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Games in Intrinsic Form with Witsenhausen Model

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UNIVERSITÉ — — PARIS-EST

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L'INFORMATION ET DE LA COMMUNICATION

THÈSE DE DOCTORAT

Spécialité : Mathématiques

Présentée par

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Pour obtenir le grade de

DOCTEUR DE L'UNIVERSITÉ PARIS-EST

JEUX SOUS FORME INTRINSÈQUE AVEC LE MODÈLE DE WITSENHAUSEN

Soutenance le 01 Octobre 2020 devant le jury composé de :

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Résumé

Dans un contexte stratégique, l'information (qui sait quoi et avant qui) joue un rôle crucial. Dans cette thèse, nous considérons des modèles de théorie des jeux avec information et nous présentons un nouveau modèle, qui ne repose pas sur les arbres. En effet, d'une part, ne pas se contraindre avec un arbre peut s'avérer intéressant pour modéliser l'information et, d'autre part, il existe des exemples de jeux qui peuvent être joués mais ne peuvent pas être écrits sur un arbre. Le manuscrit est en deux parties.

Dans la première partie, nous nous concentrons sur trois modèles où le concept d'information est présent : le modèle d'arbre extensif de Kuhn (K-modèle), le modèle d'arbre infini d'Alós-Ferrer et Ritzberger (AFR-modèle) et le modèle de Witsenhausen (W-modèle). Alors qu'un arbre est donné à la fois dans les K- et AFR-modèles comme une des primitives, dans le W-modèle il s'agit plutôt d'un objet qui peut éventuellement être induit par une structure d'information adéquate. Nous montrons, d'une part, que les W-modèles finis et causaux peuvent être plongés dans les AFR-modèles et, d'autre part, qu'une classe restreinte d'AFR-modèles finis peut être plongée dans les W-modèles. En outre, nous traduisons les définitions de mémoire parfaite et de mémoire des informations passées dans le langage du W-modèle, puis nous formulons des conjectures sur leurs relations avec les structures d'information correspondantes dans l'AFR-modèle.

Dans la deuxième partie, nous discutons des W-jeux. Lorsqu'ils sont équipés de joueurs et de relations de préférence, les trois modèles ci-dessus deviennent des jeux. Nous introduisons les W-jeux, c'est-à-dire les W-modèles avec une partition de l'ensemble des agents en des ensembles d'agents exécutants des joueurs et où chaque joueur est muni d'une relation de préférence (par exemple, une fonction de gain et une croyance). Nous donnons une définition de l'équilibre de Nash pour les W-jeux. Enfin, pour une sous-classe de jeux de type Principal-Agent, que nous appelons W-jeux avec Agent suffisamment informé, nous fournissons des conditions sous lesquelles un équilibre de Nash peut être obtenu par récurrence rétrograde.

En conclusion, nous discutons de plusieurs pistes ouvertes, telles que l'extension à des ensembles ou à des joueurs infinis et l'étude des équilibres parfaits en sous-jeux pour les W-jeux.

Abstract

In a strategic context, information (who knows what and before whom) plays a crucial role. In this thesis, we consider game theory models with information, and we present a new model which does not rely on trees. Indeed, on the one hand, working without a tree proves interesting in the context of modelling information and, on the other hand, there are examples of games that can be played but cannot be written on a tree. The manuscript is in two parts.

In the first part, we focus on three models where the concept of information is present: Kuhn extensive tree model (K-model), Alós-Ferrer and Ritzberger infinite tree model (AFR-model) and Witsenhausen model (W-model). Whereas a tree is given in both K- and AFR-models among the primitives, in the W-model it is rather an object that can possibly be induced by a proper information structure. We prove that, on the one hand, causal and finite W-models can be embedded into AFR-models and, on the other hand, that a restricted class of finite AFR-models can be embedded into W-models. Moreover, we translate definitions of perfect recall and memory of past information into the language of W-model, and we formulate conjectures relating them to the corresponding information structures in the AFR-model.

In the second part, we discuss W-games. When supplied with players and preferences, any of the three above models becomes a game. We introduce W-games, that is, W-models with a partition of the set of agents into sets of players' representative agents, supplying each player with a preference relation (for instance, a payoff function and a belief). We give a definition of the Nash equilibrium for W-games. Then, for a subclass of Principal-Agent games that we call Enough Informed Agent W-games, we provide conditions under which a Nash equilibrium can be obtained by backward induction.

In conclusion, we discuss several open leads, such as extension to infinite sets or players and the study of subgame perfect equilibrium in W-games.

Chapter 1

Introduction

1.1 Context of the thesis

This manuscript presents the research that I have done during three years of PhD, from October 2017 to October 2020, under the supervision of Michel De Lara and in close collaboration with Jean-Philippe Chancelier. I was financed by École Nationale des Ponts et Chaussées and joined Centre d'Enseignement et de Recherche en Mathématiques et Calcul Scientifique (CERMICS).

In this thesis, we consider Game Theory models with information.

1.2 Game Theory models with information

In this thesis, we focus on three Game Theory models, namely, K-, AFR- and W-models, but the global picture is broader. Indeed, there are three approaches in the existing literature to model the tree structure in extensive form games.

- *Set-tree* formulation in games was initiated by von Neumann and Morgenstern in [36] and then extensively developed by Alós-Ferrer and Ritzberger in [2, 3] (AFR-model as in Definition 2.15). A recent approach of Streufert, the so-called *choice-set model* (CS-model, see Section 2 in [34]), is yet another set-tree formulation
- *Graph-tree* formulation in games is pioneered by Kuhn in [18] and developed by Selten in [31] (K-model as in Definition 2.4). By far, this is the most widely used way to speak of extensive form games in the existing literature.
- *Sequence-tree* formulation in games is initiated by Harris in [12] and generalized by Osborne and Rubinstein in [23] (so-called OR-model, see Definition 200.1 in §11.1.2 of [23]).

The first embedding of a tree model of one type into another was made by Alós-Ferrer and Ritzberger (see [2] or Example 6.5 in [3]). The authors showed that the sequence-tree

OR-model is a particular case of so-called *simple trees* model (*ST-model*, another graph-tree model introduced by Alós-Ferrer and Ritzberger that is proved to be equivalent to the AFR-model as in Definition 2.15, see Theorems 6.1, 6.2 in [3]). The first equivalence between sequence- and graph-trees appeared in [17], where the authors proved that the OR-model is equivalent to the KS-model by giving an explicit method to transform a sequence-tree into a graph-tree and vice versa.

Finally, Streufert built, in a series of articles [33, 34], a complete chain of equivalences between the existing tree models (KS-, AFR-, OR, CS-models and simple trees, see Figure 1.2 taken from [34] illustrating the so-called *Selten's horse* tree structure) and went further classifying them using the machinery of category theory. Besides, the CS-model is a relaxation of the OR-model. Informally speaking, the main idea of the sequence-tree OR-model is that each node is a *sequence* of past choices, where the choices themselves are abstract and form the underlying set. In the CS-model, each node is an (unordered) *set* of past choices rather than sequence as in the OR-model. Pretty formal and technical proofs of equivalences between primitives of each model with a corresponding equivalent element of the chain can be also found in [34]. This shows that all the five extensive forms with the trees of three types introduced above are of roughly equal generality.

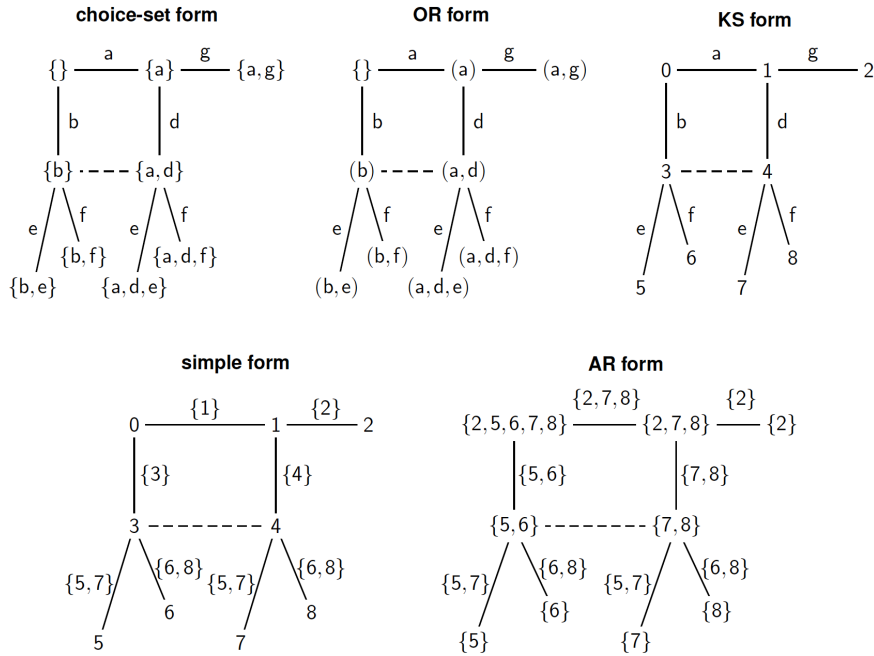


Figure 1.1: Five equivalent tree forms of the Selten's "horse" taken from [34]. The dashed lines here define the only non-singleton information set in each model.

In the following Table 1.1, we compare six Game Theory models and give references where the corresponding equivalences were proved. Five of the six models listed in the Table 1.1 were proved to be equivalent in a series of papers, namely, [3, 17, 34], published in recent years. One of the contributions of this thesis is adding the W-model to this chain

of equivalences, by showing equivalence between a certain subclass of W-models and the AFR-model in Chapters 3 and 4.

| | CS-model | OR-model | K-model | ST-model | AFR-model | W-model |
|----------------------|--|--|--|--|--|--------------------------|
| Year | 2018 | 1994 | 1953 | 2016 | 2005 | 1971 |
| Nodes | choice sets | choice sequences | abstract | abstract | outcome sets | (no tree) |
| Choices | abstract | abstract | abstract | node sets | outcome sets | outcome sets |
| Information | node sets | sets of sequences | node sets | node sets | node sets | outcome σ -fields |
| Number of primitives | 2 primitives | 4 primitives | 4 primitives | 2 primitives | 2 primitives | 2 primitives |
| Number of axioms | 4 axioms | 2 axioms | 4 axioms | 4 axioms | 2 axioms | 1 axiom |
| Equivalence | <i>ORtoCS</i> [34] Theorem 3.1 <i>CStoOR</i> [34] Theorem 3.2 | <i>KtoOR</i> [17] Theorem 4.1 <i>ORtoK</i> [17] Theorem 4.2 | <i>STtoK</i> [34] Theorem 5.1 <i>KtoST</i> [34] Theorem 5.2 | <i>AFRtoST</i> [3] Theorem 6.1 <i>STtoAFR</i> [3] Theorem 6.2 | <i>WtoAFR</i> Chapter 3 <i>AFRtoW</i> Chapter 4 | |

Table 1.1: Six models in Game Theory.

In most applications of extensive form games, players are required to enjoy both the memory of past information and actions. This condition, known as perfect recall, was introduced by Kuhn in [18] as “equivalent to the assertion that each player is allowed by the rules of the game to remember everything he knew at previous moves and all of his choices at those moves”. This condition is a strong one, not reflecting the capabilities of real players in complex strategic settings. A big advantage of the W-model is that many classical information patterns, such as perfect recall, imperfect memory of past actions, imperfect memory of past information, can be expressed in a short and clear way in terms of W-model. This clarity is achieved thanks to the information structure of the agents being defined as σ -fields over the underlying set of outcomes (configurations). In Chapter 5, we translate definitions of perfect recall and memory of past information into the language of W-model and formulate conjectures relating them to the corresponding information structures of the AFR-model.

When supplied with players and preferences, any of the three above models becomes complete and is traditionally called a *game* in the literature. To this end, in Chapter 6, we introduce W-games, that is, W-models with a partition of the set of agents into sets of players’ representative agents, supplying each player with a preference relation (for instance, a payoff function and a belief). We give a definition of the Nash equilibrium for W-games in Chapter 6. We finish by introducing the concept of subsystem.

Then, in Chapter 7, we develop a particular case of W-games, so-called *Principal-Agent* models where there are two decision-makers — that are both players, as in Game Theory, and agents, as in the W-model.

There are two main strands in the existing literature about the Principal-Agent models: the first is devoted to finding Nash equilibria in various Principal-Agent games (see, for example, [20]) and the second is devoted to establishing conditions under which an equilibrium exists (by whether imposing restrictions on the primitives of the model, for example, see [8, 11, 14, 16], or imposing restrictions on the allowed mechanisms, for example, see [6, 13, 24, 25, 28] with the most general framework given in [15]). Existence conditions are outside the scope of this thesis, in Chapter 7, we concentrate on the conditions under which a solution of a Principal-Agent game is yielded as a result of backward induction assuming that the solution exists.

1.3 Outline of the thesis

1. In Part I, we focus on existing models with information and prove equivalence between them.
 - In Chapter 2, we give exposition of three game theory models with information, namely, the classical Kunh's extensive tree model (K-model), the Alós-Ferrer, Ritzberger infinite tree model (AFR-model), which is a generalization of the first and, finally, the Witsenhausen intrinsic model (W-model), which is the main topic of this thesis.
 - In Chapter 3, we establish a one-way implication between the W- and AFR-models; we construct the primitives of the latter out of the primitives of the first and deduce the axioms imposed on the primitives of the AFR-model.
 - In Chapter 4, we go in the opposite direction constructing primitives of the W-models out of the primitives of the AFR-model in the spirit of Chapter 3.
 - In Chapter 5, we discuss different information structures and the way they are expressed in the language of the three models. We cover classical information patterns, such as perfect recall and perfect memory of past information. As we show, the W-model allows for precise and concise descriptions of many classical information patterns from Game Theory.
2. In Part II, we develop the so-called W-games in Intrinsic Form, that is, W-models equipped with a partition of the agents into players, where each player is endowed with either a preference relation or a payoffs function and a belief probability distribution.
 - In Chapter 6, we introduce a number of essential concepts, namely, W-game, Nash equilibrium, subsystems, which will be used later in Chapter 7 to define a subclass of the Principal-Agent game and in Chapter 8 when speaking of the future avenues of research.
 - In Chapter 7, for a subclass of Principal-Agent games, namely, the so-called Enough informed Agent W-games, we provide conditions under which a Nash equilibrium of this game can be obtained by backward induction.

3. In Conclusion, we give possible avenues of research not developed in this thesis: among these are clarifying the notions of mixed and behavioral strategies in the W-model, proving an analogue of the Kuhn's theorem about the equivalence of mixed and behavioral strategies and introducing the notion of subgames, subgame perfect equilibrium and backward induction. Also, there is a transversal avenue of research in developing the W-model for a non-finite number of agents.

Part I

Information Models for Game Theory

Chapter 2

Game theory models with information

2.1 Introduction

In our exposition of game theory models, we focus on the representation of information, leaving aside preferences and beliefs, that will be considered in Part II. We will stick to the following roadmap:

- in Section 2.2 we introduce the Kuhn's extensive tree model,
- in Section 2.3 we introduce the Alós-Ferrer, Ritzberger infinite tree model,
- in Section 2.4 we introduce the Witsenhausen intrinsic model,
- in Section 2.5 we illustrate the models on advanced examples, expressing the concepts of forgetfulness and absent-mindedness.

2.2 Kuhn's tree model (K-model)

Here, we present the classical Kuhn's tree model (K-model) or model in extensive form as *four objects* (aside of the mixed strategy prescribed to Nature and the payoff function) with *four axioms* imposed on them. In this exposition, we rely on [22] and on the original paper of Selten [31]. We also illustrate the model on simple examples. To this end,

- in §2.2.1 we give basics on graph theory and introduce the main notations in order to work with trees afterwards,
- in §2.2.2 we introduce the basic objects of the K-model,
- in §2.2.3 we give the definition of the K-model, listing the basic objects defined above and the axioms imposed on them,

- in §2.2.4 we define the K-strategies,
- in §2.2.5 we illustrate the K-model on simple examples.

2.2.1 Prerequisites: basics on graph theory and notations

A *finite directed graph* is a couple (V, E) , where

- V is a finite set, whose elements are called *vertices*,
- $E \subset V \times V$ is a finite set of couples of vertices, whose elements are called (*directed*) *edges*; for any directed edge $e \in E$ there is a unique couple of vertices $v, v' \in V$, giving the two ends of the edge, such that $e = (v, v')$.

Let v and v' be two vertices in a graph (V, E) . A *path* from v to v' is a finite sequence $(v_1, v_2, \dots, v_i, v_{i+1}, \dots, v_{N+1})$ of vertices, with $v_1 = v$ and $v_{N+1} = v'$, such that for any two vertices v_i, v_{i+1} with consecutive indices, there is an edge $e_i \in E$, such that $e_i = (v_i, v_{i+1})$. A path is called a *simple path* if all vertices in the sequence $(v_1, v_2, \dots, v_i, v_{i+1}, \dots, v_{N+1})$ are distinct, that is, $v_j \neq v_k$ for every $i \neq j$, $1 \leq i \leq N + 1$. A path from v_1 to v_{N+1} can be also written as a sequence of edges as follows

$$\left((v_1, v_2), (v_2, v_3), \dots, (v_i, v_{i+1}), \dots, (v_N, v_{N+1}) \right) = (e_1, e_2, \dots, e_i, \dots, e_N).$$

We will employ the following two distinct notations for the path from vertex v to vertex v' : $(v, v')_V$ stands for the path written as a subset of vertices V and $(v, v')_E$ stands for the path written as a subset of edges E .

Definition 2.1. *A tree is a triple $T = (V, E, v^0)$, where (V, E) is a connected acyclic directed graph with a specific vertex $v^0 \in V$ called the root of the tree.*

Definition 2.1 implies that every path in a tree is a simple path as there are no cycles. Therefore, as we consider trees in the manuscript, by “path” we will always mean “simple path” since the two notions coincide in this context. Moreover, for every vertex $v \in V$ of the tree V , there is a unique path from the root v^0 to the vertex v , denoted by $(v^0, v)_V$ (if represented by vertices) or by $(v^0, v)_E$ (if represented by edges).

2.2.2 Primitives of the K-model

What we will call a K-model is made of four basic objects (primitives)

$$\left((V, E, v^0), (X^i)_{i \in \mathcal{P}_0}, (I^i)_{i \in \mathcal{P}}, \left\{ C^{i,G} \right\}_{\substack{i \in \mathcal{P}, \\ G \in I^i}} \right).$$

Before providing a formal definition of a K-model in §2.2.3, we now give descriptions of these primitives.

2.2.2.1 The game tree

The first basic object of the K-model is a *tree* (V, E, v^0) as defined in Definition 2.1.

Definition 2.2. *Given a tree (V, E, v^0) , for any vertex $v \in V$, we define the set of edges emanating from the vertex v by*

$$E_v = \{e \in E \mid \exists v' \in V \text{ such that } e = (v, v')\} \subset E. \quad (2.1a)$$

A vertex is a leaf or a terminal point if there are no directed edges emanating from it. The set of terminal points with such property is called the set of leaves and is formally defined as

$$Z = \{v \in V \mid E_v = \emptyset\}. \quad (2.1b)$$

The set of all vertices that are not leaves is called the set of moves and is denoted by

$$X = V \setminus Z. \quad (2.1c)$$

For any leaf $z \in Z$, the unique path

$$w_z = (v^0, z)_V \quad (2.1d)$$

from the root to leaf z is called a play.

There is a one-to-one correspondence between the set of plays and the set of leaves.

2.2.2.2 The players partition

In Definition 9.2, we recall that a partition of a set S is a subset $\Pi \subset 2^S$ made of subsets \wp ($\wp \in \Pi$) of S that are nonempty, two by two disjoint, and whose union is S . To denote that Π is a partition, we use the notation

$$(\forall \Pi \subset 2^S) \quad S = \bigsqcup_{\wp \in \Pi} \wp \iff \begin{cases} S = \bigcup_{\wp \in \Pi} \wp, \\ \wp \cap \wp' = \emptyset, \quad \forall \wp \neq \wp', \\ \wp \neq \emptyset, \quad \forall \wp \in \Pi. \end{cases} \quad (2.2)$$

Let \mathcal{P}_0 be a finite set representing all players, where $0 \in \mathcal{P}_0$ is the Nature player (Nature has no preferences, hence is a singular player) and $\mathcal{P} = \mathcal{P}_0 \setminus \{0\}$ is the (finite) set of *individual players*.

The second basic object of the K-model is a given *partition* $\{X^i\}_{i \in \mathcal{P}_0}$ of the set X of moves in Equation (2.1c), named the *players partition* and written as

$$X = \bigsqcup_{i \in \mathcal{P}_0} X^i, \quad (2.3)$$

where the set X^i of moves is called player i 's *player set*.

2.2.2.3 The information partition

For player $i \in \mathcal{P}_0$, the third basic object of the K-model is the so-called *information partition* $I^i \subset 2^{X^i}$ of each player set X^i , written as

$$X^i = \bigsqcup_{G \in I^i} G . \quad (2.4)$$

An element $G \in I^i$ of this partition is called *information set* of the player i .

In the K-model, we define the *K-information structure* Ξ_K as a collection of all information sets of the players

$$\Xi_K = \{G \in I^i \mid i \in \mathcal{P}\} . \quad (2.5)$$

2.2.2.4 The choice partition

For any individual player $i \in \mathcal{P}$ and any information set $G \in I^i$ of the player i , we define the set of *edges emanating from the atom* G , by

$$E_G = \bigcup_{v \in G} E_v \subset E , \quad (2.6)$$

where the set E_v of edges emanating from a given vertex $v \in V$ is defined in Equation (2.1a).

Definition 2.3. For any player $i \in \mathcal{P}$ and any information set $G \in I^i$ of the player i , the choice partition assigned to the information set G of the player i is a partition of the set E_G of edges emanating from the information set G (as defined in (2.6)) that we denote $C^{i,G} \subset 2^{E_G}$, that is,

$$E_G = \bigsqcup_{c \in C^{i,G}} c . \quad (2.7)$$

An element $c \in C^{i,G}$ of the choice partition assigned to the information set G of the player i , that is, a subset $c \subset E_G$, is called a choice assigned to the information set G of the player i .

A choice $c \in C^{i,G}$ assigned to the information set G of the player i , is called an *eligible choice* if it contains exactly one edge from the set E_v of edges emanating from any vertex $v \in G$. Written formally,

$$\forall i \in \mathcal{P} , \forall G \in I^i , c \in C^{i,G} \text{ is eligible} \iff \forall v \in G , |c \cap E_v| = 1 , \quad (2.8)$$

where $|\cdot|$ denotes the cardinal of a finite set.

2.2.3 Definition of the K-model (4 basic objects, 4 axioms)

Now, we are ready to formally define the K-model, which is the information part of the celebrated Kuhn extensive form in Game Theory.

Definition 2.4. A K-model $\left((V, E, v^0), \{X^i\}_{i \in \mathcal{P}_0}, \{I^i\}_{i \in \mathcal{P}_0}, \{C^{i,G}\}_{\substack{i \in \mathcal{P}_0, \\ G \in I^i}} \right)$ consists of four basic objects, namely,

(K-BO1) a finite tree (V, E, v^0) ,

(K-BO2) a player partition of the set X of moves, $\{X^i\}_{i \in \mathcal{P}_0}$, as in Equation (2.3),

(K-BO3) an information partition of each element of the player partition, $\{I^i\}_{i \in \mathcal{P}_0}$, as in Equation (2.4),

(K-BO4) a choice partition $\{C^{i,G}\}_{i \in \mathcal{P}_0, G \in I^i}$, as in Equation (2.7),

and the following four axioms imposed on them:

(K-Axiom1) for the Nature player, each information set is a singleton, that is,

$$I^0 = \{\{x_0\} \mid x_0 \in X^0\} \text{ and } X^0 = \bigsqcup_{x_0 \in I^0} \{x_0\},$$

(K-Axiom2) for any player $i \in \mathcal{P}$ and any i 's information set $G \in I^i$, any play in Equation (2.1d) intersects G at most once; formally speaking, for any player $i \in \mathcal{P}$, any leaf $z \in Z$ and any information set $G \in I^i$ of the player i , if $w_z \cap G \neq \emptyset$, then $w_z \cap G$ is a singleton, where the corresponding play w_z is as in (2.1d), that is,

$$(\forall i \in \mathcal{P}, \forall G \in I^i, \forall z \in Z) \quad |(v^0, z)_V \cap G| \leq 1,$$

(K-Axiom3) for any player $i \in \mathcal{P}$ and any information set $G \in I^i$ of the player i , the number of emanating edges (as defined in (2.1a)) is the same at every vertex of this atom, that is, for any $v, v' \in G$, $|E_v| = |E_{v'}|$,

(K-Axiom4) For any player $i \in \mathcal{P}$, any information set $G \in I^i$ of the player i , any choice $c \in C^{i,G}$ assigned to the information set $G \in I^i$ of the player i is eligible as defined in (2.8).

2.2.4 K-strategies

Now that information and choices have been properly defined in (2.4) and (2.7) respectively, we can turn to the notion of K-strategies.

For any player $i \in \mathcal{P}$, we define the set of all choices available to i by

$$C^i = \{c \in C^{i,G} \mid G \in I^i\}, \quad (2.9)$$

where the information partition I^i of the player i has been defined in Equation (2.4), and the choice partition $C^{i,G}$ assigned to the information set G of the player i has been defined in Equation (2.7).

Definition 2.5. A pure K-strategy of a player $i \in \mathcal{P}$ is a mapping $s^i : X^i \rightarrow C^i$ from the player's moves X^i in (K-BO2) of Definition 2.4 and (2.3) (vertices where the player is active) into the player's choices C^i in Equation (2.17b) (actions the player takes), such that

- for any move $v \in X^i$, and for the unique information set $G \in I^i$ such that $v \in G$ (existence and uniqueness result from I^i being a partition of the set X^i of moves of the player i as in (2.4)), the choice $s^i(v)$ is such that $s^i(v) \in C^{i,G}$,
- we have $s^i(v) = s^i(v')$ for any $v' \in G$.

A profile of pure strategies is an element of

$$s = \{s^i\}_{i \in \mathcal{P}} \in \prod_{i \in \mathcal{P}} S^i,$$

where S^i is the set of pure K-strategies of the player i .

The following proposition is well-known, see, for example, [22, p.45, §3.3]. We give its proof for the sake of completeness.

Proposition 2.6. In a finite K-model as in Definition 2.4, any profile $s = \{s^i\}_{i \in \mathcal{P}}$ of pure K-strategies, as in Definition 2.5, induces a unique leaf $z(s)$ (or, equivalently, a unique play $w_{z(s)}$ as in (2.1d)).

Proof. We suppose given a profile $s = \{s^i\}_{i \in \mathcal{P}}$ of pure K-strategies, as in Definition 2.5. For the root vertex v_0 , there exists a unique player $i(v_0) \in \mathcal{P}$, such that $v_0 \in X^{i(v_0)}$, by the players partition in (K-BO2) of Definition 2.4 and (2.3). Take the image of the strategy $s^{i(v_0)}(v_0) \in C^{i(v_0),\{v_0\}}$ of this chosen player¹. By (K-Axiom4) of Definition 2.4, any choice is eligible as in (2.8), that is, $s^{i(v_0)}(v_0) \cap E_{v_0} = (v_0, v_1) \in E_{v_0}$ and the vertex v_1 is uniquely defined (as the edge (v_0, v_1) is uniquely defined), where E_{v_0} is the set of edges emanating from the root vertex v_0 as in (2.1a). Thus, the first step of the algorithm results in the vertex v_1 .

Let the Step k of the algorithm results in a vertex v_k . If the vertex v_k is a leaf, the algorithm stops. Else, there exists a unique player $i(v_k) \in \mathcal{P}$, such that $v_k \in X^{i(v_k)}$, by the players partition in (K-BO2) of Definition 2.4 and (2.3), and the unique information set $G_{v_k} \in I^{i(v_k)}$, such that $v_k \in G_{v_k}$ by the information partition in (K-BO3) of Definition 2.4 and (2.4). Take the image of the strategy $s^{i(v_k)}(v_k) \in C^{i(v_k),G_{v_k}}$ of this chosen player. By (K-Axiom4) of Definition 2.4, any choice is eligible as in (2.8), that is, $s^{i(v_k)}(v_k) \cap E_{v_k} = (v_k, v_{k+1}) \in E_{v_k}$ and the vertex v_{k+1} is uniquely defined², where E_{v_k} is the set of edges emanating from the root vertex v_k as in (2.1a). Thus, the Step k of the algorithm results in the vertex v_{k+1} .

As the tree is finite, the process terminates at some point giving a unique vertex $z(s)$, which is a leaf.

This finishes the proof. □

¹the set $C^{i(v_0),\{v_0\}}$ of choices of the player $i(v_0)$ at information set $\{v_0\}$ is well-defined as the root vertex forms a singleton information set $\{v_0\}$ by (K-Axiom2) of Definition 2.4

²as the edge (v_k, v_{k+1}) is uniquely defined

2.2.5 Examples

To illustrate the K-formalism presented above, we present three examples with two players: first, playing simultaneously (§2.2.5.1); second, one playing after another (§2.2.5.2); third, after the Nature's move (§2.2.5.3). At this stage, we keep the examples simple for the sake of comparison between the K-model and the other two models (AFR and W) to come.

2.2.5.1 Two players playing simultaneously

Here, we give what certainly is the most simple (interesting) example of game (without payoffs, as in this whole Part I). There are two players playing simultaneously, each of them making a decision among two possible options, irrespective of the other player decision. This example describes a simultaneous move game on a two by two square in strategic form, and covers a whole range of classical examples of games: matching pennies, battle of the sexes, stag hunt, hawk and doves, chicken game, The only difference between them is the payoff structure (that we do not elaborate in this Part I).

The story of Alice and Bob. As this simple model (a game without payoffs) will serve as a prototype to illustrate the three K-, AFR- and W-models, we formulate it as a short story common to all three forthcoming mathematical formulations:

- Alice chooses between “Top” and “Bottom” (T and B for short);
- Bob chooses between “Left” and “Right” (L and R for short);
- Both Alice and Bob choose simultaneously, irrespective of the other.

Mathematical formulation of the story of Alice and Bob (traditional notations of the K-model). Now, we propose a relevant mathematical formulation of the above story, in the language of the K-model, especially of Definition 2.4.

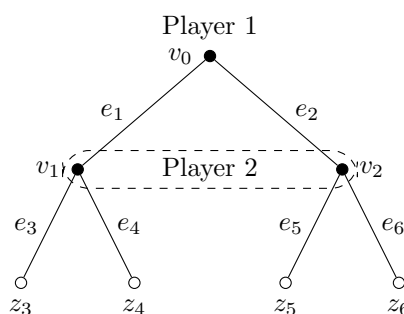


Figure 2.1: Two players playing simultaneously (K-model)

The following is the list of the primitives of a K-model (see also Figure 2.1).

- The set of vertices

$$V = \{v_0, v_1, v_2, z_3, z_4, z_5, z_6\}.$$

- The set of edges

$$E = \left\{ \underbrace{(v_0, v_1)}_{e_1}, \underbrace{(v_0, v_2)}_{e_2}, \underbrace{(v_1, z_3)}_{e_3}, \underbrace{(v_1, z_4)}_{e_4}, \underbrace{(v_2, z_5)}_{e_5}, \underbrace{(v_2, z_6)}_{e_6} \right\}.$$

With the set V of vertices and the set E of edges, choosing the vertex v_0 as the root, we define the basic object (K-BO1) of Definition 2.4.

- The set of edges determines
 1. the set of moves $X = \{v_0, v_1, v_2\}$ by (2.1c)
(as $E_{v_0} = \{e_1, e_2\}$, $E_{v_1} = \{e_3, e_4\}$ and $E_{v_2} = \{e_5, e_6\}$),
 2. the set of leaves $Z = \{z_3, z_4, z_5, z_6\}$ by (2.1b)
(as $E_{z_3} = E_{z_4} = E_{z_5} = E_{z_6} = \emptyset$).
- The set of players is $\mathcal{P} = \{1, 2\}$ (Player 1 is playing at the root followed by Player 2), and we partition the set X of moves into players' moves as

$$X = X^1 \sqcup X^2, \text{ where } X^1 = \{v_0\}, X^2 = \{v_1, v_2\}.$$

With this partition, we define the basic object (K-BO2) of Definition 2.4.

- We define the players' information partitions of players' sets of moves by

$$I^1 = \{\{v_0\}\}, I^2 = \{\{v_1, v_2\}\}.$$

With this partition, we define the basic object (K-BO3) of Definition 2.4.

- We define the players' choice partitions by

$$C^1 = \{\{e_1\}, \{e_2\}\}, C^2 = \{\{e_3, e_5\}, \{e_4, e_6\}\}.$$

With this partition, we define the basic object (K-BO4) of Definition 2.4.

The four Axioms in Definition 2.4 are easily seen to be satisfied.

(K-Axiom1) Satisfied automatically as there is no Nature player.

(K-Axiom2) The set of leaves $Z = \{z_3, z_4, z_5, z_6\}$ defines four plays $w_{z_3}, w_{z_4}, w_{z_5}, w_{z_6}$ as in (2.1d). It is immediately seen that each play intersects information sets $\{v_0\}$ and $\{v_1, v_2\}$ at most once.

(K-Axiom3) The Axiom follows immediately for the information set $\{v_0\}$ as it is a singleton. For the other information set $\{v_1, v_2\}$, by construction of the sets E_{v_1} and E_{v_2} of emanating edges from vertices v_1 and v_2 respectively, we have that $|E_{v_1}| = |E_{v_2}|$.

(K-Axiom4) By construction of the choices in C^1 and C^2 , we immediately have that each choice of the players is eligible.

Mathematical formulation of the story of Alice and Bob (with notations common to all three K-, AFR- and W-models). Finally, we can give names for the players' choices. We choose Alice as the first (row) player, and we set $\{e_1\} = T$ ("Top") and $\{e_2\} = B$ ("Bottom"); we choose Bob as the second (column) player, and we set $\{e_3, e_5\} = L$ ("Left") and $\{e_4, e_6\} = R$ ("Right").

The set of basic object of this example is given by

(K-BO1) the game tree (V, E, v^0) ,

(K-BO2) the players partition X^1 and X^2 ,

(K-BO3) the players' information partitions I^1 and I^2 ,

(K-BO4) the players' choice partitions $C^1 = \{T, B\}$ and $C^2 = \{L, R\}$.

2.2.5.2 Two ordered players

Here, we let Player 1 play first, and Player 2 makes his or her decision knowing the Player 1's action.

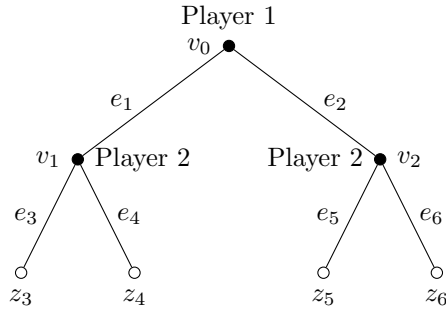


Figure 2.2: Two ordered players (K-model)

In this example, most of the objects coincide with the previous model in §2.2.5.1; however, as the information structure is nontrivial, we explicitly write the players' information and choice partitions.

- The players' information partitions of players' sets of moves are

$$I^1 = \{\{v_0\}\}, I^2 = \{\{v_1\}, \{v_2\}\}.$$

With this partition, we define the basic object (K-BO3) of Definition 2.4.

- The players' choice partitions are

$$C^1 = \{\{e_1\}, \{e_2\}\}, C^2 = \{\{e_3\}, \{e_5\}, \{e_4\}, \{e_6\}\}.$$

With this partition, we define the basic object (K-BO4) of Definition 2.4.

The four Axioms in Definition 2.4 are easily seen to be satisfied.

2.2.5.3 Two ordered players with Nature

Here, Nature plays at the root and then Player 1 makes his or her action knowing the Nature's move; Player 2 knows both the Nature's move and the one of the Player 1.

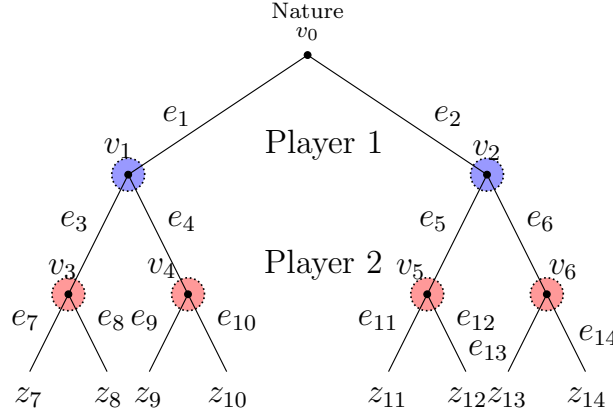


Figure 2.3: Two ordered players with Nature (K-model)

The following is the list of the primitives of a K-model.

- The set of vertices $V = \{v_0, v_1, v_2, v_3, v_4, v_5, v_6, z_7, z_8, z_9, z_{10}, z_{11}, z_{12}, z_{13}, z_{14}\}$.
- The set of edges

$$E = \left\{ \underbrace{(v_0, v_1)}_{e_1}, \underbrace{(v_0, v_2)}_{e_2}, \underbrace{(v_1, v_3)}_{e_3}, \underbrace{(v_1, v_4)}_{e_4}, \underbrace{(v_2, v_5)}_{e_5}, \underbrace{(v_2, v_6)}_{e_6}, \underbrace{(v_3, z_7)}_{e_7}, \underbrace{(v_3, z_8)}_{e_8}, \right. \\ \left. \underbrace{(v_4, z_9)}_{e_9}, \underbrace{(v_4, z_{10})}_{e_{10}}, \underbrace{(v_5, z_{11})}_{e_{11}}, \underbrace{(v_5, z_{12})}_{e_{12}}, \underbrace{(v_6, z_{13})}_{e_{13}}, \underbrace{(v_6, z_{14})}_{e_{14}} \right\}.$$

With the set V of vertices and the set E of edges, choosing the vertex v_0 as the root, we define the basic object (K-BO1) of Definition 2.4.

- The set of edges determines
 1. the set of moves $X = \{v_0, v_1, v_2, v_3, v_4, v_5, v_6\}$,
 2. the set of leaves $Z = \{z_7, z_8, z_9, z_{10}, z_{11}, z_{12}, z_{13}, z_{14}\}$.
- The set of moves X is partitioned into players' moves as follows

$$X = X^0 \sqcup X^1 \sqcup X^2, \text{ where } X^0 = \{v_0\}, X^1 = \{v_1, v_2\}, X^2 = \{v_3, v_4, v_5, v_6\}.$$

With this partition, we define the basic object (K-BO2) of Definition 2.4.

- The players' information partitions of players' sets of moves are

$$I^0 = \{\{v_0\}\}, \quad I^1 = \{\{v_1\}, \{v_2\}\}, \quad I^2 = \{\{v_3\}, \{v_4\}, \{v_5\}, \{v_6\}\}.$$

With this partition, we define the basic object (K-BO3) of Definition 2.4.

- The players' choice partitions are

$$\begin{aligned} C^0 &= \{\{e_1\}, \{e_2\}\}, \\ C^1 &= \{\{e_3\}, \{e_4\}, \{e_5\}, \{e_6\}\}, \\ C^2 &= \{\{e_7\}, \{e_8\}, \{e_9\}, \{e_{10}\}, \{e_{11}\}, \{e_{12}\}, \{e_{13}\}, \{e_{14}\}\}. \end{aligned}$$

With this partition, we define the basic object (K-BO4) of Definition 2.4.

The four Axioms in Definition 2.4 are easily seen to be satisfied.

2.3 Alós-Ferrer, Ritzberger infinite tree model (AFR-model)

The *extensive decision problem* (EDP) by Alós-Ferrer and Ritzberger (short, AFR) is certainly the most general way of speaking about so-called extensive form games in the modern literature. The authors develop the refined-partitions approach of von Neumann and Morgenstein. This framework covers continuous and differential games, and requires two primitive objects and four axioms. Despite of its generality, some problems arise when it comes to define strategies on an EDP. In order to have well-defined strategies, the authors restrict their framework to “discrete” trees on which every vertex has a distinct immediate predecessor, thus satisfying three “desiderata” that are, in their own words:

- (A0) every play can be induced by some strategy profile,
- (A1) every strategy profile induces some (unique) play,
- (A2) the play induced by a given strategy profile is unique.

They call this restricted framework *discrete extensive forms* (DEF). It requires the same two primitive objects as EDP, but only two axioms instead of four.

In this manuscript, we present the discrete extensive form that we refer to as the *AFR-model* (we also consider a version of it, called Nature-rooted AFR-model). For this end,

- in §2.3.1 we give basics on set theory, which is the language the authors are using to describe their model,
- in §2.3.2 we introduce the basic objects of the AFR-model,

- in §2.3.3 we give the definition of the AFR-model, listing the basic objects defined above and the axioms imposed on them,
- in §2.3.4 we define the AFR-strategies,
- in §2.3.5 we illustrate the AFR-model on simple examples.

2.3.1 Prerequisites: set theory basics

A *preordered set* is a pair (N, \succeq) consisting of a nonempty set N and a reflexive and transitive binary relation \succeq on N . A preordered set (N, \succeq) for which the relation \succeq is antisymmetric is called a *partially ordered set* or *poset*. A nonempty subset $\mathfrak{c} \subset N$ is a *chain* if, for all $x, y \in \mathfrak{c}$, either $x \succeq y$ or $y \succeq x$, i.e. if the preorder on \mathfrak{c} induced by \succeq is complete (on \mathfrak{c}).

Given a preordered set (N, \succeq) and an element $x \in N$, we define the *up-set* $\uparrow x$ and the *down-set* $\downarrow x$ by

$$\uparrow x = \{y \in N \mid y \succeq x\} \text{ and } \downarrow x = \{y \in N \mid x \succeq y\}. \quad (2.10)$$

With this, we will present a notion of tree that is more general than the one in Definition 2.1.

Definition 2.7. *A tree is a poset (N, \succeq) such that, for any $x \in N$, the upset $\uparrow x$, as in (2.10), is a chain, and there is a distinguished vertex x^0 , called root, with the property that $x^0 \succeq x$ for any $x \in N$.*

Elements of a tree are called vertices. A vertex $x \in N$ of a tree is called a finite vertex if $\uparrow x \setminus \{x\}$ has a minimum, that is, there exists an element $v \in \uparrow x \setminus \{x\}$, such that, for any $v' \in \uparrow x \setminus \{x\}$, we have that $v' \succeq v$. The set of all finite vertices of a tree (N, \succeq) is denoted by

$$F(N) = \{x \in N \mid \uparrow x \setminus \{x\} \text{ has a minimum}\}. \quad (2.11a)$$

On the set $F(N)$ of finite vertices, supplied with the root x^0 , we define the general parent mapping by

$$p : F(N) \cup \{x^0\} \rightarrow F(N) \cup \{x^0\} \text{ s.t. } \begin{cases} p(x) = \min(\uparrow x \setminus \{x\}), \forall x \in F(N), \\ p(x^0) = x^0. \end{cases} \quad (2.11b)$$

For any vertex $v \in N$, we define the set of children of the vertex v by

$$p^{-1}(v) = \{v' \in N \mid p(v') = v\} \subset N. \quad (2.11c)$$

A vertex $z \in N$ is called a leaf if the set of children of the vertex z is empty, and $Z \subset N$ is the set of leaves, that is,

$$z \in Z \iff p^{-1}(z) = \emptyset, \quad (2.11d)$$

where the set $p^{-1}(z)$ of children vertices is defined in Equation (2.11c).

Note that the root x^0 is, formally speaking, not a finite vertex as the set $\uparrow x^0 \setminus \{x^0\}$ is empty. When the set V is finite, the tree in Definition 2.7 is equivalent to the one in Definition 2.1.

2.3.2 Primitives of the AFR-model

To define an AFR-model, one needs an underlying set \mathbb{W} of plays. Then, an AFR-model is made of two basic objects $((V, \supset), \{C^i\}_{i \in \mathcal{P}})$ defined on the underlying set \mathbb{W} of plays. Here, we give descriptions of them; then we will give a definition with the list of two axioms imposed on them.

2.3.2.1 Game tree

A game tree belongs to a special class of trees, among those presented in §2.3.1. We need a definition.

Definition 2.8. *Let \mathbb{W} be a set. A \mathbb{W} -poset is a poset (partially ordered set) (V, \supset) , where $V \subset 2^{\mathbb{W}} \setminus \emptyset$ is made of nonempty subsets of \mathbb{W} and the binary relation \supset is set inclusion. A \mathbb{W} -poset (V, \supset) is rooted if $\{\mathbb{W}\} \in V$.*

As $V \subset 2^{\mathbb{W}}$, the \mathbb{W} -poset V inherits from $2^{\mathbb{W}}$ several set operations on the subsets of \mathbb{W} . Thus, for any two elements $v, v' \in V$, their difference $v \setminus v' \subset \mathbb{W}$ makes sense as a subset of \mathbb{W} (though $v \setminus v'$ may or may not be in V). The same holds true for their intersection $v \cap v'$ and their union $v \cup v'$.

The first element of an AFR-model is a discrete game tree.

Definition 2.9. *A game tree is a rooted \mathbb{W} -poset (V, \supset) , whose elements are called vertices, that satisfies the following two properties:*

- (GT1) *A subset $\mathfrak{c} \subset V$ is a chain if and only if there exists a play $w \in \mathbb{W}$ such that, for any vertex $v \in \mathfrak{c}$, we have $w \in v$,*
- (GT2) *If $w, w' \in \mathbb{W}$ with $w \neq w'$, then there exist two distinct vertices $v, v' \in V$ such that $w \in v \setminus v'$ and $w' \in v' \setminus v$.*

A game tree (V, \supset) is up-discrete if all nonempty chains in V have maxima. A game tree (V, \supset) is down-discrete if the chain $\uparrow v \setminus \{v\}$ has an infimum in $Z \cap (\uparrow v \setminus \{v\})$ for all non-root vertices $v \in V \setminus \{\mathbb{W}\}$, where the set Z of the terminal leaves is defined in (2.11d). A game tree (V, \supset) is discrete if it is up-discrete and down-discrete.

We will need the following result, valid on discrete game trees, which significantly relies on [3, Theorem 6.1].

Proposition 2.10. *Let (V, \supset) be a discrete game tree. Then, for any finite vertex $v \in F(V)$ in Equation (2.11a), the upset $\uparrow v$ is a finite set (that is, the cardinal $|\uparrow v| < +\infty$ is finite) and has the expression*

$$\uparrow v = \{v\} \sqcup \uparrow p(v). \quad (2.13a)$$

Moreover, we have

$$\uparrow v = \{v, p(v), p^2(v), \dots, p^{|\uparrow v|-1}(v)\}, \quad (2.13b)$$

where $p^{|\uparrow v|-1}(v) = \{\mathbb{W}\}$.

Proof. Let a finite vertex $v \in F(V)$ be given.

► We start by proving (2.13a) in two steps.

• First, we show that $\uparrow v \supset \{v\} \sqcup \uparrow p(v)$, for any finite vertex $v \in F(V)$. Indeed, on the one hand, we have that $v \in \uparrow v$ by (2.10) and, on the other hand, the inclusion $v \subset p(v)$ (as $p(v) \in \uparrow v \setminus \{v\} \subset \uparrow v$ by (2.11b)) implies that $\uparrow p(v) \subset \uparrow v$ by (2.10). Here, we restrict ourselves to non-root finite vertices as, formally speaking, $\uparrow \{\mathbb{W}\} = \{\mathbb{W}\}$, thus, $\uparrow \{\mathbb{W}\} \setminus \{\mathbb{W}\} = \emptyset$. As, on a game tree, the root vertex is the only one to coincide with its parent, we conclude that $\uparrow v \supset \{v\} \sqcup \uparrow p(v)$, for any finite vertex $v \in F(V)$.

• Second, we show that $\uparrow v \subset \{v\} \sqcup \uparrow p(v)$, for any finite vertex $v \in F(V)$. We write

$$\begin{aligned} \uparrow v &= \{v\} \sqcup (\uparrow v \setminus \{v\}) \\ &= \{v\} \sqcup \{v' \in (\uparrow v \setminus \{v\}) \mid v' \supset \min(\uparrow v \setminus \{v\})\} \\ &\hspace{15em} \text{(by definition of the minimum of a poset)} \\ &\subset \{v\} \sqcup \{v' \in V \mid v' \supset \min(\uparrow v \setminus \{v\})\} \hspace{5em} \text{(as } (\uparrow v \setminus \{v\}) \subset V) \\ &= \{v\} \sqcup \uparrow p(v), \hspace{10em} \text{(by definition of the parent as in (2.11b))} \end{aligned}$$

which proves the inverse inclusion, and, thus, gives (2.13a).

► Finally, we deduce (2.13b) as a corollary of representation (2.13a). For any finite vertex $v \in F(V)$ and any index k , such that $p^k(v) \neq \{\mathbb{W}\}$, we write

$$\begin{aligned} \uparrow v &= \{v\} \sqcup \uparrow p(v) \hspace{15em} \text{(by (2.13a))} \\ &= \{v\} \sqcup \{p(v)\} \sqcup \uparrow p^2(v) \hspace{5em} \text{(by (2.13a) applied for } p(v)) \\ &= \bigsqcup_{\ell \in \{0, \dots, k-1\}} p^\ell(v) \sqcup \uparrow p^k(v), \end{aligned}$$

by repeatedly applying (2.13a) for $v = p^\ell\{v\}$, where $\ell \in \{0, \dots, k-1\}$ and k is such that $p^k(v) \neq \{\mathbb{W}\}$.

As, for any finite vertex $v \in F(V)$, the upset $\uparrow v$ is a finite set, that is, $|\uparrow v| < +\infty$, by the item (d) on page 136 in the proof of [3, Theorem 6.1] and $\{\mathbb{W}\} \in \uparrow v$, for any vertex $v \in V$, then there exists \tilde{k} , such that $p^{\tilde{k}}(v) = \{\mathbb{W}\}$. Setting $n = \min\{k \mid p^k(v) = \{\mathbb{W}\}\}$, we conclude that

$$\uparrow v = \bigsqcup_{\ell \in \{0, \dots, n\}} p^\ell(v),$$

where $n = |\uparrow v| - 1$. This gives (2.13b) and ends the proof. \square

The following definition is a particular case of the discrete tree that will be used in Chapters 3 and 4 to establish equivalences between the AFR- and W-models.

Definition 2.11. *A finite game tree is a game tree (V, \supset) such that the set of vertices V is finite (that is, finite cardinal $|V| < +\infty$).*

Note that any finite game tree is a particular case of a discrete game tree.

The following property, for a poset to be a tree, holds true [3, Lemma 2.1]: a poset (N, \succeq) is a tree if and only if, for all $x, y, z \in N$,

$$\text{if } y \succeq x \text{ and } z \succeq x \text{ then } z \succeq y \text{ or } y \succeq z .$$

This property, rewritten in set-theoretic terms, has the following alternative translation for \mathbb{W} -posets; it will be widely used in the sequel.

Definition 2.12. A \mathbb{W} -poset (V, \supset) satisfies trivial intersection (in short, TrIP) if, for all $v, v' \in V$

$$\text{if } v \cap v' \neq \emptyset \text{ then } v \supset v' \text{ or } v' \supset v . \quad (2.14)$$

In Definition 2.9, condition (GT1) is equivalent to TrIP in Equation (2.14), and condition (GT2) is called *irreducibility*.

In a game tree, vertices that are properly followed by other vertices are called *moves*, and the set of moves is denoted by

$$X = \{v \in V \mid \downarrow v \setminus \{v\} \neq \emptyset\} . \quad (2.15)$$

Vertices that are not followed by any vertex are called *terminal vertices* and denoted by

$$Z = V \setminus X . \quad (2.16)$$

The authors show [3, Lemma 4.1, Item (b)] that a vertex $v \in V$ is terminal if and only if there is $w \in \mathbb{W}$ such that $v = \{w\}$ or, in other words, if and only

$$Z = \{\{w\}\}_{w \in \mathbb{W}} .$$

2.3.2.2 The system of choices

The second primitive object of the AFR-model is a so-called *system of collections of choices* indexed by the set \mathcal{P} of AFR-players.

Definition 2.13. For any player $i \in \mathcal{P}$, a nonempty collection

$$C^i \subset 2^{\mathbb{W}} \quad (2.17a)$$

of subsets of plays is called a collection of choices of the player i . Any element $c \in C^i$ of this collection, that is, a subset $c \subset \mathbb{W}$ of the underlying set \mathbb{W} of plays, is called a choice of the player i .

We also define a system \mathcal{C} of collections of choices as follows

$$\mathcal{C} = \{C^i\}_{i \in \mathcal{P}}, \text{ with } C^i \subset 2^{\mathbb{W}}, \forall i \in \mathcal{P} . \quad (2.17b)$$

In the AFR-model, a choice of a player combines both what the player knows at the moment of making an action, and the action made. We introduce the following mapping in order to map a given choice to the (unique) information set this choice is attached to.

Definition 2.14. For any subset $W \subset \mathbb{W}$ of plays, let

$$\uparrow W = \{v \in V \mid v \supset W\} \text{ and } \downarrow W = \{v \in V \mid v \subset W\}. \quad (2.18a)$$

be the upset and the downset of the set W respectively. Then, we define the immediate predecessor mapping P as

$$P : 2^{\mathbb{W}} \rightarrow 2^V \quad (2.18b)$$

$$\mathbb{W} \supset W \mapsto \{v' \in V \mid \exists v \in \downarrow W, \uparrow v' = \uparrow v \setminus \downarrow W\} \subset V. \quad (2.18c)$$

For any choice $c \in \mathcal{C}$, we define the *information set to which the choice c is assigned* by

$$P(c) \subset V. \quad (2.19a)$$

We also define the *AFR-information structure* Ξ_{AFR} as a collection of all information sets as in (2.19a) to which some choice $c \in \mathcal{C}$ is assigned, that is,

$$\Xi_{AFR} = \{P(c) \mid c \in \mathcal{C}\} \subset 2^V. \quad (2.19b)$$

As each choice $c \in \mathcal{C}$ is a subset of plays, and as its image under the immediate predecessor mapping $P(c)$ yields the corresponding information set (a subset of vertices) this choice is assigned to, we can draw a parallel between the objects of the K- and AFR-models. Indeed, in the K-model (see Definition 2.4), choices are primitives constructed out of the tree and information partition, as each choice is an element of the partition of the set of edges emanating out of the corresponding information set. Thus, in the K-model, first come information sets, second come choices. By contrast, in the AFR-model, we see that choices come first, and that information sets come second (they are not listed among the primitives, but are a derived object).

2.3.3 Definition of the AFR-model (2 basic objects, 2 axioms)

Now, we are ready to formally define the AFR-model, which is the most general extension of the Kuhn extensive form in Game Theory.

Definition 2.15. An AFR-model (T, \mathcal{C}) consists of two basic objects, namely

(AFR-BO1) a discrete game tree $T = (V, \supset)$ (as in Definition 2.9),

(AFR-BO2) a system $\mathcal{C} = \{C^i\}_{i \in \mathcal{P}}$ of collections of choices (as in (2.17b) in Definition 2.13)

and the following two axioms imposed on them

(AFR-Axiom1) For any player $i \in \mathcal{P}$ and any two choices $c, c' \in C^i$ of the player, if $P(c) \cap P(c') \neq \emptyset$ and $c \neq c'$ then $P(c) = P(c')$ and $c \cap c' = \emptyset$,

(AFR-Axiom2) For any move $x \in X$, the set of the children vertices of the move x (as in (2.11c)) is given by

$$p^{-1}(x) = \left\{ x \cap \left(\bigcap_{i \in J(x)} c^i \right) \mid \{c^i\}_{i \in J(x)} \in \prod_{i \in J(x)} A_i(x) \right\},$$

where the immediate predecessor mapping P is defined in (2.18b), the parent mapping p is defined in (2.11b), $A_i(x) = \{c \in C^i \mid x \in P(c)\}$ is the set of choices available to player i at a move x for all $i \in \mathcal{P}$, and $J(x) = \{i \in \mathcal{P} \mid A_i(x) \neq \emptyset\}$ is the set of so-called players active at move x (that is required to be nonempty for all $x \in X$).

When listing the basic objects of an AFR-model in Definition 2.15, neither the Nature player nor the sample set are defined explicitly. Most of the models of extensive form games [18, 19, 21] have, among the axioms imposed on their basic objects, one explicitly saying that each information set of the Nature player is a singleton, i.e. the Nature player has perfect information.

To prepare future results in Chapters 3 and 4, we introduce a subclass of AFR-models, as in Definition 2.15. In this subclass,

- only one player is active at each move, which is reflected in (AFR-Axiom1'), the original (AFR-Axiom-1) in Definition 2.15 is formulated for $i = j$,
- for any move, the set of children in (AFR-Axiom2') is also rewritten as there is only one player active at each move,
- the Nature player acts once and for all, at the root of the tree, which is reflected in the third added axiom (AFR-NR) below (“NR” for “Nature rooted”).

We call such form a *Nature-rooted AFR-model (NR AFR)*. Under the restricted assumption that only one player is active at any move, (AFR-Axiom1) and (AFR-Axiom2) are both equivalent to (AFR-Axiom1') and (AFR-Axiom2').

Definition 2.16. A Nature-rooted AFR-model (T, \mathcal{C}) consists of two basic objects, as in Definition 2.15, and the following three axioms imposed on them:

(AFR-Axiom1') For any players $i, j \in \mathcal{P}$ and any choices $c \in C^i$, $c' \in C^j$ of the players, if $P(c) \cap P(c') \neq \emptyset$ and $c \neq c'$ then $i = j$, $P(c) = P(c')$ and $c \cap c' = \emptyset$,

(AFR-Axiom2') For any move $x \in X$, the set of the children vertices of the move x (as in (2.11c)) is given by

$$p^{-1}(x) = \{x \cap c \mid c \in A_{i(x)}(x)\}, \quad (2.20)$$

where the parent mapping p has been defined in Equation (2.11b), the player $i(x) \in \mathcal{P}$ is such that $\{i(x)\} = J(x)$, as the set $J(x)$ of players active in the move x , defined in (AFR-Axiom2), is reduced a singleton by (AFR-Axiom1'), and the set $A_{i(x)}(x)$ of actions of the player $i(x)$ in the move x is given by

$$A_{i(x)}(x) = \{c \in C^{i(x)} \mid x \in P(c)\}. \quad (2.21)$$

(AFR-NR) For any choice $c \in C^0$ of Nature, its immediate predecessors are reduced to the root, that is, $P(c) = \{\mathbb{W}\}$,

where P is the immediate predecessor mapping as in (2.18b).

2.3.4 AFR-strategies

Now that information and choices have been properly defined in (2.19b) and in (2.17a) of Definition 2.13 respectively, we turn to the notion of AFR-strategies.

For each player $i \in \mathcal{P}$, we define the set X^i of the moves of the player i by

$$X^i = \{x \in X \mid \exists c \in C^i : x \in P(c)\}, \quad (2.22a)$$

which induces a partition

$$X = \bigsqcup_{i \in \mathcal{P}} X^i, \quad (2.22b)$$

by (AFR-Axiom1') in Definition 2.16.

Definition 2.17. For each player $i \in \mathcal{P}$, a pure AFR-strategy of the player i is a mapping $s^i : X^i \rightarrow C^i$, from player's moves as in (2.22a) to player's choices as in (2.17a), that satisfies

$$(s^i)^{-1}(c) = P(c) \text{ for any } c \in s^i(X^i).$$

A pure AFR-strategy profile is an element

$$s = \{s^i\}_{i \in \mathcal{P}} \in S = \prod_{i \in \mathcal{P}} S^i,$$

where S^i is the set of pure AFR-strategies of the player i .

In this framework, unlike the most general one of EDP discussed in the introduction of §2.3, it can be proved that each strategy induces a unique play [3, Theorem 5.4]. For the purpose of a more precise statement, we introduce, for every strategy profile $s \in S$, the correspondence $R_s : \mathbb{W} \rightrightarrows \mathbb{W}$ (“ R ” for “reaching”) defined by

$$R_s(w) = \cap \{s^i(x) \mid w \in x \in X, i \in J(x)\}, \quad (2.23)$$

where $J(x)$ is the set of players active in move x as in Definition (2.15).

Proposition 2.18. In an AFR-model as in Definition 2.15, the two following properties hold true.

(A1) For every strategy profile $s \in S$, there is some play $w \in \mathbb{W}$ such that $w \in R_s(w)$.

(A2) If, for every strategy profile $s \in S$, there is a play $w \in \mathbb{W}$ such that $w \in R_s(w)$, then the mapping R_s in Equation (2.23) has no other fixed point and $R_s(w) = \{w\}$.

The detailed proof of this statement can be found in [3, Theorem 5.4].

2.3.5 Examples

To illustrate the AFR-formalism presented above, we present three examples with two players: first, playing simultaneously (§2.3.5.1); second, one playing after another (§2.3.5.2); third playing after the Nature’s move (§2.3.5.3).

These three examples are the same as in §2.2.5 to make the comparison between the K-formalism and the AFR-formalism (and the W-formalism to come) easier.

2.3.5.1 Two players playing simultaneously

As in §2.2.5.1, we give one of the simplest and the most common examples with two players playing simultaneously or, making two decisions irrespective of each other.

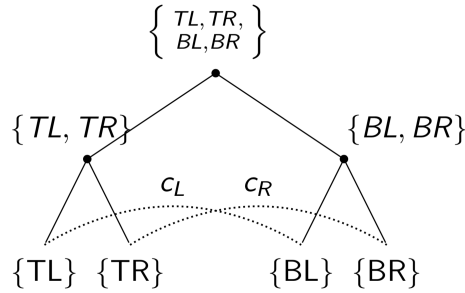


Figure 2.4: Two players playing simultaneously (AFR-model)

Mathematical formulation of the story of Alice and Bob (traditional notations of the AFR-model). The following is the list of the primitives of an AFR-model.

- *Set of plays*

$$\mathbb{W} = \{TL, TR, BL, BR\} .$$

- *Set of vertices*

$$V = \left\{ \underbrace{\{TL, TR, BL, BR\}}_{\text{root}}, \underbrace{\{TL, TR\}, \{BL, BR\}}_{\text{non-root moves}}, \underbrace{\{TL\}, \{TR\}, \{BL\}, \{BR\}}_{\text{leaves}} \right\} \subset 2^{\mathbb{W}} .$$

With the set V of vertices, we define the basic object (AFR-BO1) of Definition 2.15.

- *Choices of the players* (collections of nonempty union of vertices)

$$\begin{aligned} C^1 &= \{c_T, c_B\}, C^2 = \{c_L, c_R\}, \text{ where} \\ c_T &= \{TL, TR\}, c_B = \{BL, BR\}, \\ c_L &= \{BL, TL\}, c_R = \{TR, BR\} . \end{aligned}$$

With the collection $\mathcal{C} = \{C^i\}_{i=1,2}$ of choices, we define the basic object (AFR-BO2) of Definition 2.15.

These choices define the following information sets of each player:

$$\begin{aligned} \text{Player 1: } P(c_T) &= P(c_B) = \{\{TL, TR, BL, BR\}\} \subset V, \\ \text{Player 2: } P(c_L) &= P(c_R) = \{\{TL, TR\}, \{BL, BR\}\} \subset V. \end{aligned}$$

The two Axioms in Definition 2.15 are easily seen to be satisfied. Indeed,

(AFR-Axiom1) Satisfied automatically for Player 1 (respectively, Player 2) as the choices c_T and c_B (respectively, c_L and c_R) are disjoint.

(AFR-Axiom2) As there is only one player active at each vertex, (AFR-Axiom2) in Definition 2.15 is replaced by (AFR-Axiom2') in Definition 2.16, which is immediately satisfied for all three moves. We check illustrate it for $\{TL, TR\}$.

$$\begin{aligned} p^{-1}(\{TL, TR\}) &= \{\{TL\}, \{TR\}\} \\ &\quad \text{(by definition of the set of children as in (2.11c))} \\ &= \{\{TL, TR\} \cap c_L, \{TL, TR\} \cap c_R\}. \\ &\quad \text{(by construction of the choices } c_L \text{ and } c_R) \end{aligned}$$

Mathematical formulation of the story of Alice and Bob (with notations common to all three K-, AFR- and W-models). Finally, we can give names for the players' choices. We choose Alice as the first (row) player, and we set $c_T = T$ ("Top") and $c_B = B$ ("Bottom"); we choose Bob as the second (column) player, and we set $c_L = L$ ("Left") and $c_R = R$ ("Right").

The set of basic object of this example is given by

(AFR-BO1) the set V of vertices,

(AFR-BO2) the collections of the choices of the players $C^1 = \{T, B\}, C^2 = \{L, R\}$.

2.3.5.2 Two ordered players

We consider the tree in Figure 2.2 described using the formalism of Kuhn as in §2.2.5.2 and rewrite it as a poset, where each vertex corresponds to the set of plays that pass by it and where every upset is a chain (totally ordered by inclusion).

The following is the list of the primitives of an AFR-model.

- *Set of plays*

$$\mathbb{W} = \{TL, TR, BL, BR\}.$$

- *Set of vertices*

$$V = \left\{ \underbrace{\{TL, TR, BL, BR\}}_{\text{root}}, \underbrace{\{TL, TR\}, \{BL, BR\}}_{\text{moves}}, \underbrace{\{TL\}, \{TR\}, \{BL\}, \{BR\}}_{\text{leaves}} \right\} \subset 2^{\mathbb{W}}.$$

With the set V of vertices, we define the basic object (AFR-BO1) of Definition 2.15.

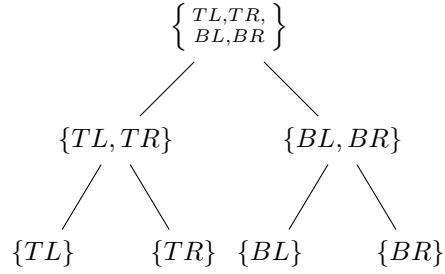


Figure 2.5: Two players playing simultaneously (AFR-model)

- *Choices of the players* (collections of nonempty unions of vertices)

$$\begin{aligned}
 C^1 &= \{c_T, c_B\}, C^2 = \{c_L^T, c_L^B, c_R^T, c_R^B\}, \text{ where} \\
 c_T &= \{TL, TR\}, c_B = \{BL, BR\}, \\
 c_L^B &= \{BL\}, c_L^T = \{TL\}, c_R^T = \{TR\}, c_R^B = \{BR\}.
 \end{aligned}$$

With the collection $\mathcal{C} = \{C^i\}_{i=1,2}$ of choices, we define the basic object (AFR-BO2) of Definition 2.15.

These choices define the following information sets:

$$\begin{aligned}
 \text{Player 1: } P(c_T) &= P(c_B) = \{\{TL, TR, BL, BR\}\} \subset V, \\
 \text{Player 2: } P(c_L) &= P(c_R) = \{\{TL, TR\}, \{BL, BR\}\} \subset V.
 \end{aligned}$$

The two Axioms in Definition 2.15 are easily seen to be satisfied.

2.3.5.3 Two ordered players with Nature

We consider the tree in Figure 2.3 described using the formalism of Kuhn as in §2.2.5.3. and rewrite it as a poset, where each vertex corresponds to the set of plays that pass by it and where every upset is a chain (totally ordered by inclusion).

The following is the list of the primitives of an AFR-model.

- *Set of plays*

$$\mathbb{W} = \{\omega^+TL, \omega^+TR, \omega^+BL, \omega^+BR, \omega^-TL, \omega^-TR, \omega^-BL, \omega^-BR\}.$$

- *Set of vertices*

$$\begin{aligned}
V = & \left\{ \underbrace{\{\omega^+TL, \omega^+TR, \omega^+BL, \omega^+BR, \omega^-TL, \omega^-TR, \omega^-BL, \omega^-BR\}}_{\text{root, move of the Nature}}, \right. \\
& \underbrace{\{\omega^+TL, \omega^+TR, \omega^+BL, \omega^+BR\}, \{\omega^-TL, \omega^-TR, \omega^-BL, \omega^-BR\}}_{\text{moves of Player 1}}, \\
& \underbrace{\{\omega^+TL, \omega^+TR\}, \{\omega^+BL, \omega^+BR\}, \{\omega^-TL, \omega^-TR\}, \{\omega^-BL, \omega^-BR\}}_{\text{moves of Player 2}}, \\
& \left. \underbrace{\{\omega^+TL\}, \{\omega^+TR\}, \{\omega^+BL\}, \{\omega^+BR\}, \{\omega^-TL\}, \{\omega^-TR\}, \{\omega^-BL\}, \{\omega^-BR\}}_{\text{leaves}} \right\} \subset 2^{\mathbb{W}}.
\end{aligned}$$

With the set V of vertices, we define the basic object (AFR-BO1) of Definition 2.15.

- *Choices of the players* (collections of nonempty unions of vertices)

$$\begin{aligned}
C^0 &= \{\{\omega^+TL, \omega^+TR, \omega^+BL, \omega^+BR\}, \{\omega^-TL, \omega^-TR, \omega^-BL, \omega^-BR\}\}, \\
C^1 &= \{\{\omega^+TL, \omega^+TR\}, \{\omega^+BL, \omega^+BR\}, \{\omega^-TL, \omega^-TR\}, \{\omega^-BL, \omega^-BR\}\}, \\
C^2 &= \{\{\omega^+TL\}, \{\omega^+TR\}, \{\omega^+BL\}, \{\omega^+BR\}, \{\omega^-TL\}, \{\omega^-TR\}, \{\omega^-BL\}, \{\omega^-BR\}\}.
\end{aligned}$$

With the collection $\mathcal{C} = \{C^i\}_{i=1,2}$ of choices, we define the basic object (AFR-BO2) of Definition 2.15.

The two Axioms in Definition 2.15 are easily seen to be satisfied.

2.4 Witsenhausen intrinsic model (W-model)

After the K- and AFR-models, we finally present the Witsenhausen's intrinsic model [37, 38], that we call *W-model*. This W-model enables what certainly is the most general representation of the information available to so-called agents and of its relative influence on the other agents' decisions. Originally, Witsenhausen introduced this model for control theory issues [37, 38]. The connection with games was briefly discussed in his paper [37] but, to our knowledge, has not been developed since. To present the main object of our research,

- in §2.4.1 we introduce the basic objects of the W-model,
- in §2.4.2 we give the definition of the W-model listing the basic objects defined above and the axiom imposed on them,
- in §2.4.3 we define the W-strategies,
- in §2.4.4 we define configuration orderings and causality discussing in the end solvability under causality,
- in §2.4.5 we illustrate the W-model on simple examples.

2.4.1 Primitives of the W-model

A W-model is a collection $(\mathbb{A}, (\Omega, \mathcal{F}), \{\mathbb{U}_a, \mathcal{U}_a\}_{a \in \mathbb{A}}, \{\mathcal{J}_a\}_{a \in \mathbb{A}})$.

The W-model [37, 38] consists of a finite set of agents, of a collection of decision sets, with a corresponding collection of σ -fields, and of a single sample set equipped with a σ -field, and representing uncertainties, or states of Nature. This model does not suppose any temporal ordering of decisions, in particular, there is no tree structure *a priori* given.

2.4.1.1 Agent's action set and the sample space of Nature

In the W-model, the introduction of σ -fields from the start is the key to handle information. We refer the reader to §9.1 for background on partitions and partition fields, σ -fields and algebras, as they are widely used afterwards,

As the first primitive, Witsenhausen introduces the following elements

- a finite set \mathbb{A} , whose elements are called *agents*; each agent $a \in \mathbb{A}$ is supposed to make one decision $u_a \in \mathbb{U}_a$ where the set \mathbb{U}_a is the *set of decisions* for agent a ,
- for each agent $a \in \mathbb{A}$, a σ -field \mathcal{U}_a on the set \mathbb{U}_a of decisions,
- a set Ω , the *sample space* or *Nature set*, which represents all uncertainties (any $\omega \in \Omega$ is called a *state of Nature*),
- a σ -field \mathcal{F} on the sample space Ω .

The notion of agent is not present in the K- and AFR-models, whereas it is central in the W-model. Loosely speaking, if an individual is taking an action today and an action tomorrow, he is to be represented by two agents, as, first, he or she may have different information before taking the actions and, second, because decision sets of today and tomorrow may differ. In the same vein, an individual who makes a sequence of decisions – one for each period $t = 0, 1, 2, \dots, T-1$ – is represented by T agents, labelled $t = 0, 1, 2, \dots, T-1$.

The *configuration space* is the product space (called *hybrid space* by Witsenhausen, hence the \mathbb{H} notation)

$$\mathbb{H} = \Omega \times \prod_{a \in \mathbb{A}} \mathbb{U}_a, \quad (2.24a)$$

equipped with the product *configuration field*

$$\mathcal{H} = \mathcal{F} \otimes \bigotimes_{a \in \mathbb{A}} \mathcal{U}_a. \quad (2.24b)$$

For the following exposition, we will need a bit of notations. For any subset $\mathbb{B} \subset \mathbb{A}$ of agents, and the complementary set of agents

$$- \mathbb{B} = \mathbb{A} \setminus \mathbb{B} \quad (2.25)$$

we define the corresponding product sets of actions³

$$\bar{\mathbb{U}}_{\mathbb{B}} = \prod_{a \in \mathbb{B}} \mathbb{U}_a, \quad \bar{\mathbb{U}}_{-\mathbb{B}} = \prod_{a \in -\mathbb{B}} \mathbb{U}_a, \quad (2.26a)$$

the following control field

$$\mathcal{U}_{\mathbb{B}} = \bigotimes_{a \in \mathbb{B}} \mathcal{U}_a \otimes \bigotimes_{b \in -\mathbb{B}} \{\emptyset, \mathbb{U}_b\} \subset \bigotimes_{a \in \mathbb{A}} \mathcal{U}_a, \quad (2.26b)$$

and the following configuration field

$$\mathcal{H}_{\mathbb{B}} = \mathcal{F} \otimes \mathcal{U}_{\mathbb{B}} = \mathcal{F} \otimes \bigotimes_{a \in \mathbb{B}} \mathcal{U}_a \otimes \bigotimes_{b \in -\mathbb{B}} \{\emptyset, \mathbb{U}_b\} \subset \mathcal{H}. \quad (2.26c)$$

In the sequel, we will often use the special cases when $\mathbb{B} = \mathbb{A}$ in Equation (2.26a) and (2.26b), that is,

$$\bar{\mathbb{U}}_{\mathbb{A}} = \mathbb{U}_{\mathbb{A}} = \prod_{a \in \mathbb{A}} \mathbb{U}_a, \quad \mathcal{U}_{\mathbb{A}} = \bigotimes_{a \in \mathbb{A}} \mathcal{U}_a, \quad \mathcal{H}_{\mathbb{A}} = \mathcal{F} \otimes \mathcal{U}_{\mathbb{A}} = \mathcal{H}, \quad (2.26d)$$

when $\mathbb{B} = \{a\}$ is a singleton

$$\bar{\mathbb{U}}_{\{a\}} = \mathbb{U}_a, \quad \mathcal{U}_{\{a\}} = \mathcal{U}_a \otimes \bigotimes_{b \in -\{a\}} \{\emptyset, \mathbb{U}_b\} \subset \mathcal{U}_{\mathbb{A}}, \quad \mathcal{H}_{\{a\}} = \mathcal{F} \otimes \mathcal{U}_{\{a\}} \subset \mathcal{H}, \quad (2.26e)$$

where $-\{a\} = \mathbb{A} \setminus \{a\}$ by (2.25), and when, analogously,

$$\bar{\mathbb{U}}_{-\{a\}} = \prod_{a \in -\{a\}} \mathbb{U}_a, \quad \mathcal{U}_{-\{a\}} = \bigotimes_{b \in -\{a\}} \mathcal{U}_b \otimes \{\emptyset, \mathbb{U}_a\} \subset \mathcal{U}_{\mathbb{A}}, \quad \mathcal{H}_{-\{a\}} = \mathcal{F} \otimes \mathcal{U}_{-\{a\}} \subset \mathcal{H}. \quad (2.26f)$$

2.4.1.2 The information fields

As the second primitive, Witsenhausen introduces the *information field* of each agent $a \in \mathbb{A}$ as a σ -field

$$\mathcal{J}_a \subset \mathcal{H}, \quad \forall a \in \mathbb{A}. \quad (2.27)$$

This representation means that the information of agent a may depend on the states of Nature and on all agents' decisions (including itself in case of self-information, as we will define below).

The *W-information structure* Ξ_W is given by the collection of the information fields of all agents, that is,

$$\Xi_W = \{\mathcal{J}_a\}_{a \in \mathbb{A}}. \quad (2.28)$$

An example is given in Figure 2.6.

Remark 2.19. Analogously to the *K*- and *AFR*-information structures, as defined in (2.5) and in (2.19b) respectively, we derive the collection of all atoms of the agents

$$\Xi'_W = \{G\}_{\substack{G \in \mathcal{J}_a \\ a \in \mathbb{A}}}.$$

³The upper bar stands for “reduced” sets, that are not subsets of \mathbb{H} .

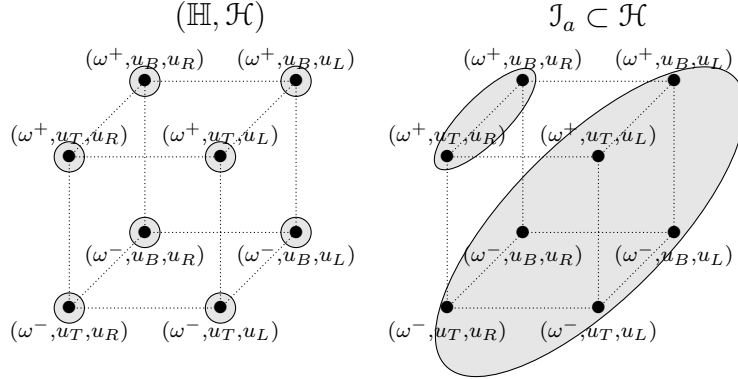


Figure 2.6: Example of configuration set \mathbb{H} , configuration field \mathcal{H} and information subfield \mathcal{J}_a (represented by a partition) in a W-model

2.4.2 Definition of the W-model (2 basic objects, 1 axiom)

Now, we give the following definition of the W-model.

Definition 2.20. A W-model $(\mathbb{A}, (\Omega, \mathcal{F}), \{\mathbb{U}_a, \mathcal{U}_a\}_{a \in \mathbb{A}}, \{\mathcal{J}_a\}_{a \in \mathbb{A}})$ consists of two basic objects, namely,

(W-BO1a) a sample space (Ω, \mathcal{F}) equipped with a σ -field,

(W-BO1b) a collection $\{\mathbb{U}_a, \mathcal{U}_a\}_{a \in \mathbb{A}}$ of agents' actions sets equipped with σ -fields, where the set \mathbb{A} of agents is finite,

(W-BO2) a collection $\{\mathcal{J}_a\}_{a \in \mathbb{A}}$ of agents' information subfields of the configuration field \mathcal{H} in Equation (2.24b),

and one axiom imposed on them

(W-Axiom1) for all agent $a \in \mathbb{A}$, the absence of self-information holds true, namely,

$$\mathcal{J}_a \subset \mathcal{H}_{-\{a\}} = \mathcal{F} \otimes \mathcal{U}_{-\{a\}}, \quad (2.29)$$

where the subfields $\mathcal{H}_{-\{a\}}$ and $\mathcal{U}_{-\{a\}}$ are defined in Equation (2.26f).

Possibly, the axiom of absence of self-information can be replaced by the stronger axiom of solvability (Definition 2.22) or the even stronger axiom of causality (Definition 2.24), both to be discussed later.

2.4.3 W-strategies

Now that the underlying space of configurations has been built in (2.24b) out of the agents' action sets and of the sample set, and that the agents' information σ -fields are given on the corresponding configuration σ -field in (2.24b), we turn to the notion of strategy.

In the Witsenhausen framework, strategies are carried by the agents, and make reference to their information field.

Definition 2.21. A pure W-strategy of agent $a \in \mathbb{A}$

$$\lambda_a : (\mathbb{H}, \mathcal{H}) \rightarrow (\mathbb{U}_a, \mathcal{U}_a) \quad (2.30a)$$

is a mapping from the underlying set \mathbb{H} of configurations into the set \mathbb{U}_a of actions of the agent a , which is measurable w.r.t. the information field \mathcal{J}_a of the agent a , that is,

$$\lambda_a^{-1}(\mathcal{U}_a) \subset \mathcal{J}_a . \quad (2.30b)$$

We denote by Λ_a the set of all pure W-strategies of the agent $a \in \mathbb{A}$. A pure W-strategy profile λ is an element of the following set

$$\lambda = \{\lambda_a\}_{a \in \mathbb{A}} \in \prod_{a \in \mathbb{A}} \Lambda_a = \Lambda . \quad (2.31)$$

Condition (2.30b) expresses the property that the admissible W-strategy of agent a may only depend upon the information \mathcal{J}_a available to the agent.

2.4.4 Solvability, causality

Now, we speak of solvability (§2.4.4.1), configuration orderings and causality (§2.4.4.2), and we finish by the algorithm that provides, under causality, a unique outcome (configuration), for any given pure W-strategy profile and any given state of Nature (§2.4.4.3).

2.4.4.1 Solvability

Witsenhausen defines the property of solvability as follows.

Definition 2.22 ([38]). *The solvability property holds true for the W-model (as in Definition 2.20) when, for any pure W-strategy profile $\lambda = \{\lambda_a\}_{a \in \mathbb{A}} \in \prod_{a \in \mathbb{A}} \Lambda_a$ (as in Definition 2.21) and any state of nature $\omega \in \Omega$, there exists one, and only one, decision profile $u = \{u_b\}_{b \in \mathbb{A}} \in \mathbb{U}_{\mathbb{A}}$, which is a solution of the closed-loop equations*

$$u = \lambda(\omega, u) , \quad (2.32a)$$

that is,

$$\forall \omega \in \Omega , \exists ! u = \{u_b\}_{b \in \mathbb{A}} \in \mathbb{U}_{\mathbb{A}} \text{ s.t. } u_a = \lambda_a(\omega, \{u_b\}_{b \in \mathbb{A}}) , \forall a \in \mathbb{A} , \quad (2.32b)$$

where $\mathbb{U}_{\mathbb{A}}$ is defined in Equation (2.26d). Denoting $M_\lambda(\omega)$ this unique $u \in \mathbb{U}_{\mathbb{A}}$, we obtain a mapping $M_\lambda : \Omega \rightarrow \mathbb{U}_{\mathbb{A}}$. Thanks to the mapping M_λ we define the solution mapping

$$S_\lambda : \Omega \rightarrow \mathbb{H} \quad (2.32c)$$

by

$$S_\lambda(\omega) = (\omega, u) \iff u = \lambda(\omega, u) , \quad (2.32d)$$

that is, $S_\lambda(\omega) = (\omega, M_\lambda(\omega))$, $\forall \omega \in \Omega$.

2.4.4.2 Configuration orderings and causality

In his articles [37,38], Witsenhausen introduces a notion of causality which will prove to be useful to compare his model with the K- and AFR-models. In a causal system, agents are ordered, one playing after the other with available information depending only on agents acting earlier, but the order may depend upon the configuration.

To properly define causality, we will now outline a series of notations and definitions, that will prove quite convenient to handle, in a compact way, causal systems.

Configuration orderings. For $k \in \{1, \dots, |\mathbb{A}|\}$, let Σ^k denote the set of k -orderings, that is, injective mappings from $\{1, \dots, k\}$ to the finite set \mathbb{A} of agents:

$$\Sigma^k = \{ \kappa : \{1, \dots, k\} \rightarrow \mathbb{A} \mid \kappa \text{ is an injection} \} . \quad (2.33a)$$

Notice that $\Sigma^{|\mathbb{A}|}$ is the set of *total orderings* of agents in \mathbb{A} , that is, bijective mappings from $\{1, \dots, |\mathbb{A}|\}$ to \mathbb{A} . We extend this definition to the case of $k = 0$ by formally setting

$$\Sigma^0 = \{\emptyset\} . \quad (2.33b)$$

We define the *set of all orderings* by

$$\Sigma = \bigcup_{k \in \{0, \dots, |\mathbb{A}|\}} \Sigma^k . \quad (2.33c)$$

For any $k \in \{1, \dots, |\mathbb{A}|\}$, we define the *cardinal* $|\kappa|$ of any ordering $\kappa \in \Sigma^k$ by

$$|\kappa| = k \in \{1, \dots, |\mathbb{A}|\} , \quad \forall \kappa \in \Sigma^k , \quad (2.33d)$$

and the *set* $\Sigma^{(\geq k)}$ of orderings with cardinal greater or equal than k by

$$\Sigma^{(\geq k)} = \bigcup_{\ell \geq k} \Sigma^\ell \subset \Sigma . \quad (2.33e)$$

For any $k \in \{1, \dots, |\mathbb{A}|\}$, any ordering $\kappa \in \Sigma^k$, and any integer $\ell \leq k$, the projection of the ordering κ on the first ℓ components is well defined

$$\kappa_{|\{1, \dots, \ell\}} = (1 \mapsto \kappa(1), \dots, \ell \mapsto \kappa(\ell)) \in \Sigma^\ell . \quad (2.33f)$$

For any $k \in \{1, \dots, |\mathbb{A}|\}$, there is a natural mapping ψ_k

$$\begin{aligned} \psi_k : \Sigma^{(\geq k)} &\rightarrow \Sigma^k \\ \kappa &\mapsto \kappa_{|\{1, \dots, k\}} \end{aligned} \quad (2.33g)$$

which is the restriction, as defined in Equation (2.33f), of any ordering $\kappa \in \Sigma^{(\geq k)}$ defined in Equation (2.33e) to the domain set $\{1, \dots, k\}$.

For any $k \in \{1, \dots, |\mathbb{A}|\}$, and any partial k -ordering $\kappa \in \Sigma^k$, we define the *range* $\|\kappa\|$ of the ordering κ by

$$\|\kappa\| = \{\kappa(1), \dots, \kappa(k)\} \subset \mathbb{A}, \quad \forall \kappa \in \Sigma^k, \quad (2.33h)$$

the *last element* κ^* of the ordering κ as the agent

$$\kappa^* = \kappa(k) \in \mathbb{A}, \quad \forall \kappa \in \Sigma^k, \quad (2.33i)$$

the *restriction* κ^- of the ordering κ to the first $k - 1$ elements by

$$\kappa^- = \psi_{k-1}(\kappa) = (1 \mapsto \kappa(1), \dots, k-1 \mapsto \kappa(k-1)) \in \Sigma^{k-1}, \quad \forall \kappa \in \Sigma^k, \quad (2.33j)$$

where $\kappa^- = \emptyset$ when $k = 1$, by the convention (2.33b), and where the restriction mapping ψ_{k-1} is defined in Equation (2.33g). With the notations introduced above, any nonempty ordering $\kappa \in \Sigma \setminus \{(\emptyset)\}$, can be written as

$$\kappa = (\kappa^-, \kappa^*), \quad \forall \kappa \in \Sigma \setminus \{(\emptyset)\}, \quad (2.33k)$$

and, for any nontotal ordering $\kappa \in \Sigma \setminus \Sigma^{|\mathbb{A}|}$, we have

$$(\kappa, a)^- = \kappa \text{ and } (\kappa, a)^* = a, \quad \forall \kappa \in \Sigma \setminus \Sigma^{|\mathbb{A}|}, \quad \forall a \in \mathbb{A} \setminus \|\kappa\|. \quad (2.33l)$$

Definition 2.23. A configuration ordering is a mapping $\varphi : \mathbb{H} \rightarrow \Sigma^{|\mathbb{A}|}$ from configurations towards total orderings: along each configuration $h \in \mathbb{H}$, the agents are ordered by the total ordering $\varphi(h) \in \Sigma^{|\mathbb{A}|}$.

With any configuration-ordering φ and any partial ordering $\kappa \in \Sigma$, we associate the subset $\mathbb{H}_\kappa^\varphi \subset \mathbb{H}$ of configurations given by

$$\mathbb{H}_\kappa^\varphi = \left\{ h \in \mathbb{H} \mid \psi_{|\kappa|}(\varphi(h)) = \kappa \right\} = (\psi_{|\kappa|} \circ \varphi)^{-1}(\{\kappa\}), \quad \forall \kappa \in \Sigma. \quad (2.34)$$

By convention, we put $\mathbb{H}_\emptyset^\varphi = \mathbb{H}$.

The set $\mathbb{H}_\kappa^\varphi$ contains all the configurations such that the agent $\kappa(1) = \psi_1(\kappa)$ is acting first, the agent $\kappa(2) = (\psi_2(\kappa))^*$ is acting second, ..., the last agent κ^* is acting at stage $|\kappa|$.

Finally, we give two more notations that are used later in Chapter 3.

For any configuration ordering $\varphi : \mathbb{H} \rightarrow \Sigma^{|\mathbb{A}|}$, and any $k \in \{1, \dots, |\mathbb{A}|\}$, we define the following set $\psi_k(\varphi(\mathbb{H})) \subset \Sigma^k$ of k -orderings compatible with the configuration ordering φ by

$$\psi_k(\varphi(\mathbb{H})) = \{\kappa \in \Sigma^k \mid \exists h \in \mathbb{H}, \kappa = \psi_k(\varphi(h))\} \subset \Sigma^k, \quad (2.35a)$$

and we extend the definition to the case $k = 0$ by formally setting $\psi_0(\varphi(\mathbb{H})) = \varphi(\mathbb{H})$, where

$$\varphi(\mathbb{H}) = \{\rho \in \Sigma^{|\mathbb{A}|} \mid \exists h \in \mathbb{H}, \rho = \varphi(h)\} \subset \Sigma^{|\mathbb{A}|}. \quad (2.35b)$$

Then, the set $\Sigma^\varphi \subset \Sigma$ of all orderings compatible with the configuration ordering φ is given by

$$\Sigma^\varphi = \bigcup_{0 \leq k \leq |\mathbb{A}|} \psi_k(\varphi(\mathbb{H})) \subset \Sigma. \quad (2.35c)$$

Causality. Now, we introduce the notion of causality following the exposition [37] of Witsenhausen. Before this, we need to fix some notations.

Let a partial ordering $\kappa \in \Sigma$ be given. Replacing in Equation (2.26a) and in Equation (2.26b) the subset of agents \mathbb{B} by $\|\kappa\|$ defined in Equation (2.33h) and $\|\kappa^-\|$ defined in Equation (2.33j), we obtain the following product sets of agents given together with the generic elements and subfields of $\mathcal{U}_{\mathbb{A}}$ and \mathcal{H} :

$$u_{\|\kappa^-\|} \in \overline{\mathcal{U}}_{\|\kappa^-\|} = \prod_{a \in \|\kappa^-\|} \mathcal{U}_a, \quad (2.36a)$$

$$u_{\|\kappa\|} \in \overline{\mathcal{U}}_{\|\kappa\|} = \prod_{a \in \|\kappa\|} \mathcal{U}_a = \overline{\mathcal{U}}_{\|\kappa^-\|} \times \mathcal{U}_{\kappa^*}, \quad (2.36b)$$

$$\mathcal{U}_{\|\kappa\|} = \bigotimes_{a \in \|\kappa\|} \mathcal{U}_a \otimes \bigotimes_{b \in -\|\kappa\|} \{\emptyset, \mathbb{U}_b\} \subset \mathcal{U}_{\mathbb{A}}, \quad (2.36c)$$

$$\mathcal{U}_{\|\kappa^-\|} = \bigotimes_{a \in \|\kappa^-\|} \mathcal{U}_a \otimes \bigotimes_{b \in -\|\kappa^-\|} \{\emptyset, \mathbb{U}_b\} \subset \mathcal{U}_{\|\kappa\|}, \quad (2.36d)$$

$$\mathcal{H}_{\|\kappa\|} = \mathcal{F} \otimes \mathcal{U}_{\|\kappa\|} \subset \mathcal{H}, \quad (2.36e)$$

$$\mathcal{H}_{\|\kappa^-\|} = \mathcal{F} \otimes \mathcal{U}_{\|\kappa^-\|} \subset \mathcal{H}_{\|\kappa\|}. \quad (2.36f)$$

After having fixed the notations, we turn to the formal definition of causality.

Definition 2.24. (*[37]*) *A W-model (as in Definition 2.20) is causal if there exists (at least one) configuration-ordering φ from \mathbb{H} towards $\Sigma^{|\mathbb{A}|}$ (as in Definition 2.23), with the property that*

$$\mathbb{H}_{\kappa}^{\varphi} \cap G \in \mathcal{H}_{\|\kappa^-\|}, \quad \forall G \in \mathcal{J}_{\kappa^*}, \quad \forall \kappa \in \Sigma, \quad (2.37)$$

where the field $\mathcal{H}_{\|\kappa^-\|}$ is defined in Equation (2.36f) and the “last” agent κ^* is defined in Equation (2.33i).

In other words, when the first agents are known and given by κ^- , the information \mathcal{J}_{κ^*} of the last agent κ^* depends at most on the decisions of the predecessor agents κ^- and Nature.

We propose the following definition.

Definition 2.25. *We say that a mapping $\varphi : \mathbb{H} \rightarrow \Sigma^{|\mathbb{A}|}$ from configurations towards total orderings is a causal configuration-ordering if Equation (2.37) holds true.*

With this definition of causal configuration-ordering, the Definition 2.24 of a causal W-model is one for which there exists (at least one) causal configuration-ordering.

In the sequel, we will make use of the following characterization of the sets $\mathbb{H}_{\kappa}^{\varphi}$ of configurations defined, for any nonempty ordering $\kappa \in \Sigma \setminus \{(\emptyset)\}$, in Equation (2.34).

Proposition 2.26. *If a configuration-ordering $\varphi : \mathbb{H} \rightarrow \Sigma^{|\mathbb{A}|}$ is causal (as in Definition 2.24), then for any $\kappa \in \Sigma \setminus \Sigma^1$, the corresponding set $\mathbb{H}_\kappa^\varphi$ of configurations is an element of the field $\mathcal{H}_{\|\kappa^-\|}$ defined in Equation (2.36f), hence, is a cylinder of the form*

$$\mathbb{H}_\kappa^\varphi = \overline{\mathbb{H}}_\kappa^\varphi \times \overline{\mathbb{U}}_{-\|\kappa^-\|}, \quad \forall \kappa \in \Sigma \setminus \Sigma^1, \quad (2.38a)$$

where

$$\overline{\mathbb{H}}_\kappa^\varphi \subset \Omega \times \overline{\mathbb{U}}_{\|\kappa^-\|}, \quad \forall \kappa \in \Sigma \setminus \Sigma^1, \quad (2.38b)$$

and, for any 1-ordering $(a) \in \Sigma^1$, where (a) denotes the partial 1-ordering $(1 \mapsto a) \in \psi_1(\varphi(\mathbb{H}))$ the corresponding set $\mathbb{H}_{(a)}^\varphi$ is a cylinder of the form

$$\mathbb{H}_{(a)}^\varphi = \Omega_a \times \mathbb{U}_\mathbb{A}, \quad \forall (a) \in \Sigma^1, \quad (2.38c)$$

where

$$\Omega_a \subset \Omega. \quad (2.38d)$$

Proof. For any causal configuration-ordering $\varphi : \mathbb{H} \rightarrow \Sigma^{|\mathbb{A}|}$ and ordering $\kappa \in \Sigma \setminus \Sigma^1$ (the proof for any 1-ordering $(a) \in \Sigma^1$ is analogous), the corresponding set $\mathbb{H}_\kappa^\varphi$ defined in Equation (2.34) is an element of $\mathcal{F} \otimes \mathcal{U}_{\|\kappa^-\|}$ by taking $G = \mathbb{H}$ in Equation (2.37), where the field $\mathcal{U}_{\|\kappa^-\|}$ is defined in Equation (2.36d), thus, $\mathbb{H}_\kappa^\varphi$ is a cylinder of the form

$$\mathbb{H}_\kappa^\varphi = \overline{\mathbb{H}}_\kappa^\varphi \times \overline{\mathbb{U}}_{-\|\kappa^-\|}, \quad \text{where } \overline{\mathbb{H}}_\kappa^\varphi \subset \Omega \times \overline{\mathbb{U}}_{\|\kappa^-\|}, \quad \forall \kappa \in \Sigma \setminus \Sigma^1.$$

This finishes the proof. \square

2.4.4.3 Solvability for pure W-strategies under causality

Finally, we discuss solvability for pure W-strategies under causality. In what follows, we give the algorithm yielding, for any given pure W-strategies profile, a unique outcome under causality.

Algorithm 2.27 (Solving the closed-loop equations). *Let a causal W-model as in Definition 2.20 and a profile $\lambda \in \Lambda$ of W-strategies as in (2.31) be given. For any $\omega \in \Omega$, the unique solution to the closed-loop equations $u = \lambda(\omega, u)$ in (2.32a) that is,*

$$\forall \omega \in \Omega, \quad \exists! u = \{u_b\}_{b \in \mathbb{A}} \in \mathbb{U}_\mathbb{A} \text{ s.t. } u_a = \lambda_a(\omega, \{u_b\}_{b \in \mathbb{A}}), \quad \forall a \in \mathbb{A}, \quad (2.39)$$

is given by the following algorithm.

Step 1 *The first agent and his decision are determined uniquely by ω as follows.*

a) *There is a unique agent $a_1 \in \mathbb{A}$ such that*

$$(\omega, u) \in \mathbb{H}_{(a_1)}^\varphi, \quad \forall u \in \mathbb{U}_\mathbb{A}, \quad (2.40a)$$

where (a_1) denotes the partial 1-ordering $(1 \mapsto a_1) \in \psi_1(\varphi(\mathbb{H}))$ and $\mathbb{H}_{(a_1)}^\varphi$ is defined in (2.34).

b) There is a unique decision $\hat{u}_{a_1} \in \mathbb{U}_{a_1}$ such that

$$\hat{u}_{a_1} = \lambda_{a_1}(\omega, u), \quad \forall u \in \mathbb{U}_{\mathbb{A}}. \quad (2.40b)$$

Then, at the end of Step 1 of the algorithm, we obtain the 1-ordering $(a_1) \in \Sigma_1$ and the sequence $(\omega, \hat{u}_{a_1}) \in \Omega \times \mathbb{U}_{a_1}$.

Step (k+1) Let Step k of the algorithm result in a partial ordering $\kappa \in \Sigma_k$ and a sequence $(\omega, \hat{u}_{\|\kappa\|}) = (\omega, \hat{u}_{\kappa(1)}, \hat{u}_{\kappa(2)}, \dots, \hat{u}_{\kappa(k)}) \in \Omega \times \overline{\mathbb{U}}_{\|\kappa\|}$.

Then, the agent a_{k+1} and his decision are determined uniquely as follows.

a) There is a unique agent a_{k+1} such that

$$(\omega, \hat{u}_{\|\kappa\|}) \in \overline{\mathbb{H}}_{(\kappa, a_{k+1})}^{\varphi}, \quad (2.41a)$$

where the $(k+1)$ -ordering (κ, a_{k+1}) belongs to $\psi_{k+1}(\varphi(\mathbb{H}))$ and $\mathbb{H}_{(\kappa, a_{k+1})}^{\varphi}$ is defined in (2.34).

b) There is a unique decision $\hat{u}_{a_{k+1}} \in \mathbb{U}_{a_{k+1}}$ such that

$$\hat{u}_{a_{k+1}} = \lambda_{a_{k+1}}(\omega, \hat{u}_{\|\kappa\|}, u_{-\|\kappa\|}), \quad \forall u_{-\|\kappa\|} \in \mathbb{U}_{-\|\kappa\|}. \quad (2.41b)$$

Then, at the end of Step $k+1$ of the algorithm, we obtain the partial $(k+1)$ -ordering $(\kappa, a_{k+1}) \in \Sigma_{k+1}$ and the sequence

$$(\omega, \hat{u}_{\|(\kappa, a_{k+1})\|}) = (\omega, \hat{u}_{\kappa(1)}, \hat{u}_{\kappa(2)}, \dots, \hat{u}_{\kappa(k)}, \hat{u}_{a_{k+1}}) \in \Omega \times \overline{\mathbb{U}}_{\|\kappa\|} \times \mathbb{U}_{a_{k+1}}.$$

Proof. Let $\omega \in \Omega$ be given.

Step 1. First, we prove Item a in Step 1. We write the set of configurations as

$$\begin{aligned} \Omega \times \mathbb{U}_{\mathbb{A}} = \mathbb{H} &= \bigsqcup_{(a) \in \psi_1(\varphi(\mathbb{H}))} \mathbb{H}_{(a)}^{\varphi} && \text{(by (3.50b) for } k=1) \\ &= \bigsqcup_{(a) \in \psi_1(\varphi(\mathbb{H}))} \Omega_a \times \mathbb{U}_{\mathbb{A}}. && \text{(as } \mathbb{H}_{(a)}^{\varphi} = \bigsqcup_{(a) \in \psi_1(\varphi(\mathbb{H}))} \Omega_a \times \mathbb{U}_{\mathbb{A}} \text{ by (2.38c))} \end{aligned}$$

As no set is empty, cancelling $\mathbb{U}_{\mathbb{A}}$ on both sides yields the following partition of the sample space

$$\Omega = \bigsqcup_{(a) \in \psi_1(\varphi(\mathbb{H}))} \Omega_a.$$

Therefore, there is a unique agent $a_1 \in \mathbb{A}$ such that $\omega \in \overline{\mathbb{H}}_{(a_1)}^{\varphi} \subset \Omega$. Then, as $\mathbb{H}_{(a_1)}^{\varphi}$ is a cylinder of the form $\Omega_{a_1} \times \mathbb{U}_{\mathbb{A}}$ by (2.38c), we deduce that $(\omega, u) \in \mathbb{H}_{(a_1)}^{\varphi}$ for any $u \in \mathbb{U}_{\mathbb{A}}$.

Second, we prove Item b in Step 1. We fix some $\bar{u} \in \mathbb{U}_\mathbb{A}$. There is a unique atom $\bar{G} \in \mathcal{J}_{a_1}$ such that $(\omega, \bar{u}) \in \bar{G}$ (by Definition 9.2 of the atoms of an information partition). From the Definition 2.24 of a causal ordering, and especially (2.37), we have that

$$\mathbb{H}_{(a_1)}^\varphi \cap \bar{G} \in \mathcal{F} \otimes \prod_{a \in \mathbb{A}} \{\emptyset, \mathbb{U}_a\}.$$

Therefore, $(\omega, u) \in \mathbb{H}_{(a_1)}^\varphi \cap \bar{G}$, for any $u \in \mathbb{U}_\mathbb{A}$, hence $(\omega, u) \in \bar{G}$, for any $u \in \mathbb{U}_\mathbb{A}$. Because the strategy λ_{a_1} of the agent a_1 is \mathcal{J}_{a_1} -measurable, it is constant on the atom $\bar{G} \in \mathcal{J}_{a_1}$, hence $\lambda_{a_1}(\omega, \bar{u}) = \lambda_{a_1}(\omega, u)$, for any $u \in \mathbb{U}_\mathbb{A}$. This is why we can set

$$\hat{u}_{a_1} = \lambda_{a_1}(\omega, u), \quad \forall u \in \mathbb{U}_\mathbb{A}.$$

Step (k+1) First, we prove Item a in Step $(k+1)$. Let Step k of the algorithm result in a partial ordering $\kappa \in \Sigma_k$ and a sequence $(\omega, \hat{u}_{\|\kappa\|}) \in \Omega \times \bar{\mathbb{U}}_{\|\kappa\|}$.

For $\kappa \in \Sigma_k$, we write the corresponding subset of configurations $\mathbb{H}_\kappa^\varphi$ as

$$\begin{aligned} \bar{\mathbb{H}}_\kappa^\varphi \times \bar{\mathbb{U}}_{-\|\kappa\|} &= \mathbb{H}_\kappa^\varphi && \text{(as } \mathbb{H}_\kappa^\varphi = \bar{\mathbb{H}}_\kappa^\varphi \times \bar{\mathbb{U}}_{-\|\kappa\|} \text{ by (2.38a))} \\ &= \bigsqcup_{\substack{\kappa' \in \psi_{k+1}(\varphi(\mathbb{H})) \\ (\kappa')^- = \kappa}} \mathbb{H}_{\kappa'}^\varphi && \text{(by (3.50c))} \\ &= \bigsqcup_{\substack{\kappa' \in \psi_{k+1}(\varphi(\mathbb{H})) \\ (\kappa')^- = \kappa}} \underbrace{\bar{\mathbb{H}}_{\kappa'}^\varphi \times \bar{\mathbb{U}}_{-(\kappa')^-}}_{=\bar{\mathbb{U}}_{-\|\kappa\|}}, && \text{(as } \mathbb{H}_{\kappa'}^\varphi = \bar{\mathbb{H}}_{\kappa'}^\varphi \times \bar{\mathbb{U}}_{-(\kappa')^-} \text{ by (2.38a))} \end{aligned}$$

where, for any $(k+1)$ -ordering $\kappa' \in \psi_{k+1}(\varphi(\mathbb{H}))$, the corresponding set $\bar{\mathbb{H}}_{\kappa'}^\varphi$ is a subset of $\Omega \times \bar{\mathbb{U}}_{\|\kappa\|}$. As no set is empty, cancelling $\bar{\mathbb{U}}_{-\|\kappa\|}$ on both sides yields the following partition

$$\bar{\mathbb{H}}_\kappa^\varphi = \bigsqcup_{\substack{\kappa' \in \psi_{k+1}(\varphi(\mathbb{H})) \\ (\kappa')^- = \kappa}} \bar{\mathbb{H}}_{\kappa'}^\varphi.$$

Therefore, there is a unique $(k+1)$ -ordering $\kappa' \in \Sigma_{k+1}$, hence, a unique agent $a_{k+1} \in \mathbb{A} \setminus \|\kappa\|$, such that $(\omega, \hat{u}_{\|\kappa\|}) \in \bar{\mathbb{H}}_{\kappa'}^\varphi \subset \Omega \times \bar{\mathbb{U}}_{\|\kappa\|}$, where $(\kappa')^- = \kappa$ and $(\kappa')^* = a_{k+1}$. Thus, this unique $(k+1)$ -ordering κ' is of the form $(\kappa, a_{k+1}) \in \Sigma_{k+1}$. Then, as $\mathbb{H}_{(\kappa, a_{k+1})}^\varphi$ is a cylinder $\bar{\mathbb{H}}_{(\kappa, a_{k+1})}^\varphi \times \bar{\mathbb{U}}_{-\|\kappa\|}$ by (3.50b), we deduce that $(\omega, \hat{u}_{\|\kappa\|}, u_{-\|\kappa\|}) \in \mathbb{H}_{(\kappa, a_{k+1})}^\varphi$ for any $u_{-\|\kappa\|} \in \bar{\mathbb{U}}_{-\|\kappa\|}$, hence, there is a unique agent a_{k+1} such that $(\omega, \hat{u}_{\|\kappa\|}) \in \bar{\mathbb{H}}_{(\kappa, a_{k+1})}^\varphi$ and (2.41a) is proved.

Second, we prove Item b in Step $(k+1)$. We fix some $\bar{u}_{-\|\kappa\|} \in \bar{\mathbb{U}}_{-\|\kappa\|}$. There is a unique atom $\bar{G} \in \mathcal{J}_{a_{k+1}}$ such that $(\omega, \hat{u}_{\|\kappa\|}, \bar{u}_{-\|\kappa\|}) \in \bar{G}$. From the Definition 2.24 of a causal ordering, and especially (2.37), we have that

$$\mathbb{H}_{(\kappa, a_{k+1})}^\varphi \cap \bar{G} \in \mathcal{F} \otimes \mathcal{U}_{\|\kappa\|}.$$

Therefore, $(\omega, \hat{u}_{\|\kappa\|}, u_{-\|\kappa\|}) \in \mathbb{H}_{(\kappa, a_{k+1})}^{\varphi} \cap \overline{G}$, for any $u_{-\|\kappa\|} \in \overline{\mathbb{U}}_{-\|\kappa\|}$, hence $(\omega, \hat{u}_{\|\kappa\|}, u_{-\|\kappa\|}) \in \overline{G}$, for any $u_{-\|\kappa\|} \in \overline{\mathbb{U}}_{-\|\kappa\|}$. Because the strategy $\lambda_{a_{k+1}}$ of the agent a_{k+1} is $\mathcal{J}_{a_{k+1}}$ -measurable, it is constant on the atom \overline{G} , hence $\lambda_{a_{k+1}}(\omega, \hat{u}_{\|\kappa\|}, \bar{u}_{-\|\kappa\|}) = \lambda_{a_{k+1}}(\omega, \hat{u}_{\|\kappa\|}, u_{-\|\kappa\|})$, for any $u_{-\|\kappa\|} \in \overline{\mathbb{U}}_{-\|\kappa\|}$. This is why we can set

$$\hat{u}_{a_{k+1}} = \lambda_{a_{k+1}}(\omega, \hat{u}_{\|\kappa\|}, u_{-\|\kappa\|}) , \quad \forall u_{-\|\kappa\|} \in \overline{\mathbb{U}}_{-\|\kappa\|} .$$

This ends the proof. \square

2.4.5 Examples

To illustrate the W-formalism presented above, we give here three examples with two players: first, playing simultaneously (§2.4.5.1); second, one playing after another (§2.4.5.2); third playing after the Nature's move (§2.4.5.3).

These three examples are the same as in §2.2.5 and §2.3.5 to make the comparison between the K-formalism, the AFR-formalism and the W-formalism easier.

2.4.5.1 Two unordered players (trivial information)

Mathematical formulation of the story of Alice and Bob (traditional notations of the W-model). The simplest model with two agents is the following situation: two agents a (Alice) and b (Bob) having two actions each

$$\mathbb{U}_a = \{u_T, u_B\}, \quad \mathbb{U}_b = \{u_L, u_R\} . \quad (2.42)$$

With the control sets \mathbb{U}_a and \mathbb{U}_b we define the basic object (W-BO1) in Definition 2.20.

Thus, the configuration space consists of four elements

$$\mathbb{H} = \{u_T, u_B\} \times \{u_L, u_R\} , \quad (2.43)$$

and no information about each other's decisions (see Figure 2.7)

$$\mathcal{J}_a = \mathcal{J}_b = \{\emptyset, \mathbb{U}_a\} \otimes \{\emptyset, \mathbb{U}_b\} ,$$

which can be interpreted as them playing simultaneously. With the information fields \mathcal{J}_a and \mathcal{J}_b we define the basic object (W-BO2) in Definition 2.20.

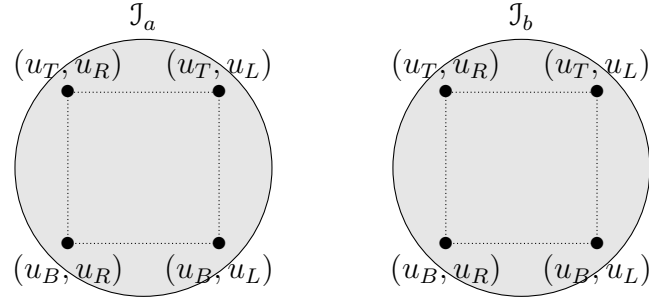
The (W-Axiom1) in Definition 2.20 is easily seen to be satisfied from the way the information fields \mathcal{J}_a and \mathcal{J}_b are defined: $\mathcal{J}_a \subset \mathcal{H}_{-\{a\}}$ and $\mathcal{J}_b \subset \mathcal{H}_{-\{b\}}$.

Mathematical formulation of the story of Alice and Bob (with notations common to all three K-, AFR- and W-models). Finally, we can give names for the players' actions that are common to all three models. We choose Alice as the first (row) player, and we set $u_T = T$ ("Top") and $u_B = B$ ("Bottom"); we choose Bob as the second (column) player, and we set $u_L = L$ ("Left") and $u_R = R$ ("Right").

The set of basic object of this example is given by

(W-BO1) the control sets $\mathbb{U}_a = \{T, B\}$ and $\mathbb{U}_b = \{L, R\}$ of the agents,

(W-BO2) the (trivial) information fields $\mathcal{J}_a = \mathcal{J}_b = \{\emptyset, \{T, B\} \times \{L, R\}\}$ of the agents.

Figure 2.7: Atoms of the information fields of the agents a and b playing simultaneously

2.4.5.2 Two ordered players (without Nature)

As in the previous example, there are two agents a (Alice) and b (Bob) and no Nature, having two possible actions each (see (2.45)) and the underlying configuration space consists of four elements (see (2.46)).

Suppose that Bob's information field is trivial (Bob knows nothing of Alice's decisions), that is,

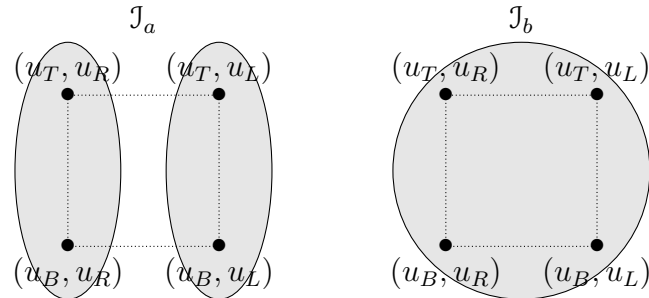
$$\mathcal{J}_b = \{\emptyset, \{u_T, u_B\}\} \otimes \{\emptyset, \{u_L, u_R\}\},$$

and that Alice knows what Bob does (Alice can distinguish between u_L and u_R)

$$\mathcal{J}_a = \{\emptyset, \{u_T, u_B\}\} \otimes \{\emptyset, \{u_L\}, \{u_R\}, \{u_L, u_R\}\}.$$

With the information fields \mathcal{J}_a and \mathcal{J}_b we define the basic object (W-BO2) in Definition 2.20.

In this example, the agents are naturally ordered: Bob plays first, Alice plays second. Had the order been inverted, then there would have been a sort of paradox – Alice would play first, before Bob, and would know Bob's action that has not been yet taken by him.

Figure 2.8: Atoms of the information fields of the ordered agents a and b (without Nature)

The (W-Axiom1) in Definition 2.20 is easily seen to be satisfied from the way the information fields \mathcal{J}_a and \mathcal{J}_b are defined: $\mathcal{J}_a \subset \mathcal{H}_{-\{a\}}$ and $\mathcal{J}_b \subset \mathcal{H}_{-\{b\}}$.

2.4.5.3 Two ordered players (with Nature)

In this example, there are two agents a (Alice) and b (Bob) and two states of Nature $\Omega = \{\omega^+, \omega^-\}$ (say, heads or tails). As in the previous examples, agents have two possible actions each (see (2.45)). Thus, the configuration space consists of eight elements:

$$\mathbb{H} = \{\omega^+, \omega^-\} \times \{u_T, u_B\} \times \{u_L, u_R\}.$$

We consider the following information structure:

$$\mathcal{J}_b = \overbrace{\{\emptyset, \{\omega^+\}, \{\omega^-\}, \{\omega^+, \omega^-\}\}}^{\text{Bob knows Nature's move}} \otimes \overbrace{\{\emptyset, \{u_T, u_B\}\}}^{\text{Bob does not know what Alice does}} \otimes \{\emptyset, \mathbb{U}_b\}, \quad (2.44a)$$

$$\mathcal{J}_a = \overbrace{\{\emptyset, \{\omega^+\}, \{\omega^-\}, \{\omega^+, \omega^-\}\}}^{\text{Alice knows Nature's move}} \otimes \{\emptyset, \mathbb{U}_a\} \otimes \overbrace{\{\emptyset, \{u_L\}, \{u_R\}, \{u_L, u_R\}\}}^{\text{Alice knows what Bob does}}. \quad (2.44b)$$

With the information fields \mathcal{J}_a and \mathcal{J}_b we define the basic object (W-BO2) in Definition 2.20. Again, here agents are naturally ordered: Bob plays first, Alice plays second.

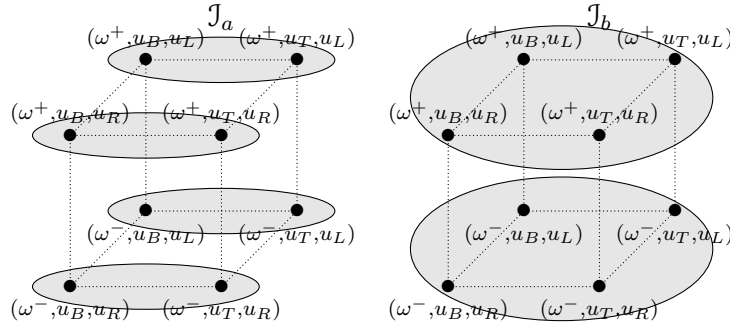


Figure 2.9: Atoms of the information fields of the ordered agents a and b (with Nature)

The (W-Axiom1) in Definition 2.20 is easily seen to be satisfied from the way the information fields \mathcal{J}_a and \mathcal{J}_b are defined: $\mathcal{J}_a \subset \mathcal{H}_{-\{a\}}$ and $\mathcal{J}_b \subset \mathcal{H}_{-\{b\}}$.

2.5 Advanced examples: forgetfulness and absent-mindedness

Here, we illustrate K-, AFR- and W-models on more advanced examples of absent-mindedness (§2.5.1) and of forgetfulness (§2.5.2). Besides illustrating the formalism of each model, this §2.5 prepares the discussion about information structures and possible classifications in Chapter 5.

2.5.1 Absent-mindedness

We now present a classic example dubbed the *paradox of the absent minded driver* by Piccione and Rubinstein who introduced it in [27].

As the story goes, a tired driver plans the trip back home on a highway with two exits. During the course of the game, he gets to take an action at most twice from the two-element action set $\{S, T\}$, where “S” stands for “Stay” (on the highway) and “T” stands for “Turn” (off the highway). Taking the first exit ends up in a bad neighbourhood, whereas the second exit is the one the driver would like to take, leading to the sweet home. In order to arrive there, the driver needs to stay at the first exit and turn at the second. If both exits are ignored, the driver ends up at a hotel. The absent-mindedness of the driver is rooted in the fact that both of the exits from the highway are indistinguishable for him.

2.5.1.1 Absent-mindedness in the K-model

We model the absent-minded driver situation in the K-model as follows.

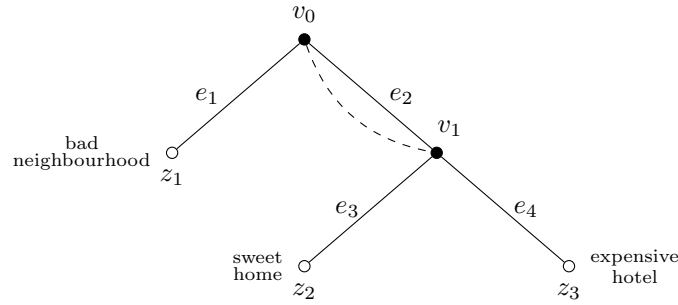


Figure 2.10: Absent-mindedness in the K-model

The following is the list of primitives.

- Set of vertices $V = \{v_0, v_1, z_1, z_2, z_3\}$.
- Set of edges $E = \left\{ \underbrace{(v_0, z_1)}_{e_1}, \underbrace{(v_0, v_1)}_{e_2}, \underbrace{(v_1, z_2)}_{e_3}, \underbrace{(v_1, z_3)}_{e_4} \right\}$.
- Set of player's moves $X = \{v_0, v_1\}$ and the set of leaves $Z = \{z_2, z_3\}$.
- Player's information partition $I = \{v_0, v_1\}$.
- Set of choices $C = \{\{e_1, e_3\}, \{e_2, e_4\}\}$.

The model per se violates (K-Axiom2) in Definition 2.4 stating that no play can cross any information set more than once. Indeed, the play $w_{z_2} = \{v_0, v_1, z_2\}$, as defined in (2.1d), crosses the only information set $I = \{v_0, v_1\}$ twice (i.e., at two vertices):

$$w_{z_2} \cap I = \{v_0, v_1, z_2\} \cap \{v_0, v_1\} = \{v_0, v_1\} \subset V .$$

Making relation to the general description of the model, we give names to the choices $\{e_1, e_3\} = T$ and $\{e_2, e_4\} = S$ of the player.

As presented above, the absent-minded driver model violates one of the axioms of the K-model. Despite of this, it can be depicted as the following extensive form game in Figure (2.10). In the literature (see, for example, [10]), extensive forms modelling absent-minded players are sometimes called *non-Kuhn trees*.

2.5.1.2 Absent-mindedness in the AFR-model

Straightforward embedding of the absent-minded driver example using the language of the AFR-model will, as in §2.5.1.1, end up in violating one of the two axioms of the AFR-model in Definition 2.15, namely, (AFR-Axiom1) that builds information sets from choices and imposes a condition that choices available simultaneously cannot intersect.

There is a player that has two decision locii and two possible decisions at each of the points: *turn* (T) or *stay* (S). Denote three possible outcomes as w_1, w_2 and w_3 . The tree structure is shown in Figure 2.11 followed by the list of objects.

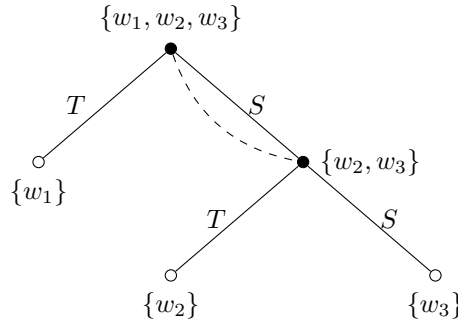


Figure 2.11: Absent-mindedness in the AFR-model

The following is the list of primitives of an AFR-model.

- Set of plays

$$\mathbb{W} = \{w_1, w_2, w_3\} .$$

- Set of vertices

$$V = \left\{ \underbrace{\{w_1, w_2, w_3\}}_{\text{root}}, \underbrace{\{w_2, w_3\}}_{\text{move}}, \underbrace{\{w_1\}, \{w_2\}, \{w_3\}}_{\text{leaves}} \right\} \subset 2^{\mathbb{W}} .$$

- Choices of the player

$$C = \{c_T, c_S\}, \text{ where} \\ c_T = \{w_1, w_2\}, c_S = \{w_1, w_3\} .$$

These two choices define an information set

$$P(c_T) = P(c_S) = \left\{ \{w_1, w_2, w_3\}, \{w_2, w_3\} \right\} ,$$

but, as $c_T \cap c_S = w_1$ and $c_T \neq c_S$, it violates (AFR-Axiom 1) in Definition 2.15.

So, the straightforward embedding of the absent minded driver into AFR-model fails because of the (AFR-Axiom 1) ruling out the absent-mindedness.

2.5.1.3 Absent-mindedness in the W-model

We will build here a W-model with two agents having a certain information structure that proves to be equivalent to the absent minded driver problem; this approach coincides with the one proposed by Gilboa. In his paper [10], he replaced the absent-minded player by two agents and added a move of Nature at the root that determines which of the two agents moves first. In Figure 2.12 is the illustration of the absent-minded driver problem (as in [10]) on a tree and with fictive moves added.

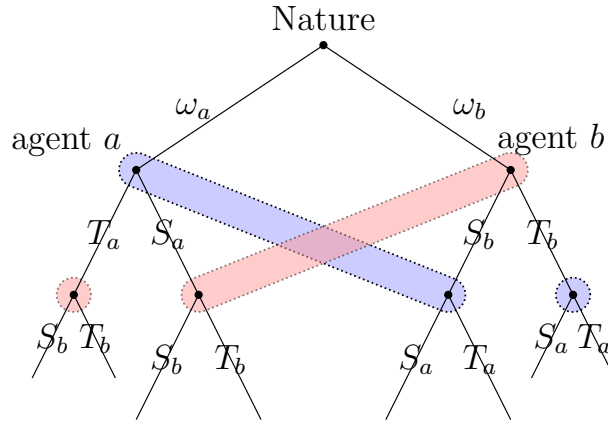


Figure 2.12: Absent-mindedness on the tree with two symmetric agents and added fictive vertices

Coming back to the absent minded driver example in the W-model, assume that the player is represented by two agents, a and b , having two actions *Turn* (T) or *Stay* (S) each indexed according to the agent's name:

$$\mathbb{U}_a = \{T_a, S_a\}, \quad \mathbb{U}_b = \{T_b, S_b\}.$$

Also, there is Nature taking its actions in the set $\Omega = \{\omega_a, \omega_b\}$, which selects an agent that will move first.

The configuration space is then given by

$$\mathbb{H} = \Omega \times \mathbb{U}_a \times \mathbb{U}_b = \{\omega_a, \omega_b\} \times \{T_a, S_a\} \times \{T_b, S_b\}.$$

The main idea behind the AM-property is that, at some point, an absent-minded player loses the public clock. This can be easily seen through the prism of agents representing him: neither agent a nor agent b know the order in which they got to take actions, i.e. each of them lacks knowledge about the Nature's move. The only thing an agent knows is that

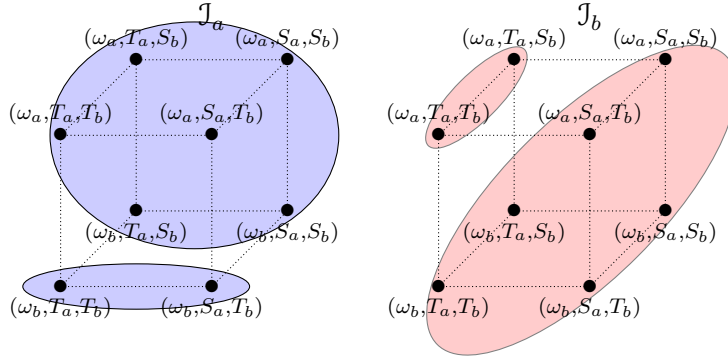


Figure 2.13: Atoms of the information fields of the absent-minded player's agents

if he got to play, had the other agent acted before, this other agent chose the action “Stay”. This ambiguity can be expressed through the following agents' information partition fields:

$$\begin{aligned}
 \mathcal{J}_a &= \left\{ \emptyset, \overbrace{\{\omega_a\} \times \mathbb{U}_a \times \mathbb{U}_b \cup \{\omega_b\} \times \{S_b\} \times \mathbb{U}_a}^{\text{agent } a \text{ makes a move}}, \overbrace{\{\omega_b\} \times \{T_b\} \times \mathbb{U}_a}^{\text{agent } a \text{ does not make a move}}, \mathbb{W} \right\}, \\
 &\quad \text{agent } a \text{ is whether} \quad \text{or he acts second after} \quad \text{agent } b \text{ chose } T \\
 &\quad \text{the first one to act} \quad \text{agent } b \text{ has chosen } S \quad \text{and finished the game} \\
 \mathcal{J}_b &= \left\{ \emptyset, \{\omega_b\} \times \mathbb{U}_a \times \mathbb{U}_b \cup \{\omega_a\} \times \{S_a\} \times \mathbb{U}_b, \{\omega_a\} \times \{T_a\} \times \mathbb{U}_b, \mathbb{W} \right\}.
 \end{aligned}$$

2.5.2 Forgetfulness

The second advanced example is the so-called *imperfect recall of past actions*, when a player makes one decision followed by another and forgets the one made first.

The story behind this example is as follows. A player makes two consecutive decisions, first is needed to choose between “Top” and “Bottom”, and afterwards between “Left” and “Right”, but the outcome at the first stage is being forgotten.

2.5.2.1 Forgetfulness in the K-model

This example is a particular case of the example with two unordered players (playing simultaneously) in §2.2.5.1,

- The set of vertices $V = \{v_0, v_1, v_2, z_3, z_4, z_5, z_6\}$.
- The set of edges $E = \left\{ \underbrace{(v_0, v_1)}_{e_1}, \underbrace{(v_0, v_2)}_{e_2}, \underbrace{(v_1, z_3)}_{e_3}, \underbrace{(v_1, z_4)}_{e_4}, \underbrace{(v_2, z_5)}_{e_5}, \underbrace{(v_2, z_6)}_{e_6} \right\}$.

With the set V of vertices and the set E of edges, choosing the vertex v_0 as the root, we define the basic object (K-BO1) of Definition 2.4.

- The set of edges determines

1. the set of moves $X = \{v_0, v_1, v_2\}$ by (2.1c)
(as $E_{v_0} = \{e_1, e_2\}$, $E_{v_1} = \{e_3, e_4\}$ and $E_{v_2} = \{e_5, e_6\}$),
 2. the set of leaves $Z = \{z_3, z_4, z_5, z_6\}$ by (2.1b)
(as $E_{z_3} = E_{z_4} = E_{z_5} = E_{z_6} = \emptyset$).
- The set of players is a singleton $\mathcal{P} = \{1\}$ and we partition the set X of moves into players' moves is trivial

$$X^1 = X .$$

With this partition, we define the basic object (K-BO2) of Definition 2.4.

- We define the player's information partition of (player's) set of moves by

$$I = \{\{v_0\}, \{v_1, v_2\}\} .$$

With this partition, we define the basic object (K-BO3) of Definition 2.4.

- We define the player's choice partitions by

$$C = \{\{e_1\}, \{e_2\}, \{e_3, e_5\}, \{e_4, e_6\}\} .$$

With this partition, we define the basic object (K-BO4) of Definition 2.4.

The four Axioms in Definition 2.4 are easily seen to be satisfied.

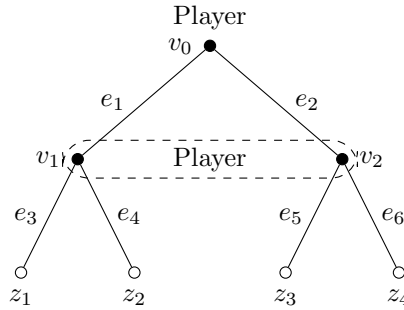


Figure 2.14: Forgetfulness in the K-model

The forgetfulness of the player is cyphered in the information structure. As vertices in the information set $\{v_1, v_2\}$ are indistinguishable for the player, and their parent vertex, the root v_0 , forms the other (singleton) information set, then the edges (v_0, v_1) and (v_0, v_2) , which represent the actions made in the information set $\{v_0\}$, are indistinguishable for the player as well. Thus, the player in the second information set $\{v_1, v_2\}$ has no memory of the (past) action made in the information set $\{v_0\}$.

To draw a parallel with the main storyline, we name the player's choices as follows

$$\{e_1\} = T , \quad \{e_2\} = B , \quad \{e_3, e_5\} = L , \quad \{e_4, e_6\} = R .$$

2.5.2.2 Forgetfulness in the AFR-model

To model forgetfulness in the AFR-model, we take the simplest case of two players playing simultaneously and having two actions each. Seen as one player, the following exposition will explain why this united player has no memory of the past actions.

The following is the list of the primitives of an AFR-model.

- *Set of plays*

$$\mathbb{W} = \{TL, TR, BL, BR\} .$$

- *Set of vertices*

$$V = \left\{ \underbrace{\{TL, TR, BL, BR\}}_{\text{root}}, \underbrace{\{TL, TR\}, \{BL, BR\}}_{\text{non-root moves}}, \underbrace{\{TL\}, \{TR\}, \{BL\}, \{BR\}}_{\text{leaves}} \right\} \subset 2^{\mathbb{W}} .$$

- *Choices of the player*

$$\begin{aligned} C &= \{c_T, c_B, c_L, c_R\}, \text{ where} \\ c_T &= \{TL, TR\}, c_B = \{BL, BR\}, \\ c_L &= \{BL, TL\}, c_R = \{TR, BR\} . \end{aligned}$$

These choices define the following information sets of the player:

$$P(c_T) = P(c_B) = \{\{TL, TR, BL, BR\}\} \subset V , \quad P(c_L) = P(c_R) = \{\{TL, TR\}, \{BL, BR\}\} \subset V .$$

The two Axioms in Definition 2.15 are easily seen to be satisfied.

Forgetfulness of the player is cyphered in the system of choices, namely, in the information sets the choices are assigned to. In this model, the player has two information sets $\{\{TL, TR, BL, BR\}\}$ and $\{\{TL, TR\}, \{BL, BR\}\}$. When choosing between $c_T = \{TL, TR\}$ and $c_B = \{BL, BR\}$ that are assigned to the information set $\{\{TL, TR, BL, BR\}\}$, by doing so, the player decides, which outcomes to discard: if the choice c_T is selected, then the discarded outcomes are BL and BR that constitute the other choice c_B assigned to the same information set $\{\{TL, TR, BL, BR\}\}$, and vice versa. Suppose, the choice c_T is taken. When it comes to choosing between $c_L = \{BL, TL\}$ and $c_R = \{TR, BR\}$ assigned to the other information set $\{\{TL, TR\}, \{BL, BR\}\}$, we notice that each of them contains one outcome that was supposed to be discarded upon making the choice c_T at the previous step. Thus, at the second information set $\{\{TL, TR\}, \{BL, BR\}\}$, the player does not remember the choice that was made at the first information set $\{\{TL, TR, BL, BR\}\}$.

2.5.2.3 Forgetfulness in the W-model

In the W-formalism, the simplest example of a forgetful player can be given by two agents a and b representing the player and having two possible actions each, thus, the underlying configuration space consists of four elements.

As in §2.4.5.1, the agents have no information about each other's decisions. The fact that the agents belong to the same player can be represented by them having the same criterion.

Stated formally, the control sets of the agents a (Alice) and b (Bob) are

$$\mathbb{U}_a = \{u_T, u_B\}, \quad \mathbb{U}_b = \{u_L, u_R\}. \quad (2.45)$$

With the control sets \mathbb{U}_a and \mathbb{U}_b we define the basic object (W-BO1) in Definition 2.20.

Thus, the configuration space consists of four elements

$$\mathbb{H} = \{u_T, u_B\} \times \{u_L, u_R\}, \quad (2.46)$$

and no information about each other's decisions (see Figure 2.7)

$$\mathcal{J}_a = \mathcal{J}_b = \{\emptyset, \mathbb{U}_a\} \otimes \{\emptyset, \mathbb{U}_b\}.$$

With the information fields \mathcal{J}_a and \mathcal{J}_b we define the basic object (W-BO2) in Definition 2.20.

The forgetfulness in this framework can be interpreted as, say, Bob playing second after Alice playing first, without knowing the outcome of her action. Once the two agents belong to the same player, the player can be seen as a forgetful one, once the second agent of the player has no information about the action made by the first.

2.6 Conclusion

The AFR-model is a powerful generalization of the K-model, obtained by extending the notion of tree to possibly uncountable or continuous decision-makers and actions. Both the K- and AFR-models require basic objects and axioms that are rather technical.

The W-model is not an extension of the AFR-model as it does not rely on a tree structure. The basic objects display a precise and concise description and the axiomatics is limited and transparent — except with the notion of causality, which is technical, and close to the notion of tree. By contrast with the AFR-model, the W-model is defined with a finite number of agents — whereas the AFR-model was originally developed to handle countable and partly uncountable and continuous decision-makers (like in continuous, stochastic and differential games).

Now, in Chapters 3 and 4, we will discuss connections between the three models we have presented.

Chapter 3

From W-model to AFR-model

3.1 Introduction

In this Chapter 3, we show that a causal W-model with complete π -fields, as described in §2.4.2, can be embedded into an AFR-model, as described in §2.3.3. For this purpose, we will stick to the following roadmap.

- In Section 3.2, we give general results about two tree structures in partial orderings, build the set of plays on which all subsequent constructed objects are supported, a set of so-called WtoAFR-vertices supported on this set of plays. We prove that the set of WtoAFR-vertices is ordered by set inclusion, thus, possesses a tree structure, which yields the so-called WtoAFR-tree, the first primitive of an AFR-model,
- In Section 3.3, we build the collection of so-called WtoAFR-choices, which is the second primitive of an AFR-model, and prove that, for the constructed primitives, the two AFR-axioms hold true,
- In Section 3.4, we construct AFR-strategies, as defined in §2.3.4, from W-strategies, as defined in Section 2.4.3,
- In Section 3.5, we state and prove technical results used in the previous sections.

In the end of this introductory Section 3.1, we state the summarizing Theorem 3.1, which is the main result of this Chapter 3.

Theorem 3.1. *Let be given*

- *a set \mathbb{A} of agents, each agent $a \in \mathbb{A}$ having a set of actions \mathbb{U}_a equipped with complete partition field \mathcal{U}_a and Nature taking actions in the sample space Ω equipped with complete partition field \mathcal{F} ,*
- *an information π -field $\mathcal{J}_a \subset \mathcal{H}$ for each agent $a \in \mathbb{A}$,*
- *a causal configuration ordering $\varphi : \mathbb{H} \rightarrow \Sigma$ as in (2.37).*

For any partition $\mathbb{A} = \bigsqcup_{i \in \mathcal{P}} \mathbb{A}^i$ of the agents into players, we construct an AFR-model as in Definition 2.15 with

- the set \mathcal{P} of AFR-players,
- the set $\mathbb{W} = \mathbb{H}$ of plays,
- the WtoAFR-tree (V, \supset) , where the set V of WtoAFR-vertices is defined in (3.20d),
- the collection $C^i = \{C_a\}_{a \in \mathbb{A}^i}$ of WtoAFR-choices of the player $i \in \mathcal{P}$, where for any representative agent $a \in \mathbb{A}^i$ of the player i , the corresponding partition C_a of WtoAFR-choices is as in (3.29).

The constructed objects satisfy (AFR-Axiom1') and (AFR-Axiom2') stated in Definition 2.16.

Proof. The theorem is a consequence of Propositions 3.14 and 3.18. □

3.2 WtoAFR-tree

In this Section 3.2, we give some general results about tree structures in partial orderings, namely,

- in §3.2.1, we introduce an order relation \succeq on the set Σ of orderings (from integers towards agents), and we prove that the poset (Σ, \succeq) is a tree,
- in §3.2.2, we prove that the $\Sigma^{|\mathbb{A}|}$ -poset $(\{\Sigma_\kappa\}_{\kappa \in \Sigma \setminus \Sigma^{|\mathbb{A}|}}, \supset)$ is a tree,
- in §3.2.3, we build the set of plays on which all subsequent constructed objects are supported and the set of so-called WtoAFR-vertices supported on this set of plays. We prove that the set of WtoAFR-vertices is ordered by set inclusion, thus, possesses a tree structure, which yields the so-called WtoAFR-tree, the first primitive of an AFR-model.

Some of the results proved in this Section 3.2 rely on technical results postponed to Section 3.5.

3.2.1 The tree of partial orderings

We introduce a relation on the set Σ of partial orderings, defined in (2.33c). For any two partial orderings $\kappa, \kappa' \in \Sigma$,

$$\kappa' \succeq \kappa \iff \begin{cases} |\kappa'| \leq |\kappa| \\ \psi_{|\kappa'|}(\kappa) = \kappa' \end{cases} \quad (3.1)$$

where the cardinal $|\kappa|$ is as in (2.33d) and the restriction mapping ψ_κ is as in (2.33g) applied for $k = |\kappa|$. Thus, $\kappa' \succeq \kappa$ means that κ' is the restriction of κ to the first $|\kappa'|$ integers. The choice of notation may seem counterintuitive, but the logic is as follows: the “shorter” an ordering is, the “bigger” it is. Indeed, the “shortest” empty ordering $\{(\emptyset)\}$ is then defined as the “biggest” root vertex. Similarly, the set of the “longest” total orderings $\Sigma^{|\mathbb{A}|}$ forms the set of the “smallest” vertices, that is, leaves of the poset (Σ, \succeq) , which is proved to be a tree in Proposition 3.2 .

Note that, $\kappa^- \succeq \kappa$, for any nonempty ordering $\kappa \in \Sigma \setminus \{(\emptyset)\}$. Indeed, for any such nonempty ordering κ , we have $\kappa^- = \psi_{|\kappa|-1}(\kappa) = \psi_{|\kappa^-|}(\kappa)$, where the first elements κ^- are as in (2.33j). Combined with the fact that $|\kappa^-| \leq |\kappa|$ and the Equivalence 3.1, we obtain that

$$\kappa^- \succeq \kappa, \quad \forall \kappa \in \Sigma \setminus \{(\emptyset)\}. \quad (3.2)$$

In the following Proposition 3.2, we show that (Σ, \succeq) is a poset and, moreover, possesses a tree structure.

Proposition 3.2. *The set Σ of orderings as in (2.33c), supplied with the binary relation \succeq defined in (3.1), is*

- a poset (Σ, \succeq) , that is, the binary relation \succeq is reflexive, antisymmetric and transitive,
- a tree according to Definition 2.7, that is, for any vertex $\kappa \in \Sigma$, the upset

$$\uparrow \kappa = \{\kappa' \in \Sigma \mid \kappa' \succeq \kappa\} \quad (3.3)$$

as in (2.10), is a chain.

Moreover, the image of any nonempty ordering $\kappa \in \Sigma \setminus \{(\emptyset)\}$ under the parent mapping p , as in (2.11b), is $\kappa^- \in \Sigma^{|\kappa|-1}$, that is,

$$p(\kappa) = \kappa^-, \quad \forall \kappa \in \Sigma \setminus \{(\emptyset)\}. \quad (3.4)$$

Proof. The proof is in three points.

► First, we prove that the set Σ supplied with the binary relation \succeq is a poset.

Reflexivity. We prove that for any $\kappa \in \Sigma$, the relation $\kappa \succeq \kappa$ holds true.

$$\begin{aligned} \psi_{|\kappa|}(\kappa) &= \kappa_{\{1, \dots, |\kappa|\}} \\ \text{(by definition of the restriction mapping } \psi_k, \text{ as in (2.33g) applied for } k = |\kappa|) \\ &= \kappa \\ \text{(by projection on the first components as in (2.33f) applied for } \ell = |\kappa|) \end{aligned}$$

Thus, by definition of the relation \succeq as in (3.1), $\kappa \succeq \kappa$.

Antisymmetry. We prove that, for any $\kappa, \kappa' \in \Sigma$, such that $\kappa \neq \kappa'$ and that $\kappa \succeq \kappa'$, then necessarily we have that $\kappa' \not\preceq \kappa$.

The proof is by contradiction. For this purpose, we suppose that $\kappa' \succeq \kappa$ holds true together with $\kappa \succeq \kappa'$ and $\kappa \neq \kappa'$. From (3.1), we have simultaneously that $|\kappa'| \leq |\kappa|$ and $|\kappa| \leq |\kappa'|$, thus, $|\kappa'| = |\kappa|$. Then, we write

$$\begin{aligned} \kappa' &= \psi_{|\kappa'|}(\kappa) && \text{(by definition of the relation } \succeq \text{ as in (3.1), as } \kappa' \succeq \kappa) \\ &= \psi_{|\kappa|}(\kappa) && \text{(as } |\kappa'| = |\kappa|) \\ &= \kappa, && \text{(by projection on the first } \ell \text{ components as in (2.33f) applied for } \ell = |\kappa|) \end{aligned}$$

which contradicts the assumption $\kappa \neq \kappa'$.

Transitivity. We consider three orderings $\kappa, \kappa', \kappa'' \in \Sigma$, such that $\kappa'' \succeq \kappa'$ and $\kappa' \succeq \kappa$. We prove that $\kappa'' \succeq \kappa$.

By definition of the relation \succeq as in (3.1), we have $|\kappa''| \leq |\kappa'| \leq |\kappa|$, thus, $|\kappa''| \leq |\kappa|$. Then,

$$\begin{aligned} \kappa'' &= \psi_{|\kappa''|}(\kappa') && \text{(by definition of the relation } \succeq \text{ as in (3.1), as } \kappa'' \succeq \kappa') \\ &= \psi_{|\kappa''|}(\psi_{|\kappa'|}(\kappa)) && \text{(by definition of the relation } \succeq \text{ as in (3.1), as } \kappa' \succeq \kappa) \\ &= \psi_{|\kappa''|}(\kappa). && \text{(by Lemma 3.27 applied for } k'' = |\kappa''| \text{ and } k' = |\kappa'|) \end{aligned}$$

► Second, we prove that the poset (Σ, \succeq) is a tree. To this end, we show that, for any element $\kappa \in \Sigma$, the upset $\uparrow \kappa$, as defined in (3.3), is a chain. For this purpose, we consider two orderings $\kappa', \kappa'' \in \uparrow \kappa$ and we prove that either $\kappa' \succeq \kappa''$ or $\kappa'' \succeq \kappa'$. We assume without loss of generality that $|\kappa''| \leq |\kappa'|$.

As $\kappa', \kappa'' \in \uparrow \kappa = \{\kappa''' \in \Sigma \mid \kappa''' \succeq \kappa\}$ by definition (3.3), and as $\kappa''' \succeq \kappa \implies |\kappa''| \leq |\kappa|$ by definition (3.1), then $|\kappa''| \leq |\kappa'| \leq |\kappa|$ and we have that

$$\begin{aligned} \kappa'' &= \psi_{|\kappa''|}(\kappa) && \text{(by definition of the binary relation } \succeq \text{ as in (3.1), as } \kappa'' \succeq \kappa) \\ &\succeq \psi_{|\kappa'|}(\kappa) && \text{(by (3.47c) in Lemma 3.27 applied for } k'' = |\kappa''| \text{ and } k' = |\kappa'|) \\ &= \kappa'. && \text{(by definition of the binary relation } \succeq \text{ in (3.1), as } \kappa' \succeq \kappa) \end{aligned}$$

► Third, for any $\kappa \in \Sigma \setminus \{\emptyset\}$, we prove Equation (3.4). We write

$$\begin{aligned} p(\kappa) &= \min(\uparrow \kappa \setminus \{\kappa\}) && \text{(by definition of the parent mapping } p \text{ as in (2.11b))} \\ &= \min(\{\kappa' \in \Sigma \mid \kappa' \succeq \kappa\} \setminus \{\kappa\}) && \text{(by definition of the upset } \uparrow \kappa \text{ as in (3.3))} \\ &= \min\{\kappa' \in \Sigma \mid \kappa' \succeq \kappa^-\} \end{aligned}$$

as $\kappa' \succeq \kappa \iff \kappa' = \kappa$ or $\kappa' \succeq \kappa^-$ by (3.7), to be found in the postponed Lemma 3.4

$$= \kappa^-. \quad \text{(as } \kappa^- \in \{\kappa' \in \Sigma \mid \kappa' \succeq \kappa^-\} \text{ and } \kappa^- \text{ is a lower bound)}$$

This ends the proof. □

The Figure 3.1 is a tree representation of the poset (Σ, \succeq) for the set $\mathbb{A} = \{a, b, c\}$ of agents. The empty ordering $\Sigma^0 = (\emptyset)$ as in (2.33b) is the root of the tree. Focusing on the lower part of Figure 3.1, we can observe that any total ordering in Σ^3 is the only child vertex of the corresponding parent. In fact, we will now establish in Corollary 3.3 that all trees (Σ, \succeq) have the property that the set of only children coincides with the set of total orderings. In other words, the parents of total orderings are the only parents to have an only child; informally speaking, total orderings are the only vertices that have no “sibling” vertices. This property plays its role in the sequel.

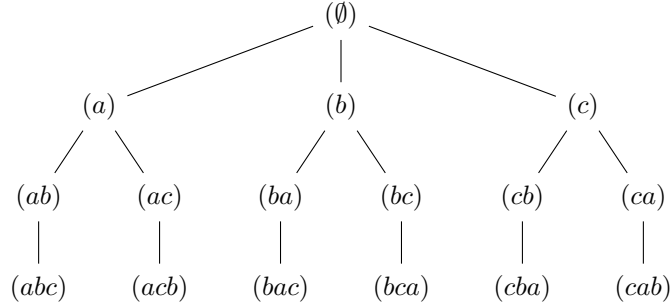


Figure 3.1: Example of a tree (Σ, \succeq) of orderings in a three agents case

Corollary 3.3. *The poset $(\Sigma \setminus \Sigma^{|\mathbb{A}|}, \succeq)$ is a tree.*

Moreover, for any nonempty ordering $\kappa \in \Sigma \setminus \{(\emptyset)\}$, the set $p^{-1}(\kappa^-)$ of children vertices (as in (2.11c)) of κ^- is a singleton iff $\kappa \in \Sigma^{|\mathbb{A}|}$, that is,

$$p^{-1}(\kappa^-) = \{\kappa\} \iff \kappa \in \Sigma^{|\mathbb{A}|}. \quad (3.5)$$

Proof. The set $(\Sigma \setminus \Sigma^{|\mathbb{A}|}, \succeq)$ is deduced from the tree (Σ, \succeq) by removing the leaves. It is readily checked that the resulting set is a poset and also a tree.

• In order to prove Equivalence (3.5), we establish yet another useful relation: for any nontotal ordering $\kappa \in \Sigma \setminus \Sigma^{|\mathbb{A}|}$, the set of children vertices, as in (2.11c), is given by

$$p^{-1}(\kappa) = \{(\kappa, a) \mid a \in \mathbb{A} \setminus \|\kappa\|\} \subset \Sigma^{|\kappa|+1}. \quad (3.6)$$

By Equation (3.4), we have that $\kappa' \in p^{-1}(\kappa) \implies \kappa'^- = p(\kappa') = \kappa$. We get that $|\kappa'| = |\kappa| + 1$ by definition (2.33j) of κ'^- . Thus, we have obtained the inclusion $p^{-1}(\kappa) \subset \Sigma^{|\kappa|+1}$, for any partial ordering $\kappa \in \Sigma \setminus \Sigma^{|\mathbb{A}|}$. Then, to prove the equality in Equation (3.6), we now show two inclusions.

On the one hand, we show that $\{(\kappa, a) \mid a \in \mathbb{A} \setminus \|\kappa\|\} \subset p^{-1}(\kappa)$. For any agent $a \in \mathbb{A} \setminus \|\kappa\|$, the corresponding ordering (κ, a) is a mapping $(\kappa, a) : \{1, \dots, |\kappa| + 1\} \rightarrow \mathbb{A}$, which is an injection and hence belongs to $\Sigma^{|\kappa|+1}$. Moreover, we have

$$\begin{aligned} p((\kappa, a)) &= (\kappa, a)^- && \text{(by (3.4))} \\ &= \kappa. && \text{(by (2.33l))} \end{aligned}$$

Thus, we have obtained that $\{(\kappa, a) \mid a \in \mathbb{A} \setminus \|\kappa\|\} \subset p^{-1}(\kappa)$.

On the other hand, we show that $p^{-1}(\kappa) \subset \{(\kappa, a) \mid a \in \mathbb{A} \setminus \|\kappa\|\}$. We write

$$p^{-1}(\kappa) \subset \{\kappa' \in \Sigma^{|\kappa|+1} \mid p(\kappa') = \kappa\}$$

by definition of the set $p^{-1}(\kappa)$ of children vertices as in (2.11c) and the fact that, for any nontotal ordering $\kappa \in \Sigma \setminus \Sigma^{|\mathbb{A}|}$, $p^{-1}(\kappa) \subset \Sigma^{|\kappa|+1}$

$$= \{\kappa' \in \Sigma^{|\kappa|+1} \mid \kappa'^- = \kappa\}$$

as, for any $\kappa' \in \Sigma^{|\kappa|+1}$, we have $\kappa' \in \Sigma \setminus \{(\emptyset)\}$, and thus Equation (3.4) gives that $p(\kappa') = \kappa'^-$

$$= \{(\kappa, a) \in \Sigma^{|\kappa|+1} \mid (\kappa, a) : \{1, \dots, |\kappa| + 1\} \rightarrow \mathbb{A} \text{ is an injection}\}$$

(by definition of the set Σ^k of k -orderings as in (2.33a) applied for $k = |\kappa| + 1$)

$$= \{(\kappa, a) \in \Sigma^{|\kappa|+1} \mid a \in \mathbb{A} \setminus \|\kappa\|\},$$

(as, if $a \in \|\kappa\|$, then the mapping $(\kappa, a) : \{1, \dots, |\kappa| + 1\} \rightarrow \mathbb{A}$ is not an injection)

which proves the inverse inclusion and, thus, the equality in Equation (3.6).

• Now, we prove Equivalence (3.5).

(\Leftarrow) We consider a total ordering $\kappa \in \Sigma^{|\mathbb{A}|}$. Then $\kappa^- \in \Sigma \setminus \Sigma^{|\mathbb{A}|}$ and using (3.6), we obtain that

$$p^{-1}(\kappa^-) = \{(\kappa^-, a) \mid a \in \mathbb{A} \setminus \|\kappa^-\|\}.$$

As $\kappa^- \in \Sigma^{|\mathbb{A}|-1}$ by (2.33j), then $\mathbb{A} \setminus \|\kappa^-\|$ is reduced to a singleton. Therefore, $p^{-1}(\kappa^-)$ is also reduced to a singleton and since $\kappa^* \in \mathbb{A} \setminus \|\kappa^-\|$ we obtain the equality $p^{-1}(\kappa^-) = \{(\kappa^-, \kappa^*)\}$. We therefore obtain that $p^{-1}(\kappa^-) = \{\kappa\}$.

(\Rightarrow) Assume that $p^{-1}(\kappa^-) = \{\kappa\}$ for some $\kappa \in \Sigma \setminus \{(\emptyset)\}$. Then, $\kappa^- \in \Sigma \setminus \Sigma^{|\mathbb{A}|}$ and by (3.6), we have

$$\{(\kappa^-, a) \mid a \in \mathbb{A} \setminus \|\kappa^-\|\} = \{\kappa\}.$$

Thus, $\mathbb{A} \setminus \|\kappa^-\|$ is a singleton, which is only possible when $|\kappa^-| = |\mathbb{A}| - 1$, that is, when $|\kappa| = |\mathbb{A}|$. Thus, we have that $\kappa \in \Sigma^{|\mathbb{A}|}$.

This ends the proof. \square

We finish this §3.2.1 by establishing an, already used and extensively used in the sequel, equivalence between any two orderings related through the binary relation \succeq .

Lemma 3.4. *For any two partial orderings $\kappa, \kappa' \in \Sigma$, we have the following equivalence*

$$\kappa' \succeq \kappa \iff \kappa' = \kappa \text{ or } (\kappa' \succeq \kappa^- \text{ and } \kappa \neq \{(\emptyset)\}). \quad (3.7)$$

Proof. We consider two partial orderings $\kappa, \kappa' \in \Sigma$. We prove the two implications of the Equivalence (3.7).

► (\Leftarrow) We suppose that $\kappa' = \kappa$ or that $(\kappa' \succeq \kappa^- \text{ and } \kappa \neq \{(\emptyset)\})$, and we consider the two possible cases one by one. If $\kappa' = \kappa$, we trivially have that $\kappa' \succeq \kappa' = \kappa$ by reflexivity of the

relation \succeq established in Proposition 3.2. If $\kappa' \succeq \kappa^-$ and $\kappa \neq \{(\emptyset)\}$, we get that $\kappa' \succeq \kappa$, by transitivity of the relation \succeq , established in Proposition 3.2, because $\kappa^- \succeq \kappa$ by (3.2) since $\kappa \in \Sigma \setminus \{(\emptyset)\}$.

► (\Rightarrow) We suppose that $\kappa' \succeq \kappa$. Then, we get that $|\kappa'| \leq |\kappa|$, by definition of the relation \succeq as in (3.1), and we consider the two possible cases $|\kappa'| = |\kappa|$ or $|\kappa'| < |\kappa|$ one by one. If we suppose that $|\kappa'| = |\kappa|$, then we have that

$$\begin{aligned} \kappa' &= \psi_{|\kappa'|}(\kappa) && \text{(by definition of the relation } \succeq \text{ as in (3.1))} \\ &= \psi_{|\kappa|}(\kappa) && \text{(as } |\kappa'| = |\kappa| \text{ by assumption)} \\ &= \kappa, \end{aligned}$$

and we obtain that $\kappa' = \kappa$. If we suppose that $|\kappa'| \leq |\kappa| - 1$, then we have that $\kappa \neq \{(\emptyset)\}$ since $|\kappa| \geq 1$, and we have that

$$\begin{aligned} \kappa' &= \psi_{|\kappa'|}(\kappa) && \text{(by definition of the relation } \succeq \text{ as in (3.1))} \\ &\succeq \psi_{|\kappa|-1}(\kappa) && \text{(by postponed (3.47c) applied for } k' = |\kappa| - 1 \text{ and } k'' = |\kappa'|) \\ &= \kappa^-, && \text{(by (2.33j))} \end{aligned}$$

and we get that $\kappa' \succeq \kappa^-$.

This ends the proof. \square

3.2.2 The $\Sigma^{|\mathbb{A}|}$ -poset tree of total orderings

For any partial ordering $\kappa \in \Sigma$, we define the following subset $\Sigma_\kappa \subset \Sigma^{|\mathbb{A}|}$ of total orderings coinciding with κ on the first $\{1, \dots, |\kappa|\}$ elements by¹

$$\Sigma_\kappa = \{\rho \in \Sigma^{|\mathbb{A}|} \mid \psi_{|\kappa|}(\rho) = \kappa\} \subset \Sigma^{|\mathbb{A}|}, \quad \forall \kappa \in \Sigma. \quad (3.8)$$

The main result of this §3.2.2 is the following Proposition 3.5.

Proposition 3.5. *The mapping*

$$\Sigma \setminus \Sigma^{|\mathbb{A}|} \rightarrow \{\Sigma_\kappa\}_{\kappa \in \Sigma \setminus \Sigma^{|\mathbb{A}|}}, \quad \Sigma \setminus \Sigma^{|\mathbb{A}|} \ni \kappa \mapsto \Sigma_\kappa \subset \Sigma^{|\mathbb{A}|}, \quad (3.9)$$

where the subset Σ_κ of total orderings is defined in (3.8), is an order isomorphism between the poset $(\Sigma \setminus \Sigma^{|\mathbb{A}|}, \succeq)$ and the $\Sigma^{|\mathbb{A}|}$ -poset $(\{\Sigma_\kappa\}_{\kappa \in \Sigma \setminus \Sigma^{|\mathbb{A}|}}, \supset)$.

As a consequence, the $\Sigma^{|\mathbb{A}|}$ -poset $(\{\Sigma_\kappa\}_{\kappa \in \Sigma \setminus \Sigma^{|\mathbb{A}|}}, \supset)$ is a tree and, for any nonempty and nontotal ordering $\kappa \in \Sigma \setminus \{(\emptyset) \cup \Sigma^{|\mathbb{A}|}\}$, the parent of the corresponding vertex Σ_κ is given by the vertex Σ_{κ^-} , that is,

$$p(\Sigma_\kappa) = \Sigma_{\kappa^-}, \quad \forall \kappa \in \Sigma \setminus \{(\emptyset) \cup \Sigma^{|\mathbb{A}|}\}. \quad (3.10)$$

¹Notice the lower index for orderings, and the upper index for cardinals.

Proof. The proof is in three steps.

► First, we show that the mapping defined in (3.9) is an order morphism between the poset $(\Sigma \setminus \Sigma^{|\mathbb{A}|}, \succeq)$ and the $\Sigma^{|\mathbb{A}|}$ -poset $(\{\Sigma_\kappa\}_{\kappa \in \Sigma \setminus \Sigma^{|\mathbb{A}|}}, \supset)$, that is, that we have

$$\left(\forall \kappa, \kappa' \in \Sigma \setminus \Sigma^{|\mathbb{A}|} \right) \quad \kappa' \succeq \kappa \iff \Sigma_{\kappa'} \supset \Sigma_\kappa. \quad (3.11)$$

We prove the two implications of the Equivalence (3.11).

(\Rightarrow) We consider two nontotal orderings $\kappa, \kappa' \in \Sigma \setminus \Sigma^{|\mathbb{A}|}$, such that $\kappa' \succeq \kappa$, and we show that $\Sigma_{\kappa'} \supset \Sigma_\kappa$. For any total ordering $\rho \in \Sigma_\kappa$, we have that

$$\psi_{|\kappa'|}(\rho) = \psi_{|\kappa'|}(\psi_{|\kappa|}(\rho))$$

by (3.47b) since $\kappa' \succeq \kappa \implies |\kappa'| \leq |\kappa|$, by definition of the relation \succeq as in (3.1)

$$\begin{aligned} &= \psi_{|\kappa'|}(\kappa) && \text{(as } \psi_{|\kappa|}(\rho) = \kappa \text{ by (3.8) since } \rho \in \Sigma_\kappa) \\ &= \kappa'. && \text{(as } \kappa' \succeq \kappa \text{ by definition of the relation } \succeq \text{ in (3.1))} \end{aligned}$$

Thus, we have obtained that $\rho \in \Sigma_{\kappa'}$, as $\psi_{|\kappa'|}(\rho) = \kappa'$. Hence the inclusion $\Sigma_{\kappa'} \supset \Sigma_\kappa$ is proved.

(\Leftarrow) We consider two nontotal orderings $\kappa, \kappa' \in \Sigma \setminus \Sigma^{|\mathbb{A}|}$ such that $\Sigma_{\kappa'} \supset \Sigma_\kappa$ and we show that $\kappa' \succeq \kappa$.

From $\Sigma_{\kappa'} \supset \Sigma_\kappa$, we deduce that $|\Sigma_{\kappa'}| = (|\mathbb{A}| - |\kappa'|)! \geq |\Sigma_\kappa| = (|\mathbb{A}| - |\kappa|)!$, hence that $|\kappa'| \leq |\kappa|$. Then, for any total ordering $\rho \in \Sigma_\kappa \subset \Sigma^{|\mathbb{A}|}$, we have that

$$\begin{aligned} \psi_{|\kappa'|}(\kappa) &= \psi_{|\kappa'|}(\psi_{|\kappa|}(\rho)) && \text{(as } \kappa = \psi_{|\kappa|}(\rho) \text{ by (3.8) since } \rho \in \Sigma_\kappa) \\ &= \psi_{|\kappa'|}(\rho) && \text{(by (3.47b) since } |\kappa'| \leq |\kappa|) \\ &= \kappa' \end{aligned}$$

because $\rho \in \Sigma_\kappa$ and $\Sigma_{\kappa'} \supset \Sigma_\kappa$ imply that $\rho \in \Sigma_{\kappa'}$, hence that $\psi_{|\kappa'|}(\rho) = \kappa'$ by (3.8). As we had $|\kappa| \geq |\kappa'|$, and as we have obtained that $\psi_{|\kappa'|}(\kappa) = \kappa'$, we conclude that $\kappa \succeq \kappa'$ by definition of the relation \succeq in (3.1).

► Second, we show that the mapping (3.9) is a bijection. As it is surjective by definition, there remains to show that it is injective. Now, if $\Sigma_{\kappa'} = \Sigma_\kappa$, we deduce from (3.11) that both $\kappa' \succeq \kappa$ and $\kappa \succeq \kappa'$, hence that $\kappa' = \kappa$ by antisymmetry of the relation \succeq .

► Third, we prove the two remaining points. As, by Corollary 3.3, the poset $(\Sigma \setminus \Sigma^{|\mathbb{A}|}, \succeq)$ is a tree, its image by the order isomorphism (3.9) is a tree as an order isomorphism preserves chains. As a consequence, the $\Sigma^{|\mathbb{A}|}$ -poset $(\{\Sigma_\kappa\}_{\kappa \in \Sigma \setminus \Sigma^{|\mathbb{A}|}}, \supset)$ is a tree. Equation (3.10) is a straightforward consequence of (3.4) thanks to the order isomorphism (3.9). Indeed, as the minimum of a finite set is preserved under an order isomorphism, then the minimum operation commutes with the order isomorphism, which gives the result.

This ends the proof. \square

We give the following useful Corollary 3.6.

Corollary 3.6. *For any two partial orderings $\kappa, \kappa' \in \Sigma$, such that $|\kappa'| \leq |\kappa|$, we have the following equivalence*

$$\Sigma_\kappa = \Sigma_{\kappa'} \iff \kappa = \kappa' \text{ or } (\kappa \in \Sigma^{|\mathbb{A}|} \text{ and } \kappa' = \kappa^-). \quad (3.12)$$

Proof. We prove the two implications of the Equivalence (3.12). We consider $\kappa, \kappa' \in \Sigma$ such that $|\kappa'| \leq |\kappa|$.

(\Leftarrow) Suppose that $\kappa = \kappa'$ or $(\kappa \in \Sigma^{|\mathbb{A}|}$ and $\kappa' = \kappa^-)$.

If $\kappa = \kappa'$, we obviously get that $\Sigma_\kappa = \Sigma_{\kappa'}$.

It is readily observed that, if $\kappa \in \Sigma^{|\mathbb{A}|}$, we have that $\Sigma_\kappa = \Sigma_{\kappa^-}$, because the last value $\kappa(|\kappa|)$ is uniquely determined by κ^- (with $\{\kappa(|\kappa|)\} = \mathbb{A} \setminus \|\kappa^-\|$). Therefore, we get that $\kappa \in \Sigma^{|\mathbb{A}|}$ and $\kappa' = \kappa^-$ jointly imply that $\Sigma_\kappa = \Sigma_{\kappa'}$.

(\Rightarrow) Suppose that $\Sigma_\kappa = \Sigma_{\kappa'}$. We consider three cases, namely, $(\kappa \in \Sigma^{|\mathbb{A}|}$ and $\kappa' = \kappa^-)$ or $(\kappa \in \Sigma^{|\mathbb{A}|}$ and $\kappa' = \kappa)$ or $(\kappa \notin \Sigma^{|\mathbb{A}|}$ and $\kappa' = \kappa)$.

If $\kappa \notin \Sigma^{|\mathbb{A}|}$, then $\kappa' \notin \Sigma^{|\mathbb{A}|}$ since $|\kappa'| \leq |\kappa|$. As, by Proposition 3.5, the mapping $\Sigma \setminus \Sigma^{|\mathbb{A}|} \ni \kappa \mapsto \Sigma_\kappa$ is bijective, we deduce from $\Sigma_\kappa = \Sigma_{\kappa'}$ that $\kappa = \kappa'$.

If $\kappa \in \Sigma^{|\mathbb{A}|}$, we have that $\Sigma_\kappa = \Sigma_{\kappa^-}$, as seen just above. There are two subcases.

If $\kappa' \in \Sigma^{|\mathbb{A}|}$, we have that $\Sigma_{\kappa'} = \Sigma_{\kappa'^-}$, and therefore that $\Sigma_{\kappa^-} = \Sigma_\kappa = \Sigma_{\kappa'} = \Sigma_{\kappa'^-}$. Now, as $\kappa^- \notin \Sigma^{|\mathbb{A}|}$ and $\kappa'^- \notin \Sigma^{|\mathbb{A}|}$, we deduce that $\kappa^- = \kappa'^-$, as shown above. Since $\kappa \in \Sigma^{|\mathbb{A}|}$ and $\kappa' \in \Sigma^{|\mathbb{A}|}$, we obtain that $\kappa = \kappa'$.

If $\kappa' \notin \Sigma^{|\mathbb{A}|}$, we have that $\Sigma_{\kappa^-} = \Sigma_\kappa = \Sigma_{\kappa'}$. Now, as $\kappa^- \notin \Sigma^{|\mathbb{A}|}$ and $\kappa' \notin \Sigma^{|\mathbb{A}|}$, we deduce that $\kappa^- = \kappa'$, as shown above.

This ends the proof. \square

In Figure (3.2), we give an example of a Σ^3 -poset $(\{\Sigma_\kappa\}_{\kappa \in \Sigma \setminus \Sigma^3}, \supset)$, which is the image of the poset in Figure (3.1) under the order isomorphism (3.9), supplied with the elements $\{\Sigma_\rho\}_{\rho \in \Sigma^3}$, where

$$\Sigma^3 = \{(abc), (acb), (bac), (bca), (cba), (cab)\}.$$

Notice that $\Sigma_{(ac)} = \Sigma_{(acb)} = (acb)$, by (3.12), and analogous relations hold for the other five total orderings in Σ^3 . Hence, we observe that, for any total ordering $\rho \in \Sigma^3$, the corresponding set Σ_ρ coincides with the set Σ_{ρ^-} , and there are no other vertices with such property by Corollary 3.6. That is, had we added the singletons $\{\Sigma_\rho\}_{\rho \in \Sigma^3}$ as potential leaf vertices, they would have duplicated some in the tree constructed in Proposition 3.5.

Finally, we prove the following useful implication that will be used for the proof of Proposition 3.14.

Lemma 3.7. *Let a causal ordering $\varphi : \mathbb{H} \rightarrow \Sigma^{|\mathbb{A}|}$ as in (2.37) be given. For any two orderings $\kappa, \kappa' \in \Sigma$, we have the following implication*

$$(\forall \kappa, \kappa' \in \Sigma) \quad \kappa' \succeq \kappa \Rightarrow \mathbb{H}_{\kappa'}^\varphi \supset \mathbb{H}_\kappa^\varphi. \quad (3.13)$$

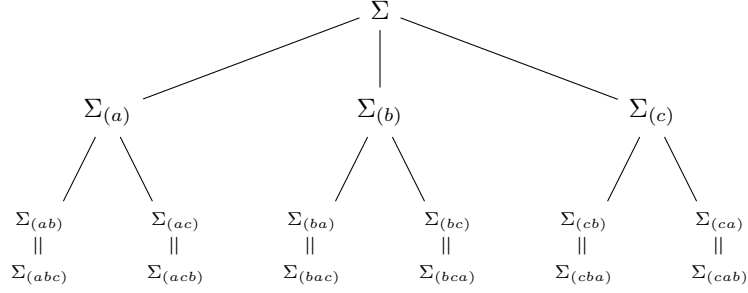


Figure 3.2: Σ^3 -poset $(\{\Sigma_\kappa\}_{\kappa \in \Sigma \setminus \Sigma^3}, \supseteq)$, which is the image in leaves of the poset in Figure (3.1)

| tree | Σ -poset and $\Sigma^{ \mathbb{A} }$ -poset “image in leaves” | | WtoAFR-tree |
|--------|--|---|---|
| | (Σ, \succeq) | $(\{\Sigma_\kappa\}_{\kappa \in \Sigma}, \supseteq)$ | (V, \supseteq) |
| moves | $\{\kappa \mid \kappa \in \Sigma \setminus \Sigma^{ \mathbb{A} }\}$ | $\{\Sigma_\kappa \mid \kappa \in \Sigma^{(\geq \mathbb{A} -2)}\}$ | $\{x_{\kappa, h_{\kappa^-}} \mid \kappa \in \Sigma\}$ |
| leaves | $\{\rho \mid \rho \in \Sigma^{ \mathbb{A} }\}$ | $\{\{\kappa\} \mid \kappa \in \Sigma^{ \mathbb{A} -1}\}$ | $\{\{h\} \mid h \in \mathbb{H}\}$ |
| upset | $\uparrow \kappa = \{\kappa' \in \Sigma \mid \kappa' \succeq \kappa\}$ | $\uparrow \Sigma_\kappa = \{\Sigma_{\kappa'} \mid \kappa' \succeq \kappa\}$ | (3.25), (3.26) |
| parent | $p(\kappa) = \kappa^-$ | $p(\Sigma_\kappa) = \Sigma_{\kappa^-}$ | (3.27) |

Table 3.1: Comparison table between three trees

Proof. We consider two orderings $\kappa, \kappa' \in \Sigma$, such that $\kappa' \succeq \kappa$, and we show that $\mathbb{H}_{\kappa'}^\varphi \supset \mathbb{H}_\kappa^\varphi$. For any configuration $h \in \mathbb{H}_\kappa^\varphi$, we have that

$$\psi_{|\kappa'|}(\varphi(h)) = \psi_{|\kappa'|}(\psi_{|\kappa|}(\varphi(h)))$$

by (3.47b) since $\kappa' \succeq \kappa \implies |\kappa'| \leq |\kappa|$, by definition of the relation \succeq as in (3.1)

$$\begin{aligned} &= \psi_{|\kappa'|}(\kappa) && \text{(as } \psi_{|\kappa|}(\varphi(h)) = \kappa \text{ by (2.34) since } h \in \mathbb{H}_\kappa^\varphi) \\ &= \kappa'. && \text{(as } \kappa' \succeq \kappa \text{ by definition of the relation } \succeq \text{ in (3.1))} \end{aligned}$$

Thus, for every $h \in \mathbb{H}_\kappa^\varphi$, we have that $\psi_{|\kappa'|}(\varphi(h)) = \kappa'$, which implies that $h \in \mathbb{H}_{\kappa'}^\varphi$, hence the inclusion $\mathbb{H}_{\kappa'}^\varphi \supset \mathbb{H}_\kappa^\varphi$ is proved. \square

3.2.3 Constructing WtoAFR-tree

Besides the two trees (Σ, \succeq) and $(\{\Sigma_\kappa\}_{\kappa \in \Sigma \setminus \Sigma^{|\mathbb{A}|}}, \supseteq)$ discussed in §3.2.1 and §3.2.2 respectively, there is yet another poset closely related to these two, namely, the \mathbb{H} -poset $(\{\mathbb{H}_\kappa^\varphi\}_{\kappa \in \Sigma^\varphi}, \supseteq)$, where the set $\Sigma^\varphi \subset \Sigma$ is defined in (2.35c). We prove in Lemma 3.7 one useful implication for the \mathbb{H} -poset $(\{\mathbb{H}_\kappa^\varphi\}_{\kappa \in \Sigma^\varphi}, \supseteq)$. In this §3.2.3, we are going to construct the WtoAFR-tree as a refinement of the \mathbb{H} -poset $(\{\mathbb{H}_\kappa^\varphi\}_{\kappa \in \Sigma^\varphi}, \supseteq)$ by intersecting it with the so-called vertex partition. In Table 3.1, we draw a comparison between these trees.

We call *set of WtoAFR-plays* the underlying set of plays for the WtoAFR-model given by

$$\mathbb{W} = \mathbb{H} . \quad (3.14)$$

From now on, in this Chapter 3, we will work with the set \mathbb{H} of configurations instead of the set \mathbb{W} of plays. Stating the main results requires a bit of notations.

First, we introduce the following list of notations that will be extensively used in the sequel when speaking about moves and constructing the WtoAFR-tree.

Definition 3.8. *For any nonempty subset $\mathbb{B} \subset \mathbb{A}$ of agent, the projection $\text{proj}_{\mathbb{B}}$ from the product configuration set $\mathbb{H} = \Omega \times \bar{\mathbb{U}}_{\mathbb{B}} \times \bar{\mathbb{U}}_{-\mathbb{B}}$ to the product set $\bar{\mathbb{U}}_{\mathbb{B}}$ of actions of agents in \mathbb{B} is defined by*

$$\text{proj}_{\mathbb{B}} : \mathbb{H} \rightarrow \bar{\mathbb{U}}_{\mathbb{B}} \quad (3.15a)$$

$$h = (\omega, \{u_b\}_{b \in \mathbb{A}}) \mapsto h_{\mathbb{B}} = u_{\mathbb{B}} = \{u_b\}_{b \in \mathbb{B}} . \quad (3.15b)$$

Then, for the empty ordering (\emptyset) we define the projection by

$$\text{proj}_{(\emptyset)} : \mathbb{H} \rightarrow \Omega \quad (3.16)$$

$$h = (\omega, \{u_b\}_{b \in \mathbb{A}}) \mapsto h_{(\emptyset)} = \omega \in \Omega .$$

Then, for any nonempty partial ordering $\kappa \in \Sigma \setminus \{(\emptyset)\}$, the projection proj_{κ} from the product configuration set $\mathbb{H} = \Omega \times \bar{\mathbb{U}}_{\|\kappa\|} \times \bar{\mathbb{U}}_{-\|\kappa\|}$ to the product set $\Omega \times \bar{\mathbb{U}}_{\|\kappa\|}$ of Nature with the actions of agents in $\|\kappa\|$ is defined by

$$\text{proj}_{\kappa} : \mathbb{H} \rightarrow \Omega \times \bar{\mathbb{U}}_{\|\kappa\|} \quad (3.17a)$$

$$h = (\omega, \{u_b\}_{b \in \mathbb{A}}) \mapsto h_{\kappa} = \left(\underbrace{\text{proj}_{(\emptyset)}(h)}_{\omega}, \underbrace{\text{proj}_{\|\kappa\|}(h)}_{u_{\|\kappa\|} = \{u_b\}_{b \in \|\kappa\|}} \right) \in \Omega \times \bar{\mathbb{U}}_{\|\kappa\|} , \quad (3.17b)$$

where the range $\|\kappa\|$ is as in (2.33h).

In the sequel, we will also make use of the following auxiliary coordinate mapping. For any two orderings $\kappa, \kappa' \in \Sigma$, such that $\kappa' \succeq \kappa$ as in Equation (3.1), the following projection is well-defined:

$$\begin{aligned} \text{proj}_{\|\kappa\| \setminus \|\kappa'\|} : \mathbb{H} &\rightarrow \bar{\mathbb{U}}_{\|\kappa\| \setminus \|\kappa'\|} \\ h &\mapsto h_{\|\kappa\| \setminus \|\kappa'\|} . \end{aligned}$$

Indeed, as $\kappa' \succeq \kappa$, we immediately get that $\|\kappa'\| \subset \|\kappa\|$ and $\text{proj}_{\|\kappa\| \setminus \|\kappa'\|}$ is the projection defined in Equation (3.15) for $\mathbb{B} = \|\kappa\| \setminus \|\kappa'\|$.

For any configuration $h \in \mathbb{H}$, we have

$$\kappa' \succeq \kappa \Rightarrow h_{\kappa} = \left(\begin{array}{c} h_{\kappa'} \\ \cap \\ \Omega \times \bar{\mathbb{U}}_{\|\kappa\|} \end{array} , \begin{array}{c} h_{\|\kappa\| \setminus \|\kappa'\|} \\ \cap \\ \Omega \times \bar{\mathbb{U}}_{\|\kappa'\|} \\ \cap \\ \bar{\mathbb{U}}_{\|\kappa\| \setminus \|\kappa'\|} \end{array} \right) \quad (3.18)$$

where the longer κ -prefix $h_\kappa = (\omega, u_{\|\kappa\|}) \in \Omega \times \overline{\mathbb{U}}_{\|\kappa\|}$ has been factorized with the shorter κ' -prefix $h_{\kappa'} = (\omega, u_{\|\kappa'\|}) \in \Omega \times \overline{\mathbb{U}}_{\|\kappa'\|}$. By an abuse of notation, we will also denote by $\text{proj}_{\kappa'}$ any of the following projection mappings, when $\kappa' \succeq \kappa$:

$$\text{proj}_{\kappa'} : \Omega \times \overline{\mathbb{U}}_{\|\kappa\|} \rightarrow \Omega \times \overline{\mathbb{U}}_{\|\kappa'\|} \quad (3.19a)$$

$$(\omega, \{u_b\}_{b \in \|\kappa\|}) \mapsto (\omega, \{u_b\}_{b \in \|\kappa'\|}) . \quad (3.19b)$$

Now, we are ready to define the candidate set of WtoAFR-vertices.

Definition 3.9. *Let a causal configuration ordering $\varphi : \mathbb{H} \rightarrow \Sigma^{|\mathbb{A}|}$ be given as in Definition 2.24. For any nonempty ordering $\kappa \in \Sigma \setminus \{(\emptyset)\}$ and any² $\mathfrak{h}_{\kappa^-} \in \Omega \times \overline{\mathbb{U}}_{\|\kappa^-\|}$, as in (3.17), we define,*

$$x_{\kappa, \mathfrak{h}_{\kappa^-}} = \mathbb{H}_\kappa^\varphi \cap (\{\mathfrak{h}_{\kappa^-}\} \times \overline{\mathbb{U}}_{-\|\kappa^-\|}) \subset \mathbb{H} , \quad (3.20a)$$

where the set $\mathbb{H}_\kappa^\varphi$ of configurations is as in (2.34). Then, we define

$$X = \{x_{\kappa, \mathfrak{h}_{\kappa^-}} \in 2^\mathbb{H} \mid x_{\kappa, \mathfrak{h}_{\kappa^-}} = \mathbb{H}_\kappa^\varphi \cap (\{\mathfrak{h}_{\kappa^-}\} \times \overline{\mathbb{U}}_{-\|\kappa^-\|}) , \text{ s.t. } x_{\kappa, \mathfrak{h}_{\kappa^-}} \neq \emptyset\} , \quad (3.20b)$$

and

$$Z = \{\{h\} \mid h \in \mathbb{H}\} . \quad (3.20c)$$

Finally, define

$$V = X \cup Z \cup \{\mathbb{H}\} . \quad (3.20d)$$

First of all, in the following Lemmas 3.10 and 3.11, we give simple representations of any element $x_{\kappa, \mathfrak{h}_{\kappa^-}}$ of the set X defined in (3.20b).

Lemma 3.10. *Under the assumptions of Theorem 3.1, any element $x_{\kappa, \mathfrak{h}_{\kappa^-}} \in X$, where the set X is defined in (3.20b), is a cylinder of the form*

$$x_{\kappa, \mathfrak{h}_{\kappa^-}} = (\overline{\mathbb{H}}_\kappa^\varphi \cap \{\mathfrak{h}_{\kappa^-}\}) \times \overline{\mathbb{U}}_{-\|\kappa^-\|} , \quad (3.21)$$

with $\overline{\mathbb{H}}_\kappa^\varphi \cap \{\mathfrak{h}_{\kappa^-}\}$ nonempty, where the set $\overline{\mathbb{H}}_\kappa^\varphi$ is given by (2.38a).

Proof. Let $x_{\kappa, \mathfrak{h}_{\kappa^-}}$ as defined Equation (3.20a) be given. We write

$$\begin{aligned} x_{\kappa, \mathfrak{h}_{\kappa^-}} &= \mathbb{H}_\kappa^\varphi \cap (\{\mathfrak{h}_{\kappa^-}\} \times \overline{\mathbb{U}}_{-\|\kappa^-\|}) \quad (\text{by definition of the element } x_{\kappa, \mathfrak{h}_{\kappa^-}} \text{ as in (3.20a)}) \\ &= (\overline{\mathbb{H}}_\kappa^\varphi \times \overline{\mathbb{U}}_{-\|\kappa^-\|}) \cap (\{\mathfrak{h}_{\kappa^-}\} \times \overline{\mathbb{U}}_{-\|\kappa^-\|}) \\ &\quad (\text{under causality, as in (2.37), the set } \mathbb{H}_\kappa^\varphi \text{ is a cylinder by (2.38a)}) \\ &= (\overline{\mathbb{H}}_\kappa^\varphi \cap \{\mathfrak{h}_{\kappa^-}\}) \times \overline{\mathbb{U}}_{-\|\kappa^-\|} . \end{aligned}$$

Since the elements of X satisfy Equation (3.20a), Equation (3.21) is valid for any $x_{\kappa, \mathfrak{h}_{\kappa^-}} \in X$. This ends the proof. \square

²We do not use the notation h_{κ^-} , because the latter will be defined in (3.17) as the projection of a configuration $h \in \mathbb{H}$.

Proof. For any element $x_{\kappa, \mathfrak{h}_{\kappa^-}} \in X$, we write

$$\begin{aligned} x_{\kappa, \mathfrak{h}_{\kappa^-}} &= \mathbb{H}_{\kappa}^{\varphi} \cap (\{\mathfrak{h}_{\kappa^-}\} \times \overline{\mathbb{U}}_{-\|\kappa^-\|}) \quad (\text{by definition of the element } x_{\kappa, \mathfrak{h}_{\kappa^-}} \text{ as in (3.20a)}) \\ &= (\overline{\mathbb{H}}_{\kappa}^{\varphi} \times \overline{\mathbb{U}}_{-\|\kappa^-\|}) \cap (\{\mathfrak{h}_{\kappa^-}\} \times \overline{\mathbb{U}}_{-\|\kappa^-\|}) \\ &\quad (\text{under causality, as in (2.37), the set } \mathbb{H}_{\kappa}^{\varphi} \text{ is a cylinder by (2.38a)}) \\ &= (\overline{\mathbb{H}}_{\kappa}^{\varphi} \cap \{\mathfrak{h}_{\kappa^-}\}) \times \overline{\mathbb{U}}_{-\|\kappa^-\|}. \end{aligned}$$

Since the elements of X satisfy Equation (3.20a), then Equation (3.21) is valid for any $x_{\kappa, \mathfrak{h}_{\kappa^-}} \in X$. As any element of X is nonempty by (3.20b), then $x_{\kappa, \mathfrak{h}_{\kappa^-}} \neq \emptyset$, which implies that $\overline{\mathbb{H}}_{\kappa}^{\varphi} \cap \mathfrak{h}_{\kappa^-}$ is also nonempty. This ends the proof. \square

Lemma 3.11. *For any $x_{\kappa, \mathfrak{h}_{\kappa^-}} \in X$, we have that*

$$x_{\kappa, \mathfrak{h}_{\kappa^-}} = \{h \in \mathbb{H} \mid \kappa \succeq \varphi(h) \text{ and } \mathfrak{h}_{\kappa^-} = h_{\kappa^-}\}. \quad (3.22)$$

Proof.

• (C). Let $x_{\kappa, \mathfrak{h}_{\kappa^-}} \in X$ and $h \in x_{\kappa, \mathfrak{h}_{\kappa^-}}$ be given, noting that $x_{\kappa, \mathfrak{h}_{\kappa^-}} \neq \emptyset$ when $x_{\kappa, \mathfrak{h}_{\kappa^-}} \in X$. By definition of the WtoAFR-move $x_{\kappa, \mathfrak{h}_{\kappa^-}}$, as in (3.20a), we have that

$$x_{\kappa, \mathfrak{h}_{\kappa^-}} = \mathbb{H}_{\kappa}^{\varphi} \cap (\{\mathfrak{h}_{\kappa^-}\} \times \overline{\mathbb{U}}_{-\|\kappa^-\|}). \quad (3.23)$$

Thus, we have that $h \in \mathbb{H}_{\kappa}^{\varphi}$ and by definition of the set $\mathbb{H}_{\kappa}^{\varphi}$ of configurations, as in (2.34), we have that $\psi_{|\kappa|} \circ \varphi(h) = \kappa$, which implies the relation $\kappa \succeq \varphi(h)$ by definition of the order \succeq , as in (3.1). Now, using again Equation (3.23), we also have that $h \in \{\mathfrak{h}_{\kappa^-}\} \times \overline{\mathbb{U}}_{-\|\kappa^-\|}$ which immediately gives the second restriction $\mathfrak{h}_{\kappa^-} = h_{\kappa^-}$.

• (D). Let be given $\kappa \in \Sigma \setminus \{(\emptyset)\}$, $\mathfrak{h}_{\kappa^-} \in \Omega \times \overline{\mathbb{U}}_{\|\kappa^-\|}$ and $h \in \mathbb{H}$ such that $\kappa \succeq \varphi(h)$ and $\mathfrak{h}_{\kappa^-} = h_{\kappa^-}$.

Using the fact that $\kappa \in \Sigma \setminus \{(\emptyset)\}$, and that $\kappa \succeq \varphi(h)$, we have $\kappa^- \succeq \kappa$ by (3.2) and, thus, $\kappa^- \succeq \varphi(h)$ by transitivity of the relation \succeq established in Proposition 3.2. Now, we write

$$\begin{aligned} \{h\} &= \mathbb{H}_{\varphi(h)}^{\varphi} \cap \{h\} && (\text{as, } h \in \mathbb{H}_{\varphi(h)}^{\varphi}, \text{ by definition of } \mathbb{H}_{\kappa}^{\varphi} \text{ in (2.34)}) \\ &= \mathbb{H}_{\kappa}^{\varphi} \cap \{h\} && (\text{as } \kappa \succeq \varphi(h) \text{ implies } \mathbb{H}_{\kappa}^{\varphi} \supset \mathbb{H}_{\varphi(h)}^{\varphi} \text{ by (3.13)}) \\ &= \mathbb{H}_{\kappa}^{\varphi} \cap \{h_{\varphi(h)}\} && (\text{as } h = h_{\varphi(h)} \text{ for any } h \in \mathbb{H}) \\ &= \mathbb{H}_{\kappa}^{\varphi} \cap \{(h_{\kappa^-}, h_{\mathbb{A} \setminus \|\kappa^-\|})\} && (\text{by the factorization relation in (3.18) with } \kappa^- \succeq \varphi(h)) \\ &= \mathbb{H}_{\kappa}^{\varphi} \cap \{(\mathfrak{h}_{\kappa^-}, h_{\mathbb{A} \setminus \|\kappa^-\|})\} && (\text{using the assumption } \mathfrak{h}_{\kappa^-} = h_{\kappa^-}) \\ &\subset \mathbb{H}_{\kappa}^{\varphi} \cap (\{\mathfrak{h}_{\kappa^-}\} \times \overline{\mathbb{U}}_{-\|\kappa^-\|}) && (\text{as } h_{\mathbb{A} \setminus \|\kappa^-\|} \in \overline{\mathbb{U}}_{-\|\kappa^-\|}) \\ &= x_{\kappa, \mathfrak{h}_{\kappa^-}} \end{aligned}$$

by definition of the WtoAFR-move $x_{\kappa, \mathfrak{h}_{\kappa^-}}$ as in (3.20a).

Thus, the Equation (3.22) is proved. \square

In Lemma 3.12 we establish an equivalence between an inclusion between any two WtoAFR-moves and some technical condition on the orderings and prefixes that define them.

Lemma 3.12. *For any two WtoAFR-moves $x_{\kappa, \mathfrak{h}_{\kappa^-}}, x_{\kappa', \mathfrak{h}_{\kappa'^-}} \in X$, we have the following equivalence*

$$(\kappa' \succeq \kappa \text{ and } \mathfrak{h}_{\kappa'^-} = \text{proj}_{\kappa'^-}(\mathfrak{h}_{\kappa^-})) \iff x_{\kappa', \mathfrak{h}_{\kappa'^-}} \supset x_{\kappa, \mathfrak{h}_{\kappa^-}} .$$

Proof.

• (\Rightarrow). Let be given $\kappa' \succeq \kappa$ and $\mathfrak{h}_{\kappa^-} \in \Omega \times \overline{\mathbb{U}}_{\|\kappa^-\|}$. Let $\mathfrak{h}_{\kappa'^-} \in \Omega \times \overline{\mathbb{U}}_{\|\kappa'^-\|}$ be given by $\mathfrak{h}_{\kappa'^-} = \text{proj}_{\kappa'^-}(\mathfrak{h}_{\kappa^-})$. We have that

$$\begin{aligned} x_{\kappa, \mathfrak{h}_{\kappa^-}} &= \mathbb{H}_{\kappa}^{\varphi} \cap (\{\mathfrak{h}_{\kappa^-}\} \times \overline{\mathbb{U}}_{-\|\kappa^-\|}) && \text{(using Equation (3.20a) defining } x_{\kappa, \mathfrak{h}_{\kappa^-}}) \\ &\subset \mathbb{H}_{\kappa'}^{\varphi} \cap (\{\mathfrak{h}_{\kappa^-}\} \times \overline{\mathbb{U}}_{-\|\kappa^-\|}) && \text{(since } \mathbb{H}_{\kappa'}^{\varphi} \supset \mathbb{H}_{\kappa}^{\varphi} \text{ when } \kappa' \succeq \kappa \text{ by (3.13))} \\ &= \mathbb{H}_{\kappa'}^{\varphi} \cap (\{(\text{proj}_{\kappa'^-}(\mathfrak{h}_{\kappa^-}), \mathfrak{h}_{\|\kappa^-\| \setminus \|\kappa'^-\|})\} \times \overline{\mathbb{U}}_{-\|\kappa^-\|}) && \text{(by definition of } \text{proj}_{\kappa'^-}) \\ &= \mathbb{H}_{\kappa'}^{\varphi} \cap (\{(\mathfrak{h}_{\kappa'^-}, \mathfrak{h}_{\|\kappa^-\| \setminus \|\kappa'^-\|})\} \times \overline{\mathbb{U}}_{-\|\kappa^-\|}) && \text{(as } \mathfrak{h}_{\kappa'^-} = \text{proj}_{\kappa'^-}(\mathfrak{h}_{\kappa^-})) \\ &= \mathbb{H}_{\kappa'}^{\varphi} \cap (\{\mathfrak{h}_{\kappa'^-}\} \times (\mathfrak{h}_{\|\kappa^-\| \setminus \|\kappa'^-\|} \times \overline{\mathbb{U}}_{-\|\kappa^-\|})) \\ &\subset \mathbb{H}_{\kappa'}^{\varphi} \cap (\{\mathfrak{h}_{\kappa'^-}\} \times \overline{\mathbb{U}}_{-\|\kappa'^-\|}) && \text{(as } (\mathfrak{h}_{\|\kappa^-\| \setminus \|\kappa'^-\|} \times \overline{\mathbb{U}}_{-\|\kappa^-\|}) \subset \overline{\mathbb{U}}_{-\|\kappa'^-\|}) \\ &= x_{\kappa', \mathfrak{h}_{\kappa'^-}} . \end{aligned}$$

• (\Leftarrow). Let $x_{\kappa, \mathfrak{h}_{\kappa^-}}, x_{\kappa', \mathfrak{h}_{\kappa'^-}} \in X$ by given with $x_{\kappa, \mathfrak{h}_{\kappa^-}} \subset x_{\kappa', \mathfrak{h}_{\kappa'^-}}$. Consider $h \in x_{\kappa, \mathfrak{h}_{\kappa^-}}$, then we also have that $h \in x_{\kappa', \mathfrak{h}_{\kappa'^-}}$ and by using the node caraterization given by Equation (3.22) we have that both $\kappa \succeq \varphi(h)$ and $\kappa' \succeq \varphi(h)$. Thus, by definition of an upset as in (3.3), we deduce that $\kappa, \kappa' \in \uparrow\{\varphi(h)\}$. As (Σ, \succeq) is a tree by Proposition 3.2, the upset $\uparrow\{\varphi(h)\}$ is a chain and we obtain that whether $\kappa' \succeq \kappa$ or $\kappa \succeq \kappa'$.

First, we show that $\kappa \succeq \kappa'$ implies that $\kappa = \kappa'$ and $\mathfrak{h}_{\kappa^-} = \mathfrak{h}_{\kappa'^-}$. We successively have that

$$\begin{aligned} \text{proj}_{\kappa^-} \mathfrak{h}_{\kappa'^-} &= \text{proj}_{\kappa^-} h_{\kappa'^-} && \text{(we have that } \mathfrak{h}_{\kappa'^-} = h_{\kappa'^-} \text{ by (3.22) as } h \in x_{\kappa', \mathfrak{h}_{\kappa'^-}}) \\ &= \text{proj}_{\kappa^-} (h_{\kappa^-}, h_{\|\kappa^-\| \setminus \|\kappa'^-\|}) && \text{(by factorization (3.18) as } \kappa \succeq \kappa') \\ &= h_{\kappa^-} && \text{(by (3.19))} \\ &= \mathfrak{h}_{\kappa^-} . && \text{(we have that } \mathfrak{h}_{\kappa^-} = h_{\kappa^-} \text{ by (3.22) as } h \in x_{\kappa, \mathfrak{h}_{\kappa^-}}) \end{aligned}$$

Then, using the first part of the proof (\Rightarrow), we obtain that $x_{\kappa, \mathfrak{h}_{\kappa^-}} \supset x_{\kappa', \mathfrak{h}_{\kappa'^-}}$. We therefore have that $x_{\kappa, \mathfrak{h}_{\kappa^-}} = x_{\kappa', \mathfrak{h}_{\kappa'^-}}$, which implies that $\kappa = \kappa'$ and $\mathfrak{h}_{\kappa^-} = \mathfrak{h}_{\kappa'^-}$ by (3.24). Indeed, since $\kappa = \kappa'$, we have therefore obtained that $\text{proj}_{\kappa'^-}(\mathfrak{h}_{\kappa^-}) = \text{proj}_{\kappa^-}(\mathfrak{h}_{\kappa^-}) = \mathfrak{h}_{\kappa^-} = \mathfrak{h}_{\kappa'^-}$ which gives the reverse implication in this first case.

Second, if $\kappa' \succeq \kappa$, then $\kappa'^- \succeq \kappa^-$ and then, for the κ^- -prefix \mathfrak{h}_{κ^-} , the projection $\text{proj}_{\kappa'^-} \mathfrak{h}_{\kappa^-}$

as in (3.19) is well-defined. We have that

$$\begin{aligned}
\text{proj}_{\kappa'^-} \mathfrak{h}_{\kappa^-} &= \text{proj}_{\kappa'^-} h_{\kappa^-} && \text{(we have that } \mathfrak{h}_{\kappa^-} = h_{\kappa^-} \text{ by (3.22) as } h \in x_{\kappa, \mathfrak{h}_{\kappa^-}} \text{)} \\
&= \text{proj}_{\kappa'^-} (h_{\kappa'^-}, h_{\|\kappa'^-\| \setminus \|\kappa^-\|}) && \text{(by factorization (3.18))} \\
&= h_{\kappa'^-} && \text{(by (3.19))} \\
&= \mathfrak{h}_{\kappa'^-}, && \text{(we have that } \mathfrak{h}_{\kappa'^-} = h_{\kappa'^-} \text{ by (3.22) as } h \in x_{\kappa', \mathfrak{h}_{\kappa'^-}} \text{)}
\end{aligned}$$

which proves that $\mathfrak{h}_{\kappa'^-} = \text{proj}_{\kappa'^-}(\mathfrak{h}_{\kappa^-})$. Thus, the reverse implication is proved in this second case.

This concludes the proof. \square

As an immediate corollary of Lemma 3.12 we give the following result.

Corollary 3.13. *For any two WtoAFR-moves $x_{\kappa, \mathfrak{h}_{\kappa^-}}, x_{\kappa', \mathfrak{h}_{\kappa'^-}} \in X$, we have the following equivalence*

$$x_{\kappa, \mathfrak{h}_{\kappa^-}} = x_{\kappa', \mathfrak{h}_{\kappa'^-}} \iff (\kappa = \kappa' \text{ and } \mathfrak{h}_{\kappa^-} = \mathfrak{h}_{\kappa'^-}). \quad (3.24)$$

Proof. The Equivalence 3.24 follows immediately by applying Equivalence 3.12 twice, first, for $x_{\kappa, \mathfrak{h}_{\kappa^-}} \supset x_{\kappa', \mathfrak{h}_{\kappa'^-}}$ and, second, for $x_{\kappa, \mathfrak{h}_{\kappa^-}} \subset x_{\kappa', \mathfrak{h}_{\kappa'^-}}$. \square

In this Chapter 3, we construct a WtoAFR-model thanks to causality and the configuration ordering that it provides. The property of causality comes into play twice, first, when we build a tree and, second, when we speak of choices and information of the WtoAFR-model.

In Proposition 3.14, we prove that the \mathbb{H} -poset (V, \supset) is a tree.

Proposition 3.14. *Under the assumptions of Theorem 3.1, the \mathbb{H} -poset (V, \supset) is a tree, where the set V of so-called WtoAFR-vertices is defined in (3.20d) with the root vertex $\{\mathbb{H}\}$, the set X of so-called WtoAFR-moves as in (3.20a), and the set Z of so-called WtoAFR-leaves as in (3.20c). We call the \mathbb{H} -poset (V, \supset) the WtoAFR-tree.*

Proof. In order to prove that (V, \supset) is a tree, we show that the upset of any candidate WtoAFR-vertex, as defined in (3.20d), is a chain. Since the upset of the root vertex is indeed a chain reduced to one vertex, it remains to consider the WtoAFR-moves and the WtoAFR-leaves.

► We consider first the case of WtoAFR-moves. Let an ordering $\kappa \in \Sigma \setminus \{(\emptyset)\}$ and a (κ^-) -prefix $\mathfrak{h}_{\kappa^-} \in \Omega \times \mathbb{U}_{\kappa^-}$ be given such that $x_{\kappa, \mathfrak{h}_{\kappa^-}} \neq \emptyset$ giving a move $x_{\kappa, \mathfrak{h}_{\kappa^-}} \in X$. Using the definition of the upset of a vertex in (2.10) we have that

$$\uparrow x_{\kappa, \mathfrak{h}_{\kappa^-}} = \{x_{\kappa', \mathfrak{h}_{\kappa'^-}} \in X \mid x_{\kappa', \mathfrak{h}_{\kappa'^-}} \supset x_{\kappa, \mathfrak{h}_{\kappa^-}}\} \cup \{\mathbb{H}\}.$$

Which gives, using Lemma 3.12 that

$$\uparrow x_{\kappa, \mathfrak{h}_{\kappa^-}} = \{x_{\kappa', \mathfrak{h}_{\kappa'^-}} \in X \mid \kappa' \succeq \kappa \text{ and } \mathfrak{h}_{\kappa'^-} = \text{proj}_{\kappa'^-}(\mathfrak{h}_{\kappa^-})\} \cup \{\mathbb{H}\}. \quad (3.25)$$

Now, using Equation (3.25), we prove that the upset of any WtoAFR-move is a chain. To this end, we consider two moves $x_{\kappa', \mathfrak{h}_{\kappa'-}}, x_{\kappa'', \mathfrak{h}_{\kappa''-}} \in \uparrow x_{\kappa, \mathfrak{h}_{\kappa-}}$ and we show that either $x_{\kappa', \mathfrak{h}_{\kappa'-}} \supset x_{\kappa'', \mathfrak{h}_{\kappa''-}}$ or $x_{\kappa', \mathfrak{h}_{\kappa'-}} \subset x_{\kappa'', \mathfrak{h}_{\kappa''-}}$. If at least one of the two moves coincides with the root vertex $\{\mathbb{H}\}$, the desired inclusion follows immediately. Now, we suppose that the two moves are distinct from the root vertex $\{\mathbb{H}\}$. By the expression (3.25) of the upset $\uparrow x_{\kappa, \mathfrak{h}_{\kappa-}}$, we get that, on the one hand, $\kappa' \succeq \kappa$ and $\mathfrak{h}_{\kappa'-} = \text{proj}_{\kappa'-}(\mathfrak{h}_{\kappa-})$, and, on the other hand, $\kappa'' \succeq \kappa$ and $\mathfrak{h}_{\kappa''-} = \text{proj}_{\kappa''-}(\mathfrak{h}_{\kappa-})$. Thus, $\kappa', \kappa'' \in \uparrow \kappa$ by definition of the upset $\uparrow \kappa$ as in (3.3). As the upset $\uparrow \kappa$ is a chain by Proposition 3.2, we can assume, without loss of generality, that $\kappa' \succeq \kappa'' \succeq \kappa$ and, thus, $\mathfrak{h}_{\kappa'-} = \text{proj}_{\kappa'-}(\mathfrak{h}_{\kappa''-})$ by (3.19). Applying (3.12) for $\kappa = \kappa''$, we get that $\kappa' \succeq \kappa''$ and $\mathfrak{h}_{\kappa'-} = \text{proj}_{\kappa'-}(\mathfrak{h}_{\kappa''-})$ together imply the inclusion $x_{\kappa'', \mathfrak{h}_{\kappa''-}} \subset x_{\kappa', \mathfrak{h}_{\kappa'-}}$, which proves that the upset $\uparrow x_{\kappa, \mathfrak{h}_{\kappa-}}$ is a chain.

► Now, we consider the case WtoAFR-leaves. As a preliminary we prove that for any $h \in \mathbb{H}$ the upset of a WtoAFR-leaf is given by the following expression

$$\uparrow \{h\} = \{h\} \cup \uparrow x_{\varphi(h), \mathfrak{h}_{\varphi(h)-}}. \quad (3.26)$$

- (\supset). Using Lemma 3.11, we obtain that $\{h\} \subset x_{\varphi(h), \mathfrak{h}_{\varphi(h)-}}$ and therefore $\uparrow x_{\varphi(h), \mathfrak{h}_{\varphi(h)-}} \subset \uparrow \{h\}$ which added to the fact that we also have $\{h\} \subset \uparrow \{h\}$ and $\{\mathbb{H}\} \subset \uparrow \{h\}$ gives that $\{h\} \cup \uparrow x_{\varphi(h), \mathfrak{h}_{\varphi(h)-}} \cup \{\mathbb{H}\} \subset \uparrow \{h\}$.
- (\subset). By definition (3.20d) of $V = X \cup Z \cup \{\mathbb{H}\}$ and as, for any $h' \in \mathbb{H}$ such that $h' \neq h$, we have $\{h'\} \cap \{h\} = \emptyset$ we have that $\uparrow \{h\} \subset \{h\} \cup X' \cup \{\mathbb{H}\}$ where $X' \subset X$ is the subset of moves which contains $\{h\}$. Consider $x_{\kappa, \mathfrak{h}_{\kappa-}} \in X'$, then, using Equation (3.22) we must have that $\kappa \succeq \varphi(h)$ and $\mathfrak{h}_{\kappa-} = \mathfrak{h}_{\varphi(h)-}$. Thus, we have that $\mathfrak{h}_{\kappa-} = \mathfrak{h}_{\varphi(h)-} = \text{proj}_{\kappa-} h = \text{proj}_{\kappa-}(h_{\varphi(h)-})$, where the last equality comes from Equation (3.19) using the fact that $\kappa \succeq \varphi(h)$. We have therefore obtained that both $\kappa \succeq \varphi(h)$ and $\mathfrak{h}_{\kappa-} = \text{proj}_{\kappa-}(h_{\varphi(h)-})$ which implies that $x_{\kappa, \mathfrak{h}_{\kappa-}} \supset x_{\varphi(h), \mathfrak{h}_{\varphi(h)-}}$ and therefore $x_{\kappa, \mathfrak{h}_{\kappa-}} \in \uparrow x_{\varphi(h), \mathfrak{h}_{\varphi(h)-}}$. We conclude that $X' \subset \uparrow x_{\varphi(h), \mathfrak{h}_{\varphi(h)-}}$ and therefore $\uparrow \{h\} \subset \{h\} \cup \uparrow x_{\varphi(h), \mathfrak{h}_{\varphi(h)-}} \cup \{\mathbb{H}\} = \{h\} \cup \uparrow x_{\varphi(h), \mathfrak{h}_{\varphi(h)-}}$.

Now, using Equation (3.26), we conclude that $\uparrow \{h\}$ is a chain as we have already shown that $\uparrow x_{\varphi(h), \mathfrak{h}_{\varphi(h)-}}$ is a chain itself and $\{h\}$ belongs to every element of it by Lemma 3.12.

Thus, by showing that, for any WtoAFR-vertex $v \in V$ as in (3.20d), the corresponding upset is a chain, we have proved that (V, \supset) is a tree.

This finishes the proof. □

Finally, in Proposition 3.15, for every WtoAFR-vertex, we give its image under the parent mapping p defined in (2.11b).

Proposition 3.15. *Under the assumptions of Theorem 3.1, for any nonroot WtoAFR-vertex $v \in V \setminus \{\mathbb{H}\}$, where the set V of WtoAFR-vertices is as in (3.20d), the image $p(v)$*

under the parent mapping p as in (2.11b), is given by

$$p(v) = \begin{cases} \{\mathbb{H}\} , & \text{if } v = x_{(a), \mathfrak{h}(\emptyset)} \text{ for some } (a) \in \Sigma^1 , \mathfrak{h}(\emptyset) \in \Omega , \\ x_{\kappa^-, \mathfrak{h}_{(\kappa^-)^-}} , & \text{if } v = x_{\kappa, \mathfrak{h}_{\kappa^-}} \text{ for some } \kappa \in \Sigma^{(\geq 2)} , \mathfrak{h}_{\kappa^-} \in \Omega \times \overline{\mathbb{U}}_{\|\kappa^-\|} \\ & \text{and } \text{proj}_{(\kappa^-)^-}(\mathfrak{h}_{\kappa^-}) = \mathfrak{h}_{(\kappa^-)^-} , \\ x_{\varphi(h), \mathfrak{h}_{\varphi(h)^-}} , & \text{if } v = \{h\} \in Z \text{ for some } h \in \mathbb{H} \text{ and } \text{proj}_{\varphi(h)^-} h = \mathfrak{h}_{\varphi(h)^-} . \end{cases} \quad (3.27)$$

We set, by definition, $p(\{\mathbb{H}\}) = \{\mathbb{H}\}$ for the sake of completeness.

Proof. We prove Equation (3.27) giving, for any nonroot WtoAFR-vertex, the image under the parent mapping defined in (2.11b). We distinguish three cases, namely, WtoAFR-moves of the first level, the rest of the WtoAFR-moves and the WtoAFR-leaves.

- We prove the first case in Equation (3.27).

For any agent $a \in \mathbb{A}$ and any (\emptyset) -prefix $\mathfrak{h}(\emptyset) \in \Omega$, we write the upset of the corresponding move $x_{(a), \mathfrak{h}(\emptyset)} \in X$ as

$$\begin{aligned} \uparrow x_{(a), \mathfrak{h}(\emptyset)} &= \{x_{\kappa', \mathfrak{h}_{\kappa'^-}} \mid \kappa' \succeq (a) \text{ and } \text{proj}_{\kappa'^-}(\mathfrak{h}(\emptyset)) = \mathfrak{h}_{\kappa'^-}\} \cup \{\mathbb{H}\} \\ &\quad \text{(by the upset of any WtoAFR-move as (3.25))} \\ &= \{x_{(a), \mathfrak{h}(\emptyset)}\} \cup \{\mathbb{H}\} , \end{aligned}$$

as, by Lemma 3.4, $\kappa' \succeq (a)$ implies that $\kappa' = (a)$ or $\kappa' \succeq (a)^- = (\emptyset)$, and the latter implies that $\kappa' = (\emptyset)$ and then κ'^- is not defined (see Equation (2.33j)). From this, the image of move $x_{(a), \mathfrak{h}(\emptyset)}$ under the parent mapping p follows immediately:

$$p(x_{(a), \mathfrak{h}(\emptyset)}) = \min(\uparrow x_{(a), \mathfrak{h}(\emptyset)} \setminus \{x_{(a), \mathfrak{h}(\emptyset)}\}) = \min\{\mathbb{H}\} = \{\mathbb{H}\} .$$

- We prove the second case in Equation (3.27).

We consider a move $x_{\kappa, \mathfrak{h}_{\kappa^-}} \in X$ such that $\kappa \in \Sigma^{(\geq 2)}$ (thus, the existence of $(\kappa^-)^- \in \Sigma^{|\kappa|-2}$ is ensured). We have that

$$\begin{aligned} p(x_{\kappa, \mathfrak{h}_{\kappa^-}}) &= \min(\uparrow x_{\kappa, \mathfrak{h}_{\kappa^-}} \setminus \{x_{\kappa, \mathfrak{h}_{\kappa^-}}\}) \quad \text{(by definition of the parent mapping } p \text{ in (2.11b))} \\ &= \min(\{x_{\kappa', \mathfrak{h}_{\kappa'^-}} \mid \kappa' \succeq \kappa \text{ and } \text{proj}_{\kappa'^-}(\mathfrak{h}_{\kappa^-}) = \mathfrak{h}_{\kappa'^-}\} \cup \{\mathbb{H}\} \setminus \{x_{\kappa, \mathfrak{h}_{\kappa^-}}\}) \\ &\quad \text{(by (3.25) that provides an expression of the upset } \uparrow x_{\kappa, \mathfrak{h}_{\kappa^-}} \text{ of a WtoAFR-move } x_{\kappa, \mathfrak{h}_{\kappa^-}}) \\ &= \min(\{x_{\kappa', \mathfrak{h}_{\kappa'^-}} \mid \kappa' \succeq \kappa^- \text{ and } \text{proj}_{\kappa'^-}(\mathfrak{h}_{\kappa^-}) = \mathfrak{h}_{\kappa'^-}\} \cup \{\mathbb{H}\}) \\ &\quad \text{(by (3.7) in Lemma 3.4)} \\ &= x_{\kappa^-, \mathfrak{h}_{(\kappa^-)^-}} \cdot \text{(as } x_{\kappa^-, \mathfrak{h}_{(\kappa^-)^-}} \in \{x_{\kappa', \mathfrak{h}_{\kappa'^-}} \mid \kappa' \succeq \kappa^-\} \cup \{\mathbb{H}\} \text{ and is a lower bound)} \end{aligned}$$

- We prove the third case in Equation (3.27). We have

$$\begin{aligned}
p(\{h\}) &= \min(\uparrow\{h\} \setminus \{h\}) && \text{(by definition of the parent mapping as in (2.11b))} \\
&= \min \uparrow x_{\varphi(h), h_{\varphi(h)-}} && \text{(by the expression (3.26) of the upset } \uparrow\{h\}\text{)} \\
&= x_{\varphi(h), h_{\varphi(h)-}} \cdot && \text{(as } x_{\varphi(h), h_{\varphi(h)-}} \in \uparrow x_{\varphi(h), h_{\varphi(h)-}} \text{ and } x_{\varphi(h), h_{\varphi(h)-}} \text{ is a lower bound)}
\end{aligned}$$

This ends the proof. \square

3.3 WtoAFR-choices

The system of choices \mathcal{C} is one of the two basic objects in the AFR-model, as seen in Section 2.3. Here, we construct choices from the agents' action sets and information fields $\{\mathbb{U}_a, \mathcal{J}_a\}_{a \in \mathbb{A}}$ that are the basic objects of the W-model, as seen in Section 2.4. To this end,

- In §3.3.1, we define, for any agent a , the choice partition as the least upper bound of two relevant partitions,
- In §3.3.2, we give details about the atoms of the information and choice partitions,
- In §3.3.3, we prove that the constructed primitives satisfy (AFR-Axiom1),
- In §3.3.4, we prove that the constructed primitives satisfy (AFR-Axiom2').

3.3.1 Definition of the WtoAFR-choices

First, we introduce two equivalence relations on the set \mathbb{H} of configurations and the partitions that they induce. Second, we define choices as the intersection of the atoms of these partitions (any choice constructed in such way is a subset of plays).

Definition 3.16. *For any agent $a \in \mathbb{A}$, we call*

- *the information relation of the agent a the equivalence relation \mathfrak{I}_a on the set \mathbb{H} of configurations as in (3.14) defined by*

$$(\forall h, h' \in \mathbb{H}) \quad h \mathfrak{I}_a h' \iff \exists H \in \mathcal{J}_a \text{ such that } h, h' \in H. \quad (3.28a)$$

With the equivalence relation \mathfrak{I}_a as in (3.28a), we define the so-called information partition $\mathbb{H}/\mathfrak{I}_a$.

- *the action relation of the agent a the equivalence relation \mathfrak{A}_a on the set \mathbb{H} of configurations defined by*

$$(\forall h, h' \in \mathbb{H}) \quad h \mathfrak{A}_a h' \iff h_a = h'_a, \quad (3.28b)$$

where the projection from the set \mathbb{H} of configurations to the set \mathbb{U}_a of actions of the agent a is defined³ by

$$\begin{aligned} \text{proj}_a : \mathbb{H} &\rightarrow \mathbb{U}_a \\ h = (\omega, \{u_b\}_{b \in \mathbb{A}}) &\mapsto u_a . \end{aligned} \tag{3.28c}$$

With the equivalence relation \mathfrak{U}_a as in (3.28b), we define the so-called action partition $\mathbb{H}/\mathfrak{U}_a$.

Definition 3.17. For any agent $a \in \mathbb{A}$, we define the set of choices of the agent a by

$$C_a = \mathbb{H}/\mathfrak{J}_a \bigvee \mathbb{H}/\mathfrak{U}_a , \quad \forall a \in \mathbb{A} , \tag{3.29}$$

where the equivalence relations \mathfrak{J}_a and \mathfrak{U}_a are as in (3.28a) and (3.28b) respectively and the least upper bound of two partitions in (9.6). By definition, the set C_a of choices is a partition.

Note that, for any agent $a \in \mathbb{A}$, the corresponding choice partition C_a is the smallest partition among the ones that are finer (as defined in (9.5)) than both $\mathbb{H}/\mathfrak{J}_a$ and $\mathbb{H}/\mathfrak{U}_a$.

Proposition 3.18. For any partition $\mathbb{A} = \bigsqcup_{i \in \mathcal{P}} \mathbb{A}^i$ of the set of W -agents into players, the collection

$$C^i = \{C_a\}_{a \in \mathbb{A}^i} , \tag{3.30}$$

satisfies (AFR-Axiom1') and (AFR-Axiom2') stated in Definition 2.16, where, for every representative agent $a \in \mathbb{A}^i$ of the player i , the choice partition C_a is as in (3.29) and the WtoAFR-tree (V, \supset) is as in (3.20d). We call C^i the collection of WtoAFR-choices of the player i .

The postponed Proposition 3.24 establishes that the collection C^i of WtoAFR-choices satisfies (AFR-Axiom1'). The same goes with Proposition 3.25 and (AFR-Axiom2').

3.3.2 Details about the information and choice partitions

In this §3.3.2, we introduce atoms of the information and choice partitions in §3.3.2.1, define the auxiliary move equivalence and show that the WtoAFR-information atoms are made of WtoAFR-moves in §3.3.2.2, and, finally, deduce a closed form of any WtoAFR-information atom and choice in §3.3.2.3.

³projection proj_a is a particular case of the auxiliary coordinate mapping as in (3.15), namely, when $\mathbb{B} = \{a\}$ for some agent $a \in \mathbb{A}$. Here, we omit curly brackets of the singleton $\{a\}$ in the subscript in order to make notation less heavy

3.3.2.1 Atoms of the information and choice partitions

In order to prove that, for any player i equipped with the set \mathbb{A}^i of representative agents, the collection C^i of WtoAFR-choices of the player i , as in (3.30), satisfies both AFR-axioms, we need to detail what is an atom of the choice partition C_a introduced in (3.29) for any agent a . As, for any agent a , the choice partition C_a is defined as the least upper bound of partitions \mathbb{H}/\mathcal{I}_a and \mathbb{H}/\mathcal{U}_a , we first introduce the atoms of these two partitions.

For any agent $a \in \mathbb{A}$,

- the partition \mathbb{H}/\mathcal{I}_a induced by the information relation \mathcal{I}_a as in (3.28a) splits the set \mathbb{H} of configurations into subsets of configurations indistinguishable for the agent a , namely,

$$\mathbb{H}/\mathcal{I}_a = \{G \mid G \in \mathcal{I}_a\}, \quad (3.31)$$

A generic *information atom* $G \in \mathbb{H}/\mathcal{I}_a$ of this partition is a subset of configurations, that is, $G \subset \mathbb{H}$;

- the partition \mathbb{H}/\mathcal{U}_a induced by the action relation \mathcal{U}_a as in (3.28b) splits the set \mathbb{H} of configurations into atoms comprised of subsets of configurations with the same actions of W-agent a , namely,

$$\mathbb{H}/\mathcal{U}_a = \{[u_a] \mid u_a \in \mathbb{U}_a\}, \quad (3.32a)$$

where, for any $u_a \in \mathbb{U}_a$, a generic equivalence class $[u_a] \in \mathbb{H}/\mathcal{U}_a$ is given by

$$[u_a] = \Omega \times \{u_a\} \times \bar{\mathbb{U}}_{-a}. \quad (3.32b)$$

For any information atom $G \in \mathbb{H}/\mathcal{I}_a$ and any action $u_a \in \mathbb{U}_a$ of the agent a inducing the corresponding equivalence class $[u_a] \in \mathbb{H}/\mathcal{U}_a$ as in (3.32b), we define the corresponding generic element c_{G,u_a} by

$$c_{G,u_a} = G \cap [u_a]. \quad (3.33)$$

Lemma 3.19. *For any agent $a \in \mathbb{A}$, the set C_a of choices is given by*

$$C_a = \{c_{G,u_a} \mid G \in \mathbb{H}/\mathcal{I}_a, u_a \in \mathbb{U}_a\}. \quad (3.34a)$$

Moreover, the following mapping is an injection

$$\begin{aligned} \mathbf{i} : \mathbb{H}/\mathcal{I}_a \times \mathbb{H}/\mathcal{U}_a &\rightarrow \mathbb{H}/\mathcal{I}_a \bigvee \mathbb{H}/\mathcal{U}_a, \\ (G, u_a) &\mapsto c_{G,u_a} = G \cap [u_a]. \end{aligned} \quad (3.34b)$$

Proof. To prove (3.34a), we write

$$\begin{aligned}
C_a &= \mathbb{H}/\mathfrak{I}_a \bigvee \mathbb{H}/\mathfrak{U}_a, \\
&\text{(by definition of the set } C_a \text{ of the WtoAFR-choices of the agent } a \text{ as in (3.29))} \\
&= \{G \cap K \mid G \in \mathbb{H}/\mathfrak{I}_a, K \in \mathbb{H}/\mathfrak{U}_a\} \\
&\text{(by property of the least upper bound of two partitions in Equation (9.6))} \\
&= \{G \cap [u_a] \mid G \in \mathbb{H}/\mathfrak{I}_a, u_a \in \mathbb{U}_a\} \quad \text{(by (3.32a))} \\
&= \{c_{G,u_a} \mid G \in \mathbb{H}/\mathfrak{I}_a, u_a \in \mathbb{U}_a\}. \\
&\text{(by definition of the WtoAFR-choice of the agent } a \text{ as in (3.33))}
\end{aligned}$$

We now show that $\mathbf{i} : (G, u_a) \mapsto c_{G,u_a}$ is an injection.

For this purpose, we consider two information atoms $G, G' \in \mathbb{H}/\mathfrak{I}_a$ and two actions $u_a, u'_a \in \mathbb{U}_a$ of the agent a , such that $\mathbf{i}(G, u_a) = \mathbf{i}(G', u'_a)$ and we prove that $G = G'$ and $u_a = u'_a$. Using the definition of the WtoAFR-choice of the agent a as in (3.33), we have

$$\mathbf{i}(G, u_a) = \mathbf{i}(G', u'_a) \iff c_{G,u_a} = c_{G',u'_a} \iff G \cap [u_a] = G' \cap [u'_a].$$

We consider a configuration $\tilde{h} \in G \cap [u_a]$.

On the one hand, as $\tilde{h} \in [u_a]$, we have that $\tilde{h}_a = u_a$, where the projection $\text{proj}_a : \mathbb{H} \rightarrow \mathbb{U}_a$ is as in (3.28c). On the other hand, as $\tilde{h} \in G' \cap [u'_a]$, we get that $\tilde{h} \in [u'_a]$ and $\tilde{h}_a = u'_a$. From this follows that $u_a = u'_a$.

As $\tilde{h} \in G$ and $\tilde{h} \in G'$, we get that $G \cap G' \neq \emptyset$, hence that $G = G'$ as they are atoms of the information partition $\mathbb{H}/\mathfrak{I}_a$ as in (3.32a).

This ends the proof. \square

In the following Proposition 3.23, postponed to §3.3.2.3, for any agent a , we establish a closed form expression for any information atom $G \in \mathbb{H}/\mathfrak{I}_a$ in (3.39a) and, for any action $u_a \in \mathbb{U}_a$ of the agent a , a closed form expression for the corresponding choice c_{G,u_a} in (3.39b). To this end, we introduce in §3.3.2.2 the sets of WtoAFR-moves of agents, which partition the set X of WtoAFR-moves defined above in (3.20b). We show that each information atom $G \in \mathbb{H}/\mathfrak{I}_a$ of the agent a is made out of WtoAFR-moves of agent a . This result, proved in Proposition 3.22, is the main ingredient to get the representation (3.39a).

3.3.2.2 Auxiliary move equivalence

Before, in (3.20b), we have defined the set X of WtoAFR-moves. Here, by introducing an auxiliary *moves partition* we will be more specific, defining for each agent $a \in \mathbb{A}$, the set X_a of *WtoAFR-moves of the agent* a , a subset of WtoAFR-moves X , where the agent acts. We will show in Lemma 3.21 that the sets $\{X_a\}_{a \in \mathbb{A}}$ form a partition of the set X of WtoAFR-moves and prove in Proposition 3.22 that any information set of an agent is a union of moves of this agent.

Definition 3.20. For any agent $a \in \mathbb{A}$, the move relation \mathfrak{M}_a^φ of the agent a is defined on the set \mathbb{H} of configurations in the following way

$$(\forall h, h' \in \mathbb{H}) \quad h\mathfrak{M}_a^\varphi h' \iff \begin{cases} \exists \kappa \in \Sigma^\varphi \setminus \{(\emptyset)\} : \kappa^* = a, \\ h_{\kappa^-} = h'_{\kappa^-} \\ h, h' \in \mathbb{H}_\kappa^\varphi, \end{cases} \quad (3.35a)$$

where the set Σ^φ of orderings is defined in (2.35c) and, for any nonempty ordering $\kappa \in \Sigma^\varphi \setminus \{(\emptyset)\}$, the last element κ^* in (2.33i), for any configuration $h \in \mathbb{H}$, the corresponding κ^- -prefix in (3.17), and the set $\mathbb{H}_\kappa^\varphi$ in (2.34).

For any agent $a \in \mathbb{A}$, we denote by X_a and we call set of WtoAFR-moves of the agent a the partition⁴ of the set \mathbb{H} of configurations induced by relation \mathfrak{M}_a^φ :

$$X_a = \mathbb{H}/\mathfrak{M}_a^\varphi. \quad (3.35b)$$

In the following Lemma 3.21, we give an explicit formula for the set of WtoAFR-moves X_a of any agent $a \in \mathbb{A}$ defined in Equation 3.35b as an element of the partition of the underlying set \mathbb{H} of configurations and we show that the collection $\{X_a\}_{a \in \mathbb{A}}$ of agents' WtoAFR-moves forms a partition of the set X of WtoAFR-moves.

Lemma 3.21. The set X of WtoAFR-moves in Equation (3.20b) is partitioned as

$$X = \bigsqcup_{a \in \mathbb{A}} X_a, \quad (3.36a)$$

where, for any agent $a \in \mathbb{A}$, the set of WtoAFR-moves X_a of the agent a is as in (3.35b).

Moreover, for any agent $a \in \mathbb{A}$, the corresponding set X_a of WtoAFR-moves of the agent a is given by

$$X_a = \{x_{\kappa, \mathfrak{h}_{\kappa^-}} \mid \kappa \in \Sigma^\varphi \setminus \{(\emptyset)\}, \kappa^* = a, \mathfrak{h}_{\kappa^-} \in \Omega \times \overline{\mathbb{U}}_{\|\kappa^-\|}, \text{ s.t. } x_{\kappa, \mathfrak{h}_{\kappa^-}} \neq \emptyset\}, \quad (3.36b)$$

where the WtoAFR-move $x_{\kappa, \mathfrak{h}_{\kappa^-}}$ is as in (3.20a).

Proof. The proof is in two steps. First, we prove Equation 3.36b. Let $a \in \mathbb{A}$ and $x \in X_a$ as defined in (3.35b) be given. Then, by (3.35a), there exist a nonempty partial ordering $\kappa \in \Sigma^\varphi \setminus \{(\emptyset)\}$, such that $\kappa^* = a$ associated to the move x . Consider $h \in x$, then we indeed have that $h_{\kappa^-} \in \Omega \times \overline{\mathbb{U}}_{\|\kappa^-\|}$. Now, for any configuration $h' \in x$ we have by (3.35a), that $\kappa \succeq \varphi(h')$ since $h' \in \mathbb{H}_\kappa^\varphi$ and $h'_{\kappa^-} = h_{\kappa^-}$ thus we have that $x = x_{\kappa, \mathfrak{h}_{\kappa^-}}$ and thus x belongs to the set defined by the right hand side of equation 3.36b. The converse inclusion is immediate by considering the representation of the moves given in Equation (3.22). Second, to prove partition in (3.36a), we write

⁴this definition precludes empty elements as a partition contains no empty elements

$$\begin{aligned}
X &= \{x_{\kappa, \mathfrak{h}_{\kappa^-}} \mid \kappa \in \Sigma^\varphi \setminus \{(\emptyset)\}, \mathfrak{h}_{\kappa^-} \in \Omega \times \overline{\mathbb{U}}_{\|\kappa^-\|}, \text{ s.t. } x_{\kappa, \mathfrak{h}_{\kappa^-}} \neq \emptyset\} \\
&\quad (\text{by definition of the set } X \text{ of WtoAFR-moves in (3.20b)}) \\
&= \bigsqcup_{a \in \mathbb{A}} \{x_{\kappa, \mathfrak{h}_{\kappa^-}} \mid \kappa \in \Sigma^\varphi \setminus \{(\emptyset)\}, \kappa^* = a, \mathfrak{h}_{\kappa^-} \in \Omega \times \overline{\mathbb{U}}_{\|\kappa^-\|}, \text{ s.t. } x_{\kappa, \mathfrak{h}_{\kappa^-}} \neq \emptyset\} \\
&\quad (\text{as, trivially, } \Sigma^\varphi \setminus \{(\emptyset)\} = \bigsqcup_{a \in \mathbb{A}} \{\kappa \mid \kappa^* = a\}) \\
&= \bigsqcup_{a \in \mathbb{A}} X_a. \quad (\text{by definition of WtoAFR-move as in (3.36b)})
\end{aligned}$$

This ends the proof. \square

Note that Equation 3.36b can be rewritten as the following equivalence

$$x \in X_a \iff \begin{cases} \exists \kappa \in \Sigma^\varphi \setminus \{(\emptyset)\} & \text{s.t. } \kappa^* = a, \\ \exists \mathfrak{h}_{\kappa^-} \in \Omega \times \overline{\mathbb{U}}_{\|\kappa^-\|} & \text{s.t. } x = x_{\kappa, \mathfrak{h}_{\kappa^-}} \neq \emptyset. \end{cases} \quad (3.37)$$

Proposition 3.22. *Under the assumptions of Theorem 3.1, for any agent $a \in \mathbb{A}$, we have the implication*

$$(\forall h, h' \in \mathbb{H}) \quad h\mathfrak{M}_a^\varphi h' \Rightarrow h\mathfrak{I}_a h', \quad (3.38a)$$

where the equivalence relations \mathfrak{M}_a^φ and \mathfrak{I}_a are as in Equations (3.35a) and (3.28a) respectively.

Thus, the moves partition $\mathbb{H}/\mathfrak{M}_a^\varphi$ is finer than the information partition $\mathbb{H}/\mathfrak{I}_a$, that is,

$$\mathbb{H}/\mathfrak{I}_a \preceq \mathbb{H}/\mathfrak{M}_a^\varphi \quad (3.38b)$$

as in (9.5), and, for any move $x \in \mathbb{H}/\mathfrak{M}_a^\varphi$ of the agent a and for any information atom $G \in \mathbb{H}/\mathfrak{I}_a$ of the agent a ,

$$x \cap G = \begin{cases} x, & \text{if } x \subset G \\ \emptyset, & \text{else.} \end{cases} \quad (3.38c)$$

Proof. Let an agent $a \in \mathbb{A}$ and two configurations $h, h' \in \mathbb{H}$, such that $h\mathfrak{M}_a^\varphi h'$, be given. Then, by definition of the moves equivalence relation \mathfrak{M}_a^φ as in (3.35a), there exists a nonempty ordering $\kappa \in \Sigma^\varphi \setminus \{(\emptyset)\}$ such that $\kappa^* = a$, $h, h' \in \mathbb{H}_\kappa^\varphi$ and $h_{\kappa^-} = h'_{\kappa^-}$. We prove that $h\mathfrak{I}_a h'$, where the information relation \mathfrak{I}_a is as in (3.28a), that is, we prove that there exists an information atom $G \in \mathbb{H}/\mathfrak{I}_a$ such that $h, h' \in G$.

Let $G_h, G_{h'} \in \mathbb{H}/\mathfrak{I}_a$ be the information atoms containing configurations h and h' respectively (uniqueness of the atoms G_h and $G_{h'}$ follows as $\mathbb{H}/\mathfrak{I}_a$ is a partition of the underlying set \mathbb{H} of configurations). From causality, as in (2.37), as $G_h \in \mathfrak{I}_a = \mathfrak{I}_{\kappa^*}$, we deduce that $\mathbb{H}_\kappa^\varphi \cap G_h \in \mathcal{H}_{\|\kappa^-\|}$ is a cylinder. Hence, by definition of the field $\mathcal{H}_{\|\kappa^-\|}$ as in (2.36f), there exists a set $\overline{U} \subset \Omega \times \overline{\mathbb{U}}_{\|\kappa^-\|}$ such that

$$G_h \cap \mathbb{H}_\kappa^\varphi = \overline{U} \times \overline{\mathbb{U}}_{-\|\kappa^-\|}.$$

As $h \in G_h$ and $h \in \mathbb{H}_\kappa^\varphi$ by assumption, we get that $h \in \bar{U} \times \bar{U}_{-\|\kappa-\|}$ and $h_{\kappa-} \in \bar{U}$. As $h_{\kappa-} = h'_{\kappa-}$, by assumption, we obtain that $h'_{\kappa-} \in \bar{U}$ and, hence, that $h' \in G_h \cap \mathbb{H}_\kappa^\varphi$. From this follows that $G_h \cap G_{h'} \neq \emptyset$ and then that $G_h = G_{h'}$, as both are atoms of the information partition $\mathbb{H}/\mathfrak{I}_a$. This proves that $h\mathfrak{I}_a h'$.

This ends the proof. \square

3.3.2.3 Closed form representations of information atoms and choices

Now, we are ready to state and prove announced in §3.3.2.1 closed form representations of information atoms and choices.

Proposition 3.23. *Under the assumptions of Theorem 3.1, for any agent $a \in \mathbb{A}$, any information atom $G \in \mathbb{H}/\mathfrak{I}_a$ and any action $u_a \in \mathbb{U}_a$ of the agent,*

- the atom G has the following closed form

$$G = \bigsqcup_{\substack{x \in X_a: \\ x \subset G}} x, \quad (3.39a)$$

where X_a has been defined in (3.35b)

- the choice $c_{G,u_a} \in \mathbb{H}/\mathfrak{I}_a \vee \mathbb{H}/\mathfrak{U}_a$, induced by the atom $G \in \mathbb{H}/\mathfrak{I}_a$ and the action $u_a \in \mathbb{U}_a$ as in Equation (3.33), has the following closed form

$$c_{G,u_a} = \bigsqcup_{\substack{x \in X_a: \\ x \subset G}} (x \cap [u_a]), \quad (3.39b)$$

where none of the above sets is empty because the following property holds true

$$x \cap [u_a] \neq \emptyset, \quad \forall x \in X_a, \quad \forall u_a \in \mathbb{U}_a. \quad (3.39c)$$

Proof. Let an agent $a \in \mathbb{A}$, an information atom $G \in \mathbb{H}/\mathfrak{I}_a$ and an action $u_a \in \mathbb{U}_a$ of the agent be given.

► We prove Equation (3.39a). We write

$$\begin{aligned} G &= \mathbb{H} \cap G && \text{(as } G \subset \mathbb{H}\text{)} \\ &= \left(\bigsqcup_{x \in \mathbb{H}/\mathfrak{M}_a^\varphi} x \right) \cap G && \text{(since } \mathbb{H}/\mathfrak{M}_a^\varphi \text{ is a partition of } \mathbb{H} \text{ induced by relation } \mathfrak{M}_a^\varphi\text{)} \\ &= \bigsqcup_{\substack{x \in \mathbb{H}/\mathfrak{M}_a^\varphi \\ (x \cap G) \neq \emptyset}} (x \cap G) && \text{(using distributive property and removing empty sets)} \\ &= \bigsqcup_{\substack{x \in \mathbb{H}/\mathfrak{M}_a^\varphi: \\ x \subset G}} x, && \text{(by 3.38c, } (x \cap G) = x \text{ if } x \subset G \text{ or } (x \cap G) = \emptyset \text{ if } x \not\subset G\text{)} \end{aligned}$$

which gives Equation (3.39a) using the equality $X_a = \mathbb{H}/\mathfrak{M}_a^\varphi$ given by Equation (3.35b).

► We prove Equation (3.39b). We write

$$\begin{aligned}
c_{G,u_a} &= G \cap [u_a] && \text{(by definition of WtoAFR-choice } c_{G,u_a} \text{ as in (3.33))} \\
&= \left(\bigsqcup_{\substack{x \in X_a: \\ x \subset G}} x \right) \cap [u_a] && \text{(by partition of the atom } G \text{ as in (3.39a))} \\
&= \bigsqcup_{\substack{x \in X_a: \\ x \subset G}} (x \cap [u_a]) . && \text{(distributive property)}
\end{aligned}$$

► We prove Equation (3.39c). Let a move $x \in X_a$ of agent $a \in \mathbb{A}$ be given. Then, by definition of the WtoAFR-move as in (3.20a), there exists a nonempty ordering $\kappa \in \Sigma^\varphi \setminus \{(\emptyset)\}$, such that $\kappa^* = a$, and some (κ^-) -prefix $\mathfrak{h}_{\kappa^-} \in \Omega \times \overline{\mathbb{U}}_{\|\kappa^-\|}$ such that $x = x_{\kappa, \mathfrak{h}_{\kappa^-}}$ and such that $x_{\kappa, \mathfrak{h}_{\kappa^-}} \neq \emptyset$. Using Lemma 3.10 the move $x_{\kappa, \mathfrak{h}_{\kappa^-}}$ is a cylinder of the form $x_{\kappa, \mathfrak{h}_{\kappa^-}} = (\overline{\mathbb{H}}_\kappa^\varphi \cap \{\mathfrak{h}_{\kappa^-}\}) \times \overline{\mathbb{U}}_{-\|\kappa^-\|}$ and therefore we have that $\overline{\mathbb{H}}_\kappa^\varphi \cap \mathfrak{h}_{\kappa^-} \neq \emptyset$. Now, we write

$$\begin{aligned}
x \cap [u_a] &= x_{\kappa, \mathfrak{h}_{\kappa^-}} \cap [u_a] \\
&= x_{\kappa, \mathfrak{h}_{\kappa^-}} \cap (\Omega \times \overline{\mathbb{U}}_{\|\kappa^-\|} \times \{u_a\} \times \overline{\mathbb{U}}_{-\|\kappa\|}) \\
&\quad \text{(by definition of the equivalence class } [u_a] \in \mathbb{H}/\mathfrak{U}_a \text{ as in (3.32b))} \\
&= \left((\overline{\mathbb{H}}_\kappa^\varphi \cap \{\mathfrak{h}_{\kappa^-}\}) \times \overline{\mathbb{U}}_{-\|\kappa^-\|} \right) \cap \left(\Omega \times \overline{\mathbb{U}}_{\|\kappa^-\|} \times \{u_a\} \times \overline{\mathbb{U}}_{-\|\kappa\|} \right) \\
&\quad \text{(by representation of the WtoAFF-move } x_{\kappa, \mathfrak{h}_{\kappa^-}} \text{ as in (3.21))} \\
&= \left((\overline{\mathbb{H}}_\kappa^\varphi \cap \{\mathfrak{h}_{\kappa^-}\}) \times \mathbb{U}_a \times \overline{\mathbb{U}}_{-\|\kappa\|} \right) \cap \left(\Omega \times \overline{\mathbb{U}}_{\|\kappa^-\|} \times \{u_a\} \times \overline{\mathbb{U}}_{-\|\kappa\|} \right) \\
&\quad \text{(as } a = \kappa^* \notin \|\kappa^-\| \text{ by (2.33j))} \\
&= (\overline{\mathbb{H}}_\kappa^\varphi \cap \{\mathfrak{h}_{\kappa^-}\}) \times \{u_a\} \times \overline{\mathbb{U}}_{-\|\kappa\|} \quad \text{(as } u_a \in \mathbb{U}_a \text{ and } (\overline{\mathbb{H}}_\kappa^\varphi \cap \mathfrak{h}_{\kappa^-}) \subset \Omega \times \overline{\mathbb{U}}_{\|\kappa^-\|}) \\
&\neq \emptyset . \quad \text{(as it was noted above that } \overline{\mathbb{H}}_\kappa^\varphi \cap \mathfrak{h}_{\kappa^-} \neq \emptyset)
\end{aligned}$$

This ends the proof. \square

3.3.3 WtoAFR-choices satisfy (AFR-Axiom1')

We are going to prove that, for any player $i \in \mathcal{P}$, the constructed set C^i of WtoAFR-choices in Proposition 3.18, and the WtoAFR-tree (V, \supset) , where the set V of vertices is defined in Proposition 3.14, satisfy (AFR-Axiom1') and (AFR-Axiom2') stated in Definition 2.16. Before doing this, we establish Proposition 3.32 that gives a closed form for the immediate predecessor of any WtoAFR-choice.

We establish (AFR-Axiom1') for WtoAFR-choices in the following Proposition 3.24 relying on postponed Proposition 3.32 to be found in §3.5.3, where we prove, for any WtoAFR-choice c_{G,u_a} , the following result

$$P(c_{G,u_a}) = \{x \in X_a \mid x \subset G\} \subset X_a ,$$

where the immediate predecessor mapping P is as in (2.18b).

Proposition 3.24. *For any partition $\mathbb{A} = \bigsqcup_{i \in \mathcal{P}} \mathbb{A}^i$ of the set of W-agents into players, the collection $\{C^i\}_{i \in \mathcal{P}}$ satisfies (AFR-Axiom1') stated in Definition 2.16, that is, for any two choices $c \in C^i$ and $c' \in C^j$ of the players,*

$$\left(\forall i, j \in \mathcal{P}, \forall c \in C^i, c' \in C^j \right) \quad \left(P(c) \cap P(c') \neq \emptyset, c \neq c' \right) \Rightarrow \left(i = j, P(c) = P(c'), c \cap c' = \emptyset \right), \quad (3.40)$$

where for every player $i \in \mathcal{P}$, the corresponding collection C^i of choices is given in (3.30).

Proof. The proof is in two steps. Before proving the stronger (AFR-Axiom1') stated in Definition 2.16, we show that (AFR-Axiom1) in Definition 2.15 holds true, namely, for any two choices $c, c' \in C^i$ of the player i ,

$$\left(\forall c, c' \in C^i \right) \quad \left(P(c) \cap P(c') \neq \emptyset, c \neq c' \right) \Rightarrow \left(P(c) = P(c'), c \cap c' = \emptyset \right). \quad (3.41)$$

► First, we prove (AFR-Axiom1) for a given player $i \in \mathcal{P}$ in two steps.

- For any representative agent $a \in \mathbb{A}^i$ of the player i we prove the restricted version of (AFR-Axiom1) for the corresponding collection of choices C_a .

To this end, we consider two distinct WtoAFR-choices $c \neq c' \in C_a$ of the representative agent $a \in \mathbb{A}^i$. As WtoAFR-choices $c, c' \in C_a$ are atoms of the choice partition C_a as in (3.29), then $c \neq c'$ implies $c \cap c' = \emptyset$.

By characterization (3.34a) of the set C_a of WtoAFR-choices of the agent a , the choices c and c' are uniquely determined by the corresponding information atoms $G, G' \in \mathbb{H}/\mathfrak{I}_a$ of the agent a and actions $u_a, u'_a \in \mathbb{U}_a$ of the agent a respectively, that is,

$$c = c_{G, u_a}, \quad c' = c_{G', u'_a}.$$

We now prove that, if $c_{G, u_a} \neq c_{G', u'_a}$ and $P(c_{G, u_a}) \cap P(c_{G', u'_a}) \neq \emptyset$, then $P(c_{G, u_a}) = P(c_{G', u'_a})$ and $c_{G, u_a} \cap c_{G', u'_a} = \emptyset$.

By the image of a WtoAFR-choice under the immediate predecessor mapping P as in (3.53), we write

$$P(c_{G, u_a}) = \{x \in X_a \mid x \subset G\}, \quad P(c_{G', u'_a}) = \{x \in X_a \mid x \subset G'\}.$$

Note that for any information atom $G'' \in \mathbb{H}/\mathfrak{I}_a$ of the agent a , we have that $P(c_{G'', \bar{u}_a}) = P(c_{G'', \bar{\bar{u}}_a})$, for any two disjoint $\bar{u}_a, \bar{\bar{u}}_a \in \mathbb{U}_a$ by (3.53).

As $P(c_{G, u_a}) \cap P(c_{G', u'_a}) \neq \emptyset$ by the assumption, there exists $\tilde{x} \in X_a$ such that $\tilde{x} \in P(c_{G, u_a}) \cap P(c_{G', u'_a})$. As a consequence, we get that $\tilde{x} \subset G \cap G'$, hence that $G \cap G' \neq \emptyset$. We deduce that $G = G'$, as they are atoms of the information partition $\mathbb{H}/\mathfrak{I}_a$, which immediately implies that $P(c_{G, u_a}) = \{x \in X_a \mid x \subset G\} = P(c_{G', u'_a})$. This proves the restricted version of (AFR-Axiom1) for the set C_a of choices of the agent a .

- Second, we extend (AFR-Axiom1) to the whole set of WtoAFR-choices $C^i = \{C_a\}_{a \in \mathbb{A}^i}$ of the player i equipped with the set \mathbb{A}^i of representative agents. For this purpose, we are going to show that, for any two choices $c, c' \in C^i$, their immediate predecessors $P(c)$ and $P(c')$ are always disjoint.

Indeed, take any two agents $a, b \in \mathbb{A}^i$ and any choices $c_a \in C_a$ and $c_b \in C_b$. By characterization (3.34a) of the set of choices, there are $u_a \in \mathbb{U}_a$, $G_a \in \mathbb{H}/\mathfrak{J}_a$ and $u_b \in \mathbb{U}_b$, $G_b \in \mathbb{H}/\mathfrak{J}_b$ such that $c_a = c_{G_a, u_a}$ and $c_b = c_{G_b, u_b}$ respectively.

By the image of a WtoAFR-choice under the immediate predecessor mapping p as in (3.53), we write

$$P(c_{G_a, u_a}) = \{x \in X_a \mid x \subset G_a\}, \quad P(c_{G_b, u_b}) = \{x \in X_a \mid x \subset G_b\}.$$

As before, we note that for any information atom $G'' \in \mathbb{H}/\mathfrak{J}_a$ of the agent a , we have that $P(c_{G'', \bar{u}_a}) = P(c_{G'', \bar{\bar{u}}_a})$, for any two disjoint $\bar{u}_a, \bar{\bar{u}}_a \in \mathbb{U}_a$ by (3.53). If $P(c_{G, u_a}) \cap P(c_{G', u'_a}) \neq \emptyset$, then there is a move $x \in X$, such that $x \in X_a$ and $x \in X_b$, thus, $X_a \cap X_b \neq \emptyset$, which implies that $a = b$ as $X = \bigsqcup_{a' \in \mathbb{A}} X_{a'}$ by (3.36a).

If $a = b$, the result is already proved in the first step of the proof, which establishes (AFR-Axiom1) for the whole set C^i of WtoAFR-choices of the player i .

► Finally, we prove the restricted (AFR-Axiom1').

Let any two players $i, j \in \mathcal{P}$ be given. Take any two choices $c \in C^i$ and $c' \in C^j$ of the players i and j respectively. By characterization (3.34a) of the set of choices, there are $u_a \in \mathbb{U}_a$, $G_a \in \mathbb{H}/\mathfrak{J}_a$ for some representative agent $a \in \mathbb{A}^i$ of the player i and $u_b \in \mathbb{U}_b$, $G_b \in \mathbb{H}/\mathfrak{J}_b$ for some representative agent $b \in \mathbb{A}^j$ of the player j such that $c = c_{G_a, u_a}$ and $c' = c_{G_b, u_b}$ respectively.

The rest of the proof repeats the previous step. If $P(c_{G_a, u_a}) \cap P(c_{G_b, u_b}) \neq \emptyset$, then, $a = b$ and, thus, $i = j$ as $\mathbb{A} = \bigsqcup_{i \in \mathcal{P}} \mathbb{A}^i$ is assumed to be a partition.

This finishes the proof. □

3.3.4 WtoAFR-choices satisfy (AFR-Axiom2')

Now, we will establish the second and the last (AFR-Axiom2) that characterizes the inverse of the parent mapping (2.11b). We will show that its restricted version holds true, namely, (AFR-Axiom2') as in (2.20) in Definition 2.16. It slightly differs from the one given in the original AFR-setting as, in our setting, we let only one player be active at a non-leaf vertex. Before stating Proposition 3.43, we formalize the notion of a player being active at a WtoAFR-move.

For any WtoAFR-move $x \in X$ and any player $i \in \mathcal{P}$ equipped with the set \mathbb{A}^i of representative agents, the player i is said to be *active at the move* x if

$$x \in \bigcup_{a \in \mathbb{A}^i} X_a. \quad (3.42)$$

The following Proposition 3.25 relies on the postponed Lemmas 3.38 and 3.39 to be found in §3.5.4.

Proposition 3.25. *For any WtoAFR-move $x \in X$ and any player $i \in \mathcal{P}$ active at the WtoAFR-move x as in (3.42), the condition (AFR-Axiom2'), as in (2.20) in Definition 2.16, holds true, that is,*

$$p^{-1}(x) = \{x \cap c \mid c \in C^i : x \in P(c)\}. \quad (3.43)$$

Proof. Let a player $i \in \mathcal{P}$ equipped with the set \mathbb{A}^i of representative agents be given. If the player i is active at a WtoAFR-move $x \in X$ as in (3.42), there exists a unique agent $a = a(x) \in \mathbb{A}^i$, such that $x \in X_a$, by partition (3.36a) of the set X of WtoAFR-moves in Lemma 3.21. Then, for this agent a we are going to prove that

$$p^{-1}(x) = \{x \cap c \mid c \in C_a : x \in P(c)\} . \quad (3.44)$$

For this purpose, we write

$$\begin{aligned} p^{-1}(x) &= \{x \cap [u_a] \mid u_a \in \mathbb{U}_a\} \\ &\quad \text{(by definition of the set } p^{-1}(x) \text{ of children of the move } x \text{ as in (3.57))} \\ &= \{x \cap (G \cap [u_a]) \mid u_a \in \mathbb{U}_a\} \end{aligned}$$

where $G = G(x)$ is the unique information atom such that $x \subset G$; indeed, as shown in Proposition 3.22, the moves partition X_a of the agent a defined in (3.35b) is finer than the information partition $\mathbb{H}/\mathfrak{J}_a$ (where the relation \mathfrak{J}_a is defined in (3.28a)), so that, for any move $x \in X_a$ of the agent a , there exists a unique information atom $G = G(x)$ such that $x \subset G$

$$\begin{aligned} &= \{x \cap c_{G,u_a} \mid u_a \in \mathbb{U}_a\} \quad \text{(by definition of the WtoAFR-choice } c_{G,u_a} \text{ as in (3.33))} \\ &= \{x \cap c \mid c \in C_a \text{ such that } c = c_{G,u_a} \text{ for some } u_a \in \mathbb{U}_a\} \\ &= \{x \cap c \mid c \in C_a \text{ such that } x \cap c \neq \emptyset\} \quad \text{(by (3.66) in Lemma 3.38)} \\ &= \{x \cap c \mid c \in C_a \text{ such that } x \in P(c)\} , \quad \text{(by (3.67) in Lemma 3.39)} \end{aligned}$$

which gives the desired relation (3.44) and, thus, proves (3.43).

This ends the proof. \square

We have built the primitives of the AFR-model – the game tree and the collection of players' choices – out of the primitives of the W-model – sample space of Nature, action sets of agents and the information partition of the agents. The existence of a causal configuration ordering was a key element in that construction and in the obtention of the two AFR-Axioms on the primitives.

We will elucidate in the following Section 3.4 how a W-strategy induces a so-called WtoAFR-strategy.

3.4 WtoAFR-strategies

Strategies in the W-model are attached to the agents: as defined in Definition 2.21, for any agent $a \in \mathbb{A}$, a W-strategy $\lambda_a : (\mathbb{H}, \mathcal{H}) \rightarrow (\mathbb{U}_a, \mathcal{U}_a)$ is a mapping — from the underlying set \mathbb{H} of configurations, as in (2.24a), equipped with the complete π -field \mathcal{H} , as in (2.24b),

into the set \mathbb{U}_a of actions of the agent a , equipped with the complete π -field \mathcal{U}_a — with the imposed condition, as in (2.30b), that the strategy is a measurable mapping with respect to the information π -field \mathcal{J}_a of the agent a , that is, $\lambda_a^{-1}(\mathcal{U}_a) \subset \mathcal{J}_a$.

Strategies in the AFR-model are attached to the players: as defined in Definition 2.17, for any player $i \in \mathcal{P}$, an AFR-strategy $s^i : X^i \rightarrow C^i$ is a mapping — from the set of moves X^i of the player i , as in (2.22a) into the set of choices C^i of the player i , as in §2.3.2.2 — with the imposed condition that the inverse strategy $(s^i)^{-1} : C^i \rightarrow 2^{X^i}$ acts on choices as the immediate predecessor mapping, that is, $(s^i)^{-1}(c) = P(c)$ for any $c \in s^i(X^i)$.

In this Section 3.4, we construct, from a W-strategy $\lambda_a : \mathbb{H} \rightarrow \mathbb{U}_a$ (hence such that the measurability condition as in (2.30b) holds true), a mapping $s_a : X_a \rightarrow C_a$ such that the measurability condition as in (2.17) holds true.

Proposition 3.26. *For any agent $a \in \mathbb{A}$, and any W-strategy $\lambda_a : (\mathbb{H}, \mathcal{H}) \rightarrow (\mathbb{U}_a, \mathcal{U}_a)$ of the agent a , as in Definition 2.21, there exists a mapping*

$$s_a : X_a \rightarrow C_a, \quad (3.45a)$$

where the set X_a of WtoAFR-moves of the agent a is as in (3.35b) and the set C_a of WtoAFR-choices of the agent a is as in (3.29), which is defined as the following composition

$$s_a = (\pi_{\mathfrak{M}_a^\varphi}^{-1} \circ \pi_{\mathcal{J}_a}) \circ (\pi_{\mathcal{J}_a}^{-1} \circ \lambda_a, id) \circ \mathbf{i}, \quad (3.45b)$$

such that the Diagram (3.45c) commutes

$$\begin{array}{ccc} \mathbb{H}/\mathcal{J}_a & \xrightarrow{(\tilde{s}_a, id)} & \mathbb{H}/\mathcal{U}_a \times \mathbb{H}/\mathcal{J}_a \\ \pi_{\mathfrak{M}_a^\varphi}^{-1} \circ \pi_{\mathcal{J}_a} \uparrow & & \downarrow \mathbf{i} \\ \mathbb{H}/\mathfrak{M}_a^\varphi = X_a & \xrightarrow{s_a} & C_a = \mathbb{H}/\mathcal{U}_a \vee \mathbb{H}/\mathcal{J}_a \end{array} \quad (3.45c)$$

where the injection $\mathbf{i} : \mathbb{H}/\mathcal{J}_a \times \mathbb{H}/\mathcal{U}_a \rightarrow \mathbb{H}/\mathcal{J}_a \vee \mathbb{H}/\mathcal{U}_a$ is an intersection operation as in (3.34b), and the auxiliary mapping $\tilde{s}_a = \pi_{\mathcal{J}_a}^{-1} \circ \lambda_a$ is such that the Diagram (3.45d) commutes

$$\begin{array}{ccc} \mathbb{H} & \xrightarrow{\lambda_a} & \mathbb{U}_a \equiv \mathbb{H}/\mathcal{U}_a \\ \pi_{\mathcal{J}_a}^{-1} \uparrow & \nearrow \tilde{s}_a = \pi_{\mathcal{J}_a}^{-1} \circ \lambda_a & \\ \mathbb{H}/\mathcal{J}_a & & \end{array} \quad (3.45d)$$

Thus defined, the mapping s_a satisfies the following measurability constraint for an AFR-strategy as in Definition 2.17

$$s_a^{-1}(c) = P(c), \quad \forall c \in C_a, \quad (3.45e)$$

where P is the immediate predecessor mapping as in (2.18b). We call the mapping $s_a : X_a \rightarrow C_a$ a WtoAFR-strategy of the agent a .

For any subset $\mathbb{A}^i \subset \mathbb{A}$ of agents, we define

$$s^i : X^i \rightarrow C^i, \quad \text{s.t. } \forall a \in \mathbb{A}^i, \quad (s^i)|_{X_a} = s_a : X_a \rightarrow C_a, \quad (3.45f)$$

where, $X^i = \bigcup_{a \in \mathbb{A}^i} X_a$ is the set of WtoAFR-moves of the player i , $C^i = \{C_a\}_{a \in \mathbb{A}^i}$ is the set of WtoAFR-choices of the player i , and, for any $a \in \mathbb{A}^i$, the WtoAFR-strategy $s_a : X_a \rightarrow C_a$ of the agent a is as in (3.45a). Thus defined, the mapping s^i also satisfies the following measurability constraint for an AFR-strategy as in Definition 2.17

$$(s^i)^{-1}(c) = P(c), \quad \forall c \in C_a, \quad \forall a \in \mathbb{A}^i. \quad (3.45g)$$

We call the mapping $s^i : X^i \rightarrow C^i$ a WtoAFR-strategy of the player i equipped with the set \mathbb{A}^i of representative agents.

Proof. We build a WtoAFR-strategy in two steps.

As a first step, for any W-agent $a \in \mathbb{A}$, and any W-strategy $\lambda_a : (\mathbb{H}, \mathcal{H}) \rightarrow (\mathbb{U}_a, \mathcal{U}_a)$ as in (2.30a) such that the measurability constraint $\lambda_a^{-1}(\mathcal{U}_a) \subset \mathcal{J}_a$, as in (2.30b), holds true, we construct a mapping $s_a : X_a \rightarrow C_a$ and show that the measurability condition as in (3.45e) is satisfied for the mapping s_a . As a second step, for any group of representative W-agents $\mathbb{A}^i \subset \mathbb{A}$ we build a player $i \in \mathcal{P}$, endow the player i with a mapping $s_i : X_i \rightarrow C^i$ and show that the measurability condition as in (3.45e) is satisfied for the mapping s_i .

► First, for any W-agent $a \in \mathbb{A}$, and any measurable W-strategy $\lambda_a : (\mathbb{H}, \mathcal{H}) \rightarrow (\mathbb{U}_a, \mathcal{U}_a)$ as in (2.30a), we construct the candidate WtoAFR-strategy $s_a : X_a \rightarrow C_a$ for the agent a , as in (3.45a), and show that the Diagram (3.45c) commutes.

By Lemma 9.3, we define an auxiliary mapping $\tilde{s}_a : \mathbb{H}/\mathcal{J}_a \rightarrow \mathbb{U}_a$. We prove that the Diagram (3.45d) commutes. Let an information atom $G \in \mathbb{H}/\mathcal{J}_a$ of the agent a is given. Then, we write

$$\pi_{\mathcal{J}_a}^{-1}(G) = \{h \in \mathbb{H} \mid h \in G\}$$

and, for any $h \in \pi_{\mathcal{J}_a}^{-1}(G)$, that is, for any $h \in G$, we have that $\lambda_a(h) = u_a$, for some $u_a \in \mathbb{U}_a$, as the W-strategy λ_a is constant on each information atom by the measurability condition as in (2.30b).

With the help of the auxiliary mapping $\tilde{s}_a : \mathbb{H}/\mathcal{J}_a \rightarrow \mathbb{U}_a$, we construct the candidate WtoAFR-strategy $s_a : X_a \rightarrow C_a$ from the set X_a of WtoAFR-moves of agent a as in (3.35b) into the set of WtoAFR-choices C_a of the agent a , as in (3.29).

The Diagram (3.45c) commutes as, under the property of causality stated in (2.37), the moves partition $X_a = \mathbb{H}/\mathfrak{M}_a^\varphi$ as in (3.35b) is finer than the information partition \mathbb{H}/\mathcal{J}_a , that is, $\mathbb{H}/\mathcal{J}_a \preceq \mathbb{H}/\mathfrak{M}_a^\varphi$, by (3.38b), thus,

$$\begin{aligned} (\pi_{\mathfrak{M}_a^\varphi}^{-1} \circ \pi_{\mathcal{J}_a}) : \mathbb{H}/\mathfrak{M}_a^\varphi &\rightarrow \mathbb{H}/\mathcal{J}_a \\ x &\mapsto G, \quad \forall x \in \{x \in X_a \mid x \subset G\} \end{aligned}$$

is well-defined. Thus, the candidate WtoAFR-strategy $s_a : X_a \rightarrow C_a$, is also well-defined as the composition in (3.46)

$$s_a = (\pi_{\mathfrak{M}_a^\varphi}^{-1} \circ \pi_{\mathcal{J}_a}) \circ (\pi_{\mathcal{J}_a}^{-1} \circ \lambda_a, \text{id}) \circ \mathbf{i}, \quad (3.46)$$

where the injection $i : \mathbb{H}/\mathcal{J}_a \times \mathbb{H}/\mathcal{U}_a \rightarrow \mathbb{H}/\mathcal{J}_a \vee \mathbb{H}/\mathcal{U}_a$ is as in (3.34b). Indeed, following the Diagram (3.45c) from the bottom left, then up, then right and then down, we see that the mapping $s_a : X_a \rightarrow C_a$ acts on any WtoAFR-move $x \in \{x \in X_a \mid x \subset G\}$ of the agent a as the following chain

$$x \xrightarrow{\pi_{\mathfrak{M}_a^{\mathcal{U}_a}}^{-1} \circ \pi_{\mathcal{J}_a}} G \xrightarrow{\pi_{\mathcal{J}_a}^{-1} \circ \lambda_a, \text{id}} G \times u_a \xrightarrow{i} c_{G, u_a} = G \cap [u_a].$$

► Second, we prove the measurability condition (3.45e) for the candidate WtoAFR-strategy $s_a : X_a \rightarrow C_a$ constructed on the previous step.

Let a WtoAFR-choice $c \in C_a$ of the agent a be given. By (3.34a), it is uniquely determined by an information atom $G' \in \mathbb{H}/\mathcal{J}_a$ of the agent a and an action $u'_a \in \mathcal{U}_a$ of the agent a , that is, $c = c_{G', u'_a}$. We prove that $s_a^{-1}(c_{G', u'_a}) = P(c_{G', u'_a})$.

Following the Diagram (3.45c) from the bottom right, then up, then left and then down, we see that the mapping $s_a^{-1} : 2^{C_a} \rightarrow 2^{X_a}$ acts on the WtoAFR-choice $c_{G', u'_a} \in C_a$ of the agent a as the following chain

$$c_{G', u'_a} = G' \cap [u'_a] \xrightarrow{i^{-1}} G \times u_a \xrightarrow{(\pi_{\mathcal{J}_a}^{-1} \circ \lambda_a, \text{id})^{-1}} G' \xrightarrow{(\pi_{\mathfrak{M}_a^{\mathcal{U}_a}}^{-1} \circ \pi_{\mathcal{J}_a})^{-1}} \{x \in X_a \mid x \subset G'\}.$$

On the other hand, the image of the WtoAFR-choice c_{G', u'_a} under the immediate predecessor mapping P is given by $P(c_{G', u'_a}) = \{x \in X_a \mid x \subset G'\}$, by (3.53), which proves the measurability condition $s_a^{-1}(c_{G', u'_a}) = P(c_{G', u'_a})$.

► Finally, we check the measurability condition (3.45g) for the candidate WtoAFR-strategy $s^i : X^i \rightarrow C^i$ of the player i equipped with the set $\mathbb{A}^i \subset \mathbb{A}$ of the representative agents, as defined in (3.45f), where, $X^i = \bigcup_{a \in \mathbb{A}^i} X_a$ is the set of WtoAFR-moves of the player i and $C^i = \{C_a\}_{a \in \mathbb{A}^i}$ is the set of WtoAFR-choices of the player i .

Let a WtoAFR-choice $c \in \{C_a\}_{a \in \mathbb{A}^i}$ of the player i be given. We show that there exists a unique representative agent $a \in \mathbb{A}^i$ of the player i , such that $(s^i)^{-1}(c) = s_a^{-1}(c)$.

Assume, that the WtoAFR-choice $c \in \{C_a\}_{a \in \mathbb{A}^i}$ belongs to two representative agents $a, b \in \mathbb{A}^i$ of the player i , that is, $c \in C_a$ and $c \in C_b$, where the corresponding choice partitions are as in (3.29). By characterization (3.34a) of the set of choices, there are $u_a \in \mathcal{U}_a$, $G_a \in \mathbb{H}/\mathcal{J}_a$ and $u_b \in \mathcal{U}_b$, $G_b \in \mathbb{H}/\mathcal{J}_b$ such that $c = c_{G_a, u_a}$ and $c = c_{G_b, u_b}$ respectively, that is, $c_{G_a, u_a} = c = c_{G_b, u_b}$. Then, we write

$$\{x \in X_a \mid x \subset G_a\} = P(c_{G_a, u_a})$$

$$\begin{aligned} \text{(by the image of a WtoAFR-choice under the immediate predecessor mapping } P \text{ as in (3.53))} \\ &= P(c_{G_b, u_b}) \quad \text{(by the assumption that } c_{G_a, u_a} = c = c_{G_b, u_b}\text{)} \\ &= \{x \in X_b \mid x \subset G_b\}. \end{aligned}$$

$$\text{(by the image of a WtoAFR-choice under the immediate predecessor mapping } P \text{ as in (3.53))}$$

This immediately implies that $X_a \cap X_b \neq \emptyset$, and $a = b$ as X_a and X_b are partitions by (3.35b). Thus, for any WtoAFR-choice $c \in C^i$ of player i , there exists a unique representative agent $a \in \mathbb{A}^i$, such that $c \in C_a$, and $(s^i)^{-1}(c) = s_a^{-1}(c)$.

As the measurability condition for the WtoAFR-strategy $s_a : X_a \rightarrow C_a$ of the agent a is established on the previous step in (3.45e), we get that $s_a^{-1}(c) = P(c)$ and conclude

$$\forall c \in C^i, \exists! a \in \mathbb{A}^i, (s^i)^{-1}(c) = s_a^{-1}(c),$$

where, $s_a^{-1}(c) = P(c)$ by (3.45e).

This proves that the WtoAFR-strategy $s^i : X^i \rightarrow C^i$ of the player i also satisfies the measurability constraint for an AFR-strategy as in Definition 2.17.

This ends the proof. \square

3.5 Proofs and technicalities

To prove Theorem 3.1 — that is, to show that, for any player $i \in \mathcal{P}$, the constructed set C^i of WtoAFR-choices and the WtoAFR-tree (V, \supset) satisfy (AFR-Axiom1) and (AFR-Axiom2') — we need a bit of preparation. To this end,

- in §3.5.1, we prove Lemma 3.27 that has been used in §3.2.1 and §3.2.2,
- in §3.5.2, we give three partitions of the underlying set \mathbb{H} of configurations and prove that some relevant subsets of the set \mathbb{H} are cylinders,
- in §3.5.3, we give postponed technical results used for the proof of Proposition 3.24,
- in §3.5.4, we give postponed technical results used for the proof of Proposition 3.25.

3.5.1 Lemma about restrictions of an ordering (§3.2.1, §3.2.2)

In this §3.5.1, we prove the result that has been used in §3.2.1 and §3.2.2, namely, we establish technical relations for restrictions of partial orderings.

Lemma 3.27. *For any two indices $0 \leq k'' \leq k' \leq |\mathbb{A}|$, the composition $\psi_{k''} \circ \psi_{k'}$ of the restriction mappings, as in (2.33g), is well-defined as the mapping*

$$(\psi_{k''} \circ \psi_{k'}) : \Sigma^{(\geq k')} \rightarrow \Sigma^{k''}, \quad (3.47a)$$

where the set of orderings $\Sigma^{(\geq k')}$ is as in (2.33e), and the composition $\psi_{k''} \circ \psi_{k'}$ coincides with $\psi_{k''} : \Sigma^{(\geq k'')} \rightarrow \Sigma^{k''}$ restricted to $\Sigma^{(\geq k')}$, that is,

$$(\psi_{k''} \circ \psi_{k'}) (\kappa) = \psi_{k''} (\kappa), \quad \forall \kappa \in \Sigma^{(\geq k')}. \quad (3.47b)$$

Moreover, we have that

$$\psi_{k''} (\kappa) \succeq \psi_{k'} (\kappa), \quad \forall 0 \leq k'' \leq k' \leq |\mathbb{A}|, \quad \forall \kappa \in \Sigma^{(\geq k')}. \quad (3.47c)$$

Proof. Let two indices k' and k'' be given such that $0 \leq k'' \leq k' \leq |\mathbb{A}|$. First, we prove that the composition in Equation (3.47a) is well-defined. Using Equation (2.33g), we have that $\psi_{k'} : \Sigma^{(\geq k')} \rightarrow \Sigma^{k'}$ and $\psi_{k''} : \Sigma^{(\geq k'')} \rightarrow \Sigma^{k''}$. Now, since $k'' \leq k'$ we have that $\psi_{k'}(\Sigma^{(\geq k')}) \subset \Sigma^{k'} \subset \Sigma^{(\geq k'')}$, the codomain of $\psi_{k'}$ is included in the domain of $\psi_{k''}$. Thus, the composition $\psi_{k''} \circ \psi_{k'} : \Sigma^{(\geq k')} \rightarrow \Sigma^{k''}$ is well defined.

Second, Equation (3.47b) follows from

$$\begin{aligned} (\psi_{k''} \circ \psi_{k'})(\kappa) &= (\kappa|_{\{1, \dots, k'\}})|_{\{1, \dots, k''\}} && \text{(by (2.33g) applied for } k = k' \text{ and } k = k'') \\ &= \psi_{k''}(\kappa) . && \text{(by (2.33g) applied for } k = k'') \end{aligned}$$

Third, we prove Equation (3.47c). Let $\kappa \in \Sigma^{(\geq k')}$ be given. Let $\kappa'' = \psi_{k''}(\kappa)$ and $\kappa' = \psi_{k'}(\kappa)$, we want to show that $\kappa'' \succeq \kappa'$. First, we have that $|\kappa''| = |\psi_{k''}(\kappa)| = k'' \leq k' = |\psi_{k'}(\kappa)| = |\kappa'|$. Second, we successively have,

$$\begin{aligned} \psi_{|\kappa''|}(\kappa') &= \psi_{k''}(\kappa') && \text{(as } |\kappa''| = k'') \\ &= \psi_{k''}(\psi_{k'}(\kappa)) && \text{(as } \kappa' = \psi_{k'}(\kappa)) \\ &= \psi_{k''}(\kappa) && \text{(by (3.47b) as } \kappa \in \Sigma^{(\geq k')}) \\ &= \kappa'' . && \text{(as } \kappa'' = \psi_{k''}(\kappa)) \end{aligned}$$

We have thus obtained that $\kappa'' \succeq \kappa'$ by Equation (3.1) which gives Equation (3.47c). This ends the proof. \square

3.5.2 Partitions of the \mathbb{H} of configurations and cylindric subsets

In this §3.5.2, we give three partitions of the underlying set \mathbb{H} of configurations in (2.24a), and we also prove that some relevant subsets of the set \mathbb{H} are cylinders, because of the causality assumption.

Lemma 3.28. *Let be given two orderings $\kappa, \kappa' \in \Sigma^\varphi \setminus \{(\emptyset)\}$, where the set Σ^φ is defined in (2.35c). Then we have the following implications:*

$$\mathbb{H}_\kappa^\varphi \cap \mathbb{H}_{\kappa'}^\varphi \neq \emptyset \implies \kappa' \succeq \kappa \text{ or } \kappa \succeq \kappa' , \quad (3.48a)$$

$$\mathbb{H}_\kappa^\varphi \cap \mathbb{H}_{\kappa'}^\varphi \neq \emptyset \text{ and } |\kappa'| = |\kappa| \implies \kappa = \kappa' , \quad (3.48b)$$

$$\mathbb{H}_\kappa^\varphi \cap \mathbb{H}_{\kappa'}^\varphi \neq \emptyset \text{ and } \kappa^* = \kappa'^* \implies \kappa = \kappa' , \quad (3.48c)$$

$$\mathbb{H}_\kappa^\varphi \cap \mathbb{H}_{\kappa'}^\varphi \neq \emptyset \text{ and } \kappa' \in \Sigma^{|\kappa|+1} \implies \kappa = \kappa'^- . \quad (3.48d)$$

Proof.

• (3.48a). Assume that we have that $\mathbb{H}_\kappa^\varphi \cap \mathbb{H}_{\kappa'}^\varphi \neq \emptyset$. For any configuration $h \in \mathbb{H}_\kappa^\varphi \cap \mathbb{H}_{\kappa'}^\varphi$, we have by (2.34) that $\psi_{|\kappa|}(\varphi(h)) = \kappa$ and, that $\psi_{|\kappa'|}(\varphi(h)) = \kappa'$. Therefore, by definition of the relation \succeq as in (3.1), we have that $\kappa \succeq \varphi(h)$ and $\kappa' \succeq \varphi(h)$. Thus, by definition (3.3) of an upset, we deduce that $\kappa, \kappa' \in \uparrow \{\varphi(h)\}$. As (Σ, \succeq) is a tree by Proposition 3.2, the upset $\uparrow \{\varphi(h)\}$ is a chain and we obtain that whether $\kappa' \succeq \kappa$ or $\kappa \succeq \kappa'$.

• (3.48b). Assume that we have that $\mathbb{H}_\kappa^\varphi \cap \mathbb{H}_{\kappa'}^\varphi \neq \emptyset$ and $|\kappa'| = |\kappa|$. Using Implication (3.48a) we must have that $\kappa \succeq \kappa'$ or $\kappa' \succeq \kappa$. Assume first that $\kappa \succeq \kappa'$, then by lemma 3.4, $\kappa \succeq \kappa'$ is equivalent to $\kappa = \kappa'$ or $\kappa \succeq \kappa'^-$. Since $\kappa \succeq \kappa'^-$ is not possible by cardinality assumption $|\kappa'| = |\kappa|$ we have that $\kappa = \kappa'$. The second case $\kappa' \succeq \kappa$ is treated exactly in the same way and left to the reader.

• (3.48c). Assume that we have that $\mathbb{H}_\kappa^\varphi \cap \mathbb{H}_{\kappa'}^\varphi \neq \emptyset$ and $\kappa^* = \kappa'^*$. Using Implication (3.48a) we must have that $\kappa \succeq \kappa'$ or $\kappa' \succeq \kappa$. Suppose, without loss of generality, that, $\kappa' \succeq \kappa$ (the reasoning for the case $\kappa \succeq \kappa'$ is identical, one needs to swap κ and κ' in what follows). By Lemma 3.4, $\kappa' \succeq \kappa$ is equivalent to $\kappa' = \kappa$ or $\kappa' \succeq \kappa^-$. We prove that $\kappa' \succeq \kappa^-$ is impossible.

Suppose indeed that $\kappa' \succeq \kappa^-$ and denote by $a \in \mathbb{A}$ the common value of $\kappa^* = \kappa'^*$. On the one hand, from $\kappa^* = \kappa'^* = a$, we would deduce that $\kappa(|\kappa|) = \kappa'(|\kappa'|) = a$. On the other hand, from $\kappa' \succeq \kappa$, we would deduce that $\kappa_{|1, \dots, |\kappa'|} = \kappa'$, hence that $\kappa(|\kappa'|) = a$. Combining both results, we would get $\kappa(|\kappa|) = \kappa(|\kappa'|) = a$. But this is impossible because $\kappa' \succeq \kappa^- \implies |\kappa'| \leq |\kappa| - 1$. Therefore, we have obtained that $\kappa' = \kappa$,

• (3.48d) Assume that we have that $\mathbb{H}_\kappa^\varphi \cap \mathbb{H}_{\kappa'}^\varphi \neq \emptyset$ and $\kappa' \in \Sigma^{|\kappa|+1}$. Using Implication (3.48a) we must have $\kappa \succeq \kappa'$ since the other case $\kappa' \succeq \kappa$ would imply that $|\kappa'| \leq |\kappa|$ which is not compatible with $\kappa' \in \Sigma^{|\kappa|+1}$. Now, by lemma 3.4, $\kappa \succeq \kappa'$ is equivalent to $\kappa = \kappa'$ or $\kappa \succeq \kappa'^-$, but $\kappa' = \kappa$ is not possible by cardinality assumption. We therefore have $\kappa \succeq \kappa'^-$ which combined with the fact that $|\kappa'^-| = |\kappa'| - 1 = |\kappa|$ implies again by Lemma 3.4 that $\kappa = \kappa'^-$. \square

Lemma 3.29. *For any agent $a \in \mathbb{A}$, we have the following partition of the configuration space \mathbb{H}*

$$\mathbb{H} = \bigsqcup_{\substack{\kappa \in \Sigma^\varphi \setminus \{(\emptyset)\} \\ \kappa^* = a}} \mathbb{H}_\kappa^\varphi, \quad (3.49)$$

where the set Σ^φ is defined in (2.35c).

Proof. Let $h \in \mathbb{H}$ be given, there always exists $k \in \{1, \dots, |\mathbb{A}|\}$ such that $\kappa = \psi_k(\varphi(h))$ with $\kappa^* = a$. Thus, there always exists $\kappa \in \Sigma^\varphi \setminus \{(\emptyset)\}$ such that $\kappa^* = a$ and $h \in \mathbb{H}_\kappa^\varphi$. We therefore have that $\mathbb{H} = \bigcup_{\kappa \in \Sigma^\varphi \setminus \{(\emptyset)\}, \kappa^* = a} \mathbb{H}_\kappa^\varphi$. Moreover, when $\kappa \in \Sigma^\varphi \setminus \{(\emptyset)\}$ we have that $\mathbb{H}_\kappa^\varphi \neq \emptyset$. It remains to prove that we have a partition. For this purpose we show that if two orderings $\kappa, \kappa' \in \Sigma^\varphi \setminus \{(\emptyset)\}$ are such that $\mathbb{H}_\kappa^\varphi \cap \mathbb{H}_{\kappa'}^\varphi \neq \emptyset$ then we have that $\kappa' = \kappa$ and hence that $\mathbb{H}_\kappa^\varphi = \mathbb{H}_{\kappa'}^\varphi$. This statement is indeed a direct consequence of Implication (3.48c) in Lemma 3.28.

This ends the proof. \square

Proposition 3.30. *Let $\varphi : \mathbb{H} \rightarrow \Sigma_{|\mathbb{A}|}$ be a configuration-ordering as in Definition 2.23. The configuration space \mathbb{H} in (2.24a) can be partitioned as*

$$\mathbb{H} = \bigsqcup_{\rho \in \varphi(\mathbb{H})} \mathbb{H}_\rho^\varphi. \quad (3.50a)$$

Moreover, for any $k \in \{1, \dots, |\mathbb{A}|\}$, we have the following partition

$$\left(\forall k \in \{1, \dots, |\mathbb{A}|\} \right) \quad \mathbb{H} = \bigsqcup_{\kappa \in \psi_k(\varphi(\mathbb{H}))} \mathbb{H}_\kappa^\varphi, \quad (3.50b)$$

where, for any nontotal ordering $\kappa \in \Sigma^\varphi \setminus \varphi(\mathbb{H})$ compatible with the configuration ordering $\varphi : \mathbb{H} \rightarrow \Sigma_{|\mathbb{A}|}$, the corresponding set $\mathbb{H}_\kappa^\varphi$ defined in (2.34) can itself be partitioned in

$$\mathbb{H}_\kappa^\varphi = \bigsqcup_{\substack{\kappa' \in \psi_{|\kappa|+1}(\varphi(\mathbb{H})) \\ (\kappa')^- = \kappa}} \mathbb{H}_{\kappa'}^\varphi, \quad (3.50c)$$

or, equivalently, in

$$\mathbb{H}_\kappa^\varphi = \bigsqcup_{\rho \in \downarrow \kappa \cap \varphi(\mathbb{H})} \mathbb{H}_\rho^\varphi. \quad (3.50d)$$

Proof. In the formulas above, we have restricted the range of the partitions to $\kappa \in \psi_k(\varphi(\mathbb{H}))$ to get rid of the possible empty sets in the family $\{\mathbb{H}_\kappa^\varphi \mid \kappa \in \Sigma_k\}$. Indeed, by definition, a partition does not contain empty sets.

- We start by proving Equation (3.50b). Let $k \in \{1, \dots, |\mathbb{A}|\}$ be fixed. First, we show that $\mathbb{H}_\kappa^\varphi \cap \mathbb{H}_{\kappa'}^\varphi \neq \emptyset$ with $\kappa, \kappa' \in \psi_k(\varphi(\mathbb{H}))$ implies that $\kappa = \kappa'$. Indeed, this is an immediate consequence of Implication (3.48b) in Lemma 3.28 as we have $|\kappa| = |\kappa'| = k$. Second, if $\kappa \in \psi_k(\varphi(\mathbb{H}))$, then there exists $h \in \mathbb{H}$ such that $\kappa = \psi_k(\varphi(h))$ and thus $\mathbb{H}_\kappa^\varphi \neq \emptyset$. We therefore have obtained a partition $\bigsqcup_{\kappa \in \psi_k(\varphi(\mathbb{H}))} \mathbb{H}_\kappa^\varphi$ included in \mathbb{H} . Now, given $h \in \mathbb{H}$, we have that $h \in \mathbb{H}_\kappa$ with $\kappa = \psi_k(\varphi(h))$ and we conclude that the previous inclusion is an equality, that is Equation (3.50b) is established.

- We prove Equation (3.50a). Equation (3.50a) is obtained from Equation (3.50b) by considering the case $k = |\mathbb{A}|$ as we have that $\psi_{|\mathbb{A}|}(\varphi(\mathbb{H})) = \varphi(\mathbb{H})$.

- We prove (3.50c). For any $\kappa \in \Sigma^\varphi \setminus \varphi(\mathbb{H})$ and the corresponding set of configurations $\mathbb{H}_\kappa^\varphi$, we write

$$\begin{aligned} \mathbb{H}_\kappa^\varphi &= \mathbb{H}_\kappa^\varphi \cap \mathbb{H} \\ &= \mathbb{H}_\kappa^\varphi \cap \bigsqcup_{\kappa' \in \psi_{|\kappa|+1}(\varphi(\mathbb{H}))} \mathbb{H}_{\kappa'}^\varphi && \text{(by (3.50b) applied for } k = |\kappa| + 1 \in \{2, \dots, |\mathbb{A}|\}) \\ &= \bigsqcup_{\kappa' \in \psi_{|\kappa|+1}(\varphi(\mathbb{H}))} (\mathbb{H}_{\kappa'}^\varphi \cap \mathbb{H}_\kappa^\varphi) && \text{(distributive property)} \\ &= \bigsqcup_{\substack{\kappa' \in \psi_{|\kappa|+1}(\varphi(\mathbb{H})) \\ (\kappa')^- = \kappa}} (\mathbb{H}_{\kappa'}^\varphi \cap \mathbb{H}_\kappa^\varphi) && \text{(as, using (3.48d), } \mathbb{H}_{\kappa'}^\varphi \cap \mathbb{H}_\kappa^\varphi = \emptyset \text{ when } (\kappa')^- \neq \kappa) \\ &= \bigsqcup_{\substack{\kappa' \in \psi_{|\kappa|+1}(\varphi(\mathbb{H})) \\ (\kappa')^- = \kappa}} \mathbb{H}_{\kappa'}^\varphi. && \text{(as } \mathbb{H}_{\kappa'}^\varphi \cap \mathbb{H}_\kappa^\varphi = \mathbb{H}_{\kappa'}^\varphi \text{ by (3.13) since } \kappa = (\kappa')^- \succeq \kappa') \end{aligned}$$

This ends the proof. \square

In the sequel, we will sometimes use the following reformulation of Equation (3.50c) writing explicitly the “last” agents (as in Equation (2.33i)) in the index of the union

$$\mathbb{H}_\kappa^\varphi = \bigsqcup_{\substack{b \in -\|\kappa\| \\ (\kappa, b) \in \psi_{|\kappa|+1}(\varphi(\mathbb{H}))}} \mathbb{H}_{(\kappa, b)}^\varphi. \quad (3.51)$$

The last statement of this §3.5.2, is Corollary 3.31 giving a result analogous to Equation (3.51) taking into account the cylindric structure of the set $\mathbb{H}_\kappa^\varphi$ of configurations.

Corollary 3.31. *If a configuration-ordering $\varphi : \mathbb{H} \rightarrow \Sigma_{|\mathbb{A}|}$ is causal (as in Definition 2.24), then, for any nonempty ordering $\kappa \in \Sigma^\varphi \setminus \{(\emptyset)\}$, we have the following partition*

$$\overline{\mathbb{H}}_\kappa^\varphi \times \mathbb{U}_{\kappa^*} = \bigsqcup_{\substack{b \in -\|\kappa\| \\ (\kappa, b) \in \psi_{|\kappa|+1}(\varphi(\mathbb{H}))}} \overline{\mathbb{H}}_{(\kappa, b)}^\varphi, \quad (3.52)$$

where, for any nonempty ordering $\kappa \in \Sigma^\varphi \setminus \{(\emptyset)\}$, the corresponding set $\overline{\mathbb{H}}_\kappa^\varphi \subset \Omega \times \overline{\mathbb{U}}_{\kappa^-}$ is as in (2.38a).

Proof. For any nonempty ordering $\kappa \in \Sigma^\varphi \setminus \{(\emptyset)\}$, we write

$$\begin{aligned} (\overline{\mathbb{H}}_\kappa^\varphi \times \mathbb{U}_{\kappa^*}) \times \overline{\mathbb{U}}_{-\|\kappa\|} &= \overline{\mathbb{H}}_\kappa^\varphi \times \overline{\mathbb{U}}_{-\|\kappa\|} \\ &= \mathbb{H}_\kappa^\varphi \quad (\text{by the cylindrical factorization (2.38a) valid by causality}) \\ &= \bigsqcup_{\substack{b \in -\|\kappa\| \\ (\kappa, b) \in \psi_{|\kappa|+1}(\varphi(\mathbb{H}))}} \mathbb{H}_{(\kappa, b)}^\varphi \quad (\text{by (3.51)}) \\ &= \bigsqcup_{\substack{b \in -\|\kappa\| \\ (\kappa, b) \in \psi_{|\kappa|+1}(\varphi(\mathbb{H}))}} (\overline{\mathbb{H}}_{(\kappa, b)}^\varphi \times \overline{\mathbb{U}}_{-\|(\kappa, b)^-\|}) \\ &\quad (\text{by the cylindrical factorization (2.38a) valid by causality}) \\ &= \bigsqcup_{\substack{b \in -\|\kappa\| \\ (\kappa, b) \in \psi_{|\kappa|+1}(\varphi(\mathbb{H}))}} (\overline{\mathbb{H}}_{(\kappa, b)}^\varphi \times \overline{\mathbb{U}}_{-\|\kappa\|}) \\ &\quad (\text{as, for } b \in -\|\kappa\|, (\kappa, b)^- = \kappa \text{ by (2.33l)}) \\ &= \left(\bigsqcup_{\substack{b \in -\|\kappa\| \\ (\kappa, b) \in \psi_{|\kappa|+1}(\varphi(\mathbb{H}))}} \overline{\mathbb{H}}_{(\kappa, b)}^\varphi \right) \times \overline{\mathbb{U}}_{-\|\kappa\|}. \end{aligned}$$

Cancelling out $\overline{\mathbb{U}}_{-\|\kappa\|}$ on both sides of last equality gives Equation (3.52).

This ends the proof. □

3.5.3 Technical results used for the proof of (AFR-Axiom1)

Here, we give postponed technical results used for the proof of Proposition 3.24. The main result of this §3.5.3 is the following Proposition 3.32 giving for any WtoAFR-choice c_{G,u_a} as in (3.33) the image $P(c_{G,u_a})$ under the immediate predecessor mapping P as in (2.18b). The proof relies on a sequence of lemmas that are stated and proved after the proposition.

Proposition 3.32. *Let an agent $a \in \mathbb{A}$ be given. For any information atom $G \in \mathbb{H}/\mathfrak{I}_a$ and any action $u_a \in \mathbb{U}_a$, the immediate predecessor $P(c_{G,u_a})$ of the corresponding choice c_{G,u_a} , as in (3.33) is given by*

$$\left(\forall G \in \mathbb{H}/\mathfrak{I}_a, \forall u_a \in \mathbb{U}_a \right) \quad P(c_{G,u_a}) = \{x \in X_a \mid x \subset G\}, \quad (3.53)$$

where the immediate predecessor mapping P is defined in (2.18b) and the set X_a of WtoAFR-moves of the agent a is defined in (3.35b).

Proof. Let an agent $a \in \mathbb{A}$, an information atom $G \in \mathbb{H}/\mathfrak{I}_a$ of the agent and an action $u_a \in \mathbb{U}_a$ of the agent be given.

By definition of the immediate predecessor mapping P , as in (2.18b) applied for $W = c_{G,u_a}$, we have that

$$P(c_{G,u_a}) = \{x \in V \mid \exists v \in \downarrow c_{G,u_a}, \uparrow x = \uparrow v \setminus \downarrow c_{G,u_a}\},$$

where the downset $\downarrow c_{G,u_a}$ of the choice c_{G,u_a} is given by

$$\downarrow c_{G,u_a} = \{v \in V \mid v \subset c_{G,u_a}\}. \quad (3.54)$$

► First, we prove in Lemma 3.35, that the downset $\downarrow c_{G,u_a}$ is of the form

$$\downarrow c_{G,u_a} = \bigsqcup_{\substack{x \in X_a: \\ x \subset G}} \downarrow (x \cap [u_a]),$$

where, for any move $x \in X_a$ and any action $u_a \in \mathbb{U}_a$, the corresponding element $x \cap [u_a]$ is a vertex, that is, $x \cap [u_a] \in V$ by Lemma 3.33.

► Second, we prove in Lemma 3.36 that the set of maximal elements $\max(\downarrow c_{G,u_a})$ of the downset of the corresponding choice c_{G,u_a} is given by

$$\max(\downarrow c_{G,u_a}) = \{x \cap [u_a] \mid x \in X_a, \text{ s.t. } x \subset G\}.$$

► Finally, in Lemma 3.37, we prove that

$$P(c_{G,u_a}) = \{p(v) \mid v \in \max(\downarrow c_{G,u_a})\},$$

which immediately gives the desired relation (3.53) for any WtoAFR-move $x \in X_a$ of the agent a , we have that $p(x \cap [u_a]) = x \in X_a$ by (3.57) in Lemma 3.34.

This finishes the proof. □

Now, we prove the five postponed lemmas that gave us the proof of Proposition 3.32.

In the following Lemma 3.33, for any information atom $G \in \mathbb{H}/\mathcal{I}_a$ of the agent a , any WtoAFR-move $x \subset G$, and any action $u_a \in \mathbb{U}_a$ of the agent a , we describe explicitly a generic building block $(x \cap [u_a])$ for the choice $c \in C_a$ of the agent a .

Lemma 3.33. *Let a causal configuration ordering $\varphi : \mathbb{H} \rightarrow \Sigma$ be given. For any nonempty ordering $\kappa \in \Sigma^\varphi \setminus \{(\emptyset)\}$ compatible with the causal configuration ordering φ , such that $\kappa^* = a$, any κ^- -prefix $\mathfrak{h}_{\kappa^-} \in \Omega \times \bar{\mathbb{U}}_{\|\kappa^-\|}$, and any action $u_a \in \mathbb{U}_a$ of the agent a , there exists a unique agent $b \in -\|\kappa\|$, such that*

$$x_{\kappa, \mathfrak{h}_{\kappa^-}} \cap [u_a] = \begin{cases} x_{(\kappa, b), (\mathfrak{h}_{\kappa^-}, u_a)} \in X, & \text{if } \kappa \notin \varphi(\mathbb{H}), \\ (\mathfrak{h}_{\kappa^-}, u_a) \in Z, & \text{if } \kappa \in \varphi(\mathbb{H}). \end{cases} \quad (3.55)$$

As the agent b is uniquely determined by the WtoAFR-move $x_{(\kappa, \mathfrak{h}_{\kappa^-})} \in X_a$ and the action $u_a \in \mathbb{U}_a$, we denote it by $B(x_{\kappa, \mathfrak{h}_{\kappa^-}}, u_a)$, that is,

$$b = B(x_{\kappa, \mathfrak{h}_{\kappa^-}}, u_a) \iff x_{\kappa, \mathfrak{h}_{\kappa^-}} \cap [u_a] = x_{(\kappa, b), (\mathfrak{h}_{\kappa^-}, u_a)}. \quad (3.56)$$

Proof. \blacktriangleright Let a move $x \in X_a$ be given and $x = x_{\kappa, \mathfrak{h}_{\kappa^-}}$ for some nonempty ordering $\kappa \in \Sigma^\varphi \setminus \{(\emptyset)\}$, such that $\kappa^* = a$, and for some κ^- -prefix $\mathfrak{h}_{\kappa^-} \in \Omega \times \bar{\mathbb{U}}_{\|\kappa^-\|}$.

- First, if the ordering $\kappa \in \varphi(\mathbb{H})$ is total, we write

$$\begin{aligned} x_{\kappa, \mathfrak{h}_{\kappa^-}} \cap [u_a] &= \mathbb{H}_\kappa^\varphi \cap (\{\mathfrak{h}_{\kappa^-}\} \times \bar{\mathbb{U}}_{-\|\kappa^-\|}) \cap [u_a] \\ &\quad \text{(by definition of WtoAFR-move as in (3.20a))} \\ &= \mathbb{H}_\kappa^\varphi \cap (\{\mathfrak{h}_{\kappa^-}\} \times \mathbb{U}_a) \cap [u_a] \\ &\quad \text{(as } \kappa^* = a \text{ and, for any total ordering } \kappa \in \varphi(\mathbb{H}), \kappa^* = -\|\kappa^-\| \text{ by (2.33j))} \\ &= \mathbb{H}_\kappa^\varphi \cap \{(\mathfrak{h}_{\kappa^-}, u_a)\} \quad \text{(as } u_a \in \mathbb{U}_a \text{ and } \mathfrak{h}_{\kappa^-} \in \Omega \times \bar{\mathbb{U}}_{-a}) \\ &= \{(\mathfrak{h}_{\kappa^-}, u_a)\}. \quad \text{(as } x_{\kappa, \mathfrak{h}_{\kappa^-}} \cap [u_a] \neq \emptyset \text{ by (3.39c))} \end{aligned}$$

- Second, if the ordering $\kappa \notin \Sigma^{|\mathbb{A}|}$ is nontotal, we write

$$\begin{aligned} x_{\kappa, \mathfrak{h}_{\kappa^-}} \cap [u_a] &= \left(\mathbb{H}_\kappa^\varphi \cap (\{\mathfrak{h}_{\kappa^-}\} \times \bar{\mathbb{U}}_{-\|\kappa^-\|}) \right) \cap [u_a] \\ &\quad \text{(by definition of WtoAFR-move as in (3.20a))} \\ &= \mathbb{H}_\kappa^\varphi \cap (\{(\mathfrak{h}_{\kappa^-}, u_a)\} \times \bar{\mathbb{U}}_{-\|\kappa\|}) \quad \text{(as } a = \kappa^* \notin \|\kappa^-\| \text{ by (2.33j))} \\ &= (\bar{\mathbb{H}}_\kappa^\varphi \times \mathbb{U}_a \times \bar{\mathbb{U}}_{\|\kappa^-\|}) \cap (\{(\mathfrak{h}_{\kappa^-}, u_a)\} \times \bar{\mathbb{U}}_{-\|\kappa\|}) \\ &\quad \text{(as } \mathbb{H}_\kappa^\varphi \text{ is a cylinder by (2.38a))} \\ &= \left(\bigsqcup_{\substack{b \in -\|\kappa\| \\ (\kappa, b) \in \psi_{|\kappa|+1}(\varphi(\mathbb{H}))}} \bar{\mathbb{H}}_{(\kappa, b)}^\varphi \cap \{(\mathfrak{h}_{\kappa^-}, u_a)\} \right) \times \bar{\mathbb{U}}_{-\|\kappa\|} \\ &\quad \text{(by partition of } \bar{\mathbb{H}}_\kappa^\varphi \text{ as in (3.52) applied for } \kappa^* = a) \\ &= \left(\bar{\mathbb{H}}_{(\kappa, B(x_{\kappa, \mathfrak{h}_{\kappa^-}}, u_a))}^\varphi \cap \{(\mathfrak{h}_{\kappa^-}, u_a)\} \right) \times \bar{\mathbb{U}}_{-\|\kappa\|} \end{aligned}$$

where the agent $B(x_{\kappa, \mathfrak{h}_{\kappa^-}}, u_a) \in -\|\kappa\|$ is uniquely defined as $(\mathfrak{h}_{\kappa^-}, u_a) \in \overline{\mathbb{H}}_{\kappa}^{\varphi}$ and because of the partition of the set $\overline{\mathbb{H}}_{\kappa}^{\varphi}$ given in (3.52), there is only one atom of this partition containing the κ -prefix $(\mathfrak{h}_{\kappa^-}, u_a)$

$$\begin{aligned} &= \mathbb{H}_{(\kappa, B(x_{\kappa, \mathfrak{h}_{\kappa^-}}, u_a))}^{\varphi} \cap (\{(\mathfrak{h}_{\kappa^-}, u_a)\} \times \mathbb{U}_{-\|\kappa\|}) \\ &\quad \text{(as } \mathbb{H}_{(\kappa, B(x_{\kappa, \mathfrak{h}_{\kappa^-}}, u_a))}^{\varphi} \text{ is a cylinder by (2.38a))} \\ &= x_{(\kappa, B(x_{\kappa, \mathfrak{h}_{\kappa^-}}, u_a)), (\mathfrak{h}_{\kappa^-}, u_a)} \cdot \quad \text{(by definition of WtoAFR-move as in (3.20a))} \end{aligned}$$

This finishes the proof. \square

Now, using the closed forms of a WtoAFR-atom and a WtoAFR-choice established in Lemma 3.23, in the following Lemma 3.34, we deduce a closed form representation of the set of children for any WtoAFR-move.

Lemma 3.34. *Let a causal W -model as in Definition 2.24 be given. For any agent $a \in \mathbb{A}$ and any WtoAFR-move $x \in X_a$ of the agent a , as in (3.36b), the set $p^{-1}(x)$ of children vertices of the move x , as in (2.11c), is given by*

$$p^{-1}(x) = \{x \cap [u_a] \mid u_a \in \mathbb{U}_a\}, \quad (3.57)$$

where for any action $u_a \in \mathbb{U}_a$, the corresponding equivalence class $[u_a] \in \mathbb{H}/\mathfrak{U}_a$ is defined in (3.32b).

Proof. The proof is in two steps.

► (\supset). Let an agent $a \in \mathbb{A}$ and a WtoAFR-move $x \in X_a$ of the agent a be given. We prove that $\{x \cap [u_a] \mid u_a \in \mathbb{U}_a\} \subset p^{-1}(x)$. To obtain the inclusion we prove that $p(x \cap [u_a]) = x$. As $x \in X_a$, then exists a nonempty ordering $\kappa \in \Sigma^{\varphi} \setminus \{(\emptyset)\}$, where $\kappa^* = a$, and a κ^- -prefix $\mathfrak{h}_{\kappa^-} \in \Omega \times \overline{\mathbb{U}}_{\|\kappa^-\|}$, such that $x = x_{\kappa, \mathfrak{h}_{\kappa^-}}$, by (3.36b) and $x_{\kappa, \mathfrak{h}_{\kappa^-}} \neq \emptyset$ by (3.20b).

By Lemma 3.33, we need to consider two cases.

- If the ordering $\kappa \notin \varphi(\mathbb{H})$ is nontotal, we write

$$\begin{aligned} p(x_{\kappa, \mathfrak{h}_{\kappa^-}} \cap [u_a]) &= p(x_{(\kappa, B(x_{\kappa, \mathfrak{h}_{\kappa^-}}, u_a)), (\mathfrak{h}_{\kappa^-}, u_a)}) && \text{(by the first case in (3.55))} \\ &= x_{(\kappa, B(x_{\kappa, \mathfrak{h}_{\kappa^-}}, u_a))^{-}, (\mathfrak{h}_{\kappa^-}, u_a)_{((\kappa, B(x_{\kappa, \mathfrak{h}_{\kappa^-}}, u_a))^{-})^{-}}} \end{aligned}$$

by the second case in (3.27), as the ordering $\kappa \in \Sigma^{\varphi} \setminus \{(\emptyset)\}$ is assumed to be nonempty by the assumption and, thus, $(\kappa, B(x_{\kappa, \mathfrak{h}_{\kappa^-}}, u_a)) \in \Sigma^{(\geq 2)}$

$$\begin{aligned} &= x_{\kappa, (\mathfrak{h}_{\kappa^-}, u_a)_{\kappa^-}} \\ &\quad \text{(as } B(x_{\kappa, \mathfrak{h}_{\kappa^-}}, u_a) \in \mathbb{A}, \text{ then } (\kappa, B(x_{\kappa, \mathfrak{h}_{\kappa^-}}, u_a))^{-} = \kappa \text{ by (2.33l))} \\ &= x_{\kappa, \mathfrak{h}_{\kappa^-}} \quad \text{(as } \kappa^- \succeq (\kappa^-, a), \text{ then } \text{proj}_{\kappa^-}(\mathfrak{h}_{\kappa^-}, u_a) = \mathfrak{h}_{\kappa^-} \text{ by (3.19a))} \end{aligned}$$

- If the ordering $\kappa \in \varphi(\mathbb{H})$ is total, we write

$$\begin{aligned}
p(x_{\kappa, \mathfrak{h}_{\kappa^-}} \cap [u_a]) &= p((\mathfrak{h}_{\kappa^-}, u_a)) && \text{(by the second case in (3.55))} \\
&= x_{(\varphi(\mathfrak{h}_{\kappa^-}, u_a), \mathfrak{h}_{(\varphi(\mathfrak{h}_{\kappa^-}, u_a))^-})} && \text{(by the third case in (3.27) as } (\mathfrak{h}_{\kappa^-}, u_a) \in Z) \\
&= x_{\kappa, \mathfrak{h}_{\kappa^-}} && \text{(as } \kappa^* = a, \text{ then } \varphi(\mathfrak{h}_{\kappa^-}, u_a) = \kappa)
\end{aligned}$$

Thus, we have proved that $p(x_{\kappa, \mathfrak{h}_{\kappa^-}} \cap [u_a]) = x_{\kappa, \mathfrak{h}_{\kappa^-}}$.

- (C). Now, we prove the inverse inclusion.

First, we show that the elements of the set $\{x \cap [u_a] \mid u_a \in \mathbb{U}_a\}$ form a partition of the WtoAFR-move $x \in X_a$ of the agent a , that is, $x = \bigsqcup_{u_a \in \mathbb{U}_a} (x \cap [u_a])$. This immediately follows as $\mathbb{H}/\mathfrak{U}_a$ is a partition of the set \mathbb{H} and, we write

$$x = x \cap \mathbb{H} = x \cap \bigsqcup_{u_a \in \mathbb{U}_a} [u_a] = \bigsqcup_{u_a \in \mathbb{U}_a} (x \cap [u_a]) .$$

Assume that $x' \in p^{-1}(x)$, then we have $x' \subset x = \bigsqcup_{u_a \in \mathbb{U}_a} (x \cap [u_a])$. Using the fact that x' and $x \cap [u_a]$ for $u_a \in \mathbb{U}_a$ are tree nodes, there must exist $u'_a \in \mathbb{U}_a$ such that $x' \subset x \cap [u'_a]$. We therefore have that $x' \subset x \cap [u'_a] \subset x$. As we have assumed that $p(x') = x$ and since $x \cap [u'_a] \neq x$ we must have that $x' = x \cap [u'_a]$. We therefore have that $p^{-1}(x) \subset \{x \cap [u_a] \mid u_a \in \mathbb{U}_a\}$.

This finishes the proof. □

In the following Lemma 3.35, we give a partition of the downset $\downarrow c_{G, u_a}$ of the WtoAFR-choice c_{G, u_a} .

Lemma 3.35. *Let an agent $a \in \mathbb{A}$ be given. For any information atom $G \in \mathbb{H}/\mathfrak{J}_a$ and any action $u_a \in \mathbb{U}_a$, the downset $\downarrow c_{G, u_a}$ of the corresponding WtoAFR-choice c_{G, u_a} is of the form*

$$\downarrow c_{G, u_a} = \bigsqcup_{\substack{x \in X_a: \\ x \subset G}} \downarrow (x \cap [u_a]) . \quad (3.58)$$

Proof. Let an agent $a \in \mathbb{A}$ be given. For any information atom $G \in \mathbb{H}/\mathfrak{J}_a$ and any action $u_a \in \mathbb{U}_a$, consider the corresponding choice c_{G, u_a} as in (3.33) and prove that the downset $\downarrow c_{G, u_a}$ is partitioned as in (3.58) in two steps. First, we show that the candidate atoms of this partition are disjoint. Second, we show that they are maximal.

- For any WtoAFR-move $x \in X_a$ of the agent a , such that $x \subset G$, we have that $x \cap [u_a] \subset c_{G, u_a}$ by representation of the choice c_{G, u_a} , as in (3.39b). Then, for any vertex $v \in \downarrow (x \cap [u_a])$, by definition of the downset as in (2.10), we have that $v \subset x \cap [u_a]$ and, thus, $v \subset c_{G, u_a}$, which proves that

$$\downarrow (x \cap [u_a]) \subset \downarrow c_{G, u_a} , \quad \forall x \in X_a , \quad x \subset G .$$

Now, we show that, for any two distinct WtoAFR-moves $x \neq x' \in X_a$ of the agent a , and any action $u_a \in \mathbb{U}_a$ of the agent a , the corresponding downsets of the moves $x \cap [u_a]$ and $x \cap [u_{a'}]$ are disjoint, that is,

$$\left(\forall x \neq x' \in X_a, \forall u_a \in \mathbb{U}_a \right) \quad \downarrow(x \cap [u_a]) \cap \downarrow(x' \cap [u_a]) = \emptyset. \quad (3.59)$$

Take any two distinct WtoAFR-moves $x, x' \in X_a$ of the agent a . As X_a is a partition of the underlying set \mathbb{H} of configurations, as defined in (3.35b), then $x \cap x' = \emptyset$ and, thus, $(x \cap [u_a]) \cap (x' \cap [u_a]) = \emptyset$.

Now, we prove that the downsets $\downarrow(x \cap [u_a])$ and $\downarrow(x' \cap [u_a])$ are disjoint. As $(x \cap [u_a]) \cap (x' \cap [u_a]) = \emptyset$, then for any $v \in \downarrow(x \cap [u_a])$ and any $v' \in \downarrow(x' \cap [u_a])$ we conclude that $v \cap v' = \emptyset$ as, $v \subset \downarrow(x \cap [u_a])$ and $v' \subset \downarrow(x' \cap [u_a])$ by definition of the downset in (2.10). This proves that $\downarrow(x \cap [u_a]) \cap \downarrow(x' \cap [u_a]) = \emptyset$ and, thus, we have (3.59).

• Finally, for any WtoAFR-move $x \in X_a$ of the agent a , such that $x \subset G$, and any action $u_a \in \mathbb{U}_a$ of the agent a , we prove that

$$(\exists v \in V, \text{ s.t. } x \cap [u_a] \subset v \subset c_{G, u_a}) \Rightarrow v = x \cap [u_a]. \quad (3.60)$$

Take an WtoAFR-move $x \in X_a$ of the agent a . Then, by (3.20b), there exists a nonempty ordering $\kappa \in \Sigma^\varphi \setminus \{(\emptyset)\}$, $\kappa^* = a$ and a κ^- -prefix $\mathfrak{h}_{\kappa^-} \in \Omega \times \overline{\mathbb{U}}_{\|\kappa^-\|}$, such that $x = x_{\kappa, \mathfrak{h}_{\kappa^-}} \neq \emptyset$ by (3.36b).

Assume there is a WtoAFR-vertex $v \in V$ such that $v \supset (x_{\kappa, \mathfrak{h}_{\kappa^-}} \cap [u_a])$. We write

$$\begin{aligned} v &\in \uparrow(x_{\kappa, \mathfrak{h}_{\kappa^-}} \cap [u_a]) \\ \text{(by definition of the upset of a vertex as in (2.10), as } v &\subset (x_{\kappa, \mathfrak{h}_{\kappa^-}} \cap [u_a]) \text{ by the assumption)} \\ &= (x_{\kappa, \mathfrak{h}_{\kappa^-}} \cap [u_a]) \cup \uparrow x. \\ \text{(by representation of the upset as in (2.13a), as } p(x_{\kappa, \mathfrak{h}_{\kappa^-}} \cap [u_a]) &= x \text{ by Lemma 3.34)} \end{aligned}$$

If $v \in \uparrow x_{\kappa, \mathfrak{h}_{\kappa^-}}$, then, by definition of the upset in (2.10), we get that $x_{\kappa, \mathfrak{h}_{\kappa^-}} \subset v$.

We write

$$\begin{aligned} x_{\kappa, \mathfrak{h}_{\kappa^-}} &= (\overline{\mathbb{H}}_\kappa^\varphi \cap \{\mathfrak{h}_{\kappa^-}\}) \times \overline{\mathbb{U}}_{-\|\kappa^-\|} \quad \text{(by representation of the WtoAFR-move as in (3.21))} \\ &= (\overline{\mathbb{H}}_\kappa^\varphi \cap \{\mathfrak{h}_{\kappa^-}\}) \times \mathbb{U}_a \times \overline{\mathbb{U}}_{-\|\kappa\|} \quad \text{(as } a = \kappa^* \notin \|\kappa^-\| \text{ by (2.33j))} \end{aligned}$$

As $x_{\kappa, \mathfrak{h}_{\kappa^-}} \subset v$, then there exists a configuration $\tilde{h} \in x_{\kappa, \mathfrak{h}_{\kappa^-}} \subset v$, such that $\tilde{h}_a = \tilde{u}_a \neq u_a$. This contradicts the fact that $v \subset c_{G, u_a}$, as, for any $h \in c_{G, u_a}$, $h_a = u_a$, by definition of the WtoAFR-choice of the agent a as in (3.33).

Thus, $v = (x_{\kappa, \mathfrak{h}_{\kappa^-}} \cap [u_a]) = (x \cap [u_a])$, which proves (3.60) and, thus, the representation (3.58).

This finishes the proof. \square

In the following Lemma 3.36, we give the expression for the set $\max(\downarrow c_{G, u_a})$ of maximal elements of the downset $\downarrow c_{G, u_a}$.

Lemma 3.36. *Let an agent $a \in \mathbb{A}$ be given. For any information atom $G \in \mathbb{H}/\mathfrak{I}_a$ and any action $u_a \in \mathbb{U}_a$, the set of maximal elements $\max(\downarrow c_{G,u_a})$ of the downset of the corresponding choice c_{G,u_a} is given by*

$$\max(\downarrow c_{G,u_a}) = \{x \cap [u_a] \mid x \in X_a, \text{ s.t. } x \subset G\}. \quad (3.61)$$

Proof. We prove Equation (3.61). We successively have

$$\begin{aligned} \max(\downarrow c_{G,u_a}) &= \max\left(\bigsqcup_{\substack{x \in X_a: \\ x \subset G}} \downarrow(x \cap [u_a])\right) && \text{(by (3.58) in Lemma 3.35)} \\ &= \bigsqcup_{\substack{x \in X_a: \\ x \subset G}} \max(\downarrow(x \cap [u_a])) \\ && \text{(as, in the finite case, max commutes with partition)} \\ &= \bigsqcup_{\substack{x \in X_a: \\ x \subset G}} (x \cap [u_a]), \end{aligned}$$

as, for any WtoAFR-vertex $v \in V$, $v \in \downarrow v$ and for any $v' \in \downarrow v$, $v' \subset v$ by Definition 2.10, thus, it immediately follows that $\max \downarrow v = v$. As, for any move $x \in X_a$ and any action $u_a \in \mathbb{U}_a$, the corresponding element $x \cap [u_a]$ is a vertex, that is, $x \cap [u_a] \in V$ by Lemma 3.33, this gives the last equality and proves (3.61).

This finishes the proof. \square

Finally, in Lemma 3.37, we write the immediate predecessor image $P(c_{G,u_a})$ of the WtoAFR-choice c_{G,u_a} in terms of the parent mapping p is as in (2.11b).

Lemma 3.37. *Let an agent $a \in \mathbb{A}$ be given. For any information atom $G \in \mathbb{H}/\mathfrak{I}_a$ and any action $u_a \in \mathbb{U}_a$, the immediate predecessor $P(c_{G,u_a})$ of the corresponding choice c_{G,u_a} is given by*

$$P(c_{G,u_a}) = \{p(v) \mid v \in \max(\downarrow c_{G,u_a})\}, \quad (3.62)$$

where the parent mapping p is as in (2.11b).

Proof. By definition of the immediate predecessor mapping P , as in (2.18b) applied for $W = c_{G,u_a}$, we have that

$$P(c_{G,u_a}) = \{x \in V \mid \exists v \in \downarrow c_{G,u_a}, \uparrow x = \uparrow v \setminus \downarrow c_{G,u_a}\}, \quad (3.63)$$

where the downset $\downarrow c_{G,u_a}$ of the choice c_{G,u_a} is given by (3.54).

The proof is in two steps. First, we show that

$$\forall x \in X_a, x \subset G \implies x \not\subset c_{G,u_a}. \quad (3.64)$$

Second, we prove (3.62).

► Let a WtoAFR-choice $c_{G,u_a} \in C_a$ of the agent a be given. First, we prove (3.64). For any WtoAFR-move $x \in X_a$ of the agent a , such that $x \subset G$, we write

$$x = x_{\kappa, \mathfrak{h}_{\kappa^-}}$$

as any WtoAFR-move $x \in X_a$, is of the form $x = x_{\kappa, \mathfrak{h}_{\kappa^-}}$, for some nonempty ordering $\kappa \in \Sigma^\varphi \setminus \{(\emptyset)\}$, such that $\kappa^* = a$, and a κ^- -prefix $\mathfrak{h}_{\kappa^-} \in \overline{\mathbb{U}}_{\|\kappa^-\|}$, by (3.36b)

$$\begin{aligned} &= (\overline{\mathbb{H}}_\kappa^\varphi \cap \mathfrak{h}_{\kappa^-}) \times \overline{\mathbb{U}}_{-\|\kappa^-\|} && \text{(by representation of the WtoAFR-move as in 3.21)} \\ &= (\overline{\mathbb{H}}_\kappa^\varphi \cap \mathfrak{h}_{\kappa^-}) \times \mathbb{U}_a \times \overline{\mathbb{U}}_{-\|\kappa^-\|} . && \text{(as } a = \kappa^* \notin \|\kappa^-\| \text{ by (2.33j))} \end{aligned}$$

Thus, there exists a configuration $\tilde{h} \in x_{\kappa, \mathfrak{h}_{\kappa^-}}$, such that $\tilde{h}_a = \tilde{u}_a \neq u_a$. This proves that $x_{\kappa, \mathfrak{h}_{\kappa^-}} \not\subset c_{G,u_a}$ as $c_{G,u_a} = G \cap [u_a]$, by definition of the WtoAFR-choice of the agent a as in (3.33).

► Finally, we prove (3.62) in two substeps.

• (C) Any vertex $v' \in \max(\downarrow c_{G,u_a})$, is of the form $v' = x \cap [u_a]$ for some WtoAFR-move $x \in X_a$ of the agent a such that $x \subset G$, by (3.61).

We show that there exists a vertex $v \in \downarrow c_{G,u_a}$, such that $\uparrow x = \uparrow v \setminus \downarrow c_{G,u_a}$, that is, $x \in P(c_{G,u_a})$. Show that $v = x \cap [u_a]$ gives the result. We write

$$\begin{aligned} \uparrow(x \cap [u_a]) \setminus \downarrow c_{G,u_a} &= \left((x \cap [u_a]) \cup \uparrow p(x \cap [u_a]) \right) \setminus \downarrow c_{G,u_a} \\ &\text{(as, for any nonroot vertex } v, \text{ representation } \uparrow v = \{v\} \cup \uparrow p(v) \text{ holds by (2.13a))} \\ &= \left((x \cap [u_a]) \cup \uparrow x \right) \setminus \downarrow c_{G,u_a} && \text{(as } p(x \cap [u_a]) = x, \text{ by (3.57))} \\ &= \uparrow x , \end{aligned}$$

as $(x \cap [u_a]) \in \downarrow c_{G,u_a}$ by assumption and $x \not\subset c_{G,u_a}$ by (3.64), thus, $x \notin \downarrow c_{G,u_a}$ and neither of the elements of the upset $\uparrow x$, as, for any $x' \in \uparrow x$, we have the inclusion $x \subset x'$ by (2.10).

This proves that $x \in P(c_{G,u_a})$, and, thus, $\{p(v) \mid v \in \max(\downarrow c_{G,u_a})\} \subset P(c_{G,u_a})$.

• (D) In order to prove the inverse inclusion, we take a vertex $\tilde{x} \in P(c_{G,u_a})$ and we show that there exists a vertex $\tilde{v} \in \max \downarrow c_{G,u_a}$ such that $p(\tilde{v}) = \tilde{x}$.

By definition of the immediate predecessor $P(c_{G,u_a})$ of the WtoAFR-choice c_{G,u_a} as in (3.63), for $\tilde{x} \in P(c_{G,u_a})$, there exists a vertex $v \in \downarrow c_{G,u_a}$, such that

$$\uparrow \tilde{x} = \uparrow v \setminus \downarrow c_{G,u_a} . \tag{3.65}$$

As the downset $\downarrow c_{G,u_a}$ of the WtoAFR-choice c_{G,u_a} is of the form

$$\downarrow c_{G,u_a} = \bigsqcup_{\substack{x \in X_a: \\ x \subset G}} \downarrow(x \cap [u_a]) .$$

by (3.58), then, for $v \in \downarrow c_{G,u_a}$, there exists a WtoAFR-move $x \in X_a$ of the agent a , such that $x \subset G$ and $v \in \downarrow(x \cap [u_a])$.

We show that $\uparrow v \setminus \downarrow c_{G,u_a} = \uparrow x$, then, $\uparrow x = \uparrow \tilde{x}$ and, thus, $x = \tilde{x}$ by (3.24). Then, for $(x \cap [u_a]) \in \max \downarrow c_{G,u_a}$ (which holds true by (3.61)), we have that $p(x \cap [u_a]) = \tilde{x}$ by (3.57).

On the one hand, as $v \in \downarrow c_{G,u_a}$, then $v \subset (x \cap [u_a])$ by definition of the downset of a vertex as in (2.10). As $p(x \cap [u_a]) = x$ by (3.57), then $v \subset x$ (as, for any vertex $v \in V$, $p(v) \in \uparrow v$ by (2.11b) and, thus, $v \subset p(v)$ by definition of the upset as in (2.10)), that is, $x \in \uparrow v$.

On the other hand, $x \in X_a$ and $x \subset G$, then, $x \not\subset c_{G,u_a}$ by (3.64) and, by definition of the downset $\downarrow c_{G,u_a}$ of the WtoAFR-choice c_{G,u_a} as in (3.54), $x \notin \downarrow c_{G,u_a}$.

Then, we write

$$\begin{aligned} x \in \uparrow v \setminus \downarrow c_{G,u_a} & & (\text{as } x \in \uparrow v \text{ and } x \notin \downarrow c_{G,u_a}) \\ = \uparrow \tilde{x}, & & (\text{by assumption (3.65)}) \end{aligned}$$

which implies that $\tilde{x} \subset x$ by definition of the upset if a vertex as in (2.10).

If $\tilde{x} \subset (x \cap [u_a]) \subset x$, then $\tilde{x} \subset c_{G,u_a}$, as $(x \cap [u_a]) \subset c_{G,u_a}$ by (3.61), that is, $\tilde{x} \in \downarrow c_{G,u_a}$ by definition of the downset $\downarrow c_{G,u_a}$ of the WtoAFR-choice c_{G,u_a} as in (3.54), which contradicts (3.65) ($\tilde{x} \in \uparrow \tilde{x}$, but $\uparrow \tilde{x} \cap \downarrow c_{G,u_a} = \emptyset$).

If $(x \cap [u_a]) \subsetneq \tilde{x} \subset x$, then, by definition of the upset of a vertex as in (2.10), $\tilde{x} \in \uparrow (x \cap [u_a])$, and, as $(x \cap [u_a]) \neq \tilde{x}$, then $\tilde{x} \in \uparrow (x \cap [u_a]) \setminus \{(x \cap [u_a])\}$. At the same time, $p(x \cap [u_a]) = x$ by (3.57), that is,

$$x = \min \left(\uparrow (x \cap [u_a]) \setminus \{(x \cap [u_a])\} \right),$$

by definition of the parent as in (2.11b), then, as $\tilde{x} \subset x$, we conclude that $\tilde{x} = x$, which proves the inverse inclusion and, thus, yields (3.62).

This ends the proof. □

3.5.4 Technical results used for the proof of (AFR-Axiom2)

In this §3.5.4, we give two lemmas justifying the two last transitions in the proof of Proposition 3.25.

Lemma 3.38. *For any WtoAFR-move $x \in X_a$ of the agent a , and the unique information atom $G \in \mathbb{H}/\mathfrak{I}_a$ of the agent a containing it, that is, $x \subset G$, we have the following relation*

$$\begin{aligned} \{x \cap c \mid c \in C_a \text{ such that } c = c_{G,u_a} \text{ for some } u_a \in \mathbb{U}_a\} & & (3.66) \\ = \{x \cap c \mid c \in C_a \text{ such that } x \cap c \neq \emptyset\}, & & \end{aligned}$$

where, for any $u_a \in \mathbb{U}_a$, the WtoAFR-choice c_{G,u_a} is as in (3.33).

Proof. For any WtoAFR-move $x \in X_a$ of the agent a , and the unique information atom $G \in \mathbb{H}/\mathfrak{I}_a$, such that $x \subset G$, we prove (3.66). The proof is in two step as we need to show two inclusions.

(\subset) As the WtoAFR-move $x \in X_a$ of the agent a is a subset of G , that is, $x \subset G$, we write

$$\begin{aligned} x \cap c_{G,u_a} &= x \cap (G \cap [u_a]) \quad (\text{by definition of the WtoAFR-choice } c = c_{G,u_a} \text{ as in (3.33)}) \\ &= (x \cap G) \cap (x \cap [u_a]) \\ &\neq \emptyset. \quad (\text{as } x \subset G \text{ and for any } x \in X_a \text{ and } u \in \mathbb{U}_a, x \cap [u] \neq \emptyset \text{ by (3.39c)}) \end{aligned}$$

(\supset) As the WtoAFR-move $x \in X_a$ of the agent a is a subset of G , that is, $x \subset G$ and assuming that $x \cap c \neq \emptyset$, for some WtoAFR-choice $c \in C_a$ of the agent a , we write

$$x \cap c = x \cap c_{G',u_a}$$

as $c \in C_a$, then $c = c_{G',u_a}$ for some $G' \in \mathbb{H}/\mathcal{I}_a$ and $u_a \in \mathbb{U}_a$, by characterization (3.34a) of the set C_a of WtoAFR-choices of the agent a

$$\begin{aligned} &= x \cap (G' \cap [u_a]) \quad (\text{by definition of the WtoAFR-choice } c = c_{G',u_a} \text{ as in (3.33)}) \\ &= (x \cap G') \cap (x \cap [u_a]), \end{aligned}$$

where $x \cap [u_a] \neq \emptyset$ for any $u_a \in \mathbb{U}_a$ by (3.39c), and, as $x \cap c \neq \emptyset$ by assumption, we conclude that $x \cap G' \neq \emptyset$. From this follows that $x \subset G'$ as $\mathbb{H}/\mathcal{I}_a \preceq \mathbb{H}/\mathfrak{M}_a^\varphi$ by (3.38b). At the same time, $x \subset G$ and, thus, $G = G'$ as they are atoms of the information partition \mathbb{H}/\mathcal{I}_a of the agent a . This proves the inverse inclusion and, thus, the relation (3.66).

This finishes the proof. \square

Lemma 3.39. *For any WtoAFR-move $x \in X_a$ of the agent a , and the unique information atom $G \in \mathbb{H}/\mathcal{I}_a$ of the agent a containing it, that is, $x \subset G$, we have the following relation*

$$\{x \cap c \mid c \in C_a : x \cap c \neq \emptyset\} = \{x \cap c \mid c \in C_a : x \in P(c)\}, \quad (3.67)$$

where the immediate predecessor P is as in (2.18b).

Proof. For any WtoAFR-move $x \in X_a$ of the agent a , we prove (3.67).

(\subset) As, for a given WtoAFR-move $x \in X_a$, the unique information atom $G \in \mathbb{H}/\mathcal{I}_a$ containing it, and, for a given WtoAFR-choice $c \in C_a$, $x \cap c \neq \emptyset$, then, by (3.66), $c = c_{G,u_a}$ for some action $u_a \in \mathbb{U}_a$.

Moreover, as $x \cap c \neq \emptyset$, then $x \cap (G \cap [u_a]) \neq \emptyset$ by definition of the WtoAFR-choice c_{G,u_a} as in (3.33). As, for any $x \in X_a$ and $u \in \mathbb{U}_a$, $x \cap [u] \neq \emptyset$ by (3.39c), then, $x \cap G \neq \emptyset$ and $x \subset G$ as $\mathbb{H}/\mathcal{I}_a \preceq \mathbb{H}/\mathfrak{M}_a^\varphi$ by (3.38b). At the same time, by (3.53) we have that

$$P(c_{G,u_a}) = \{x' \in X_a \mid x' \subset G\},$$

hence we get $x \in P(c_{G,u_a})$, and the following inclusion is proved

$$\{x \cap c \mid c \in C_a : x \cap c \neq \emptyset\} \subset \{x \cap c \mid c \in C_a : x \in P(c)\}.$$

(\supset) For any choice $c \in \mathbb{H}/\mathcal{I}_a \vee \mathbb{H}/\mathcal{U}_a$, there exists an information atom $G' \in \mathbb{H}/\mathcal{I}_a$ and an action $u_a \in \mathbb{H}/\mathcal{U}_a$, such that $c = c_{G',u_a}$ by characterization (3.34a) of the set C_a of WtoAFR-choices of the agent a . Then, by (3.53), we get that

$$P(c_{G',u_a}) = \{x' \in X_a \mid x' \subset G'\}.$$

As $x \in P(c)$ by assumption, we get that $x \subset G'$. At the same time, we have that $x \subset G$, which implies that $G = G'$ as they are both atoms of the partition \mathbb{H}/\mathcal{I}_a . Then, we write

$$\begin{aligned} x \cap c &= x \cap c_{G,u_a} \\ &= x \cap (G \cap [u_a]) && \text{(by definition of the WtoAFR-choice } c_{G',u_a} \text{ as in (3.33))} \\ &= x \cap [u_a] && \text{(as } x \subset G) \\ &\neq \emptyset, && \text{(as, for any } x \in X_a \text{ and } u \in \mathbb{U}_a, x \cap [u_a] \neq \emptyset \text{ by (3.39c))} \end{aligned}$$

which proves the inverse inclusion and, thus, the relation (3.67). \square

3.6 Example: absent-minded driver

The absent-minded driver example was thoroughly discussed in §2.5.1. Here, we are going to build an equivalent AFR-model with a single player $\mathcal{P} = \{\{a, b\}\}$, specifying its two main objects: the set of vertices (V, \supset) and set $C = C_a \cup C_b$ of WtoAFR-choices of the player respectively.

3.6.1 WtoAFR-tree

In the Gilboa's doubling agent trick [10], Nature decides on the agents' order in the initial W-model

$$\Omega = \{\omega^+, \omega^-\},$$

and the agents' action sets are as follows

$$\mathbb{U}_a = \{T_a, S_a\}, \quad \mathbb{U}_b = \{T_b, S_b\},$$

thus, the set \mathbb{H} of configurations is

$$\mathbb{H} = \Omega \times \mathbb{U}_a \times \mathbb{U}_b = \{\omega^+, \omega^-\} \times \{T_a, S_a\} \times \{T_b, S_b\},$$

and, finally, the set of agents' orderings is

$$\Sigma = \{(ab), (ba)\}$$

The configuration ordering is then

$$\varphi(h) = \begin{cases} (a, b), & \text{if } h \in \{\omega^+\} \times \mathbb{U}_a \times \mathbb{U}_b, \\ (b, a), & \text{if } h \in \{\omega^-\} \times \mathbb{U}_a \times \mathbb{U}_b \end{cases}$$

and the set of WtoAFR-plays is given by the set \mathbb{H} of configurations.

The following is the list of primitives of the constructed WtoAFR-model.

- Root $\{\mathbb{H}\}$,
- Vertices of level 1:
 $\{\{\omega^+\} \times \{S_a, T_a\} \times \{S_b, T_b\} \times \{(ab)\}, \{\omega^-\} \times \{S_a, T_a\} \times \{S_b, T_b\} \times \{(ba)\}\}$
- Vertices of level 2:
 $\{\{\omega^+\} \times \{S_a\} \times \{S_b, T_b\} \times \{(ab)\}, \{\omega^+\} \times \{T_a\} \times \{S_b, T_b\} \times \{(ab)\}, \{\omega^-\} \times \{S_a, T_a\} \times \{S_b\} \times \{(ba)\}, \{\omega^-\} \times \{S_a, T_a\} \times \{T_b\} \times \{(ba)\}\}$

3.6.2 WtoAFR-choices and information sets

From the way the agents' information partition fields were defined in §2.5.1.3, we deduce the partitions of the set \mathbb{H} of configurations.

$$\mathbb{H}/\mathfrak{I}_a = \{\emptyset, \{\omega^+\} \times \mathbb{U}_a \times \mathbb{U}_b \times \{ab\} \cup \{\omega^-\} \times \mathbb{U}_a \times \{S_b\} \times \{ba\}, \{\omega^-\} \times \mathbb{U}_a \times \{T_b\} \times \{ba\}, \mathbb{H}\},$$

and the information partition $\mathbb{H}/\mathfrak{I}_b$ of agent b is defined in the same way.

Then, we build the two auxiliary action-projection partitions:

$$\mathbb{H}/\mathfrak{U}_a = \{\Omega \times \{T_a\} \times \mathbb{U}_b, \Omega \times \{S_a\} \times \mathbb{U}_b\},$$

$$\mathbb{H}/\mathfrak{U}_b = \{\Omega \times \mathbb{U}_a \times \{T_b\}, \Omega \times \mathbb{U}_a \times \{S_b\}\}.$$

With these auxiliary partitions, the choices of the agents a and b are $C_a = \mathbb{H}/\mathfrak{U}_a \vee \mathbb{H}/\mathfrak{I}_a$ and $C_b = \mathbb{H}/\mathfrak{U}_b \vee \mathbb{H}/\mathfrak{I}_b$, respectively. The player represented by these two agents has the following collection of choices:

$$C = C_a \cup C_b.$$

3.7 Conclusion

In this Chapter 3, we have shown that a causal W-model with complete π -fields can be embedded into an AFR-model. The embedding is performed thanks to the information defined as a partition field of the underlying product set of configurations and the axiom of causality imposed on the basic objects.

Here is a summary sketch of the embedding. We deduce the tree structure of the AFR-model out of a causal configuration ordering of the W-model (assumed to exist by the causality axiom). For this purpose, we take the set of configurations as the underlying set of plays. The tree is being built in two steps: first, we define the so-called skeleton poset of configurations; second, we refine the skeleton poset intersecting the vertices with atoms of the so-called vertex partition. Finally, we define choices as the intersections of the atoms of two π -fields: the information π -field, that is straightforwardly inherited from the W-model, and the action π -field, obtained by projecting the set of configurations onto the action set of each agent.

Chapter 4

From AFR-model to W-model

4.1 Introduction

In this Chapter, we show that an AFR-model, as described in §2.3.3, can be embedded into a causal W-model with partition-fields, as described in §2.4.2. We will stick to the following roadmap.

- In Section 4.2, we construct the three primitives of a W-model from an AFR-model:
 - a set of agents,
 - actions sets, one for each agent, equipped with the complete π -field; a sample space of Nature, equipped with the complete π -field; the product of the sample space with the action sets forms the set of configurations, equipped with the corresponding configuration product field,
 - agent information π -fields that are subfields of the configuration product field.
- In Section 4.3, we construct W-strategies, as defined in §2.4.3, from AFR-strategies, as defined in §2.3.4. We are leaving open the questions of causality and solvability of the constructed model, see §9.2.1 and §9.2.2 of the Appendix respectively for more details.

4.2 From AFR-primitives to W-primitives

Using two objects of the AFR-model the *tree* (V, \supset) supported on the set \mathbb{W} of plays and the *collection of players' choices* $\mathcal{C} = \{C^i\}_{i \in \mathcal{P}}$, we build a W-model first by constructing *agents* together with their *action sets* and a *sample space* that corresponds to the set of all possible uncertainties, thus yielding the underlying product *set of configurations*. Each of the constructed set of actions and the sample space is equipped with the corresponding complete π -field, which all together form a product *configuration field*. Each agent *information π -field* is a subfield of the constructed configuration field.

4.2.1 Recalls on AFR-primitives

Given below is the list of objects of the AFR-model as in §2.3.2: BO are basic objects and CO are constructed ones and mappings defined on them. For the sake of clarity, we assume that the AFR-model we are working with is finite.

(AFR-BO0) \mathbb{W} is the underlying set of *plays*.

(AFR-BO1) (V, \supset) is a finite game *tree* with a finite *set of vertices* $V \subset 2^{\mathbb{W}}$ (see Definition 2.11).

(AFR-BO2) $C^i \subset 2^{\mathbb{W}}$ is the *set of choices* available to player i ; each choice is a union of vertices.

(AFR-CO1) $P : 2^{\mathbb{W}} \rightarrow 2^V$ is the *immediate predecessor mapping*.

For a subset of plays $W \subset \mathbb{W}$, its *set of immediate predecessors* is $P(W) = \{v \in V : \exists v' \in \downarrow W : \uparrow v = \uparrow v' \setminus \downarrow W\}$ as in Definition 2.14. In the same vein can be defined the *immediate successor mapping* $P^{-1} : 2^V \rightarrow 2^{\mathbb{W}}$

(AFR-CO2) $X^i = \{v \in V \mid \exists c \in C^i : v \in P(c)\}$ is the *set of moves* (decision points) of player i .

4.2.2 Constructing AFRtoW-primitives

As, in a W-model, we can say that the Nature player moves once and for all “before” the agents, we will start from a Nature-rooted AFR-model (as in Definition 2.16) in order to build an equivalent AFRtoW-model. We construct AFRtoW-objects in two steps: first, for each individual player, we construct a set of so-called *Selten-agents* and their information sets, roughly speaking, inhabiting each information set with a separate Selten-agent. In fact, it is possible to do better than constructing agents in such a lavish way, and we will define an equivalence relation on the set of Selten-agents and then “slice” each element of this partition into AFRtoW-agents defining their action afterwards.

From now on, we write the set of players $\mathcal{P}_0 = \{0\} \cup \mathcal{P}$, separating the Nature player 0 from the set of individual players \mathcal{P} .

4.2.2.1 Constructing AFRtoW-agents, AFRtoW-action sets, AFRtoW-sample space and AFRtoW-information π -fields

We start by defining agents, their action sets together with the sample space and the product set of configurations.

- We define the set of AFRtoW-agents \mathbb{A} by

$$\mathbb{A} = \{a \mid a = P(c), c \in C^i, i \in \mathcal{P}\} = \bigcup_{i \in \mathcal{P}} P(C^i). \quad (4.1a)$$

Also, for every player $i \in \mathcal{P}$, we define the set \mathbb{A}^i of representative AFRtoW-agents of the player i by

$$\mathbb{A}^i = P(C^i) . \quad (4.1b)$$

- For the Nature player 0 and the collection C^0 of Nature's choices, the *AFRtoW-sample space* Ω is defined by

$$\Omega = C^0 . \quad (4.2)$$

- For any AFRtoW-agent $a \in \mathbb{A}$, the set \mathbb{U}_a of agent a 's actions is defined by

$$\mathbb{U}_a = P^{-1}(a) . \quad (4.3)$$

- The set \mathbb{H} of *AFRtoW-configurations* we defined by

$$\mathbb{H} = \Omega \times \prod_{a \in \mathbb{A}} \mathbb{U}_a = C^0 \times \prod_{i \in \mathcal{P}} \prod_{a \in P(C^i)} P^{-1}(a) . \quad (4.4)$$

To define information, we need more notations.

- For any vertex $v \in V$, the set $\mathbb{A}_v \subset \mathbb{A}$ of agents that played before the vertex v was reached we define by

$$\mathbb{A}_v = \{a \in \mathbb{A} \mid \exists i' \in \mathcal{P}, a \in \mathbb{A}_{i'}, \text{ and } \exists v' \in \uparrow v \setminus \{v\}, v' \in a\} , \quad (4.5)$$

where the upset $\uparrow v \subset V$ is defined in (2.10).

For any vertex $v \in V$, we define the subset $H_v \subset \mathbb{H}$ of AFRtoW-configurations as the following cylinder

$$H_v = \{\omega(v)\} \times \prod_{a \in \mathbb{A}_v} \{u_a(v)\} \times \prod_{b \in \mathbb{A} \setminus \mathbb{A}_v} \mathbb{U}_b , \quad (4.6)$$

where the set \mathbb{A}_v of agents is defined in (4.5) and where, for any agent $a \in \mathbb{A}_v$, there is a unique action $u_a(v) \in \mathbb{U}_a$ of the agent a containing the vertex v as in (4.10) (the existence and uniqueness of such $u_a(v)$ is proved in the postponed Lemma 4.1) and, analogously, $\omega(v) \in \Omega$ is the unique element of the sample space containing the vertex v , that is, characterized by $v \subset u_a(v)$ and $v \subset \omega(v)$ respectively.

- For any agent $a \in \mathbb{A}$ (which, as defined in (4.1a), is a set of vertices, that is, $a \subset V$) we define the agent a 's *AFRtoW-information π -field* by

$$\mathcal{I}_a = \left\{ \emptyset, \bigcup_{v \in a} H_v, \mathbb{H} \setminus \bigcup_{v \in a} H_v, \mathbb{H} \right\} , \quad (4.7)$$

where the set H_v of configurations is defined in (4.6).

Note the information of a W-agent, in general, is richer than the one of a AFRtoW-agent. Each AFRtoW-agent is endowed with a very special binary information partition – the agent knows the information atom defining him and can only distinguish it from the rest of the world.

We finish by giving the following postponed Lemma 4.1 that made it possible to write (4.6).

Lemma 4.1. *Let a Nature-rooted AFR-model be given as in Definition 2.16. For any vertex $v \in V$ and any move $x \in \uparrow v \setminus \{v\}$,*

- *there is a unique AFRtoW-agent $a(x) \in \mathbb{A}_v$, where the set \mathbb{A} of AFRtoW-agents is as in (4.1a), such that $x \in a(x)$, that is,*

$$(\forall v \in V, \forall x \in \uparrow v \setminus \{v\}) \quad \exists! a(x) \in \mathbb{A}_v, \quad x \in a(x), \quad (4.8a)$$

and is characterized by

$$a(x) = P(c), \quad \forall c \in C^{i(x)}, \quad x \in P(c) \quad (4.8b)$$

and that satisfies

$$P^{-1}(a(x)) = \{c \in C^{i(x)} \mid P(c) = a(x)\} = \{c \in C^{i(x)} \mid x \in P(c)\}. \quad (4.8c)$$

- *the set $A_{i(x)}(x)$ of actions of the player $i(x)$ active at the move x as in Definition 2.16, coincides with the set $\mathbb{U}_{a(x)}$ of actions of the AFRtoW-agent $a(x)$ active at the move x , as defined in (4.3), that is,*

$$(\forall x \in X) \quad A_{i(x)}(x) = \mathbb{U}_{a(x)}, \quad (4.9)$$

- *for any AFRtoW-agent $a \in \mathbb{A}_v$, where the set \mathbb{A}_v is defined in (4.5), there exists a unique action $u_a(v) \in \mathbb{U}_a$, such that $v \subset u_a(v)$, that is,*

$$(\forall v \in V, \forall a \in \mathbb{A}_v) \quad \exists! u_a(v) \in \mathbb{U}_a, \quad v \subset u_a(v). \quad (4.10)$$

Proof. The proof is in tree steps. Let a vertex $v \in V$ and a move $x \in \uparrow v \setminus \{v\}$ be given.

► It follows from (AFR-Axiom1'), as in Definition 2.16, that, for each player $i \in \mathcal{P}$, the corresponding set X^i of moves, as defined in (2.22a), is partitioned by

$$X^i = \bigsqcup_{a \in \mathbb{A}^i} a, \quad (4.11)$$

where the set \mathbb{A}^i of representative AFRtoW-agents of the player i is defined in (4.1b). Indeed, for any two representative AFRtoW-agents $a, b \in \mathbb{A}^i$ of the player i , there exist

choices $c_a, c_b \in C^i$, such that $a = P(c_a)$ and $b = P(c_b)$ by (2.22a). If $a \cap b \neq \emptyset$, that is, $P(c_a) \cap P(c_b) \neq \emptyset$, then, by (AFR-Axiom1'), we conclude that $P(c_a) = P(c_b)$ and $a = b$.

Thus, for the move $x \in \uparrow v \setminus \{v\} \subset X$, there exists a unique player $i(x)$, such that $x \in X^{i(x)}$ by (2.22b), and there exists a unique AFRtoW-agent $a(x) \in \mathbb{A}^{i(x)}$, such that $x \in a(x)$ by (4.11) applied for $i(x)$.

► We write

$$\begin{aligned} A_{i(x)}(x) &= \{c \in C^{i(x)} \mid x \in P(c)\} && \text{(by definition of the set } A_{i(x)}(x) \text{ as in (2.21))} \\ &= P^{-1}(a(x)) && \text{(by characterization of the agent } a(x) \text{ as in (4.8c))} \\ &= \mathbb{U}_{a(x)}. && \text{(by (4.3) applied for } a(x)) \end{aligned}$$

► First, we show that $p^{-1}(x) \cap \uparrow v$ is a singleton. Second, we use (AFR-Axiom2') to prove (4.10).

• Assume there are two vertices $v', v'' \in p^{-1}(x) \cap \uparrow v$.

As $v', v'' \in \uparrow v$, and $\uparrow v$ is a chain, then $v'' \supset v'$ or $v' \supset v''$. Without loss of generality, we assume that $v' \supset v''$, that is, $v' \in \uparrow v''$ by definition of the upset as in (2.10). We prove that $v' \neq v''$ leads to a contradiction.

To this end, we write

$$\begin{aligned} v' \in \uparrow v'' \setminus \{v''\} &&& \text{(as } v' \in \uparrow v'', \text{ but } v' \neq v'' \text{ by the assumption)} \\ &= \uparrow x && \text{(by (2.13a) as } x = p(v'')\text{)} \\ &= \uparrow v' \setminus \{v'\}, && \text{(by (2.13a) as } x = p(v')\text{)} \end{aligned}$$

which brings us to a contradiction and proves that $v' = v''$. Thus, there exists a unique $v' = v'(v) \in p^{-1}(x)$, such that $p^{-1}(x) \cap \uparrow v = \{v'(v)\}$.

• Finally, we write

$$\begin{aligned} p^{-1}(x) &= \{x \cap c \mid c \in A_{i(x)}(x)\} && \text{(by (AFR-Axiom2') in Definition 2.16)} \\ &= \{x \cap u_{a(x)} \mid u_{a(x)} \in \mathbb{U}_{a(x)}\}, \end{aligned}$$

where the agent $a(x) \in \mathbb{A}_v$ is uniquely defined by (4.8a) and $A_{i(x)}(x) = \mathbb{U}_{a(x)}$ by (4.9).

On the one hand, we consider the vertex $v'(v) \in p^{-1}(x)$, which is uniquely defined by $p^{-1}(x) \cap \uparrow v = \{v'(v)\}$, thus, $v \subset v'(v)$. On the other hand, there exists unique $u_{a(x)}(v') \in \mathbb{U}_{a(x)}$, such that $v'(v) = x \cap u_{a(x)}(v')$ as $p^{-1}(x) = \{x \cap u_{a(x)} \mid u_{a(x)} \in \mathbb{U}_{a(x)}\}$, that is, $v'(v) \subset u_{a(x)}(v')$.

As the vertex $v'(v)$ is uniquely determined by v , we set $u_{a(x)}(v') = u_{a(x)}(v)$ and conclude that $v \subset v'(v) \subset u_{a(x)}(v)$. As the Nature player is listed in the set \mathcal{P} of players, then, automatically, there also exists unique element of the sample space $\omega(v) \in \Omega$ such that $v \subset \omega(v)$.

This finishes the proof. □

We are leaving open the question of causality for the constructed AFRtoW-model. See §9.2.1 of the Appendix for more details.

4.3 AFRtoW-strategies

In this Section 4.3, we construct, from an AFR-strategy $s_a : X_a \rightarrow C_a$ (hence, such that the measurability condition as in Definition 2.17 holds true) a mapping $\lambda_a : \mathbb{H} \rightarrow \mathbb{U}_a$ such that the measurability condition as in (2.30b) holds true.

4.3.1 Building AFRtoW-strategies

Using partition (4.11), we can write any AFR-strategy s^i in §2.3.4 as a family of local strategies attached to each AFRtoW-agent of the player i .

Proposition 4.2. *Let $i \in \mathcal{P}$ be a player and $s^i : X^i \rightarrow C^i$ be an AFR-strategy. Then, there exists a family of mappings $\{s_{i,a}\}_{a \in \mathbb{A}^i}$, where $s_{i,a} : X^i \rightarrow P^{-1}(a)$ such that, for any player i 's move $x \in X^i$, there exists a AFRtoW-agent $a \in \mathbb{A}^i$ (containing the move $x \in a$), and $s^i(x) = s_{i,a}(x) \in P^{-1}(a)$.*

Proof. Let a move $x \in X^i$ be given. It directly follows from representation (4.11), that there exists a unique AFRtoW-agent $a = a(x) \in \mathbb{A}^i$ containing it, that is, $x \in a$. On the one hand, as the image of the move x under the AFR-strategy s^i of the player i is a choice, that is, $s^i(x) \in C^i$, then, by (AFR-Axiom1'), there exists a unique AFRtoW-agent $a' \in \mathbb{A}^i$, such that $P(s^i(x)) \in a'$. On the other hand, by the measurability condition in Definition 2.17, the action of the inverse mapping $(s^i)^{-1}$ coincides with the immediate predecessor mapping P action, giving

$$x \in (s^i)^{-1}(s^i(x)) = P(s^i(x)) = a' .$$

From this, it follows that $x \in a \cap a'$ and $a = a'$ as they are atoms of the partition (4.11). Thus, we set $s_{i,a} = (s_i)|_a$, where

$$(s_i)|_a : X^i \cap a \rightarrow P^{-1}(a) .$$

This ends the proof. □

Now we are ready to build AFRtoW-strategies from W-strategies.

Proposition 4.3. *Let a player $i \in \mathcal{P}$ and an AFR-strategy $s^i : X^i \rightarrow C^i$ be given. Then, there exists a family of mappings $(\lambda_a)_{a \in \mathbb{A}^i}$, such that, for any agent $a \in \mathbb{A}^i$,*

$$\lambda_a = \mathbf{i}_a \circ s_{i,a} , \tag{4.12}$$

that is, such that the diagram in Figure (4.1) commutes, where \mathbf{i}_a maps every configuration $h \in H_a$ into the corresponding vertex $v \in a$ that contains h and the local strategy $s_{i,a}$ is defined in Proposition 4.2.

$$\begin{array}{ccc}
 \{v \mid v \in a\} & \xrightarrow{s_{i,a}} & P^{-1}(a) \\
 \uparrow \mathbf{i}_a & \nearrow \lambda_a & \\
 H_a & &
 \end{array}$$

Figure 4.1: Constructing the mapping λ_a

Proof. Let a player $i \in \mathcal{P}$ and a strategy s^i be given. For any i 's W-agent $a \in \mathbb{A}^i$ and the corresponding local strategy $s_{i,a}$ defined in Proposition 4.2, we build the AFRtoW-strategy λ_a . First, we concentrate on the subset of histories $H_a = \bigcup_{v \in G} H_v \subset \mathbb{H}$, where H_v is defined in (4.6). The set H_a is the domain of the strategy λ_a we want to build. For any configuration $h \in H_a$, there exists a unique $v \in G$ such that $h \in H_v$ (as any pair of vertices $v, v' \in G$ is disjoint, then $H_a = \bigsqcup_{v \in G} H_v$). We denote this unique vertex for each configuration $h \in H$ as $\mathbf{i}_a(h)$. Thus, $\mathbf{i}_a : \mathbb{H} \rightarrow X^i$ is defined and the composition $\lambda_a = \mathbf{i}_a \circ s_{i,a}$ as well. As the local strategy $s_{i,a}$ is constant for each move $x \in a$, then the constructed strategy λ_a is constant for each configuration $h \in H_a$, thus, is \mathcal{J}_a -measurable, where the information π -field \mathcal{J}_a is defined in (4.7).

This ends the proof. \square

We are leaving open the question of solvability for an AFRtoW-strategy. See §9.2.2 of the Appendix for more details.

4.4 Conclusion

In this Chapter, we have shown that an AFR-model can be embedded into a causal W-model with π -fields.

Here is a summary sketch of the embedding. We treat each information set of an AFR-player as a so-called AFRtoW-agent, then build the corresponding action set and information π -field thanks to the immediate predecessor mapping. The sample space of the Nature is derived immediately out of the so-called Nature-rooted AFR-model, which is a relaxation of the AFR-model in which Nature plays at the root once and for all.

The questions of causality and solvability of the constructed model are left open; two conjectures are formulated in §9.2.1 and §9.2.2 of the Appendix respectively.

Chapter 5

Classification of information structures

5.1 Introduction

In this Chapter 5, we discuss classical information patterns in K-, AFR- and W-models. We generalize definitions introduced by Witsenhausen in [38] and make conjectures on their relations with the classical patterns from Game Theory, such as perfect recall and memory of the past information.

5.2 Information structures in the K-model

The notion of perfect recall was introduced by Kuhn in [18] as “equivalent to the assertion that each player is allowed by the rules of the game to remember everything he knew at previous moves and all of his choices at those moves”.

This concept, expressed in the K-language – in terms of both vertices and edges – turns out to be rather cumbersome and difficult to work with. It is expressed with a natural order on the set of vertices V induced by the tree structure: vertex v precedes v' ($v > v'$) if there exists a path from v to v' . The root precedes any non-root vertex: $v^0 > v$ for any non-root vertex $v \in V \setminus \{v^0\}$.

The core idea of perfect recall is simple, namely, if a player has perfect recall, then at each of his information sets he remembers what he knew and what he did in the past, which is stated formally in the following way.

Definition 5.1. *A player $i \in \mathcal{P}$ endowed with an information structure I_i is said to have K-perfect recall if*

$$\begin{aligned} \forall G \in I_i, \forall v_1, v_2 \in G \text{ if } \exists v'_1 \in G' \in I_i \text{ s.t. } v'_1 > v_1, \\ \text{then } \exists v'_2 \in G' \text{ s.t. } v'_2 > v_2, \exists c \in C^i(G) \text{ s.t.} \\ (v'_1, v_1)_E \cap E_{v'_1} \in c, (v'_2, v_2)_E \cap E_{v'_2} \in c, \end{aligned}$$

where the choice c is eligible (see §2.2.2.4).

The Kuhn's framework, besides formulating the notion of perfect recall in a cumbersome way, does not really offer any other classification apart from the binary one – see Figure 5.1 – telling only if a player has or has not perfect recall.

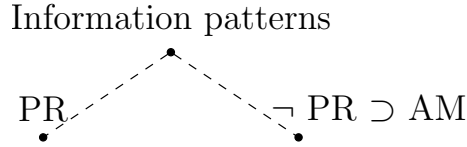


Figure 5.1: K-classification of information patterns

The notion of perfect recall is especially useful for applications, say, it is a necessary and sufficient condition for the Kuhn's theorem to hold true (see, for instance, [3]).

There is a finer classification of imperfect recall in the K-model given by Bonanno in [7], where the notion is decomposed into two independent properties: *memory of past actions* and *memory of past information*, which coincides with the one given in Figure 5.2.

5.3 Information structures in AFR-model

As, in AFR-model, the information is ciphered inside the structure of choices, a possible classification of the information structures can be expressed in terms of choices.

The following property proves to be equivalent to perfect recall of Kuhn (see Corollary 6.3 in [3] for details).

Definition 5.2. *A player $i \in \mathcal{P}$ displays the choice trivial intersection property (C-TrIP for short), if for any pair c, c' of his choices in C^i , it holds that*

$$\text{if } c \cap c' \neq \emptyset \text{ then either } c \subset c' \text{ or } c' \subset c .$$

Before formulating an analogous property for information sets in the AFR-model, we need a bit of preparations

Mapping an information set into a corresponding subset of plays is made by the mapping $\mathcal{W} : 2^V \rightarrow 2^{\mathbb{W}}$ defined by

$$\mathcal{W}(G) = \bigcup_{v \in G} v \subset \mathbb{W} , \quad \forall G \subset V . \quad (5.1)$$

The image $\mathcal{W}(G)$ is called the *image in plays* of the subset G of vertices. As in the prequel, for any player $i \in \mathcal{P}$, the set of his informations sets is

$$P(C^i) = \{P(c) \mid c \in C^i\} \subset 2^V .$$

Now, we take the image in plays of each information set yielding

$$\mathcal{W}(P(C^i)) = \{\mathcal{W}(P(c)) \mid c \in C^i\} \subset 2^{\mathbb{W}} . \quad (5.2)$$

Definition 5.3. A player $i \in \mathcal{P}$ displays the information trivial intersection property (I-TrIP), if for any pair $\mathcal{W}(P(c)), \mathcal{W}(P(c'))$ of his information sets' images in plays, it holds that

if $\mathcal{W}(P(c)) \cap \mathcal{W}(P(c')) \neq \emptyset$ then either $\mathcal{W}(P(c)) \subset \mathcal{W}(P(c'))$ or $\mathcal{W}(P(c')) \subset \mathcal{W}(P(c))$.

As it is shown in §6.4.2 of [3], C-TrIP implies I-TrIP. Thus, the AFR-classification – see Figure 5.2 – is richer than the one of Kuhn.

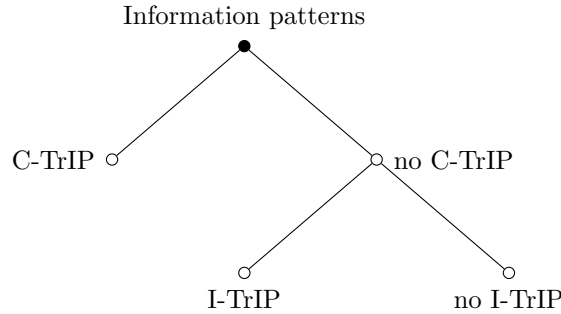


Figure 5.2: AFR-classification of information patterns

This classification of information structures given in Figure (5.2) is equivalent to the one of the AFR-model.

5.4 Information structures in the W-model

Using the W-model for information representation, Witsenhausen provided a typology in [38]. Some elements of the typology concern a subset \mathbb{B} of the agents \mathbb{A} of a W-model (station), whereas others concern the whole set \mathbb{A} of agents (sequential and strictly classical systems).

In this Section, we will generalize part of the typology — namely, station, sequential and strictly classical systems — in two directions. On the one hand, we will provide new definitions of sequential and strictly classical subsets of agents. On the other hand, we will propose “localized” versions of the definitions of station, sequential and strictly classical *subsets of agents*.

For the purpose of expliciting what we mean by “localization”, we recall the notion of trace field.

Definition 5.4. For any field \mathcal{G} over \mathbb{H} and any (nonempty) subset $H \subset \mathbb{H}$, we define the trace field $H \cap \mathcal{G}$ as the field over H given by

$$H \cap \mathcal{G} = \{H \cap G \mid G \in \mathcal{G}\} \subset 2^H, \tag{5.3}$$

that is, made of the traces on H of the elements of the field \mathcal{G} .

| | constant configuration ordering $\varphi : \mathbb{H} \rightarrow \Sigma^{ \mathbb{A} }$ | non-constant configuration ordering $\varphi : \mathbb{H} \rightarrow \Sigma^{ \mathbb{A} }$ |
|------------------------------------|--|--|
| $\mathbb{B} = \mathbb{A}$ | W-station W-sequential W-classical system | causality |
| $\mathbb{B} \subsetneq \mathbb{A}$ | | W-perfect recall W-information memory |

Table 5.1: Extension of W-information patterns

Beware that $H \cap \mathcal{G}$ is not a subfield of $2^{\mathbb{H}}$ (it does not contain \mathbb{H} except when $H = \mathbb{H}$), but is a subfield of 2^H .

We will need the following slight variation of the definition (2.33a) of configuration orderings. For any subset $\mathbb{B} \subset \mathbb{A}$ of agents, we denote by

$$\Sigma_{\mathbb{B}} = \{ \varrho : \{1, \dots, |\mathbb{B}|\} \rightarrow \mathbb{B} \mid \varrho \text{ is an injection} \} \quad (5.4)$$

the set of (*total*) *orderings* of agents in \mathbb{B} , that is, bijective mappings from $\{1, \dots, |\mathbb{B}|\}$ to \mathbb{B} .

5.4.1 Sequential systems and causality

In Witsenhausen's formalism, a system is *sequential* [38] if there is a total ordering $\rho \in \Sigma^{|\mathbb{A}|}$ of agents \mathbb{A} such that each agent is influenced at most by the previous agents, that is,

$$\begin{aligned} \mathcal{J}_{\rho(1)} &\subset \mathcal{H}_{(\emptyset)} \\ \mathcal{J}_{\rho(2)} &\subset \mathcal{H}_{\|\psi_{1\rho}\|} \\ &\dots \\ \mathcal{J}_{\rho(k)} &\subset \mathcal{H}_{\|\psi_{k-1\rho}\|} \quad (k = 2, \dots, |\mathbb{A}| - 1) \\ &\dots \\ \mathcal{J}_{\kappa^*} &\subset \mathcal{H}_{\|\psi_{|\mathbb{A}|-1\rho}\|} = \mathcal{H}_{\|\kappa^-\|}, \end{aligned}$$

where for any index $k = 1, \dots, |\mathbb{A}|$, the corresponding field $\mathcal{H}_{\|\psi_{k-1\rho}\|}$ is as in (2.26c) applied for the corresponding subset $\|\psi_{k-1\rho}\|$ of agents.

A sequential system defined in such way is a particular case of causality as stated in Definition 2.24, under the assumption of a constant configuration ordering $\varphi : \mathbb{H} \rightarrow \Sigma^{|\mathbb{A}|}$. In Conjecture 5.5, we hypothesize the following six equivalent characterizations of causality. Here, we reformulate the classical Definition 2.24 of causality given in §2.4.4.2, for the subset $\Sigma^\varphi \subset \Sigma$ of orderings that are compatible with a given configuration ordering $\varphi : \mathbb{H} \rightarrow \Sigma$ as defined in (2.35c), thus, incompatible orderings $\Sigma \setminus \Sigma^\varphi$ are eliminated from consideration.

Conjecture 5.5. *The following statements are equivalent.*

1. There exists (at least one) configuration-ordering φ from \mathbb{H} towards $\Sigma^{|\mathbb{A}|}$ (as in Definition 2.23), with the property that

$$\left(\forall \kappa \in \Sigma^\varphi, \forall G \in \mathcal{J}_{\kappa^*} \right) \quad \mathbb{H}_\kappa^\varphi \cap G \in \mathcal{H}_{\|\kappa^-\|}, \quad (5.5)$$

where the field $\mathcal{H}_{\|\kappa^-\|}$ is defined in Equation (2.36f), the “last” agent κ^* is defined in Equation (2.33i) and the subset $\Sigma^\varphi \subset \Sigma$ of orderings that are compatible with the given configuration ordering $\varphi : \mathbb{H} \rightarrow \Sigma$ is as in (2.35c).

2. There exists a configuration-ordering $\varphi : \mathbb{H} \rightarrow \Sigma^{|\mathbb{A}|}$, with the property that

$$\left(\forall \kappa \in \Sigma^\varphi \right) \quad \mathbb{H}_\kappa^\varphi \cap \mathcal{J}_{\kappa^*} \subset \mathcal{H}_{\|\kappa^-\|}, \quad (5.6)$$

where the field $\mathcal{H}_{\|\kappa^-\|}$ is defined in Equation (2.36f) and the “last” agent κ^* is defined in Equation (2.33i).

3. There exists a configuration-ordering $\varphi : \mathbb{H} \rightarrow \Sigma^{|\mathbb{A}|}$, with the property that

$$\left(\forall \kappa \in \Sigma^\varphi \right) \quad \mathbb{H}_\kappa^\varphi \cap \mathcal{J}_{\kappa^*} \subset \mathbb{H}_\kappa^\varphi \cap \mathcal{H}_{\|\kappa^-\|} \text{ and } \mathbb{H}_\kappa^\varphi \in \mathcal{H}_{\|\kappa^-\|}. \quad (5.7)$$

4. There exists a configuration-ordering $\varphi : \mathbb{H} \rightarrow \Sigma^{|\mathbb{A}|}$, with the property that

$$\left(\forall k = 1, \dots, |\mathbb{A}|, \forall \rho \in \varphi(\mathbb{H}) \right) \quad (5.8)$$

$$\mathbb{H}_{\psi_{k\rho}}^\varphi \cap \mathcal{J}_{\rho^{(k)}} \subset \mathbb{H}_{\psi_{k\rho}}^\varphi \cap \mathcal{H}_{\|\psi_{k-1\rho}\|} \text{ and } \mathbb{H}_{\psi_{k\rho}}^\varphi \in \mathcal{H}_{\|\psi_{k-1\rho}\|},$$

where for the empty ordering (\emptyset) we set $\mathcal{H}_{(\emptyset)} = \mathcal{F} \otimes \bigotimes_{a \in \mathbb{A}} \{\emptyset, \mathbb{U}_a\}$.

5. There exists a partition

$$\mathbb{H} = \bigsqcup_{\rho \in \Gamma} \mathbb{H}_\rho, \quad (5.9a)$$

where $\Gamma \subset \Sigma^{|\mathbb{A}|}$ such that

$$\left(\forall k = 1, \dots, |\mathbb{A}|, \forall \rho \in \Gamma \right) \quad \mathbb{H}_\rho \cap \mathcal{J}_{\rho^{(k)}} \subset \mathbb{H}_\rho \cap \mathcal{H}_{\|\psi_{k-1\rho}\|}, \quad (5.9b)$$

and

$$\left(\forall \kappa \in \Sigma^\Gamma \right) \quad \bigcup_{\rho \in \downarrow \kappa \cap \Gamma} \mathbb{H}_\rho \in \mathcal{H}_{\|\kappa^-\|}, \quad (5.9c)$$

where the set Σ^Γ of orderings is given by

$$\Sigma^\Gamma = \bigsqcup_{k \in \{1, \dots, |\mathbb{A}|\}} \psi_k(\Gamma) \subset \Sigma. \quad (5.9d)$$

6. There exists a partition like in (5.9a) with the property that, for any total ordering $\rho \in \Gamma$, the set \mathbb{A} of agents is a sequential system restricted to \mathbb{H}_ρ , for the total ordering ρ , and that (5.9c) holds true.

We finish by giving the following definition of the so-called W-sequential system that generalizes causality (originally defined for the whole set \mathbb{A} of agents) to a case of a set $\mathbb{B} \subset \mathbb{A}$ of agents.

Definition 5.6. A subset $\mathbb{B} \subset \mathbb{A}$ of agents is said to be a W-sequential system if, for any ordering $\kappa \in \Sigma$, such that, $\kappa^* \in \mathbb{B}$,

$$(\forall \kappa \in \Sigma, \kappa^* \in \mathbb{B}) \quad \mathbb{H}_\kappa^\varphi \cap \mathcal{J}_{\kappa^*} \subset \mathcal{H}_{\|\kappa^-\| \cap \mathbb{B}}, \quad (5.10)$$

where the subset $\mathbb{H}_\kappa^\varphi \subset \mathbb{H}$ of configurations has been defined in (2.34), κ^* in (2.33i), κ^- in (2.33j), $\|\kappa^-\|$ in (2.33h), and where the subfield $\mathcal{H}_{\|\kappa^-\| \cap \mathbb{B}} \subset \mathcal{H}$ is as in (2.26c) applied for $\|\kappa^-\| \cap \mathbb{B}$.

5.4.2 Station and W-information memory

In Witsenhausen's formalism, a *station* [37] is a subset of agents such that the set of information fields of these agents is totally ordered under inclusion (i.e., nested). In other words, a subset \mathbb{B} of agents in \mathbb{A} is a station iff there exists an ordering $\varrho \in \Sigma_{\mathbb{B}}$ of \mathbb{B} such that

$$\mathcal{J}_{\varrho(1)} \subset \cdots \subset \mathcal{J}_{\varrho(k)} \subset \mathcal{J}_{\varrho(k+1)} \subset \cdots \subset \mathcal{J}_{\varrho(|\mathbb{B}|)}. \quad (5.11)$$

Here, we propose a more general definition, namely, W-information memory which is the main definition of this §5.4.2. W-information memory is conjectured to be equivalent to an analogue of the I-TrIP of AFR in Conjecture 5.8.

Definition 5.7. Let a causal W-model with the configuration-ordering $\varphi : \mathbb{H} \rightarrow \Sigma^{|\mathbb{A}|}$ be given as in Definition 2.24. A subset $\mathbb{B} \subset \mathbb{A}$ of agents is said to have W-information memory if, for any ordering $\kappa \in \Sigma$, such that, $\kappa^* \in \mathbb{B}$,

$$(\forall \kappa \in \Sigma, \kappa^* \in \mathbb{B}) \quad \mathbb{H}_\kappa^\varphi \cap \mathcal{J}_{\|\kappa^-\| \cap \mathbb{B}} \subset \mathcal{J}_{\kappa^*}, \quad (5.12a)$$

where the subset $\mathbb{H}_\kappa^\varphi \subset \mathbb{H}$ of configurations has been defined in (2.34), κ^* in (2.33i), κ^- in (2.33j), $\|\kappa^-\|$ in (2.33h), and where

$$\mathcal{J}_{\|\kappa^-\| \cap \mathbb{B}} = \bigvee_{\substack{a \in \|\kappa^-\| \\ a \in \mathbb{B}}} \mathcal{J}_a. \quad (5.12b)$$

The station of Witsenhausen is a particular case of Definition 5.7 for a constant configuration ordering $\varphi : \mathbb{H} \rightarrow \Sigma^{|\mathbb{A}|}$ and $\mathbb{B} = \mathbb{A}$.

We finish by giving the following conjectured equivalence between W-information memory and I-TrIP.

Conjecture 5.8. *Let a W-model as in Definition 2.20 be given with a causal configuration ordering $\varphi : \mathbb{H} \rightarrow \Sigma^{|\mathbb{A}|}$ as in (2.37). Then, for any set $\mathbb{B} \subset \mathbb{A}$ of agents, the corresponding collection $\Xi_{\mathbb{B}}$ of information sets given by*

$$\Xi_{\mathbb{B}} = \{G \subset \mathbb{H} \mid G \text{ is an atom of } \mathcal{J}_a, a \in \mathbb{B}\}, \quad (5.13)$$

satisfies TriP, (that is, for any two $G, G' \in \Xi_{\mathbb{B}}$, if $G \cap G' \neq \emptyset$, then $G \subset G'$ or $G' \subset G$) iff the set \mathbb{B} of agents has W-information memory as in Definition 5.7.

5.4.3 Strictly classical system and W-perfect recall

Definition 5.9. *A system is called classical if it is sequential and the set \mathbb{A} of agents forms a station.*

Hence, in a classical system, there exists an ordering such that any agent is influenced at most by the previous agents and knows what they know.

Here, following [38], we introduce the notion of strictly classical system.

Definition 5.10 ([38]). *A system is called strictly classical if there exists an ordering $(a_1, \dots, a_{|\mathbb{A}|})$ of agents \mathbb{A} such that*

$$\mathcal{J}_{a_k} \bigvee \mathcal{U}_{a_k} \subset \mathcal{J}_{a_{k+1}} \subset \mathcal{H}_{\{a_1, \dots, a_k\}}, \quad \forall k = 1, \dots, |\mathbb{A}| - 1, \quad (5.14)$$

where, for any agent $a \in \mathbb{A}$, the choice subfield $\mathcal{C}_a \subset \mathcal{H}$ is given by

$$\mathcal{C}_a = \mathcal{J}_a \bigvee \mathcal{U}_a \quad (5.15)$$

Hence, in a strictly classical system, there exists an ordering such that any agent is influenced at most by the previous agents, and knows what they know and the previous agent's decision.

The following Lemma gives a closed form expression for the information structure of a strictly classical system.

Lemma 5.11. *A system is strictly classical iff there exists an ordering $(a_1, \dots, a_{|\mathbb{A}|})$ and a sequence $\{\mathcal{F}_{a_k}\}_{k \in \{1, \dots, |\mathbb{A}|\}}$ of subfields of the sample field \mathcal{F} such that*

$$\mathcal{F}_{a_k} \subset \mathcal{F}_{a_{k+1}} \quad \text{and} \quad \mathcal{J}_{a_{k+1}} = \mathcal{F}_{a_{k+1}} \otimes \bigotimes_{l=1}^k \mathcal{U}_{a_l} \otimes \bigotimes_{m=k+1}^{|\mathbb{A}|} \{\emptyset, \mathbb{U}_{a_m}\}, \quad \forall k = 1, \dots, |\mathbb{A}| - 1.$$

Proof. We proceed by induction. Apply (5.14) for $k = 1$:

$$\mathcal{J}_{a_1} \bigvee \mathcal{U}_{\{a_1\}} \subset \mathcal{J}_{a_2} \subset \mathcal{F} \otimes \mathcal{U}_{a_1} \otimes \bigotimes_{m=2}^{|\mathbb{A}|} \{\emptyset, \mathbb{U}_{a_m}\}. \quad (5.16)$$

As agent a_1 is the first to act, he knows at most the move of Nature, that is, there exists a subfield $\mathcal{F}_{a_1} \subset \mathcal{F}$ such that

$$\mathcal{J}_{a_1} = \mathcal{F}_{a_1} \otimes \bigotimes_{m=1}^{|\mathbb{A}|} \{\emptyset, \mathbb{U}_{a_m}\},$$

which, combined with (5.16), implies that

$$\mathcal{J}_{a_1} \bigvee \mathcal{U}_{\{a_1\}} = \mathcal{F}_{a_1} \otimes \mathcal{U}_{a_1} \otimes \bigotimes_{m=2}^{|\mathbb{A}|} \{\emptyset, \mathbb{U}_{a_m}\} \subset \mathcal{J}_{a_2} \subset \mathcal{F} \otimes \mathcal{U}_{a_1} \otimes \bigotimes_{m=2}^{|\mathbb{A}|} \{\emptyset, \mathbb{U}_{a_m}\},$$

and, thus, for agent a_2 there exists a subfield $\mathcal{F}_{a_2} \subset \mathcal{F}_{a_1} \subset \mathcal{F}$ such that

$$\mathcal{J}_{a_2} = \mathcal{F}_{a_2} \otimes \mathcal{U}_{a_1} \otimes \bigotimes_{m=2}^{|\mathbb{A}|} \{\emptyset, \mathbb{U}_{a_m}\}.$$

Assume now that the statement holds true for all agents a_1, \dots, a_k and there exists a subfield $\mathcal{F}_{a_{k-1}} \subset \mathcal{F}_{a_k}$ such that

$$\mathcal{J}_{a_k} = \mathcal{F}_{a_k} \otimes \bigotimes_{l=1}^{k-1} \mathcal{U}_{a_l} \otimes \bigotimes_{m=k}^{|\mathbb{A}|} \{\emptyset, \mathbb{U}_{a_m}\},$$

which, inserted into (5.14), yields

$$\mathcal{J}_{a_k} \bigvee \mathcal{U}_{\{a_k\}} = \mathcal{F}_{a_k} \otimes \bigotimes_{l=1}^k \mathcal{U}_{a_l} \otimes \bigotimes_{m=k+1}^{|\mathbb{A}|} \{\emptyset, \mathbb{U}_{a_m}\} \subset \mathcal{J}_{a_{k+1}} \subset \mathcal{F} \otimes \bigotimes_{l=1}^k \mathcal{U}_{a_l} \otimes \bigotimes_{m=k+1}^{|\mathbb{A}|} \{\emptyset, \mathbb{U}_{a_m}\}$$

Thus, for agent a_{k+1} , there exists a subfield $\mathcal{F}_{a_{k+1}} \subset \mathcal{F}_{a_k} \subset \mathcal{F}$ such that

$$\mathcal{J}_{a_{k+1}} = \mathcal{F}_{a_{k+1}} \otimes \bigotimes_{l=1}^k \mathcal{U}_{a_l} \otimes \bigotimes_{m=k+1}^{|\mathbb{A}|} \{\emptyset, \mathbb{U}_{a_m}\}.$$

This finishes the proof. \square

The following example motivates Definition 5.12. Consider the game with two agents a and b as in §2.4.5.3 with the only difference that the order of agents depends on the move of Nature. Information structure is given by

$$\mathcal{J}_a = \{\emptyset, \{\omega^+\}\} \otimes \{\emptyset, \mathbb{U}_a\} \otimes \{\emptyset, \mathbb{U}_b\}, \{\omega^-\} \otimes \{\emptyset, \mathbb{U}_a\} \otimes \{u_b^+\}, \{\omega^-\} \otimes \{\emptyset, \mathbb{U}_a\} \otimes \{u_b^-\}, \mathbb{H}\}, \quad (5.17)$$

$$\mathcal{J}_b = \{\emptyset, \{\omega^-\}\} \otimes \{\emptyset, \mathbb{U}_a\} \otimes \{\emptyset, \mathbb{U}_b\}, \{\omega^+\} \otimes \{u_a^+\} \otimes \{\emptyset, \mathbb{U}_b\}, \{\omega^+\} \otimes \{u_a^-\} \otimes \{\emptyset, \mathbb{U}_b\}, \mathbb{H}\}.$$

The agents' choice fields are given by

$$\begin{aligned}\mathcal{C}_a &= \mathcal{J}_a \bigvee \mathcal{U}_a = \{\emptyset, \{\omega^+\} \otimes \mathcal{U}_a \otimes \{\emptyset, \mathbb{U}_b\}, \{\omega^-\} \otimes \mathcal{U}_a \otimes \mathcal{U}_b, \mathbb{H}\}, \\ \mathcal{C}_b &= \mathcal{J}_b \bigvee \mathcal{U}_b = \{\emptyset, \{\omega^+\} \otimes \mathcal{U}_a \otimes \mathcal{U}_b, \{\omega^-\} \otimes \{\emptyset, \mathbb{U}_a\} \otimes \mathcal{U}_a, \mathbb{H}\}.\end{aligned}$$

If we build the tree of this game as in §2.2.5.3 (see Figure 5.3), we notice that the player represented by agents a and b has perfect recall in the sense of Definition 5.1. But, as we

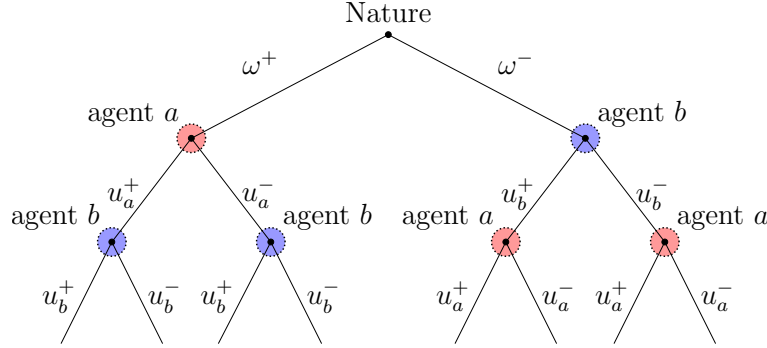


Figure 5.3: Perfect recall with non-nested agents' information fields

can see, neither the fields \mathcal{J}_a and \mathcal{C}_b are nested, nor \mathcal{J}_b and \mathcal{C}_a . Despite of this fact, there is still “local” nestedness. Consider the causal ordering $\varphi : \mathbb{H} \rightarrow \Sigma$

$$\varphi(h) = \begin{cases} (ab), & \text{if } h \in \mathbb{H}^+ = \{\omega^+\} \times \mathbb{U}_a \times \mathbb{U}_b \\ (ba), & \text{if } h \in \mathbb{H}^- = \{\omega^-\} \times \mathbb{U}_a \times \mathbb{U}_b \end{cases} \quad (5.18)$$

induced by the information structure (5.17). The local nestedness holds when the fields are projected on either of the two elements of the partition $\mathbb{H} = \mathbb{H}^+ \sqcup \mathbb{H}^-$:

$$\mathcal{C}_a \cap \mathbb{H}^+ \subset \mathcal{J}_b \cap \mathbb{H}^+, \quad \mathcal{C}_b \cap \mathbb{H}^- \subset \mathcal{J}_a \cap \mathbb{H}^-.$$

In the following definition, we propose a generalization of Definition 5.10 in two directions: formulating it for a subset $\mathbb{B} \subset \mathbb{A}$ of agents and allowing for the property to hold locally.

Definition 5.12. *We consider a causal W-model with the configuration-ordering $\varphi : \mathbb{H} \rightarrow \Sigma^{|\mathbb{A}|}$ as in Definition 2.24.*

We say that a subset $\mathbb{B} \subset \mathbb{A}$ of agents enjoys W-perfect recall if

$$\mathbb{H}_\kappa^\varphi \cap \mathcal{C}_{\|\kappa^-\| \cap \mathbb{B}} \subset \mathcal{J}_{\kappa^*}, \quad \forall \kappa \in \Sigma, \quad (5.19)$$

where the subset $\mathbb{H}_\kappa^\varphi \subset \mathbb{H}$ of configurations has been defined in (2.34), κ^ in (2.33i), κ^- in (2.33j), $\|\kappa^-\|$ in (2.33h), and where*

$$\mathcal{C}_{\|\kappa^-\| \cap \mathbb{B}} = \bigvee_{\substack{a \in \|\kappa^-\| \\ a \in \mathbb{B}}} \mathcal{C}_a, \quad (5.20)$$

with the choice subfield $\mathcal{C}_a \subset \mathcal{H}$ given by (5.15).

We finish by giving the following conjectured equivalence between W -perfect recall and C - TrIP .

Conjecture 5.13. *Let a W -model as in Definition 2.20 be given with a causal configuration ordering $\varphi : \mathbb{H} \rightarrow \Sigma^{|\mathbb{A}|}$ as in (2.37). Then, for any set $\mathbb{B} \subset \mathbb{A}$ of agents, the corresponding collection $\mathfrak{C}_{\mathbb{B}}$ of information sets given by*

$$\mathfrak{C}_{\mathbb{B}} = \{G \subset \mathbb{H} \mid G \text{ is an atom of } \mathcal{C}_a, a \in \mathbb{B}\}, \quad (5.21)$$

satisfies TrIP , (that is, for any two $c, c' \in \mathfrak{C}_{\mathbb{B}}$, if $c \cap c' \neq \emptyset$, then $c \subset c'$ or $c' \subset c$) iff the set \mathbb{B} of agents enjoys W -perfect recall as in Definition 5.12.

5.4.4 Noncausal systems

One of the examples of a noncausal system is a so-called *deadlock*, where two agents $\mathbb{A} = \{a, b\}$ have information fields given by:

$$\mathcal{J}_a = \{\emptyset, \mathbb{U}_a\} \otimes \mathcal{U}_b \otimes \{\emptyset, \Omega\}, \quad \mathcal{J}_b = \mathcal{U}_a \otimes \{\emptyset, \mathbb{U}_b\} \otimes \{\emptyset, \Omega\}.$$

In *noncausal systems*, the decision process may have no solution, one solution, or multiple solutions, which can be already seen in the simplest deterministic case with two agents a and b having two-element action sets each $\mathbb{U}_a = \{u_a^+, u_a^-\}$ and $\mathbb{U}_b = \{u_b^+, u_b^-\}$, thus, the information fields are (see Figure 5.4)

$$\mathcal{J}_a = \{\emptyset, \mathbb{U}_a\} \otimes \mathcal{U}_b, \quad \mathcal{J}_b = \mathcal{U}_a \otimes \{\emptyset, \mathbb{U}_b\}.$$

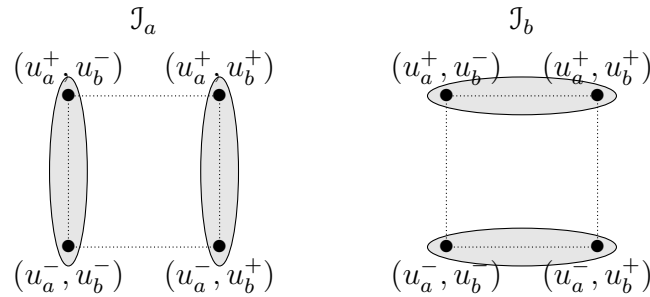


Figure 5.4: Deadlock: information partitions of the agents a and b

For each agent, the choice field in (5.15) is given by the complete configuration field

$$\mathcal{C}_a = \mathcal{J}_a \bigvee \mathcal{U}_a = \mathcal{U}_a \otimes \mathcal{U}_b = \mathcal{C}_b.$$

The measurability condition imposed on the agents' strategies implies that each agent

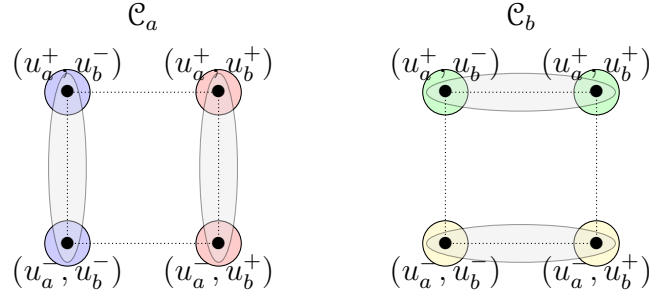


Figure 5.5: Deadlock: choice partitions of the agents a and b

must choose one choice of each colour (see Figure 5.5). Thus, one can easily find the strategies to guaranty two, one and zero solutions.

In his paper [37], Witsenhausen gives the following example of a noncausal solvable system. Let the set of three agents $\mathbb{A} = \{a, b, c\}$ be equipped with the following action sets

$$\mathbb{U}_a = \mathbb{U}_b = \mathbb{U}_c = \{0, 1\} ,$$

and there is no Nature. In this deterministic setting, the set $\mathbb{H} = \{0, 1\}^3$ of configurations has 8 elements. The agents are equipped with the following information fields

$$\begin{aligned} \mathcal{J}_a &= \sigma(u_b(1 - u_c)) = \{\emptyset, \{(u_a, u_b, u_c) \mid u_b(1 - u_c) = 1\}, \{(u_a, u_b, u_c) \mid u_b(1 - u_c) = 0\}, \mathbb{H}\} , \\ \mathcal{J}_b &= \sigma(u_c(1 - u_a)) = \{\emptyset, \{(u_a, u_b, u_c) \mid u_c(1 - u_a) = 1\}, \{(u_a, u_b, u_c) \mid u_c(1 - u_a) = 0\}, \mathbb{H}\} , \\ \mathcal{J}_c &= \sigma(u_a(1 - u_b)) = \{\emptyset, \{(u_a, u_b, u_c) \mid u_a(1 - u_b) = 1\}, \{(u_a, u_b, u_c) \mid u_a(1 - u_b) = 0\}, \mathbb{H}\} . \end{aligned}$$

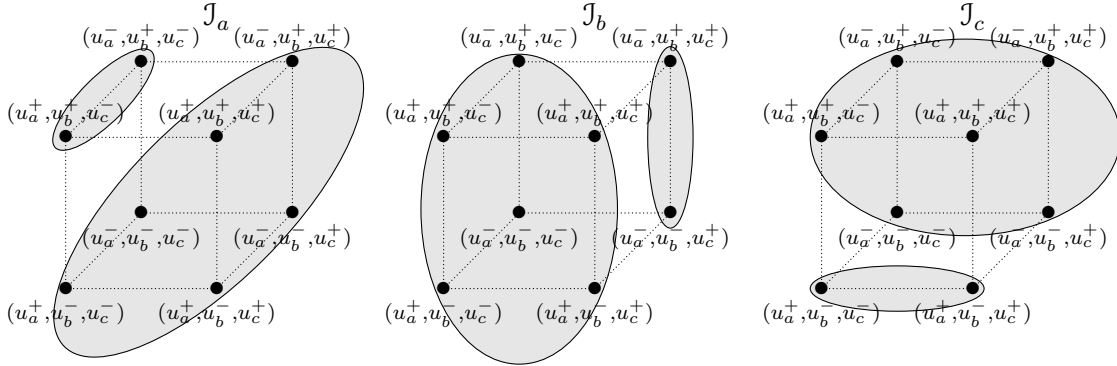


Figure 5.6: Noncausal solvable system: information partitions of the three agents.

The W-strategies of the agents have the form

$$\begin{aligned} \lambda_a(u_a, u_b, u_c) &= \tilde{\lambda}_a(u_b(1 - u_c)) , \\ \lambda_b(u_a, u_b, u_c) &= \tilde{\lambda}_b(u_c(1 - u_a)) , \\ \lambda_c(u_a, u_b, u_c) &= \tilde{\lambda}_c(u_a(1 - u_b)) , \end{aligned}$$

where $\tilde{\lambda}_a, \tilde{\lambda}_b, \tilde{\lambda}_c : \{0, 1\} \rightarrow \{0, 1\}$. Each agent a is endowed with 4 adapted strategies

$$\begin{aligned} \tilde{\lambda}_a^{\text{Id,Id}}(u_b, u_c) &= \begin{cases} 0, & \text{if } u_b(1 - u_c) = 0 \\ 1, & \text{if } u_b(1 - u_c) = 1 \end{cases} & \tilde{\lambda}_a^{\text{Id,1-Id}}(u_b, u_c) &= \begin{cases} 0, & \text{if } u_b(1 - u_c) = 0 \\ 0, & \text{if } u_b(1 - u_c) = 1 \end{cases} \\ \tilde{\lambda}_a^{\text{1-Id,Id}}(u_b, u_c) &= \begin{cases} 1, & \text{if } u_b(1 - u_c) = 0 \\ 1, & \text{if } u_b(1 - u_c) = 1 \end{cases} & \tilde{\lambda}_a^{\text{1-Id,1-Id}}(u_b, u_c) &= \begin{cases} 1, & \text{if } u_b(1 - u_c) = 0 \\ 0, & \text{if } u_b(1 - u_c) = 1 \end{cases} . \end{aligned}$$

Both agents b and c have four W-strategies each, which yields $4^3 = 64$ different strategies profiles $\lambda = (\lambda_a, \lambda_b, \lambda_c)$.

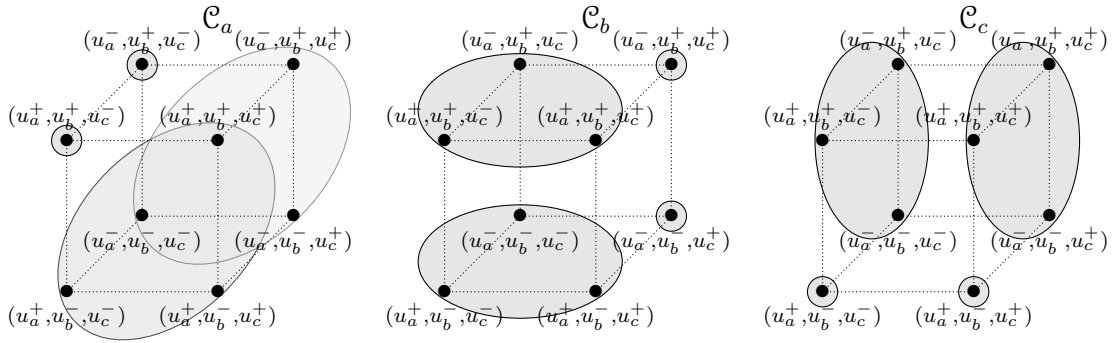


Figure 5.7: Noncausal solvable system: choice partitions of the three agents.

5.5 Conclusion

In this Chapter, we have presented classical information patterns from Game Theory and gave extensions of the information patterns introduced by Witsenhausen in [38] generalizing them in two directions: first, we have formulated these definitions for any subset of agents, second, the definitions allow for non-constant configuration orderings.

We conjectured that the so-called W-information memory (respectively, W-perfect recall) is equivalent to the so-called I-TrIP (respectively, C-TrIP) introduced in [3].

Part II

Games in Intrinsic Form

Chapter 6

W-games

6.1 Introduction

None of the three models seen in Chapter 2 is complete until we define individual players, the players' preferences (payoffs) and attitudes towards Nature (beliefs). In this Chapter 6, we define W-games adding to the Definition 2.20 of the W-model, the partition of the set \mathbb{A} of agents into sets of representative agents of the players, and the collection of players' preferences. Then, we discuss a special case of payoffs and beliefs. Also, we introduce a number of essential concepts, which will be used later in Chapter 7 to define a subclass of the Principal-Agent game and in Chapter 8 when speaking of the future avenues of research. To this end,

- in Section 6.2 we give two definitions of the W-game and formulate a proper notion of the Nash equilibrium for either of the two W-games,
- in Section 6.3, we define the concept of subsystem and establish a useful property of co-cycle, a particular case of which is used later in Chapter 7.

6.2 Nash equilibrium in the W-game

In this Section 6.2, we give a definition of a W-game, which is a W-model as in Definition 2.20 with players and players' preferences on the set of random variables as defined in §6.2.1. Then, in §6.2.2, we define a W-game for the case of payoffs and beliefs. Both definitions are followed by a corresponding notion of a Nash equilibrium.

Randomized strategies are easily defined over finite sets of strategies. Although, when the sets of strategies are infinite, some delicate questions arise (see §5.2.1 in [3] or [4, 5] for more details). Thus, we do not discuss the notions of mixed and behavioral strategies, leaving them among the open questions.

6.2.1 The general case with preferences over random variables

First, we define a W-game.

Definition 6.1. A W-game $\left(((\Omega, \mathcal{F}), \{\mathcal{U}_a, \mathcal{U}_a, \mathcal{J}_a\}_{a \in \mathbb{A}}), \{\mathbb{A}^i\}_{i \in \mathcal{P}}, \{\succsim^i\}_{i \in \mathcal{P}} \right)$, with players' preferences, is made of

- a W-model $((\Omega, \mathcal{F}), \{\mathcal{U}_a, \mathcal{U}_a, \mathcal{J}_a\}_{a \in \mathbb{A}})$ as in Definition 2.20, which is solvable as in Definition 2.22,
- a partition $\mathbb{A} = \bigsqcup_{i \in \mathcal{P}} \mathbb{A}^i$ of the W-agents, where each atom \mathbb{A}^i is made of the executives of the so-called individual player $i \in \mathcal{P}$,
- for each player $i \in \mathcal{P}$, a preference relation \succsim^i on the set of measurable mappings $H : \Omega \rightarrow \mathbb{H}$ from (Ω, \mathcal{F}) to the configuration space $(\mathbb{H}, \mathcal{H})$ as in (2.24a).

Second, we define the concept of Nash equilibrium for a solvable W-game with players' preferences over random variables.

Definition 6.2. A pure strategy profile $\lambda^* = (\lambda_i^*)_{i \in \mathcal{P}} \in \prod_{i \in \mathcal{P}} \Lambda_{\mathbb{A}^i}$ is a (Bayesian) Nash Equilibrium if

$$(\forall i \in \mathcal{P}, \forall \lambda_i \in \Lambda_i) \quad S_{(\lambda_i, \lambda_{-i}^*)} \succsim^i S_{(\lambda_i^*, \lambda_{-i}^*)},$$

where, for any strategy profile $\lambda \in \Lambda_{\mathbb{A}}$, $S_\lambda : \Omega \rightarrow \mathbb{H}$ is a measurable solution map as in (2.32d).

6.2.2 The special case of payoffs and beliefs

The preference relation \succsim^i in Definition 6.1 can be obtained from the numerical representation $\mathbb{E}_{\mathbb{P}^i}[j^i \circ H]$, for any measurable mapping $H : \Omega \rightarrow \mathbb{H}$, any measurable function $j^i : (\mathbb{H}, \mathcal{H}) \rightarrow [0, +\infty]$, and any probability distribution $\mathbb{P}^i : \mathcal{F} \rightarrow [0, 1]$ over the states of Nature (Ω, \mathcal{F}) . This observation allows us to formulate the second definition of the W-game.

Definition 6.3. A W-game $\left(((\Omega, \mathcal{F}), \{\mathcal{U}_a, \mathcal{U}_a, \mathcal{J}_a\}_{a \in \mathbb{A}}), \{\mathbb{A}^i\}_{i \in \mathcal{P}}, \{j^i\}_{i \in \mathcal{P}}, \{\mathbb{P}^i\}_{i \in \mathcal{P}} \right)$, with players' criteria and beliefs, is made of

1. a W-model $((\Omega, \mathcal{F}), \{\mathcal{U}_a, \mathcal{U}_a, \mathcal{J}_a\}_{a \in \mathbb{A}})$ as in Definition 2.20, which is solvable as in Definition 2.22,
2. a partition $\mathbb{A} = \bigsqcup_{i \in \mathcal{P}} \mathbb{A}^i$ of the W-agents, where each atom \mathbb{A}^i is made of the executives of the so-called individual player $i \in \mathcal{P}$,
3. for each player $i \in \mathcal{P}$,
 - a criterion j^i , which is a measurable function $j^i : (\mathbb{H}, \mathcal{H}) \rightarrow [0, +\infty]$,

- a belief \mathbb{P}^i , which is a probability distribution $\mathbb{P}^i : \mathcal{F} \rightarrow [0, 1]$ over the states of Nature (Ω, \mathcal{F}) .

As the W-model is solvable by assumption, for any admissible strategy $\lambda \in \Lambda_{\mathbb{A}}$, there is a measurable solution map $S_\lambda : \Omega \rightarrow \mathbb{H}$. A composition of the solution map with a player i 's criterion provides a random variable

$$j^i \circ S_\lambda : \Omega \rightarrow \mathbb{R} ,$$

which can be integrated w.r.t. the player's belief \mathbb{P}^i , yielding

$$\mathbb{E}_{\mathbb{P}^i}[j^i \circ S_\lambda] \in \mathbb{R} ,$$

where $\mathbb{E}_{\mathbb{P}^i}$ denotes the mathematical expectation w.r.t. the probability \mathbb{P}^i on (Ω, \mathcal{F}) .

With all this in hand, we can equip every player $i \in \mathcal{P}$ with

- a strategy set made of pure strategies $\Lambda_{\mathbb{A}^i} = \prod_{a \in \mathbb{A}^i} \Lambda_a$,
- the player's payoff from W-strategies to reals

$$\prod_{i \in \mathcal{P}} \Lambda_{\mathbb{A}^i} \ni \lambda \mapsto \mathbb{E}_{\mathbb{P}^i}[j^i \circ S_\lambda] . \quad (6.1)$$

Finally, we define the concept of Nash equilibrium for a W-game with players' criteria and beliefs.

Definition 6.4. A pure strategy profile $\lambda^* = (\lambda_i^*)_{i \in \mathcal{P}} \in \prod_{i \in \mathcal{P}} \Lambda_{\mathbb{A}^i}$ is a (Bayesian) Nash Equilibrium if, for all players $i \in \mathcal{P}$,

$$\mathbb{E}_{\mathbb{P}^i}[j^i \circ S_{(\lambda_i^*, \lambda_{-i}^*)}] \geq \mathbb{E}_{\mathbb{P}^i}[j^i \circ S_{(\lambda_i, \lambda_{-i}^*)}] , \quad \forall \lambda_i \in \Lambda_{\mathbb{A}^i} . \quad (6.2)$$

6.3 Subsystems in the W-model

In what follows, we will make the assumption that all the σ -fields contain the singletons. The following definition is due to Witsenhausen (1975).

Definition 6.5. A nonempty subset of agents $\mathbb{B} \subset \mathbb{A}$ is a subsystem if the information field

$$\mathcal{J}_{\mathbb{B}} = \bigvee_{b \in \mathbb{B}} \mathcal{J}_b$$

at most depends on the decisions of the agents in \mathbb{B} , that is,

$$\mathcal{J}_{\mathbb{B}} \subset \mathcal{F} \otimes \mathcal{U}_{\mathbb{B}} \otimes \{\emptyset, \bar{\mathbb{U}}_{-\mathbb{B}}\} = \mathcal{F} \otimes \mathcal{U}_{\mathbb{B}} \otimes \bigotimes_{c \in -\mathbb{B}} \{\emptyset, \mathcal{U}_c\} ,$$

where $-\mathbb{B}$ denotes the complementary of the subset \mathbb{B} as in (2.25).

Thus, the information received by agents in the subsystem \mathbb{B} depends upon states of Nature and decisions of members of \mathbb{B} only.

The next proposition shows that subsystems make it possible to decompose solution maps.

Proposition 6.6. *Let a W-model as in Definition 2.20 be given and a subset $\mathbb{B} \subset \mathbb{A}$ of agents form a subsystem as in Definition 6.5. Then, any W-strategy profile $\lambda \in \Lambda_{\mathbb{A}}$, can be factorized as $\lambda = (\lambda_{\mathbb{B}}, \lambda_{-\mathbb{B}})$ where*

$$\lambda_{\mathbb{B}} : \Omega \times \bar{U}_{\mathbb{B}} \times \bar{U}_{-\mathbb{B}} \rightarrow \bar{U}_{\mathbb{B}}, \quad \lambda_{-\mathbb{B}} : \Omega \times \bar{U}_{\mathbb{B}} \times \bar{U}_{-\mathbb{B}} \rightarrow \bar{U}_{-\mathbb{B}}, \quad (6.3a)$$

and there exists a unique reduced W-strategy profile $\bar{\lambda}_{\mathbb{B}} = \{\bar{\lambda}_a\}_{a \in \mathbb{B}}$, where the corresponding reduced strategy $\bar{\lambda}_a : \Omega \times \bar{U}_{\mathbb{B}} \rightarrow U_a$, of any agent $a \in \mathbb{B}$, is such that

$$(\forall a \in \mathbb{B}, \forall (\omega, u_{\mathbb{B}}, u_{-\mathbb{B}}) \in \Omega \times \bar{U}_{\mathbb{B}} \times \bar{U}_{-\mathbb{B}}) \quad \lambda_a(\omega, u_{\mathbb{B}}, u_{-\mathbb{B}}) = \bar{\lambda}_a(\omega, u_{\mathbb{B}}). \quad (6.3b)$$

Moreover, if the W-model is solvable, for any profile of W-strategies $\lambda = (\lambda_{\mathbb{B}}, \lambda_{-\mathbb{B}})$ decomposed as in (6.3a), with solution map

$$S_{(\lambda_{\mathbb{B}}, \lambda_{-\mathbb{B}})} : (\Omega, \mathcal{F}) \rightarrow (\Omega \times \bar{U}_{\mathbb{B}} \times \bar{U}_{-\mathbb{B}}, \mathcal{F} \otimes \mathcal{U}_{\mathbb{B}} \otimes \mathcal{U}_{-\mathbb{B}}),$$

there exist two reduced solution maps,

$$\bar{S}_{\lambda_{\mathbb{B}}} : (\Omega, \mathcal{F}) \rightarrow (\Omega \times \bar{U}_{\mathbb{B}}, \mathcal{F} \otimes \mathcal{U}_{\mathbb{B}}), \quad \bar{S}_{\lambda_{-\mathbb{B}}} : (\Omega \times \bar{U}_{\mathbb{B}}, \mathcal{F} \otimes \mathcal{U}_{\mathbb{B}}) \rightarrow (\Omega \times \bar{U}_{\mathbb{B}} \times \bar{U}_{-\mathbb{B}}, \mathcal{F} \otimes \mathcal{U}_{\mathbb{B}} \otimes \mathcal{U}_{-\mathbb{B}}),$$

which satisfy the following co-cycle property

$$S_{(\lambda_{\mathbb{B}}, \lambda_{-\mathbb{B}})} = \bar{S}_{\lambda_{-\mathbb{B}}} \circ \bar{S}_{\lambda_{\mathbb{B}}}. \quad (6.4)$$

We do not give a proof of this Proposition, instead, we discuss of a particular case in Proposition 7.4, to be found in Section 7.3, of a so-called Enough-informed Agent W-game, where the Principal player is represented by one W-agent and forms a subsystem.

6.4 Conclusion

In this Chapter 6, we have formulated the notions of W-game, Nash equilibrium and subsystems. Definitions of mixed and behavioral strategies in the W-model are left as open questions. Another open question is to clarify the notion of subgames in the W-model and connect it to the notion of subsystems. Once this is done, we expect the notions of subgame perfect equilibrium and backward induction to be formulated in the W-model.

Chapter 7

Principal-Agent models

7.1 Introduction

A branch of Economics studies so-called *Principal-Agent* models where there are two decision-makers — that are both players, as in Game Theory, and agents, as in the Witsenhausen Intrinsic Model. There are two main strands in the existing literature about the Principal-Agent models: the first is devoted to finding Nash equilibria in Principal-Agent games (see, for example, [20]) and the second – establishing conditions under which an equilibrium exists (by whether imposing restrictions on the primitives of the model, for example, see [8, 11, 14, 16], or imposing restrictions on the allowed mechanisms, for example, see [6, 13, 24, 25, 28] with the most general framework given in [15]). Existence conditions are outside the scope of the Thesis, in this Chapter 7, we concentrate on the conditions under which a solution of a Principal-Agent game is yielded as a result of backward induction assuming that it exists. For this purpose, we will stick to the following roadmap.

- In Section 7.2, we define the Principal-Agent W-game and illustrate it on classic examples from Economics.
- In Section 7.3, we introduce the key concept of the so-called enough informed Agent, give essential background on integrands to prove the two main results of this Chapter 7.

In the rest of this Chapter 7, in order to distinguish a generic W-agent from the specific Agent player in Principal-Agent models, we will employ the upper case for the latter.

7.2 Nash equilibrium in the Principal-Agent W-game

In this Section 7.2 we introduce the main objects, namely,

- in §7.2.1, we introduce the so-called Principal-Agent W-game, which is a particular case of the W-model as defined in §2.4.2 and define the concept of Nash equilibrium for a Principal-Agent W-game,

- in §7.2.2, we illustrate different information structures of the Principal-Agent W-model on classic examples from Economics.

7.2.1 Definition of the Principal-Agent W-game

In the W-model, we will represent a Principal-Agent model by the following two decision-makers (or two W-agents)

- the Principal \mathbf{Pr} (leader), who makes decisions $u_{\mathbf{Pr}} \in \mathbb{U}_{\mathbf{Pr}}$, where the set $\mathbb{U}_{\mathbf{Pr}}$ of decisions is equipped with a σ -field $\mathcal{U}_{\mathbf{Pr}}$,
- the Agent \mathbf{Ag} (follower), who makes decisions $u_{\mathbf{Ag}} \in \mathbb{U}_{\mathbf{Ag}}$, where the set $\mathbb{U}_{\mathbf{Ag}}$ of decisions is equipped with a σ -field $\mathcal{U}_{\mathbf{Ag}}$,

and by

- Nature that selects $\omega \in \Omega$, where Ω is equipped with a σ -field \mathcal{F} .

Thus, by (2.24a), we obtain the following configuration space \mathbb{H} and configuration field \mathcal{H}

$$\mathbb{H} = \Omega \times \prod_{a \in \{\mathbf{Pr}, \mathbf{Ag}\}} \mathbb{U}_a, \quad (7.1a)$$

$$\mathcal{H} = \mathcal{F} \otimes \bigotimes_{a \in \{\mathbf{Pr}, \mathbf{Ag}\}} \mathcal{U}_a. \quad (7.1b)$$

Following the Definition 6.3 of a W-game, we define a Principal-Agent W-game as one with two W-agents in the set $\mathbb{A} = \{\mathbf{Pr}, \mathbf{Ag}\}$, and two players in the set $\mathcal{P} = \{\{\mathbf{Pr}\}, \{\mathbf{Ag}\}\}$ made of the two atoms $\{\mathbf{Pr}\}$ and $\{\mathbf{Ag}\}$ of the complete partition of \mathbb{A} . The two players each have a belief, $\mathbb{P}^{\{\mathbf{Pr}\}}$ and $\mathbb{P}^{\{\mathbf{Ag}\}}$ over Nature, and a criterion, $j^{\{\mathbf{Pr}\}}$ and $j^{\{\mathbf{Ag}\}}$, that they aim at maximizing.

Definition 7.1. *A Principal-Agent W-game is a collection*

$$\left((\Omega, \mathcal{F}), \{\mathbb{U}_a, \mathcal{U}_a, \mathcal{J}_a\}_{a \in \{\mathbf{Pr}, \mathbf{Ag}\}}, \mathbb{A} = \{\mathbf{Pr}\} \sqcup \{\mathbf{Ag}\}, \{j^i\}_{i \in \{\{\mathbf{Pr}\}, \{\mathbf{Ag}\}\}}, \{\mathbb{P}^i\}_{i \in \{\{\mathbf{Pr}\}, \{\mathbf{Ag}\}\}} \right), \quad (7.2)$$

which is made of

- a W-model $\left((\Omega, \mathcal{F}), \{\mathbb{U}_a, \mathcal{U}_a, \mathcal{J}_a\}_{a \in \{\mathbf{Pr}, \mathbf{Ag}\}} \right)$, as in Definition 2.20, supposed to be solvable (hence that displays absence of self-information),
- the partition $\mathbb{A} = \{\mathbf{Pr}\} \sqcup \{\mathbf{Ag}\}$ of the W-agents into two players, $\{\mathbf{Pr}\}$ and $\{\mathbf{Ag}\}$,
- beliefs $\mathbb{P}^{\{\mathbf{Pr}\}}$ and $\mathbb{P}^{\{\mathbf{Ag}\}}$, that are probability distributions over (Ω, \mathcal{F}) ,
- criteria $j^{\{\mathbf{Pr}\}}$ and $j^{\{\mathbf{Ag}\}}$, that are measurable functions $(\mathbb{H}, \mathcal{H}) \rightarrow \overline{\mathbb{R}}$, where the configuration space \mathbb{H} equipped with the σ -field \mathcal{H} are given in (7.1).

The set of W-strategies for a Principal-Agent W-game is made of the two sets of W-strategies, one per agent, of the underlying W-model $\left((\Omega, \mathcal{F}), \{\mathbb{U}_a, \mathcal{U}_a, \mathcal{J}_a\}_{a \in \{\text{Pr}, \text{Ag}\}} \right)$, that is,

$$\Lambda_{\text{Pr}} = \left\{ \lambda_{\text{Pr}} : (\mathbb{H}, \mathcal{H}) \rightarrow (\mathbb{U}_{\text{Pr}}, \mathcal{U}_{\text{Pr}}) \mid \lambda_{\text{Pr}}^{-1}(\mathcal{U}_{\text{Pr}}) \subset \mathcal{J}_{\text{Pr}} \right\}, \quad (7.3a)$$

$$\Lambda_{\text{Ag}} = \left\{ \lambda_{\text{Ag}} : (\mathbb{H}, \mathcal{H}) \rightarrow (\mathbb{U}_{\text{Ag}}, \mathcal{U}_{\text{Ag}}) \mid \lambda_{\text{Ag}}^{-1}(\mathcal{U}_{\text{Ag}}) \subset \mathcal{J}_{\text{Ag}} \right\}. \quad (7.3b)$$

By the solvability assumption, for any pure strategy profile $\lambda = (\lambda_a)_{a \in \{\text{Pr}, \text{Ag}\}} \in \prod_{a \in \{\text{Pr}, \text{Ag}\}} \Lambda_a$, there exists a measurable solution mapping $S_\lambda : (\Omega, \mathcal{F}) \rightarrow (\mathbb{H}, \mathcal{H})$ by (2.32c), that is:

$$\begin{aligned} S_\lambda &: (\Omega, \mathcal{F}) \rightarrow (\mathbb{H}, \mathcal{H}) \\ \forall \lambda = \{\lambda_a\}_{a \in \{\text{Pr}, \text{Ag}\}} \in \prod_{a \in \{\text{Pr}, \text{Ag}\}} \Lambda_a, \quad \forall (\omega, \{u_a\}_{a \in \{\text{Pr}, \text{Ag}\}}) \in \mathbb{H}, \\ (\omega, \{u_a\}_{a \in \{\text{Pr}, \text{Ag}\}}) = S_\lambda(\omega) &\iff \begin{cases} u_{\text{Pr}} = \lambda_{\text{Pr}}(\omega, \{u_a\}_{a \in \{\text{Pr}, \text{Ag}\}}), \\ u_{\text{Ag}} = \lambda_{\text{Ag}}(\omega, \{u_a\}_{a \in \{\text{Pr}, \text{Ag}\}}). \end{cases} \end{aligned} \quad (7.4)$$

Now we are ready to specify the concept of Nash equilibrium for a Principal-Agent W-game.

Definition 7.2. A (pure) W-strategy profile $\lambda^* = \{\lambda_a^*\}_{a \in \{\text{Pr}, \text{Ag}\}} \in \prod_{a \in \{\text{Pr}, \text{Ag}\}} \Lambda_a$ is a Nash equilibrium for the Principal-Agent game if and only if

$$\mathbb{E}_{\mathbb{P}^{\{\text{Pr}\}}} [j^{\{\text{Pr}\}} \circ S_{(\lambda_{\text{Pr}}^*, \lambda_{\text{Ag}}^*)}] \geq \mathbb{E}_{\mathbb{P}^{\{\text{Pr}\}}} [j^{\{\text{Pr}\}} \circ S_{(\lambda_{\text{Pr}}, \lambda_{\text{Ag}}^*)}], \quad \forall \lambda_{\text{Pr}} \in \Lambda_{\text{Pr}}, \quad (7.5a)$$

$$\mathbb{E}_{\mathbb{P}^{\{\text{Ag}\}}} [j^{\{\text{Ag}\}} \circ S_{(\lambda_{\text{Pr}}^*, \lambda_{\text{Ag}}^*)}] \geq \mathbb{E}_{\mathbb{P}^{\{\text{Ag}\}}} [j^{\{\text{Ag}\}} \circ S_{(\lambda_{\text{Pr}}^*, \lambda_{\text{Ag}})], \quad \forall \lambda_{\text{Ag}} \in \Lambda_{\text{Ag}}. \quad (7.5b)$$

7.2.2 Examples

Here below, we emphasize the information structure of Principal-Agent economic models in the Witsenhausen framework.

7.2.2.1 Stackelberg leadership model

In the Stackelberg leadership model of game theory, the leader **Pr** observes at most (partly) the state of Nature, which we model as

$$\mathcal{J}_{\text{Pr}} \subset \underbrace{\{\emptyset, \mathbb{U}_{\text{Ag}}\}}_{\substack{\text{Pr does not know} \\ \text{Ag's action}}} \otimes \{\emptyset, \mathbb{U}_{\text{Pr}}\} \otimes \mathcal{F}, \quad (7.6a)$$

whereas the follower **Ag** may partly observe the action of the leader **Pr**, represented by

$$\mathcal{J}_{\text{Ag}} \subset \{\emptyset, \mathbb{U}_{\text{Ag}}\} \otimes \mathbb{U}_{\text{Pr}} \otimes \mathcal{F}. \quad (7.6b)$$

As a consequence, the system is sequential with the Principal Pr as first W-agent (leader) and the Agent Ag as second W-agent (follower).

As we will see in §7.3.1, the Stackelberg leadership model belongs to the class of so-called *enough informed Agent* Principal-Agent W-games.

7.2.2.2 Hidden type and signalling

The Agent Ag knows the state of nature (her type), that is,

$$\{\emptyset, \mathcal{U}_{\text{Ag}}\} \otimes \{\emptyset, \mathcal{U}_{\text{Pr}}\} \otimes \underbrace{\mathcal{F}}_{\text{known inner type}} \subset \mathcal{J}_{\text{Ag}}, \quad (7.7a)$$

but the Principal Pr does not know the agent type, that is,

$$\mathcal{J}_{\text{Pr}} \subset \mathcal{U}_{\text{Ag}} \otimes \{\emptyset, \mathcal{U}_{\text{Pr}}\} \otimes \underbrace{\{\emptyset, \Omega\}}_{\text{unknown Ag type}}. \quad (7.7b)$$

This formalism covers the market for lemons by Akerlof [1] and the costly signalling examples (peacock's tail and diplomas on the job market by Spence [32]).

In the case where $\mathcal{J}_{\text{Ag}} = \{\emptyset, \mathcal{U}_{\text{Ag}}\} \otimes \{\emptyset, \mathcal{U}_{\text{Pr}}\} \otimes \mathcal{F}$, the system is sequential with the Principal Pr as first W-agent (leader) and the Agent Ag as second W-agent (follower).

7.2.2.3 Hidden action

An insurance company (the Principal Pr) cannot observe the efforts of the insured (the Agent Ag) to avoid risky behaviour. The firm faces the hazard that insured persons behave “immorally” (playing with matches at home). Moral hazard or hidden action occurs when the decisions of the Agent Ag are hidden to the Principal Pr , namely,

$$\mathcal{J}_{\text{Pr}} \subset \underbrace{\{\emptyset, \mathcal{U}_{\text{Ag}}\}}_{\text{hidden action}} \otimes \{\emptyset, \mathcal{U}_{\text{Pr}}\} \otimes \mathcal{F}. \quad (7.8)$$

In case of moral hazard, the system is sequential with the Principal Pr as first W-agent, (which does not preclude to choose the Agent Ag as first W-agent in some special cases, as in a static team situation)

7.3 Enough informed Agent W-games

We now focus on a restricted class of Principal-Agent W-games and formulate conditions under which a Nash equilibrium of a Principal-Agent game can be found as a result of backward induction. To this end,

- in §7.3.1, we define the restricted class of the so-called Enough informed Agent W-games,

- in §7.3.2, we introduce technicalities, namely, lower semicontinuous integrands, the conditional expectation and measurable selections of lower semicontinuous integrands,
- in §7.3.3 we state and prove the main results of this Chapter 7, namely, Proposition 7.13 and Theorem 7.14.

7.3.1 Definition of Enough informed Agent W-games

First, we define the restricted class of the so-called Enough informed Agent W-games, which is extensively used in the sequel.

Enough informed Agent W-games.

Definition 7.3. *We call Enough informed Agent W-game a collection (7.2) which is made of*

1. a W-model $\left((\Omega, \mathcal{F}), \{\mathcal{U}_a, \mathcal{U}_a, \mathcal{J}_a\}_{a \in \{\text{Pr}, \text{Ag}\}} \right)$, as in Definition 2.20,

(a) such that all the σ -fields \mathcal{F} and $\{\mathcal{U}_a, \mathcal{J}_a\}_{a \in \{\text{Pr}, \text{Ag}\}}$ contain the singletons,

(b) with the following information structure:

- the Principal knows at most the state of Nature, that is,

$$\mathcal{J}_{\text{Pr}} \subset \mathcal{F} \otimes \{\emptyset, \mathcal{U}_{\text{Pr}}\} \otimes \{\emptyset, \mathcal{U}_{\text{Ag}}\}, \quad (7.9a)$$

- the Agent knows at least what the Principal knows and what the Principal does, that is,

$$\mathcal{J}_{\text{Pr}} \vee \mathcal{U}_{\text{Pr}} \subset \mathcal{J}_{\text{Ag}}, \quad (7.9b)$$

where \mathcal{U}_{Pr} is a shorthand for $\{\emptyset, \Omega\} \otimes \mathcal{U}_{\text{Pr}} \otimes \{\emptyset, \mathcal{U}_{\text{Ag}}\}$,

2. the partition $\mathbb{A} = \{\text{Pr}\} \sqcup \{\text{Ag}\}$ of the W-agents into two players,
3. beliefs $\mathbb{P}^{\{\text{Pr}\}}$ and $\mathbb{P}^{\{\text{Ag}\}}$, that are probability distributions over (Ω, \mathcal{F}) ,
4. criteria $j^{\{\text{Pr}\}}$ and $j^{\{\text{Ag}\}}$ that are functions $\Omega \times \mathcal{U}_{\text{Pr}} \times \mathcal{U}_{\text{Ag}} \rightarrow \overline{\mathbb{R}}$.

By the information structure in Item 1b of Definition 7.3, the Principal plays first by (7.9a), followed by the Agent player by (7.9b). This is write the configuration space \mathbb{H} and configuration field \mathcal{H} in (7.1) under the form

$$\mathbb{H} = \Omega \times \prod_{a \in \{\text{Pr}, \text{Ag}\}} \mathcal{U}_a = \Omega \times \mathcal{U}_{\text{Pr}} \times \mathcal{U}_{\text{Ag}},$$

$$\mathcal{H} = \mathcal{F} \otimes \bigotimes_{a \in \{\text{Pr}, \text{Ag}\}} \mathcal{U}_a = \mathcal{F} \otimes \mathcal{U}_{\text{Pr}} \otimes \mathcal{U}_{\text{Ag}}.$$

In these two latter expressions, there is an abuse of notation in the sense that $\prod_{a \in \{\text{Pr}, \text{Ag}\}} \mathbb{U}_a$ is identified with $\mathbb{U}_{\text{Pr}} \times \mathbb{U}_{\text{Ag}}$, and the same for $\bigotimes_{a \in \{\text{Pr}, \text{Ag}\}} \mathcal{U}_a$ identified with $\mathcal{U}_{\text{Pr}} \otimes \mathcal{U}_{\text{Ag}}$, to stress the fact that the Principal plays before the Agent.

Proposition 7.4. *An Enough informed Agent W-game is a Principal-Agent W-game, as in Definition 7.1. In particular, there exist*

1. reduced information fields $\bar{\bar{\mathcal{J}}}_{\text{Pr}}$ in (7.15a) and $\bar{\mathcal{J}}_{\text{Ag}}$ in (7.16a), such that

$$\bar{\bar{\mathcal{J}}}_{\text{Pr}} \subset \mathcal{F}, \quad \mathcal{J}_{\text{Pr}} = \bar{\bar{\mathcal{J}}}_{\text{Pr}} \otimes \{\emptyset, \mathbb{U}_{\text{Pr}}\} \otimes \{\emptyset, \mathbb{U}_{\text{Ag}}\}, \quad (7.10a)$$

$$\bar{\mathcal{J}}_{\text{Ag}} \subset \mathcal{F} \otimes \mathcal{U}_{\text{Pr}}, \quad \mathcal{J}_{\text{Ag}} = \bar{\mathcal{J}}_{\text{Ag}} \otimes \{\emptyset, \mathbb{U}_{\text{Ag}}\}, \quad (7.10b)$$

and that

$$(\bar{\bar{\mathcal{J}}}_{\text{Pr}} \otimes \{\emptyset, \mathbb{U}_{\text{Pr}}\}) \bigvee (\{\emptyset, \Omega\} \otimes \mathcal{U}_{\text{Pr}}) \subset \bar{\mathcal{J}}_{\text{Ag}}, \quad (7.10c)$$

2. reduced W-strategy sets in (7.15c) and in (7.16c)

$$\bar{\bar{\Lambda}}_{\text{Pr}} = \{\mu_{\text{Pr}} : (\Omega, \mathcal{F}) \rightarrow (\mathbb{U}_{\text{Pr}}, \mathcal{U}_{\text{Pr}}) \mid (\mu_{\text{Pr}})^{-1}(\mathcal{U}_{\text{Pr}}) \subset \bar{\bar{\mathcal{J}}}_{\text{Pr}}\}, \quad (7.11a)$$

$$\bar{\Lambda}_{\text{Ag}} = \{\mu_{\text{Ag}} : ((\Omega \times \mathbb{U}_{\text{Pr}}), (\mathcal{F} \otimes \mathcal{U}_{\text{Pr}})) \rightarrow (\mathbb{U}_{\text{Ag}}, \mathcal{U}_{\text{Ag}}) \mid (\mu_{\text{Ag}})^{-1}(\mathcal{U}_{\text{Ag}}) \subset \bar{\mathcal{J}}_{\text{Ag}}\}, \quad (7.11b)$$

3. reduction mappings in (7.15b) and in (7.16b), that are bijections

$$\lambda_{\text{Pr}} \in \Lambda_{\text{Pr}} \mapsto \bar{\bar{\lambda}}_{\text{Pr}} \in \bar{\bar{\Lambda}}_{\text{Pr}} \quad (7.12a)$$

$$\lambda_{\text{Ag}} \in \Lambda_{\text{Ag}} \mapsto \bar{\lambda}_{\text{Ag}} \in \bar{\Lambda}_{\text{Ag}} \quad (7.12b)$$

4. reduced solution maps in (7.15f) and in (7.16f), that are measurable mappings

$$\begin{aligned} \bar{\bar{S}}_{\mu_{\text{Pr}}} &: (\Omega, \mathcal{F}) \rightarrow (\Omega \times \mathbb{U}_{\text{Pr}}, \mathcal{F} \otimes \mathcal{U}_{\text{Pr}}) \\ \omega &\mapsto (\omega, \mu_{\text{Pr}}(\omega)), \quad \forall \mu_{\text{Pr}} \in \bar{\bar{\Lambda}}_{\text{Pr}}, \end{aligned} \quad (7.13a)$$

$$\begin{aligned} \bar{S}_{\mu_{\text{Ag}}} &: (\Omega \times \mathbb{U}_{\text{Pr}}, \mathcal{F} \otimes \mathcal{U}_{\text{Pr}}) \rightarrow (\Omega \times \mathbb{U}_{\text{Pr}} \times \mathbb{U}_{\text{Ag}}, \mathcal{F} \otimes \mathcal{U}_{\text{Pr}} \otimes \mathcal{U}_{\text{Ag}}) \\ (\omega, u_{\text{Pr}}) &\mapsto (\omega, u_{\text{Pr}}, \mu_{\text{Ag}}(\omega, u_{\text{Pr}})), \quad \forall \mu_{\text{Ag}} \in \bar{\Lambda}_{\text{Ag}}, \end{aligned} \quad (7.13b)$$

such that the W-model in Definition 7.3 is solvable, with measurable solution map

$$S_{(\lambda_{\text{Pr}}, \lambda_{\text{Ag}})} : (\Omega, \mathcal{F}) \rightarrow (\Omega \times \mathbb{U}_{\text{Pr}} \times \mathbb{U}_{\text{Ag}}, \mathcal{F} \otimes \mathcal{U}_{\text{Pr}} \otimes \mathcal{U}_{\text{Ag}}), \quad (7.14a)$$

given, for any W-strategy profile $\lambda = (\lambda_{\text{Pr}}, \lambda_{\text{Ag}}) \in \Lambda_{\text{Pr}} \times \Lambda_{\text{Ag}}$, by

$$S_{(\lambda_{\text{Pr}}, \lambda_{\text{Ag}})} = \bar{S}_{\bar{\lambda}_{\text{Ag}}} \circ \bar{\bar{S}}_{\bar{\bar{\lambda}}_{\text{Pr}}}. \quad (7.14b)$$

The proof of the Proposition is broken into a series of steps to be found here below.

Reduced subfields, W-strategies and solutions map for the Principal Pr. We define the *reduced information field of the Principal Pr* by

$$\bar{\mathcal{J}}_{\text{Pr}} = \{ \bar{G} \in \mathcal{F} \mid \bar{G} \otimes \{\emptyset, \mathbb{U}_{\text{Pr}}\} \otimes \{\emptyset, \mathbb{U}_{\text{Ag}}\} \in \mathcal{J}_{\text{Pr}} \} \subset \mathcal{F}. \quad (7.15a)$$

Now, because $\mathcal{J}_{\text{Pr}} \subset \mathcal{F} \otimes \{\emptyset, \mathbb{U}_{\text{Pr}}\} \otimes \{\emptyset, \mathbb{U}_{\text{Ag}}\}$ by (7.9a), we immediately obtain the equality (7.10a), namely $\mathcal{J}_{\text{Pr}} = \bar{\mathcal{J}}_{\text{Pr}} \otimes \{\emptyset, \mathbb{U}_{\text{Pr}}\} \otimes \{\emptyset, \mathbb{U}_{\text{Ag}}\}$.

By (7.3a) and (7.9a), and because all the σ -fields \mathcal{F} and $\{\mathcal{U}_a, \mathcal{J}_a\}_{a \in \{\text{Pr}, \text{Ag}\}}$ contain the singletons by Item 1a in Definition 7.3, with any W-strategy $\lambda_{\text{Pr}} \in \Lambda_{\text{Pr}}$ for the Principal Pr, we can associate a mapping, the *reduced W-strategy* $\bar{\lambda}_{\text{Pr}}$ (of the Principal Pr), by

$$\bar{\lambda}_{\text{Pr}} : (\Omega, \mathcal{F}) \rightarrow (\mathbb{U}_{\text{Pr}}, \mathcal{U}_{\text{Pr}}) \text{ such that } \begin{cases} \bar{\lambda}_{\text{Pr}}(\omega) = \lambda_{\text{Pr}}(\omega, u_{\text{Pr}}, u_{\text{Ag}}), \\ \forall (\omega, u_{\text{Pr}}, u_{\text{Ag}}) \in \Omega \times \mathbb{U}_{\text{Pr}} \times \mathbb{U}_{\text{Ag}}, \\ (\bar{\lambda}_{\text{Pr}})^{-1}(\mathcal{U}_{\text{Pr}}) \subset \bar{\mathcal{J}}_{\text{Pr}}. \end{cases} \quad (7.15b)$$

Indeed, if we had $\lambda_{\text{Pr}}(\omega, u_{\text{Pr}}, u_{\text{Ag}}) \neq \lambda_{\text{Pr}}(\omega, u'_{\text{Pr}}, u'_{\text{Ag}})$, we would have $(\omega, u_{\text{Pr}}, u_{\text{Ag}}) \in G = \lambda_{\text{Pr}}^{-1}(\{\lambda_{\text{Pr}}(\omega, u_{\text{Pr}}, u_{\text{Ag}})\})$ and $(\omega, u'_{\text{Pr}}, u'_{\text{Ag}}) \in G' = \lambda_{\text{Pr}}^{-1}(\{\lambda_{\text{Pr}}(\omega, u'_{\text{Pr}}, u'_{\text{Ag}})\})$, with $G \cap G' = \emptyset$. Now, since $\lambda_{\text{Pr}} \in \Lambda_{\text{Pr}}$, the mapping λ_{Pr} is \mathcal{J}_{Pr} -measurable, so that $G \in \mathcal{J}_{\text{Pr}}$ and $G' \in \mathcal{J}_{\text{Pr}}$, because by assumption in Item 1a in Definition 7.3, the σ -field \mathcal{J}_{Pr} contain the singletons. But since $G \in \mathcal{J}_{\text{Pr}} \subset \mathcal{F} \otimes \{\emptyset, \mathbb{U}_{\text{Pr}}\} \otimes \{\emptyset, \mathbb{U}_{\text{Ag}}\}$, G is a cylinder such that $(\omega, u''_{\text{Pr}}, u''_{\text{Ag}}) \in G$, for all $(u''_{\text{Pr}}, u''_{\text{Ag}})$. In particular, we get that $(\omega, u'_{\text{Pr}}, u'_{\text{Ag}}) \in G$, and we arrive at a contradiction because $(\omega, u'_{\text{Pr}}, u'_{\text{Ag}}) \in G'$ and $G \cap G' = \emptyset$ by definition of G and G' .

If we define the *reduced W-strategy set (of the Principal Pr)* by

$$\bar{\Lambda}_{\text{Pr}} = \{ \mu_{\text{Pr}} : (\Omega, \mathcal{F}) \rightarrow (\mathbb{U}_{\text{Pr}}, \mathcal{U}_{\text{Pr}}) \mid (\mu_{\text{Pr}})^{-1}(\mathcal{U}_{\text{Pr}}) \subset \bar{\mathcal{J}}_{\text{Pr}} \}, \quad (7.15c)$$

then it is easily seen that the following mappings are in bijection:

$$\lambda_{\text{Pr}} \in \Lambda_{\text{Pr}} \mapsto \bar{\lambda}_{\text{Pr}} \in \bar{\Lambda}_{\text{Pr}} \text{ defined by (7.15b),} \quad (7.15d)$$

$$\mu_{\text{Pr}} \in \bar{\Lambda}_{\text{Pr}} \mapsto \lambda_{\text{Pr}} \in \Lambda_{\text{Pr}} \text{ defined by } \lambda_{\text{Pr}}(\omega, u_{\text{Pr}}, u_{\text{Ag}}) = \mu_{\text{Pr}}(\omega), \quad (7.15e)$$

$$\forall (\omega, u_{\text{Pr}}, u_{\text{Ag}}) \in \Omega \times \mathbb{U}_{\text{Pr}} \times \mathbb{U}_{\text{Ag}}.$$

We define the *reduced solution map (of the Principal Pr)* $\bar{S}_{\mu_{\text{Pr}}}$, for any $\mu_{\text{Pr}} \in \bar{\Lambda}_{\text{Pr}}$, by

$$\bar{S}_{\mu_{\text{Pr}}} : \Omega \rightarrow \Omega \times \mathbb{U}_{\text{Pr}}, \quad \omega \mapsto (\omega, \mu_{\text{Pr}}(\omega)). \quad (7.15f)$$

By (7.15f), (7.15c) and (7.15a), the reduced solution map $\bar{S}_{\mu_{\text{Pr}}}$ is \mathcal{F} -measurable when $\mu_{\text{Pr}} \in \bar{\Lambda}_{\text{Pr}}$, giving

$$\bar{S}_{\mu_{\text{Pr}}} : (\Omega, \mathcal{F}) \rightarrow (\Omega \times \mathbb{U}_{\text{Pr}}, \mathcal{F} \otimes \mathcal{U}_{\text{Pr}}). \quad (7.15g)$$

Reduced subfields, W-strategies and solutions map for the Agent Ag. We define the *reduced information field of the Agent Ag* by

$$\bar{\mathcal{J}}_{\text{Ag}} = \{\bar{G} \in \mathcal{F} \otimes \mathcal{U}_{\text{Pr}} \mid \bar{G} \otimes \{\emptyset, \mathbb{U}_{\text{Ag}}\} \in \mathcal{J}_{\text{Ag}}\} \subset \mathcal{F} \otimes \mathcal{U}_{\text{Pr}}. \quad (7.16a)$$

By absence of self-information — namely, $\mathcal{J}_{\text{Ag}} \subset \mathcal{F} \otimes \mathcal{U}_{\text{Pr}} \otimes \{\emptyset, \mathbb{U}_{\text{Ag}}\}$ — we immediately obtain the equality (7.10b), namely $\mathcal{J}_{\text{Ag}} = \bar{\mathcal{J}}_{\text{Ag}} \otimes \{\emptyset, \mathbb{U}_{\text{Ag}}\}$.

By (7.3b) and absence of self-information, and because all the σ -fields \mathcal{F} and $\{\mathcal{U}_a, \mathcal{J}_a\}_{a \in \{\text{Pr}, \text{Ag}\}}$ contain the singletons by Item 1a in Definition 7.3, with any W-strategy $\lambda_{\text{Ag}} \in \Lambda_{\text{Ag}}$ for the Agent Ag, we can associate a mapping, the *reduced W-strategy* $\bar{\lambda}_{\text{Ag}}$ (of the Agent Ag), by

$$\bar{\lambda}_{\text{Ag}} : ((\Omega \times \mathbb{U}_{\text{Pr}}), (\mathcal{F} \otimes \mathcal{U}_{\text{Pr}})) \rightarrow (\mathbb{U}_{\text{Ag}}, \mathcal{U}_{\text{Ag}}) \text{ such that } \begin{cases} \bar{\lambda}_{\text{Ag}}(\omega, u_{\text{Pr}}) = \lambda_{\text{Ag}}(\omega, u_{\text{Pr}}, u_{\text{Ag}}), \\ \forall (\omega, u_{\text{Pr}}, u_{\text{Ag}}) \in \Omega \times \mathbb{U}_{\text{Pr}} \times \mathbb{U}_{\text{Ag}} \\ (\bar{\lambda}_{\text{Ag}})^{-1}(\mathcal{U}_{\text{Ag}}) \subset \bar{\mathcal{J}}_{\text{Ag}}. \end{cases} \quad (7.16b)$$

The proof follows the same lines than the proof in the Principal case.

If we define the *reduced W-strategy set (of the Agent Ag)* by

$$\bar{\Lambda}_{\text{Ag}} = \{\mu_{\text{Ag}} : ((\Omega \times \mathbb{U}_{\text{Pr}}), (\mathcal{F} \otimes \mathcal{U}_{\text{Pr}})) \rightarrow (\mathbb{U}_{\text{Ag}}, \mathcal{U}_{\text{Ag}}) \mid (\mu_{\text{Ag}})^{-1}(\mathcal{U}_{\text{Ag}}) \subset \bar{\mathcal{J}}_{\text{Ag}}\}, \quad (7.16c)$$

then it is easily seen that the following mappings are in bijection:

$$\lambda_{\text{Ag}} \in \Lambda_{\text{Ag}} \mapsto \bar{\lambda}_{\text{Ag}} \in \bar{\Lambda}_{\text{Ag}} \text{ defined by (7.16b)}, \quad (7.16d)$$

$$\mu_{\text{Ag}} \in \bar{\Lambda}_{\text{Ag}} \mapsto \lambda_{\text{Ag}} \in \Lambda_{\text{Ag}} \text{ defined by } \lambda_{\text{Ag}}(\omega, u_{\text{Pr}}, u_{\text{Ag}}) = \mu_{\text{Ag}}(\omega, u_{\text{Pr}}), \quad (7.16e)$$

$$\forall (\omega, u_{\text{Pr}}, u_{\text{Ag}}) \in \Omega \times \mathbb{U}_{\text{Pr}} \times \mathbb{U}_{\text{Ag}}.$$

We define the *reduced solution map (of the Agent Ag)* $\bar{S}_{\mu_{\text{Ag}}}$, for any $\mu_{\text{Ag}} \in \bar{\Lambda}_{\text{Ag}}$, by

$$\bar{S}_{\mu_{\text{Ag}}} : \Omega \times \mathbb{U}_{\text{Pr}} \rightarrow \Omega \times \mathbb{U}_{\text{Pr}} \times \mathbb{U}_{\text{Ag}}, \quad (\omega, u_{\text{Pr}}) \mapsto (\omega, u_{\text{Pr}}, \mu_{\text{Ag}}(\omega, u_{\text{Pr}})). \quad (7.16f)$$

By (7.16f), (7.16c) and (7.16a), the reduced solution map $\bar{S}_{\mu_{\text{Ag}}}$ is $(\mathcal{F} \otimes \mathcal{U}_{\text{Pr}})$ -measurable, when $\mu_{\text{Ag}} \in \bar{\Lambda}_{\text{Ag}}$, giving

$$\bar{\bar{S}}_{\mu_{\text{Ag}}} : (\Omega \times \mathbb{U}_{\text{Pr}}, \mathcal{F} \otimes \mathcal{U}_{\text{Pr}}) \rightarrow (\Omega \times \mathbb{U}_{\text{Pr}} \times \mathbb{U}_{\text{Ag}}, \mathcal{F} \otimes \mathcal{U}_{\text{Pr}} \otimes \mathcal{U}_{\text{Ag}}). \quad (7.16g)$$

The Agent knows at least what the Principal knows and what the Principal does. We have

$$\begin{aligned} \mathcal{J}_{\text{Pr}} \bigvee \mathcal{U}_{\text{Pr}} \subset \mathcal{J}_{\text{Ag}} &\iff \mathcal{J}_{\text{Pr}} \bigvee (\{\emptyset, \Omega\} \otimes \mathcal{U}_{\text{Pr}} \otimes \{\emptyset, \mathbb{U}_{\text{Ag}}\}) \subset \mathcal{J}_{\text{Ag}} \\ &\iff \mathcal{J}_{\text{Pr}} \subset \mathcal{J}_{\text{Ag}} \text{ and } \{\emptyset, \Omega\} \otimes \mathcal{U}_{\text{Pr}} \otimes \{\emptyset, \mathbb{U}_{\text{Ag}}\} \subset \mathcal{J}_{\text{Ag}} \\ &\iff \bar{\bar{\mathcal{J}}}_{\text{Pr}} \otimes \{\emptyset, \mathbb{U}_{\text{Pr}}\} \otimes \{\emptyset, \mathbb{U}_{\text{Ag}}\} \subset \bar{\mathcal{J}}_{\text{Ag}} \otimes \{\emptyset, \mathbb{U}_{\text{Ag}}\} \\ &\text{and } \{\emptyset, \Omega\} \otimes \mathcal{U}_{\text{Pr}} \otimes \{\emptyset, \mathbb{U}_{\text{Ag}}\} \subset \bar{\mathcal{J}}_{\text{Ag}} \otimes \{\emptyset, \mathbb{U}_{\text{Ag}}\} \quad (\text{by (7.10a) and (7.10b)}) \\ &\iff \bar{\bar{\mathcal{J}}}_{\text{Pr}} \otimes \{\emptyset, \mathbb{U}_{\text{Pr}}\} \subset \bar{\mathcal{J}}_{\text{Ag}} \text{ and } \{\emptyset, \Omega\} \otimes \mathcal{U}_{\text{Pr}} \subset \bar{\mathcal{J}}_{\text{Ag}} \\ &\iff (\bar{\bar{\mathcal{J}}}_{\text{Pr}} \otimes \{\emptyset, \mathbb{U}_{\text{Pr}}\}) \bigvee (\{\emptyset, \Omega\} \otimes \mathcal{U}_{\text{Pr}}) \subset \bar{\mathcal{J}}_{\text{Ag}}. \end{aligned}$$

Thus, we have shown that the inclusion (7.9b) implies the inclusion (7.10c).

Solvability and reduced solvability.

Lemma 7.5. *For any W-strategy profile $\lambda = (\lambda_{\text{Pr}}, \lambda_{\text{Ag}}) \in \Lambda_{\text{Pr}} \times \Lambda_{\text{Ag}}$, the corresponding solution mapping $S_\lambda = S_{(\lambda_{\text{Pr}}, \lambda_{\text{Ag}})} : \Omega \rightarrow \Omega \times \mathbb{U}_{\text{Pr}} \times \mathbb{U}_{\text{Ag}}$ is well-defined as in (2.32c) and has the expression*

$$S_{(\lambda_{\text{Pr}}, \lambda_{\text{Ag}})} = \bar{S}_{\bar{\lambda}_{\text{Ag}}} \circ \bar{\bar{S}}_{\bar{\lambda}_{\text{Pr}}}, \quad (7.17)$$

where the reduced solution map $\bar{S}_{\bar{\lambda}_{\text{Ag}}}$ has been defined in (7.16f), and the reduced solution map $\bar{\bar{S}}_{\bar{\lambda}_{\text{Pr}}}$ in (7.15f). Moreover, the solution mapping $S_{(\lambda_{\text{Pr}}, \lambda_{\text{Ag}})} : (\Omega, \mathcal{F}) \rightarrow (\Omega \times \mathbb{U}_{\text{Pr}} \times \mathbb{U}_{\text{Ag}}, \mathcal{F} \otimes \mathcal{U}_{\text{Pr}} \otimes \mathcal{U}_{\text{Ag}})$ is measurable.

Proof. Consider a W-strategy profile $\lambda = (\lambda_{\text{Pr}}, \lambda_{\text{Ag}}) \in \Lambda_{\text{Pr}} \times \Lambda_{\text{Ag}}$. The closed-loop equation (2.32a) is here

$$\begin{aligned} & u_{\text{Pr}} = \lambda_{\text{Pr}}(\omega, u_{\text{Pr}}, u_{\text{Ag}}) \text{ and } u_{\text{Ag}} = \lambda_{\text{Ag}}(\omega, u_{\text{Pr}}, u_{\text{Ag}}) \\ \iff & u_{\text{Pr}} = \bar{\lambda}_{\text{Pr}}(\omega) \text{ and } u_{\text{Ag}} = \bar{\lambda}_{\text{Ag}}(\omega, u_{\text{Pr}}) \quad (\text{by (7.15b) and (7.16b)}) \\ \iff & (\omega, u_{\text{Pr}}) = \bar{\bar{S}}_{\bar{\lambda}_{\text{Pr}}}(\omega) \text{ and } (\omega, u_{\text{Pr}}, u_{\text{Ag}}) = \bar{S}_{\bar{\lambda}_{\text{Ag}}}(\omega, u_{\text{Pr}}) \quad (\text{by (7.15f) and (7.16f)}) \\ \Rightarrow & (\omega, u_{\text{Pr}}, u_{\text{Ag}}) = \bar{S}_{\bar{\lambda}_{\text{Ag}}}(\bar{\bar{S}}_{\bar{\lambda}_{\text{Pr}}}(\omega)) = (\bar{S}_{\bar{\lambda}_{\text{Ag}}} \circ \bar{\bar{S}}_{\bar{\lambda}_{\text{Pr}}})(\omega). \end{aligned}$$

By definition (2.32c) of the solution mapping, this means that $S_{(\lambda_{\text{Pr}}, \lambda_{\text{Ag}})} = \bar{S}_{\bar{\lambda}_{\text{Ag}}} \circ \bar{\bar{S}}_{\bar{\lambda}_{\text{Pr}}}$.

Moreover, as just proved above, we have the following measurability properties

$$(\Omega, \mathcal{F}) \xrightarrow{\bar{\bar{S}}_{\bar{\lambda}_{\text{Pr}}}} (\Omega \times \mathbb{U}_{\text{Pr}}, \mathcal{F} \otimes \mathcal{U}_{\text{Pr}}) \quad (7.18a)$$

$$(\Omega \times \mathbb{U}_{\text{Pr}}, \mathcal{F} \otimes \mathcal{U}_{\text{Pr}}) \xrightarrow{\bar{S}_{\bar{\lambda}_{\text{Ag}}}} (\Omega \times \mathbb{U}_{\text{Pr}} \times \mathbb{U}_{\text{Ag}}, \mathcal{F} \otimes \mathcal{U}_{\text{Pr}} \otimes \mathcal{U}_{\text{Ag}}). \quad (7.18b)$$

By composition, we conclude that the solution mapping $S_{(\lambda_{\text{Pr}}, \lambda_{\text{Ag}})} = \bar{S}_{\bar{\lambda}_{\text{Ag}}} \circ \bar{\bar{S}}_{\bar{\lambda}_{\text{Pr}}}$ is measurable since

$$(\Omega, \mathcal{F}) \xrightarrow{\bar{\bar{S}}_{\bar{\lambda}_{\text{Pr}}}} (\Omega \times \mathbb{U}_{\text{Pr}}, \mathcal{F} \otimes \mathcal{U}_{\text{Pr}}) \xrightarrow{\bar{S}_{\bar{\lambda}_{\text{Ag}}}} (\Omega \times \mathbb{U}_{\text{Pr}} \times \mathbb{U}_{\text{Ag}}, \mathcal{F} \otimes \mathcal{U}_{\text{Pr}} \otimes \mathcal{U}_{\text{Ag}}). \quad (7.19)$$

□

7.3.2 Background on integrands

In what follows, we work on a product space $\Omega \times \mathbb{U}$ and consider a function $f : \Omega \times \mathbb{U} \rightarrow \mathbb{R} \cup \{+\infty\}$. On the one hand, we need to define conditional expectations of f on Ω , parametrically in $u \in \mathbb{U}$. On the other hand, we need a measurable selection in $\arg \min f(\omega, u)$. For this purpose, we introduce lower semicontinuous (l.s.c.) integrands $\int_{u \in \mathbb{U}}$ and normal integrands.

Recall that, in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a subset Ω' of Ω is said to be \mathbb{P} -negligible if there exists $\Omega'' \in \mathcal{F}$ such that $\Omega' \subset \Omega''$ and $\mathbb{P}(\Omega'') = 0$. A subfield $\mathcal{G} \subset \mathcal{F}$ is said to be \mathbb{P} -complete if \mathcal{G} contains all the \mathbb{P} -negligible subsets.

Conditional expectation of a lower semicontinuous integrand. We first introduce lower semicontinuous integrands. Notice that all functions take their values in $\mathbb{R} \cup \{+\infty\}$.

Definition 7.6. (*[35, Definition 1]*) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. Let $(\mathbb{U}, \mathcal{B}(\mathbb{U}))$ be a separable Banach space, equipped with its Borel σ -field. An extended real-valued function $f : \Omega \times \mathbb{U} \rightarrow \mathbb{R} \cup \{+\infty\}$ is said to be a \mathcal{F} -lower semicontinuous integrand if the function f is $\mathcal{F} \otimes \mathcal{B}(\mathbb{U})$ -measurable, and if

$$\mathbb{P}\{\omega \in \Omega \mid u \in \mathbb{U} \mapsto f(\omega, u) \text{ is lower semicontinuous}\} = 1 .$$

In order to define the conditional expectation of a lower semicontinuous integrand with respect to a \mathbb{P} -complete subfield $\mathcal{G} \subset \mathcal{F}$, we introduce the notion of \mathcal{F} -quasi integrable function following [35] (see also [9, § 8.3.4.]).

Definition 7.7. (*[35, Definition 2]*) Let a measurable space (Ω, \mathcal{F}) and a separable Banach space $(\mathbb{U}, \mathcal{B}(\mathbb{U}))$, equipped with the Borel σ -field, be given. An extended real-valued function $f : \Omega \times \mathbb{U} \rightarrow \mathbb{R} \cup \{+\infty\}$ is said to be \mathcal{F} -quasi integrable w.r.t. to the probability \mathbb{P} if the function f is $\mathcal{F} \otimes \mathcal{B}(\mathbb{U})$ -measurable, and if there exists a sequence $\{\alpha^{(n)}\}_{n \in \mathbb{N}}$, with $\alpha^{(n)} \in L^1(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{R})$, such that

$$\mathbb{P}\left\{\omega \in \Omega \mid \alpha^{(n)}(\omega) \leq \inf_{\|u\| \leq n} f(\omega, u), \quad \forall n \in \mathbb{N}\right\} = 0 .$$

With \mathcal{F} -quasi integrable lower semicontinuous integrands, we can define conditional expectations as follows.

Proposition 7.8. (*[35, Proposition 12]*) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. Let $(\mathbb{U}, \mathcal{B}(\mathbb{U}))$ be a separable Banach space, equipped with its Borel σ -field.

Let \mathcal{G} be a \mathbb{P} -complete subfield of \mathcal{F} and $f : \Omega \times \mathbb{U} \rightarrow \mathbb{R} \cup \{+\infty\}$ be a \mathcal{F} -lower semicontinuous integrand (as in Definition 7.6) which is \mathcal{F} -quasi integrable (as in Definition 7.7).

Then, there exists a $\mathcal{G} \otimes \mathcal{B}(\mathbb{U})$ -measurable function $f_{\mathcal{G}} : \Omega \times \mathbb{U} \rightarrow \mathbb{R} \cup \{+\infty\}$, such that

- the function $f_{\mathcal{G}} : \Omega \times \mathbb{U} \rightarrow \mathbb{R} \cup \{+\infty\}$ is a \mathcal{G} -lower semicontinuous integrand,
- for all \mathcal{G} -measurable bounded mapping $\lambda : \Omega \rightarrow \mathbb{U}$, we have that

$$\mathbb{P}\left\{\omega \in \Omega \mid \mathbb{E}_{\mathbb{P}}[f(\cdot, \lambda(\cdot)) \mid \mathcal{G}](\omega) = f_{\mathcal{G}}(\omega, \lambda(\omega))\right\} = 1 . \quad (7.20)$$

The function $f_{\mathcal{G}} : \Omega \times \mathbb{U} \rightarrow \mathbb{R} \cup \{+\infty\}$ is unique — in the sense that two candidates coincide on a subset $\Omega' \times \mathbb{U}$ where $\mathbb{P}(\Omega') = 1$ — and called the \mathcal{G} -conditional expectation of the lower semicontinuous integrand f .

It is an easy consequence of Proposition 7.8 that, under the same assumptions, for any closed subset U of \mathbb{U} , the function $f + \delta_{\Omega \times U}$ is a \mathcal{F} -lower semicontinuous integrand which is \mathcal{F} -quasi integrable, and that

$$(f + \delta_{\Omega \times U})_{\mathcal{G}} = f_{\mathcal{G}} + \delta_{\Omega \times U} . \quad (7.21)$$

Under the assumptions of Proposition 7.8, and using [35, Lemme 6], if the function $h : \Omega \times \mathbb{U} \rightarrow \mathbb{R} \cup \{+\infty\}$ satisfies the same assumptions as the function f , then it holds that

$$f \geq h \Rightarrow f_{\mathcal{G}} \geq h_{\mathcal{G}} . \quad (7.22)$$

Measurable selection of a normal integrand. We give a formal definition of a normal integrand following [26] (see also [30, Chapter 14, Section D] in the case of \mathbb{R}^d spaces).

Notice that all functions take their values in $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$.

Definition 7.9. *Let a measurable space (Ω, \mathcal{F}) and a Borel space $(\mathbb{U}, \mathcal{B}(\mathbb{U}))$ be given.*

- *A set-valued mapping $F : \Omega \rightrightarrows \mathbb{U}$ is said to be \mathcal{F} -measurable if, for every open set $U \subset \mathbb{U}$, the inverse image satisfies:*

$$F^{-1}(U) = \{\omega \in \Omega \mid F(\omega) \cap U \neq \emptyset\} \in \mathcal{F} .$$

- *An extended real-valued function $f : \Omega \times \mathbb{U} \rightarrow \overline{\mathbb{R}}$ is said to be a normal integrand if the epigraphical mapping*

$$\begin{aligned} \text{epi } f : \Omega \rightrightarrows \mathbb{U} \times \mathbb{R} \\ \omega \mapsto \text{epi } f(\omega, \cdot) = \{(u, \alpha) \in \mathbb{U} \times \mathbb{R} \mid f(\omega, u) \leq \alpha\} \end{aligned}$$

is closed-valued and \mathcal{F} -measurable. When needed, we say that f is a \mathcal{F} -normal integrand.

Normal integrands and lower semicontinuous integrands are related as follows. When $\mathbb{U} = \mathbb{R}^d$, any \mathcal{F} -normal integrand is a \mathcal{F} -lower semicontinuous integrand. To obtain the reverse statement, we need more hypothesis.

Proposition 7.10. *([30, Corollary 14.34]) Let $f : \Omega \times \mathbb{R}^d \rightarrow \overline{\mathbb{R}}$ be a \mathcal{F} -normal integrand. Then, the function $f : \Omega \times \mathbb{R}^d \rightarrow \overline{\mathbb{R}}$ is $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}^d)$ -measurable, and the function $f(\omega, \cdot) : \mathbb{R}^d \rightarrow \overline{\mathbb{R}}$ is l.s.c., for any $\omega \in \Omega$.*

Let $f : \Omega \times \mathbb{R}^d \rightarrow \overline{\mathbb{R}}$ be such that the function $f : \Omega \times \mathbb{R}^d \rightarrow \overline{\mathbb{R}}$ is $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}^d)$ -measurable, and that the function $f(\omega, \cdot) : \mathbb{R}^d \rightarrow \overline{\mathbb{R}}$ is l.s.c., for any $\omega \in \Omega$. If (Ω, \mathcal{F}) is complete w.r.t. some σ -finite measure, then the function f is a \mathcal{F} -normal integrand.

We will need the following measurable selection result.

Proposition 7.11. (*[30, Theorem 14.37]*) For any \mathcal{F} -normal integrand $f : \Omega \times \mathbb{R}^d \rightarrow \overline{\mathbb{R}}$, where \mathbb{R}^d is equipped with the Borel σ -field, the function $\omega \in \Omega \mapsto \inf_{u \in \mathbb{R}^d} f(\omega, u)$ is \mathcal{F} -measurable and the set-valued mapping $\omega \in \Omega \mapsto \arg \min_{u \in \mathbb{R}^d} f(\omega, u)$ is closed-valued and \mathcal{F} -measurable.

In particular, therefore, the set $\Omega' = \{\omega \in \Omega \mid \arg \min_{u \in \mathbb{R}^d} f(\omega, u) \neq \emptyset\}$ is \mathcal{F} -measurable, and, for each $\omega \in \Omega'$, it is possible to select a minimizing point $\lambda^*(\omega)$ in such a manner that the mapping $\omega \in \Omega' \mapsto \lambda(\omega)$ is \mathcal{F} -measurable.

Measurable selection of a conditional expectation. Now, we establish the existence of a measurable selection, with values in a compact subset of \mathbb{R}^d , for the argmin of a conditional expectation.

Lemma 7.12. Let U_0 be a bounded and closed subset of \mathbb{R}^d . Let $h : \Omega \times U_0 \rightarrow \mathbb{R} \cup \{+\infty\}$. We define the function $f : \Omega \times \mathbb{R}^d \rightarrow \mathbb{R} \cup \{+\infty\}$ by

$$\forall (\omega, u) \in \Omega \times \mathbb{R}^d, \quad f(\omega, u) = \begin{cases} h(\omega, u) & \text{if } u \in U_0, \\ +\infty & \text{if } u \notin U_0. \end{cases} \quad (7.23)$$

Suppose that $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space, and let \mathcal{G} be a \mathbb{P} -complete subfield of \mathcal{F} . Suppose that the function $h : \Omega \times U_0 \rightarrow \mathbb{R} \cup \{+\infty\}$ is a \mathcal{F} -normal integrand, and is bounded below.

Then, the following statements hold true.

- The function $f : \Omega \times \mathbb{R}^d \rightarrow \mathbb{R} \cup \{+\infty\}$ in (7.23) is a \mathcal{F} -lower semicontinuous integrand, is \mathcal{F} -quasi integrable (see Definition 7.7) and admits a \mathcal{G} -conditional expectation $f_{\mathcal{G}} : \Omega \times \mathbb{R}^d \rightarrow \overline{\mathbb{R}}$ (see Proposition 7.8). Moreover, we have, for all \mathcal{G} -measurable mapping $\lambda : \Omega \rightarrow U_0$,

$$\mathbb{P}\{\omega \in \Omega \mid \mathbb{E}_{\mathbb{P}}[h(\cdot, \lambda(\cdot)) \mid \mathcal{G}](\omega) = f_{\mathcal{G}}(\omega, \lambda(\omega))\} = 1. \quad (7.24a)$$

and

$$\mathbb{P}\{\omega \in \Omega \mid f_{\mathcal{G}}(\omega, u) = +\infty, \quad \forall u \notin U_0\} = 1, \quad (7.24b)$$

- If, in addition, there exists $u_0 \in U_0$ such that $h(\cdot, u_0)$ is \mathbb{P} -integrable, then there exists a \mathcal{G} -measurable mapping $\lambda^* : \Omega \rightarrow U_0$ such that

$$\mathbb{P}\left\{\omega \in \Omega \mid \lambda^*(\omega) \in \arg \min_{u \in \mathbb{R}^d} f_{\mathcal{G}}(\omega, u)\right\} = 1. \quad (7.24c)$$

Proof. By Proposition 7.10, when defined over $\Omega \times \mathbb{R}^d$ with values in $\mathbb{R} \cup \{+\infty\}$, \mathcal{F} -normal integrands are \mathcal{F} -lower semicontinuous integrands.

- First, we establish that the function $f : \Omega \times \mathbb{R}^d \rightarrow \mathbb{R} \cup \{+\infty\}$ in (7.23) is a \mathcal{F} -normal integrand, hence is a \mathcal{F} -lower semicontinuous integrand.

It is easily seen that

$$\begin{aligned} \text{epi } f(\omega, \cdot) &= \{(u, \alpha) \in \mathbb{R}^d \times \mathbb{R} \mid f(\omega, u) \leq \alpha\} \\ &= \{(u, \alpha) \in U_0 \times \mathbb{R} \mid h(\omega, u) \leq \alpha\} = \text{epi } h(\omega, \cdot). \end{aligned}$$

Therefore, the epigraphical (set-valued) mappings $\text{epi } h : \Omega \rightrightarrows U_0 \times \mathbb{R}$ and $\text{epi } f : \Omega \rightrightarrows \mathbb{R}^d \times \mathbb{R}$ differ only by their image sets, $U_0 \times \mathbb{R}$ or $\mathbb{R}^d \times \mathbb{R}$. As U_0 is a closed subset of \mathbb{R}^d , the closed subsets of $U_0 \times \mathbb{R}$ coincide with the trace on U_0 of the closed subsets of $\mathbb{R}^d \times \mathbb{R}$, hence are closed subsets of $\mathbb{R}^d \times \mathbb{R}$.

As the function $h : \Omega \times U_0 \rightarrow \mathbb{R} \cup \{+\infty\}$ is a \mathcal{F} -normal integrand, the function $h(\omega, \cdot) : U_0 \rightarrow \overline{\mathbb{R}}$ is l.s.c., for any $\omega \in \Omega$, hence the set-valued mapping $\text{epi } h$ takes values in the closed subsets of $U_0 \times \mathbb{R}$, hence in the closed subsets of $\mathbb{R}^d \times \mathbb{R}$. Therefore, the set-valued mapping $\text{epi } f$ is closed-valued.

Now, let H be an open subset of $\mathbb{R}^d \times \mathbb{R}$. We have

$$\begin{aligned} \text{epi } f^{-1}(H) &= \{\omega \in \Omega \mid \text{epi } f(\omega) \cap H \neq \emptyset\} \\ &= \{\omega \in \Omega \mid \text{epi } h(\omega) \cap H \neq \emptyset\} \quad (\text{as } \text{epi } f(\omega, \cdot) = \text{epi } h(\omega, \cdot), \text{ for all } \omega \in \Omega) \\ &= \{\omega \in \Omega \mid \text{epi } h(\omega) \cap (H \cap (U_0 \times \mathbb{R})) \neq \emptyset\} \\ &\quad (\text{as } \text{epi } h(\omega, \cdot) \subset U_0 \times \mathbb{R}, \text{ for all } \omega \in \Omega) \\ &\in \mathcal{F}. \quad (\text{because } H \cap (U_0 \times \mathbb{R}) \text{ is an open subset of } U_0 \times \mathbb{R}) \end{aligned}$$

Therefore, the set-valued mapping $\text{epi } f$ is \mathcal{F} -measurable.

We conclude that the function $f : \Omega \times \mathbb{R}^d \rightarrow \mathbb{R} \cup \{+\infty\}$ in (7.23) is a \mathcal{F} -normal integrand.

Second, we prove that the function $f : \Omega \times \mathbb{R}^d \rightarrow \mathbb{R} \cup \{+\infty\}$ in (7.23) is a \mathcal{F} -lower semicontinuous integrand, is \mathcal{F} -quasi integrable and admits a \mathcal{G} -conditional expectation $f_{\mathcal{G}} : \Omega \times \mathbb{R}^d \rightarrow \overline{\mathbb{R}}$.

Since the function $f : \Omega \times \mathbb{R}^d \rightarrow \mathbb{R} \cup \{+\infty\}$ in (7.23) is a \mathcal{F} -normal integrand, it is a \mathcal{F} -lower semicontinuous integrand by Proposition 7.10. Since the function $h : \Omega \times U_0 \rightarrow \mathbb{R} \cup \{+\infty\}$ is bounded below, so is the function $f : \Omega \times \mathbb{R}^d \rightarrow \mathbb{R} \cup \{+\infty\}$ hence this latter function is \mathcal{F} -quasi integrable (see Definition 7.7). By Proposition 7.8, we get that the function $f : \Omega \times \mathbb{R}^d \rightarrow \mathbb{R} \cup \{+\infty\}$ admits a \mathcal{G} -conditional expectation $f_{\mathcal{G}} : \Omega \times \mathbb{R}^d \rightarrow \overline{\mathbb{R}}$.

Third, we show that (7.24a) holds true. But this is a straightforward consequence of (7.20), of the very definition (7.23) of the function $f : \Omega \times \mathbb{R}^d \rightarrow \mathbb{R} \cup \{+\infty\}$, and of the fact that any \mathcal{G} -measurable mapping $\lambda : \Omega \rightarrow U_0$ induces a \mathcal{G} -measurable mapping $\lambda : \Omega \rightarrow \mathbb{R}^d$ with values in the bounded and closed subset U_0 of \mathbb{R}^d .

Fourth, we show that (7.24b) holds true.

We define, for all $n \in \mathbb{N}^*$, $V_n = \{u \in \mathbb{R}^d \mid 1/n \leq d(u, U_0)\}$, which is nonempty and closed. The sequence $\{V_n\}_{n \geq 1}$ is increasing with n and such that $\bigcup_{n \geq 1} V_n = U_0^c$, the complementary set of U_0 in \mathbb{R}^d , because U_0 is closed. By the definition (7.23) of the function $f : \Omega \times \mathbb{R}^d \rightarrow \mathbb{R} \cup \{+\infty\}$, for all $n \in \mathbb{N}^*$ we have $+\infty = f(\omega, u) \geq n$, for all $(\omega, u) \in \Omega \times V_n$,

that is, $f + \delta_{\Omega \times V_n} \geq n + \delta_{\Omega \times V_n}$. By the two claims right after the Proposition 7.8, using the fact that the set V_n is closed, we get that $f_{\mathcal{G}} + \delta_{\Omega \times V_n} \geq n + \delta_{\Omega \times V_n}$, for all $n \in \mathbb{N}^*$, where this inequality has to be understood as $f_{\mathcal{G}}(\omega, u) \geq n$ for all $(\omega, u) \in \Omega_n \times V_n$ where $\mathbb{P}(\Omega_n) = 1$. Therefore, on the set $\Omega' = \bigcap_{n \geq 1} \Omega_n$, of probability 1, we have $f_{\mathcal{G}}(\omega, u) \geq n$ for all $(\omega, u) \in \Omega' \times V_n$. As the sequence $\{V_n\}_{n \geq 1}$ is increasing with n and such that $\bigcup_{n \geq 1} V_n = U_0^c$, we obtain that $f_{\mathcal{G}}(\omega, u) = +\infty$ for all $(\omega, u) \in \Omega' \times U_0^c$.

We conclude that (7.24b) holds true.

- We prove (7.24c).

On the one hand, since the function $f_{\mathcal{G}} : \Omega \times U_0 \rightarrow \mathbb{R} \cup \{+\infty\}$ is a \mathcal{G} -l.s.c. integrand, the function $f_{\mathcal{G}}(\omega, \cdot) : U_0 \rightarrow \overline{\mathbb{R}}$ is l.s.c., for \mathbb{P} -almost any $\omega \in \Omega$. As the set U_0 is a bounded and closed subset of \mathbb{R}^d , it is compact, and we get that¹

$$\mathbb{P}\{\omega \in \Omega \mid f_{\mathcal{G}}(\omega, \cdot) \text{ is l.s.c.}\} = \mathbb{P}\{\omega \in \Omega \mid \arg \min_{u \in \mathbb{R}^d} f_{\mathcal{G}}(\omega, u) \neq \emptyset\} = 1.$$

On the other hand, by Proposition 7.11

$$\Omega_1 = \{\omega \in \Omega \mid \arg \min_{u \in \mathbb{R}^d} f_{\mathcal{G}}(\omega, u) \neq \emptyset\} \in \mathcal{G}.$$

Therefore, we get that $\mathbb{P}(\Omega_1) = 1$. By the same Proposition 7.11, it is possible, for each $\omega \in \Omega_1$, to select a minimizing point $\lambda^*(\omega) \in \arg \min_{u \in \mathbb{R}^d} f_{\mathcal{G}}(\omega, u)$ in such a manner that the mapping $\omega \in \Omega_1 \mapsto \lambda^*(\omega)$ is \mathcal{G} -measurable. We take an element b in U_0 , and we define $\lambda^*(\omega) = b$ when $\omega \notin \Omega_1$. Thus doing, the mapping $\lambda^* : \Omega \rightarrow \mathbb{R}^d$ is \mathcal{G} -measurable.

We consider the two following \mathcal{G} -measurable sets

$$\Omega_0 = \{\omega \in \Omega \mid \lambda^*(\omega) \notin U_0\} \text{ and } \Omega_\infty = \{\omega \in \Omega \mid f_{\mathcal{G}}(\omega, u) = +\infty, \forall u \notin U_0\},$$

that are such that $\Omega_0 \subset \Omega_1$ and that $\mathbb{P}(\Omega_\infty) = 1$ by (7.24b). We are going to show that $\mathbb{P}(\Omega_0) = 0$. On the one hand, when $\omega \in \Omega_0 \cap \Omega_\infty$, we have that $+\infty = f_{\mathcal{G}}(\omega, \lambda^*(\omega))$, by definition of Ω_0 and of Ω_∞ . On the other hand, we have $f_{\mathcal{G}}(\omega, \lambda^*(\omega)) \leq f_{\mathcal{G}}(\omega, u)$, for all $u \in \mathbb{R}^d$, by definition of the minimizing point $\lambda^*(\omega)$, when $\omega \in \Omega_1$. Therefore, as $\Omega_1 \supset \Omega_0 \supset \Omega_0 \cap \Omega_\infty$, for all $\omega \in \Omega_0 \cap \Omega_\infty$, we have that $f_{\mathcal{G}}(\omega, u) = +\infty$ for all $u \in \mathbb{R}^d$, hence in particular $f_{\mathcal{G}}(\omega, u_0) = +\infty$ where $u_0 \in U_0$ is such that $h(\cdot, u_0)$ is \mathbb{P} -integrable, by assumption in Item 7.12.

On the other hand, by (7.20) in Proposition 7.8, we get that

$$\mathbb{E}_{\mathbb{P}}[f(\cdot, u_0) \mid \mathcal{G}] = f_{\mathcal{G}}(\cdot, u_0) \quad \mathbb{P} - \text{a.s.}$$

By the definition (7.23) of the function $f : \Omega \times \mathbb{R}^d \rightarrow \mathbb{R} \cup \{+\infty\}$, we have that $f(\cdot, u_0) = h(\cdot, u_0)$, and thus

$$\mathbb{E}_{\mathbb{P}}[f(\cdot, u_0) \mid \mathcal{G}] = \mathbb{E}_{\mathbb{P}}[h(\cdot, u_0) \mid \mathcal{G}] < +\infty \quad \mathbb{P} - \text{a.s.}$$

¹In case $\text{dom } f_{\mathcal{G}}(\omega, \cdot) = \emptyset$, hence $f_{\mathcal{G}}(\omega, \cdot) \equiv +\infty$, we take the convention that $\arg \min_{u \in \mathbb{R}^d} f_{\mathcal{G}}(\omega, u) = \mathbb{R}^d$.

since $h(\cdot, u_0)$ is \mathbb{P} -integrable. Therefore, on the intersection of $\Omega_0 \cap \Omega_\infty$ with a subset of Ω of probability 1, we have both $\mathbb{E}_{\mathbb{P}}[f(\cdot, u_0) | \mathcal{G}] = f_{\mathcal{G}}(\cdot, u_0)$ and $\mathbb{E}_{\mathbb{P}}[f(\cdot, u_0) | \mathcal{G}] < +\infty$, hence $f_{\mathcal{G}}(\cdot, u_0) < +\infty$. We conclude that $\mathbb{P}(\Omega_0 \cap \Omega_\infty) = 0$ and, as $\mathbb{P}(\Omega_\infty) = 1$, that $\mathbb{P}(\Omega_0) = 0$.

Thus, we obtain that

$$\Omega_0^c \cap \Omega_1 = \left\{ \omega \in \Omega \mid \lambda^*(\omega) \in U_0 \cap \arg \min_{u \in \mathbb{R}^d} f_{\mathcal{G}}(\omega, u) \right\} \text{ and } \mathbb{P}(\Omega_0^c \cap \Omega_1) = 1 .$$

As the mapping $\lambda^* : \Omega \rightarrow \mathbb{R}^d$ is \mathcal{G} -measurable, as U_0 is closed and as $\mathbb{P}\{\omega \in \Omega \mid \lambda^*(\omega) \in U_0\} = 1$, it is straightforward that the mapping $\lambda^* : \Omega \rightarrow U_0$ is \mathcal{G} -measurable. Thus, we have proved (7.24c).

This ends the proof. \square

7.3.3 Main result (backward induction for the Enough informed Agent W-games)

Now, we are ready to state and prove the main result of this Section.

Proposition 7.13. *Suppose given an Enough informed Agent W-game, as in Definition 7.3, with the additional following technical assumptions:*

1. *The space \mathbb{U}_{Ag} is a nonempty bounded and closed subset of \mathbb{R}^d , equipped with the Borel σ -field $\mathcal{U}_{\text{Ag}} = \mathcal{B}(\mathbb{U}_{\text{Ag}})$.*

2. *There exists a probability $\overline{\mathbb{Q}}_{\{\text{Ag}\}}$ on $(\Omega \times \mathbb{U}_{\text{Pr}}, \mathcal{F} \otimes \mathcal{U}_{\text{Pr}})$ such that,*

(a) *the probability space $(\Omega \times \mathbb{U}_{\text{Pr}}, \mathcal{F} \otimes \mathcal{U}_{\text{Pr}}, \overline{\mathbb{Q}}_{\{\text{Ag}\}})$ is complete and the σ -field $\overline{\mathcal{J}}_{\text{Ag}}$ is $\overline{\mathbb{Q}}_{\{\text{Ag}\}}$ -complete,*

(b) *for any $\mu_{\text{Pr}} \in \overline{\Lambda}_{\text{Pr}}$ in (7.15c), the pushforward probability $\mathbb{P}^{\{\text{Ag}\}} \circ \overline{S}_{\mu_{\text{Pr}}}^{-1}$ on $(\Omega \times \mathbb{U}_{\text{Pr}}, \mathcal{F} \otimes \mathcal{U}_{\text{Pr}})$ has a density (Radon-Nikodim derivative)*

$$\overline{T}_{\mu_{\text{Pr}}} : \Omega \times \mathbb{U}_{\text{Pr}} \rightarrow \mathbb{R}_+ \tag{7.25a}$$

w.r.t. the probability $\overline{\mathbb{Q}}_{\{\text{Ag}\}}$, that is,

$$\mathbb{P}^{\{\text{Ag}\}} \circ \overline{S}_{\mu_{\text{Pr}}}^{-1} = \overline{T}_{\mu_{\text{Pr}}} \overline{\mathbb{Q}}_{\{\text{Ag}\}} , \tag{7.25b}$$

(c) *the density $\overline{T}_{\mu_{\text{Pr}}}$ in (7.25a) is measurable w.r.t. $(\overline{\mathcal{J}}_{\text{Pr}} \otimes \{\emptyset, \mathbb{U}_{\text{Pr}}\}) \vee (\{\emptyset, \Omega\} \otimes \mathcal{U}_{\text{Pr}})$, where $\overline{\mathcal{J}}_{\text{Pr}}$ is the reduced information field of the Principal Pr as defined in (7.15a).*

3. *Regarding the criterion $j^{\{\text{Ag}\}} : (\Omega \times \mathbb{U}_{\text{Pr}} \times \mathbb{U}_{\text{Ag}}, \mathcal{F} \otimes \mathcal{U}_{\text{Pr}} \otimes \mathcal{U}_{\text{Ag}}) \rightarrow \overline{\mathbb{R}}$ of the Agent player $\{\text{Ag}\}$,*

- (a) the function $j^{\{\text{Ag}\}} : \Omega \times \mathbb{U}_{\text{Pr}} \times \mathbb{U}_{\text{Ag}} \rightarrow \overline{\mathbb{R}}$ is bounded above,
 (b) the function $-j^{\{\text{Ag}\}} : (\Omega \times \mathbb{U}_{\text{Pr}}) \times \mathbb{U}_{\text{Ag}} \rightarrow \overline{\mathbb{R}}$ is a $\mathcal{F} \otimes \mathcal{U}_{\text{Pr}}$ -normal integrand, as in Definition 7.9.

Then, there exists a W -strategy $\lambda_{\text{Ag}}^* \in \Lambda_{\text{Ag}}$ such that

$$\mathbb{E}_{\mathbb{P}^{\{\text{Ag}\}}} [j^{\{\text{Ag}\}} \circ S_{(\lambda_{\text{Pr}}, \lambda_{\text{Ag}}^*)}] \geq \mathbb{E}_{\mathbb{P}^{\{\text{Ag}\}}} [j^{\{\text{Ag}\}} \circ S_{(\lambda_{\text{Pr}}, \lambda_{\text{Ag}})}], \quad \forall \lambda_{\text{Pr}} \in \Lambda_{\text{Pr}}, \quad \forall \lambda_{\text{Ag}} \in \Lambda_{\text{Ag}}, \quad (7.26)$$

where the solution map $S_{(\lambda_{\text{Pr}}, \lambda_{\text{Ag}})}$ is defined in (7.14a)–(7.14b).

Proof. By Proposition 7.10, when defined over $(\Omega \times \mathbb{U}_{\text{Pr}}) \times \mathbb{R}^d$ with values in $\mathbb{R} \cup \{+\infty\}$, \mathcal{F} -normal integrands are \mathcal{F} -lower semicontinuous integrands.

- We introduce the function $-\bar{j}^{\{\text{Ag}\}} : (\Omega \times \mathbb{U}_{\text{Pr}}) \times \mathbb{R}^d \rightarrow \mathbb{R} \cup \{+\infty\}$ by

$$-\bar{j}^{\{\text{Ag}\}}((\omega, u_{\text{Pr}}), u_{\text{Ag}}) = \begin{cases} -j^{\{\text{Ag}\}}(\omega, u_{\text{Pr}}, u_{\text{Ag}}) & \text{if } (\omega, u_{\text{Pr}}, u_{\text{Ag}}) \in \Omega \times \mathbb{U}_{\text{Pr}} \times \mathbb{U}_{\text{Ag}}, \\ +\infty & \text{if } (\omega, u_{\text{Pr}}, u_{\text{Ag}}) \in \Omega \times \mathbb{U}_{\text{Pr}} \times (\mathbb{R}^d \setminus \mathbb{U}_{\text{Ag}}). \end{cases} \quad (7.27a)$$

The function $-\bar{j}^{\{\text{Ag}\}}$ indeed takes values in $\mathbb{R} \cup \{+\infty\}$, as the function $j^{\{\text{Ag}\}} : \Omega \times \mathbb{U}_{\text{Pr}} \times \mathbb{U}_{\text{Ag}} \rightarrow \overline{\mathbb{R}}$ is bounded above by assumption in Item 3a

By assumption in Item 3b, and by Lemma 7.12, the function $-\bar{j}^{\{\text{Ag}\}} : (\Omega \times \mathbb{U}_{\text{Pr}}) \times \mathbb{R}^d \rightarrow \mathbb{R} \cup \{+\infty\}$ in (7.27a) is a $\mathcal{F} \otimes \mathcal{U}_{\text{Pr}}$ -lower semicontinuous integrand, which is $\overline{\mathbb{Q}}_{\{\text{Ag}\}}$ -quasi integrable. As $\bar{\mathcal{J}}_{\text{Ag}}$ is $\overline{\mathbb{Q}}_{\{\text{Ag}\}}$ -complete by assumption in Item 2a, Lemma 7.12 asserts the existence of a $\bar{\mathcal{J}}_{\text{Ag}}$ -lower semicontinuous integrand

$$-\hat{j}^{\{\text{Ag}\}} : (\Omega \times \mathbb{U}_{\text{Pr}}) \times \mathbb{R}^d \rightarrow \mathbb{R} \cup \{+\infty\}, \quad (7.27b)$$

which is an $\bar{\mathcal{J}}_{\text{Ag}}$ -conditional expectation of $-\bar{j}^{\{\text{Ag}\}}$, in the sense that,

$$\overline{\mathbb{Q}}_{\{\text{Ag}\}} \{(\omega, u_{\text{Pr}}) \in \Omega \times \mathbb{U}_{\text{Pr}} \mid \mathbb{E}_{\overline{\mathbb{Q}}_{\{\text{Ag}\}}} [-j^{\{\text{Ag}\}}(\cdot, \lambda(\cdot)) \mid \mathcal{G}](\omega, u_{\text{Pr}}) = -\hat{j}^{\{\text{Ag}\}}((\omega, u_{\text{Pr}}), \lambda(\omega, u_{\text{Pr}}))\} = 1,$$

for all \mathcal{G} -measurable mapping $\lambda : \Omega \times \mathbb{U}_{\text{Pr}} \rightarrow \mathbb{U}_{\text{Ag}}$, by (7.24a). Therefore, for all $\mu_{\text{Ag}} \in \bar{\Lambda}_{\text{Ag}}$, it holds that

$$\mathbb{E}_{\overline{\mathbb{Q}}_{\{\text{Ag}\}}} [j^{\{\text{Ag}\}} \circ \bar{S}_{\mu_{\text{Ag}}} \mid \bar{\mathcal{J}}_{\text{Ag}}] = \hat{j}^{\{\text{Ag}\}} \circ \bar{S}_{\mu_{\text{Ag}}}, \quad \overline{\mathbb{Q}}_{\{\text{Ag}\}}\text{-a.s.}, \quad (7.27c)$$

by using the expression $\bar{S}_{\mu_{\text{Ag}}} = (\text{I}_{\Omega \times \mathbb{U}_{\text{Pr}}}, \mu_{\text{Ag}})$ for the reduced solution map $\bar{S}_{\mu_{\text{Ag}}}$ in (7.16f).

By the one-to-one correspondence (7.16d), we associate with any W -strategy $\lambda_{\text{Ag}} \in \Lambda_{\text{Ag}}$ the reduced strategy $\bar{\lambda}_{\text{Ag}} : \Omega \times \mathbb{U}_{\text{Pr}} \rightarrow \mathbb{U}_{\text{Ag}}$ such that $\bar{\lambda}_{\text{Ag}}(\omega, u_{\text{Pr}}) = \lambda_{\text{Ag}}(\omega, u_{\text{Pr}}, u_{\text{Ag}})$, for all $(\omega, u_{\text{Pr}}, u_{\text{Ag}}) \in \Omega \times \mathbb{U}_{\text{Pr}} \times \mathbb{U}_{\text{Ag}}$, by (7.16e). As a consequence, from (7.27c), we immediately obtain that

$$\mathbb{E}_{\overline{\mathbb{Q}}_{\{\text{Ag}\}}} [j^{\{\text{Ag}\}} \circ \bar{S}_{\bar{\lambda}_{\text{Ag}}} \mid \bar{\mathcal{J}}_{\text{Ag}}] = \hat{j}^{\{\text{Ag}\}} \circ \bar{S}_{\bar{\lambda}_{\text{Ag}}}, \quad \overline{\mathbb{Q}}_{\{\text{Ag}\}}\text{-a.s.}, \quad \forall \lambda_{\text{Ag}} \in \Lambda_{\text{Ag}}. \quad (7.27d)$$

• By Lemma 7.12, we obtain the existence of a $\bar{\mathcal{J}}_{\text{Ag}}$ -measurable mapping $\mu_{\text{Ag}}^* : \Omega \times \mathbb{U}_{\text{Pr}} \rightarrow \mathbb{U}_{\text{Ag}}$ such that

$$\bar{\mathbb{Q}}_{\{\text{Ag}\}}(\bar{\mathcal{D}}^*) = 1, \quad (7.28a)$$

where the $\mathcal{F} \otimes \mathcal{U}_{\text{Pr}}$ -measurable set $\bar{\mathcal{D}}^*$ is given by

$$\bar{\mathcal{D}}^* = \{(\omega, u_{\text{Pr}}) \in \Omega \times \mathbb{U}_{\text{Pr}} \mid \mu_{\text{Ag}}^*(\omega, u_{\text{Pr}}) \in \arg \max_{u_{\text{Ag}} \in \mathbb{U}_{\text{Ag}}} \hat{\mathcal{J}}^{\{\text{Ag}\}}((\omega, u_{\text{Pr}}), u_{\text{Ag}})\}.$$

As the mapping μ_{Ag}^* is $\bar{\mathcal{J}}_{\text{Ag}}$ -measurable, it belongs to $\bar{\Lambda}_{\text{Ag}}$ in (7.16c). By the one-to-one correspondence (7.16d), we associate with $\mu_{\text{Ag}}^* \in \bar{\Lambda}_{\text{Ag}}$ the W-strategy $\lambda_{\text{Ag}}^* \in \Lambda_{\text{Ag}}$ defined by $\lambda_{\text{Ag}}^*(\omega, u_{\text{Pr}}, u_{\text{Ag}}) = \mu_{\text{Ag}}^*(\omega, u_{\text{Pr}})$, for all $(\omega, u_{\text{Pr}}, u_{\text{Ag}}) \in \Omega \times \mathbb{U}_{\text{Pr}} \times \mathbb{U}_{\text{Ag}}$, by (7.16e). By construction, the reduced W-strategy $\bar{\lambda}_{\text{Ag}}^* \in \bar{\Lambda}_{\text{Ag}}$ is obviously the mapping μ_{Ag}^* , so that

$$\bar{\mathcal{D}}^* = \{(\omega, u_{\text{Pr}}) \in \Omega \times \mathbb{U}_{\text{Pr}} \mid \hat{\mathcal{J}}^{\{\text{Ag}\}}((\omega, u_{\text{Pr}}), \bar{\lambda}_{\text{Ag}}^*(\omega, u_{\text{Pr}})) \geq \hat{\mathcal{J}}^{\{\text{Ag}\}}((\omega, u_{\text{Pr}}), u_{\text{Ag}}), \forall u_{\text{Ag}} \in \mathbb{U}_{\text{Ag}}\}. \quad (7.28b)$$

We define, for any $\lambda_{\text{Ag}} \in \Lambda_{\text{Ag}}$,

$$\bar{\mathcal{D}}_{\lambda_{\text{Ag}}}^* = \{(\omega, u_{\text{Pr}}) \in \Omega \times \mathbb{U}_{\text{Pr}} \mid (\hat{\mathcal{J}}^{\{\text{Ag}\}} \circ \bar{S}_{\lambda_{\text{Ag}}}^*)(\omega, u_{\text{Pr}}) \geq (\hat{\mathcal{J}}^{\{\text{Ag}\}} \circ \bar{S}_{\bar{\lambda}_{\text{Ag}}})(\omega, u_{\text{Pr}})\}. \quad (7.28c)$$

By the expression $\bar{S}_{\bar{\lambda}_{\text{Ag}}} = (\text{I}_{\Omega \times \mathbb{U}_{\text{Pr}}}, \bar{\lambda}_{\text{Ag}})$ for the reduced solution map $\bar{S}_{\bar{\lambda}_{\text{Ag}}}$ in (7.16f), we get that

$$\bar{\mathcal{D}}_{\lambda_{\text{Ag}}}^* = \{(\omega, u_{\text{Pr}}) \in \Omega \times \mathbb{U}_{\text{Pr}} \mid \hat{\mathcal{J}}^{\{\text{Ag}\}}((\omega, u_{\text{Pr}}), \bar{\lambda}_{\text{Ag}}^*(\omega, u_{\text{Pr}})) \geq \hat{\mathcal{J}}^{\{\text{Ag}\}}((\omega, u_{\text{Pr}}), \bar{\lambda}_{\text{Ag}}(\omega, u_{\text{Pr}}))\},$$

so that, by (7.28b), we obtain that $\bar{\mathcal{D}}^* \subset \bar{\mathcal{D}}_{\lambda_{\text{Ag}}}^*$, where $\bar{\mathbb{Q}}_{\{\text{Ag}\}}(\bar{\mathcal{D}}^*) = 1$ by (7.28a). As the probability space $(\Omega \times \mathbb{U}_{\text{Pr}}, \mathcal{F} \otimes \mathcal{U}_{\text{Pr}}, \bar{\mathbb{Q}}_{\{\text{Ag}\}})$ is complete by assumption in Item 2a, this gives

$$\bar{\mathcal{D}}_{\lambda_{\text{Ag}}}^* \in \mathcal{F} \otimes \mathcal{U}_{\text{Pr}}, \quad \bar{\mathbb{Q}}_{\{\text{Ag}\}}(\bar{\mathcal{D}}_{\lambda_{\text{Ag}}}^*) = 1, \quad \forall \lambda_{\text{Ag}} \in \Lambda_{\text{Ag}}. \quad (7.28d)$$

We will prove, in two steps, that λ_{Ag}^* satisfies (7.26).

• By the one-to-one correspondence (7.15d), we associate with any W-strategy $\lambda_{\text{Pr}} \in \Lambda_{\text{Pr}}$ the reduced strategy $\bar{\lambda}_{\text{Pr}} : \Omega \rightarrow \Lambda_{\text{Pr}}$ such that $\lambda_{\text{Pr}}(\omega, u_{\text{Pr}}, u_{\text{Ag}}) = \bar{\lambda}_{\text{Pr}}(\omega)$ for all $(\omega, u_{\text{Pr}}, u_{\text{Ag}})$, by (7.15e). By the assumption (7.25a), it holds that the pushforward probability $\mathbb{P}^{\{\text{Ag}\}} \circ \bar{S}_{\lambda_{\text{Pr}}}^{-1}$ is absolutely continuous w.r.t. the probability $\bar{\mathbb{Q}}_{\{\text{Ag}\}}$. Therefore, from (7.28d), we get

$$\mathbb{P}^{\{\text{Ag}\}}(\bar{S}_{\lambda_{\text{Pr}}}^{-1}(\bar{\mathcal{D}}_{\lambda_{\text{Ag}}}^*)) = \left(\mathbb{P}^{\{\text{Ag}\}} \circ \bar{S}_{\lambda_{\text{Pr}}}^{-1}\right)(\bar{\mathcal{D}}_{\lambda_{\text{Ag}}}^*) = 1, \quad \forall \lambda_{\text{Ag}} \in \Lambda_{\text{Ag}}, \quad (7.29a)$$

where

$$\bar{S}_{\lambda_{\text{Pr}}}^{-1}(\bar{\mathcal{D}}_{\lambda_{\text{Ag}}}^*) = \{\omega \in \Omega \mid (\hat{\mathcal{J}}^{\{\text{Ag}\}} \circ \bar{S}_{\lambda_{\text{Ag}}}^* \circ \bar{S}_{\lambda_{\text{Pr}}}^{-1})(\omega) \geq (\hat{\mathcal{J}}^{\{\text{Ag}\}} \circ \bar{S}_{\lambda_{\text{Ag}}} \circ \bar{S}_{\lambda_{\text{Pr}}}^{-1})(\omega)\} \quad (\text{by (7.28c)})$$

$$= \{\omega \in \Omega \mid (\hat{\mathcal{J}}^{\{\text{Ag}\}} \circ S_{(\lambda_{\text{Pr}}, \lambda_{\text{Ag}}^*)})(\omega) \geq (\hat{\mathcal{J}}^{\{\text{Ag}\}} \circ S_{(\lambda_{\text{Pr}}, \lambda_{\text{Ag}})})(\omega)\} \quad (7.29b)$$

by the expression (7.17) for the solution map $S_{(\lambda_{\text{Pr}}, \lambda_{\text{Ag}})} = \bar{S}_{\bar{\lambda}_{\text{Ag}}} \circ \bar{S}_{\bar{\lambda}_{\text{Pr}}}$.

In the expression (7.29b) above, both functions $\hat{j}^{\{\text{Ag}\}} \circ S_{(\lambda_{\text{Pr}}, \lambda_{\text{Ag}}^*)}$ and $\hat{j}^{\{\text{Ag}\}} \circ S_{(\lambda_{\text{Pr}}, \lambda_{\text{Ag}})}$ are \mathcal{F} -measurable. Indeed, as the integrand $\hat{j}^{\{\text{Ag}\}}$ in (7.27b) is a $\bar{\mathcal{J}}_{\text{Ag}}$ -normal integrand, it is a $(\mathcal{F} \otimes \mathcal{U}_{\text{Pr}}) \otimes \mathcal{U}_{\text{Ag}}$ -normal integrand since $\bar{\mathcal{J}}_{\text{Ag}} \subset \mathcal{F} \otimes \mathcal{U}_{\text{Pr}}$ by (7.16a). Hence, the function $\hat{j}^{\{\text{Ag}\}}$ is $\mathcal{F} \otimes \mathcal{U}_{\text{Pr}} \otimes \mathcal{U}_{\text{Ag}}$ -measurable. Now, since $S_{(\lambda_{\text{Pr}}, \lambda_{\text{Ag}}^*)}$ is \mathcal{F} -measurable by (7.14a), we conclude that the function $\hat{j}^{\{\text{Ag}\}} \circ S_{(\lambda_{\text{Pr}}, \lambda_{\text{Ag}}^*)}$ is \mathcal{F} -measurable. The same holds true for $\hat{j}^{\{\text{Ag}\}} \circ S_{(\lambda_{\text{Pr}}, \lambda_{\text{Ag}})}$. Moreover, both functions $\hat{j}^{\{\text{Ag}\}} \circ S_{(\lambda_{\text{Pr}}, \lambda_{\text{Ag}}^*)}$ and $\hat{j}^{\{\text{Ag}\}} \circ S_{(\lambda_{\text{Pr}}, \lambda_{\text{Ag}})}$ are bounded above because the function $j^{\{\text{Ag}\}}$ is bounded above by assumption in Item 3a, and by (7.22).

Thus, from (7.29a) and (7.29b), we deduce that

$$\mathbb{E}_{\mathbb{P}^{\{\text{Ag}\}}} \left[\hat{j}^{\{\text{Ag}\}} \circ S_{(\lambda_{\text{Pr}}, \lambda_{\text{Ag}}^*)} \right] \geq \mathbb{E}_{\mathbb{P}^{\{\text{Ag}\}}} \left[\hat{j}^{\{\text{Ag}\}} \circ S_{(\lambda_{\text{Pr}}, \lambda_{\text{Ag}})} \right], \quad \forall \lambda_{\text{Pr}} \in \Lambda_{\text{Pr}}, \quad \forall \lambda_{\text{Ag}} \in \Lambda_{\text{Ag}}. \quad (7.29c)$$

• We now prove that the Inequality (7.29c) is still valid when we replace the criterion $\hat{j}^{\{\text{Ag}\}}$ in (7.29c) by the original criterion $j^{\{\text{Ag}\}}$. We proceed as follows:

$$\mathbb{E}_{\mathbb{P}^{\{\text{Ag}\}}} \left[j^{\{\text{Ag}\}} \circ S_{(\lambda_{\text{Pr}}, \lambda_{\text{Ag}})} \right]$$

is well-defined, being bounded above because the function $j^{\{\text{Ag}\}}$ is bounded above by assumption in Item 3a, and being \mathcal{F} -measurable; indeed, on the one hand, the function $j^{\{\text{Ag}\}} : (\Omega \times \mathbb{U}_{\text{Pr}}) \times \mathbb{U}_{\text{Ag}} \rightarrow \bar{\mathbb{R}}$ is a $\mathcal{F} \otimes \mathcal{U}_{\text{Pr}}$ -normal integrand, by assumption in Item 3b, hence $j^{\{\text{Ag}\}}$ is $\mathcal{F} \otimes \mathcal{U}_{\text{Pr}} \otimes \mathcal{U}_{\text{Ag}}$ -measurable and, on the other hand, $S_{(\lambda_{\text{Pr}}, \lambda_{\text{Ag}})}$ is \mathcal{F} -measurable by (7.14a), so that the function $\hat{j}^{\{\text{Ag}\}} \circ S_{(\lambda_{\text{Pr}}, \lambda_{\text{Ag}})}$ is \mathcal{F} -measurable

$$\begin{aligned} &= \mathbb{E}_{\mathbb{P}^{\{\text{Ag}\}}} \left[j^{\{\text{Ag}\}} \circ \bar{S}_{\bar{\lambda}_{\text{Ag}}} \circ \bar{S}_{\bar{\lambda}_{\text{Pr}}} \right] && \text{(by (7.17))} \\ &= \mathbb{E}_{\mathbb{P}^{\{\text{Ag}\}} \circ \bar{S}_{\bar{\lambda}_{\text{Pr}}}^{-1}} \left[j^{\{\text{Ag}\}} \circ \bar{S}_{\bar{\lambda}_{\text{Ag}}} \right] && \text{(pushforward measure under } \bar{S}_{\bar{\lambda}_{\text{Pr}}}^{-1} \text{)} \\ &= \mathbb{E}_{\bar{\mathbb{Q}}^{\{\text{Ag}\}}} \left[\bar{T}_{\lambda_{\text{Pr}}} \times (j^{\{\text{Ag}\}} \circ \bar{S}_{\bar{\lambda}_{\text{Ag}}}) \right] \\ &\quad \text{(by (7.25a), } \mathbb{P}^{\{\text{Ag}\}} \circ \bar{S}_{\bar{\lambda}_{\text{Pr}}}^{-1} \text{ has density } \bar{T}_{\lambda_{\text{Pr}}} \text{ w.r.t. } \bar{\mathbb{Q}}^{\{\text{Ag}\}} \text{ on } \Omega \times \mathbb{U}_{\text{Pr}} \text{)} \\ &= \mathbb{E}_{\bar{\mathbb{Q}}^{\{\text{Ag}\}}} \left[\mathbb{E}_{\bar{\mathbb{Q}}^{\{\text{Ag}\}}} \left[\bar{T}_{\lambda_{\text{Pr}}} \times (j^{\{\text{Ag}\}} \circ \bar{S}_{\bar{\lambda}_{\text{Ag}}}) \middle| \bar{\mathcal{J}}_{\text{Ag}} \right] \right] && \text{(tower property)} \\ &= \mathbb{E}_{\bar{\mathbb{Q}}^{\{\text{Ag}\}}} \left[\bar{T}_{\lambda_{\text{Pr}}} \mathbb{E}_{\bar{\mathbb{Q}}^{\{\text{Ag}\}}} \left[j^{\{\text{Ag}\}} \circ \bar{S}_{\bar{\lambda}_{\text{Ag}}} \middle| \bar{\mathcal{J}}_{\text{Ag}} \right] \right] \end{aligned}$$

because, by assumption in Item 2c, the Radon-Nikodim derivative $\bar{T}_{\lambda_{\text{Pr}}}$ is $(\bar{\mathbb{J}}_{\text{Pr}} \otimes \{\emptyset, \mathbb{U}_{\text{Pr}}\}) \vee (\{\emptyset, \Omega\} \otimes \mathbb{U}_{\text{Pr}})$ -measurable, hence $\bar{\mathbb{J}}_{\text{Ag}}$ -measurable by (7.10c)

$$\begin{aligned}
&= \mathbb{E}_{\bar{\mathbb{Q}}_{\{\text{Ag}\}}} \left[\bar{T}_{\lambda_{\text{Pr}}} \times (\hat{j}^{\{\text{Ag}\}} \circ \bar{S}_{\lambda_{\text{Ag}}}) \right] && \text{(by (7.27d))} \\
&= \mathbb{E}_{\mathbb{P}^{\{\text{Ag}\}} \circ \bar{S}_{\lambda_{\text{Pr}}}^{-1}} \left[\hat{j}^{\{\text{Ag}\}} \circ \bar{S}_{\lambda_{\text{Ag}}} \right] \\
&\quad \text{(by (7.25a), } \mathbb{P}^{\{\text{Ag}\}} \circ \bar{S}_{\lambda_{\text{Pr}}}^{-1} \text{ has density } \bar{T}_{\lambda_{\text{Pr}}} \text{ w.r.t. } \bar{\mathbb{Q}}_{\{\text{Ag}\}} \text{ on } \Omega \times \mathbb{U}_{\text{Pr}}) \\
&= \mathbb{E}_{\mathbb{P}^{\{\text{Ag}\}}} \left[\hat{j}^{\{\text{Ag}\}} \circ \bar{S}_{\lambda_{\text{Ag}}} \circ \bar{S}_{\lambda_{\text{Pr}}} \right] && \text{(pushforward measure under } \bar{S}_{\lambda_{\text{Pr}}} \text{)} \\
&= \mathbb{E}_{\mathbb{P}^{\{\text{Ag}\}}} \left[\hat{j}^{\{\text{Ag}\}} \circ S_{(\lambda_{\text{Pr}}, \lambda_{\text{Ag}})} \right]. && \text{(by (7.17))}
\end{aligned}$$

Therefore, the Inequality (7.29c) is still valid when we replace the function $\hat{j}^{\{\text{Ag}\}}$ in (7.29c) by the original criterion $j^{\{\text{Ag}\}}$, giving thus (7.26).

This ends the proof. \square

In [39], Witsenhausen proves that, when the set \mathbb{U}_{Pr} is countable, there exists a probability $\bar{\mathbb{Q}}_{\{\text{Ag}\}}$ on $(\Omega \times \mathbb{U}_{\text{Pr}}, \mathcal{F} \otimes \mathbb{U}_{\text{Pr}})$ such that Item 2b and Item 2c in Proposition 7.13 hold true.

Theorem 7.14. *Suppose given an Enough informed Agent W-game, as in Definition 7.3, which satisfies the technical assumptions of Proposition 7.13, and let $\lambda_{\text{Ag}}^* \in \Lambda_{\text{Ag}}$ be a W-strategy for the Agent Ag such that (7.26) holds true.*

Suppose also that the function $j^{\{\text{Pr}\}} : \Omega \times \mathbb{U}_{\text{Pr}} \times \mathbb{U}_{\text{Ag}} \rightarrow \bar{\mathbb{R}}$ is bounded above.

If there exists a W-strategy $\lambda_{\text{Pr}}^ \in \Lambda_{\text{Pr}}$ for the Principal Pr such that*

$$\mathbb{E}_{\mathbb{P}^{\{\text{Pr}\}}} [j^{\{\text{Pr}\}} \circ S_{(\lambda_{\text{Pr}}^*, \lambda_{\text{Ag}}^*)}] \geq \mathbb{E}_{\mathbb{P}^{\{\text{Pr}\}}} [j^{\{\text{Pr}\}} \circ S_{(\lambda_{\text{Pr}}, \lambda_{\text{Ag}}^*)}], \quad \forall \lambda_{\text{Pr}} \in \Lambda_{\text{Pr}}, \quad (7.30)$$

then, the profile $(\lambda_{\text{Pr}}^, \lambda_{\text{Ag}}^*)$ constitutes a Nash equilibrium of the Principal-Agent W-game as in Definition 7.2.*

Proof. The Agent's W-strategy $\bar{\lambda}_{\text{Ag}}^* : \Omega \times \mathbb{U}_{\text{Pr}} \rightarrow \mathbb{U}_{\text{Ag}}$ is such that (7.5b) holds true for any Principal's W-strategy $\lambda_{\text{Pr}} \in \Lambda_{\text{Pr}}$, hence for the specific strategy λ_{Pr}^* of the Principal, giving

$$\mathbb{E}_{\mathbb{P}^{\{\text{Ag}\}}} [j^{\{\text{Ag}\}} \circ S_{(\lambda_{\text{Pr}}^*, \lambda_{\text{Ag}}^*)}] \geq \mathbb{E}_{\mathbb{P}^{\{\text{Ag}\}}} [j^{\{\text{Ag}\}} \circ S_{(\lambda_{\text{Pr}}, \lambda_{\text{Ag}}^*)}], \quad \forall \lambda_{\text{Pr}} \in \Lambda_{\text{Pr}}. \quad (7.31)$$

Combined with the assumption (7.30), we get (7.5). \square

7.4 Conclusion

In this Chapter 7, we consider a subclass of Principal-Agent games, namely, the Enough informed Agent W-games, and assuming the existence of the Nash equilibria of such game, we provide technical conditions under which this solution can be yielded as a result of

backward induction. Among these conditions, we assume that the action set of the Agent is nonempty bounded and closed subset of \mathbb{R}^d , that there exists a measurable density for some probability distribution, and that the criterion of Agent is bounded, whereas the negative criterion is a normal integrand. Note that when we do not assume that the set of actions of the Agent is finite, the completeness of some subfield w.r.t. to some probability is needed, which can be restrictive.

We leave open the question if the main result can be weakened by having assumptions on the measurability of the criterions of Principal and Agent.

Part III

Conclusion and Appendix

Chapter 8

Conclusion

8.1 Contribution of this thesis

This thesis is a contribution to Game Theory with information. We have presented links between different models of games with information, studied different information patterns and given conditions under which some Principal-Agent games can be solved by backward induction.

- In Chapter 2, we have presented connections and differences between three classes of games with information.
- In Chapter 3, we have shown that the class of causal finite W -models can be embedded into AFR-models.
- In Chapter 4, we have gone in the opposite direction, by showing that a restricted class of so-called Nature-rooted finite AFR-models can be embedded into W -models.
- In Chapter 5, we have expressed classical information patterns in the framework of the W -model.
- In Chapter 6, we have introduced W -games.
- In Chapter 7, we have studied a subclass of Principal-Agent games, namely, the so-called Enough informed Agent W -games, and we have provided conditions under which a Nash equilibrium can be obtained by backward induction.

8.2 Future avenues of research

Here, we recapitulate further possible work and questions that remain open.

Developing the W-model beyond finite sets (Chapters 2, 3 and 4)

In both Chapters 3 and 4, the equivalence between AFR- and W-models is shown under the assumptions of finiteness and completeness of the corresponding fields, although both models allow for both countable and non-countable primitives. A transversal avenue of research would be lifting the assumption of finiteness of the set of agents and the sets of agents' actions.

Besides, one could explore the equivalence between AFR- and W-models in the infinite sets and players case. As in the most general case of the AFR-model, a strategy may not induce a unique outcome, the same problem may arise for the infinite causal W-model. Then, it can be studied if some restricted conditions may still grant the property of solvability.

Information structures (Chapter 5)

We conjectured that the so-called W-information memory (respectively, W-perfect recall) is equivalent to the so-called I-TrIP (respectively, C-TrIP) introduced in [3].

Kuhn's equivalence theorem (Chapter 6)

As said in the conclusion of Chapter 6, we leave it as an open question of how to define mixed and behavioral strategies in the W-model. Once these notions are properly formulated, one might be able to prove an analogue of the Kuhn's theorem about the equivalence between mixed and behavioral strategies under the assumption of perfect recall.

To this end, first, for any behavioral strategy and the corresponding probability distribution over the set of outcomes that it induces, one needs to build a mixed strategy that induces the same probability distribution. Second, in a similar way, for any mixed strategy and the corresponding probability distribution, one can build a behavioral strategy giving the same probability distribution over the set of outcomes. Once both steps are done, this would yield an analogue of the Kuhn's theorem. It is proved in [18,29] that, for any player, mixed strategies are equivalent to behavioral strategies if and only if the player possesses perfect recall. Thus, the analogue of perfect recall in the W-model stated in Chapter 5 will be undoubtedly used in the proof of the Kuhn's theorem.

Subgames and Backward Induction (Chapter 6)

As it was said in the conclusion of Chapter 6, we leave it as an open question to define the notion of subgame in the W-model and to relate it to the notion of subsystem as in Definition 6.5. Once subgames are properly defined, the concepts of backward induction and subgame perfect equilibrium might possibly be formulated in the W-model.

Principal-Agent games (Chapter 7)

In Chapter 7, we considered a subclass of Enough informed Agent W-games, and assuming the existence of the Nash equilibria of such game, we provided technical conditions under

which an equilibrium can be found by backward induction. It remains an open question if the assumptions could be weakened by imposing some extra assumptions, for instance, on the measurability of the criterions $j_{\{\text{Ag}\}}$ and $j_{\{\text{Pr}\}}$ with respect to proper σ -fields related to the information structure.

Another interesting avenue of research may be developing a multi-agent setting, where both Principal and Agent players are each endowed with a group of representative W-agents.

Chapter 9

Appendix

9.1 Basics on partitions, σ - and π -fields

As in this manuscript, we describe the concept of information in the language of σ -, π -fields and partitions, we present basic material following [9].

In Chapters 3 and 4, we work with finite sets when establishing equivalences between the AFR- and W-models. In that finite sets case, the three (more and more restricted) notions — algebra, σ -field, π -field — collapse into a single one. After formally defining σ -field, we go on with detailing the notion of π -field (it should be remembered that all notions below remain valid for algebras and σ -fields when the underlying set Ω is finite).

Algebras and σ -fields

A (*Boolean*) algebra on Ω is a subset $\mathcal{G} \subset 2^\Omega$, containing Ω , and which is stable under complementation and finite union (hence under finite intersection).

Less restricted than algebras, are σ -fields. A σ -field on Ω is a subset $\mathcal{G} \subset 2^\Omega$, containing Ω , and which is stable under complementation and countable union (hence under countable intersection). Consider two σ -fields \mathcal{G} and \mathcal{G}' . We say that the field \mathcal{G} is *finer* than the field \mathcal{G}' if every element of \mathcal{G} is included in an element of \mathcal{G}' . We also say that \mathcal{G}' is a *subfield* of \mathcal{G} .

π -fields and partitions

More restricted than σ -fields is the class of partition fields (or π -fields).

First, we define a π -field as a collection of subsets of a set S , which is stable under arbitrary union and intersection (countable or not).

Definition 9.1. *Let a set S be given. A partition field (or π -field) of the set S is a nonempty collection \mathcal{G} of subsets of S , that is, $\mathcal{G} \subset 2^S$, which is stable under complementation and unlimited union (hence, under unlimited intersection).*

The complete π -field is made of all subsets of the set S .

We say that a partition field \mathcal{G}' is *finer* than a partition \mathcal{G} if every element of \mathcal{G} is also an element of \mathcal{G}' , that is,

$$\mathcal{G} \subset \mathcal{G}' \iff \forall G \in \mathcal{G}, G \in \mathcal{G}' . \quad (9.1)$$

After defining a restricted subclass of σ -fields, the so-called π -fields, we give some facts about partitions that are closely related to π -fields.

Definition 9.2. *Let a set S be given. A collection Π of subsets of the set S is a partition, which we denote as*

$$S = \bigsqcup_{G \in \Pi} G , \quad (9.2)$$

if it consists of mutually disjoint nonempty subsets G ($G \in \Pi$) whose union is S :

$$\Pi \subset 2^S \text{ s.t. } \begin{cases} S = \bigcup_{G \in \Pi} G, \\ G \cap G' = \emptyset, \forall G \neq G', \\ G \neq \emptyset, \forall G \in \Pi. \end{cases} \quad (9.3)$$

An atom of partition Π is a nonempty subset $G \in \Pi$ such that $G' \in \Pi$ and $G' \subset G$ imply that $G' = \emptyset$ or $G' = G$.

Consider a partition $\Pi = \{G\}_{G \in \Pi}$ of a set S . Two elements of the set S are said to be *indistinguishable* w.r.t. Π if they belong to the same element $G \in \Pi$ of the partition. We define the *indistinguishability relation* \mathfrak{R}_Π on the set S as follows:

$$s \mathfrak{R}_\Pi s' \text{ iff } \exists G \in \Pi \text{ s.t. } s \in G \text{ and } s' \in G .$$

One can see that the equivalence classes S/\mathfrak{R}_Π of the underlying equivalence relation \mathfrak{R}_Π are the elements of the partition Π :

$$S/\mathfrak{R}_\Pi = \Pi . \quad (9.4)$$

On the other hand, the classes of any equivalence relation on S form a partition. Hence, partitions Π on S are in one-to-one correspondence with equivalence relations \mathfrak{R} on S by the mappings

$$\Pi \mapsto \mathfrak{R}_\Pi \text{ and } \mathfrak{R} \mapsto S/\mathfrak{R} .$$

We say that a partition Π' is *finer* than a partition Π if every element of Π' is included in an element of Π , that is,

$$\Pi \preceq \Pi' \iff \forall G' \in \Pi', \exists G \in \Pi, \text{ s.t. } G' \subset G . \quad (9.5)$$

Consider two partitions Π and Π' of a given set S . The *least upper bound* of the partitions Π and Π' is the partition given by

$$\Pi \vee \Pi' = \{G'' \subset S \mid G'' \neq \emptyset, \exists G \in \Pi, \exists G' \in \Pi', G'' = G \cap G'\} . \quad (9.6)$$

The following Lemma 9.3 is used in Section 3.4 when building WtoAFR-strategies.

Lemma 9.3. *Let two sets A and B with corresponding equivalence relations \mathfrak{A} and \mathfrak{B} , and a mapping $f : (A, A/\mathfrak{A}) \rightarrow (B, B/\mathfrak{B})$ be given. We have the following equivalence*

$$\left(f^{-1}(B/\mathfrak{B}) \subset A/\mathfrak{A} \right) \iff \left(\exists \tilde{f} : A/\mathfrak{A} \rightarrow B \text{ such that } f = \tilde{f} \circ \pi_{\mathfrak{A}} \right),$$

where $\tilde{f} : A/\mathfrak{A} \rightarrow B$ is such that the following diagram commutes

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow \pi_{\mathfrak{A}} & \nearrow \tilde{f} & \\ A/\mathfrak{A} & & \end{array} \quad (9.7)$$

9.2 Additional material for Chapter 4

As we have left open two questions in Chapter 4, here we state both as conjectures and propose a tentative way of solving them.

9.2.1 Conjecture: constructing a causal AFRtoW-ordering

Conjecture 9.4. *Let a Nature-rooted AFR-model as in Definition 2.16 be given. Then there exists an AFRtoW-ordering φ as in (9.11), which is causal as in Definition 2.25.*

Tentative proof. The following, is a chain of conjectured technical steps of a tentative proof of the conjecture.

- We consider the set Z of terminal vertices as in (2.16). With each terminal vertex $z \in Z$ we associate the set of configurations H_z as in (4.6). We conjecture that the family $\{H_z\}_{z \in Z}$ is made of disjoint subsets of the set \mathbb{H} of AFRtoW-configurations as defined in (4.4), that is,

$$(\forall z, z' \in Z) \quad H_z \cap H_{z'} \neq \emptyset \iff z = z', \quad (9.8)$$

where the set H_z of AFRtoW-configurations is defined in (4.6).

- For each terminal vertex $z \in Z$, we consider the upset $\uparrow z$ as in (2.10) and the set \mathbb{A}_z as in (4.5) of agents who played before the vertex z was reached. As the tree (V, \supset) is finite (see Definition 2.11), then each leaf z is a finite vertex, that is, $z \in F(V)$, where the set $F(V)$ of finite vertices is defined in Equation (2.11a). Thus, for any $z \in Z$, the upset $\uparrow z$ is of the form (see Equation (2.13b))

$$\uparrow z = \{z, p(z), p^2(z), \dots, p^{|\uparrow z|-1}(z)\}.$$

As a consequence of (2.13a), we have the following chain of strict inclusions

$$z = p^0(z) \subsetneq p(z) \subsetneq p^2(z) \subsetneq \dots \subsetneq p^{|\uparrow z|-1}(z) = \mathbb{W},$$

where p is the parent mapping as in (2.11b). This implies another chain of strict inclusions

$$\mathbb{A}_z \supsetneq \mathbb{A}_{p(z)} \supsetneq \mathbb{A}_{p^2(z)} \supsetneq \dots \supsetneq \mathbb{A}_{p^{|\uparrow z|-1}(z)} = \emptyset, \quad (9.9)$$

by Equation (4.5). As in each non-terminal vertex there is only one AFR-player that is active (see (AFR-Axiom1') in Definition 2.16), for any two sets $\mathbb{A}_{p^{|\uparrow z|-(j+1)}(z)}$ and $\mathbb{A}_{p^{|\uparrow z|-j}(z)}$ with consecutive numbers, where $1 \leq j \leq |\uparrow z|$, we have that $\mathbb{A}_{p^{|\uparrow z|-(j+1)}(z)} \setminus \mathbb{A}_{p^{|\uparrow z|-j}(z)}$ is a singleton of \mathbb{A} and we denote

$$\{a_z^{(j)}\} = \mathbb{A}_{p^{|\uparrow z|-(j+1)}(z)} \setminus \mathbb{A}_{p^{|\uparrow z|-j}(z)}. \quad (9.10)$$

Let a Nature-rooted AFR-model as in Definition 2.16 be given. We call *AFRtoW-ordering* a mapping $\varphi : \bigsqcup_{z \in Z} H_z \rightarrow \Sigma^{|\mathbb{A}|}$ such that, for any terminal vertex $z \in Z$ as in (2.16) and the corresponding subset of AFRtoW-configurations H_z as in (4.6), we have

$$(\forall h \in H_z) \quad \varphi(h) = (a_z^{(1)}, \dots, a_z^{(|\uparrow z|-1)}, \bar{\kappa}) \in \Sigma^{|\mathbb{A}|}, \text{ for some } \bar{\kappa} \in \Sigma_{\mathbb{A} \setminus \mathbb{A}_z}, \quad (9.11)$$

where, for any index $1 \leq j \leq |\uparrow z|$, the corresponding agent $a_z^{(j)}$ is defined in (9.10).

- Finally, we conjecture that the constructed ordering $\varphi : \mathbb{H} \rightarrow \Sigma^{|\mathbb{A}|}$ is causal as in Definition 2.24.

This finishes the tentative proof. □

9.2.2 Conjecture: solvability for AFRtoW-strategies

In this §9.2.2 we state the second conjecture, namely, that the constructed AFRtoW-model is solvable, that is, any AFRtoW-strategy as in §4.3 induces a unique outcome.

Conjecture 9.5. *Any profile $\{\lambda_a\}_{a \in \mathbb{A}}$ of AFRtoW-strategies induces a unique outcome on the set $\bigsqcup_{z \in Z} H_z$ of AFRtoW-configurations.*

Tentative proof. The following, is a chain of conjectured technical steps of a tentative proof of the conjecture.

- For any vertex $v \in V$, the corresponding set H_v of AFRtoW-configurations as in (4.6), is partitioned by

$$H_v = \bigsqcup_{z \in Z \cap \downarrow v} H_z. \quad (9.12)$$

- The following, is a simple corollary of the conjectured representation 9.12.

For any leaf $z \in Z$ and any AFRtoW-agent $b \in \mathbb{A} \setminus \mathbb{A}_z$,

$$H_b \cap H_z = \emptyset,$$

where $H_b = \bigcup_{v \in b} H_v$.

Here, we give a short proof of this corollary. Let a leaf $z \in Z$ and a AFRtoW-agent $b \in \mathbb{A} \setminus \mathbb{A}_z$ be given. As $b \in \mathbb{A} \setminus \mathbb{A}_z$, then $\uparrow z \cap b = \emptyset$ (by construction of the set \mathbb{A}_z of AFRtoW-agents as in (4.5)). Then, we write

$$\begin{aligned}
H_b \cap H_z &= \left(\bigcup_{v \in b} H_v \right) \cap H_z && \text{(by definition of the set } H_b) \\
&= \left(\bigcup_{v \in b} \bigsqcup_{z' \in Z \cap \downarrow v} H_{z'} \right) \cap H_z && \text{(as } H_v = \bigsqcup_{z' \in Z \cap \downarrow v} H_{z'} \text{ by (9.12))} \\
&= \bigcup_{v \in b} \bigsqcup_{z' \in Z \cap \downarrow v} (H_{z'} \cap H_z) && \text{(by associativity)} \\
&= \emptyset,
\end{aligned}$$

as, otherwise, for any vertex $v \in b$ and for any leaf $z' \in Z \cap \downarrow v$, if $H_{z'} \cap H_z \neq \emptyset$, then $H_{z'} = H_z$ (as they are atoms of the partition as in (9.8)), thus, $z' = z$ and $z \in Z \cap \downarrow v$. This implies that $z \subset v$ and, thus, $v \in \uparrow z$ by definition of the upset as in (2.10). As a consequence, $\uparrow z \cap b \neq \emptyset$, which brings us to a contradiction.

This finishes the tentative proof. □

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