{3.41}$$

and

$$\begin{aligned}
\mathbb{E}\left[\left\|\mathbf{D}_N^{(1)}\Delta\mathbf{W}_{l+1}^{Q,N}\right\|_2^2\right] &= N\mathbb{E}\left[\sum_{i=1}^{N}\left(\sum_{k\geq 1}\left(\sigma_{i+1}^k-\sigma_i^k\right)\left(W^k((l+1)\Delta t)-W^k(l\Delta t)\right)\right)^2\right]\\
&= N\sum_{i=1}^{N}\sum_{k\geq 1}\left(\sigma_{i+1}^k-\sigma_i^k\right)^2\mathbb{E}\left[\left(W^k((l+1)\Delta t)-W^k(l\Delta t)\right)^2\right]\\
&= \Delta t\sum_{k\geq 1}\left\|\mathbf{D}_N^{(1)}\boldsymbol{\sigma}^k\right\|_2^2.
\end{aligned}$$

Now, from the definition of $\boldsymbol{\sigma}^k$, the Jensen inequality and (3.5), we have

$$\begin{aligned}
\left\|\mathbf{D}_N^{(1)}\boldsymbol{\sigma}^k\right\|_2^2 &= N\sum_{i=1}^{N}\sum_{k\geq 1}\left(N\int_{\frac{i-1}{N}}^{\frac{i}{N}}\left(g_k\left(x+\frac{1}{N}\right)-g_k(x)\right)\mathrm{d}x\right)^2\\
&\leq N^2\sum_{i=1}^{N}\sum_{k\geq 1}\int_{\frac{i-1}{N}}^{\frac{i}{N}}\left(g_k\left(x+\frac{1}{N}\right)-g_k(x)\right)^2\mathrm{d}x\\
&= N^2\sum_{i=1}^{N}\sum_{k\geq 1}\int_{\frac{i-1}{N}}^{\frac{i}{N}}\left(\int_x^{x+\frac{1}{N}}\partial_x g_k(y)\mathrm{d}y\right)^2\mathrm{d}x\\
&\leq N^2\sum_{i=1}^{N}\int_{\frac{i-1}{N}}^{\frac{i}{N}}\frac{1}{N}\int_x^{x+\frac{1}{N}}\sum_{k\geq 1}\partial_x g_k(y)^2\mathrm{d}y\mathrm{d}x\\
&\leq D_0.
\end{aligned} \tag{3.42}$$

Thus, we have

$$\mathbb{E}\left[\left\|\mathbf{D}_N^{(1)}\Delta\mathbf{W}_{l+1}^{Q,N}\right\|_2^2\right] \leq \Delta t D_0. \tag{3.43}$$

Injecting (3.43) into (3.41), and (3.41) into (3.40), we get

$$\frac{1}{n}\sum_{l=0}^{n-1}\mathbb{E}\left[\left\|\mathbf{D}_N^{(1)}\mathbf{U}_{l+1}\right\|_2^2\right] \leq \frac{1}{2n\nu\Delta t}\|\mathbf{u}_0\|_2^2 + \frac{D_0}{2\nu} + \Delta t D_0. \tag{3.44}$$

Since $\|\mathbf{D}_N^{(1)}\cdot\|_2$ defines a norm on $\mathbb{R}_0^N$ and since from Corollary 3.26, the process $(\mathbf{U}_n)_{n\in\mathbb{N}}$ is Feller, the result follows from Markov's inequality and the Krylov-Bogoliubov theorem [36, Theorem 3.1.1].

Using the same arguments as for the end of the proof of Proposition 3.19, Inequalities (3.44) and (3.40) yield respectively (3.33) and (3.34). □

We now proceed to the proof of uniqueness of the invariant measure $\nu_{N,\Delta t}$.

Lemma 3.28 (Hitting any neighbourhood of 0 with positive probability)**.** *Let Assumptions 3.1 and 3.3 hold. Let $(\mathbf{U}_n)_{n\in\mathbb{N}}$ and $(\mathbf{V}_n)_{n\in\mathbb{N}}$ be two solutions of (3.15) driven by the same Wiener process $\mathbf{W}^{Q,N}$. For any $\varepsilon>0$ and any $M>0$, there exists $n_{\varepsilon,M}\in\mathbb{N}$ such that*

$$p_{\varepsilon,M} := \inf_{\|\mathbf{u}_0\|_2\vee\|\mathbf{v}_0\|_2\leq M}\mathbb{P}_{(\mathbf{u}_0,\mathbf{v}_0)}\left(\left\|\mathbf{U}_{n_{\varepsilon,M}}\right\|_1+\left\|\mathbf{V}_{n_{\varepsilon,M}}\right\|_1\leq\varepsilon\right)>0.$$

Proof. First, let $\varepsilon > 0$ and let us fix $\mathbf{u}_0, \mathbf{v}_0 \in \mathbb{R}_0^N$ such that $\|\mathbf{u}_0\|_2 \leq M$ and $\|\mathbf{v}_0\|_2 \leq M$.

Let $(\mathbf{u}_n)_{n\in\mathbb{N}}$ and $(\mathbf{v}_n)_{n\in\mathbb{N}}$ denote the noiseless counterparts of the processes $(\mathbf{U}_n)_{n\in\mathbb{N}}$ and $(\mathbf{V}_n)_{n\in\mathbb{N}}$, *i.e.*

$$\begin{cases} \mathbf{u}_{n+1} = \mathbf{u}_n + \Delta t \mathbf{b}\left(\mathbf{u}_{n+1}\right) \\ \mathbf{v}_{n+1} = \mathbf{v}_n + \Delta t \mathbf{b}\left(\mathbf{v}_{n+1}\right), \end{cases} \tag{3.45}$$

with initial conditions $\mathbf{u}_0$ and $\mathbf{v}_0$. Then $(\mathbf{u}_n)_{n\in\mathbb{N}}$ and $(\mathbf{v}_n)_{n\in\mathbb{N}}$ are subject to non-perturbed dissipativity, and consequently the sum of their energies decreases to 0 over time. Indeed, we have

$$\begin{aligned} &\|\mathbf{u}_n\|_2^2 + \|\mathbf{v}_n\|_2^2 = \|\mathbf{u}_{n+1} - \Delta t \mathbf{b}\left(\mathbf{u}_{n+1}\right)\|_2^2 + \|\mathbf{v}_{n+1} - \Delta t \mathbf{b}\left(\mathbf{v}_{n+1}\right)\|_2^2 \\ &= \|\mathbf{u}_{n+1}\|_2^2 + \|\mathbf{v}_{n+1}\|_2^2 + (\Delta t)^2 \left(\|\mathbf{b}\left(\mathbf{u}_{n+1}\right)\|_2^2 + \|\mathbf{b}\left(\mathbf{v}_{n+1}\right)\|_2^2\right) - 2\Delta t \left(\langle \mathbf{u}_{n+1}, \mathbf{b}\left(\mathbf{u}_{n+1}\right)\rangle + \langle \mathbf{v}_{n+1}, \mathbf{b}\left(\mathbf{v}_{n+1}\right)\rangle\right) \end{aligned}$$

therefore, using successively Lemma 3.13.(ii) and Lemma 3.10,

$$\begin{aligned} \|\mathbf{u}_{n+1}\|_2^2 + \|\mathbf{v}_{n+1}\|_2^2 - \left(\|\mathbf{u}_n\|_2^2 + \|\mathbf{v}_n\|_2^2\right) &\leq 2\Delta t \left(\langle \mathbf{u}_{n+1}, \mathbf{b}\left(\mathbf{u}_{n+1}\right)\rangle + \langle \mathbf{v}_{n+1}, \mathbf{b}\left(\mathbf{v}_{n+1}\right)\rangle\right) \\ &\leq -2\Delta t \nu \left(\left\|\mathbf{D}_N^{(1)} \mathbf{u}_{n+1}\right\|_2^2 + \left\|\mathbf{D}_N^{(1)} \mathbf{v}_{n+1}\right\|_2^2\right) \\ &\leq -2\Delta t \nu \left(\|\mathbf{u}_{n+1}\|_2^2 + \|\mathbf{v}_{n+1}\|_2^2\right) \end{aligned}$$

so that

$$\|\mathbf{u}_{n+1}\|_2^2 + \|\mathbf{v}_{n+1}\|_2^2 \leq \frac{1}{1 + 2\Delta t \nu} \left(\|\mathbf{u}_n\|_2^2 + \|\mathbf{v}_n\|_2^2\right),$$

by induction, we get for all $n \in \mathbb{N}$,

$$\|\mathbf{u}_n\|_2^2 + \|\mathbf{v}_n\|_2^2 \leq \left(\frac{1}{1 + 2\Delta t \nu}\right)^n \left(\|\mathbf{u}_0\|_2^2 + \|\mathbf{v}_0\|_2^2\right).$$

It appears now that if we fix the value

$$n_{\varepsilon,M} := \left\lceil \frac{-1}{\log(1 + 2\Delta t \nu)} \log\left(\frac{\varepsilon^2}{16M^2}\right) \right\rceil,$$

we get for all $n \geq n_{\varepsilon,M}$,

$$\|\mathbf{u}_n\|_1 + \|\mathbf{v}_n\|_1 \leq \|\mathbf{u}_n\|_2 + \|\mathbf{v}_n\|_2 \leq \frac{\varepsilon}{2}.$$

Now, we fix $\delta_\varepsilon := \varepsilon/(4n_{\varepsilon,M})$ and we restrict ourselves to the event

$$\left\{ \sup_{n=1,\dots,n_{\varepsilon,M}} \left\|\Delta \mathbf{W}_n^{Q,N}\right\|_1 \leq \delta_\varepsilon \right\}. \tag{3.46}$$

Let $(\mathbf{U}_n)_{n\in\mathbb{N}}$ and $(\mathbf{V}_n)_{n\in\mathbb{N}}$ be two solutions of (3.15) with the deterministic initial conditions $\mathbf{u}_0$ and $\mathbf{v}_0$ respectively. With similar arguments as for the proof of Proposition 3.17, we get from (3.15), (3.45) and Lemma 3.13.(ii), for all $n \in \mathbb{N}$,

$$\begin{aligned} \|\mathbf{U}_{n+1} - \mathbf{u}_{n+1}\|_1 + \|\mathbf{V}_{n+1} - \mathbf{v}_{n+1}\|_1 &\leq \left\|\mathbf{U}_{n+\frac{1}{2}} - \mathbf{u}_{n+1}\right\|_1 + \left\|\mathbf{V}_{n+\frac{1}{2}} - \mathbf{v}_{n+1}\right\|_1 + 2\left\|\Delta \mathbf{W}_{n+1}^{Q,N}\right\|_1 \\ &= \left\langle \mathbf{sign}\left(\mathbf{U}_{n+\frac{1}{2}} - \mathbf{u}_{n+1}\right), \mathbf{U}_n - \mathbf{u}_n \right\rangle \\ &\quad + \Delta t \left\langle \mathbf{sign}\left(\mathbf{U}_{n+\frac{1}{2}} - \mathbf{u}_{n+1}\right), \mathbf{b}\left(\mathbf{U}_{n+\frac{1}{2}}\right) - \mathbf{b}\left(\mathbf{u}_{n+1}\right) \right\rangle \\ &\quad + \left\langle \mathbf{sign}\left(\mathbf{V}_{n+\frac{1}{2}} - \mathbf{v}_{n+1}\right), \mathbf{V}_n - \mathbf{v}_n \right\rangle \\ &\quad + \Delta t \left\langle \mathbf{sign}\left(\mathbf{V}_{n+\frac{1}{2}} - \mathbf{v}_{n+1}\right), \mathbf{b}\left(\mathbf{V}_{n+\frac{1}{2}}\right) - \mathbf{b}\left(\mathbf{v}_{n+1}\right) \right\rangle + 2\left\|\Delta \mathbf{W}_{n+1}^{Q,N}\right\|_1 \\ &\leq \|\mathbf{U}_n - \mathbf{u}_n\|_1 + \|\mathbf{V}_n - \mathbf{v}_n\|_1 + 2\left\|\Delta \mathbf{W}_{n+1}^{Q,N}\right\|_1. \end{aligned}$$

On the event (3.46), we have for all $n = 1, \dots, n_{\varepsilon,M}$,

$$\|\mathbf{U}_{n+1} - \mathbf{u}_{n+1}\|_1 + \|\mathbf{V}_{n+1} - \mathbf{v}_{n+1}\|_1 \le \|\mathbf{U}_n - \mathbf{u}_n\|_1 + \|\mathbf{V}_n - \mathbf{v}_n\|_1 + 2\delta_\varepsilon.$$

In particular, by induction, we have

$$\left\|\mathbf{U}_{n_{\varepsilon,M}} - \mathbf{u}_{n_{\varepsilon,M}}\right\|_1 + \left\|\mathbf{V}_{n_{\varepsilon,M}} - \mathbf{v}_{n_{\varepsilon,M}}\right\|_1 \le 2n_{\varepsilon,M}\delta_\varepsilon = \frac{\varepsilon}{2}.$$

Thus,

$$\begin{aligned}\left\|\mathbf{U}_{n_{\varepsilon,M}}\right\|_1 + \left\|\mathbf{V}_{n_{\varepsilon,M}}\right\|_1 &\le \left\|\mathbf{U}_{n_{\varepsilon,M}} - \mathbf{u}_{n_{\varepsilon,M}}\right\|_1 + \left\|\mathbf{V}_{n_{\varepsilon,M}} - \mathbf{v}_{n_{\varepsilon,M}}\right\|_1 + \left\|\mathbf{u}_{n_{\varepsilon,M}}\right\|_1 + \left\|\mathbf{v}_{n_{\varepsilon,M}}\right\|_1 \\ &\le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

We just have shown that

$$\mathbb{P}_{(\mathbf{u}_0,\mathbf{v}_0)}\left(\left\|\mathbf{U}_{n_{\varepsilon,M}}\right\|_1 + \left\|\mathbf{V}_{n_{\varepsilon,M}}\right\|_1 \le \varepsilon\right) \ge \mathbb{P}\left(\sup_{n=1,\dots,n_{\varepsilon,M}} \left\|\Delta \mathbf{W}_n^{Q,N}\right\|_1 \le \delta_\varepsilon\right) > 0.$$

Since the event (3.46) does not depend on $\mathbf{u}_0$ nor $\mathbf{v}_0$, we get the result. $\square$

For two solutions $(\mathbf{U}_n)_{n\in\mathbb{N}}$ and $(\mathbf{V}_n)_{n\in\mathbb{N}}$ of (3.15) driven by the same Wiener process $\mathbf{W}^{Q,N}$, we define the entrance time

$$\eta_M := \inf\left\{n \in \mathbb{N} : \|\mathbf{U}_{n+1}\|_2 \vee \|\mathbf{V}_{n+1}\|_2 \le M\right\}.$$

The following lemma is the time-discrete version of Lemma 3.22. The proof is omitted as it is very similar to its time-continuous counterpart.

Lemma 3.29 (Almost sure entrance in some ball)**.** *Under Assumptions 3.1 and 3.3, there exists $M > 0$ such that for any initial conditions $\mathbf{u}_0$, $\mathbf{v}_0 \in \mathbb{R}_0^N$ for the processes $(\mathbf{U}_n)_{n\in\mathbb{N}}$ and $(\mathbf{V}_n)_{n\in\mathbb{N}}$, $\eta_M < +\infty$ almost surely.*

Proof of Theorem 1, Assertion (ii). Given Lemmas 3.24, 3.28 and 3.29, the proof is done in exactly the same way as for Assertion (i).

Finally, the fact that ν_N and $\nu_{N,\Delta t}$ belong to $\mathcal{P}_2(\mathbb{R}_0^N)$ come from Propositions 3.19 and 3.27 respectively (for $\nu_{N,\Delta t}$, we use in particular the fact that $\|\mathbf{D}_N^{(1)} \cdot \|_2$ defines a norm on $\mathbb{R}_0^N$). $\square$

3.3 Convergence of invariant measures: semi-discrete scheme towards SPDE

The purpose of this section is to prove that $W_2(\mu_N^{(m)}, \mu) \longrightarrow 0$, $N \to +\infty$, $m = 0, 1, 2$, which will be the first part of the proof of Theorem 3.7. In Subsection 3.3.1, we provide a result ensuring that it is sufficient that the convergence holds for only one $m \in \{0, 1, 2\}$, in which case it will hold for the three of them. Then, we show that $(\mu_N^{(m)})_{N\ge1}$ is relatively compact in $\mathcal{P}_2(L_0^2(\mathbb{T}))$, and in Subsection 3.3.2, we present a procedure to identify any subsequential limit of $(\mu_N^{(m)})_{N\ge1}$ as the invariant measure μ for the solution of (3.1), which leads to the proof the first assertion of Theorem 3.7. Subsection 3.3.3 contain the proofs of the lemmas from Subsections 3.3.1 and 3.3.2.

3.3.1 Notations and preliminary results

For $m = 0, 1, 2$, we define the *interpolation operators* $\Psi_N^{(m)} : \mathbb{R}_0^N \to W_0^{m,\infty}(\mathbb{T})$ by

$$\Psi_N^{(m)}\mathbf{v}(x) = \sum_{i=1}^{N} v_i \phi_N^{(m)}\left(x - \frac{i}{N}\right), \quad \mathbf{v} = (v_1, \dots, v_N) \in \mathbb{R}_0^N,$$

where

$$\phi_N^{(0)}(x) = \mathbf{1}_{(-\frac{1}{N},0]}(x),$$

$$\phi_N^{(1)}(x) = N\left(x + \frac{1}{N}\right)\mathbf{1}_{(-\frac{1}{N},0]}(x) + N\left(\frac{1}{N} - x\right)\mathbf{1}_{(0,\frac{1}{N}]}(x),$$

$$\begin{aligned}\phi_N^{(2)}(x) = \frac{N^2}{2}\left(x + \frac{1}{N}\right)\left(x + \frac{2}{N}\right)\mathbf{1}_{(-\frac{2}{N},-\frac{1}{N}]}(x) - N^2\left(x + \frac{1}{N}\right)\left(x - \frac{1}{N}\right)\mathbf{1}_{(-\frac{1}{N},0]}(x) \\ + \frac{N^2}{2}\left(x - \frac{1}{N}\right)\left(x - \frac{2}{N}\right)\mathbf{1}_{(0,\frac{1}{N}]}(x),\end{aligned}$$

so that $\Psi_N^{(0)}\mathbf{v}$, $\Psi_N^{(1)}\mathbf{v}$ and $\Psi_N^{(2)}\mathbf{v}$ are the respectively piecewise constant, linear and quadratic interpolations of the values v_i at the points i/N. In this regard, note that for $\mathbf{v} \in \mathbb{R}_0^N$, $i \in \mathbb{Z}/N\mathbb{Z}$ and $m = 0, 1, 2$, we have $\Psi_N^{(m)}\mathbf{v}(\frac{i}{N}) = v_i$.

We recall that these operators allowed to define the sequences of embedded invariant measures $\mu_N^{(m)} = \nu_N \circ (\Psi_N^{(m)})^{-1}$, where $\mu_N^{(m)}$ is here considered as an element of $\mathcal{P}(L_0^2(\mathbb{T}))$.

We prove the following lemma in the appendix:

Lemma 3.30. *The following properties hold:*

(i) for $\mathbf{v} \in \mathbb{R}_0^N$, any $p \in [1, +\infty]$ and any $m = 0, 1, 2$,

$$\left\|\Psi_N^{(m)}\mathbf{v}\right\|_{W_0^{m,p}(\mathbb{T})} = \left\|\mathbf{D}_N^{(m)}\mathbf{v}\right\|_p,$$

(ii) for any $\mathbf{v} \in \mathbb{R}_0^N$,

$$\left\|\Psi_N^{(1)}\mathbf{v} - \Psi_N^{(0)}\mathbf{v}\right\|_{L_0^2(\mathbb{T})}^2 = \frac{1}{3N^2}\left\|\mathbf{D}_N^{(1)}\mathbf{v}\right\|_2^2$$

and

$$\left\|\Psi_N^{(2)}\mathbf{v} - \Psi_N^{(0)}\mathbf{v}\right\|_{L_0^2(\mathbb{T})}^2 \le \frac{3}{20N^4}\left\|\mathbf{D}_N^{(2)}\mathbf{v}\right\|_2^2 + \frac{1}{2N^2}\left\|\mathbf{D}_N^{(1)}\mathbf{v}\right\|_2^2.$$

The proof of the following Lemma is given below in Subsection 3.3.3.

Lemma 3.31 (Discrete H_0^1 and H_0^2 bounds)**.** *Let Assumptions 3.1 and 3.3 hold and let $\mathbf{V}$ be a random variable in $\mathbb{R}_0^N$ with distribution ν_N, then $\mathbf{V}$ satisfies*

$$\mathbb{E}\left[\left\|\mathbf{D}_N^{(1)}\mathbf{V}\right\|_2^2\right] \le \frac{D_0}{2\nu} =: C^{1,2}. \tag{3.47}$$

Furthermore, there exists a positive constant $C^{2,2}$ not depending on N such that,

$$\mathbb{E}\left[\left\|\mathbf{D}_N^{(2)}\mathbf{V}\right\|_2^2\right] \le C^{2,2}. \tag{3.48}$$

The following result ensures in particular that the estimates obtained at Lemma 3.31 and Proposition 3.19 remain true when passing to the limit $N \to +\infty$.

Lemma 3.32 (Relative compactness)**.** *Under Assumptions 3.1 and 3.3, the three families of probability measures $(\mu_N^{(m)})_{N\geq 1}$, $m = 0, 1, 2$, are relatively compact in the space $\mathcal{P}_2(L_0^2(\mathbb{T}))$. Moreover, for any $m = 0, 1, 2$ and for any subsequential limit $\mu^* \in \mathcal{P}_2(L_0^2(\mathbb{T}))$ of $(\mu_N^{(m)})_{N\geq 1}$, a random variable $v \sim \mu^*$ satisfies*

$$\mathbb{E}\left[\|v\|_{H_0^1(\mathbb{T})}^2\right] \leq C^{1,2}, \quad \mathbb{E}\left[\|v\|_{H_0^2(\mathbb{T})}^2\right] \leq C^{2,2} \quad \text{and} \quad \mathbb{E}\left[\|v\|_{L_0^p(\mathbb{T})}^p\right] \leq C^{0,p}, \quad p \in [1, +\infty). \tag{3.49}$$

Proof. **Step 1. Relative compactness of $(\mu_N^{(1)})_{N\geq 1}$ in $\mathcal{P}(L_0^2(\mathbb{T}))$.** Let $\mathbf{V}$ be an $\mathbb{R}_0^N$-valued random variable with distribution ν_N. Then, $\Psi_N^{(1)}\mathbf{V}$ has distribution $\mu_N^{(1)}$. Thanks to Lemmas 3.30 and 3.31, we have

$$\mathbb{E}\left[\left\|\Psi_N^{(1)}\mathbf{V}\right\|_{H_0^1(\mathbb{T})}^2\right] = \mathbb{E}\left[\left\|\mathbf{D}_N^{(1)}\mathbf{V}\right\|_2^2\right] \leq C^{1,2}.$$

Thus, Markov's inequality implies

$$\forall \varepsilon > 0, \qquad \mathbb{P}\left(\left\|\Psi_N^{(1)}\mathbf{V}\right\|_{H_0^1(\mathbb{T})}^2 > \frac{1}{\varepsilon}\right) \leq \varepsilon C^{1,2}.$$

The space $H_0^1(\mathbb{T})$ is compactly embedded in $L_0^2(\mathbb{T})$, so this last inequality means that the sequence $(\mu_N^{(1)})_{N\in\mathbb{N}^*}$ is tight in the space $\mathcal{P}(L_0^2(\mathbb{T}))$. As a consequence of Prokhorov's theorem [12, Theorem 5.1], any subsequence of $(\mu_N^{(1)})_{N\geq 1}$ admits itself a weakly converging subsequence in $\mathcal{P}(L_0^2(\mathbb{T}))$. In this respect, let $\mu^* \in \mathcal{P}(L_0^2(\mathbb{T}))$ be a subsequential limit of $(\mu_N^{(1)})_{N\geq 1}$ and let $(\mu_{N_j}^{(1)})_{j\in\mathbb{N}}$ be the associated subsequence.

Step 2. Relative compactness of $(\mu_N^{(0)})_{N\geq 1}$ in $\mathcal{P}_2(L_0^2(\mathbb{T}))$. Let v be a random variable with distribution μ^*. On the one hand, the sequence of random variables $(\Psi_{N_j}^{(1)}\mathbf{V})_{j\in\mathbb{N}}$ converges in distribution towards v. On the other hand, by Lemmas 3.30.(ii) and 3.31, we have

$$\mathbb{E}\left[\left\|\Psi_{N_j}^{(0)}\mathbf{V} - \Psi_{N_j}^{(1)}\mathbf{V}\right\|_{L_0^2(\mathbb{T})}^2\right] \leq \frac{1}{3N_j^2}\mathbb{E}\left[\left\|\mathbf{D}_{N_j}^{(1)}\mathbf{V}\right\|_2^2\right] \leq \frac{C^{1,2}}{3N_j^2} \xrightarrow[j\to\infty]{} 0,$$

so that $\Psi_{N_j}^{(0)}\mathbf{V} - \Psi_{N_j}^{(1)}\mathbf{V}$ converges in probability towards 0 as $j \to +\infty$. As a consequence, by Slutsky's theorem [12, Theorem 3.9], the couple $(\Psi_{N_j}^{(1)}\mathbf{V}, \Psi_{N_j}^{(0)}\mathbf{V} - \Psi_{N_j}^{(1)}\mathbf{V})$ converges in distribution towards $(v, 0)$ as $j \to +\infty$. In particular, $\Psi_{N_j}^{(0)}\mathbf{V}$ converges in distribution towards v, which means that $\mu_{N_j}^{(0)}$ converges weakly in $\mathcal{P}(L_0^2(\mathbb{T}))$ towards μ^*. Moreover, $\mu_{N_j}^{(0)}$ has uniform moment bounds with respect to j thanks to (3.2), Lemma 3.30.(i) and Proposition 3.19:

$$\mathbb{E}\left[\left\|\Psi_{N_j}^{(0)}\mathbf{V}\right\|_{L_0^2(\mathbb{T})}^p\right] \leq \mathbb{E}\left[\left\|\Psi_{N_j}^{(0)}\mathbf{V}\right\|_{L_0^p(\mathbb{T})}^p\right] = \mathbb{E}\left[\|\mathbf{V}\|_p^p\right] \leq C^{0,p}, \quad \forall p \geq 2. \tag{3.50}$$

As a consequence, $(\mu_{N_j}^{(0)})_{j\in\mathbb{N}}$ satisfies a uniform integrability condition in the sense of [102, Definition 6.8.(iii)] and thus, is converging for the Wasserstein distance in $\mathcal{P}_2(L_0^2(\mathbb{T}))$ towards μ^*.

Step 3. Relative compactness of $(\mu_{N_j}^{(1)})_{j\in\mathbb{N}}$ and $(\mu_{N_j}^{(2)})_{j\in\mathbb{N}}$ in $\mathcal{P}_2(L_0^2(\mathbb{T}))$. The sequences $(\mu_{N_j}^{(1)})_{j\in\mathbb{N}}$ and $(\mu_{N_j}^{(2)})_{j\in\mathbb{N}}$ also converge towards μ^* in $\mathcal{P}_2(L_0^2(\mathbb{T}))$ by Lemmas 3.30 and 3.31. Indeed, we have

$$W_2\left(\mu_{N_j}^{(1)}, \mu_{N_j}^{(0)}\right)^2 \leq \mathbb{E}\left[\left\|\Psi_{N_j}^{(1)}\mathbf{V} - \Psi_{N_j}^{(0)}\mathbf{V}\right\|_{L_0^2(\mathbb{T})}^2\right] \leq \frac{C^{1,2}}{3N_j^2}, \tag{3.51}$$

and

$$W_2\left(\mu_{N_j}^{(2)},\mu_{N_j}^{(0)}\right)^2 \leq \mathbb{E}\left[\left\|\Psi_{N_j}^{(2)}\mathbf{V}-\Psi_{N_j}^{(0)}\mathbf{V}\right\|_{L_0^2(\mathbb{T})}^2\right]$$
$$\leq \mathbb{E}\left[\frac{3}{20N_j^4}\left\|\mathbf{D}_{N_j}^{(2)}\mathbf{V}\right\|_2^2+\frac{1}{2N_j^2}\left\|\mathbf{D}_{N_j}^{(1)}\mathbf{V}\right\|_2^2\right]$$
$$\leq \frac{3C^{2,2}}{20N_j^4}+\frac{C^{1,2}}{2N_j^2}. \tag{3.52}$$

Step 4. Moment estimates. Finally, the estimates (3.49) follow from Portemanteau's theorem: since $\|\cdot\|_{H_0^1(\mathbb{T})}^2$ is lower semi-continuous on the space $L_0^2(\mathbb{T})$, we have

$$\mathbb{E}\left[\|v\|_{H_0^1(\mathbb{T})}^2\right] \leq \liminf_{j\to\infty}\mathbb{E}\left[\left\|\Psi_{N_j}^{(1)}\mathbf{V}\right\|_{H_0^1(\mathbb{T})}^2\right] \leq C^{1,2},$$

and the same argument applies for $\|\cdot\|_{H_0^2(\mathbb{T})}^2$ and $\|\cdot\|_{L_0^p(\mathbb{T})}^p$ using respectively the sequences of random variables $(\Psi_{N_j}^{(2)}\mathbf{V})_{j\in\mathbb{N}}$ and $(\Psi_{N_j}^{(0)}\mathbf{V})_{j\in\mathbb{N}}$. □

The three following lemmas will be useful for the proof of finite time convergence stated in the next subsection, namely Proposition 3.36. The proofs are given in Subsection 3.3.3.

Lemma 3.33 (Discrete $W_0^{1,3}$ bound)**.** *Let Assumptions 3.1 and 3.3 hold and let $\mathbf{V}$ be an $\mathbb{R}_0^N$-valued random variable distributed according to ν_N. Then, there exists a constant $C^{1,3}$ depending only on ν, $p_{\overline{A}}$ and D_0 such that*

$$\mathbb{E}\left[\left\|\mathbf{D}_N^{(1)}\mathbf{V}\right\|_3^3\right] \leq C^{1,3}.$$

Lemma 3.34 (Discrete H_0^1 bound in finite time)**.** *Let Assumptions 3.1 and 3.3 hold and let $(\mathbf{U}(t))_{t\geq 0}$ be the solution of (3.11) with an initial condition $\mathbf{U}_0\sim\nu_N$. For every $T>0$, there exists a constant $C_T^{1,2}$ not depending on N such that*

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left\|\mathbf{D}_N^{(1)}\mathbf{U}(t)\right\|_2^2\right] \leq C_T^{1,2}.$$

Lemma 3.35 (Moments on the solution of (3.1))**.** *Under Assumption 3.1, for all $p\in[2,+\infty)$ and $T>0$, there are constants $\widetilde{C}_T^{0,p}$ and $\widetilde{C}_T^{1,2}$ such that the solution $(u(t))_{t\geq 0}$ of (3.1) with initial condition $u_0\sim\mu^*$ satisfies for all $t\in[0,T]$:*

$$\sup_{t\in[0,T]}\mathbb{E}\left[\|u(t)\|_{L_0^p(\mathbb{T})}^p\right] \leq \widetilde{C}_T^{0,p} \quad \textit{and} \quad \sup_{t\in[0,T]}\mathbb{E}\left[\|u(t)\|_{H_0^1(\mathbb{T})}^2\right] \leq \widetilde{C}_T^{1,2}.$$

3.3.2 Characterisation of the limit

In Lemma 3.32 we proved the existence of subsequential limits μ^* for the sequences of embedded invariant measures $(\mu_N^{(m)})_{N\geq 1}$. Our convergence argument now consists in identifying any such limit μ^* with the unique invariant measure μ of the solution $(u(t))_{t\geq 0}$ of (3.1) (see Proposition 3.2). We proved in Lemma 3.32 that μ^* gives full weight to $H_0^2(\mathbb{T})$. As a consequence of this result, the measure μ^* can be considered as an initial distribution for $(u(t))_{t\geq 0}$. The weak convergence of the subsequence $\mu_{N_j}^{(1)}$ towards μ^* can be represented, by virtue of the Skorokhod theorem, by the almost sure $L_0^2(\mathbb{T})$-convergence, on some particular probability space, of a sequence of random variables $u_{j,0}^{(1)}$

towards u_0, where $u_{j,0}^{(1)} \sim \mu_{N_j}^{(1)}$, $\forall j \in \mathbb{N}$, and $u_0 \sim \mu^*$. We may define on this probability space, up to enlarging it, a Q-Wiener process $(W^Q(t))_{t\geq 0}$ defined as in Section 3.1.1, independent of u_0 and $u_{j,0}^{(1)}$, along with a normal filtration. In such a way, we may consider u_0 and $u_{j,0}^{(1)}$ as initial conditions for the solution of (3.1) and the embedded solutions of (3.11) respectively. More precisely, if we denote by $\mathbf{U}_0 = (U_{1,0}, \dots, U_{N_j,0})$ the $\mathbb{R}_0^{N_j}$-valued random variable such that $U_{i,0} = u_{j,0}(i/N)$, and if we define $(\mathbf{U}(t))_{t\geq 0}$ the solution of (3.11) starting at $\mathbf{U}_0$, then we define the process $(u_{N_j}^{(1)}(t))_{t\geq 0}$ by $u_{N_j}^{(1)}(t) = \Psi_{N_j}^{(1)}\mathbf{U}(t)$, for all $t \geq 0$.

Given that $(u_{N_j}^{(1)}(t))_{t\geq 0}$ is a numerical approximation of $(u(t))_{t\geq 0}$, convergence at time 0 shall lead to convergence at every finite time t:

Proposition 3.36. *Under Assumptions 3.1 and 3.3, for every $t \geq 0$, we have*

$$\lim_{j\to\infty} \mathbb{E}\left[\left\|u_{N_j}^{(1)}(t) - u(t)\right\|_{L_0^2(\mathbb{T})}^2\right] = 0.$$

This result is proved in Section 3.3.3 below. Let us explain how this finite time result leads to the convergence of $(\mu_N^{(m)})_{N\geq 1}$ towards μ in $\mathcal{P}_2(L_0^2(\mathbb{T}))$.

Proof of Theorem 3.7: part 1/2. The measure $\mu_{N_j}^{(1)}$ is invariant for the process $(u_{N_j}^{(1)}(t))_{t\geq 0}$. For all $t \geq 0$, let $\mu_t^* \in \mathcal{P}(L_0^2(\mathbb{T}))$ denote the probability distribution of $u(t)$. By Definition 3.6 and Proposition 3.36, we have

$$\forall t \geq 0, \quad \lim_{j\to\infty} W_2\left(\mu_{N_j}^{(1)}, \mu_t^*\right) = 0.$$

By continuity of the Wasserstein distance [102, Corollary 6.11] and Lemma 3.32, this leads to

$$\forall t \geq 0, \quad W_2\left(\mu^*, \mu_t^*\right) = 0. \tag{3.53}$$

From Lemma 2.20, there exists a unique probability measure in $\mathcal{P}(H_0^2(\mathbb{T}))$ coinciding with μ^* on the Borel sets of $H_0^2(\mathbb{T})$ (for convenience, we still call this measure μ^*). The meaning of (3.53) is that this measure $\mu^* \in \mathcal{P}(H_0^2(\mathbb{T}))$ is invariant for the process $(u(t))_{t\geq 0}$. However, this process already has a unique invariant measure $\mu \in \mathcal{P}(H_0^2(\mathbb{T}))$, so that necessarily $\mu^* = \mu$. As a consequence, μ is the only subsequential limit in $\mathcal{P}_2(L_0^2(\mathbb{T}))$ of the sequence $(\mu_N^{(1)})_{N\geq 1}$, and since from Lemma 3.32, $(\mu_N^{(1)})_{N\geq 1}$ is relatively compact in $\mathcal{P}_2(L_0^2(\mathbb{T}))$, we just proved (3.16) (see for instance [12, Theorem 2.6]) in the case $m = 1$. The cases $m = 0$ and $m = 2$ follow from the bounds (3.51) and (3.52). □

Remark 3.37. *This last proof shows in particular that μ^* is invariant for $(u(t))_{t\geq 0}$. Therefore it provides a second proof for the existence part in Theorem 2.7.*

3.3.3 Proofs

Proof of Lemma 3.31. Let us decompose the function $\mathbf{b}$ as the sum of $\mathbf{b}_1$ and $\mathbf{b}_2$, defined by

$$\forall \mathbf{v} \in \mathbb{R}_0^N, \quad b_i^1(\mathbf{v}) := -N\left(\overline{A}(v_i, v_{i+1}) - \overline{A}(v_{i-1}, v_i)\right) \quad \text{and} \quad \mathbf{b}^2 := \nu \mathbf{D}_N^{(2)}. \tag{3.54}$$

To prove the first inequality of the lemma, apply Corollary 3.20 with $p = 2$ and recall that we may take $C^{0,0} = 1$.

We focus now on the discrete H_0^2 estimate. Let $\mathbf{V} \sim \nu_N$ and let $(\mathbf{U}(t))_{t\geq 0}$ be the solution of (3.11) with initial distribution ν_N. We may compute the dynamics of the discrete H_0^1-norm of $(\mathbf{U}(t))_{t\geq 0}$ by using Itô's formula: for all $t \geq 0$,

$$\left\|\mathbf{D}_N^{(1)}\mathbf{U}(t)\right\|_2^2 = \left\|\mathbf{D}_N^{(1)}\mathbf{U}_0\right\|_2^2 + 2\int_0^t \left\langle \mathbf{D}_N^{(1)}\mathbf{b}(\mathbf{U}(s)), \mathbf{D}_N^{(1)}\mathbf{U}(s)\right\rangle \mathrm{d}s + 2\int_0^t \left\langle \mathbf{D}_N^{(1)}\mathbf{U}(s), \mathrm{d}\left(\mathbf{D}_N^{(1)}\mathbf{W}^{Q,N}\right)(s)\right\rangle + t\sum_{k\geq 1}\left\|\mathbf{D}_N^{(1)}\boldsymbol{\sigma}^k\right\|_2^2. \quad (3.55)$$

It appears that the third term of the right-hand side is a martingale since

$$\sum_{k\geq 1}\mathbb{E}\left[\int_0^t \left\langle \mathbf{D}_N^{(1)}\mathbf{U}(s), \mathbf{D}_N^{(1)}\boldsymbol{\sigma}^k\right\rangle^2 \mathrm{d}s\right] \leq t\left(\sum_{k\geq 1}\left\|\mathbf{D}_N^{(1)}\boldsymbol{\sigma}^k\right\|_2^2\right)\mathbb{E}\left[\left\|\mathbf{D}_N^{(1)}\mathbf{V}\right\|_2^2\right] \leq tD_0C^{1,2} < +\infty, \quad (3.56)$$

where we used the stationarity of $(\mathbf{U}(t))_{t\geq 0}$, Inequality (3.42), and the first inequality from this lemma. Thus, taking the expectation and expanding the drift term, we get

$$\mathbb{E}\left[\left\|\mathbf{D}_N^{(1)}\mathbf{U}(t)\right\|_2^2\right] = \mathbb{E}\left[\left\|\mathbf{D}_N^{(1)}\mathbf{U}_0\right\|_2^2\right] + 2\int_0^t \mathbb{E}\left[\left\langle \mathbf{D}_N^{(1)}\mathbf{U}(s), \mathbf{D}_N^{(1)}\mathbf{b}(\mathbf{U}(s))\right\rangle\right]\mathrm{d}s + t\sum_{k\geq 1}\left\|\mathbf{D}_N^{(1)}\boldsymbol{\sigma}^k\right\|_2^2.$$

Since the process starts from its invariant measure, the left-hand side cancels with the first term of the right-hand side. Besides, we may drop the time index. Using the decomposition $\mathbf{b} = \mathbf{b}^1 + \mathbf{b}^2$, we may sum by parts both the viscous and the flux term, and after dividing by t, it remains

$$2\nu\mathbb{E}\left[\left\|\mathbf{D}_N^{(2)}\mathbf{V}\right\|_2^2\right] = -2\mathbb{E}\left[\left\langle \mathbf{D}_N^{(2)}\mathbf{V}, \mathbf{b}^1(\mathbf{V})\right\rangle\right] + \sum_{k\geq 1}\left\|\mathbf{D}_N^{(1)}\boldsymbol{\sigma}^k\right\|_2^2 \leq 2\sqrt{\mathbb{E}\left[\left\|\mathbf{D}_N^{(2)}\mathbf{V}\right\|_2^2\right]}\sqrt{\mathbb{E}\left[\|\mathbf{b}^1(\mathbf{V})\|_2^2\right]} + D_0, \quad (3.57)$$

where we used in particular the Cauchy-Schwarz inequality. We can bound the term in the second square root thanks to Assumption 3.3:

$$\begin{aligned}
\mathbb{E}\left[\|\mathbf{b}^1(\mathbf{V})\|_2^2\right] &= \mathbb{E}\left[N\sum_{i=1}^N \left(\overline{A}(V_i, V_{i+1}) - \overline{A}(V_{i-1}, V_i)\right)^2\right] \\
&= \mathbb{E}\left[N\sum_{i=1}^N\left(\int_{V_i}^{V_{i+1}} \partial_2\overline{A}(V_i, z)\mathrm{d}z + \int_{V_{i-1}}^{V_i}\partial_1\overline{A}(z, V_i)\mathrm{d}z\right)^2\right] \quad \text{(by (3.13))} \\
&\leq 2\mathbb{E}\left[N\sum_{i=1}^N (V_{i+1} - V_i)\int_{V_i}^{V_{i+1}}\partial_2\overline{A}(V_i, z)^2\mathrm{d}z\right] + 2\mathbb{E}\left[N\sum_{i=1}^N (V_i - V_{i-1})\int_{V_{i-1}}^{V_i}\partial_1\overline{A}(z, V_i)^2\mathrm{d}z\right] \\
&\qquad \text{(by Jensen)} \\
&\leq 4C_{\overline{A}}^2\mathbb{E}\left[N\sum_{i=1}^N (V_i - V_{i-1})\int_{V_{i-1}}^{V_i}(1 + |z|^{p_{\overline{A}}})^2\,\mathrm{d}z\right] \quad \text{(by (3.14))} \\
&\leq 8C_{\overline{A}}^2\left(\mathbb{E}\left[N\sum_{i=1}^N (V_i - V_{i-1})^2\right] + \mathbb{E}\left[N\sum_{i=1}^N (V_i - V_{i-1})\int_{V_{i-1}}^{V_i}|z|^{2p_{\overline{A}}}\mathrm{d}z\right]\right) \\
&= 8C_{\overline{A}}^2\left(\mathbb{E}\left[\left\|\mathbf{D}_N^{(1)}\mathbf{V}\right\|_2^2\right] + \frac{1}{2p_{\overline{A}}+1}\mathbb{E}\left[\left\langle \mathbf{D}_N^{(1)}\left(\mathbf{V}^{2p_{\overline{A}}+1}\right), \mathbf{D}_N^{(1)}\mathbf{V}\right\rangle\right]\right) \\
&\leq 8C_{\overline{A}}^2\left(C^{1,2} + \frac{D_0}{2\nu}C^{0,2p_{\overline{A}}}\right),
\end{aligned} \quad (3.58)$$

where we used Lemma 3.31 and Corollary 3.20 with $p = 2p_{\overline{A}} + 2$ at the last line.

Injecting (3.58) into (3.57), we get

$$2\nu\mathbb{E}\left[\left\|\mathbf{D}_N^{(2)}\mathbf{U}\right\|_2^2\right] \leq 2\sqrt{\mathbb{E}\left[\left\|\mathbf{D}_N^{(2)}\mathbf{U}\right\|_2^2\right]}\sqrt{4C_{\overline{A}}^2\frac{D_0}{\nu}\left(1 + C^{0,2p_{\overline{A}}}\right)} + D_0.$$

Applying Young's inequality on the right-hand side, we get

$$2\nu\mathbb{E}\left[\left\|\mathbf{D}_N^{(2)}\mathbf{U}\right\|_2^2\right] \leq \nu\mathbb{E}\left[\left\|\mathbf{D}_N^{(2)}\mathbf{U}\right\|_2^2\right] + 4C_{\overline{A}}^2\frac{D_0}{\nu^2}\left(1 + C^{0,2p_{\overline{A}}}\right) + D_0,$$

which rewrites

$$\mathbb{E}\left[\left\|\mathbf{D}_N^{(2)}\mathbf{U}\right\|_2^2\right] \leq 4C_{\overline{A}}^2\frac{D_0}{\nu^3}\left(1 + C^{0,2p_{\overline{A}}}\right) + \frac{D_0}{\nu}.$$

Since the right-hand side does not depend on N, we get the result. □

Proof of Lemma 3.33. From summations by parts and Hölder's inequality, we establish a discrete Gagliardo-Nirenberg inequality in the following way (similar inequalities in the multi-dimensional case are given for instance in [18, Lemma 6] or [11, Theorem 4]):

$$\begin{aligned}
\mathbb{E}\left[\sum_{i=1}^N |V_{i+1} - V_i|^3\right] &= \mathbb{E}\left[\sum_{i=1}^N (V_{i+1} - V_i)^2|V_{i+1} - V_i|\right] \\
&= -\mathbb{E}\left[\sum_{i=1}^N V_i\left((V_{i+1} - V_i)|V_{i+1} - V_i| - (V_i - V_{i-1})|V_i - V_{i-1}|\right)\right] \\
&= -\mathbb{E}\left[\sum_{i=1}^N V_i\left((V_{i+1} - 2V_i + V_{i-1})|V_{i+1} - V_i| + (V_i - V_{i-1})\left(|V_{i+1} - V_i| - |V_i - V_{i-1}|\right)\right)\right] \\
&\leq \mathbb{E}\left[\sum_{i=1}^N |V_i||V_{i+1} - V_i||V_{i+1} - 2V_i + V_{i-1}|\right] + \mathbb{E}\left[\sum_{i=1}^N |V_i||V_i - V_{i-1}||V_{i+1} - 2V_i + V_{i-1}|\right] \\
&\leq 2\mathbb{E}\left[\sum_{i=1}^N |V_i|^6\right]^{\frac{1}{6}}\mathbb{E}\left[\sum_{i=1}^N |V_{i+1} - V_i|^3\right]^{\frac{1}{3}}\mathbb{E}\left[\sum_{i=1}^N |V_{i+1} - 2V_i + V_{i-1}|^2\right]^{\frac{1}{2}}.
\end{aligned}$$

Dividing on both sides by $\mathbb{E}[\sum_i |V_{i+1} - V_i|^3]^{1/3}$ and then passing to the power $3/2$, we obtain

$$\mathbb{E}\left[\sum_{i=1}^N |V_{i+1} - V_i|^3\right] \leq 2^{3/2}\mathbb{E}\left[\sum_{i=1}^N |V_i|^6\right]^{\frac{1}{4}}\mathbb{E}\left[\sum_{i=1}^N |V_{i+1} - 2V_i + V_{i-1}|^2\right]^{\frac{3}{4}}.$$

Multiplying on both sides by N^2, we derive the inequality

$$\mathbb{E}\left[\left\|\mathbf{D}_N^{(1)}\mathbf{V}\right\|_3^3\right] \leq 2\sqrt{2}\mathbb{E}\left[\|\mathbf{V}\|_6^6\right]^{1/4}\mathbb{E}\left[\left\|\mathbf{D}_N^{(2)}\mathbf{V}\right\|_2^2\right]^{3/4},$$

and we conclude thanks to Proposition 3.19 and Lemma 3.31:

$$\mathbb{E}\left[\left\|\mathbf{D}_N^{(1)}\mathbf{V}\right\|_3^3\right] \leq 2\sqrt{2}\left(C^{0,6}\right)^{1/4}\left(C^{2,2}\right)^{3/4} =: C^{1,3}.$$ □

Proof of Lemma 3.34. Let $\mathbf{U}_0 \sim \nu_N$. Using the decomposition $\mathbf{b} = \mathbf{b}^1 + \mathbf{b}^2$ introduced at (3.54), we may expand Equation (3.55):

$$\left\|\mathbf{D}_N^{(1)}\mathbf{U}(t)\right\|_2^2 = \left\|\mathbf{D}_N^{(1)}\mathbf{U}_0\right\|_2^2 + 2\int_0^t \left\langle \mathbf{D}_N^{(1)}\mathbf{b}^1(\mathbf{U}(s)), \mathbf{D}_N^{(1)}\mathbf{U}(s)\right\rangle \mathrm{d}s + 2\nu\int_0^t \left\langle \mathbf{D}_N^{(1)}\mathbf{D}_N^{(2)}\mathbf{U}(s), \mathbf{D}_N^{(1)}\mathbf{U}(s)\right\rangle \mathrm{d}s$$
$$+ 2\int_0^t \left\langle \mathbf{D}_N^{(1)}\mathbf{U}(s), \mathrm{d}\left(\mathbf{D}_N^{(1)}\mathbf{W}^{Q,N}\right)(s)\right\rangle + t\left\|\mathbf{D}_N^{(1)}\boldsymbol{\sigma}^k\right\|_2^2. \quad (3.59)$$

We shall address the second term of the right-hand side by applying (3.18) and Young's inequality:

$$\begin{aligned} 2\int_0^t \left\langle \mathbf{D}_N^{(1)}\mathbf{b}^1(\mathbf{U}(s)), \mathbf{D}_N^{(1)}\mathbf{U}(s)\right\rangle \mathrm{d}s &= -2\int_0^t \left\langle \mathbf{b}^1(\mathbf{U}(s)), \mathbf{D}_N^{(2)}\mathbf{U}(s)\right\rangle \mathrm{d}s \\ &\leq \frac{1}{2\nu}\int_0^t \left\|\mathbf{b}^1(\mathbf{U}(s))\right\|_2^2 \mathrm{d}s + 2\nu\int_0^t \left\|\mathbf{D}_N^{(2)}\mathbf{U}(s)\right\|_2^2 \mathrm{d}s. \end{aligned} \quad (3.60)$$

As for the viscous term in (3.59), Equation (3.18) leads to

$$2\nu\int_0^t \left\langle \mathbf{D}_N^{(1)}\mathbf{D}_N^{(2)}\mathbf{U}(s), \mathbf{D}_N^{(1)}\mathbf{U}(s)\right\rangle \mathrm{d}s = -2\nu\int_0^t \left\|\mathbf{D}_N^{(2)}\mathbf{U}(s)\right\|_2^2 \mathrm{d}s. \quad (3.61)$$

Thus, injecting (3.61) and (3.60) into (3.59) and using the bound (3.42) results in:

$$\left\|\mathbf{D}_N^{(1)}\mathbf{U}(t)\right\|_2^2 \leq \left\|\mathbf{D}_N^{(1)}\mathbf{U}_0\right\|_2^2 + \frac{1}{2\nu}\int_0^t \left\|\mathbf{b}^1(\mathbf{U}(s))\right\|_2^2 \mathrm{d}s + 2\int_0^t \left\langle \mathbf{D}_N^{(1)}\mathbf{U}(s), \mathrm{d}\left(\mathbf{D}_N^{(1)}\mathbf{W}^{Q,N}\right)(s)\right\rangle + tD_0. \quad (3.62)$$

Taking the supremum in time and the expectation over the second term of the right-hand side, by stationarity of $(\mathbf{U}(t))_{t\geq 0}$, we get the bound

$$\begin{aligned} \mathbb{E}\left[\sup_{t\in[0,T]} \frac{1}{2\nu}\int_0^t \left\|\mathbf{b}^1(\mathbf{U}(s))\right\|_2^2 \mathrm{d}s\right] &\leq \frac{1}{2\nu}\mathbb{E}\left[\int_0^T \left\|\mathbf{b}^1(\mathbf{U}(s))\right\|_2^2 \mathrm{d}s\right] \\ &= \frac{1}{2\nu}\int_0^T \mathbb{E}\left[\left\|\mathbf{b}^1(\mathbf{U}(s))\right\|_2^2\right] \mathrm{d}s = \frac{T}{2\nu}\mathbb{E}\left[\left\|\mathbf{b}^1(\mathbf{U}_0)\right\|_2^2\right]. \end{aligned}$$

Applying now inequality (3.58), we get

$$\mathbb{E}\left[\sup_{t\in[0,T]} \frac{1}{2\nu}\int_0^t \left\|\mathbf{b}^1(\mathbf{U}(s))\right\|_2^2 \mathrm{d}s\right] \leq \frac{2C_{\overline{A}}^2 T D_0}{\nu^2}\left(1 + C^{0,2p_{\overline{A}}}\right). \quad (3.63)$$

We now turn our attention to the third term of the right-hand side in (3.62). Recall that by (3.56), the process $(\int_0^t \langle \mathbf{D}_N^{(1)}\mathbf{U}(s), \mathrm{d}\left(\mathbf{D}_N^{(1)}\mathbf{W}^{Q,N}\right)(s)\rangle)_{t\geq 0}$ is a martingale. Therefore, applying successively the Jensen and the Doob inequalities, the Itô isometry, the Cauchy-Schwarz inequality and Lemma 3.31, we get

$$\begin{aligned}\mathbb{E}\left[\sup_{t\in[0,T]}\left|\int_0^t\left\langle \mathbf{D}_N^{(1)}\mathbf{U}(s),\mathrm{d}\left(\mathbf{D}_N^{(1)}\mathbf{W}^{Q,N}\right)(s)\right\rangle\right|\right] &\leq \mathbb{E}\left[\sup_{t\in[0,T]}\left|\int_0^t\left\langle \mathbf{D}_N^{(1)}\mathbf{U}(s),\mathrm{d}\left(\mathbf{D}_N^{(1)}\mathbf{W}^{Q,N}\right)(s)\right\rangle\right|^2\right]^{1/2}\\ &\leq 2\mathbb{E}\left[\left|\int_0^T\left\langle \mathbf{D}_N^{(1)}\mathbf{U}(s),\mathrm{d}\left(\mathbf{D}_N^{(1)}\mathbf{W}^{Q,N}\right)(s)\right\rangle\right|^2\right]^{1/2}\\ &= 2\mathbb{E}\left[\sum_{k\geq 1}\int_0^T\left\langle \mathbf{D}_N^{(1)}\mathbf{U}(s),\mathbf{D}_N^{(1)}\boldsymbol{\sigma}^k\right\rangle^2\mathrm{d}s\right]^{1/2}\\ &\leq 2\sqrt{T}\mathbb{E}\left[\left\|\mathbf{D}_N^{(1)}\mathbf{U}_0\right\|_2^2\right]^{1/2}\left(\sum_{k\geq 1}\left\|\mathbf{D}_N^{(1)}\boldsymbol{\sigma}^k\right\|_2^2\right)^{1/2}\\ &\leq 2\sqrt{TC^{1,2}D_0}=D_0\sqrt{\frac{2T}{\nu}}. \end{aligned}\tag{3.64}$$

Now, taking the supremum in time and the expectation in (3.62) and injecting (3.47), (3.63) and (3.64), we end up with

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left\|\mathbf{D}_N^{(1)}\mathbf{U}(t)\right\|_2^2\right]\leq C^{1,2}+\frac{2C_{\overline{A}}^2TD_0}{\nu^2}\left(1+C^{0,2p_{\overline{A}}}\right)+D_0\sqrt{\frac{2T}{\nu}}+TD_0.$$

Since the right-hand side of the above inequality does not depend on N, the result follows. □

Proof of Lemma 3.35. Let $p\in 2\mathbb{N}^*$ and let us repeat the proof of Lemma 2.15 up to (2.23). When the initial condition u_0 is random and has distribution μ^*, this equation writes

$$\begin{aligned}\mathbb{E}\left[\|u(t\wedge T_r)\|^p_{L^p_0(\mathbb{T})}\right]&=\mathbb{E}\left[\|u_0\|^p_{L^p_0(\mathbb{T})}\right]-p\mathbb{E}\left[\int_0^{t\wedge T_r}\int_{\mathbb{T}}\partial_x A(u(s))u(s)^{p-1}\mathrm{d}x\mathrm{d}s\right]\\ &-\nu p(p-1)\mathbb{E}\left[\int_0^{t\wedge T_r}\int_{\mathbb{T}}\partial_x u(s)^2u(s)^{p-2}\mathrm{d}x\mathrm{d}s\right]+\frac{p(p-1)}{2}\sum_{k\geq 1}\mathbb{E}\left[\int_0^{t\wedge T_r}\int_{\mathbb{T}}u(s)^{p-2}g_k^2\mathrm{d}x\mathrm{d}s\right],\end{aligned}$$

for all $t\in[0,T]$ and $r\geq 0$, where T_r is a stopping time converging almost surely towards $+\infty$ as $r\to+\infty$ (by Corollary 2.18). Using (2.24), the non-positivity of the third term of the right-hand side, and bounding the g_k's by their supremum, we get the inequality

$$\mathbb{E}\left[\|u(t\wedge T_r)\|^p_{L^p_0(\mathbb{T})}\right]\leq\mathbb{E}\left[\|u_0\|^p_{L^p_0(\mathbb{T})}\right]+\frac{p(p-1)}{2}\left(\sum_{k\geq 1}\|g_k\|^2_{L^\infty_0(\mathbb{T})}\right)\mathbb{E}\left[\int_0^{t\wedge T_r}\|u(s)\|^{p-2}_{L^{p-2}_0(\mathbb{T})}\mathrm{d}s\right].$$

Using now Lemma 3.32, (3.3), (3.5), and Inequality (2.18), we get

$$\mathbb{E}\left[\|u(t\wedge T_r)\|^p_{L^p_0(\mathbb{T})}\right]\leq C^{0,p}+\frac{p(p-1)}{2}D_0\left(C_5^{(p-2)}\left(1+\mathbb{E}\left[\|u_0\|^{p-2}_{L^{p-2}_0(\mathbb{T})}\right]\right)+C_6^{(p-2)}t\right),$$

where the constants $C_5^{(p-2)}$ and $C_6^{(p-2)}$, defined in Chapter 2, depend only on ν, p and D_0. Using once again Lemma 3.32, letting $r\to+\infty$ and bouding t by T, we obtain

$$\limsup_{r\to\infty}\mathbb{E}\left[\|u(t\wedge T_r)\|^p_{L^p_0(\mathbb{T})}\right]\leq C^{0,p}+\frac{p(p-1)}{2}D_0\left(C_5^{(p-2)}\left(1+C^{0,p-2}\right)+C_6^{(p-2)}T\right)=:\widetilde{C}_T^{0,p}.$$

Applying Fatou's lemma on the left-hand side, we get

$$\mathbb{E}\left[\|u(t)\|^p_{L^p_0(\mathbb{T})}\right] \le \widetilde{C}^{0,p}_T,$$

and since the right-hand side does not depend on t, we get the first wanted inequality in the case $p \in 2\mathbb{N}^*$. The general case $p \in [2,+\infty)$ then follows from the Jensen inequality.

To prove the second inequality, we start from Lemma 2.17 which, when u_0 is random, gives the estimate

$$\mathbb{E}\left[\|u(t\wedge T_r)\|^2_{H^1_0(\mathbb{T})}\right] \le \mathbb{E}\left[\|u_0\|^2_{H^1_0(\mathbb{T})}\right] + C_7\left(1+\mathbb{E}\left[\|u_0\|^{2p_A+2}_{L^{2p_A+2}_0(\mathbb{T})}\right]\right) + C_8 t,$$

from which we deduce, by applying Fatou's lemma on the left-hand side and Lemma 3.32 on the right-hand side:

$$\mathbb{E}\left[\|u(t)\|^2_{H^1_0(\mathbb{T})}\right] \le C^{1,2} + C_7\left(1+C^{0,2p_A+2}\right) + C_8 T =: \widetilde{C}^{1,2}_T. \qquad \square$$

We now turn to the proof of the main lemma in this section:

Proof of Proposition 3.36. In all this proof, for notational convenience, the subsequence $(u^{(1)}_{N_j})_{j\in\mathbb{N}}$ will be denoted by $(u^{(1)}_N)_{N\ge 1}$.

Step 0. Decomposition of the error. Let us fix a time horizon $T > 0$. We introduce the stopping time

$$\tau_{M,N} := \inf\left\{t \ge 0 : \|u(t)\|_{H^1_0(\mathbb{T})} \vee \left\|u^{(1)}_N(t)\right\|_{H^1_0(\mathbb{T})} \ge M\right\},$$

and we split the expectation in two parts: for all $t \in [0,T]$,

$$\mathbb{E}\left[\left\|u^{(1)}_N(t)-u(t)\right\|^2_{L^2_0(\mathbb{T})}\right] = \mathbb{E}\left[\left\|u^{(1)}_N(t)-u(t)\right\|^2_{L^2_0(\mathbb{T})}\mathbf{1}_{t\le\tau_{M,N}}\right] + \mathbb{E}\left[\left\|u^{(1)}_N(t)-u(t)\right\|^2_{L^2_0(\mathbb{T})}\mathbf{1}_{t>\tau_{M,N}}\right]. \tag{3.65}$$

We will address the first term of the RHS in the steps 1 to 6, and the second one in the step 7.

We have

$$\mathbb{E}\left[\left\|u^{(1)}_N(t)-u(t)\right\|^2_{L^2_0(\mathbb{T})}\mathbf{1}_{t\le\tau_{M,N}}\right] \le \mathbb{E}\left[\left\|u^{(1)}_N(\tau_{M,N}\wedge t)-u(\tau_{M,N}\wedge t)\right\|^2_{L^2_0(\mathbb{T})}\right], \tag{3.66}$$

and we will use this localization argument to take benefit from the local Lipschitz continuity of the non-linear term which, by use of the Grönwall lemma, will lead us to show that for any fixed $M > 0$,

$$\mathbb{E}\left[\left\|u^{(1)}_N(t)-u(t)\right\|^2_{L^2_0(\mathbb{T})}\mathbf{1}_{t\le\tau_{M,N}}\right] \underset{N\to\infty}{\longrightarrow} 0.$$

Applying Itô's formula [35, Theorem 4.32] and taking the expectation, we have

$$
\begin{aligned}
&\mathbb{E}\left[\left\|u_N^{(1)}(\tau_{M,N}\wedge t)-u(\tau_{M,N}\wedge t)\right\|^2_{L^2_0(\mathbb{T})}\right]\\
&=\mathbb{E}\left[\left\|u_N^{(1)}(0)-u(0)\right\|^2_{L^2_0(\mathbb{T})}\right]\\
&-2\mathbb{E}\Bigg[\int_0^{\tau_{M,N}\wedge t}\int_{\mathbb{T}}\left(u_N^{(1)}(s,x)-u(s,x)\right)\\
&\qquad\left(\sum_{i=1}^N N\left(\overline{A}\left(U_i(s),U_{i+1}(s)\right)-\overline{A}\left(U_{i-1}(s),U_i(s)\right)\right)\phi_N^{(1)}\left(x-\frac{i}{N}\right)-\partial_x A\left(u_N^{(1)}(s,x)\right)\right)\mathrm{d}x\mathrm{d}s\Bigg]\\
&-2\mathbb{E}\left[\int_0^{\tau_{M,N}\wedge t}\int_{\mathbb{T}}\left(u_N^{(1)}(s,x)-u(s,x)\right)\left(\partial_x A\left(u_N^{(1)}(s,x)\right)-\partial_x A(u(s,x))\right)\mathrm{d}x\mathrm{d}s\right]\\
&+2\nu\mathbb{E}\Bigg[\int_0^{\tau_{M,N}\wedge t}\int_{\mathbb{T}}\left(u_N^{(1)}(s,x)-u(s,x)\right)\\
&\qquad\left(N^2\sum_{i=1}^N\left(U_{i+1}(s)-2U_i(s)+U_{i-1}(s)\right)\phi_N^{(1)}\left(x-\frac{i}{N}\right)-\partial_{xx}u(s,x)\right)\mathrm{d}x\mathrm{d}s\Bigg]\\
&+\sum_{k\geq 1}\mathbb{E}\left[\int_0^{\tau_{M,N}\wedge t}\int_{\mathbb{T}}\left(\sum_{i=1}^N\sigma_i^k\phi_N^{(1)}\left(x-\frac{i}{N}\right)-g_k(x)\right)^2\mathrm{d}x\mathrm{d}s\right]\\
&=: I_1^N+I_2^N(t)+I_3^N(t)+I_4^N(t)+I_5^N(t),
\end{aligned}
$$

where the local martingale term vanished thanks to the localisation. From Step 1 to Step 5, we will get an upper bound over each of the terms I_l^N, $l=1,\dots,5$. More precisely, for all $l=1,2,4,5$, we will show that there exists a sequence $(\varepsilon_l^N)_{N\in\mathbb{N}}$ of non-negative real numbers not depending on t such that $\lim_{N\to\infty}\varepsilon_l^N=0$ and such that the following inequalities are satisfied for all $N\in\mathbb{N}$ and all $t\in[0,T]$:

$$I_1^N\leq\varepsilon_1^N, \tag{3.67}$$

$$I_2^N(t)\leq\mathbb{E}\left[\int_0^{\tau_{M,N}\wedge t}\left\|u_N^{(1)}(s)-u(s)\right\|^2_{L^2_0(\mathbb{T})}\mathrm{d}s\right]+\varepsilon_2^N, \tag{3.68}$$

$$I_4^N(t)\leq -2\nu\mathbb{E}\left[\int_0^{\tau_{M,N}\wedge t}\left\|u_N^{(1)}(s)-u(s)\right\|^2_{H^1_0(\mathbb{T})}\mathrm{d}s\right]+\varepsilon_4^N, \tag{3.69}$$

$$I_5^N(t)\leq\varepsilon_5^N. \tag{3.70}$$

In the case $l=3$, we will show that there exists a constant $\gamma_M>0$ not depending on N nor t such that

$$I_3^N(t)\leq 2\nu\mathbb{E}\left[\int_0^{\tau_{M,N}\wedge t}\left\|u_N^{(1)}(s)-u(s)\right\|^2_{H^1_0(\mathbb{T})}\mathrm{d}s\right]+\gamma_M\mathbb{E}\left[\int_0^{\tau_{M,N}\wedge t}\left\|u_N^{(1)}(s)-u(s)\right\|^2_{L^2_0(\mathbb{T})}\mathrm{d}s\right].$$

Step 1. The initial condition. By the construction of the sequence $(u_N^{(1)})_{N\geq 1}$ in Section 3.3.2, we have almost surely

$$\lim_{N\to\infty}\left\|u_N^{(1)}(0)-u(0)\right\|_{L^2_0(\mathbb{T})}=0. \tag{3.71}$$

Moreover, Proposition 3.19 and Lemma 3.32 ensure the uniform bound with respect to N over the following fourth order moment:

$$\begin{aligned}\mathbb{E}\left[\left\|u_N^{(1)}(0)-u(0)\right\|_{L_0^2(\mathbb{T})}^4\right] &\leq 8\mathbb{E}\left[\left\|u_N^{(1)}(0)\right\|_{L_0^2(\mathbb{T})}^4\right]+8\mathbb{E}\left[\|u(0)\|_{L_0^2(\mathbb{T})}^4\right]\\ &\leq 8\mathbb{E}\left[\left\|u_N^{(1)}(0)\right\|_{L_0^4(\mathbb{T})}^4\right]+8\mathbb{E}\left[\|u(0)\|_{L_0^4(\mathbb{T})}^4\right] \quad \text{(by (3.2))}\\ &\leq 16C^{0,4} \quad \text{(by Proposition 3.19 and Lemma 3.32)}, \end{aligned} \tag{3.72}$$

and the convergence of I_1^N towards 0 follows (see for instance [12, Theorem 3.5]). Thus, since I_1^N does not depend on t, we may take $\varepsilon_1^N := I_1^N$.

Step 2. The flux-numerical flux approximation. Using Young's inequality, we have

$$\begin{aligned}I_2^N(t) &\leq \mathbb{E}\left[\int_0^{\tau_{M,N}\wedge t}\left\|u_N^{(1)}(s)-u(s)\right\|_{L_0^2(\mathbb{T})}^2\mathrm{d}s\right]\\ &+\mathbb{E}\left[\int_0^{\tau_{M,N}\wedge t}\int_{\mathbb{T}}\left(\sum_{i=1}^N N\left(\overline{A}\left(U_i(s),U_{i+1}(s)\right)-\overline{A}\left(U_{i-1}(s),U_i(s)\right)\right)\phi_N^{(1)}\left(x-\frac{i}{N}\right)-\partial_x A\left(u_N^{(1)}(s,x)\right)\right)^2\mathrm{d}x\mathrm{d}s\right].\end{aligned} \tag{3.73}$$

We focus on the second term of the right-hand side which we can rewrite by

$$\begin{aligned}\mathbb{E}\Bigg[\int_0^{\tau_{M,N}\wedge t}\sum_{i=1}^N\int_{\frac{i-1}{N}}^{\frac{i}{N}}\Bigg(&\left(N\left(\overline{A}(U_{i-1}(s),U_i(s))-\overline{A}(U_{i-2}(s),U_{i-1}(s))\right)-\partial_x A\left(u_N^{(1)}(s,x)\right)\right)\phi_N^{(1)}\left(x-\frac{i-1}{N}\right)\\ &+\left(N\left(\overline{A}\left(U_i(s),U_{i+1}(s)\right)-\overline{A}\left(U_{i-1}(s),U_i(s)\right)\right)-\partial_x A\left(u_N^{(1)}(s,x)\right)\right)\phi_N^{(1)}\left(x-\frac{i}{N}\right)\Bigg)^2\mathrm{d}x\mathrm{d}s\Bigg],\end{aligned}$$

which we control by the following upper bound

$$\begin{aligned}2\mathbb{E}\Bigg[\int_0^{\tau_{M,N}\wedge t}\sum_{i=1}^N\int_{\frac{i-1}{N}}^{\frac{i}{N}}&\left(N\left(\overline{A}(U_{i-1}(s),U_i(s))-\overline{A}(U_{i-2}(s),U_{i-1}(s))\right)-\partial_x A\left(u_N^{(1)}(s,x)\right)\right)^2\\ &+\left(N\left(\overline{A}(U_i(s),U_{i+1}(s))-\overline{A}(U_{i-1}(s),U_i(s))\right)-\partial_x A\left(u_N^{(1)}(s,x)\right)\right)^2\mathrm{d}x\mathrm{d}s\Bigg].\end{aligned}$$

By definition of $u_N^{(1)}$, we have for all $s\geq 0$ and all $x\in(\frac{i-1}{N},\frac{i}{N}]$, $\partial_x u_N^{(1)}(s,x)=N(U_i(s)-U_{i-1}(s))$. Let us now focus on the second term of the above integrand and observe that by symmetry, the left one may be treated in exactly the same way. We have thanks to (3.12):

$$\begin{aligned}&\left(N\left(\overline{A}(U_i(s),U_{i+1}(s))-\overline{A}(U_{i-1}(s),U_i(s))\right)-N(U_i(s)-U_{i-1}(s))A'\left(u_N^{(1)}(s,x)\right)\right)^2\\ &=\Big(N\left(\overline{A}(U_i(s),U_{i+1}(s))-\overline{A}(U_{i-1}(s),U_i(s))\right)\\ &\qquad -N(U_i(s)-U_{i-1}(s))\left(\partial_1\overline{A}\left(u_N^{(1)}(s,x),u_N^{(1)}(s,x)\right)+\partial_2\overline{A}\left(u_N^{(1)}(s,x),u_N^{(1)}(s,x)\right)\right)\Big)^2\\ &\leq 2\left(N\int_{U_{i-1}(s)}^{U_i(s)}\left(\partial_1\overline{A}(z,U_i(s))-\partial_1\overline{A}\left(u_N^{(1)}(s,x),u_N^{(1)}(s,x)\right)\right)\mathrm{d}z\right)^2\\ &\qquad +2N^2\left(\int_{U_i(s)}^{U_{i+1}(s)}\partial_2\overline{A}(U_i(s),z)\mathrm{d}z-(U_i(s)-U_{i-1}(s))\,\partial_2\overline{A}\left(u_N^{(1)}(s,x),u_N^{(1)}(s,x)\right)\right)^2.\end{aligned}$$

In the following computations, we will get upper bounds on the terms

$$I_{2.1}^N(t) := \mathbb{E}\left[\int_0^{\tau_{M,N}\wedge t}\sum_{i=1}^N\int_{\frac{i-1}{N}}^{\frac{i}{N}}\left(N\int_{U_{i-1}(s)}^{U_i(s)}\left(\partial_1\overline{A}(z,U_i(s)) - \partial_1\overline{A}\left(u_N^{(1)}(s,x),u_N^{(1)}(s,x)\right)\right)\mathrm{d}z\right)^2\mathrm{d}x\mathrm{d}s\right],$$

$$I_{2.2}^N(t) := \mathbb{E}\Bigg[\int_0^{\tau_{M,N}\wedge t}\sum_{i=1}^N\int_{\frac{i-1}{N}}^{\frac{i}{N}}N^2\Bigg(\int_{U_i(s)}^{U_{i+1}(s)}\partial_2\overline{A}(U_i(s),z)\mathrm{d}z - (U_i(s)-U_{i-1}(s))\,\partial_2\overline{A}\left(u_N^{(1)}(s,x),u_N^{(1)}(s,x)\right)\Bigg)^2\mathrm{d}x\mathrm{d}s\Bigg].$$

We first look at the term $I_{2.1}^N(t)$. The function $\partial_1\overline{A}$ is uniformly continuous on $[-M,M]^2$. In particular,

$$\forall\varepsilon>0,\exists\delta_{M,\varepsilon}>0,\max(|w-x|,|y-z|)\le\delta_{M,\varepsilon},\max\{|w|,|x|,|y|,|z|\}\le M \quad\Rightarrow\quad |\partial_1\overline{A}(w,y)-\partial_1\overline{A}(x,z)|\le\varepsilon.$$

Furthermore, for every $s\le\tau_{M,N}$, by Lemma 3.30 and (3.3), we have $\sup_{i=1,\dots,N}|U_i(s)| = \|u_N^{(1)}(s)\|_{L_0^\infty(\mathbb{T})}\le \|u_N^{(1)}(s)\|_{H_0^1(\mathbb{T})}\le M$. Let $C_\partial^{(M)}$ be a local bound of $\partial_1\overline{A}$ and $\partial_2\overline{A}$ over the square $[-M,M]^2$.

Let $\varepsilon>0$. We have

$$\begin{aligned}
&I_{2.1}^N(t)\\
&= \mathbb{E}\Bigg[\int_0^{\tau_{M,N}\wedge t}\sum_{i=1}^N\int_{\frac{i-1}{N}}^{\frac{i}{N}}\left(N\int_{U_{i-1}(s)}^{U_i(s)}\left(\partial_1\overline{A}(z,U_i(s)) - \partial_1\overline{A}\left(u_N^{(1)}(s,x),u_N^{(1)}(s,x)\right)\right)\mathrm{d}z\mathbf{1}_{|U_i(s)-U_{i-1}(s)|\le\delta_{M,\varepsilon}}\right)^2\mathrm{d}x\mathrm{d}s\Bigg]\\
&+\mathbb{E}\left[\int_0^{\tau_{M,N}\wedge t}\sum_{i=1}^N\int_{\frac{i-1}{N}}^{\frac{i}{N}}\left(N\int_{U_{i-1}(s)}^{U_i(s)}\left(\partial_1\overline{A}(z,U_i(s)) - \partial_1\overline{A}\left(u_N^{(1)}(s,x),u_N^{(1)}(s,x)\right)\right)\mathrm{d}z\mathbf{1}_{|U_i(s)-U_{i-1}(s)|>\delta_{M,\varepsilon}}\right)^2\mathrm{d}x\mathrm{d}s\right]\\
&\le \mathbb{E}\left[\int_0^{\tau_{M,N}\wedge t}\sum_{i=1}^N\int_{\frac{i-1}{N}}^{\frac{i}{N}}\left(N\int_{U_{i-1}(s)}^{U_i(s)}\varepsilon\mathrm{d}z\mathbf{1}_{|U_i(s)-U_{i-1}(s)|\le\delta_{M,\varepsilon}}\right)^2\mathrm{d}x\mathrm{d}s\right]\\
&\quad+\mathbb{E}\left[\int_0^{\tau_{M,N}\wedge t}\sum_{i=1}^N\int_{\frac{i-1}{N}}^{\frac{i}{N}}\left(N\int_{U_{i-1}(s)}^{U_i(s)}2C_\partial^{(M)}\mathrm{d}z\frac{|U_i(s)-U_{i-1}(s)|^{1/2}}{\delta_{M,\varepsilon}^{1/2}}\right)^2\mathrm{d}x\mathrm{d}s\right]\\
&\le \varepsilon^2\mathbb{E}\left[\int_0^t N\sum_{i=1}^N|U_i(s)-U_{i-1}(s)|^2\mathrm{d}s\right] + \frac{4\left(C_\partial^{(M)}\right)^2}{\delta_{M,\varepsilon}}\int_0^t\mathbb{E}\left[N\sum_{i=1}^N|U_i(s)-U_{i-1}(s)|^3\right]\mathrm{d}s\\
&= \varepsilon^2 t\mathbb{E}\left[\left\|\mathbf{D}_N^{(1)}\mathbf{U}_0\right\|_2^2\right] + \frac{4\left(C_\partial^{(M)}\right)^2}{\delta_{M,\varepsilon}N}t\mathbb{E}\left[\left\|\mathbf{D}_N^{(1)}\mathbf{U}_0\right\|_3^3\right]\\
&\le \varepsilon^2TC^{1,2} + \frac{4\left(C_\partial^{(M)}\right)^2TC^{1,3}}{\delta_{M,\varepsilon}N}\qquad\text{(by Lemmas 3.31 and 3.33)}.
\end{aligned}$$

Since ε was chosen arbitrarily, it follows that $\varepsilon_{2.1}^N := \sup_{t\in[0,T]}I_{2.1}^N(t)$ satisfies $\lim_{N\to\infty}\varepsilon_{2.1}^N = 0$.

As for the term $I_{2.2}^N(t)$ we have

$$\begin{aligned}
I_{2.2}^N(t) = \mathbb{E}\Bigg[\int_0^{\tau_{M,N}\wedge t} \sum_{i=1}^N \int_{\frac{i-1}{N}}^{\frac{i}{N}} N^2 \Bigg(\int_{U_i(s)}^{U_{i+1}(s)} \left(\partial_2 \overline{A}(U_i(s), z) - \partial_2 \overline{A}\left(u_N^{(1)}(s,x), u_N^{(1)}(s,x)\right)\right) \mathrm{d}z \\
+ (U_{i+1}(s) - 2U_i(s) + U_{i-1}(s)) \partial_2 \overline{A}\left(u_N^{(1)}(s,x), u_N^{(1)}(s,x)\right) \Bigg)^2 \mathrm{d}x\mathrm{d}s \Bigg] \\
\leq 2\mathbb{E}\left[\int_0^{\tau_{M,N}\wedge t} \sum_{i=1}^N \int_{\frac{i-1}{N}}^{\frac{i}{N}} N^2 \left(\int_{U_i(s)}^{U_{i+1}(s)} \left(\partial_2 \overline{A}(U_i(s), z) - \partial_2 \overline{A}\left(u_N^{(1)}(s,x), u_N^{(1)}(s,x)\right)\right) \mathrm{d}z \right)^2 \mathrm{d}x\mathrm{d}s \right] \\
+ 2\mathbb{E}\left[\int_0^{\tau_{M,N}\wedge t} \sum_{i=1}^N \int_{\frac{i-1}{N}}^{\frac{i}{N}} N^2 (U_{i+1}(s) - 2U_i(s) + U_{i-1}(s))^2 \partial_2 \overline{A}\left(u_N^{(1)}(s,x), u_N^{(1)}(s,x)\right)^2 \mathrm{d}x\mathrm{d}s \right] \\
=: I_{2.2.1}^N(t) + I_{2.2.2}^N(t)
\end{aligned}$$

Now, the term $I_{2.2.1}^N(t)$ can be treated the same way as $I_{2.1}^N(t)$. In particular, $\varepsilon_{2.2.1}^N := \sup_{t\in[0,T]} I_{2.2.1}^N(t)$ satisfies $\lim_{N\to\infty} \varepsilon_{2.2.1}^N = 0$. As for $I_{2.2.2}^N(t)$, we have

$$\begin{aligned}
I_{2.2.2}^N(t) &\leq \left(C_\partial^{(M)}\right)^2 \mathbb{E}\left[\int_0^{\tau_{M,N}\wedge t} N \sum_{i=1}^N (U_{i+1}(s) - 2U_i(s) + U_{i-1}(s))^2 \mathrm{d}s \right] \\
&\leq 2\left(C_\partial^{(M)}\right)^2 \int_0^t \mathbb{E}\left[N \sum_{i=1}^N (U_{i+1}(s) - 2U_i(s) + U_{i-1}(s))^2 \right] \mathrm{d}s \\
&= 2\left(C_\partial^{(M)}\right)^2 \frac{t}{N^2} \mathbb{E}\left[\left\| \mathbf{D}_N^{(2)} \mathbf{U}_0 \right\|_2^2 \right] \\
&\leq 2\left(C_\partial^{(M)}\right)^2 T \frac{C^{2,2}}{N^2} \quad \text{(by Lemma 3.31)} \\
&=: \varepsilon_{2.2.2}^N.
\end{aligned}$$

At last, the sum of all these error terms amounts to an error term ε_2^N satisfying the requirements of Step 0, so that the inequality (3.73) reduces to (3.68).

Step 3. The flux term. Integrating by parts and applying Young's inequality, we get

$$\begin{aligned}
I_3^N(t) &= 2\mathbb{E}\left[\int_0^{\tau_{M,N}\wedge t} \int_{\mathbb{T}} \partial_x \left(u_N^{(1)}(s,x) - u(s,x)\right) \left(A\left(u_N^{(1)}(s,x)\right) - A(u(s,x)) \right) \mathrm{d}x\mathrm{d}s \right] \\
&\leq 2\nu\mathbb{E}\left[\int_0^{\tau_{M,N}\wedge t} \int_{\mathbb{T}} \left(\partial_x u_N^{(1)}(s,x) - \partial_x u(s,x) \right)^2 \mathrm{d}x\mathrm{d}s \right] \\
&\quad + \frac{1}{2\nu}\mathbb{E}\left[\int_0^{\tau_{M,N}\wedge t} \int_{\mathbb{T}} \left(A\left(u_N^{(1)}(s,x)\right) - A(u(s,x)) \right)^2 \mathrm{d}x\mathrm{d}s \right].
\end{aligned}$$

Denoting by L_M a local Lipschitz constant of A over the interval $[-M, M]$, we get

$$I_3^N(t) \leq 2\nu\mathbb{E}\left[\int_0^{\tau_{M,N}\wedge t} \left\| u_N^{(1)}(s) - u(s) \right\|_{H_0^1(\mathbb{T})}^2 \mathrm{d}s \right] + \frac{L_M^2}{2\nu}\mathbb{E}\left[\int_0^{\tau_{M,N}\wedge t} \left\| u_N^{(1)}(s) - u(s) \right\|_{L_0^2(\mathbb{T})}^2 \mathrm{d}s \right],$$

and we set therefore $\gamma_M := L_M^2/(2\nu)$.

Step 4. The viscous term. We shall compare the term

$$J^N(s) = \int_{\mathbb{T}} \left(u_N^{(1)}(s,x) - u(s,x)\right) \left(\sum_{i=1}^N N^2 (U_{i+1}(s) - 2U_i(s) + U_{i-1}(s)) \phi_N^{(1)}\left(x - \frac{i}{N}\right) - \partial_{xx} u(s,x) \right) \mathrm{d}x,$$

with

$$\widetilde{J}^N(s) := -\int_{\mathbb{T}} (\partial_x u_N^{(1)}(s,x) - \partial_x u(s,x))^2 \mathrm{d}x \leq 0.$$

Expanding the product in the definition of J^N, we first write

$$J^N(s) = J_1^N(s) + J_2^N(s) + J_3^N(s) + J_4^N(s),$$

where

$$\begin{aligned}
J_1^N(s) &:= \int_{\mathbb{T}} u_N^{(1)}(s,x) \sum_{i=1}^N N^2(U_{i+1}(s) - 2U_i(s) + U_{i-1}(s))\phi_N^{(1)}\left(x - \frac{i}{N}\right)\mathrm{d}x,\\
J_2^N(s) &:= -\int_{\mathbb{T}} u_N^{(1)}(s,x)\partial_{xx}u(s,x)\mathrm{d}x,\\
J_3^N(s) &:= -\int_{\mathbb{T}} u(s,x) \sum_{i=1}^N N^2(U_{i+1}(s) - 2U_i(s) + U_{i-1}(s))\phi_N^{(1)}\left(x - \frac{i}{N}\right)\mathrm{d}x,\\
J_4^N(s) &:= \int_{\mathbb{T}} u(s,x)\partial_{xx}u(s,x)\mathrm{d}x;
\end{aligned}$$

likewise,

$$\widetilde{J}^N(s) = \widetilde{J}_1^N(s) + \widetilde{J}_2^N(s) + \widetilde{J}_3^N(s) + \widetilde{J}_4^N(s),$$

where

$$\begin{aligned}
\widetilde{J}_1^N(s) &:= -\int_{\mathbb{T}} \partial_x u_N^{(1)}(s,x)\partial_x u_N^{(1)}(s,x)\mathrm{d}x,\\
\widetilde{J}_2^N(s) &:= \int_{\mathbb{T}} \partial_x u_N^{(1)}(s,x)\partial_x u(s,x)\mathrm{d}x,\\
\widetilde{J}_3^N(s) &:= \int_{\mathbb{T}} \partial_x u(s,x)\partial_x u_N^{(1)}(s,x)\mathrm{d}x,\\
\widetilde{J}_4^N(s) &:= -\int_{\mathbb{T}} \partial_x u(s,x)\partial_x u(s,x)\mathrm{d}x.
\end{aligned}$$

Integration by parts shows that $J_2^N(s) = \widetilde{J}_2^N(s)$ and $J_4^N(s) = \widetilde{J}_4^N(s)$. We now focus on the computation of $J_1^N(s) - \widetilde{J}_1^N(s)$ and $J_3^N(s) - \widetilde{J}_3^N(s)$.

Using the definition of $u_N^{(1)}$, we get

$$\begin{aligned}
J_1^N(s) - \widetilde{J}_1^N(s) = &\sum_{i,k=1}^N N^2(U_{i+1}(s) - 2U_i(s) + U_{i-1}(s))U_k(s) \int_{\mathbb{T}} \phi_N^{(1)}\left(x - \frac{i}{N}\right)\phi_N^{(1)}\left(x - \frac{k}{N}\right)\mathrm{d}x\\
&+ \sum_{i,k=1}^N U_i(s)U_k(s) \int_{\mathbb{T}} \partial_x\phi_N^{(1)}\left(x - \frac{i}{N}\right)\partial_x\phi_N^{(1)}\left(x - \frac{k}{N}\right)\mathrm{d}x.
\end{aligned}$$

Direct computation yields

$$\int_{\mathbb{T}} \phi_N^{(1)}\left(x - \frac{i}{N}\right)\phi_N^{(1)}\left(x - \frac{k}{N}\right)\mathrm{d}x = \begin{cases} \frac{2}{3N} & \text{if } k = i,\\ \frac{1}{6N} & \text{if } k = i \pm 1,\\ 0 & \text{otherwise,}\end{cases}$$

and

$$\int_{\mathbb{T}} \partial_x\phi_N^{(1)}\left(x - \frac{i}{N}\right)\partial_x\phi_N^{(1)}\left(x - \frac{k}{N}\right)\mathrm{d}x = \begin{cases} 2N & \text{if } k = i,\\ -N & \text{if } k = i \pm 1,\\ 0 & \text{otherwise.}\end{cases}$$

As a consequence,

$$\begin{aligned}
J_1^N(s) - \widetilde{J}_1^N(s) &= \sum_{i=1}^N N^2(U_{i+1}(s) - 2U_i(s) + U_{i-1}(s)) \left(\frac{1}{6N} U_{i-1}(s) + \frac{2}{3N} U_i(s) + \frac{1}{6N} U_{i+1}(s) \right) \\
&\quad + \sum_{i=1}^N U_i(s) \left(-NU_{i-1}(s) + 2NU_i(s) - NU_{i+1}(s) \right) \\
&= \frac{N}{6} \sum_{i=1}^N (U_{i+1}(s) - 2U_i(s) + U_{i-1}(s))^2 = \frac{1}{6N^2} \left\| \mathbf{D}_N^{(2)} \mathbf{U}(s) \right\|_2^2 .
\end{aligned}$$

By Lemma 3.31, we deduce that

$$\mathbb{E}\left[\left| J_1^N(s) - \widetilde{J}_1^N(s) \right| \right] \leq \frac{C^{2,2}}{6N^2}.$$

In order to compute $J_3^N(s) - \widetilde{J}_3^N(s)$, we first rewrite

$$\begin{aligned}
\widetilde{J}_3^N(s) &= \int_{\mathbb{T}} \partial_x u(s,x) \partial_x u_N^{(1)}(s,x) \mathrm{d}x \\
&= \sum_{i=1}^N U_i(s) \int_{\mathbb{T}} \partial_x u(s,x) \partial_x \phi_N^{(1)} \left(x - \frac{i}{N} \right) \mathrm{d}x \\
&= \sum_{i=1}^N N U_i(s) \left(\int_{\frac{i-1}{N}}^{\frac{i}{N}} \partial_x u(s,x) \mathrm{d}x - \int_{\frac{i}{N}}^{\frac{i+1}{N}} \partial_x u(s,x) \mathrm{d}x \right) \\
&= -\sum_{i=1}^N N U_i(s) \left(u\left(s, \frac{i+1}{N}\right) - 2u\left(s, \frac{i}{N}\right) + u\left(s, \frac{i-1}{N}\right) \right) \\
&= -\sum_{i=1}^N N u\left(s, \frac{i}{N}\right) \left(U_{i+1}(s) - 2U_i(s) + U_{i-1}(s) \right).
\end{aligned}$$

As a consequence,

$$\begin{aligned}
J_3^N(s) - \widetilde{J}_3^N(s) &= -\sum_{i=1}^N N(U_{i+1}(s) - 2U_i(s) + U_{i-1}(s)) \left(N \int_{\mathbb{T}} u(s,x) \phi_N^{(1)} \left(x - \frac{i}{N} \right) \mathrm{d}x - u\left(s, \frac{i}{N}\right) \right) \\
&= -\sum_{i=1}^N N^2(U_{i+1}(s) - 2U_i(s) + U_{i-1}(s)) \int_{\mathbb{T}} \left(u(s,x) - u\left(s, \frac{i}{N}\right) \right) \phi_N^{(1)} \left(x - \frac{i}{N} \right) \mathrm{d}x \\
&= -\sum_{i=1}^N N^2(U_{i+1}(s) - 2U_i(s) + U_{i-1}(s)) \int_{\mathbb{T}} \int_{\frac{i}{N}}^x \partial_x u(s,y) \mathrm{d}y \phi_N^{(1)} \left(x - \frac{i}{N} \right) \mathrm{d}x.
\end{aligned}$$

Using the rough bound

$$|\phi_N^{(1)}(x)| \leq \mathbf{1}_{\{-\frac{1}{N} \leq x \leq \frac{1}{N}\}},$$

we write

$$\left| \int_{\mathbb{T}} \int_{\frac{i}{N}}^x \partial_x u(s,y) \mathrm{d}y \phi_N^{(1)} \left(x - \frac{i}{N} \right) \mathrm{d}x \right| \leq \frac{2}{N} \int_{\frac{i-1}{N}}^{\frac{i+1}{N}} |\partial_x u(s,y)| \mathrm{d}y,$$

whence

$$\left| J_3^N(s) - \widetilde{J}_3^N(s) \right| \leq 2 \sum_{i=1}^N N \left| U_{i+1}(s) - 2U_i(s) + U_{i-1}(s) \right| \int_{\frac{i-1}{N}}^{\frac{i+1}{N}} |\partial_x u(s,y)| \mathrm{d}y.$$

By the Cauchy-Schwarz inequality,

$$\mathbb{E}\left[\left|J_3^N(s)-\widetilde{J}_3^N(s)\right|\right] \leq 2N\sqrt{\mathbb{E}\left[\sum_{i=1}^N |U_{i+1}(s)-2U_i(s)+U_{i-1}(s)|^2\right]}\sqrt{\mathbb{E}\left[\sum_{i=1}^N\left(\int_{\frac{i-1}{N}}^{\frac{i+1}{N}}|\partial_x u(s,y)|\,\mathrm{d}y\right)^2\right]}.$$

By Jensen's inequality and Lemma 3.35,

$$\mathbb{E}\left[\sum_{i=1}^N\left(\int_{\frac{i-1}{N}}^{\frac{i+1}{N}}|\partial_x u(s,y)|\,\mathrm{d}y\right)^2\right] \leq \frac{4}{N}\mathbb{E}\left[\|u(s)\|^2_{H_0^1(\mathbb{T})}\right] \leq \frac{4\widetilde{C}_T^{1,2}}{N}.$$

As a conclusion,

$$\mathbb{E}\left[\left|J_3^N(s)-\widetilde{J}_3^N(s)\right|\right] \leq \frac{4\sqrt{C^{2,2}\widetilde{C}_T^{1,2}}}{N}.$$

Coming back to the expression of $I_4^N(t)$, we have

$$\begin{aligned}
I_4^N(t) &\leq 2\nu\mathbb{E}\left[\int_0^{\tau_{M,N}\wedge t} J^N(s)\mathrm{d}s\right]\\
&\leq 2\nu\mathbb{E}\left[\int_0^{\tau_{M,N}\wedge t} \widetilde{J}^N(s)\mathrm{d}s\right] + 2\nu\int_0^t \mathbb{E}\left[\left|J_1^N(s)-\widetilde{J}_1^N(s)\right|\right]\mathrm{d}s + 2\nu\int_0^t \mathbb{E}\left[\left|J_3^N(s)-\widetilde{J}_3^N(s)\right|\right]\mathrm{d}s\\
&\leq -2\nu\mathbb{E}\left[\int_0^{\tau_{M,N}\wedge t}\left\|u_N^{(1)}(s)-u(s)\right\|^2_{H_0^1(\mathbb{T})}\mathrm{d}s\right] + \frac{TC^{2,2}\nu}{3N^2} + \frac{8T\nu\sqrt{C^{2,2}\widetilde{C}_T^{1,2}}}{N}\\
&=: -2\nu\mathbb{E}\left[\int_0^{\tau_{M,N}\wedge t}\left\|u_N^{(1)}(s)-u(s)\right\|^2_{H_0^1(\mathbb{T})}\mathrm{d}s\right] + \varepsilon_4^N.
\end{aligned}$$

Step 5. The noise term. We have

$$\begin{aligned}
I_5^N(t) &= \mathbb{E}\left[\int_0^{\tau_{M,N}\wedge t}\int_{\mathbb{T}}\sum_{k\geq 1}\left(\sum_{i=1}^N \sigma_i^k\phi_N^{(1)}\left(x-\frac{i}{N}\right)-g_k(x)\right)^2\mathrm{d}x\mathrm{d}s\right]\\
&\leq t\int_{\mathbb{T}}\sum_{k\geq 1}\left(\sum_{i=1}^N \sigma_i^k\phi_N^{(1)}\left(x-\frac{i}{N}\right)-g_k(x)\right)^2\mathrm{d}x.
\end{aligned}$$

Using the fact that $\phi_N^{(1)}\left(x-\frac{i-1}{N}\right)+\phi_N^{(1)}\left(x-\frac{i}{N}\right)=1$, for all $x\in[(i-1)/N, i/N]$, we get

$$I_5^N(t) \leq t\sum_{i=1}^N\int_{\frac{i-1}{N}}^{\frac{i}{N}}\sum_{k\geq 1}\left(\left(\sigma_{i-1}^k-g_k(x)\right)\phi_N^{(1)}\left(x-\frac{i-1}{N}\right)+\left(\sigma_i^k-g_k(x)\right)\phi_N^{(1)}\left(x-\frac{i}{N}\right)\right)^2\mathrm{d}x,$$

and by periodicity of the indexes, we can say that

$$\begin{aligned}
I_5^N(t) &\leq 2t\sum_{i=1}^N\int_{\frac{i-1}{N}}^{\frac{i}{N}}\sum_{k\geq 1}\left(N\int_{\frac{i-1}{N}}^{\frac{i}{N}} g_k(y)\mathrm{d}y-g_k(x)\right)^2\phi_N^{(1)}\left(x-\frac{i}{N}\right)^2\mathrm{d}x\\
&\leq 2t\sum_{i=1}^N\int_{\frac{i-1}{N}}^{\frac{i}{N}}\sum_{k\geq 1}\left(N\int_{\frac{i-1}{N}}^{\frac{i}{N}}(g_k(y)-g_k(x))\mathrm{d}y\right)^2\phi_N^{(1)}\left(x-\frac{i}{N}\right)^2\mathrm{d}x\\
&\leq 2t\sum_{i=1}^N\int_{\frac{i-1}{N}}^{\frac{i}{N}} N\int_{\frac{i-1}{N}}^{\frac{i}{N}}\sum_{k\geq 1}(g_k(y)-g_k(x))^2\mathrm{d}y\mathrm{d}x \quad \text{(since } |\phi_N^{(1)}|\leq 1 \text{ and by Jensen)}\\
&\leq 2tN\sum_{i=1}^N\int_{\frac{i-1}{N}}^{\frac{i}{N}}\int_{\frac{i-1}{N}}^{\frac{i}{N}}\frac{D_0}{N^2}\mathrm{d}y\mathrm{d}x = \frac{2tD_0}{N^2} \leq \frac{2TD_0}{N^2} =: \varepsilon_5^N.
\end{aligned}$$

Step 6. Conclusion for the "bounded" event. Summing all the I_i^N terms, we get for all $t \in [0,T]$

$$\mathbb{E}\left[\left\|u_N^{(1)}(\tau_{M,N}\wedge t)-u(\tau_{M,N}\wedge t)\right\|_{L_0^2(\mathbb{T})}^2\right] \leq (1+\gamma_M)\,\mathbb{E}\left[\int_0^{\tau_{M,N}\wedge t}\left\|u_N^{(1)}(s)-u(s)\right\|_{L_0^2(\mathbb{T})}^2\mathrm{d}s\right]+\varepsilon_1^N+\varepsilon_2^N+\varepsilon_4^N+\varepsilon_5^N.$$

We set $\varepsilon^N := \varepsilon_1^N+\varepsilon_2^N+\varepsilon_4^N+\varepsilon_5^N$. By (3.66), we have

$$\mathbb{E}\left[\left\|u_N^{(1)}(t)-u(t)\right\|_{L_0^2(\mathbb{T})}^2\mathbf{1}_{t\leq\tau_{M,N}}\right] \leq (1+\gamma_M)\int_0^t \mathbb{E}\left[\left\|u_N^{(1)}(s)-u(s)\right\|_{L_0^2(\mathbb{T})}^2\mathbf{1}_{s\leq\tau_{M,N}}\right]\mathrm{d}s+\varepsilon^N.$$

Thus, Grönwall's lemma applies and gives

$$\mathbb{E}\left[\left\|u_N^{(1)}(t)-u(t)\right\|_{L_0^2(\mathbb{T})}^2\mathbf{1}_{t\leq\tau_{M,N}}\right] \leq \varepsilon^N \mathrm{e}^{(1+\gamma_M)t}.$$

Step 7. Conclusion of the proof. We want now to take care of the second term of the right-hand side in (3.65). Using in particular the Cauchy-Schwarz inequality, Proposition 3.19, the Markov inequality, Lemma 3.35 and Lemma 3.34, we have

$$\begin{aligned}
&\mathbb{E}\left[\left\|u_N^{(1)}(t)-u(t)\right\|_{L_0^2(\mathbb{T})}^2\mathbf{1}_{t>\tau_{M,N}}\right]\\
&=\mathbb{E}\left[\left\|u_N^{(1)}(t)-u(t)\right\|_{L_0^2(\mathbb{T})}^2\mathbf{1}_{\sup_{s\in[0,t]}\|u(s)\|_{H_0^1(\mathbb{T})}\vee\sup_{s\in[0,t]}\left\|u_N^{(1)}(s)\right\|_{H_0^1(\mathbb{T})}>M}\right]\\
&\leq \mathbb{E}\left[\left\|u_N^{(1)}(t)-u(t)\right\|_{L_0^2(\mathbb{T})}^4\right]^{1/2}\mathbb{P}\left(\sup_{s\in[0,t]}\|u(s)\|_{H_0^1(\mathbb{T})}\vee\sup_{s\in[0,t]}\left\|u_N^{(1)}(s)\right\|_{H_0^1(\mathbb{T})}>M\right)^{1/2}\\
&\leq \sqrt{8}\left(\mathbb{E}\left[\|u_N^{(1)}(t)\|_{L_0^4(\mathbb{T})}^4\right]+\mathbb{E}\left[\|u(t)\|_{L_0^4(\mathbb{T})}^4\right]\right)^{1/2}\\
&\qquad\times\left(\mathbb{P}\left(\sup_{s\in[0,t]}\|u(s)\|_{H_0^1(\mathbb{T})}>M\right)+\mathbb{P}\left(\sup_{s\in[0,t]}\left\|u_N^{(1)}(s)\right\|_{H_0^1(\mathbb{T})}>M\right)\right)^{1/2}\\
&\leq \sqrt{8}\left(C^{0,4}+\widetilde{C}_T^{0,4}\right)^{1/2}\left(\mathbb{P}\left(\sup_{s\in[0,t]}\|u(s)\|_{H_0^1(\mathbb{T})}>M\right)+\frac{1}{M}\mathbb{E}\left[\sup_{s\in[0,t]}\left\|u_N^{(1)}(s)\right\|_{H_0^1(\mathbb{T})}\right]\right)^{1/2}\\
&\leq \sqrt{8}\left(C^{0,4}+\widetilde{C}_T^{0,4}\right)^{1/2}\left(\mathbb{P}\left(\sup_{s\in[0,t]}\|u(s)\|_{H_0^1(\mathbb{T})}>M\right)+\frac{1}{M}C_T^{1,2}\right)^{1/2}
\end{aligned}$$

Furthermore, as $(u(t))_{t\geq 0}$ is continuous from $[0,+\infty)$ to $H_0^1(\mathbb{T})$ (see Proposition 3.2), the random variable $\sup_{s\in[0,t]}\|u(s)\|_{H_0^1(\mathbb{T})}$ is finite almost surely. As a consequence,

$$\lim_{M\to\infty}\limsup_{N\to\infty}\mathbb{E}\left[\|u_N^{(1)}(t)-u(t)\|_{L_0^2(\mathbb{T})}^2\mathbf{1}_{t>\tau_{M,N}}\right]=0.$$

Combining this inequality with the conclusion of the step 6 yields the wanted result. □

3.4 Convergence of invariant measures: split-step scheme towards semi-discrete scheme

In this section, we aim to prove the second part of Theorem 3.7, namely Equation (3.17). The structure of the proof is the same as for the first part of Theorem 3.7. In Subsection 3.4.1, we show that the family of probability measures $\{\nu_{N,\Delta t} : \Delta t \in (0, \Delta t_{\max}]\}$ is tight in $\mathcal{P}(\mathbb{R}_0^N)$ and then relatively compact in $\mathcal{P}_2(\mathbb{R}_0^N)$. In Subsection 3.4.2, in a similar manner as in Subsection 3.3.2 for the semi-discrete case, we identify each subsequential limit of the family $\{\nu_{N,\Delta t} : \Delta t \in (0, \Delta t_{\max}]\}$, when $\Delta t \to 0$, as the invariant measure of the process $(\mathbf{U}_n)_{n\in\mathbb{N}}$, which leads to the final part of the proof of Theorem 3.7. Subsection 3.4.3 contain the proofs of the lemmas stated in Subsections 3.4.1 and 3.4.2.

3.4.1 Tightness, relative compactness and some estimates

Lemma 3.38 (Tightness)**.** *Under Assumptions 3.1 and 3.3, for any $N \geq 1$, the family of probability measures $\{\nu_{N,\Delta t} : \Delta t \in (0, \Delta t_{\max}]\}$ is tight in the space $\mathcal{P}(\mathbb{R}_0^N)$.*

Proof. We established at Proposition 3.27 that a random variable $\mathbf{V} \sim \nu_{N,\Delta t}$ satisfies the discrete H_0^1 estimate:

$$\mathbb{E}\left[\left\|\mathbf{D}_N^{(1)}\mathbf{V}\right\|_2^2\right] \leq \overline{C}^{1,2}.$$

Since $\|\mathbf{D}_N^{(1)} \cdot \|_2$ defines a norm on $\mathbb{R}_0^N$, the result follows from the Markov inequality. □

Lemma 3.39 (Fourth-order moment)**.** *Under Assumptions 3.1 and 3.3, there exists a constant $\overline{C}^{0,4} > 0$, depending only on D_0, ν and $\Delta t_{\max}$, such that for any time step $\Delta t \in (0, \Delta t_{\max}]$ and any random variable $\mathbf{V} \sim \nu_{N,\Delta t}$, we have*

$$\mathbb{E}\left[\|\mathbf{V}\|_4^4\right] \leq \overline{C}^{0,4}.$$

Corollary 3.40 (Relative compactness)**.** *Let $N \geq 1$. Under Assumptions 3.1 and 3.3, the family $\{\nu_{N,\Delta t} : \Delta t \in (0, \Delta t_{\max}]\}$ is relatively compact in $\mathcal{P}_2(\mathbb{R}_0^N)$.*

Proof. By virtue of the Prokhorov theorem [12, Theorem 5.1] and Lemma 3.38, any sequence extracted from $\{\nu_{N,\Delta t} : \Delta t \in (0, \Delta t_{\max}]\}$ admits a weakly converging subsequence in $\mathcal{P}(\mathbb{R}_0^N)$. Let ν^* be a subsequential weak limit and let $(\nu_{N,\Delta t_j})_{j\in\mathbb{N}}$ be a sequence weakly converging towards ν^*. Let $(\mathbf{V}_j)_{j\in\mathbb{N}}$ be a sequence of $\mathbb{R}_0^N$-valued random variables such that $\mathbf{V}_j \sim \nu_{N,\Delta t_j}$. By virtue of the Portemanteau theorem, since $\|\cdot\|_4^4$ is continuous (and thus lower semi-continuous) on $\mathbb{R}_0^N$, we have

$$\mathbb{E}\left[\|\mathbf{V}\|_4^4\right] \leq \liminf_{j\to\infty} \mathbb{E}\left[\|\mathbf{V}_j\|_4^4\right] \leq \overline{C}^{0,4},$$

so that ν^* admits a fourth-order moment and thus belongs to $\mathcal{P}_4(\mathbb{R}_0^N)$. Moreover, it follows also from Lemma 3.38 that the sequence $(\nu_{N,\Delta t_j})_{j\in\mathbb{N}}$ satisfies a uniform integrability condition in the sense of [102, Definition 6.8], and the result is now a consequence of [102, Theorem 6.9]. □

Lemma 3.41 (Finite time bound)**.** *Let Assumptions 3.1 and 3.3 hold, let $T > 0$ be a time horizon and let $(\mathbf{U}_n)_{n\in\mathbb{N}}$ be a solution of (3.15) with an initial condition $\mathbf{U}_0 \sim \nu_{N,\Delta t}$. There exists a constant $\overline{C}_T^{0,2}$ depending only on D_0, T, ν and $\Delta t_{\max}$, such that for any time step $\Delta t \in (0, \Delta t_{\max}]$, we have*

$$\mathbb{E}\left[\sup_{n=0,1,\dots,\lfloor \frac{T}{\Delta t}\rfloor} \|\mathbf{U}_n\|_2^2\right] \leq \overline{C}_T^{0,2}.$$

3.4.2 Characterisation of the limit

As in Subsection 3.3.2 for the semi-discrete scheme, we want to use a result of convergence in finite time of the numerical scheme (3.15) in order to identify each subsequential limit of the family $\{\nu_{N,\Delta t} : \Delta t \in (0, \Delta t_{\max}]\}$, when $\Delta t \to 0$, as the invariant measure ν_N for the solution of Equation (3.11). By virtue of Corollary 3.40, let $\nu^* \in \mathcal{P}_2(\mathbb{R}_0^N)$ and let $(\Delta t_j)_{j\in\mathbb{N}}$ be a sequence of time steps decreasing to zero such that $W_2(\nu_{N,\Delta t_j}, \nu^*)$ converges to zero as $j \to +\infty$. By virtue of the Skorokhod representation theorem, let $(\mathbf{U}_0^{(j)})_{j\in\mathbb{N}}$ be a sequence of $\mathbb{R}_0^N$-valued random variables converging almost surely to $\mathbf{U}_0$ such that

$$\mathbf{U}_0^{(j)} \sim \nu_{N,\Delta t_j}, \quad \forall j \in \mathbb{N} \qquad \text{and} \qquad \mathbf{U}_0 \sim \nu^*.$$

As we described in Subsection 3.3.2, up to an extension of the probability space, these random variables may be considered as initial conditions for the equations (3.11) and (3.15) (driven by the same Wiener process).

Lemma 3.42 (Finite time convergence)**.** *Let Assumptions 3.1 and 3.3 hold. Let $(\mathbf{U}(t))_{t\geq 0}$ be the solution of (3.11) with initial condition $\mathbf{U}_0$ and for all $j \in \mathbb{N}$, let $(\mathbf{U}_n^{(j)})_{n\in\mathbb{N}}$ be the solution of (3.15) with initial condition $\mathbf{U}_0^{(j)}$. For any $j \in \mathbb{N}$, we define the piecewise constant approximation $(\overline{\mathbf{U}}^{(j)}(t))_{t\geq 0}$ of $\mathbf{U}_n^{(j)}$ by $\overline{\mathbf{U}}^{(j)}(t) = \mathbf{U}_n^{(j)}$ if $t \in [n\Delta t_j, (n+1)\Delta t_j)$. Then, for all $T > 0$,*

$$\lim_{j\to\infty} \sup_{t\in[0,T]} \mathbb{E}\left[\left\|\overline{\mathbf{U}}^{(j)}(t) - \mathbf{U}(t)\right\|_2^2\right] = 0.$$

Proof of Theorem 3.7: Part 2/2. The arguments are identical to the proof of the first part. □

3.4.3 Proofs

Proof of Lemma 3.39. Let $(\mathbf{U}_n)_{n\in\mathbb{N}} = (U_{1,n}, \dots, U_{N,n})_{n\in\mathbb{N}}$ be a solution of (3.15) with a deterministic initial condition $\mathbf{u}_0$. By convexity of the function $v \mapsto v^4$, for any $\alpha, \beta \in \mathbb{R}$, we have $(\alpha-\beta)^4 \geq \alpha^4 - 4\alpha^3\beta$. In particular, for any $i \in \mathbb{Z}/N\mathbb{Z}$, taking $\alpha = U_{i,n+\frac{1}{2}}$ and $\beta = \Delta t b_i(\mathbf{U}_{n+\frac{1}{2}})$, we have

$$U_{i,n}^4 = \left(U_{i,n+\frac{1}{2}} - \Delta t b_i\left(\mathbf{U}_{n+\frac{1}{2}}\right)\right)^4 \geq U_{i,n+\frac{1}{2}}^4 - 4U_{i,n+\frac{1}{2}}^3 \Delta t b_i\left(\mathbf{U}_{n+\frac{1}{2}}\right).$$

Hence, expanding the drift function and summing over i, we get

$$\begin{aligned}\|\mathbf{U}_n\|_4^4 \geq \left\|\mathbf{U}_{n+\frac{1}{2}}\right\|_4^4 + \sum_{i=1}^N 4\Delta t U_{i,n+\frac{1}{2}}^3 \left(\overline{A}\left(U_{i,n+\frac{1}{2}}, U_{i+1,n+\frac{1}{2}}\right) - \overline{A}\left(U_{i-1,n+\frac{1}{2}}, U_{i,n+\frac{1}{2}}\right)\right) \\ - 4\nu\Delta t\left\langle \mathbf{U}_{n+\frac{1}{2}}^3, \mathbf{D}_N^{(2)}\mathbf{U}_{n+\frac{1}{2}}\right\rangle.\end{aligned}$$

We know thanks to Lemma 3.12 that the second term of the right-hand side is non-negative. Summing by parts the third term, we get

$$\|\mathbf{U}_n\|_4^4 \geq \left\|\mathbf{U}_{n+\frac{1}{2}}\right\|_4^4 + 4\nu\Delta t\left\langle \mathbf{D}_N^{(1)}\mathbf{U}_{n+\frac{1}{2}}^3, \mathbf{D}_N^{(1)}\mathbf{U}_{n+\frac{1}{2}}\right\rangle.$$

From Lemma 3.11, we get

$$\|\mathbf{U}_n\|_4^4 \geq \left\|\mathbf{U}_{n+\frac{1}{2}}\right\|_4^4 + 3\nu\Delta t\left\|\mathbf{U}_{n+\frac{1}{2}}\right\|_4^4. \tag{3.74}$$

On the other hand, let us look at the second step of the scheme (3.15). By the construction of the split-step scheme, the random variables $U_{i,n+\frac{1}{2}}$ and $\Delta W_{i,n+1}^{Q,N}$ are independent. Since $\Delta W_{i,n+1}^{Q,N} \sim \mathcal{N}(0, \Delta t\sum_{k\geq 1}(\sigma_i^k)^2)$ and $\sum_{k\geq 1}(\sigma_i^k)^2 \leq D_0$ (by (3.9)), we write

$$\begin{aligned}\mathbb{E}\left[\|\mathbf{U}_{n+1}\|_4^4\right] &= \mathbb{E}\left[\left\|\mathbf{U}_{n+\frac{1}{2}} + \Delta\mathbf{W}_{n+1}^{Q,N}\right\|_4^4\right] \\ &= \mathbb{E}\left[\left\|\mathbf{U}_{n+\frac{1}{2}}\right\|_4^4\right] + \frac{6}{N}\mathbb{E}\left[\sum_{i=1}^N U_{i,n+\frac{1}{2}}^2\left(\Delta W_{i,n+1}^{Q,N}\right)^2\right] + \mathbb{E}\left[\left\|\Delta\mathbf{W}_{n+1}^{Q,N}\right\|_4^4\right] \\ &\leq \mathbb{E}\left[\left\|\mathbf{U}_{n+\frac{1}{2}}\right\|_4^4\right] + 6D_0\Delta t\mathbb{E}\left[\left\|\mathbf{U}_{n+\frac{1}{2}}\right\|_2^2\right] + 3D_0^2\Delta t^2 \end{aligned} \tag{3.75}$$

Combining Inequalities (3.74) and (3.75), we get

$$\mathbb{E}\left[\|\mathbf{U}_n\|_4^4\right] \geq \mathbb{E}\left[\|\mathbf{U}_{n+1}\|_4^4\right] - 6D_0\Delta t\mathbb{E}\left[\left\|\mathbf{U}_{n+\frac{1}{2}}\right\|_2^2\right] - 3D_0^2\Delta t^2 + 3\nu\Delta t\mathbb{E}\left[\left\|\mathbf{U}_{n+\frac{1}{2}}\right\|_4^4\right]$$

from which we get a telescopic sum:

$$3\nu\Delta t\sum_{l=0}^{n-1}\mathbb{E}\left[\left\|\mathbf{U}_{l+\frac{1}{2}}\right\|_4^4\right] \leq \|\mathbf{u}_0\|_4^4 - \mathbb{E}\left[\|\mathbf{U}_n\|_4^4\right] + 6D_0\Delta t\sum_{l=0}^{n-1}\mathbb{E}\left[\left\|\mathbf{U}_{l+\frac{1}{2}}\right\|_2^2\right] + 3nD_0^2\Delta t^2.$$

Thus,

$$\frac{1}{n}\sum_{l=0}^{n-1}\mathbb{E}\left[\left\|\mathbf{U}_{l+\frac{1}{2}}\right\|_4^4\right] \leq \frac{1}{3\nu\Delta tn}\|\mathbf{u}_0\|_4^4 + \frac{2D_0}{\nu n}\sum_{l=0}^{n-1}\mathbb{E}\left[\left\|\mathbf{U}_{l+\frac{1}{2}}\right\|_2^2\right] + \frac{D_0^2\Delta t}{\nu}. \tag{3.76}$$

Recall that from Lemma 3.10 and Equation (3.40), we have

$$\frac{1}{n}\sum_{l=0}^{n-1}\mathbb{E}\left[\left\|\mathbf{U}_{l+\frac{1}{2}}\right\|_2^2\right] \leq \frac{1}{n}\sum_{l=0}^{n-1}\mathbb{E}\left[\left\|\mathbf{D}_N^{(1)}\mathbf{U}_{l+\frac{1}{2}}\right\|_2^2\right] \leq \frac{\|\mathbf{u}_0\|_2^2}{2\nu n\Delta t} + \frac{D_0}{2\nu}. \tag{3.77}$$

Injecting (3.77) into (3.76), we get

$$\frac{1}{n}\sum_{l=0}^{n-1}\mathbb{E}\left[\left\|\mathbf{U}_{l+\frac{1}{2}}\right\|_4^4\right] \leq \frac{1}{3\nu\Delta tn}\|\mathbf{u}_0\|_4^4 + \frac{2D_0}{\nu}\left(\frac{\|\mathbf{u}_0\|_2^2}{2\nu n\Delta t} + \frac{D_0}{2\nu}\right) + \frac{D_0^2\Delta t}{\nu}.$$

Using now the same arguments as for the end of the proof of Proposition 3.19, letting $\mathbf{V} \sim \nu_{N,\Delta t}$, we get

$$\mathbb{E}\left[\left\|\mathbf{V}_{\frac{1}{2}}\right\|_4^4\right] \leq \frac{D_0^2}{\nu^2} + \frac{D_0^2\Delta t}{\nu} = \frac{D_0^2}{\nu}\left(\frac{1}{\nu} + \Delta t\right).$$

To conclude, we use Inequality (3.75) once again:

$$\begin{aligned}
\mathbb{E}\left[\|\mathbf{V}\|_4^4\right] &\leq \mathbb{E}\left[\left\|\mathbf{V}_{\frac{1}{2}}\right\|_4^4\right] + 6D_0\Delta t\mathbb{E}\left[\left\|\mathbf{V}_{\frac{1}{2}}\right\|_2^2\right] + 3D_0^2\Delta t^2 \\
&\leq \frac{D_0^2}{\nu}\left(\frac{1}{\nu} + \Delta t\right) + \frac{3D_0^2\Delta t}{\nu} + 3D_0^2\Delta t^2 \\
&\leq D_0^2\left(\frac{1}{\nu} + 3\Delta t_{\max}\right)\left(\frac{1}{\nu} + \Delta t_{\max}\right).
\end{aligned}$$

□

Proof of Lemma 3.41. Let us repeat the proof of Proposition 3.27 up to Equation (3.37). For all $n = 0, 1, \ldots, \lfloor \frac{T}{\Delta t} \rfloor$, we write

$$\begin{aligned}
\|\mathbf{U}_n\|_2^2 &= \|\mathbf{U}_0\|_2^2 + \sum_{l=0}^{n-1}\left(\|\mathbf{U}_{l+1}\|_2^2 - \|\mathbf{U}_l\|_2^2\right) \\
&\leq \|\mathbf{U}_0\|_2^2 - 2\nu\Delta t\sum_{l=0}^{n-1}\left\|\mathbf{D}_N^{(1)}\mathbf{U}_{l+\frac{1}{2}}\right\|_2^2 + 2\sum_{l=0}^{n-1}\left\langle\mathbf{U}_{l+\frac{1}{2}}, \Delta\mathbf{W}_{l+1}^{Q,N}\right\rangle + \sum_{l=0}^{n-1}\left\|\Delta\mathbf{W}_{l+1}^{Q,N}\right\|_2^2.
\end{aligned}$$

The viscous term may be removed from the inequality. Taking the supremum in time and the expectation, we get

$$\mathbb{E}\left[\sup_{n=0,1,\ldots,\lfloor\frac{T}{\Delta t}\rfloor}\|\mathbf{U}_n\|_2^2\right] \leq \mathbb{E}\left[\|\mathbf{U}_0\|_2^2\right] + 2\mathbb{E}\left[\sup_{n=0,1,\ldots,\lfloor\frac{T}{\Delta t}\rfloor}\left|\sum_{l=0}^{n-1}\left\langle\mathbf{U}_{l+\frac{1}{2}}, \Delta\mathbf{W}_{l+1}^{Q,N}\right\rangle\right|\right] + \mathbb{E}\left[\sum_{l=0}^{\lfloor T/\Delta t\rfloor-1}\left\|\Delta\mathbf{W}_{l+1}^{Q,N}\right\|_2^2\right]. \tag{3.78}$$

First, by Lemma 3.10 and Proposition 3.27, we have

$$\mathbb{E}\left[\|\mathbf{U}_0\|_2^2\right] \le \mathbb{E}\left[\left\|\mathbf{D}_N^{(1)}\mathbf{U}_0\right\|_2^2\right] \le \overline{C}^{1,2}.$$

Noticing that the process $(\sum_{l=0}^{n-1}\langle \mathbf{U}_{l+\frac{1}{2}}, \Delta\mathbf{W}_{l+1}^{Q,N}\rangle)_{n\ge1}$ is a martingale, we get by applying successively Jensen's and Doob's inequalities to the second term of the right-hand side,

$$\begin{aligned}
\mathbb{E}\left[\sup_{n=0,1,\dots,\lfloor\frac{T}{\Delta t}\rfloor}\left|\sum_{l=0}^{n-1}\left\langle \mathbf{U}_{l+\frac{1}{2}}, \Delta\mathbf{W}_{l+1}^{Q,N}\right\rangle\right|\right] &\le \mathbb{E}\left[\sup_{n=0,1,\dots,\lfloor\frac{T}{\Delta t}\rfloor}\left|\sum_{l=0}^{n-1}\left\langle \mathbf{U}_{l+\frac{1}{2}}, \Delta\mathbf{W}_{l+1}^{Q,N}\right\rangle\right|^2\right]^{1/2} \\
&\le 2\mathbb{E}\left[\left|\sum_{l=0}^{\lfloor T/\Delta t\rfloor-1}\left\langle \mathbf{U}_{l+\frac{1}{2}}, \Delta\mathbf{W}_{l+1}^{Q,N}\right\rangle\right|^2\right]^{1/2}.
\end{aligned}$$

From (3.15), we may observe that each increment $\Delta\mathbf{W}_{l+1}^{Q,N}$ is independent of the family $(\mathbf{U}_{m+\frac{1}{2}}, \Delta\mathbf{W}_m^{Q,N})_{m=0,\dots,l}$. Therefore, letting $\mathbf{V}\sim\nu_{N,\Delta t}$ and letting $\mathbf{V}_{\frac{1}{2}}$ be the random variable satisfying $\mathbf{V}_{\frac{1}{2}} = \mathbf{V} + \mathbf{b}(\mathbf{V}_{\frac{1}{2}})$, we have

$$\begin{aligned}
\mathbb{E}\left[\sup_{n=0,1,\dots,\lfloor\frac{T}{\Delta t}\rfloor}\left|\sum_{l=0}^{n-1}\left\langle \mathbf{U}_{l+\frac{1}{2}}, \Delta\mathbf{W}_{l+1}^{Q,N}\right\rangle\right|\right] &\le 2\mathbb{E}\left[\sum_{l=0}^{\lfloor T/\Delta t\rfloor-1}\left|\left\langle \mathbf{U}_{l+\frac{1}{2}}, \Delta\mathbf{W}_{l+1}^{Q,N}\right\rangle\right|^2\right]^{1/2} \\
&\le 2\left(\sum_{l=0}^{\lfloor T/\Delta t\rfloor-1}\mathbb{E}\left[\left\|\mathbf{U}_{l+\frac{1}{2}}\right\|_2^2\right]\mathbb{E}\left[\left\|\Delta\mathbf{W}_{l+1}^{Q,N}\right\|_2^2\right]\right)^{1/2} \\
&\le 2\sqrt{D_0\Delta t}\left(\left\lfloor\frac{T}{\Delta t}\right\rfloor\mathbb{E}\left[\left\|\mathbf{V}_{\frac{1}{2}}\right\|_2^2\right]\right)^{1/2} \quad \text{(by (3.38))} \\
&\le D_0\sqrt{\frac{2T}{\nu}} \quad \text{(by Lemma 3.10 and (3.34))}.
\end{aligned}$$

Injecting this bound into (3.78), we get

$$\mathbb{E}\left[\sup_{n=0,1,\dots,\lfloor\frac{T}{\Delta t}\rfloor}\|\mathbf{U}_n\|_2^2\right] \le \overline{C}^{1,2} + 2D_0\sqrt{\frac{2T}{\nu}} + TD_0. \qquad \square$$

Proof of Lemma 3.42. Step 0. Decomposition of the error in two events. Let $T>0$. We start by introducing the exit time of some ball for the time-continuous and the time-discretised processes:

$$\rho_M^{(j)} := \inf\left\{t\ge0 : \|\mathbf{U}(t)\|_2^2 \vee \left\|\overline{\mathbf{U}}^{(j)}(t)\right\|_2^2 \ge M\right\}.$$

Then, we decompose the approximation error according to whether or not the discrete and continuous processes stay in the ball of radius M: for all $t\in[0,T]$,

$$\mathbb{E}\left[\left\|\overline{\mathbf{U}}^{(j)}(t)-\mathbf{U}(t)\right\|_2^2\right] = \mathbb{E}\left[\left\|\overline{\mathbf{U}}^{(j)}(t)-\mathbf{U}(t)\right\|_2^2\mathbf{1}_{t<\rho_M^{(j)}}\right] + \mathbb{E}\left[\left\|\overline{\mathbf{U}}^{(j)}(t)-\mathbf{U}(t)\right\|_2^2\mathbf{1}_{\rho_M^{(j)}\le t}\right]. \qquad (3.79)$$

Step 1. Decomposition of the error for the bounded trajectories. For any $t \in [0, T]$ and any $j \in \mathbb{N}$, let $n_{[t,M,j]} := \left\lfloor \frac{t \wedge \rho_M^{(j)}}{\Delta t_j} \right\rfloor$. From (3.15) and (3.11), we write

$$\begin{aligned}
\overline{\mathbf{U}}^{(j)}\left(t \wedge \rho_M^{(j)}\right) - \mathbf{U}\left(t \wedge \rho_M^{(j)}\right) = \mathbf{U}_0^{(j)} - \mathbf{U}_0 + \sum_{l=0}^{n_{[t,M,j]}-1} \left(\Delta t_j \mathbf{b}\left(\mathbf{U}_{l+\frac{1}{2}}^{(j)}\right) - \int_{l\Delta t_j}^{(l+1)\Delta t_j} \mathbf{b}(\mathbf{U}(s))\mathrm{d}s\right) \\
+ \left(t \wedge \rho_M^{(j)} - n_{[t,M,j]}\Delta t_j\right) \mathbf{b}\left(\mathbf{U}_{n_{[t,M,j]}+\frac{1}{2}}^{(j)}\right) - \int_{n_{[t,M,j]}\Delta t_j}^{t\wedge\rho_M^{(j)}} \mathbf{b}\left(\mathbf{U}(s)\right)\mathrm{d}s.
\end{aligned} \tag{3.80}$$

From this dynamics, we can decompose then again the first term of the right-hand side in (3.79) in three terms:

$$\begin{aligned}
&\mathbb{E}\left[\left\|\overline{\mathbf{U}}^{(j)}(t) - \mathbf{U}(t)\right\|_2^2 \mathbf{1}_{t<\rho_M^{(j)}}\right] \leq \mathbb{E}\left[\left\|\overline{\mathbf{U}}^{(j)}\left(t \wedge \rho_M^{(j)}\right) - \mathbf{U}\left(t \wedge \rho_M^{(j)}\right)\right\|_2^2\right] \\
&\leq 2\mathbb{E}\left[\left\|\mathbf{U}_0^{(j)} - \mathbf{U}_0\right\|_2^2\right] \\
&+ 2\mathbb{E}\left[\left\|\sum_{l=0}^{n_{[t,M,j]}-1} \left(\Delta t_j \mathbf{b}\left(\mathbf{U}_{l+\frac{1}{2}}^{(j)}\right) - \int_{l\Delta t_j}^{(l+1)\Delta t_j} \mathbf{b}(\mathbf{U}(s))\mathrm{d}s\right)\right.\right. \\
&\qquad \left.\left. + \left(t \wedge \rho_M^{(j)} - n_{[t,M,j]}\Delta t_j\right) \mathbf{b}\left(\mathbf{U}_{n_{[t,M,j]}+\frac{1}{2}}^{(j)}\right) - \int_{n_{[t,M,j]}\Delta t_j}^{t\wedge\rho_M^{(j)}} \mathbf{b}\left(\mathbf{U}(s)\right)\mathrm{d}s\right\|_2^2\right] \\
&\leq 2\mathbb{E}\left[\left\|\mathbf{U}_0^{(j)} - \mathbf{U}_0\right\|_2^2\right] && (3.81) \\
&\quad + 4\mathbb{E}\left[\left\|\sum_{l=0}^{n_{[t,M,j]}-1} \Delta t_j \left(\mathbf{b}\left(\mathbf{U}_{l+\frac{1}{2}}^{(j)}\right) - \mathbf{b}(\mathbf{U}_l^{(j)})\right) + (t \wedge \rho_M^{(j)} - n_{[t,M,j]}) \left(\mathbf{b}\left(\mathbf{U}_{n_{[t,M,j]}+\frac{1}{2}}^{(j)}\right) - \mathbf{b}\left(\mathbf{U}_{n_{[t,M,j]}}^{(j)}\right)\right)\right\|_2^2\right] && (3.82) \\
&\quad + 4\mathbb{E}\left[\left\|\sum_{l=0}^{n_{[t,M,j]}-1} \int_{l\Delta t_j}^{(l+1)\Delta t_j} \left(\mathbf{b}\left(\mathbf{U}_l^{(j)}\right) - \mathbf{b}(\mathbf{U}(s))\right)\mathrm{d}s + \int_{n_{[t,M,j]}\Delta t_j}^{t\wedge\rho_M^{(j)}} \left(\mathbf{b}\left(\mathbf{U}_{n_{[t,M,j]}}^{(j)}\right) - \mathbf{b}(\mathbf{U}(s))\right)\mathrm{d}s\right\|_2^2\right]. && (3.83)
\end{aligned}$$

Step 2. Bound over the term (3.83). Let L_M be a Lipschitz constant of $\mathbf{b}$ over the ball

$\{\|\cdot\|_2^2 \leq M\}$. From Jensen's inequality, we have

$$
\begin{aligned}
&4\mathbb{E}\left[\left\|\sum_{l=0}^{n_{[t,M,j]}-1}\int_{l\Delta t_j}^{(l+1)\Delta t_j}\left(\mathbf{b}\left(\mathbf{U}_l^{(j)}\right)-\mathbf{b}(\mathbf{U}(s))\right)\mathrm{d}s+\int_{n_{[t,M,j]}\Delta t_j}^{t\wedge\rho_M^{(j)}}\left(\mathbf{b}\left(\mathbf{U}_{n_{[t,M,j]}}^{(j)}\right)-\mathbf{b}(\mathbf{U}(s))\right)\mathrm{d}s\right\|_2^2\right]\\
&\leq 4T\mathbb{E}\left[\sum_{l=0}^{n_{[t,M,j]}-1}\int_{l\Delta t_j}^{(l+1)\Delta t_j}\left\|\mathbf{b}\left(\mathbf{U}_l^{(j)}\right)-\mathbf{b}(\mathbf{U}(s))\right\|_2^2\mathrm{d}s+\int_{n_{[t,M,j]}\Delta t_j}^{t\wedge\rho_M^{(j)}}\left\|\mathbf{b}\left(\mathbf{U}_{n_{[t,M,j]}}^{(j)}\right)-\mathbf{b}(\mathbf{U}(s))\right\|_2^2\mathrm{d}s\right]\\
&\leq 4TL_M^2\mathbb{E}\left[\sum_{l=0}^{n_{[t,M,j]}-1}\int_{l\Delta t_j}^{(l+1)\Delta t_j}\left\|\mathbf{U}_l^{(j)}-\mathbf{U}(s)\right\|_2^2\mathrm{d}s+\int_{n_{[t,M,j]}\Delta t_j}^{t\wedge\rho_M^{(j)}}\left\|\mathbf{U}_{n_{[t,M,j]}}^{(j)}-\mathbf{U}(s)\right\|_2^2\mathrm{d}s\right]\\
&= 4TL_M^2\mathbb{E}\left[\int_0^{t\wedge\rho_M^{(j)}}\left\|\overline{\mathbf{U}}^{(j)}(s)-\mathbf{U}(s)\right\|_2^2\mathrm{d}s\right]\\
&= 4TL_M^2\int_0^t\mathbb{E}\left[\left\|\overline{\mathbf{U}}^{(j)}(s)-\mathbf{U}(s)\right\|_2^2\mathbf{1}_{s<\rho_M^{(j)}}\right]\mathrm{d}s.
\end{aligned}
$$

Step 3. Decomposition of the term (3.82). Let C_M be a supremum of $\|\mathbf{b}\|_2^2$ over the ball $\{\|\cdot\|_2^2 \leq M\}$. Using the Jensen inequality and the locally Lipschitz continuity of $\mathbf{b}$, we write

$$
\begin{aligned}
&4\mathbb{E}\left[\left\|\sum_{l=0}^{n_{[t,M,j]}-1}\Delta t_j\left(\mathbf{b}\left(\mathbf{U}_{l+\frac{1}{2}}^{(j)}\right)-\mathbf{b}(\mathbf{U}_l^{(j)})\right)+(t\wedge\rho_M^{(j)}-n_{[t,M,j]})\left(\mathbf{b}\left(\mathbf{U}_{n_{[t,M,j]}+\frac{1}{2}}^{(j)}\right)-\mathbf{b}\left(\mathbf{U}_{n_{[t,M,j]}}^{(j)}\right)\right)\right\|_2^2\right]\\
&\leq 4TL_M^2\Delta t_j\mathbb{E}\left[\sum_{l=0}^{n_{[t,M,j]}}\left\|\mathbf{U}_{l+\frac{1}{2}}^{(j)}-\mathbf{U}_l^{(j)}\right\|_2^2\right]\\
&= 4T\Delta t_j^3L_M^2\mathbb{E}\left[\sum_{l=0}^{n_{[t,M,j]}}\left\|\mathbf{b}\left(\mathbf{U}_{l+\frac{1}{2}}^{(j)}\right)\right\|_2^2\right]\\
&\leq 4T\left(T+\Delta t_j\right)\Delta t_j^2L_M^2C_M,
\end{aligned}
$$

where at the last line, we have used the fact that for any $l=0,\ldots,n_{[t,M,j]}$, by (3.35), we have $\|\mathbf{U}_{l+\frac{1}{2}}\|_2^2 \leq \|\mathbf{U}_l\|_2^2 \leq M$.

Step 4. Conclusion for the bounded event. Summing the estimates obtained from Step 1 to Step 3, we get

$$
\begin{aligned}
\mathbb{E}\left[\left\|\overline{\mathbf{U}}^{(j)}(t)-\mathbf{U}(t)\right\|_2^2\mathbf{1}_{t<\rho_M^{(j)}}\right] &\leq 2\mathbb{E}\left[\left\|\mathbf{U}_0^{(j)}-\mathbf{U}_0\right\|_2^2\right]\\
&\quad+4TL_M^2\int_0^t\mathbb{E}\left[\left\|\overline{\mathbf{U}}^{(j)}(s)-\mathbf{U}(s)\right\|_2^2\mathbf{1}_{s<\rho_M^{(j)}}\right]\mathrm{d}s\\
&\quad+4T\left(T+\Delta t_j\right)\Delta t_j^2L_M^2C_M.
\end{aligned}
$$

By construction, the random variable $\|\mathbf{U}_0^{(j)}-\mathbf{U}_0\|_2$ tends to 0 almost surely as $j\to+\infty$. Furthermore, it has a fourth order moment uniform in j thanks to Lemma 3.39 and the Portemanteau theorem:

$$
\begin{aligned}
\mathbb{E}\left[\left\|\mathbf{U}_0^{(j)}-\mathbf{U}_0\right\|_2^4\right] &\leq 8\mathbb{E}\left[\left\|\mathbf{U}_0^{(j)}\right\|_4^4\right]+8\mathbb{E}\left[\|\mathbf{U}_0\|_4^4\right]\\
&\leq 8\overline{C}^{0,4}+8\liminf_{j\to\infty}\mathbb{E}\left[\left\|\mathbf{U}_0^{(j)}\right\|_4^4\right]\leq 16\overline{C}^{0,4}.
\end{aligned}
$$

Thus, $\mathbb{E}[\|\mathbf{U}_0^{(j)}-\mathbf{U}_0\|_2^2]$ tends to 0 as $j\to+\infty$. Now, by Grönwall's lemma, we have

$$\mathbb{E}\left[\left\|\overline{\mathbf{U}}^{(j)}(t)-\mathbf{U}(t)\right\|_2^2\mathbf{1}_{t<\rho_M^{(j)}}\right]\leq\left(2\mathbb{E}\left[\left\|\mathbf{U}_0^{(j)}-\mathbf{U}_0\right\|_2^2\right]+4T\left(T+\Delta t_j\right)\Delta t_j^2L_M^2C_M\right)\mathrm{e}^{4T^2L_M^2}.$$

As a consequence,

$$\lim_{j\to\infty}\sup_{t\in[0,T]}\mathbb{E}\left[\left\|\overline{\mathbf{U}}^{(j)}(t)-\mathbf{U}(t)\right\|_2^2\mathbf{1}_{t<\rho_M^{(j)}}\right]=0. \tag{3.84}$$

Step 5. The exiting trajectories. We want to bound the second term of the RHS in (3.79) uniformly in j. Observing that

$$\left\{\rho_M^{(j)}\leq t\right\}=\left\{\sup_{s\in[0,t]}\|\mathbf{U}(s)\|_2^2\vee\sup_{s\in[0,t]}\left\|\overline{\mathbf{U}}^{(j)}(s)\right\|_2^2\geq M\right\},$$

we get from the Cauchy-Schwarz inequality, Lemma 3.39 and Lemma 3.16.(ii):

$$\begin{aligned}\mathbb{E}\left[\left\|\overline{\mathbf{U}}^{(j)}(t)-\mathbf{U}(t)\right\|_2^2\mathbf{1}_{\rho_M^{(j)}\leq t}\right]&\leq\mathbb{E}\left[\left\|\overline{\mathbf{U}}^{(j)}(t)-\mathbf{U}(t)\right\|_2^4\right]^{1/2}\mathbb{P}\left(\sup_{s\in[0,t]}\|\mathbf{U}(s)\|_2^2\vee\sup_{s\in[0,t]}\left\|\overline{\mathbf{U}}^{(j)}(s)\right\|_2^2\geq M\right)^{1/2}\\&\leq 2\left(\overline{C}^{0,4}+c_0^{(4)}+c_1^{(4)}C^{0,4}+c_2^{(4)}T\right)^{1/2}\mathbb{P}\left(\sup_{s\in[0,T]}\|\mathbf{U}(s)\|_2^2\vee\sup_{s\in[0,T]}\left\|\overline{\mathbf{U}}^{(j)}(s)\right\|_2^2\geq M\right)^{1/2}.\end{aligned}$$

As for the second term, we have thanks to the Markov inequality and Lemma 3.41,

$$\begin{aligned}\mathbb{P}\left(\sup_{s\in[0,T]}\left\|\overline{\mathbf{U}}^{(j)}(s)\right\|_2^2\vee\sup_{s\in[0,T]}\|\mathbf{U}(s)\|_2^2\geq M\right)&\leq\mathbb{P}\left(\sup_{s\in[0,T]}\left\|\overline{\mathbf{U}}^{(j)}(s)\right\|_2^2\geq M\right)+\mathbb{P}\left(\sup_{s\in[0,T]}\|\mathbf{U}(s)\|_2^2\geq M\right)\\&\leq\frac{1}{M}\mathbb{E}\left[\sup_{s\in[0,T]}\left\|\overline{\mathbf{U}}^{(j)}(s)\right\|_2^2\right]+\mathbb{P}\left(\sup_{s\in[0,T]}\|\mathbf{U}(s)\|_2^2\geq M\right)\\&\leq\frac{\overline{C}_T^{0,2}}{M}+\mathbb{P}\left(\sup_{s\in[0,T]}\|\mathbf{U}(s)\|_2^2\geq M\right)\underset{M\to\infty}{\longrightarrow}0.\end{aligned}$$

As a consequence,

$$\lim_{M\to\infty}\limsup_{j\to\infty}\sup_{t\in[0,T]}\mathbb{E}\left[\left\|\overline{\mathbf{U}}^{(j)}(t)-\mathbf{U}(t)\right\|_2^2\mathbf{1}_{\rho_M^{(j)}\leq t}\right]=0. \tag{3.85}$$

Step 6. Conclusion. In the end, from (3.79) and (3.84), we have for every $M>0$,

$$\limsup_{j\to\infty}\sup_{t\in[0,T]}\mathbb{E}\left[\left\|\overline{\mathbf{U}}^{(j)}(t)-\mathbf{U}(t)\right\|_2^2\right]\leq\limsup_{j\to\infty}\sup_{t\in[0,T]}\mathbb{E}\left[\left\|\overline{\mathbf{U}}^{(j)}(t)-\mathbf{U}(t)\right\|_2^2\mathbf{1}_{\rho_M^{(j)}\leq t}\right].$$

Thus, letting $M\to+\infty$ and applying (3.85) yields the wanted result. $\square$

3.A Proofs

Proof of Lemma 3.10. Let $m = 0$. Let $p \in [1, +\infty)$ and let $\mathbf{u} \in \mathbb{R}^N$ be as in the statement. We have

$$\begin{aligned}
\left\| \mathbf{D}_N^{(0)} \mathbf{u} \right\|_p^p &= \frac{1}{N} \sum_{u_i \geq 0} |u_i|^p + \frac{1}{N} \sum_{u_i < 0} |u_i|^p \\
&\leq \frac{1}{N} \sum_{u_i \geq 0} |u_i - u_{i_-}|^p + \frac{1}{N} \sum_{u_i < 0} |u_i - u_{i_+}|^p \\
&\leq \frac{1}{N} \sum_{u_i \geq 0} N^{p-1} \sum_{j=1}^{N} |u_{j+1} - u_j|^p + \frac{1}{N} \sum_{u_i < 0} N^{p-1} \sum_{j=1}^{N} |u_{j+1} - u_j|^p \\
&= \frac{1}{N} \sum_{j=1}^{N} |N(u_{j+1} - u_j)|^p = \left\| \mathbf{D}_N^{(1)} \mathbf{u} \right\|_p^p,
\end{aligned}$$

where we used the Jensen inequality passing from the third to the fourth line. We just have proved the wanted inequality when $m = 0$ but the case $m = 1$ is proved in the same way. □

Proof of Lemma 3.11. For $\mathbf{u} \in \mathbb{R}_0^N$ and $p \in 2\mathbb{N}^*$, we have

$$\begin{aligned}
\left\langle \mathbf{D}_N^{(1)}(\mathbf{u}^{p-1}), \mathbf{D}_N^{(1)} \mathbf{u} \right\rangle &= N \sum_{i=1}^{N} \left(u_{i+1}^{p-1} - u_i^{p-1} \right) (u_{i+1} - u_i) \\
&= N(p-1) \sum_{i=1}^{N} (u_{i+1} - u_i) \int_{u_i}^{u_{i+1}} |z|^{p-2} \mathrm{d}z \\
&= N(p-1) \sum_{i=1}^{N} (u_{i+1} - u_i) \int_{u_i}^{u_{i+1}} \left(|z|^{p/2-1} \right)^2 \mathrm{d}z \\
&\geq N(p-1) \sum_{i=1}^{N} \left(\int_{u_i}^{u_{i+1}} |z|^{p/2-1} \mathrm{d}z \right)^2 \qquad \text{(by Jensen's inequality)} \\
&= \frac{4N(p-1)}{p^2} \sum_{i=1}^{N} \left(\int_{u_i}^{u_{i+1}} \frac{\mathrm{d}}{\mathrm{d}z} \left(\operatorname{sign}(z) |z|^{p/2} \right) \mathrm{d}z \right)^2 \\
&= \frac{4N(p-1)}{p^2} \sum_{i=1}^{N} \left(\operatorname{sign}(u_{i+1}) |u_{i+1}|^{p/2} - \operatorname{sign}(u_i) |u_i|^{p/2} \right)^2 .
\end{aligned}$$

We can now apply Lemma 3.10 to the vector $(\operatorname{sign}(u_i) |u_i|^{p/2})_{i=1}^N$ and eventually, we obtain

$$\left\langle \mathbf{D}_N^{(1)}(\mathbf{u}^{p-1}), \mathbf{D}_N^{(1)} \mathbf{u} \right\rangle \geq \frac{4(p-1)}{Np^2} \sum_{i=1}^{N} \left(\operatorname{sign}(u_i) |u_i|^{p/2} \right)^2 = \frac{4(p-1)}{p^2} \|\mathbf{u}\|_p^p. \qquad \square$$

Proof of Lemma 3.12. Let $\mathbf{u} \in \mathbb{R}_0^N$ and $q \in 2\mathbb{N}^*$. Summing by parts and using (3.12) and (3.13), we have

$$\begin{aligned}
\sum_{i=1}^{N} u_i^{q-1}\left(\overline{A}(u_i,u_{i+1})-\overline{A}(u_{i-1},u_i)\right) &= -\sum_{i=1}^{N}\left(u_{i+1}^{q-1}-u_i^{q-1}\right)\overline{A}(u_i,u_{i+1})\\
&= -\sum_{i=1}^{N}\int_{u_i^{q-1}}^{u_{i+1}^{q-1}}\overline{A}(u_i,u_{i+1})\mathrm{d}z\\
&\geq -\sum_{i=1}^{N}\int_{u_i^{q-1}}^{u_{i+1}^{q-1}}\overline{A}\left(z^{1/(q-1)},z^{1/(q-1)}\right)\mathrm{d}z\\
&= -\sum_{i=1}^{N}\int_{u_i^{q-1}}^{u_{i+1}^{q-1}}A\left(z^{1/(q-1)}\right)\mathrm{d}z\\
&= -\sum_{i=1}^{N}\int_{u_i^{q-1}}^{u_{i+1}^{q-1}}\frac{\mathrm{d}}{\mathrm{d}z}\left(\mathcal{A}_q(z)\right)\mathrm{d}z = 0,
\end{aligned}$$

where $\mathcal{A}_q$ denotes a function defined on $\mathbb{R}$ such that $\mathcal{A}_q'(z) = A(z^{1/(q-1)})$. $\square$

Proof of Lemma 3.13. (i) Let $\mathbf{u},\mathbf{v}\in\mathbb{R}_0^N$. From the definition of $\mathbf{b}$, we write

$$\begin{aligned}
\langle\mathbf{sign}(\mathbf{u}-\mathbf{v}),\mathbf{b}(\mathbf{u})-\mathbf{b}(\mathbf{v})\rangle = &-\sum_{i=1}^{N}\mathrm{sign}(u_i-v_i)\left(\overline{A}(u_i,u_{i+1})-\overline{A}(u_{i-1},u_i)-\overline{A}(v_i,v_{i+1})+\overline{A}(v_{i-1},v_i)\right)\\
&+\nu N\sum_{i=1}^{N}\mathrm{sign}(u_i-v_i)(u_{i+1}-2u_i+u_{i-1}-v_{i+1}+2v_i-v_{i-1}).
\end{aligned}$$

By periodicity, both terms of the right-hand side can be summed by parts, which leads to

$$\begin{aligned}
\langle\mathbf{sign}(\mathbf{u}-\mathbf{v}),\mathbf{b}(\mathbf{u})-\mathbf{b}(\mathbf{v})\rangle = &\sum_{i=1}^{N}\left(\mathrm{sign}(u_{i+1}-v_{i+1})-\mathrm{sign}(u_i-v_i)\right)\left(\overline{A}(u_i,u_{i+1})-\overline{A}(v_i,v_{i+1})\right)\\
&-\nu N\sum_{i=1}^{N}\left(\mathrm{sign}(u_{i+1}-v_{i+1})-\mathrm{sign}(u_i-v_i)\right)\left((u_{i+1}-v_{i+1})-(u_i-v_i)\right).
\end{aligned}$$

Observe that since the function sign : $\mathbb{R}\to\mathbb{R}$ is non-decreasing, each term of the second sum is non-negative. As for the first sum, it follows from the monotonicity property of $\overline{A}$ that each term is non-positive. Let us address for instance the case where $u_{i+1}\geq v_{i+1}$ and $u_i\leq v_i$. Then, on the one hand, we have $\mathrm{sign}(u_{i+1}-v_{i+1})-\mathrm{sign}(u_i-v_i)=2$. On the other hand, we have

$$\begin{aligned}
\overline{A}(u_i,u_{i+1})-\overline{A}(v_i,v_{i+1}) &= \left(\overline{A}(u_i,u_{i+1})-\overline{A}(u_i,v_{i+1})\right)+\left(\overline{A}(u_i,v_{i+1})-\overline{A}(v_i,v_{i+1})\right)\\
&= \int_{v_{i+1}}^{u_{i+1}}\partial_2\overline{A}(u_i,z)\mathrm{d}z-\int_{u_i}^{v_i}\partial_1\overline{A}(z,v_{i+1})\mathrm{d}z\leq 0.
\end{aligned}$$

The case where $u_{i+1}\leq v_{i+1}$ and $u_i\geq v_i$ is treated symmetrically.

(ii) Let $\mathbf{u}\in\mathbb{R}_0^N$. We have

$$\langle\mathbf{u},\mathbf{b}(\mathbf{u})\rangle = -\sum_{i=1}^{N}u_i\left(\overline{A}(u_i,u_{i+1})-\overline{A}(u_{i-1},u_i)\right)+\nu N\sum_{i=1}^{N}u_i(u_{i+1}-2u_i+u_{i-1}).$$

Lemma 3.12 with $q=2$ shows that the first term of the above decomposition is non-positive. Summing by parts the second term yields the result. $\square$

Proof of Lemma 3.30. (i) The wanted equality follows from standard computations.

(ii) Let us start with the first inequality. We have

$$\begin{aligned}\left\|\Psi_N^{(1)}\mathbf{v}-\Psi_N^{(0)}\mathbf{v}\right\|_{L_0^2(\mathbb{T})}^2 &= \int_{\mathbb{T}}\left(\sum_{i=1}^N v_i N\left(x-\frac{i-1}{N}\right)\mathbf{1}_{\left(\frac{i-1}{N},\frac{i}{N}\right]}(x)+\sum_{i=1}^N v_i N\left(\frac{i+1}{N}-x\right)\mathbf{1}_{\left(\frac{i}{N},\frac{i+1}{N}\right]}(x)\right.\\ &\qquad\left.-\sum_{i=1}^N v_i\mathbf{1}_{\left(\frac{i-1}{N},\frac{i}{N}\right]}(x)\right)^2\mathrm{d}x\\ &= \sum_{i=1}^N\int_{\frac{i-1}{N}}^{\frac{i}{N}}\left(v_iN\left(x-\frac{i-1}{N}\right)+v_{i-1}N\left(\frac{i}{N}-x\right)-v_i\right)^2\mathrm{d}x\\ &= \sum_{i=1}^N(v_i-v_{i-1})^2\int_{\frac{i-1}{N}}^{\frac{i}{N}}(Nx-i)^2\mathrm{d}x\\ &= \frac{1}{3N}\sum_{i=1}^N(v_i-v_{i-1})^2=\frac{1}{3N^2}\left\|\mathbf{D}_N^{(1)}\mathbf{v}\right\|_2^2.\end{aligned}$$

As for the second inequality, we have

$$\begin{aligned}&\left\|\Psi_N^{(2)}\mathbf{v}-\Psi_N^{(0)}\mathbf{v}\right\|_{L_0^2(\mathbb{T})}^2\\ &= \sum_{i=1}^N\int_{\frac{i-1}{N}}^{\frac{i}{N}}\left(\frac{1}{2}v_{i+1}(Nx-i)(Nx-(i-1))-v_i(Nx-(i-1))(Nx-(i+1))\right.\\ &\qquad\qquad\left.+\frac{1}{2}v_{i-1}(Nx-i)(Nx-(i+1))-v_i\right)^2\mathrm{d}x\\ &= \sum_{i=1}^N\int_{\frac{i-1}{N}}^{\frac{i}{N}}\left(\frac{1}{2}v_{i+1}(Nx-i)^2+\frac{1}{2}v_{i+1}(Nx-i)-v_i(Nx-i)^2+\frac{1}{2}v_{i-1}(Nx-i)^2-\frac{1}{2}v_{i-1}(Nx-i)\right)^2\\ &\le 3\sum_{i=1}^N\int_{\frac{i-1}{N}}^{\frac{i}{N}}\left(\left(\frac{1}{2}v_{i+1}-v_i+\frac{1}{2}v_{i-1}\right)(Nx-i)^2\right)^2\mathrm{d}x\\ &\quad+3\sum_{i=1}^N\int_{\frac{i-1}{N}}^{\frac{i}{N}}\left(\left(\frac{1}{2}v_{i+1}-\frac{1}{2}v_i\right)(Nx-i)\right)^2\mathrm{d}x\\ &\quad+3\sum_{i=1}^N\int_{\frac{i-1}{N}}^{\frac{i}{N}}\left(\left(\frac{1}{2}v_i-\frac{1}{2}v_{i-1}\right)(Nx-i)\right)^2\mathrm{d}x\\ &= \frac{3}{4}\sum_{i=1}^N(v_{i+1}-2v_i+v_{i-1})^2\int_{\frac{i-1}{N}}^{\frac{i}{N}}(Nx-i)^4\mathrm{d}x+\frac{3}{2}\sum_{i=1}^N(v_{i+1}-v_i)^2\int_{\frac{i-1}{N}}^{\frac{i}{N}}(Nx-i)^2\mathrm{d}x\\ &= \frac{3}{20N^4}\left\|\mathbf{D}_N^{(2)}\mathbf{v}\right\|_2^2+\frac{1}{2N^2}\left\|\mathbf{D}_N^{(1)}\mathbf{v}\right\|_2^2.\end{aligned}$$

□

Chapter 4

Numerical experiments

Résumé. *Ce chapitre correspond à la section 5 de [19]. On présente des tests numériques illustrant les théorèmes 3.5 et 3.7 dans le cas de l'équation de Burgers. La stationnarité du schéma totalement discrétisé est observée via un calcul de la moyenne ergodique. Des bornes supérieures sur les taux de convergence en temps et en espace, pour l'approximation en distance de Wasserstein de la mesure invariante, sont calculées empiriquement. Enfin, des simulations du spectre d'énergie et du facteur d'aplatissement (flatness) de la solution stationnaire sont comparées à des résultats théoriques relatifs à la théorie de la turbulence.*

Abstract. This chapter corresponds to the fifth section of [19]. We provide numerical experiments to illustrate Theorems 3.5 and 3.7 in the case of the Burgers equation. The stationarity of the fully discrete scheme is observed through the computation of the ergodic mean. Upper bounds on the convergence rates in space and in time, for the approximation of the invariant measure with respect to the Wasserstein distance, are derived empirically. Finally, numerical simulations of the energy spectrum and of the flatness of the stationary solution are compared to theoretical results in turbulence theory.

All the experiments in this section are performed on the Burgers equation, *i.e.* the flux function is set to be $A(u) = u^2/2$. Moreover, we will also fix the following set of parameters: $\nu = 10^{-5}$, $u_0 \equiv 0$, $g_k(x) = \cos(2\pi kx)$ for $k = 1, \dots, 4$ and $g_k \equiv 0$ for $k \geq 5$. The implicit equation in (3.15) is solved numerically by use of the Newton-Raphson method.

4.1 Stationarity

We seek here to give a numerical illustration of the stationarity of the Markov chain $(\mathbf{U}_n^{N,\Delta t})_{n\in\mathbb{N}}$ defined by (3.15) (in all this section, the number of cells and the time step will always appear as a superscript in the solutions). As already mentioned in Remark 3.9, by virtue of Birkhoff's ergodic theorem, for any test function $\varphi : \mathbb{R}_0^N \to \mathbb{R}$ such that $\varphi \in L^1(\nu_{N,\Delta t})$ and any random variable $\mathbf{V} \sim \nu_{N,\Delta t}$, the process $(\mathbf{U}_n^{N,\Delta t})_{n\in\mathbb{N}}$ shall satisfy

$$Y_n := \frac{1}{n}\sum_{l=0}^{n-1} \varphi\left(\mathbf{U}_l^{N,\Delta t}\right) \xrightarrow[n\to\infty]{a.s.} \mathbb{E}\left[\varphi\left(\mathbf{V}\right)\right].$$

In Figure 4.1, we record the values of the sequence $(Y_n)_{n\geq 1}$ up to the iteration $\overline{n} = 10^4$, with the following set of parameters: $\Delta t = 10^{-3}$, $N = 512$, $\varphi = \cos(\|\cdot\|_2)$. In particular, the time interval considered here is the interval $[0, 10]$.

The stationary state seems to be reached approximately at time $t = 3$.

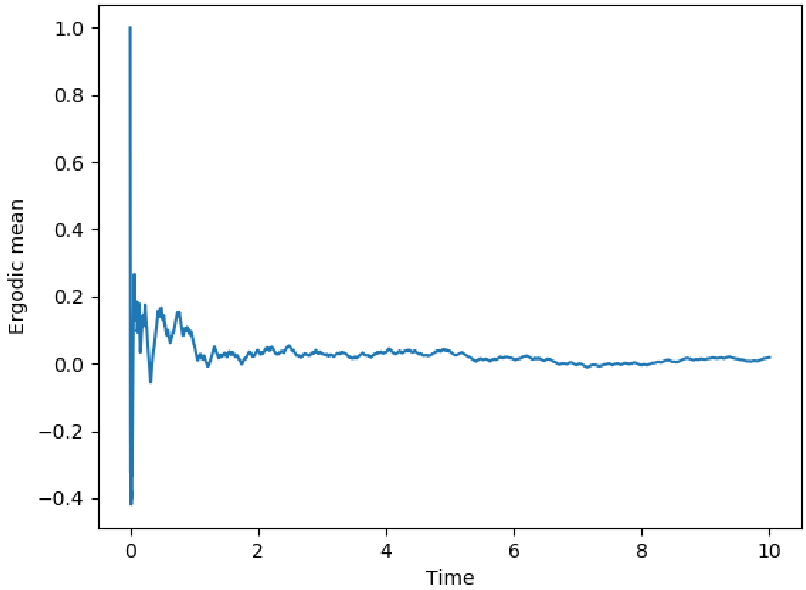

Figure 4.1: Ergodic mean of the process $\left(\varphi\left(\mathbf{U}_n^{N,\Delta t}\right)\right)_{n\in\mathbb{N}}$.

4.2 Convergence in space

In the following experiment, we aim to retrieve numerically the convergence result of Theorem 3.7 as N tends to infinity and for a fixed time step. Instead of computing directly the Wasserstein distance, we compute the strong L^2 error with respect to a reference solution computed with $N_{\mathrm{ref}} = 2^{11}$. More precisely, we record in Figure 4.2 the values of

$$\left(\frac{1}{\overline{n}}\sum_{n=0}^{\overline{n}-1}\frac{1}{N_{\mathrm{ref}}}\sum_{i=1}^{N_{\mathrm{ref}}}\left(U^{N,\Delta t}_{\left\lceil\frac{N_{\mathrm{ref}}(i-1)}{N}\right\rceil+1,n} - U^{N_{\mathrm{ref}},\Delta t}_{i,n}\right)^2\right)^{1/2} \tag{4.1}$$

as N takes values in $\{2^3, 2^4, \dots, 2^{10}\}$. For $\overline{n}$ sufficiently large, the discrete processes aim to be close to their stationary state and thus, the value (4.1) is meant to be an upper bound of the Wasserstein error approximation of the invariant measure μ:

$$\begin{aligned}
\left(\frac{1}{\overline{n}}\sum_{n=0}^{\overline{n}-1}\frac{1}{N_{\mathrm{ref}}}\sum_{i=1}^{N_{\mathrm{ref}}}\left(U^{N,\Delta t}_{\left\lceil\frac{N_{\mathrm{ref}}(i-1)}{N}\right\rceil+1,n} - U^{N_{\mathrm{ref}},\Delta t}_{i,n}\right)^2\right)^{1/2} &\approx \mathbb{E}\left[\left\|u_N^{(0)}\left(\overline{n}\Delta t\right) - u\left(\overline{n}\Delta t\right)\right\|^2_{L^2_0(\mathbb{T})}\right]^{1/2} \\
&\geq W_2\left(\mathcal{L}\left(u_N^{(0)}\left(\overline{n}\Delta t\right)\right), \mathcal{L}\left(u\left(\overline{n}\Delta t\right)\right)\right) \\
&\approx W_2\left(\mu_N^{(0)}, \mu\right).
\end{aligned}$$

Here, the other parameters are set to: $\Delta t = 10^{-3}$, $\overline{n} = 10^4$.

The result shows that the convergence in space happens at an order of at least $1/2$.

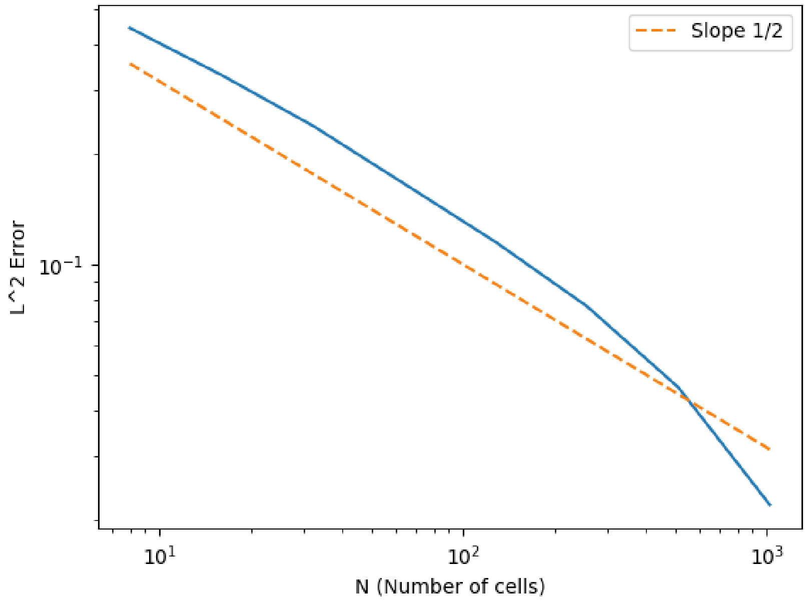

Figure 4.2: Strong error convergence at a large time with respect to N

4.3 Convergence in time

We apply the same procedure to study the convergence with respect to the time step Δt. A reference solution is computed for the time step $\Delta t_{\text{ref}} = 2^{-11}$, and for a number $\overline{n} = 10 \times 2^{11}$ of iterations, we compute the following L^2 error

$$\left(\frac{1}{\overline{n}} \sum_{n=0}^{\overline{n}-1} \left\| \mathbf{U}^{N,\Delta t}_{\lfloor n \Delta t_{\text{ref}} \Delta t^{-1} \rfloor} - \mathbf{U}^{N,\Delta t_{\text{ref}}}_{n} \right\|_2^2 \right)^{1/2},$$

which is supposed to be an upper bound of the Wasserstein distance error between the respective invariant measures for (3.11) and for (3.15):

$$\begin{aligned} \left(\frac{1}{\overline{n}} \sum_{n=1}^{\overline{n}} \left\| \mathbf{U}^{N,\Delta t}_{\lfloor n \Delta t_{\text{ref}} \Delta t^{-1} \rfloor} - \mathbf{U}^{N,\Delta t_{\text{ref}}}_{n} \right\|_2^2 \right)^{1/2} &\approx \mathbb{E} \left[\left\| \mathbf{U}^{N,\Delta t}_{\lfloor \overline{n} \Delta t_{\text{ref}} \Delta t^{-1} \rfloor} - \mathbf{U}^{N,\Delta t_{\text{ref}}}_{\overline{n}} \right\|_2^2 \right]^{1/2} \\ &\geq W_2 \left(\mathcal{L}\left(\mathbf{U}^{N,\Delta t}_{\lfloor \overline{n} \Delta t_{\text{ref}} \Delta t^{-1} \rfloor} \right), \mathcal{L}\left(\mathbf{U}^{N,\Delta t_{\text{ref}}}_{\overline{n}} \right) \right) \\ &\approx W_2 \left(\nu_{N,\Delta t}, \nu_N \right). \end{aligned}$$

This error is evaluated when Δt takes values in $\{2^{-4}, 2^{-5}, \ldots, 2^{-10}\}$ and for $N = 256$.

A rate of convergence of $1/2$ also stems from this experiment.

4.4 Burgulence estimates

Endowed with the Burgers flux function, Equation (3.1) may be interpreted as a one-dimensional and simplified version of the Navier-Stokes system, and as such, it is considered a toy model for turbulence (the so-called *burgulence*, see for instance [15, Chapter 1] or [48, 49] in this prospect). According to the turbulence theory dating back to Kolmogorov, universal properties emerge as a turbulent dynamical

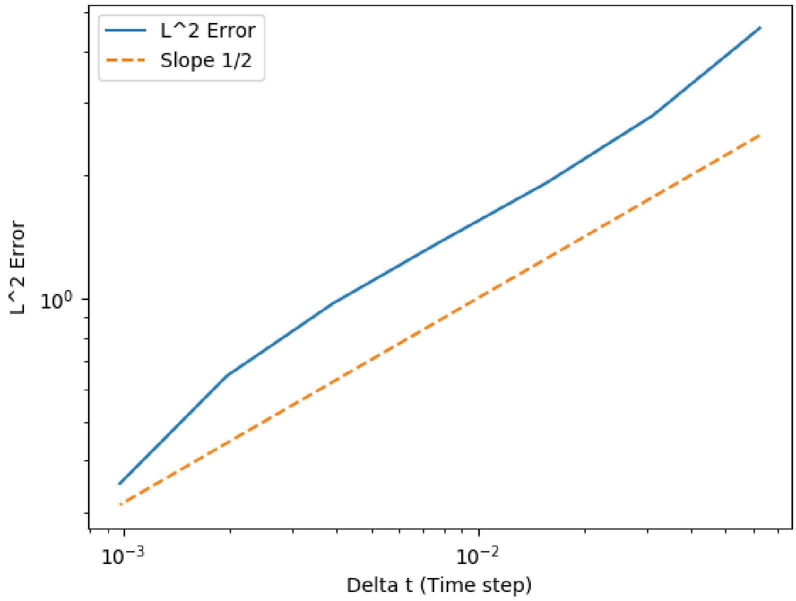

Figure 4.3: Strong error convergence at a large time with respect to Δt

system approaches its stationary state. Here, we will try to recover numerically two of these properties which, in a framework close to Equation (3.1), have been proved rigorously in Boritchev's work [15]. The first one concerns the decay rate of the energy spectrum. Let u be an $L^2_0(\mathbb{T})$-valued random variable whose distribution is the invariant measure μ of the process associated to (3.1), and let $\hat{u}_k$ denote, for any $k \geq 1$, the k-th Fourier coefficient of u:

$$\hat{u}_k := \int_{\mathbb{T}} u(x) \mathrm{e}^{-2i\pi kx} \mathrm{d}x.$$

Here, we call *energy spectrum* the function E defined for all $k \geq 1$ by $E(k) := \mathbb{E}[|\hat{u}_k|^2]$. This function satisfies a specific decay rate [15, Theorem 4.7.3] up to some averaging around the neighbour coefficients of k. For any $k \geq 1$ and any $M > 1$, we set $\mathcal{S}_{k,M} := [M^{-1}k, Mk] \cap \mathbb{N}$ which defines a set of neighbours of k. We now state the result contained in [15, Theorem 4.7.3]:

Proposition 4.1. *There exists an interval $I \subset [0,1]$, called the* inertial range, *such that*

$$\frac{1}{|\mathcal{S}_{k,M}|} \sum_{j \in \mathcal{S}_{k,M}} E(j) \sim k^{-2}, \qquad k^{-1} \in I.$$

Here, $x \sim y$ means that there exists a constant $C > 0$ such that $C^{-1}y \leq x \leq Cy$. The inertial range I is defined with more details in [15, Section 4.6]. To give a physical interpretation of this interval, it corresponds to the range of scales in which the energy of the system is transported from large scales to smaller ones. If we write $I = [\alpha, \beta]$, the inertial range is positioned between the *energy range* $[\beta, 1]$ containing the large scales, which in our case are generated by the stochastic forcing, and the *dissipation range* $[0, \alpha]$ containing the small scales dissipated by the viscous term. In particular, α depends linearly on ν.

The second universal property of interest concerns the *flatness*, that is the function F defined by

$$F(l) := \frac{\mathbb{E}\left[\int_{\mathbb{T}} |u(x+l) - u(x)|^4 \mathrm{d}x\right]}{\mathbb{E}\left[\int_{\mathbb{T}} |u(x+l) - u(x)|^2 \mathrm{d}x\right]^2}, \qquad l \in \mathbb{T},$$

where u is a random variable with distribution μ. The flatness aims to be an indicator of the spatial intermittency in the turbulent system described by (3.1). A decay rate for F in the inertial range is provided in [15, Corollary 4.6.9]:

Proposition 4.2. *Let I be the inertial range from Proposition 4.1. Then,*

$$F(l) \sim l^{-1}, \qquad l \in I.$$

From (3.15), we computed the numerical approximations of the energy spectrum and the flatness. These computations are plotted in Figure 4.4. More precisely, we used the following respective approximations:

$$E(k) \approx \frac{1}{\overline{n}} \sum_{n=0}^{\overline{n}-1} \left| \hat{U}_{k,n}^{N,\Delta t} \right|^2, \quad k = 1, \dots, \left\lceil \frac{N}{2} \right\rceil;$$

$$F\left(\frac{j}{N}\right) \approx \frac{\frac{1}{\overline{n}} \sum_{n=0}^{\overline{n}-1} \frac{1}{N} \sum_{i=1}^{N} \left| U_{i+j,n}^{N,\Delta t} - U_{i,n}^{N,\Delta t} \right|^4}{\left(\frac{1}{\overline{n}} \sum_{n=0}^{\overline{n}-1} \frac{1}{N} \sum_{i=1}^{N} \left| U_{i+j,n}^{N,\Delta t} - U_{i,n}^{N,\Delta t} \right|^2 \right)^2}, \quad j \in \mathbb{Z}/N\mathbb{Z}.$$

Here, $\hat{\mathbf{U}}_n^{N,\Delta t}$ is the discrete Fourier transform of $\mathbf{U}_n^{N,\Delta t}$, which we computed using an FFT algorithm from the Python library `numpy.fft` (v1.17). In both experiments, we have taken $N = 256$, $\Delta t = 10^{-3}$, and $\overline{n} = 10000$.

The result of Proposition 4.1 seems recovered as the slope of the energy spectrum tends to behave like k^{-2} in some sub-interval of $[0, 1]$. As regards the flatness, for intermediate scales, a decay rate varying from the order $-3/2$ to the order -1 is observed.

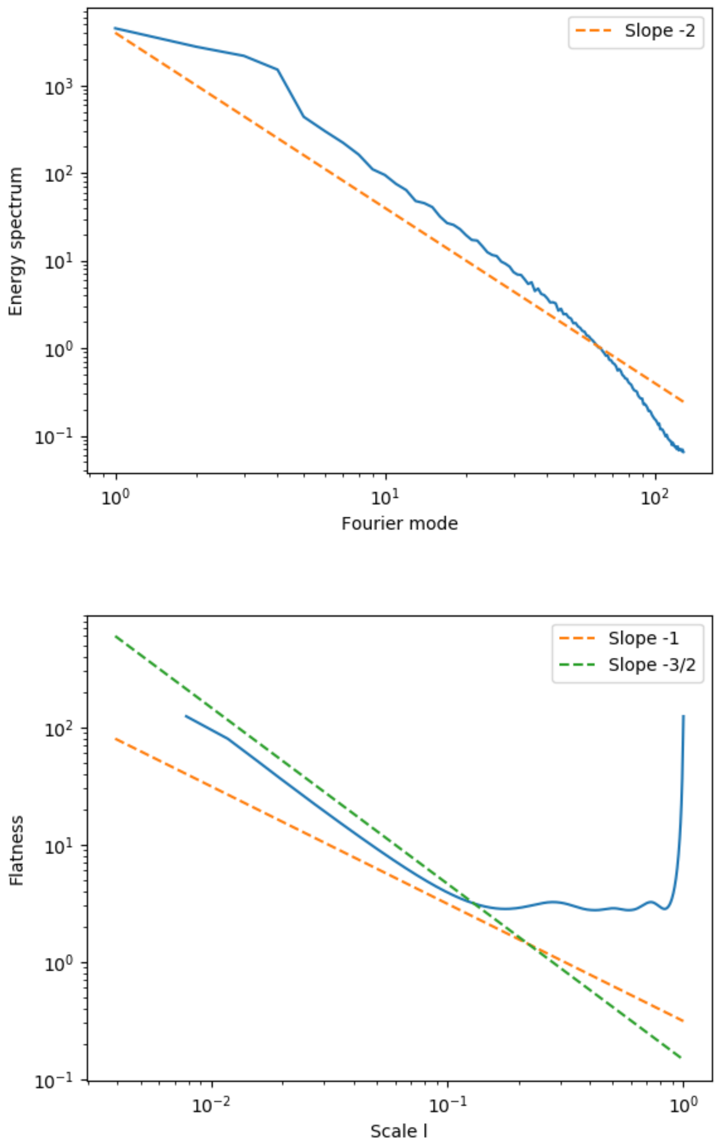

Figure 4.4: Decay rates for the energy spectrum and for the flatness.

Appendix A

Invariant measures for the numerical schemes in the inviscid multi-dimensional case

Résumé. *On se propose dans cette partie d'étudier la stationnarité des schémas semi-discret et complètement discret introduits au chapitre 3 lorsque la viscosité est nulle. Il s'agit de généraliser le résultat du théorème 3.5 au cas non-visqueux. Dans la première section, on expose comment les schémas à flux strictement monotone ont un caractère diffusif exploitable pour assurer l'existence et l'unicité d'une mesure invariante. Dans la seconde section, ce résultat est généralisé à un domaine spatial périodique de dimension quelconque.*

Abstract. In this part, we study the stationarity of the semi and fully discrete schemes introduced in Chapter 3 when the viscosity coefficient is equal to zero. We aim to generalise Theorem 3.5 to the inviscid case. In the first section, we expose how strongly monotone schemes have a diffusive nature that can be exploited to ensure the existence and uniqueness of an invariant measure. In the second section, this result is generalised to a periodic spatial domain of arbitrary dimension.

A.1 A remark on strongly monotone numerical fluxes

Let us place ourselves in the setting of Chapter 3 and let us complement Assumption 3.3 with the fact that the numerical flux $\overline{A}$ is *strongly monotone*, that is, there exist $\lambda > 0$ and a numerical flux function $\widetilde{A}$ satisfying Assumption 3.3 such that

$$\overline{A}(u,v) = \widetilde{A}(u,v) + \lambda(u-v), \qquad \forall u,v \in \mathbb{R}. \tag{A.1}$$

This notion was used for instance in [82, Definition 5]. We may notice that it is possible to construct a strongly monotone numerical flux from any numerical flux function satisfying Assumption 3.3 just by adding a term $\lambda(u-v)$. In particular, there is no restriction on the class of flux functions A that are covered here. We consider here the function $\mathbf{b}$ used to establish the semi and fully discrete approximations, respectively (3.11) and (3.15), to Equation (2.1) with such a numerical flux. Recall that this function is defined from $\mathbb{R}_0^N$ to $\mathbb{R}_0^N$ by its components

$$b_i(\mathbf{v}) := -N\left(\overline{A}(v_i,v_{i+1}) - \overline{A}(v_{i-1},v_i)\right) + \nu N^2(v_{i+1} - 2v_i + v_{i-1}), \quad i \in \mathbb{Z}/N\mathbb{Z}, \quad \mathbf{v} \in \mathbb{R}_0^N.$$

Using (A.1), we may write

$$b_i(\mathbf{v}) = -N\left(\widetilde{A}(v_i,v_{i+1}) - \widetilde{A}(v_{i-1},v_i)\right) + N(\lambda + \nu N)(v_{i+1} - 2v_i + v_{i-1}), \quad i \in \mathbb{Z}/N\mathbb{Z}.$$

Observe that the numerical scheme we obtain here is the one we would get with the numerical flux $\widetilde{A}$, but with some additional amount of viscosity (actually, with the new viscosity coefficient $\nu+\frac{\lambda}{N}$). In this respect, strongly monotone numerical fluxes provide a good example of schemes inducing *numerical diffusion*.

In this context, if we take $\nu = 0$, the function $\mathbf{b}$ still satisfies the L^1-contraction property (Lemma 3.13.(i)), whereas the dissipativity property also holds but with a dependence on N which is not of the same order as Lemma 3.13.(ii) as we have

$$\langle \mathbf{v}, \mathbf{b}(\mathbf{v})\rangle \leq -\frac{\lambda}{N}\left\|\mathbf{D}_N^{(1)}\mathbf{v}\right\|_2^2, \qquad \mathbf{v}\in\mathbb{R}_0^N.$$

As a consequence, all the qualitative properties of the processes $(\mathbf{U}(t))_{t\geq 0}$ and $(\mathbf{U}_n)_{n\in\mathbb{N}}$ obtained in Chapter 3 hold, and we have notably

Proposition A.1. *With a numerical flux of the form (A.1) and a viscosity coefficient $\nu \geq 0$, the respective solutions $(\mathbf{U}(t))_{t\geq 0}$ and $(\mathbf{U}_n)_{n\in\mathbb{N}}$ to (3.11) and (3.15) both admit a unique invariant measure in $\mathcal{P}(\mathbb{R}_0^N)$, denoted respectively ν_N and $\nu_{N,\Delta t}$.*

However, in the quantitative estimates, all of the upper bounds that relied on the viscosity have been multiplied by N. For instance, Inequality (3.47) has been replaced by

$$\mathbb{E}\left[\left\|\mathbf{D}_N^{(1)}\mathbf{V}\right\|_2^2\right] \leq \frac{ND_0}{2\lambda}, \qquad \mathbf{V}\sim\nu_N,$$

from which we cannot derive directly the tightness of the family $(\mu_N)_{N\geq 1}$, which is the first step of our proof of convergence with respect to N.

A.2 Inviscid and multi-dimensional version of Theorem 3.5

The purpose of this section is to generalise, in the inviscid case, Proposition A.1 to the case where the space variable is multi-dimensional. That is, we prove uniqueness and existence of invariant measures for semi-discrete and fully discrete numerical schemes that aim to approximate conservation laws of the form

$$\mathrm{d}u = -\mathrm{div}\left(\mathbf{A}(u)\right)\mathrm{d}t + \sum_{k\geq 1} g_k \mathrm{d}W^k(t), \qquad t\geq 0, \quad \mathbf{x}\in\mathbb{T}^d, \tag{A.2}$$

where

- $\mathbf{A}\in C^2(\mathbb{R},\mathbb{R}^d)$;
- for all $k\geq 1$, $g_k\in C(\mathbb{T}^d)$ and
$$\int_{\mathbb{T}^d} g_k(\mathbf{x})\mathrm{d}\mathbf{x} = 0;$$
- $(W^k)_{k\geq 1}$ is a family of independent real Brownian motions.

We consider a mesh $\mathcal{T}$ on $\mathbb{T}^d$ constructed as in [47], *i.e.* $\mathcal{T}$ is a finite family of disjoint open connected sets which form a partition of $\mathbb{T}^d$ up to a negligible set. For all distinct $K, L\in\mathcal{T}$, the intersection of closures $\overline{K}\cap\overline{L}$ is assumed to be contained in a hyperplane. For all $K\in\mathcal{T}$,

$$\mathcal{N}(K) := \left\{L\in\mathcal{T} : L\neq K, \overline{K}\cap\overline{L}\neq\emptyset\right\}$$

defines the set of neighbours of K.

To discretise the flux in Equation (A.2), we will not use an single function $\overline{A}$ as in the one-dimensional case, but a family of functions $(A_{K\to L})_{K\in\mathcal{T}, L\in\mathcal{N}(K)}$, where each function $A_{K\to L}:\mathbb{R}^2\to\mathbb{R}$ represents the flux of the conserved quantity at the interface $\overline{K}\cap\overline{L}$.

Definition A.2 (Monotone family of numerical fluxes). *A family of continuous functions $(\widetilde{A}_{K\to L})_{K\in\mathcal{T},L\in\mathcal{N}(K)}$ from $\mathbb{R}^2$ to $\mathbb{R}$ is said to be a* monotone family of numerical fluxes *if for all $K\in\mathcal{T}$ and all $L\in\mathcal{N}(K)$, it satisfies the following conditions :*

- *monotonicity: for all $v\in\mathbb{R}$, $\widetilde{A}_{K\to L}(\cdot,v)$ is non-decreasing and for all $u\in\mathbb{R}$, $\widetilde{A}_{K\to L}(u,\cdot)$ is non-increasing;*
- *consistency:*
$$\forall u\in\mathbb{R},\qquad \widetilde{A}_{K\to L}(u,u)=\left|\overline{K}\cap\overline{L}\right|\mathbf{A}(u)\cdot\mathbf{n}_{K,L},$$
where $|\overline{K}\cap\overline{L}|$ denotes the $(d-1)$-dimensional Hausdorff measure of $\overline{K}\cap\overline{L}$ and $\mathbf{n}_{K,L}$ is the outward unit normal to K on $\overline{K}\cap\overline{L}$;
- *symmetry:*
$$\forall u,v\in\mathbb{R},\qquad \widetilde{A}_{K\to L}(u,v)=-\widetilde{A}_{L\to K}(v,u).$$

In this section, our family $(A_{K\to L})_{K\in\mathcal{T},L\in\mathcal{N}(K)}$ is assumed to be monotone and, as in Section A.1 for the numerical flux $\overline{A}$, we make furthermore the following

Assumption A.3 (Strong monotonicity). *There exist $\lambda>0$ and a monotone family of numerical fluxes $(\widetilde{A}_{K\to L})_{K\in\mathcal{T},L\in\mathcal{N}(K)}$ such that for all $K\in\mathcal{T}$ and all $L\in\mathcal{N}(K)$,*
$$\forall u,v\in\mathbb{R},\qquad A_{K\to L}(u,v)=\widetilde{A}_{K\to L}(u,v)+\lambda(u-v).$$

In order to dicretise the noise coefficients, we define for any $K\in\mathcal{T}$ and any $k\geq 1$,
$$\sigma_K^k:=\frac{1}{|K|}\int_K g_k(\mathbf{x})\mathrm{d}\mathbf{x},$$
where $|K|$ is the Lebesgue measure of K. We can now define our semi-discrete approximation of (A.2) as the following SDE:
$$\mathrm{d}U_K(t)=\left(\frac{1}{|K|}\sum_{L\in\mathcal{N}(K)}A_{L\to K}\left(U_L(t),U_K(t)\right)\right)\mathrm{d}t+\sum_{k\geq1}\sigma_K^k\mathrm{d}W^k(t),\quad t\geq0,\quad K\in\mathcal{T}. \tag{A.3}$$

Setting $N:=\#\mathcal{T}$, Equation (A.3) turns out to be an SDE on the state space
$$\mathbb{R}_0^N:=\left\{\mathbf{u}=(u_K)_{K\in\mathcal{T}}\in\mathbb{R}^N:\sum_{K\in\mathcal{T}}|K|u_K=0\right\}.$$
Its drift function $\mathbf{b}:\mathbb{R}_0^N\to\mathbb{R}_0^N$ is defined by the components
$$b_K(\mathbf{u}):=\frac{1}{|K|}\sum_{L\in\mathcal{N}(K)}A_{L\to K}(u_L,u_K),\quad K\in\mathcal{T}.$$
Denoting $\boldsymbol{\sigma}^k:=(\sigma_K^k)_{K\in\mathcal{T}}$ for all $k\geq1$, the vectorised form of (A.3) writes
$$\mathrm{d}\mathbf{U}(t)=\mathbf{b}(\mathbf{U}(t))\mathrm{d}t+\sum_{k\geq1}\boldsymbol{\sigma}^k\mathrm{d}W^k(t),\quad t\geq0. \tag{A.4}$$
We now introduce the split-step time discretisation of (A.4) in exactly the same way as in Chapter 3. Given a time step $\Delta t>0$, and using the notation $\Delta W_n^k=W^k(n\Delta t)-W^k((n-1)\Delta t)$, we write
$$\begin{cases}\mathbf{U}_{n+\frac12}=\mathbf{U}_n+\Delta t\mathbf{b}\left(\mathbf{U}_{n+\frac12}\right),\\ \mathbf{U}_{n+1}=\mathbf{U}_{n+\frac12}+\sum_{k\geq1}\boldsymbol{\sigma}^k\Delta W_{n+1}^k.\end{cases} \tag{A.5}$$

Let us now endow the space $\mathbb{R}_0^N$ with the renormalised L^p norms and L^2 scalar product: for all $\mathbf{u} = (u_K)_{K\in\mathcal{T}}$ and $\mathbf{v} = (v_K)_{K\in\mathcal{T}}$ in $\mathbb{R}_0^N$, we define

$$\|\mathbf{u}\|_p := \left(\sum_{K\in\mathcal{T}} |K|\,|u_K|^p\right)^{1/p}, \qquad p \in [1,\infty);$$

$$\langle \mathbf{u}, \mathbf{v}\rangle := \sum_{K\in\mathcal{T}} |K| u_K v_K.$$

In this setting, we have

Proposition A.4. *Under Assumption A.3, for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}_0^N$, we have*

$$\langle \mathbf{sign}(\mathbf{u}-\mathbf{v}), \mathbf{b}(\mathbf{u}) - \mathbf{b}(\mathbf{v})\rangle \leq 0, \tag{A.6}$$

and

$$\langle \mathbf{b}(\mathbf{u}), \mathbf{u}\rangle \leq -\lambda \sum_{K\sim L} (u_L - u_K)^2, \tag{A.7}$$

where $\sum_{K\sim L}$ means the sum over each pair of neighbouring cells K and L.

Before proceeding to the proof of Proposition A.4, we will first establish a preliminary result:

Lemma A.5. *For any $\mathbf{u} \in \mathbb{R}_0^N$, we have*

$$\sum_{K\sim L} \int_{u_L}^{u_K} \widetilde{A}_{L\to K}(z,z)\mathrm{d}z = 0.$$

Proof. First of all, we have thanks to the Green formula:

$$\forall z \in \mathbb{R}, \quad \forall K \in \mathcal{T}, \qquad \sum_{L\in\mathcal{N}(K)} \widetilde{A}_{L\to K}(z,z) = 0 \tag{A.8}$$

(see [103, Equation (22)]).

Let $\mathbf{m} \in \mathbb{R}^d$ such that for all neighbouring cells K and L of $\mathcal{T}$, we have $\mathbf{n}_{K,L}\cdot\mathbf{m} \neq 0$. Since the union of all the orthogonal hyperplanes of the normal units $\mathbf{n}_{K,L}$ cannot cover $\mathbb{R}^d$, existence of such a vector $\mathbf{m}$ is forthright.

For every cell $K \in \mathcal{T}$, we split its family of neighbours $\mathcal{N}(K)$ in the two subsets

$$J_K := \{L \in \mathcal{N}(K) : \mathbf{n}_{K,L}\cdot\mathbf{m} > 0\} \quad \text{and} \quad J_K^c := \mathcal{N}(K) \setminus J_K.$$

Splitting the sum in (A.8), we have by symmetry of the numerical fluxes: for all $K \in \mathcal{T}$ and all $z \in \mathbb{R}$,

$$\sum_{L\in J_K} \widetilde{A}_{L\to K}(z,z) = \sum_{L\in J_K^c} \widetilde{A}_{K\to L}(z,z). \tag{A.9}$$

Let $\psi_{L\to K}$ denote an antiderivative of $z \mapsto \widetilde{A}_{L\to K}(z,z)$. Integrating in (A.9) up to u_K and summing over $K \in \mathcal{T}$, we get

$$\sum_{K\in\mathcal{T}}\sum_{L\in J_K} \psi_{L\to K}(u_K) = \sum_{K\in\mathcal{T}}\sum_{L\in J_K^c} \psi_{K\to L}(u_K). \tag{A.10}$$

Observe by the way that each of the double sums in (A.10) amounts to a sum over exactly all the interfaces. Let us now rewrite the left-hand side in (A.10):

$$\begin{aligned}\sum_{K\in\mathcal{T}}\sum_{L\in J_K}\psi_{L\to K}(u_K) &= \sum_{\substack{K\sim L\\ \mathbf{n}_{K,L}\cdot\mathbf{m}>0}}\psi_{L\to K}(u_K)\\ &= \sum_{\substack{K\sim L\\ \mathbf{n}_{L,K}\cdot\mathbf{m}>0}}\psi_{K\to L}(u_L)\\ &= \sum_{\substack{K\sim L\\ \mathbf{n}_{K,L}\cdot\mathbf{m}<0}}\psi_{K\to L}(u_L)\\ &= \sum_{K\in\mathcal{T}}\sum_{L\in J_K^c}\psi_{K\to L}(u_L).\end{aligned}$$

Replacing this last term back in (A.10), we obtain

$$\sum_{K\in\mathcal{T}}\sum_{L\in J_K^c}\psi_{K\to L}(u_L) = \sum_{K\in\mathcal{T}}\sum_{L\in J_K^c}\psi_{K\to L}(u_K).$$

Thus,

$$\sum_{K\in\mathcal{T}}\sum_{L\in J_K^c}\left(\psi_{K\to L}(u_K)-\psi_{K\to L}(u_L)\right)=0,$$

that is,

$$\sum_{K\sim L}\left(\psi_{K\to L}(u_K)-\psi_{K\to L}(u_L)\right)=0,$$

whence

$$\sum_{K\sim L}\int_{u_L}^{u_K}\widetilde{A}(z,z)\mathrm{d}z=0. \qquad \square$$

Proof of Proposition A.4 Notice first that for a mapping $a:\mathcal{T}^2\to\mathbb{R}$, we have

$$\sum_{K\in\mathcal{T}}\sum_{L\in\mathcal{N}(K)}a_{K,L}=\sum_{K\sim L}(a_{K,L}+a_{L,K}), \tag{A.11}$$

(see for instance [103, Equation (31)]). Let us start by proving (A.6). From (A.11) and the symmetry property of the numerical fluxes, we have

$$\begin{aligned}&\sum_{K\in\mathcal{T}}|K|\mathrm{sign}(u_K-v_K)(b_K(\mathbf{u})-b_K(\mathbf{v}))\\ &=\sum_{K\in\mathcal{T}}\sum_{L\in\mathcal{N}(K)}\mathrm{sign}(u_K-v_K)\left(A_{L\to K}(u_L,u_K)-A_{L\to K}(v_L,v_K)\right)\\ &=\sum_{K\sim L}\left(\mathrm{sign}(u_K-v_K)-\mathrm{sign}(u_L-v_L)\right)\left(A_{L\to K}(u_L,u_K)-A_{L\to K}(v_L,v_K)\right).\end{aligned}$$

By monotonicity of $\overline{A}_{L\to K}$, each term of this sum is non-positive (see the proof of Lemma 3.13).

Let us now prove (A.7). Using Assumption A.3, we write

$$\begin{aligned}
\langle \mathbf{b}(\mathbf{u}), \mathbf{u}\rangle &= \sum_{K\in\mathcal{T}} |K| b_K(\mathbf{u}) u_K \\
&= \sum_{K\in\mathcal{T}} \sum_{L\in\mathcal{N}(K)} A_{L\to K}(u_L, u_K) u_K \\
&= \sum_{K\in\mathcal{T}} \sum_{L\in\mathcal{N}(K)} \widetilde{A}_{L\to K}(u_L, u_K) u_K + \lambda \sum_{K\in\mathcal{T}} \sum_{L\in\mathcal{N}(K)} (u_L - u_K) u_K \\
&=: S_1 + S_2.
\end{aligned}$$

Let us show that $S_1 \leq 0$.

$$\begin{aligned}
S_1 &= \sum_{K\sim L} \left(\widetilde{A}_{L\to K}(u_L, u_K) u_K + \widetilde{A}_{K\to L}(u_K, u_L) u_L \right) \quad \text{(by (A.11))} \\
&= \sum_{K\sim L} \widetilde{A}_{L\to K}(u_L, u_K)(u_K - u_L) \quad \text{(by symmetry)} \\
&= \sum_{K\sim L} \int_{u_L}^{u_K} \widetilde{A}_{L\to K}(u_L, u_K) \mathrm{d}z - \sum_{K\sim L} \int_{u_L}^{u_K} \widetilde{A}(z,z) \mathrm{d}z \quad \text{(by Lemma (A.5))} \\
&= \sum_{K\sim L} \int_{u_L}^{u_K} \left(\widetilde{A}_{L\to K}(u_L, u_K) - \widetilde{A}(z,z) \right) \mathrm{d}z \\
&\leq 0 \quad \text{(by monotonicity).}
\end{aligned}$$

To finish the proof, we will show that $S_2 = -\lambda \sum_{K\sim L} (u_L - u_K)^2$.

$$\begin{aligned}
S_2 &= \lambda \sum_{K\in\mathcal{T}} \sum_{L\in\mathcal{N}(K)} (u_L - u_K) u_K \\
&= \lambda \sum_{K\sim L} ((u_L - u_K) u_K + (u_K - u_L) u_L) \quad \text{(by (A.11))} \\
&= \lambda \sum_{K\sim L} (u_L - u_K)(u_K - u_L) \\
&= -\lambda \sum_{K\sim L} (u_L - u_K)^2.
\end{aligned}$$

□

As in the one-dimensional case, the existence and uniqueness of a solution $(\mathbf{U}(t))_{t\geq 0}$ to (A.4) is ensured by (A.7), while the existence and uniqueness of a solution $(\mathbf{U}_n)_{n\in\mathbb{N}}$ to (A.4) is ensured by (A.7) and (A.6) respectively. Furthermore, the uniqueness of an invariant measure is proved exactly the same way as the one-dimensional case by the use of the L^1-contraction property for the processes $(\mathbf{U}(t))_{t\geq 0}$ and $(\mathbf{U}_n)_{n\in\mathbb{N}}$, which follows itself from (A.6). The existence is not as straightforward though. Indeed, Inequality (A.7) leads, in the semi-discrete case, to an estimate of the form

$$\frac{1}{T} \int_0^T \mathbb{E}\left[\lambda \sum_{K\sim L} (U_K(t) - U_L(t))^2 \right] \mathrm{d}t \leq C.$$

In order for the Krylov-Bogoliubov theorem to apply, one needs the level sets of the mapping $\mathbf{u} \in \mathbb{R}_0^N \mapsto \lambda \sum_{K\sim L} (u_K - u_L)^2$ to be compact, which is actually the case as we have

Lemma A.6 (Coercivity of the discrete gradient). *For every $\mathbf{u} \in \mathbb{R}_0^N$, we have*

$$\sum_{K\sim L} (u_K - u_L)^2 \geq \frac{1}{N-1} \max_{K\in\mathcal{T}} |u_K|^2.$$

Proof. First of all, the result is obvious when $\mathbf{u} = 0$. We shall therefore consider the case $\mathbf{u} \neq 0$. Let $K_{\max}$ be a cell of $\mathcal{T}$ such that $|u_{K_{\max}}| = \max_{K\in\mathcal{T}} |u_K|$. Without loss of generality, we assume that $u_{K_{\max}} > 0$. Since $\mathbf{u} \in \mathbb{R}^N_0$, there exists $K_- \in \mathcal{T}$ such that $u_{K_-} \leq 0$. Now, let $(K_1, \dots, K_l) \in \mathcal{T}^l$ be such that:

- $l \leq N$;
- $K_1 = K_-$ and $K_l = K_{\max}$;
- $K_i \in \mathcal{N}(K_{i+1})$, for all $i \in \{1, \dots, l-1\}$;
- $K_i \neq K_j$, for all $i \neq j$.

In other words, $(K_1, \dots, K_l)$ is a simple path of length l between K_- and $K_{\max}$. Since there are no repeating cells in this path, we have

$$\begin{aligned}
\sum_{K\sim L} (u_K - u_L)^2 &\geq \sum_{i=1}^{l-1} \left(u_{K_{i+1}} - u_{K_i}\right)^2 \\
&= \frac{1}{l-1} \left(\sum_{i=1}^{l-1} \left(u_{K_{i+1}} - u_{K_i}\right) \right)^2 \qquad \text{(by Cauchy-Schwarz)} \\
&\geq \frac{1}{N-1} (u_{K_l} - u_{K_1})^2 \\
&= \frac{1}{N-1} \left(u_{K_{\max}} - u_{K_-}\right)^2 \\
&\geq \frac{1}{N-1} \max_{K\in\mathcal{T}} |u_K|^2.
\end{aligned}$$

□

We can now state:

Theorem A.7. *Under Assumption A.3, the following two statements hold:*

(i) for any $N \geq 1$, the process $(\mathbf{U}(t))_{t\geq 0}$ solution of the SDE (A.4) admits a unique invariant measure $\nu_N \in \mathcal{P}(\mathbb{R}^N_0)$;

(ii) for any $N \geq 1$ and any $\Delta t > 0$, the process $(\mathbf{U}_n)_{n\in\mathbb{N}}$ defined by (A.5) admits a unique invariant measure $\nu_{N,\Delta t} \in \mathcal{P}(\mathbb{R}^N_0)$.

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