Numerical methods for the ALM

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Modélisation et méthodes numériques pour la gestion actif/passif

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C'est en faisant n'importe quoi qu'on devient n'importe qui
(Rémi Gaillard)

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Préambule (Français)

Cette thèse s'intéresse à la modélisation et aux méthodes numériques pour la gestion actif/passif (Asset and Liability Management ALM) en assurance. Dans la première partie de cette thèse, nous construisons un modèle ALM synthétique qui intègre les principales caractéristiques des contrats d'assurance-vie. Ce modèle tient compte à la fois du bilan en book et market value pour déclencher le mécanisme de participation aux bénéfices et introduit un mécanisme de détermination du taux servi proche de la pratique en déterminant un compromis entre taux règlementaire, taux concurrent et la performance générée par le portefeuille. Enfin, il considère un investissement dans un panier d'obligations permettant une couverture statique des flux de rachats sans garder en mémoire l'historique de gestion. Ce modèle est alors utilisé pour calculer le Solvency Capital Requirement (SCR) avec la formule standard. La seconde partie de cette thèse s'intéresse aux méthodes numériques efficaces pour le calcul du SCR. En particulier, nous étudions la méthode Multilevel Monte-Carlo (MLMC) dévelopée par Giles [Gil08] pour estimer l'espérance du maximum de plusieurs espérances conditionelles. Ce type de calcul apparaît notamment lorsqu'on l'on compare différents stress-tests ainsi que dans l'évaluation du module taux d'intérêt de la formule standard. Nous obtenons un résultat de convergence qui complète les travaux récents de Giles et Goda [GG19] et fournit un cadre d'application plus souple pour l'estimateur MLMC. Enfin, nous utilisons ces résultats pour l'estimation du SCR à des dates futures dans le modèle construit dans la première partie. Nous comparons les performances de l'estimateur MLMC avec les approches type Least Square Monte Carlo (LSMC) ou réseaux de neurones et démontrons la pertinence de l'approche Multilevel dans ce contexte.
Mots-clés: modèle ALM, Capital de Solvabilité Requis, Formule standard, Adossement des flux, Gap de liquidité, Risque de rachat, Book value, Participation aux bénéfices, MLMC, LSMC, Réseaux de Neurones

Preamble (English)

This thesis deals with the modeling and the construction of efficient numerical methods for the Asset and Liability Management (ALM) in insurance. The first part of this thesis introduces a synthetic ALM model that catches the key features of life insurance contracts. This model keeps track of both market and book values to apply the regulatory profit sharing rule. Second, it introduces a determination of the crediting rate to policyholders that is close to practice and is a trade-off between the regulatory rate, a competitor rate and the available profits. Third, it considers an investment in bonds that enables to match a part of the cash outflow due to surrenders, while avoiding to store the trading history. We use this model to evaluate the Solvency Capital Requirement (SCR) with the standard formula. The second part copes with efficient numerical methods to compute the SCR. More specifically, we study the Multilevel Monte-Carlo (MLMC) method developed by Giles [Gil08] to estimate the expectation of a maximum of conditional expectations. This problem arises naturally when considering many stress tests and appears in the calculation of the interest rate module of the standard formula for the SCR. We obtain theoretical convergence results that complements the recent work of Giles and Goda [GG19] and gives some additional tractability through a parameter that somehow describes regularity properties around the maximum. We then apply the MLMC estimator to the calculation of the SCR at future dates with the standard formula using the model developed in the first part. Last, we compare it with estimators obtained with Least Square Monte-Carlo or Neural Networks and show the relevance of the MLMC method in this context.
Keywords: ALM model, Solvency capital requirement, Standard formula, Cash flow matching, Liquidity gap, Surrender risk, Book value, Profit sharing, MLMC, LSMC, Neural Network

Résumé

Cette thèse se consacre à la modélisation et aux développements de méthodes numériques pour la gestion actif/passif (Asset and Liability Management ALM) des contrats d'assurance-vie. Ce type de produits joue un rôle fondamental dans le paysage de l'assurance-vie et la constitution de l'épargne des ménages. Dans la presse, le fond euro constitue un des "placements préférés des français". En 2019, les français ont déposé 11,2 milliards d'euros sur les contrats d'assurance-vie (source: [Mei19]). Toutefois, la modélisation de ce type de contrats est particulièrement complexe en raison d'options et garanties financières, ainsi que de l'évolution de l'environnement prudentiel (Solvabilité II) et comptable (IFRS).

Dans la première partie de cette thèse, nous construisons un modèle synthétique de gestion d'un fond euro qui tient compte des principales caractéristiques du business de l'assurance-vie. L'assuré souscrivant à une assurance-vie cherche à se constituer une épargne. Le contrat comprend un taux de revalorisation minimal ("taux minimum garanti") ainsi qu'une participation aux bénéfices additionnelle correspondant à un pourcentage (encadré par le code des assurances) sur les rendements générés par les investissements de l'assureurs issu du dépôt des assurés. Règlementairement, un ensemble de normes comptables ("local GAAP") contraignent l'assureur à constituer un ensemble de provisions techniques en face de ses engagements et de comptabiliser ses actifs en coût d'acquisitition (la "book value" ou valeur comptable) pour l'enregistrement de ses plus values qui elles mêmes peuvent faire l'objet d'un provisionnement spécifique. A titre d'exemple la dette de l'assureur envers l'assuré est représentée par la "provision mathématique" dans le bilan de l'assureur et correspond aux dépôts revalorisés du taux servi par l'assureur chaque année. D'autres réserves règlementaires telles que la Provision pour Participation aux Bénéfices (PPB) permettent un lissage du taux servi aux assurés dans le temps avec obligation de redistribution de la réserve constituée dans les 8 ans. La réserve de capitalisation quant à elle, encadre la redistribution des plus-moins values obligataires dans le but de constituer un coussin de sécurité contre les mouvements de taux.

Dans la première partie de ce manuscrit, nous proposons un modèle qui tient compte à la fois du bilan en market et book value pour déterminer le taux de revalorisation de l'épargne de l'assuré et développons une règle de gestion permettant un bon compromis entre actionnaires et assurés sous contrainte d'immobilisation de capital suffisant imposé par le régulateur pour que l'assureur puisse continuer ses activités. Pour déterminer ce taux, nous considérons les plus ou moins-values réalisées lors du rebalancement du portefeuille ainsi qu'un taux concurrent en fonction de l'environnement de taux actuel. Ensuite, la compagnie effectue un pilotage de ses plus-moins values latentes pour atteindre son taux cible défini comme le maximum entre le taux de la concurrence, le taux minimum garanti dans le contrat et la contrainte légale de participation aux bénéfices additionnelle. Nous intégrons également l'option de rachat du contrat par l'assuré en fonction des conditions de marché en supposant que le taux de rachat augmente si le taux proposé par l'assureur est trop bas par rapport au taux de la concurrence.

Dans la litterature, les modèles de gestion actif/passif sont souvent trop simplifiés pour être exploités directement. La pratique courante est généralement d'utiliser des modèles internes qui sont des "boîtes noires", dont les détails d'implémentation ne sont pas communiqués. Une de nos principales contributions est de proposer un modèle

réaliste intermédiaire, dans le sens où il comprend les principales caractéristiques de la gestion actif/passif (book value, market value, rachat dynamique,...), tout en étant suffisamment tractable pour effectuer des simulations Monte-Carlo dans des études ALM et servir de benchmark. Une originalité du modèle est qu'il intègre notamment, une stratégie de couverture statique du risque de taux permettant d'analyser l'impact du gap de liquidité entre flux de l'actif (revenus générés par les investissements) et les flux de passifs (paiement des prestations). En particulier, nous considérons un investissement dans un panier d'obligations de maturités allant de 1 à n, le nominal de l'obligation 1 an servant à couvrir les flux de rachats.
Le développement des normes règlementaires européenne post crise financière de 2008 ont ajouté un niveau de complexité supplémentaire dans les modèles ALM. La directive Solvabilité II entrée en application en janvier 2016 cherche à s'assurer que le niveau de fonds propre détenu par la compagnie d'assurance est suffisant par rapport au risque pris par la compagnie. La nouveauté tient au fait que ce montant à immobiliser, le capital de solvabilité requis (Solvency Capital Requirement SCR) tient compte de l'allocation d'actifs de l'assureur. Sous l'ancien régime (Solvabilité I), le coussin de sécurité mis en place pour faire face à des pertes exceptionnelles correspondait simplement à un montant forfaitaire proportionnel aux dépôts des assurés, i.e un pourcentage de la Provision Mathématique. La complexité additionnelle introduite par la mise en oeuvre de SII tient à la nouvelle méthode de valorisation du bilan de la compagnie. Dorénavant, les actifs mais aussi les passifs de la compagnies doivent être valorisées en valeur de marché. Toutefois, il n'existe pas de marché liquide pour le passif de l'assureur-vie et la valorisation des postes du bilan passe par un "Best Estimate" des engagements de la compagnie en utilisant un modèle. Ce Best Estimate des engagements correspond à la valeur actuelle (espérance conditionnelle à l'information diponible) des cash flows de passifs (paiement des prestations). Cette valeur est supposée correspondre au prix qu'une tierce partie serait prête à payée pour racheter le passif dans un marché sans arbitrage. Dans la vision Solvabilité II, le Best Estimate représente la dette de l'assureur envers ses assurés. Les fonds propres de la compagnie sont alors donnés par l'écart entre les actifs (valorisé en valeur de marché) et cette dette mesurée par le Best Estimate. Cette valeur sert de référence pour le calcul du SCR. Pour l'évaluation du SCR, l'EIOPA (European Insurance and Occupational Pensions Authority) offre deux possibilités. Soit calculer un quantile sur les pertes du portefeuille géré par l'assureur soit utiliser une formule standard consistant en une succession de stress-tests appliqués sur chaque classe d'actifs puis aggrégé via une matrice de corrélation fournie par le régulateur.
La motivation initiale du modèle ALM que nous développons dans la première partie est de pouvoir calculer le SCR en formule standard. A partir d'investigations numériques dans le modèle nous calculons les modules du SCR dans différents environements de taux et analysons l'impact du modèle de taux sur l'estimation finale du SCR. Notre étude numérique conduit également à mettre en avant certaines faiblesses de la formule standard, notamment des discontinuités dans la méthode d'aggrégation des risques qui peuvent conduire à des manipulations. Par ailleurs, la formule standard repose sur des variations de moyennes pre/post choc ignorant ainsi complètement le profil de la distribution des pertes. Nous montrons également les limites du recours à la valorisation risque-neutre dans le calcul du capital. Enfin, nous abordons un problème fondamental dans la gestion ALM à savoir la couverture du risque de taux par matching des flux et comment le faire optimalement dans le modèle en minimisant le SCR. Nous comparons

notre approche avec les outils standards d'adossement par la duration.
Dans le chapitre complémentaire de cette thèse, nous poursuivons nos investigations numériques en comparant les écarts de charge en capital en utilisant les deux méthodologies de calcul du SCR (quantile et stress-tests). Nous continuous en montrant que le modèle construit dans la première partie de cette thèse est suffisament flexible pour tenir compte des évolutions potentielles de solvabilité II, motivé par la parution du dernier document de consultation de l'EIOPA [EIOa]. Actuellement, les taux d'intérêts utilisés pour valoriser les engagements de l'assureur dans le calcul du Best Estimate sont déduit des prix de marché (swap, obligations d'états). Toutefois, pour des engagements à très long terme, le marché n'est pas suffisament liquide et une méthode d'extrapolation (Smith-Wilson) est utilisée pour projeter la courbe des taux sans-risque après la dernière maturité observable ("Last Liquid Point" LLP) et obtenir une convergence des taux d'intérêts vers un taux ultime ("Ultimate Forward Rate" UFR) qui est un paramètre exogène fournit par le régulateur pour pallier le manque de données de marché observables à très long terme. Cette "Courbe EIOPA" est un input essentiel du modèle ALM car il sert de référence à la calibration du modèle de taux et des chocs de la formule standard. En utilisant le modèle ALM, nous quantifions les impacts d'une modification de ces paramètres règlementaire (LLP, UFR) sur les provisions techniques et le Best Estimate.
Dans la seconde partie de cette thèse, nous nous intéressons aux enjeux computationnels introduits dans SII. Un problème ouvert dans l'industrie est de pouvoir calculer efficacement le SCR à des dates futures. D'un point de vue règlementaire, l'ORSA (Own Risk and Solvency Assessment) défini dans le deuxième pilier de Solvabilité II invite les assureurs à évaluer leur besoin global de solvabilité sur tout un business plan. Cela requiert non seulement le calcul du SCR à horizon 1 an mais aussi à des dates futures ($SCR_{t+1}, SCR_{t+2}, \ldots$). Par ailleurs, le calcul du coût de capital correspondant au montant que devra immobiliser l'actionnaire pour pouvoir continuer son activité est un critère important pour déterminer une allocation optimale d'actifs sur les fonds euros, l'idée étant de vérifier si les revenus futurs sont en adéquation avec les attentes de l'actionnaires en terme de coût de capital. Mentionnons également des applications au pricing de produits d'assurance-vie. Avant de lancer un nouveau produit, l'assureur doit évaluer sa rentabilité par rapport au besoin en capital généré par ce nouveau business. Actuellement, la valorisation des SCR futurs font l'objet d'estimations grossières à partir du SCR_0.
L'objectif de la seconde partie de ces travaux de thèse est de proposer des méthodes numériques efficaces pour le calcul du SCR sur un horizon de temps pluriannuel. Concrètement, l'application de la formule standard nécessite la simulation (historique) d'environnements économiques jusqu'à la date t puis d'appliquer les différents chocs sur le portefeuille pour chacun des scénarios de marché. Toutefois, la valorisation du portefeuille de l'assureur pour chaque scénarios nécessite elle même le recours à des simulations car la complexité du modèle ALM ne permet pas de disposer de formules fermées. La complexité de calcul réside dans l'imbrication des simulations appelé "simulation dans les simulations".
Des méthodes numériques efficaces pour le calcul du SCR reposent généralement sur des méthodes de regression type Least-Square-Monte-Carlo (LSMC) ou Replicating Portfolio qui représentent les standards de l'industrie. Cette famille de méthode propose d'approcher l'espérance conditionelle en procédant par des régressions par moindres carrées basées sur un très faible nombre de simulations secondaires. Le portefeuille

de l'assureur est alors approché par une combinaison linéaire de fonctions de base (LSMC) ou encore les cash flow de passif sont "répliqués" par des produits vanille dont l'espérance conditionelle est calculable par formule fermée. Les avancées récentes dans le domaine de la data science ont montré que les approches de type deep learning basée sur des réseaux de neurones peuvent être efficaces pour approcher des fonctions non-linéaire en grande dimension.
Dans le contexte assurantiel, la sélection de variables explicatives pertinentes dans la prédiction des valeurs futures du portefeuille, le choix des produits dérivés dans la construction du portefeuille répliquant comportent des problèmes opérationnels majeurs pour les compagnie d'assurance. La recherche actuarielle récente s'intéresse à l'utilisation des réseaux de neurones pour surmonter le fléau de la dimension et la difficile étape de sélection de variables des approches par régression "classique".
Dans le chapitre 4 de cette thèse, nous appliquons la méthode Multilevel Monte-Carlo (MLMC) dévelopée par Giles [Gil08] pour le calcul du SCR sur plusieurs pas de temps. Le principal intérêt de cette approche est de réduire le temps de calcul sans avoir recours à une quelconque forme d'extrapolation ou approximation fonctionnelle, mais plutôt en s'appuyant sur une allocation "intelligente" du budget de simulation. A notre connaissance, il n'existe aucune application de cette méthode dans le cadre du calcul de capital en assurance et nous contribuons à la littérature sur le sujet. Cette méthode basée sur une décomposition des espérances imbriquées comme somme téléscopique propose d'allouer le budget de simulations sur plusieurs niveaux et de répartir les simulations primaires et secondaires de façon à atteindre une précision fixée. La calibration des paramètres de l'algorithme nécessite de résoudre un problème d'optimisation sous contrainte afin d'atteindre le meilleur trade-off bias (erreur induite par la deuxième couche de simulation)/variance (erreur statistique induite par le Monte-Carlo primaire).
D'un point de vue théorique, nous améliorons un résultat de Giles et Goda [GG19] sur le sujet. Plus particulièrement nous proposons un développement du biais et de la variance pour l'estimateur MLMC associé au maximum de plusieurs espérance conditionnelles. Mathématiquement, cela revient à estimer une espérance de la forme:

$$I = \mathbb{E}\left[\max\{\mathbb{E}[Y^1|X], \ldots, \mathbb{E}[Y^P|X]\}\phi(X)\right]$$

où $\phi(X)$ représente un changement de probabilité entre univers historique et risque-neutre. Ce type de problème se retrouve également lorsque l'on cherche à estimer le choc le plus sévère sur un portefeuille financier. L'originalité de ce cadre d'étude où la régularité de la fonction payoff est intermédiaire (plus régulière qu'une indicatrice mais moins régulière qu'ue fonction de classe $\mathcal{C}^2$ complémente les investigations de [BDMGZ20] sur le sujet. Notre cadre d'étude remplace des conditions techniques difficile à vérifier dans [GG19] pour s'assurer que plusieurs espérance conditionelles ne sont pas trop proche du maximum simultanément par une condition d'intégrabilité impliquant un paramètre $\eta \in [0, 1)$ offrant une flexibilité supplémentaire pour la calibration pratique des paramètres de l'estimateur MLMC. Par exemple, dans le cas où l'on considère le maximum entre deux chocs, problème qui intervient notamment dans le calcul du module taux d'intérêt de la formule standard, cette condition revient essentiellement à la condition d'integrabilité suivante:

$$\mathbb{E}\left[\frac{1}{|\mathbb{E}[Y^2|X] - \mathbb{E}[Y^1|X]|^\eta}\right] < +\infty$$

Ce paramètre $\eta \in [0, 1)$ apparaît naturellement pour avoir une intégrale de Riemann convergente au voisinage de 0, mettant en évidence la singularité se produisant lorsque plusieurs éléments sont proche du maximum.
L'autre contribution de la seconde partie de ces travaux de thèse est d'appliquer la méthode MLMC dans le modèle ALM qui intègre les principales caractéristiques de l'assurance-vie. En particulier, la complexité de la gestion ALM (book values, mécanisme de participation aux bénéfices, taux minimum garanti...), implique une path-dependence rendant les méthodes de regressions particulièrement difficile à mettre en oeuvre (fléau de la dimension). L'estimateur MLMC évite la question de la regression et fournit un estimateur visant une précision ε avec une complexité en $O(\varepsilon^{-2})$, autrement dit aussi efficace asymptotiquement qu'un Monte-Carlo classique où l'espérance conditionelle serait connue par formule fermée.
Par ailleurs, nous comparons les performances de l'estimateur MLMC avec l'approche LSMC et réseaux de Neurones, et montrons que même si les réseaux de neurones permettent de diminuer l'erreur d'approximation observée en utilisant l'approche LSMC avec selection de features, l'approche par réseaux de neurones nécessite de stocker le jeu d'entraînement et d'entraîner le réseau (optimisation par descente de gradient) dont le temps de calcul est croissant avec le nombre de simulations primaires. L'approche MLMC présente le net avantage de ne pas stocker de données et d'éviter l'étape d'apprentissage. Ceci est particulièrement intéressant pour le contexte de la gestion ALM qui fait intervenir tout l'historique de gestion (book values, différentes réserves...). Enfin, la regression sur un faible nombre de variables explicatives induit automatiquement une perte d'information. L'estimateur MLMC en revanche convergera toujours asymptotiquement vers la vraie valeur.
Nous appliquons ensuite la méthode MLMC pour projeter les SCR futurs jusqu'à $t = 15$ ans dans nos applications numériques et nous analysons l'impact du changement de probabilité entre univers historique et risque-neutre sur chaque module des SCR futurs calculés en formule standard.
Nous terminons par une analyse sur le rôle des chocs de taux d'intérêt sur le SCR projeté sur plusieurs pas de temps. Nous montrons en particulier que le choc à la hausse des taux d'intérêts produit un effet "court terme" de baisse immédiate de la valeur de marché du portefeuille alors que sur le long terme, le réinvestissement du nominal à des taux de coupon plus élévés devient profitable pour la compagnie d'assurance. Le choc à la baisse des taux produit l'effet inverse, puisque l'assureur dispose de plus-value latente grâce au choc. Toutefois, il devra reinvestir ses obligations à un taux de rendement plus faible: C'est donc un "'effet délétère à long terme". Ainsi, lorsque le SCR est calculé à des dates t de plus en plus lointaines, l'effet "court terme" devient dominant pour expliquer les variations de valeur du SCR.

List of publications

Here is a list of articles (accepted or submitted) that were written during this thesis:

- [ACIA20a](with Aurélien Alfonsi and Jose Arturo Infante Acevedo) *A synthetic model for asset-liability management in life insurance, and analysis of the SCR with the standard formula* (published in European Actuarial Journal,(2020)).

- [ACIA20b] (with Aurélien Alfonsi and Jose Arturo Infante Acevedo) *Multilevel Monte-Carlo for computing the SCR with the standard formula and other stress tests* (submitted).

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CHAPTER 1

INTRODUCTION

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1.1 Overview of the insurance sector

Life insurance companies and pension funds are major institutional investors providing funding and financing the economy. In France and United Kingdom, policyholders' deposits on life insurance saving account exceed bank deposit (source: [Dam13]). According to the ACPR, *Autorité de contrôle prudentielle et de résolution* (the French supervision authority for banks and insurance companies), French life fund outstanding represent 1700 billion euros in 2018. Much of this funding (around 80%) goes to denominated saving contracts in euros (source : [Her19]). Insurance Firms use these funds to make large scale investment in the debt market (sovereign and corporate debt), stock market and real estate. At the end of 2014, insurance companies and pension funds accounted for 41% of the outstanding amount of euro area sovereign debt held by euro area resident (source :[DSS17]). Within the life insurance business, more than 80% of premiums come from traditional saving account, whose risk is born by the insurance company whereas the remainder come from unit-link securities which risk is born by policyholders (source: Berdin et al. [BKP16]). Hence, traditional savings still play a prominent role in the life insurance landscape and will be the primary focus of this thesis. Saving contracts are nonetheless, particularly complex products because of several embedded options (minimum guaranteed rate, profit-sharing participation,...) as well as legal constraints that make the valuation of these contracts particularly challenging. The great complexity of these products as well as the lack of convergence with banking regulation (Basel settlement) lead to adapt the prudential regulatory framework to take into account the risk profile of the company and protect the policyholder against the risk of an insurers' bankruptcy. The European Solvency regime for the insurance sector, Solvency II, while providing strong similarity with banking regulation (Basel II/III) with a pillar approach and risk-based capital differ from the banking prudential scheme since the primary focus is to protect the policyholders against an *isolated* bankruptcy of the insurance company, by enhancing financial stability of insurance companies with an incentive to harmonize insurance legislation across EU and increase competition and transparency [Hör13]. It does not make a particular emphasize on systemic risk and contagion effects, which became the primary focus of Basel III settlements in response to the financial crisis of 2007/2008 (see [GW12] for a comparative study of banking and insurance European regulation). Indeed, according to [LR04] , insurance companies are much less interdependent unlike the banking sector and its high liquidity need. Actually, banks and insurance companies have different economic model and plays a different function in the economy.

1.1.1 Distinction between Banks and Insurance

Insurance core business is to allow risk transfer from one party (the insured) to another (the insurer) and minimize those risks using mitigation techniques, spreading risk among individuals that face similar exposure. In a sense, insurers transform a risk from a single individual to one collectively supported by a large number of policyholders. The policyholder is prepared to pay the smaller premium in exchange for protection in case of unfavorable event. There is also additional functions for life insurers since they provide a long-life income stream which is at least partly guaranteed but also investment related solution (variable annuities) for those willing to accept a higher degree of risk. The business is characterized by a reverse production cycle since insurance companies receive payment for a service before having to provide it. In some sense, life insurance companies borrow money from policyholders' deposits and invest these amounts in long term assets. Usually, several years may pass before the insurance company pays a claim, since fiscal and tax regime make insurance policies advantageous after a certain time elapsed before the inception of the contract. Banks however differ substantially by the maturity of investment, their funding opportunities and risk profile (see [GW12] and [CHGJM15]). The core activity of the bank is to act as intermediary between savers, who need to deposit money in accessible account and borrowers seeking for loans. In a way, banks correct the mismatch between lender and borrowers by engaging in *maturity transformation*. The typology of risk between the banking and insurance and their role in the economy while bearing some similarities differ quite substantially and has shaped the post-crisis regulatory landscape of both sectors. Insurance companies are exposed to market-risk via the investment of policyholders deposit in the financial market but also non-financial risk (mortality, longevity risk,...). The main objective of the ALM risk management of the insurance is to handle the mismatch between assets and liabilities by ensuring that the values of assets that backed the technical provisions are fully synchronized. This is the reason why ALM focuses on duration and cash flow matching. Banks while still exposed to market risk (change in financial variable) are heavily exposed to credit-risk (the risk that a borrower do not repay its loan) and liquidity risk (the risk that an asset cannot be sold quickly). Assets of a bank consist mostly of long-term loans which cannot be transformed into cash instantaneously whereas most deposit can be withdrawn immediately, which exposes banks to liquidity risk. Hence banks needs to lend or borrow money from other bank via the interbank market which is at the heart of the interconnection in the banking sector and is the primary focus of Basel settlement. In contrast, the liquidity risk is not much a concern in the insurance sector because they have access to a stable flow of cash income (premiums, maturing assets and investment income). Moreover, the counterpart of the interbank market do not exist in the insurance sector which limits the contagion effect of liquidity issues of an insurance company to the whole financial system. Eventually, the role both sectors have in the economy is also quite different and shed some light to the paths that led to the Solvency II regulation that we will describe in the next section. Insurance firms, needs to invest the cash inflows of premiums to pay claims. They provide funding for government, business through investment in sovereign, corporate bonds and equity in financial markets. From a macroeconomic point of view, banks are the main channel for the transmission of the monetary policy of the central bank.

1.1.2 The Solvency II regulatory framework

Let us now describe, in more detail the regulatory framework specific to the insurance sector that constitute the ground basis of this thesis and motivates the need of advanced mathematical models used in risk management and ALM department of insurance companies. The Solvency II directive, that entered in force in January 2016 is the new supervisory framework for insurers and reinsurers in Europe. Its primary focus is to correct weaknesses of the previous solvency regime, among them the lack of risk sensitivity, lack of convergence with banking regulation (Basel settlement) and International standards (IASB/IFRS). It also aims at harmonizing the regulation among member states. The implementation of the reform was preceded by a series of Quantitative Impact Studies (QIS) carried out by the former insurance supervisor CEIOPS: Committee of European Insurance and Occupational Pensions Supervisors, now EIOPA: European Insurance and Occupational Pensions Authority) from 2005. The new regulatory framework puts demand on the economic capital (the Solvency Capital Requirement (SCR)) to ensure that the insurance company is able to meet its financial claim. One of the major change is the transition from a static-rule based capital requirement to a risk based capital charge. In the former Solvency regime, the basic formula for the required capital was a function of the premiums independently of the assets allocation: it targeted only the liability side of the balance sheet without taking into account the interaction between assets and liabilities. The directive share some similarity with the banking sector (Basel settlement) since it is based upon a 3 pillar approach. It is not coincidental, since the first stage of development of the SII directive took place during the period 2001 to 2003, coinciding roughly to the construction of the Basel II settlement in the banking sector. The first pillar tackle quantitative requirements and introduce two major innovations in the actuarial landscape : Market-consistent valuation of the insurer balance-sheet and risk-based capital requirements. There are two capital requirements the SCR is the key solvency control metric for the insurance company, the Minimal Capital Requirement is a lower requirement that trigger the supervisory intervention, if the capital fall below this level. The second pillar deals with qualitative requirements such as governance system covering Enterprise Risk Management, internal control and compliance. It also introduces as part of the overall internal Risk Management process the *overall solvency need* in an *Own Risk and Solvency Assessment* (ORSA) which complements qualitative requirements of Pillar 1 and is designed to provide the insurer and its stakeholder information on the risk they are exposed over a multi-year time frame horizon. More importantly it requires the computation of a multi-year solvency constraint (multi-year SCR) with respect to the firm's risk appetite (Vedani et al. [VD12]). The computational challenges introduced by the multi-year solvency constraint will be the main concern of this thesis. The third pillar cover supervisory reporting and disclosure (see [San16] for a detail study of the European Solvency System). For the computation of capital requirement, the regulator sets out two methods: a standard formula based on stress-tests on several risk modules (interest-rate risk, equity risk, spread risk...) and an internal model approach based on a quantile of the one-year loss distribution of the insurers' portfolio at a 99.5% confidence level). In the standard formula approach, the change in the own funds of the company after marginal shocks determine the necessary amount to hold the shock. Once the capital charge of each individual modules have been computed, they are combined into an overall SCR according to a specific aggregation formula (see [BSS15],[Bol]) for specific details on the standard Formula). The correlation parameters are provided by the supervision authority (EIOPA). The standard formula has been adopted by medium size player of the insurance sector despite its simplified assumption. A possible explanation lies in the difficult validation process of a full internal model approach that requires the approval of the regulator and the fact that the implementation of such model is costly and complex. In practice, insurance companies may adopt a partial internal model that substitutes the standard formula for the calculation of a particular risk module. Let us mention that risk modules for non-financial risks (mortality risk,...) do exist and enter in the computation of the Standard formula. In this thesis, we will only focus on the market-risk module.

1.1.3 An historical accounting scheme and a market consistent valuation of the balance sheet

Even though Solvency II aimed at harmonizing valuation methodologies accross EU, the plurality of local regulatory requirements and prudential schemes introduces significant differences between countries even though they basically sale the same products (see [RBB+18]) . In particular, it still remains continuation of different valuation techniques among member states namely book or market values accounting. In fact, insurance companies now have to deal with the simultaneous existence between two distinct valuation regime. The book value accounting scheme is based on historical costs. It is necessary to comply with local accounting benchmark called national *GAAP* (Generally Accepted Accounting Principles). In this regime, asset investments are recorded at their purchase price or amortized value for bond product. Even for solvency purposes historical costs accounting is necessary to compute gains or losses, monitor hidden losses (that enter in the profit distribution mechanism of saving participating policies for instance). The market-value accounting scheme tries to align the insurance balance-sheet item with the notion of fair-value. It introduces the risk-neutral valuation in actuarial models. The idea is the following: in order to assess the solvency situation, insurers needs to value their assets and liability based on objective datas. Each actor of the financial sector must have access to the same information which are provided by financial market data. For liquid asset, quoted market prices are available and the asset value of the company are provided by the market, they are "marked-to-market". However, insurance liability are not traded, and there is no liquid market for insurance liability. In that case prices are unknown. To value these technical reserve a "best estimate" based on the available information require to use a model. The modeled value is called "marked-to-model". The best estimate of liabilities (BEL) corresponds to the expected present value of future liability cash flows and is supposed to reflect the price of a third party that will be willing to pay to take over the insurer commitments on an arbitrage-free market. The conceptual background lies in arbitrage and option pricing theory. Because of embedded options (surrender option, minimum guaranteed rate) the liability portfolio can be valued as any other derivative security. In a complete market, any payoff can be perfectly hedge by a self-financing portfolio. The amount needed to initiate the hedging strategy is the market price of the liability claim payoff. Under the fundamental theorem of asset pricing, in an arbitrage-free market, it exists a unique probability measure, *the risk-neutral measure* under which discounted payoff and price processes are martingales.

1.1.4 Practical issues of market consistent valuation of assets and liability in insurance

The market consistent valuation of an insurer portfolio, taking into account every financial guarantee, discretionary rules, bonus mechanism and investment strategy is a complex task that cannot be performed by closed-form solutions and requires the use of Monte-Carlo simulations. Economic Scenario Generator (ESG) are used to produce simulations of financial variables (stock index, interest-rate, credit spreads,...). These scenarios are inputs for Asset and Liability Management Models (ALM) that are used to value the balance-sheet items of the company. The Solvency Capital assessment require the integrated use of both real-world scenarios and risk-neutral "pricing scenarios". Indeed, the SCR estimation consist of computing a risk measure on the insurers' portfolio loss over a given time horizon. Real world scenarios are used to forecasts an economic environment consistent with empirical facts observed in historical financial data. Risk-Neutral scenarios purpose is to provide an insurers' portfolio valuation consistent with the market price of derivative securities (pricing objective). Among practical difficulties introduced by the market consistency is the double constraint imposed on risk neutral ESG to be both calibrated to market prices at the evaluation date and to reproduce the term structure of interest-rate. More specifically, the computation of technical provisions require to discount cash flows over long maturities where no market data are available. The EIOPA provide a regulatory zero-coupon curve for a wide range of currency. The financial instruments selected to construct the risk free curve are interest-rate swaps and provided by Bloomberg. An interpolation method (currently Smith-Wilson) is used to project the risk-free curve after the Last-Liquid-Point-LLP, (i.e the last observable data point) to make the rate converge toward an "Ultimate-Forward-Rate" (UFR) which is an exogenous parameter set to deal with the lack of liquidity of swap rates for long maturities and the necessity for insurers to project cash flow over very long maturities (up to 150 years!). This parameter corresponds to the sum of the long-term averages of past real rates and the inflation target of the European Central Bank. In the Euro zone, it is assumed that the last liquid point is 20 years. Beyond, it is necessary to extrapolate to converge after 40 years to the UFR, so that, at the end of the convergence period (60 years), the one-year forward rate has converged to the UFR. On top of that several adjustments are considered when the risk-free curve is constructed, among them the Volatility adjustment (VA) and Credit Risk Adjustments (CRA). The VA is applied to mitigate the effect of short-term volatility of corporate bond spread on the insurer's economic own funds: the relative value of high quality corporate bonds can fall importantly compared with Government bond as investors demand more compensation for taking the liquidity risk (as in the 2008 crisis). Consequently, the insurance company may appear to have insufficient capital since the market value of its assets has decreased compared to its liability, but it might not be a problem since life insurance firms buy and hold bonds on long-term horizon, so loss of capital need not to be realized until bonds actually default. The CRA reflects the credit-risk contained in the swap-rate and act at a parallel downward shift of the market rates observed for maturities up to the last liquid point to make the curve "risk-free". In this thesis, we will not deal with this adjustment. We refer to [EIOb] for a detail description of EIOPA methodology to construct the regulatory yield-curve. The ability of the ESG model to satisfy the market-consistency constraint lead to an overcomplexification of financial models used in the Insurance sector (LMM++, G2++, Black-Karasinski...) leading to several calibration issues [VEKLP17]) and a growing literature on fair valuation of insurance contracts under Solvency II regime ([GCFG19],[AP19],[Var11]). More generally, the use of risk-neutral valuation and fair-value suffers from several criticism as pointed out by [VEKLP17] and [Thé16]. Notably, insurer uses economic valuation for solvency purposes and not for hedging. They compute capital requirement only one or two times per year. Moreover, they do not hedge their liability they mitigate their risk. In addition some risk factors are not financial risks (mortality, lapses...) and a large range of insurance risks cannot be fully hedged. Therefore, the choice made by the regulator to use the risk-neutral valuation for the SCR is questionable. We will use it however in this thesis. Nevertheless, since we are interested in by the calculation of the SCR at in the future at time $t = 1, 2, \ldots, T$ we will need to handle the portfolio under both historical and risk-neutral probabilities, respectively before and after time t.

1.1.5 The Solvency II balance sheet

We now introduce in more details a (simplified) market value balance sheet of an insurance company under the Solvency II valuation scheme. Figure 1.1 depicts the market-value balance sheet at time t. On the left-side MV_t correspond to the assets

Assets	Liabilities
MV_t	NAV_t
	RM_t
	BEL_t

Table 1.1 – SII Market-Value Balance-sheet

of the company: all income cashed in by the firm (policyholders premiums...) are invested in financial instruments (bonds, stock,real estate...) where *market values* (i.e quoted prices) are available. The insurer now owns the customers money which creates a liability for the company. SII regime assume that the company must be in *run off* situation: to estimate the overall debt of the company, no future new business is taken into account (for instance new policyholders entering the fund), the company stops selling new contracts and we derive the liability towards the current client in the insurers' portfolio.

Best Estimate of Liabilities

The largest liability item is the Best Estimate of Liabilities (BEL). It is computed as the present-value of future cash outflows of the company. Suppose that the company activities are monitored each year $t = 1, \ldots, T$ where T is the so-called *projection horizon*. For savings portfolio, T can be very large (up to 60 years). Denote $(L_t)_{t \leq T}$ the liability cash outflows that occurs at each year t (insurers claim payment...). Let $(\Omega, (\mathcal{F}_t)_{t \geq 0}, \mathbb{Q})$ a filtered probability space where $\mathbb{Q}$ is the risk-neutral pricing measure. Assuming that the company rely on a *market consistent* short-rate model $(r_t)_{t \geq 0}$ (i.e calibrated on both the EIOPA regulatory zero-coupon curve and market prices), the mathematical expression for the Best Estimate is

$$BEL_t = \mathbb{E}\left[\sum_{u=t}^{T} e^{-\int_t^u r_s ds} L_u \middle| \mathcal{F}_t\right] \tag{1.1.1}$$

Risk Margin

In technical document of the European Commission EIOPC/SEG/IM13/2010 (see also Möhr [Möh10]) the risk margin corresponds to the cost of providing an amount of eligible own funds to support the insurers obligations. More specifically, it represents the theoretical compensation for the cost of providing future regulatory requirements. From a shareholder standpoint, at $t = 0$, SCR_0 is the amount the shareholder needs to immobilize in order to pay the liability in unfavorable situations. At $t = 1$, the insurers pays the expected return γ and lends from him SCR_1 to continue to exercise its business. The procedure is repeated until the end of the business activity (time T). Denote $PV(x)$ the present-value of the future cash flow x, the mathematical expression for the risk-margin is given by:

$$RM = SCR_0 + \mathrm{PV}\left(\sum_{t=1}^{T} SCR_t - \gamma SCR_{t-1}\right) = (1-\gamma)PV\left(\sum_{t=0}^{T-1} SCR_t\right) \tag{1.1.2}$$

The factor $1-\gamma$ is called *Cost of Capital rate* (CoC) and represent the cost of locking future capital in a risk-free asset rather than just being able to invest it in any other asset classes. In Directive 2009/138/EC CoC is constant and fixed at 6%. As suggested in (Möhr [Möh10]) the present value is replaced by *the expected discounted costs of future required capital* $\mathbb{E}[e^{-\int_t^u r_s ds} SCR_u|\mathcal{F}_t]$ which leads to the following expression

$$RM_t = CoC \times \mathbb{E}\left[\sum_{u=t}^{T} e^{-\int_u^t r_u du} SCR_u \middle| \mathcal{F}_t\right] \tag{1.1.3}$$

where SCR_t denote the capital requirement for the period $[t, t+1)$ The projection of future SCR is particularly challenging and even the computation of SCR_0 is demanding since it involves nested simulation. The computation of future SCR will be the main focus of the thesis. Today, the projection of future SCR is an open problem in term of computation time. Currently, the supervisor allow simplifications and assume that future SCR are proportional to future Best Estimate. This assumption means basically that the amount to hold to be solvent is a fraction of the insurer's best estimate of its current debt/commitments towards its customers.

Net Asset Value

The sum $RM + BEL$ correspond to the overall insurers debt and is usually called *Technical Provisions*. The difference

$$NAV_t = MV_t - (RM_t + BEL_t) \tag{1.1.4}$$

correspond to the own funds of the company. The company is declared insolvent if NAV_t becomes negative as the value of its assets are lower than its technical provisions. A comprehensive introduction to the SII balance-sheet is provided in [Bol].

1.1.6 Standard Techniques to monitor Asset and Liabilities interactions

According to [EIOc],[BGK17], a representative European life insurer typically sells traditional savings contract and invest the main part of policyholders' deposits in debt and other fixed income securities (91%), 56,7% of which is invested into sovereign bonds. Hence, the main financial risk is the interest-rate risk. Only 9% of the portfolio is invested in equity product (stocks and real estate). The following figure provide the balance-sheet structure of a representative European life insurer based on [EIOc].

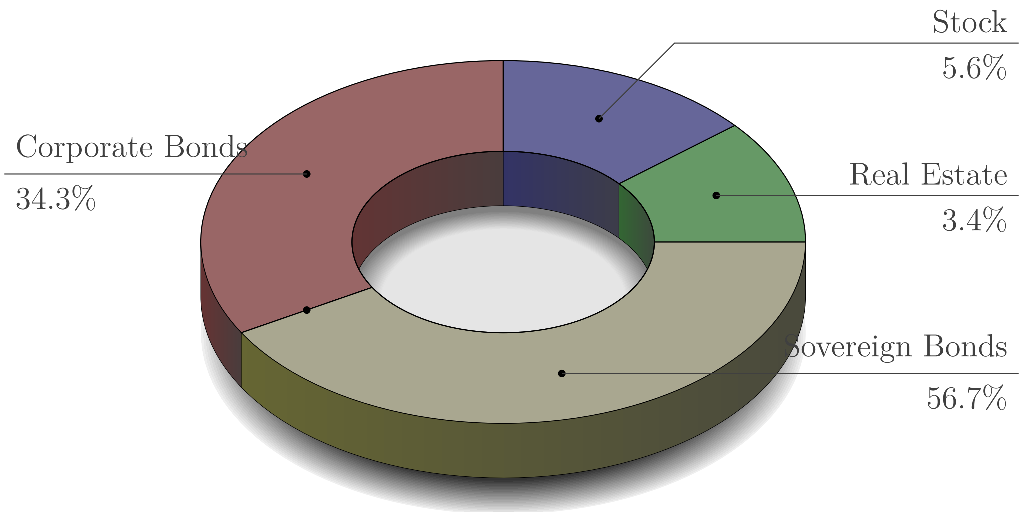

Figure 1.1 – Asset allocation of a representative European Life insurer source: EIOPA [EIOc] and Berdin et al. [BGK17]

Given the fact that bond products dominate the overall asset allocation of a life insurer, asset and liability management techniques (ALM) are important tools to manage the interest rate risk and to monitor the liquidity risk stemming from mismatches between cash inflows (coupon payments, rent, dividend...) and outflows (claim payment). The concept of duration, convexity and immunization are the fundamental instruments in ALM. Immunization methods and gap analysis aims at controlling the cash flows adjustment. Duration and convexity indicator measure how change in interest-rates affects the market values of assets and liabilities. For a first introduction of ALM techniques in the actuarial literature we refer to [Fer83].

1.1.7 Literature review on ALM model

As the interest rate risk represents one of the main risk in the ALM, much of the existing literature focused on interest rate sensitivity of insurance product and their impact on the additional distribution mechanism. Brys and De Varenne [BdV97] construct an ALM framework in which policyholders receive a minimum guaranteed rate plus a participating bonus on top of that, computed as a fraction of excess return generated by the portfolio. The model is then used to analyze the risk exposure of the company with respect to the duration of the liability. Grossen and Jørgensen [GL00], Milterssen et al. [MP03] further investigates the bonus declaration scheme by introducing a policy reserve (similar to the mathematical reserve) and a bonus account which serves as a buffer to smooth future profits. The so-called *average interest principle* (see also

Zaglauer et al. [ZB08]) aims at building up reserves in years of good returns while providing stable crediting rate to policyholders in bad years. Kling et al. [KRR07] analyze also different bonus allocation scheme to reduce the the shortfall risk of the insurance company.

Surrender and mortality effects are analyzed in Albizzati and Geman [AG94] and Gatzert [Gat08]. The former proposes a model framework to assess how changes in interest rate poses a major threat for insurers that provide a surrender option in their participating policy. The latter shows also that early death or surrenders are significant drivers of an insurers default probability. These frameworks examine separately some particular features of life insurance contracts but not the joint effect of contract parameters. Bauer et al. [BKKR06] investigate the fair valuation of participating contracts under German regulatory regime and integrate in the modeling some important accounting rules for building or liquidating legal reserves. They show that this mechanism has a significant impact on the fair value of the contract. They include management decisions under legal constraints and provide sensitivity analysis of model parameters. Fair valuation of insurance contracts are also the main focus of the framework developed by Ballota et al. [BHW06], Bacinello [Bac01], Tanskanen et al. [TL03]. Schmeisser and Wagner [SW12] proposes a model to evaluate the impact of the asset allocation and interest rate level on the solvency situation. They show that when bond portfolio returns gets closer to the guaranteed rate, the own-funds of the company approaches zero which threaten the solvency situation. Simple ALM models are also described in Bauer et al. [BRS12] and Floryszczak et al. [FCM16]. These are very simplified model used as benchmark to investigate the computational challenges of Nested simulation in the SII regime.

While most of the literature focuses on single policy contracts, Gertzner & al [GGH+08] provide a full balance sheet model incorporating a pool of non-homogeneous contracts. They construct a general setting incorporating a bonus allocation scheme, dynamic allocation and stochastic market model. They study the behavior of balance sheet items w.r.t mortality and surrender effects associated to different pool of contracts called *model points*. Berdin and Gründl [BG15] provides a more realistic ALM framework by incorporating a distinction between book and market values. They also take into account an existing stock of saving products called *legacy business* with different minimum guaranteed rate. They worked under the accounting scheme in force in Germany and investigate in two series of papers ([BG15] and [KBG19]) the impact of low (resp. high) interest rates on the insurers' solvency.

1.2 A synthetic model for the ALM

We now present the contribution of this thesis. In the literature, general frameworks for the ALM of life insurance product are generally oversimplify. Most of the time, numerical studies are performed under proprietary "black box" model where implementation details on the management procedure are not communicated. The main contribution of chapter 2 is to propose a realistic intermediate model in the sense that it comprises the main characteristics of the ALM (book and market value regime, dynamic surrender...) while keeping the model tractable for Monte Carlo simulations to perform ALM studies and be used as benchmark. The purpose of a so-called ALM study is to handle the risk coming from the mismatch between a company's asset and liability. The new standards imposed by Solvency II have further increased the complexity of those model and the need for stochastic simulations since each item of the balance sheet must be valued in a market consistent manner. The fair valuation of life insurance liability is not the only issue when computing the SCR. It has already been studied in many papers ([Bac01] [GCFG19] [BMDGL18]). In any ALM study, the firm is interested in optimally investing the policyholders' deposit in order to maximize the expected return on investment (asset management) while minimizing the SCR or at least ensure that the required capital of the selected strategy is not too high (liability management: the SCR is the regulator's metric to ensure that the obligations toward policyholders are met when using the allocation strategy). In the first part of this thesis (chapter 2) we develop a new model that both comply with the market consistent valuation of the balance sheet and book value accounting. This important distinction between market value and book value (namely selling price and purchase price) is necessary because it enter in the profit sharing mechanism when computing realized gains or losses made each year by the company. While practitioners register their gains or losses based on granular approaches like FIFO ("First In First out") accounting rules, the complexity involved by these methodologies introduce major computational overhead since one must record the whole history of trading. We adopted a macroscopic scale approach where individual contracts are not modeled individually but pooled into a group of contracts with similar features. Portfolio allocation are made on two synthetic assets (a stock index and an equally weighted basket of sovereign bonds). This macroscopic point of view enable us to reduce the computational complexity while keeping track of both book and market value buy/sell order and still maintain the model with a limited number of variables. In the literature of ALM models, one of the first model that goes beyond fair valuation and single policy analysis of insurance contract is proposed by Gertzner & al [GGH+08]. They investigate the effect of pooling non-homogeneous contract and defined management rule regarding the asset allocation and shareholders' participation. Their approach uses the crediting rate proposed by Grossen and Jorgensen [GL00] where the crediting rate is the maximum between the guaranteed rate and the profit sharing rate. However it does not take into account book value accounting scheme nor cash flow matching techniques. The investment is made on single coupon bearing bonds. Berdin and Gründl [BG15] filled the gap in the literature by modeling the balance sheet of a representative life insurer subject to the German GAAP. The calibration of the bond portfolio is obtained using historical duration data provided by EIOPA. Another original contribution brings on the computation of the crediting rate which is closer to practice. To compute this key rate, the company takes into account the realized gains or losses made during the reallocation procedure. A competitor rate that model the rate given by competing insurance companies is also considered. Then, the company tries to drive its latent gains or losses resulting from the difference between market and book value in order to reach a target rate defined as the maximum between the competing rate and the legal minimal profit-sharing distribution. If the target rate is not feasible, it tries to give the best rate possible while keeping a part of the profit sharing reserve but in any case the crediting rate is above the minimal regulatory rate. In particular, the model provides great insight on the impact of the management rule on the solvency situation of the company because if it credit a certain rate to a policyholder, we can identify in which of the four cases we are. To account for dynamic lapses, the proportion of policyholders that exit the contract is a function of the difference between the crediting rate computed using the methodology described above and the competing rate. Such an approach can be found in Floryszczak et al. [FCM16]. The last original feature of the ALM model developed in this manuscript is to consider an equally weighted portfolio of bonds with maturity ranging from 1 to n years. The dynamic of coupon rate after bond investment/divestment is also precisely described. Our method enable us to quantitatively assess the effect of shocks on the yield curve on bond returns when computing the SCR standard formula. In addition, the modeling of a basket of bonds instead of one synthetic bond obligation enable us to reproduce a cash flow matching strategy used by practitioners, where the nominal value coming from the matured 1 year bond covers the liability cash flows from surrenders and hedges at least partially the interest-rate risk statically. We illustrated this key feature of ALM using a proxy model when there is no cash flow matching and the firm invest in a single at-par coupon bearing-bond. Contrary to the work of Berdin and Gründl [BG15], we are able to calibrate the bond portfolio optimally, without historical duration data, in order to minimize the SCR. We compared our approach with Macaulay duration hedging methods, and assess the impact of mortality on cash flow matching using mortality tables. The ALM model is then used to compute the SCR using the standard formula. The result of the simulations performed in the second chapter of this thesis points out some weaknesses of the standard formula and issues related to the choice of the interest model which has not yet been tackled in the existing literature. Firstly our findings suggest that models that mean revert toward a parametric curve (Hull-White, Black-Karasinsky...) which are still very popular among practitioners are not well suited for the standard formula. The calibration of such interest rate model using the shocks provided by EIOPA leads to poorly realistic model after shock. We also discovered that the dependency between stocks and bonds introduced by the standard formula has a significant impact on the SCR for market-risk SCR_{mkt}. We observed an important variation of 50% of SCR_{mkt} half of which is contained in the ε-discontinuity in the aggregation formula

$$SCR_{mkt} = \sqrt{SCR_{eq}^2 + SCR_{int}^2 + 2\varepsilon SCR_{eq} SCR_{int}} \tag{1.2.1}$$

where the "correlation factor" $\varepsilon = 0$ if the interest-rate exposure is due to an upward shock and $\varepsilon = \frac{1}{2}$ if it is due to the downward shock. Such a discontinuity might lead to manipulation of the SCR on the edge of that discontinuity. Our suggestion for regulators is to use a continuous formula such as

$$SCR_{mkt} = \max\{\sqrt{SCR_{eq}^2 + SCR_{down}^2 + SCR_{eq} SCR_{down}}; \sqrt{SCR_{eq}^2 + SCR_{up}^2}\} \tag{1.2.2}$$

We also find out that the main driver in the standard formula is the initial allocation since it relies on shocks at time 0. The pitfall is that standard formula mostly ignores the dynamic features of ALM strategy and basically depends on the initial allocation. Namely, a static allocation strategy in stocks starting from an initial allocation w_0^s will lead to a SCR that is very close to a dynamic strategy that starts from w_0^s but gradually increases its allocation in stock. This suggests that the standard formula approach poorly reflect the risk profile of the company. Another pitfall of the standard formula is that it relies only on mean values, and does not reward if the Net Asset Values distribution is more peaked and has less variance. In the third chapter of this manuscript, we further investigate the properties of the ALM model by quantifying the impact of regulatory decision on the insurer's balance-sheet. First, we show that the model is flexible enough to take into account both methodology of computation of the SCR. Using the framework of Bauer et al. [BRS12], we derive the one-year loss distribution of the insurance company using the ALM model and compare the solvency ratios obtained with the standard formula and the quantile approach. Next, motivated by the recent consultation paper [EIOa] launch on the 2020 review of SII, we show that the ALM model can easily deal with potential changes in the regulation concerning the derivation of the interest-rate risk free curve. Currently, the EIOPA's construction of this curve that is given as an input of ALM models rely on the Smith-Wilson extrapolation from the Last-Liquid-Point (LLP) toward the Ultimate-Forward-Rate (UFR). Using the model, we discuss the impact of regulatory changes in the LLP and the UFR on the insurers' balance-sheet and the SCR.

1.3 Numerical methods for the computation of the SCR

For insurance companies the Solvency II directive introduce major computational challenges. First, Pillar I requires a computation of the VaR at a one-year risk horizon. Second, the ORSA framework require to own enough own-funds today to avoid bankruptcy over a whole time-horizon (multi-year solvency). Finally, the derivation of the risk-margin RM_t involves the computation of SCR during the whole lifetime of the insurer portfolio. In a more general setting, the problem is to compute the probability of a large loss of a financial portfolio. This task is particularly challenging in practice as complex financial portfolios do not admit a closed-form solution and the valuation require heavy Monte-Carlo simulations. More formally, this problem, involving simulations in the simulations can be framed in the so called Nested valuation setting where outer scenarios under real-world probability are used to project the risk-factors up to the risk-horizon, then, inner simulations are required, corresponding to the valuation of the portfolio, conditionally on each scenario. This brute-force task is too time-consuming to be useful and relevant for practical applications. In addition, this method is poorly accurate if the number of nested simulation is too small. The nested simulation literature is divided between two approaches. The first approach focuses on the allocation of a given computational budget. The pioneer work of Gordy and Juneja [GJ10], reveals the optimal balance between outer simulations and inner simulations to obtain accurate estimate of the tail of the target distribution. Based on this work, some authors propose to improve the direct nested simulation techniques in the context of VaR estimates. Broadie et al. [BDM11] show that ingenious allocation of relatively small computational budget can yield acceptable level of variance and bias for a given portfolio because some scenarios may have no direct impact on the final estimator. Their idea is to allocate more resources to scenarios that have high variance and a high probability of misclassifications (in the neighborhood of the Var threshold). In Devineau et al. [DL09] they construct a metric to identify adverse scenario and propose to allocate more computational effort on these scenarios. Then the algorithm select less and less adverse scenarios until no improvement of the VaR estimates has been made. Alternative techniques such has the Multilevel Monte-Carlo (MLMC) method developed by Giles [Gil08] has been applied in the context of nested expectation and tail estimation. This method is applied in a biased simulation framework, and propose to allocate resources, among several levels in order to minimize the MSE of the final estimator. This method based on telescopic aims at killing the statistical error on the first level and correct the inner bias across the different levels (see Giles et al. [GHA19], Pages et al. [LP17] for successful application in the context of Nested VaR), Bujok and al [BHR15] for application in basket credit derivatives and Bourgey et al. [BDMGZ20] for initial margin computation. The second approach tackled in the literature is to construct a so called *metamodel* based on few inner simulations. The popular approach among practitioners, based on past researches on American Option Valuations (see Longstaff et al. [LS01], Tsitsiklis et al. [TVR01]) is to regress future cash flows of the portfolio on a set of basis function that depend on state variable known at risk horizon. The Least-Square Monte-Carlo approach (LSMC) advantage is to use only few inner simulations to valuate the portfolio. It combines Monte-Carlo methods with regression techniques to approximate the pricing function. Hence, by bypassing Nested simulations, the method has potential to significantly reduce the computational time to produce the required estimates. However, this claim hold mostly in low dimension. A variant of the standard LSMC method that is widely use in practice is the so-called replicating portfolio (see Natolski et al. [NW18], Pellser and al. [PS16], Cambou et al. [CF18]) technique where the regression basis consist of derivatives with known closed form expression. The idea is to replicate the portfolio (unknown conditional expectation) with a combination of simple derivative product. The allocation weights of the derivative portfolio are obtained by solving a least-square optimization problem (see Pellser et al. [PS16] for a distinction between standard LSMC (regress-now) and portfolio replication (regress-later)). However, these approaches may lead to unsatisfactory results. First, the calibration process of the LSMC method requires proper selection of interaction between features as the number of regressors explodes with the dimensionality of the problem. Hence a careful design of the basis set of functions, well adapted to the problem at hand is require. Furthermore, the tail estimation, which is the primary focus in risk-management is very sensitive to the choice of the basis model (see Teuguia et al. [TRP14]). For instance, a model can predict well in the neighborhood of points that was used to train the algorithm but perform poorly on extreme scenarios. Recent advances in data science have shown that deep learning methods can accurately represent even highly non-linear high dimensional function [GBCB16]. The theoretical ground comes from the universal approximation theorem and the Kolmogorov-Arnold representation theorem (see Kolmogorov [Tik91], Cybenko [Cyb89], Hornik [Hor91]). Neural Networks have been successful to extract features and detect relevant patterns from large datasets. The popularity of this method compared to linear approximation models is that universal approximation is possible without specifying a particular functional relationship between inputs and outputs. The development of massive parallel computing on Gpus to speed up training of the Network made the methodology very popular among practitioners. More recently, Neural Network based algorithms have been successfully applied in financial application. Hejazi and Jackson [HJ16], Fiore and al. [CFM+18] train neural networks to compute solvency capital requirements. More recently Cheredito et al. [CEW20] employed importance sampling techniques in combination with Neural Networks to compute risk capital.

1.3.1 Nested Simulation Framework

The general problem is to estimate a risk measure of a financial portfolio at some future date τ called risk-horizon. We consider the general setting of Broadie et al. [BDM11], Bauer and Ha[BH15]. Let V_0 the current value of the portfolio (or the current level of own-funds NAV_0 in the insurance framework). The value of the portfolio at time τ

can be expressed, under no arbitrage assumption as a conditional expectation of future discounted cash flows. Let $(\Omega, \mathcal{F}, \mathcal{F}_{t\geq 0}, \mathbb{P})$ a complete filtered probability space where Ω is the space of all possible market state and $\mathbb{P}$ the historical real-world probability. Let $(X_t)_{t\geq 0} = (X_t^1, \dots, X_t^d)_{t\geq 0}, \quad d \in \mathbb{N}$, a $d-$dimensional Markov process that model the underlying risk factors of the portfolio. We assume that the filtration $\mathcal{F}_t$ that represent all relevant market information up to time t is generated by the *state process* $(X_t)_{t\geq 0}$, hence $\mathcal{F}_t = \sigma((X_u)_{u\leq t})$. We assume that it exists a risk-neutral probability $\mathbb{Q}$ under which discounted price processes are martingales. Let $D(t,u) = e^{-\int_t^u r_s ds}, \quad t \leq u$ the discount factor process where $r_s = f(X_s)_{s\geq t}$ the instantaneous interest rate. Denote Z the one-dimensional random variable that model the sum of future discounted cash flows. Hence the value of the portfolio at time τ can be express as :

$$V_\tau = \mathbb{E}^{\mathbb{Q}}\left[Z|\mathcal{F}_\tau\right] \tag{1.3.1}$$

Hence, the loss variable at time τ can be express as the change in the portfolios market-value between 0 and τ:

$$L_\tau = V_0 - V_\tau = \mathbb{E}^{\mathbb{Q}}\left[V_0 - Z|\mathcal{F}_\tau\right] \tag{1.3.2}$$

In what follows, we denote $Y = V_0 - Z$ the discounted sum of losses until the maturity of the portfolio. The goal is to compute risk-measures on the loss random variable L_τ, that quantify its risk by a scalar value $\rho(L) \in \mathbb{R}$. In a more general setting we are interesting in computing nested expectations of the form :

$$I = \mathbb{E}^{\mathbb{P}}\left[g(\mathbb{E}^{\mathbb{Q}}\left[Y|\mathcal{F}_\tau\right])\right] \tag{1.3.3}$$

For tail estimation taking $g(u) = \mathbb{1}_{u\geq 0}$ yield the following relation:

$$I = \mathbb{P}\left(\mathbb{E}^{\mathbb{Q}}\left[Y|\mathcal{F}_\tau\right]) > 0\right) = \mathbb{P}(L_\tau \geq 0) \tag{1.3.4}$$

1.3.2 Nested Monte-Carlo Estimator

The basic estimator of the Nested expectation (1.3.3) is based on approximating the inner and outer expectation using independent Monte-Carlo samples. The conditional inner expectation $E\left[Y|\mathcal{F}_\tau\right]$ (the portfolio value) is estimated for a given $(x_0, \dots, x_\tau) \in (\mathbb{R}^d)^\tau$ by a standard Monte-Carlo estimator with K simulations called *inner scenarios*

$$\widehat{E}_{j,K} = \frac{1}{K}\sum_{k=1}^{K} Y_{j,k} \tag{1.3.5}$$

where $(Y_{j,1}, \dots, Y_{j,K})$ are i.i.d sample of the conditional law of Y given that $(X_0^j, \dots, X_\tau^j) = (x_0^j, \dots, x_\tau^j)$. The outer expectation is then approximated using the standard Monte-Carlo estimator, *outer simulations* of the risk-factors $(X_0^j, \dots, X_\tau^j)_{j=1,\dots,J}$:

$$\widehat{I}_{J,K} = \frac{1}{J}\sum_{j=1}^{J} g(\widehat{E}_{j,K}) \tag{1.3.6}$$

1.3.3 Real world vs Risk neutral probability measure in the NS setting

We recall that in order to assess the solvency situation, the risk manager is interested in deriving the loss distribution of its financial portfolio over a given time horizon based on historical financial market data. To be more specific, the primary simulations of risk-factors are designed to reflect the most probable evolution of financial markets, hence outer scenarios models the "real" evolution of the risk-factors $X \in \mathbb{R}^d$ over a given time-horizon. The risk-measure (outer expectation in 1.3.4) aims at identifying the least likely scenarios that poses a threat for the solvency of the firm. The second stage of simulation (inner scenarios) aims at re-priced the portfolio of assets and liability (i.e estimating the conditional expectation) conditionally of the primary projected scenario. Mathematically speaking, it implies that outer simulations are necessarily performed under the real-world probability measure $\mathbb{P}$, while inner (pricing) scenarios are used to value the portfolio conditionally on the state of the economy X_τ (the *risk-factors*). Hence, in the insurance setting, inner simulations must be performed under the risk-neutral measure $\mathbb{Q}$ in order to value each item of the balance-sheet on a market-basis as stated by SII.

1.3.4 Machine Learning Regression based Methods

The computational burden of Nested Monte-Carlo methods for risk-capital computation has led to an investigation of alternative methods. Among popular approaches to this problem are Least-Square Monte Carlo (LSMC) and replicating portfolio methods. These family of methods take portfolio valuation (LSMC) or cash flow payoff (Replicating portfolio) as input of regression methods whose aim is to learn the conditional expectation. These methods are classified in two groups: "regress-now" or "regress-later" strategies. Regress-Now LSMC method was originally introduce by Longstaff-Schwartz [LS01] and Tsitsiklis Van Roy [TVR01] in the context of American option pricing and becomes the market standard in insurance solvency assessment involving Nested simulations (see [TRP14]). In the LSMC approach, the idea is to approximate the conditional expectation function via ordinary least-square regression based on one inner simulation called fitting scenarios (see Krah et al. [KNK20] and [KNK18]). The mathematical foundation is related to the fact that the conditional expectation is a L^2 projection for L^2 random variables and solves the least-square problem:

$$\mathbb{E}\left[(Y - \mathbb{E}[Y|\mathcal{F}_\tau])^2\right] = \min_{\phi:\text{measurable}} \mathbb{E}\left[(Y - \phi(X_0, \dots, X_\tau))^2\right]$$

Hence, a natural proxy of the conditional expectation is to find ϕ that minimizes the following empirical least-square criteria

$$\frac{1}{J}\sum_{j=1}^{J}\left(Y^j - \phi(X_0^j, \dots, X_\tau^j)\right)^2$$

The purpose of the training phase is to find the function ϕ. The replicating portfolio regress-later approach focus instead on replicating *liability cash flow payoff*. This approach has been popularized among practitioners by the market consistent vision of solvency II, where the idea is to replicate the optionality of the insurance contract using vanilla derivative that can be computed efficiently. To replicate the liability cash flow, a set of financial securities as basis functions is used instead. The conditional expectation is finally derived by applying the conditional expectation operator to each element of the basis decomposition, which require that the price of each financial instrument can be computed accurately very fast. For practical application in the life insurance sector, we refer to Devineau and Chauvigny [DC10]. A general mathematical framework for Replicating Portfolio (RP) is provided in Natolski et al. [NW18], Pellser et al. [PS16], Cambou et al. [CF18]. The distinction between the LSMC (regress-now) strategy is introduced in Glasserman and Yu [GY04] and further investigated in the insurance setting by Pelsser et al. [PS16] in the insurance setting. Among the main result in favor of the regress-later strategy is a theoretical acceleration rate of convergence compared to the LSMC strategy, as described in Pellser et al. [PS16] the projection error can be eliminated in the Replicating Portfolio approach but not in LSMC. However, finding a suitable products to replicate the payoff is by no mean an easy task, and any replicating portfolio methods will require expert judgment at some point. In addition, financial instruments do not form a structured basis of a meaningful functional space, just like orthonormal polynomial, hence feature selection involves greater computational overhead in comparison with LSMC, if we want to increase the complexity of the model. Hence, a direct application of the method for the ALM is not obvious.

Difference between LSMC for option pricing and LSMC in Solvency II

The first relevant difference between option pricing and insurance solvency assessment is that contrary to the initial application of the method, there is no early exercise strategy, hence SCR computations involves only one level of nested simulations which permit methods based on optimal budget allocation between inner and outer scenarios to be viable alternative, while completely unfeasible in the option setting (multiple inner simulations would be too costly because of the early exercise feature). Secondly, the change of probability between inner and outer scenarios influences the rate of convergence of the LSMC strategy. Thirdly, the insurance company that computes its SCR is interested in the assessment of the full loss distribution: in internal model tail estimation (SCR Value-At-Risk) is the quantity of interest or the average loss after a shock (Standard formula in the ORSA framework). In the option setting the only concern is to derive the price (average) of the derivatives . More generally, the problem considered in risk management is different in the sense that the risk-manager (insurer) is not interested only in the mean of the conditional distribution but also in the tail, but in both cases the unknown conditional expectation causes the main difficulty. Finally, the main practical difference between the option setting and the insurance setting is that the dimensionality of the problem is a major concern in SCR computation and can be significantly higher, since ALM models are path dependent. Therefore, an insurer regression model can take as input financial variables (stock, interest-rate, book values...) but also non financial ones (mortality rates, level of technical reserve, its crediting rate to policyholders...) which make the problem of exploding number of basis variable and overfitting a serious source of concern. These difficulties have played a leading role in the development of Neural Network algorithms as viable substitute to Ordinary Least-Square (OLS) based strategies in the insurance setting (see Hejazi and Jackson [HJ16], Kopczyk [Kop18], Fiore & al [CFM$^+$18], Cheredito & al [CEW20]).

1.3.5 Allocation Strategies Based on fixed Computational Budget

Now, let us describe the other main family of numerical methods to reduce the complexity of Nested Monte-Carlo. Allocation strategies analyze how a fixed computational budget can be allocated across both inner and outer scenarios to minimizes the *Mean Square Error* (MSE) of the output estimator. Among the first investigation of asymptotic properties of Nested Estimators Lee [Lee98] (1998), Lee and Gleen [LG03] (2003) and Gordy et al. [GJ10], Hong and Juneja [HJ09] analyzed uniform sampling strategies where the computational effort is spread among a constant number of primary scenarios and secondary simulations. Gordy and Juneja [GJ10] were able to assess the asymptotic complexity of the Nested Estimator when $g = \mathbb{1}_{[c,+\infty)}$ and characterize the optimal allocation between inner and outer scenarios. For a given computational budget, one look for minimizing the overall MSE of the estimator 4.2.2. One uses the bias-variance decomposition :

$$\mathrm{MSE}(\widehat{I}_{J,K}) = \mathbb{E}\left[|\widehat{I}_{J,K} - \alpha|^2\right] = \underbrace{\mathbb{E}\left[|\widehat{I}_{J,K} - \mathbb{E}[\widehat{I}_{J,K}]|^2\right]}_{\mathrm{Var}(\widehat{I}_{J,K})} + \underbrace{\left(\mathbb{E}\left[\widehat{I}_{J,K} - I\right]\right)^2}_{\mathrm{bias}^2(\widehat{I}_{J,K})} \tag{1.3.7}$$

The number of inner simulations K controls the level of bias. Allocating more inner simulations will reduce this bias. The number of outer scenarios J controls the level of variance. Based on asymptotic characterization of bias and variance, they proved the existence of an asymptotic optimal allocation $(J^\star, K^\star)$ that minimizes the MSE of the uniform Nested estimator. There result implies that to get a Root Mean Square Error (RMSE) of $O(\varepsilon)$ we require $J^\star = O(\varepsilon^{-2})$ outer scenarios and $K^\star = O(\varepsilon^{-1})$ inner samples leading to a overall complexity of $O(\varepsilon^{-3})$ which is a computationally intensive task and leaves the method irrelevant for practical applications.

Adaptive and Sequential allocation Strategies

For risk measure focusing on tail losses such as the VaR (i.e the SCR in an internal model), uniform sampling methods cannot be efficient since the most relevant scenarios concentrate on the tail of the distribution. Broadie and Moallemi [BDM11] assume that knowing a certain number of primary outer and secondary inner scenarios have been performed, more computational budget must be dedicated to primary scenarios falling on the tail since these outcome have a higher probability to affect the final estimator. The criterion they derived allocate more resources to scenarios that lies close to the threshold c (where the probability of miss-classification is high), with a high variance $\widehat{\sigma}_i$ and a low number of K_i of inner simulations. Among other Sequential Algorithms, the NS accelerator proposed by Devineau and Loisel [DL09] focuses on insurance application and the computation of the SCR for an internal model based on the VaR. Their method consists in locating *a priori* the most adverse scenarios (the situation here is different compared to the sampling of Broadie and Moallemie since no inner simulation are initially performed). The method introduce an *execution region* to locate the *a priori* extreme scenarios. The algorithm start from the worst cases (largest norm) and select an initial level h_0 such that exactly M_0 points lies in the execution region F_{h_0}. This is easily obtain using a root-search procedure. Then additional inner simulations are performed for these particular scenarios. Then "less adverse scenarios" are selected such that M_0 new points are added. The procedure stopped when no improvement of the empirical quantile have been made (i.e we stop if the quantile estimate based on a sample of size M_0 and the quantile based on $2M_0$ sample points are identical).

Multilevel Monte Carlo Methods (MLMC)

In order to reduce the computational cost in $O(\varepsilon^{-3})$ of the uniform Nested Estimator 4.2.2, Multilevel Monte-Carlo methods (MLMC) have been successfully applied to compute general nested expectations of the form 1.3.3. Before introducing MLMC techniques for Nested expectations, we give an overview of MLMC methods. MLMC algorithms was introduced by Giles [Gil08] to reduce the cost of Monte-Carlo methods in a biased setting. The general task at hand is to simulate $I = \mathbb{E}[P]$ where P cannot be sampled exactly. The most common financial application correspond to option pricing where $P = \phi\left((X_t)_{t\leq T}\right)$ is the payoff of some diffusion process $(X_t)_{t\geq 0}$ that rely on biased discretization schemes. In the insurance setting and more generally for risk management applications, the bias comes from the conditional expectation $\mathbb{E}[Y|X]$, i.e the portfolio value that require additional (inner) simulations conditionally on the realization of the risk-factor vector $X \in \mathbb{R}^d$ since no closed form solution is available. In this situation, $P = g\left(\mathbb{E}[Y|X]\right)$. Consequently, contrary to the standard (crude) Monte-Carlo framework, any estimator of I carries an inner bias. The MLMC techniques works as follows. Let $P_0, \ldots, P_L$ be a sequence of random variables which approximate P with increasing accuracy but also increasing cost. The most accurate estimator of P is at the finer (deepest) level L and we want to find an estimator such that :

$$\mathbb{E}[P] \approx \mathbb{E}[P_L]$$

The new target becomes

$$\widehat{I}_L = \mathbb{E}[P_L] \tag{1.3.8}$$

The error that comes from the approximation of P by P_L is the bias (also called *weak error*). The key idea of MLMC methods is that instead of estimating $\mathbb{E}[P_L]$ directly, it can be expanded into a telescopic sum

$$\mathbb{E}[P_L] = \mathbb{E}[P_0] + \sum_{l=1}^{L} \mathbb{E}[P_l - P_{l-1}]$$

We can then approximate each expectation by a standard Monte Carlo procedure to obtain the MLMC estimator

$$\widehat{I}^{MLMC}_{L,(J_l)_{l=0,\ldots,L}} = \frac{1}{J_0}\sum_{i_0=1}^{J_0} P_0^{(i_0)} + \sum_{l=1}^{L}\frac{1}{J_l}\sum_{i_l=1}^{J_l} P_l^{(i_l)} - P_{l-1}^{(i_l)} \tag{1.3.9}$$

One key point to note is the variable number of simulation J_l on each level. At the lowest level $l = 0$ the term $\frac{1}{J_0}\sum_{i_0=1}^{J_0} P_0^{(i_0)}$ is doing most of the job in estimating I and is used to kill the statistical error (also called *strong error*). More specifically, J_l decreases as l increases, and most of the samples are allocated on smaller level l where the computational cost of P_l is cheaper (recall that the cost of estimating P increase when l increase). The terms $P_l - P_{l-1}$ aims at correcting the bias introduce by replacing P by P_L. The important remark to make is that $P_l - P_{l-1}$ is small when l is large, if one is able to take samples that are highly correlated. So fewer samples J_l are necessary on finer level to estimate $\mathbb{E}[P_l - P_{l-1}]$ correctly. The parameter L in the MLMC estimator control the depth of the bias correction.

MLMC estimators for Nested Expectations

Let $J = (J_0, \dots, J_L) \in (\mathbb{N}^\star)^{L+1}$ and $K = (K_0, \dots, K_L) \in (\mathbb{N}^\star)^{L+1}$ the vector representing the number of primary simulations used to approximate the outer expectation (resp the inner conditional expectation) at each levels. We assume that $K_l > K_{l-1}$ and $J_l < J_{l-1}$ for any $l = 0, \dots, L$. A common choice is to consider a geometric progression on each level

$$J_l = J_0 2^{-l} \text{ and } K_l = K_0 2^{+l}, \; l = 0, \dots, L$$

we denote $\widehat{E}_{K_l}(.)$ the estimator of conditional expectation as in 1.3.5 by

$$\widehat{E}_{K_l}(x) = \frac{1}{K_l} \sum_{j=1}^{K_l} Y^j(x) \tag{1.3.10}$$

where $(Y^j(x))_{j=1,\dots,K_l}$ are i.i.d samples of Y given that $X = x$. The MLMC estimator is given by

$$\widehat{I}^{MLMC} = \frac{1}{J_0} \sum_{i_0=1}^{J_0} g\left(\widehat{E}_{K_0}(X^{i_0})\right) + \sum_{l=1}^{L} \frac{1}{J_l} \sum_{i_l=1}^{J_l} g\left(\widehat{E}_{K_l}(X^{i_l})\right) - g\left(\widehat{E}_{K_{l-1}}(X^{i_l})\right) \tag{1.3.11}$$

To make a parallel with the general setting, we reduce the variance (strong error) of the estimator by performing most primary simulations J_l for small level values l. As we progress in finer level, we correct the bias by performing more and more inner simulations K_l. Remark that in that case $P_l = g\left(\widehat{E}_{K_l}(X)\right)$.

Variance reduction using antithetic sampling

In the definition of the MLMC estimator 1.3.11, to derive $g\left(\widehat{E}_{K_{l-1}}(x)\right)$ one generates a sample $(Y^j(x))_{j=1,\dots,K_l}$ of size K_l and throw away samples from $\lceil \frac{K_{l-1}}{2} \rceil + 1$ to K_l. Setting

$$\widehat{E}'_{K_{l-1}}(x) = \frac{1}{K_{l-1}} \sum_{j=\lceil \frac{K_{l-1}}{2} \rceil + 1}^{K_l} Y^j(x) \tag{1.3.12}$$

the empirical mean over the second part of the sample. Remarking that

$$\mathbb{E}\left[g\left(\widehat{E}_{K_{l-1}}(X)\right)\right] = \mathbb{E}\left[\frac{g\left(\widehat{E}_{K_{l-1}}(X)\right) + g\left(\widehat{E}'_{K_{l-1}}(X)\right)}{2}\right] \tag{1.3.13}$$

It is possible to construct a better estimator using the MLMC antithetic version

$$\widehat{I}_A^{MLMC} = \frac{1}{J_0} \sum_{i_0=1}^{J_0} g\left(\widehat{E}_{K_0}(X^{i_0})\right) + \sum_{l=1}^{L} \frac{1}{J_l} \sum_{i_l=1}^{J_l} g\left(\widehat{E}_{K_l}(X^{i_l})\right) - \frac{g\left(\widehat{E}_{K_{l-1}}(X^{i_l})\right) + g\left(\widehat{E}'_{K_{l-1}}(X^{i_l})\right)}{2} \tag{1.3.14}$$

which reduce the variance contribution of each level (see section 9.1 of [Gil15]).

Complexity Theorem

The complexity theorem 3.1 of Giles [Gil08], adapted in the nested simulation setting shows that a MLMC estimator of type 1.3.14 can reduce the computational cost from $O(\varepsilon^{-3})$ (Uniform Nested Estimator complexity) to $O(\varepsilon^{-2})$. The remarkable result here is that provided that some regularity on the payoff function g the complexity of the Antithetic MLMC estimator can be reduced to an unbiased Monte-Carlo estimation. To be more specific, we can achieve the same complexity as if we were given a closed formula to value the conditional expectation (portfolio value) $\mathbb{E}[Y|X]$.

Theorem 1.1. *(Giles Theorem 3.1 [Gil08] adapted in the NS setting) Let P be a r.v and let P_l the corresponding approximation at level l. Denote $V_l = Var(P_l - P_{l-1})$. Let K_l is the discretization bias Parameter in the MLMC method. Assuming that it exists positive constant α, β, c_1, c_2 such that $\alpha \geq \frac{\min\{1,\beta\}}{2}$ and*

(i) (bias speed of decay) $|\mathbb{E}[P_l - P]| \leq \frac{c_1}{K_l^\alpha}$

(ii) (variance decay) $V_l \leq \frac{c_2}{K_l^\beta}$

Then it exists a constant $c_3 > 0$ and optimal parameters L, J_l for which the MLMC estimator has a MSE with bound :

$$MSE(\widehat{I}^{MLMC}) := \mathbb{E}[(\widehat{I}^{MLMC} - I)^2] \leq \varepsilon^2$$

with a computational complexity $\mathrm{Cost}(\widehat{I}^{MLMC})$ with bound:

$$\mathrm{Cost}(\widehat{I}^{MLMC}) \leq \begin{cases} c_3\varepsilon^{-2} & \text{if } \beta > 1 \\ c_3\varepsilon^{-2}\log(\varepsilon)^2 & \text{if } \beta = 1 \\ c_3\varepsilon^{-2-\frac{1-\beta}{\alpha}} & \text{if } 0 < \beta < 1 \end{cases}$$

This important result state that the MLMC method admit three possible asymptotic regimes. The best case $\beta > 1$ is when the dominant cost is on the coarsest level $l = 0$. In this situation, the MLMC method gives a complexity similar to an unbiased standard Monte-Carlo. The worst-case $0 < \beta < 1$ is when the dominant cost is on the finest level L. The dividing case $\beta = 1$ is the one for which both computational effort and the contribution to the overall variance are spread approximately evenly across all the levels. Thus, in order to achieve the optimal regime in the MLMC complexity theorem, a proper control of the speed of decays of the bias and variance needs to be determine and it mainly depend on the smoothness of the payoff function g.

Bias-Variance expansion for Nested MLMC estimators

To obtain an efficient MLMC algorithm, all efforts are devoted to the construction of a family of estimator that enter in the first regime ($\beta > 1$) in the Complexity Theorem. The major difficulty is to verify the hypothesis of the theorem, i.e obtain an estimate of the bias behavior (weak error expansion) and multilevel correction variance expansion (strong error assumption). Such results are in fact closely related to the smoothness of the payoff function g. In the case where g is sufficiently smooth $g \in \mathcal{C}^2(\mathbb{R})$ Giles [Gil15] based on Taylor series approximation, prove that the MLMC estimator 1.3.11 has a complexity in $O(\varepsilon^{-2}\log(\varepsilon)^2)$ which can be further improves to attain the optimal complexity $O(\varepsilon^{-2})$ using antithetic sampling. However, in insurance application or more generally in risk management, the regularity condition $g \in \mathcal{C}^2(\mathbb{R})$ is too strong to be relevant. Therefore, a MLMC estimator must be constructed in situation of lower regularity. However, the approximation of the bias result in a trade-off between smoothness of the payoff function g and condition on the underlying conditional distribution. More precisely, less regularity conditions on g must be compensated by stronger assumption on the distribution. Such results can be tracked back to the work of Gordy and Juneja [GJ10] for crude Nested Monte Carlo estimator in the case of limited regularity, when g is an indicator function $\mathbb{1}_{[c,+\infty)}$. This a framework relevant for tail risk measurement and SCR computation using an internal model. Giles [GHA19] proved that in that case, the standard MLMC estimator 1.3.11 provide a $O(\varepsilon^{-\frac{5}{2}})$ complexity and that antithetic sampling does not improve the overall complexity of the estimates. A thorough analysis of the bias is provided in [GLP20] when g is an indicator function. As mentioned previously, the lack of regularity of g do not permit to perform straightforward Taylor expansions to characterize the bias behavior. But assumptions such as existence of a smooth density functions for certain conditional expectations as well as conditions on their partial derivatives permit to obtain higher order expansion. Other type of function payoff g have been studied in the literature, typically situation of intermediate regularity (less than $\mathcal{C}^2$ but smoother than an indicator) is also relevant in financial applications. In Bujok et al. [BHR15], the case where g is piecewise linear is studied for the pricing of CDO tranches. Here, Y follows a Bernoulli distribution conditionally on a gaussian risk-factor X. This situation is interesting since, as pointed out by Bourgey et al. [BDMGZ20], It can be easily proved that in that case the random couple $(\mathbb{E}[Y|X], X)$ does not admit a density w.r.t the Lebesgue measure. In a very recent work, Bourgey et al. [BDMGZ20], provide a general bias expansion in a situation of *intermediate regularity* (smoother than indicator but less than twice differentiable) under some mild condition on the law of the true conditional expectation in the neighborhood of the singularity. Their result is relevant for initial margin derivation. Another situation is tackled in Giles and Goda [GG19] where the objective is to approximate $\mathbb{E}\left[\max_{p=1,\ldots,P} \mathbb{E}[Y^p|X]\right]$. Under some restrictive hypothesis that will be detailed later on, they show that the optimal complexity $O(\varepsilon^{-2})$ can be achieved using an antithetic MLMC estimator. The numerical method proposed in the second part of the thesis is closely related to the work of Bourgey et al. [BDMGZ20] and Giles-Goda [GG19]. Their result were published during the redaction of this thesis. For a Multi-year SCR projection using the standard formula, we might be typically interested in the worst loss after P shocks, i.e $g(x) = \max_{p=1,\ldots,P} x_i$. For instance, the interest-rate module in the standard formula corresponds to the worst SCR between the upward and downward shock on the interest-rate curve. In the ORSA framework, a shareholder might ask the amount to hold in order to cover the interest-rate risk until the maturity of the ALM portfolio.

Improved MLMC strategies

In the MLMC landscape, improved strategies can be developped to further reduce the bias in certain cases. In Lemaire and Pages [LP17] they combined the MLMC approach with a Richardson-Romberg extrapolation to derive a weighted multilevel estimator (ML2R) in the case were higher order weak expansion is available. In the case where $\beta < 1$ (where the MLMC cost is not close to the unbiased framework in Giles' complexity theorem 1.1), they show that the weighted ML2R estimator outperform the standard MLMC estimator. In the very recent work of Giles [GHA19], particular effort has been made to improve the cost in $O(\varepsilon^{-\frac{5}{2}})$ of the standard MLMC estimator in the case where g is an indicator function. Using the ideas of Broadie and Moallemie [BDM11] , they proposed an adaptative MLMC strategy where the number of inner simulations K_l does not follows a deterministic progression anymore but random, the algorithm adaptatively select the number of inner samples at each level. Numerically, they observe that the complexity of their strategy was close to $O(\varepsilon^{-2}\log(\varepsilon)^2)$ for the Value-At-Risk.

1.4 MLMC for the computation of future SCR and other stress-tests

In the second part of this manuscript, we tackle the computational challenges introduced by the Solvency II regulatory framework. Currently, an open problem in the industry is to compute efficiently the SCR at future dates. On the regulatory side, the ORSA framework aims at evaluating the overall solvency need related to the specific risk profile of the company. From a shareholders' standpoint, the Cost of Capital represents the target return expected by a shareholder that will lend future SCR's at each date in order to let the company pursue its activities. When assessing strategic allocation, a feasible allocation should ensure that future gains generated by the insurance portfolio meet the shareholders expectation in term of cost of capital. Chapter 4 proposes efficient numerical methods to compute the SCR at future dates using the standard formula. More specifically, it requires to simulate, using stochastic models economic environment up to time t and apply a set of stress-tests on the portfolio for each of these scenarios. However, the estimated value of this portfolio require itself a second stage of simulation because the ALM model is too complex to derive a closed-form solution. Efficient methods to deal with the computational burden of nested simulations are based on LSMC or replicating portfolios. Those two methods aims at estimating the conditional expectation (insurers' portfolio) based on very few inner (secondary) simulations. However, these two methodologies suffers from severe drawbacks in practice. In the insurance setting, the number of regressors involved in the regression step is typically high dimensional, since the ALM model is truly path-dependent when the valuation date t gets large. We may therefore include market variables (stock, interest rate level,...) as well as other (book values, crediting rate...) in the regression analysis since they take into account past management decisions as well as past policyholders' behavior. Recent researches in actuarial science are interested in the use of Neural Networks to overcome the curse of dimensionality of standard regression methods. In the fourth chapter of this thesis, we apply the MLMC method developed by Giles [Gil08] to compute the SCR at future dates. To our knowledge, there is no dedicated application of this method for SCR computation in the insurance literature. This chapter fills the gap. From a mathematical standpoint, we propose a bias and variance expansion for the MLMC estimator associated to the maximum of several conditional expectations. Our mathematical framework is also relevant when we want to compute the worst shock on a financial portfolio. The originality of this framework where the regularity of the payoff function is intermediate (smoother than an indicator payoff but only piecewise $\mathcal{C}^2$) supplements the analysis of Bourgey et al. [BDMGZ20] on the subject. The main mathematical contribution of the chapter is to construct a MLMC estimator that reach the optimal complexity $O(\varepsilon^2)$ for nested conditional expectation of the form $I = \mathbb{E}\left[\max\{E[Y^1|X], \dots, E[Y^P|X]\}\right]$. Our result improve the analysis of Giles and Goda [GG19] on this topic. It can be stated as follows:

Theorem 1.2. *Let $P \geq 2$, $\eta \in (0,1]$. Let X, Y be random variable defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that the following assumptions hold*

A-1 Y is a square integrable random variable taking value in $\mathbb{R}^P$.

A-2 ϕ is a measurable real-valued function such that $\phi(X)$ is square integrable.

for $p \in \{1, \dots, P\}$, we define $\sigma_p(X) = \sqrt{\mathrm{Var}(Y^p|X)}$, $\Sigma_p^{1+\eta}(X) = \sum_{i=1}^p \sigma_i^{1+\eta}(X)$, $\Sigma_p^2(X) = \sum_{i=1}^p \sigma_i^2(X)$, $E_X^p = \mathbb{E}[Y^p|X]$, $M_X^p = \max\{E_X^1, \dots, E_X^p\}$, $\widehat{M}_K^p = \max\{\widehat{E}_K^1, \dots, \widehat{E}_K^p\}$ and

$$C_p(X) = 2^\eta \sum_{p'=2}^{p} \frac{\Sigma_p^{1+\eta}(X)}{|E_X^{p'} - M_X^{p'}|^\eta}$$

Assuming that the following condition hold

i) $\forall p = 2, \dots, P$ $\mathbb{P}\left(M_X^{p-1} = E_X^p\right) = 0$

ii) $\Sigma^2 = \mathbb{E}\left[\Sigma_p^2(X)\phi^2(X)\right] < +\infty$ and $C = \mathbb{E}\left[C_p(X)|\phi(X)|\right] < +\infty$ then we have

$$\left|\mathbb{E}[\left(\widehat{M}_K^p - M_X\right)\phi(X)]\right| \leq \frac{C}{K^{\frac{1+\eta}{2}}} \quad \text{and} \quad \mathbb{E}[\left(\widehat{M}_K^p - M_X\right)^2 \phi(X)^2] \leq \frac{\Sigma^2}{K}$$

Besides if $V = Var(M_X^p \phi(X)) < +\infty$, we get

$$MSE(\widehat{I}_{J,K}) \leq \frac{C^2}{K^{1+\eta}} + \frac{2V}{J} + \frac{2\Sigma^2}{JK}$$

. with this upper bound, taking $K = O(\varepsilon^{-\frac{2}{1+\eta}})$ and $J = O(\varepsilon^{-2})$ is an optimal choice to get $MSE(\widehat{I}_{J,K}) = O(\varepsilon^2)$ while minimizing the computational cost JK.

Assumptions (i) and finiteness of C in assumption (ii) are necessary to handle the irregularity of the maximum when two or more conditional expectations are being equal. Our set of assumptions different from assumptions (2) and (3) in [GG19] that require technical assumptions to control the probability that two elements are too close to the maximum. These assumptions are replaced by an integrability condition (assumption ii) in our analysis involving a parameter $\eta \in (0,1)$ that provide some additional flexibility in the implementation of the MLMC estimator. Moreover, we get rid of boundedness assumption(1) in [GG19] since our analysis of the error behavior uses different arguments. The other main contribution of this chapter is to apply the MLMC method to the ALM model developed in this thesis that takes into account the main characteristic of the life insurance business. One of the main advantages of the MLMC estimator is to skip the difficult task of selecting relevant regressors by computing directly the target quantity by allocating efficiently the computational budget without a functional approximation step. Secondly, it provides an estimator with an accuracy ε with a complexity $O(\varepsilon^{-2})$, i.e as efficient as a standard Monte-Carlo procedure without a second stage of simulations. We also compare the performance of the estimator with the LSMC and Neural Network approximation. To compare with the MLMC method, we also tackled the feature engineering problem of LSMC strategies, and we developed a forward feature selection algorithm that iteratively search for the risk-factors that minimize the error on validation scenarios, representing values of the true unknown regression function. The procedure help us to rank the most relevant regressor for the LSMC method. At the first iteration, we find the one-dimensional regressor that minimizes this validation error. At the next iteration, we keep the first best attribute in memory and look for the two-dimensional function that minimizes the error and so on until we have ranked our most relevant risk-factors. To design the Neural Network framework, we fed the network with all the risk-factors that we used to design the model (no feature engineering) which lead to a high dimensional problem. Next, we compared the performance of the MLMC methods with standard LSMC strategies and Neural Network based approximation. To obtain a benchmark value, as no closed formula is available in the model, we run a brute-force Nested simulation procedure. Our findings suggests that on the one hand LSMC regression based methods will remain biased at some point. On the other hand Neural Networks requires heavy computation time for training and are less competitive than MLMC. Namely, even if we add complexity to the the proxy function or increased the number of training sample points the error with respect to the benchmark value will stop diminishing at some point while the MLMC estimation will continue to converge to this "true value". By regressing on few explanatory variables we always lose a part of the information. Another appealing feature of MLMC is that absolutely no feature engineering is required which is a major problem in practical application. Finally, another practical interest of MLMC methods is that it requires far less computational storage than regression methods especially Neural Networks. In any regression methods, we need to store the training data to train the proxy function. To ensure convergence of the regression based approximation a sufficient number of training instances must be generated which must fit in the in the computer memory. The training time and the computational storage is rapidly increasing with the number of primary simulations J. This is not a problem for the MLMC estimator that avoid any functional approximation step since, according to the formula 1.3.11, it does not need to keep training instances in memory, it just update means on each level on the fly. In addition, the MLMC method permits to compute general quantities of the form $\mathbb{E}\left[h\left(\mathbb{E}[Y|X]\right)\right]$ with the same scenarios for different functions h . Taking $h(x) = e^{iux}$ for instance provide information on the full conditional distribution. Our last result is connected to the change of probability, which is required to compute the SCR, yet rarely studied in the literature. We construct in our model an explicit change of probability for affine dynamics and projected the SCR over 15 years for different risk premiums using the MLMC estimator. In particular, with this method, we can construct a whole range of different change of probability using the exact same scenarios and adjusting with the weight associated to the change of measure.

Part I

Asset and Liability Modeling in the Solvency II Framework

CHAPTER 2

A SYNTHETIC MODEL FOR ASSET-LIABILITY MANAGEMENT IN LIFE INSURANCE

Contents

This chapter is an article written with A.Alfonsi and J.Arturo Infante Acevedo [ACIA20a] published in the European Actuarial Journal

Abstract. The aim of this paper is to introduce a synthetic ALM model that catches the key features of life insurance contracts. First, it keeps track of both market and book values to apply the regulatory profit sharing rule. Second, it introduces a determination of the crediting rate to policyholders that is close to practice and is a trade-off between the regulatory rate, a competitor rate and the available profits. Third, it considers an investment in bonds that enables to match a part of the cash outflow due to surrenders, while avoiding to store the trading history. We use this model to evaluate the Solvency Capital Requirement (SCR) with the standard formula, and show that the choice of the interest rate model is important to get a meaningful model after the regulatory shocks on the interest rate. We discuss the different values of the SCR modules first in a framework with moderate interest rates using the shocks of the present legislation, and then we consider a low interest framework with the latest recommendation of the EIOPA on the shocks. In both cases, we illustrate the importance of matching cash flows and its impact on the SCR.

Keywords: ALM model, Solvency capital requirement, Standard formula, Cash flow matching, Liquidity gap, Surrender risk, Book value, Profit sharing.

2.1 Introduction

Life insurance contracts are very popular in the world and involve very large portfolios. In 2017, the life insurer assets were about 7.5 trillions of euros in Europe (source: Insurance Europe) and 7.2 trillions of dollars in the United States (source: American Council of Life Insurers). To manage these large portfolios on a long run, insurance companies perform what is called an Asset and Liability Management (ALM). We refer to the recent paper [ABE+18] for an overview of the current topics and issues of ALM. Basically, insurance companies invest the deposit of policyholders in different asset classes (equity, sovereign bonds, corporate bonds, real estate, ...), while respecting a performance warranty with a profit sharing mechanism for the policyholders. Thus, insurance companies have to determine an appropriate allocation between the different types of asset. This allocation should be a good trade-off between risk and returns, but also with the capital requirement imposed by the regulator to handle the portfolio. To determine a suitable allocation strategy, it is worthwhile to rely on an ALM model that takes into account the main specifies of the life insurance business.

Many papers in the literature have dealt with the fair valuation of insurance liabilities, see e.g. Briys and de Varenne [BdV97], Bacinello [Bac01] or more recently Delong et al. [DDB19]. However, when handling large portfolios of life insurance liabilities over a long run, the fair valuation of the contracts under a risk neutral setting is not the only issue. The insurance company is also interested in investing the policyholders' deposit optimally, which has to be made under the real world probability, like in the pioneering work of Merton [Mer71]. Besides, the insurance company may also want to minimize or at least impose an upper bound on the Solvency Capital Requirement (SCR) related to this portfolio of liabilities. To address these questions, it is necessary to have a reliable ALM model that describes properly the life insurance business. One important specificity of these models is to keep track of both market and book values in order to determine the realized gains and losses that enter in the profit sharing mechanism. Up to our knowledge, one of the first models of this kind has been proposed by Gerstner et al. [GGH+08]. They consider an insurer that invests in bonds with a buy and hold strategy on zero-coupon bonds with a fixed maturity. The insurer keeps a constant allocation proportion in bonds, which leads sometimes to short-sell bonds. The crediting rate to policyholders is the same as the one proposed by Grosen and Jørgensen [GL00]: it is basically the maximum of the guaranteed interest rate and the profit sharing rate. However, the book value is approximated without taking into account neither the history of trading nor the difference between buy and sell orders for the updating of book values. Berdin and Grundl [BG15] fill this gap and calculate the book values according to the German GAAP (General Accepted Accounting Principles). They also consider the investment in different asset classes and across different bond maturities, which is then more precisely described by Berdin et al. [BKP16].

In this paper, we present a new ALM model that incorporates the main features of life insurance business and handles both market and book values. For simplicity of exposition, we consider only two asset classes: equity and riskless bonds, which we consider as a good approximation of top-rated sovereign bonds. The first original feature of our model is the determination of the crediting rate to policyholders. To determine this rate, we take into account the gains and losses made during the reallocation and the corresponding profit sharing rate. We consider also a competitor rate that models the rate given by competing insurance companies to their policyholders. Then, the insurance company drives its latent gain and losses and the profit sharing reserve in order to reach the targeted crediting rate, if it is possible. Otherwise, it tries to give the best rate possible while keeping a part of the profit sharing reserve, but in any case the crediting rate is above the minimal regulatory rate. Interestingly, the four cases that we distinguish to determine the crediting rate form a good indicator to monitor the ALM business. A second original feature of our model is that it takes into account dynamic surrenders (or lapses): their proportion is modeled as a function of the difference between the crediting and the competitor rates. Such a dynamic surrender rate is also considered in the model proposed by Floryszczak et al. [FCM16]. The third original point of our model is to consider an investment in an equally weighted portfolio of bonds with maturity going from 1 year to n years. The dynamic of the coupon rates of the different bonds is also precisely described. A similar but different idea is considered in [BKP16]. The nominal value of the 1-year bonds enables essentially to match the cash flow of the surrenders. This is very important for hedging a part of the risk related to interest rates. Our ALM model is written with the French GAAP, but it could quite easily be adapted to other local GAAP.

Our original motivation to design such an ALM model is to evaluate the Solvency Capital Ratio (SCR) by using the Solvency II standard formula, which is part of the regulation of the European Union [Com15]. Other papers have recently dealt with the standard formula: Gatzert and Martin [GM12] and more recently Asadi and Al Janabi [AAJ] compare the standard formula with internal models that basically use a Value-at-Risk of the basic own funds with a level of 99.5%, Boonen [Boo17] compares the Value-at-Risk and the Expected shortfall risk measures by looking at the stress factor in the standard formula that would calibrate these measures. In our numerical experiments, we evaluate the different SCR modules in our model with a constant allocation between bonds and stocks. We first examine a case with moderate interest rates around 2% and then a case with low interest rates where we use the latest recommendation of EIOPA [EIOPA18] for the shocks. We interpret the different cases corresponding to the shocks on equity and bonds. Interestingly, we find that interest rate models like the Hull and White model that mean revert toward a parametric curve are not really well-suited for the standard formula. They are able to fit the shocks but then the calibrated curve oscillates too much and the model is meaningless. Our numerical study also points some weaknesses of the standard formula, such as: the discontinuity of the formula between upward and downward shocks on the interest rates, the fact that it is computed as a difference of two expectations and thus ignores the distribution profiles, and the use of a risk-neutral valuation. Last, we illustrate the importance of matching cash flows in ALM, and discuss how to do it optimally in our model for minimizing the SCR requirement with the standard formula.

The paper is organized as follows. Section 2.2 introduces the main notation, presents the ALM model and the mechanism that determines the crediting rate to policyholders. This part is self-contained and does not rely on the model of the different assets, which is presented in Section 2.3. We discuss in particular the choice of the interest rate model in view of the application of the standard formula. Last, Section 2.4 presents and discusses the different numerical simulations.

2.2 The ALM model

We consider an insurance company handling a life insurance business with many policyholders. To be precise, we consider here a General Account (GA) guaranteed with profit contracts. We do not consider Unit-Link (UL) type of contracts where policyholders bear the risk due to market variations, which is clearly simpler for an insurance company to handle. GA contracts are mainly characterized by two drivers: the minimal guaranteed rate r^G that triggers the minimal earnings, and the participation rate $\pi_{pr} \in [0,1]$ that forces the insurer to redistribute this proportion of gain on equity assets. The French legislation imposes that $\pi_{pr} \geq 0.85$ (see [BMDGL18] p. 5). Policyholders do not receive intermediary payments: they are paid only when they exit the life insurance contract.

The insurance company then has to choose a strategic asset allocation that will enable to face up to the liabilities and provide some earnings. In particular, it is interested to assess the Solvency Capital Requirement (SCR) needed to run its strategic asset allocation. Typically, the insurance company invests in different asset classes such as equity, sovereign bonds, corporate bonds, real estate. Here, we will consider for simplicity consider two type of assets: equity and riskless bonds, the latter being a good approximation of top-rated sovereign bonds. The possibility to include (risky) corporate bonds in this model is discussed in Remark 2.3 below. We consider a time horizon $T \in \mathbb{N}^*$, usually greater than thirty years in practice. We will assume that the insurance company only make reallocations at times $t \in \mathbb{N} \cap [0,T)$ in order to reach a portfolio with respective weights $w_t^s \in [0,1]$ and $w_t^b = 1 - w_t^s$ in equity and bonds. The portfolio is assumed to be static on $(t, t+1)$, and at time T, the portfolio is liquidated. The time unit can in practice be one year or one semester: in this paper, we take a one year unit for our numerical investigations.

We denote by $(S_t)_{t\geq 0}$ the equity asset that can be thought as a stock market index to reflect that the insurance company invests in many different stocks. Concerning interest rate products, we assume that there exist riskless zero-coupon bonds and bonds of any maturity that can be bought at par. Following the notation of Brigo and Mercurio [BM06], we denote by $P(t,t')$ with $t \leq t'$ the price of a zero-coupon bond at time t with maturity t'. For simplicity, we assume that the different bonds pay a coupon at the same frequency as the portfolio reallocation: thus at time $t \in \mathbb{N}$, the value of a bond with maturity $t+n$, constant coupon c and a unit nominal value is given by

$$B(t,n,c) = \sum_{i=1}^{n} cP(t,t+i) + P(t,t+n), \tag{2.2.1}$$

and the swap rate given by

$$c_{swap}(t,n) = \frac{1 - P(t,t+n)}{\sum_{i=1}^{n} P(t,t+i)} \tag{2.2.2}$$

is the value of the coupon leading to a unit value of the bond. For $t \in \mathbb{N}$, $n \in \mathbb{N}^*$ and $\mathbf{c}_t := (c_t^i)_{i\in\{1,\dots,n\}}$, we consider a portfolio containing, for any i, $1/n$ bond with maturity $t+i$ and coupon c_t^i. We denote

$$\bar{B}(t,n,\mathbf{c}_t) = \frac{1}{n}\sum_{i=1}^{n} B(t,i,c_t^i) \tag{2.2.3}$$

the value of this combination of bonds at time t.

Before modelling the mechanism of the ALM management, we also have to specify at which rate policyholders enter or exit. Since our purpose is to evaluate the Solvency Capital Requirement, we will only consider (as recommended in Solvency II) the case where policyholder contracts run off and exclude the arrival of new contracts, even though it could be obviously added to the model. We assume that the proportion of policyholders that exit on the period $(t, t+1)$ for $t \in \mathbb{N}$ is given by $p_t^e \in (0,1)$, and that policyholders exit uniformly on $(t, t+1)$. This corresponds to the case of infinitely many policyholders that exit at a continuous rate $\lambda_t^e = -\log(1-p_t^e)$ on $(t, t+1)$. So, our model assumes that there is a large number of policyholders that exit independently, conditionally on the information they have at time t. We will assume that

$$p_t^e \geq \underline{p} > 0, \ \forall t \in \mathbb{N}. \tag{2.2.4}$$

Thus, $\underline{p}$ quantifies the structural surrenders while $p_t^e - \underline{p}$ is the proportion of surrenders that evolves along the time. This includes typically the dynamic surrenders that depend on the crediting rate and market conditions, and also the mortality variations that are usually assumed to be deterministic and calculated by using a life table. This quantity $p_t^e - \underline{p}$ will be modeled afterwards in Section 2.3.

2.2.1 Main variables and portfolio initialization at time $t = 0$

The liability of the firm is divided into different reserves that must comply with the local accounting standards. Even if the main principles of the Solvency II directive are followed by most of the countries, there are some specific features from one country to another. In our model, we will take into account the French regulation rules, and we refer to [BMDGL18] for a recent study of the French legal prudential reserve.

The Mathematical Reserve, denoted by the process $(MR_t)_{t\in\{0,\dots,T\}}$, is the main reserve in life insurance. It corresponds to the insurer's debt towards its policyholders. For sake of simplicity, we assume that the initial premium MR_0 is paid once and for all (single premium) by policyholders without fees. Thus, the initial value MR_0 of this reserve is given by the initial deposit of the policyholders. At the end of each year, the mathematical reserve is reevaluated by annual benefits (the crediting rate) paid by the insurer to the insured party.

The Capitalization Reserve, denoted by the process $(CR_t)_{t\in\{0,\dots,T\}}$, is imposed by the French legislation to buffer the capital gains obtained when selling bonds. The purpose of this reserve is twofold. First, it dissuades insurance companies from using interest rate movements to make profits on bonds, since it may impact negatively its policyholders on the long-run (typically, capital gains on bonds result in lower coupons). Second, it acts as a cushion against interest rate movements as the fund stored in the reserve can be used later on in order to absorb capital losses coming from selling bonds. The capitalization reserve is a part of the equity capital of the insurance company.

Last, the Profit-Sharing reserve denoted by the process $(PSR_t)_{t\in\{0,\dots,T\}}$ is a legal provision used as a capital buffer against stock movements in order to smooth the crediting rate. A fraction of the capital gains obtained from selling equity are stored in this reserve and is distributed the next years. This reserve belongs to the policyholders: the French legislation imposes a maximum of 8 years to redistribute the accumulated profit to policyholders.

The capital gain of the insurance company is determined by the difference between the book value (i.e. the purchase price) and the market value of the sold assets. Thus, we denote respectively by BV_t^s, BV_t^b and $BV_t = BV_t^s + BV_t^b$ the book values of the equity assets, the bonds and of the whole portfolio at time t. We similarly denote MV_t^s, MV_t^b and $MV_t = MV_t^s + MV_t^b$ the respective market values. It is clear how to evaluate market values, and we will explain later on how book values are calculated in our model.

All these quantities MR_t, CR_t, PSR_t, BV_t^s, BV_t^b, MV_t^b and MV_t^s remain nonnegative for all $t \in \{0,\dots,T\}$ in our ALM model.

At inception ($t = 0$), the insurance company receives all the policyholders' deposit MR_0 and invests this amount in a reference portfolio according to target proportions w_0^s in stocks and w_0^b in bonds. We furthermore make the following hypothesis: the company has no existing back book of contracts sold in the past. Thus, there are no capitalization and profit-sharing reserves, and book values and market values coincide:

$$CR_0 = PSR_0 = 0,\ BV_0^b = MV_0^b = w_0^b MR_0 \text{ and } BV_0^s = MV_0^s = w_0^s MR_0.$$

However, it would be easy at this stage to consider the existing back book of contracts by initializing accordingly these values. We now specify the quantity of assets in each class. Let us note that during all the time, the quantities held by the insurer are nonnegative. The initial holding ϕ_0^s in the stock asset is clearly given by

$$\phi_0^s = \frac{w_0^s MR_0}{S_0}.$$

Concerning the bonds, we assume that all bonds are bought at par during the whole strategy. We assume that the amount $w_0^b MR_0$ is invested in an asset which is an equally weighted basket (with weights $1/n$) of riskless coupon bearing bonds from maturity 1 to n with unitary face value. We thus set

$$c_0^i = c_{swap}(0,i),\ i = 1,\dots,n.$$

From the definition of the swap rate, we have $\bar{B}(0,n,\mathbf{c}_0) = 1$, with $\mathbf{c}_0 = (c_0^i)_{i\in\{1,\dots,n\}}$ and $\bar{B}$ defined by (2.2.3). Thus, the quantity of asset is simply given by

$$\phi_0^b = \frac{w_0^b MR_0}{\bar{B}(0,n,\mathbf{c}_0)} = w_0^b MR_0.$$

Here, we stress that we consider the investment in a basket of bonds with different maturities instead of only one bond. Thus, the insurance company is able to match a part of the cash outflows: the nominal value of the one-year bond is mainly used to pay back the policyholders that exit during the first year. This cash flow matching is commonly used to hedge against interest-rate risk and is known as a so-called immunization technique. In order to hedge the minimal rate of surrenders $\underline{p}$, a natural choice is to take $n = \left\lceil \frac{1}{\underline{p}} \right\rceil$. If one wants to take into account the additional market surrenders $p_t^e - \underline{p}$, it can be relevant to take more generally $n \leq \left\lceil \frac{1}{\underline{p}} \right\rceil$, n being roughly speaking an average of $1/p_t^e$. The choice of n and the influence of this parameter will be discussed in the numerical section.

2.2.2 Reallocation, claim payment and margin at time $t \in \{1, \dots, T-1\}$

This subsection presents the different steps of the portfolio reallocation at time $t \in \{1, \dots, T-1\}$. In particular, it describes the composition of cash inflows and outflows, the legal profit-sharing mechanism, a way to determine the crediting rate for policyholders and the accounting margin for shareholders. The goal of the reallocation is to end with an asset side that is allocated according to the weights w_t^s in equity and w_t^b in an equally weighted portfolio of bonds with weights $1/n$ as described in (2.2.3). This amounts to have the quantities

$$\phi_t^s = \frac{w_t^s MV_t}{S_t} \text{ and } \phi_t^b = \frac{w_t^b MV_t}{\bar{B}(t, n, \mathbf{c}_t)}$$

at the end of the the reallocation procedure, where MV_t is the market value of the assets and $\mathbf{c}_t = (c_t^i)_{i \in \{1, \dots, n\}}$ are coupon values that will be made precise afterwards. Concerning book values, since the capitalization reserve is managed with a separate accounting, the goal is to have at the end of the reallocation: $BV_t = MR_t + PSR_t$, i.e. that the liabilities exactly match the portfolio book value. The corresponding balance sheet is given in Table 2.1. For all balance sheets, the sum of the asset book values is equal to the sum of the liabilities.

Assets	Liabilities
BV_t^s	MR_t
BV_t^b	PSR_t

Table 2.1 – Book value balance sheet after the reallocation at time t.

We present the whole reallocation procedure in five steps. For convenience, we set $t_1 = t_2 = \ldots = t_5 = t$ for the different steps of the ALM management procedure at time t, the step $i + 1$ being immediately executed after the step i. Some quantities are updated only once during the five steps, and we then use the index t for these quantities. Other quantities, such as the book value are updated at different steps, and we note BV_{t_i} the book value after step i and BV_t the book value after the last update in the whole procedure. Note that these quantities (except market values) are then kept constant on $(t, t + 1)$ until the next reallocation.

Step 1: cash inflows

We recall that we assume that the portfolio is static on $(t-1,t)$ and therefore the quantity of equity (resp. bond) assets at the beginning of the reallocation is ϕ^s_{t-1} (resp. ϕ^b_{t-1}). More precisely, for all $i \in \{1,\dots,n\}$, the insurance company holds ϕ^b_{t-1}/n bonds with maturity $t-1+i$ and coupon c^i_{t-1}. The financial income corresponding to the coupon payments from each bond is thus given by

$$FI_t = \phi^b_{t-1}\left(\frac{1}{n}\sum_{i=1}^{n} c^i_{t-1}\right). \tag{2.2.5}$$

Besides, the nominal value coming from the matured bond is given by $\frac{\phi^b_{t-1}}{n}$. Thus, the insurer's overall cash inflow CIF is obtained by aggregating terms:

$$CIF_t = FI_t + \frac{\phi^b_{t-1}}{n}. \tag{2.2.6}$$

The book value in bond assets has to be updated. Following standard accounting procedures, the nominal value of the matured bonds have to be removed from the book value. We thus set

$$BV^b_{t_1} = BV^b_{t-1} - \frac{\phi^b_{t-1}}{n}.$$

In order to satisfy the bookkeeping condition, the insurer must redistribute the income FI_t on the liability side. Table 2 sums up the insurer balance-sheet after Step 1.

Assets	Liabilities
BV^s_{t-1}	MR_{t-1}
$BV^b_{t_1}$	PSR_{t-1}
CIF_t	FI_t

Table 2.2 – Book value balance sheet after Step 1.

Step 2: claim payment

Cash outflows occur when policyholders exit their contract. We recall that the proportion of policyholders that exit on $(t-1,t)$ is given by p^e_{t-1}. We assume that these policyholders are paid with the minimum guaranteed rate r^G, pro rata the time elapsed between $t-1$ and the exit. Since we assume that they exit uniformly on $(t-1,t)$, this amounts to the cash outflow

$$COF_t = p^e_{t-1} MR_{t-1}\left(1+\frac{r^G}{2}\right). \tag{2.2.7}$$

On the liability side, the liabilities corresponding to remaining policyholders are then given by

$$MR_{t_2} = (1-p^e_{t-1})MR_{t-1}.$$

On the asset side, the difference between cash inflows CIF_t and cash outflows COF_t is called the liquidity gap G_t:

$$G_t = CIF_t - COF_t = CIF_t - p^e_{t-1} MR_{t-1} - \frac{r^G}{2} p^e_{t-1} MR_{t-1}.$$

A positive gap $G_t > 0$ means that asset inflows are sufficient to cover claims. A negative gap $G_t < 0$ means that additional liquidity is necessary to pay claim-holders. To fill the funding gap, the insurer must sell asset in this situation. Table 2.3 depicts the insurer's balance after claim payment. We thus set

Assets	Liabilities
BV^s_{t-1}	MR_{t_2}
$BV^b_{t_1}$	PSR_{t-1}
G_t	$FI_t - \frac{r^G}{2} p^e_{t-1} MR_{t-1}$

Table 2.3 – Book value balance sheet after Step 2.

$$\widetilde{FI}_t = FI_t - \frac{r^G}{2} p^e_{t-1} MR_{t-1}, \tag{2.2.8}$$

which represents the coupon income corrected with the part of these earnings that are distributed to the surrendering policyholders.

Step 3: reallocation

We assume that the insurer follows a static investment strategy on $(t, t+1)$ and allocates its capital according to the portfolio weights w^s_t between stocks and bonds w^b_t at time t. The available capital, which is also the market value of the portfolio MV_t, is given by the sum of the liquidity gap G_t and the market value of each asset classes:

$$MV_t = G_t + \phi^s_{t-1} S_t + \frac{\phi^b_{t-1}}{n} \sum_{i=1}^{n-1} B(t, t+i, c^{i+1}_{t-1}), \tag{2.2.9}$$

where the function B is defined by (2.2.1). The term $\phi^s_{t-1} S_t$ is the market value of the equity and $\phi^b_{t-1} \left(\frac{1}{n} \sum_{i=1}^{n-1} B(t, t+i, c^{i+1}_{t-1}) \right)$ is the market-value of the bonds that have not reached maturity. Note that before the reallocation, the bond portfolio is made with bonds with maturity $i \in \{1, \dots, n-1\}$ and coupon c^{i+1}_{t-1}.

In what follows, we assume that $MV_t > 0$ and calculate the new quantities invested in each asset class and derive the procedure to update the book values of each asset class. The case $MV_t \leq 0$ is very unlikely but may theoretically happen, for example if the surrendering proportion p^e_{t-1} is high, the stock value has strongly decreased ($S_t/S_{t-1} << 1$) and interest rates have strongly increased on $(t-1, t)$. In this case we assume that the shareholders of the insurance company directly pay back COF_t to the surrenders. Then, COF_t does not enter in the market value, and we have $MV_t = CIF_t + \phi^s_{t-1} S_t + \frac{\phi^b_{t-1}}{n} \sum_{i=1}^{n-1} B(t, t+i, c^{i+1}_{t-1}) > 0$. We then continue the procedure as in the case $MV_t > 0$. It is certainly excessive to assume that all the COF_t is paid by the shareholders when $MV_t \leq 0$. Since this case $MV_t \leq 0$ never happens in usual conditions and in our simulations, we do not consider it necessary to model it more carefully.

We first consider the equity, where the target is to achieve a proportion w_t^s of the market value MV_t. This leads to a new position in stock given by

$$\phi_{t_3}^s = \frac{w_t^s MV_t}{S_t} > 0.$$

We note $\Delta\phi_t^s = \phi_{t_3}^s - \phi_{t-1}^s$ the variation of the number of equity assets held by the insurer. If $\Delta\phi_t^s \geq 0$ (buy order), the book value in equity is increased by the quantity of stocks that was purchased at the market-value S_t:

$$BV_{t_3}^s = BV_{t-1}^s + \Delta\phi_t^s S_t.$$

If $\Delta\phi_t^s < 0$, the insurer sells the quantity $-\Delta\phi_t^s$ of equity assets. In accounting, a standard inventory valuation method used by practitioners is the First In First Out (FIFO) method where the oldest goods purchased are sold in priority. The realized Capital Gain or Loss (CGL) is then calculated accordingly. However, this procedure requires to record the entire history of all purchases and is computationally demanding. Here, we consider the approximation

$$BV_{t_3}^s = BV_{t-1}^s \left(1 + \frac{\Delta\phi_t^s}{\phi_{t-1}^s}\right) = \frac{\phi_{t_3}^s}{\phi_{t-1}^s} BV_{t-1}^s,$$

which amounts to say that all the equity asset units held in the portfolio have the same book value. The proportional reduction factor $\frac{\Delta\phi_t^s}{\phi_{t-1}^s} \in]-1, 0]$ represents the proportion of sold stock. The capital gain or loss made by the sale is then given by $CGL_t^s = -\Delta\phi_t^s(S_t - BV_{t-1}^s/\phi_{t-1}^s)$.

Let us recall that for $x \in \mathbb{R}$, $x^+ = \max(x, 0)$ and $x^- = \max(-x, 0)$. We sum up the equity book value and the capital gain or loss (regardless whether it is a sale or a purchase):

$$BV_{t_3}^s = BV_{t-1}^s + (\Delta\phi_t^s)^+ S_t - \frac{(\Delta\phi_t^s)^-}{\phi_{t-1}^s} BV_{t-1}^s, \tag{2.2.10}$$

$$CGL_t^s = (\Delta\phi_t^s)^- \left(S_t - \frac{BV_{t-1}^s}{\phi_{t-1}^s}\right). \tag{2.2.11}$$

We now focus on the reallocation in bonds and recall that we assume that the insurer only buys bonds at par. Before the reallocation, the bond portfolio is made with bonds with time to maturity going from 1 to $n-1$. In order to continue the strategy of matching the cash flows coming from the structural surrenders, the insurer needs to invest in a basket of bonds with longest time to maturity equal to n. Thus, the insurer always has to buy the bond with longest time to maturity n. Let us introduce the following reference market value

$$\widehat{MV}_t^b = \phi_{t-1}^b \left(\frac{1}{n}\sum_{i=1}^{n-1} B(t, t+i, c_{t-1}^{i+1}) + \frac{1}{n} B(t, t+n, c_{swap}(t, n))\right). \tag{2.2.12}$$

This is the market value of the bond portfolio, if the insurer would buy exactly the same quantity ϕ_{t-1}^b/n of bonds with time to maturity n. If $w_t^b MV_t = \widehat{MV}_t^b$, the insurer thus only buys ϕ_{t-1}^b/n bonds with time to maturity n to reach the target allocation. If $w_t^b MV_t > \widehat{MV}_t^b$, he has to buy more bonds and if $w_t^b MV_t < \widehat{MV}_t^b$ he has to sell some bonds. In what follows, we describe how to do it, while keeping an equally weighted basket of bonds with time to maturity going from 1 to n.

Purchase of bonds ($w_t^b MV_t \geq \widehat{MV}_t^b$)

In this case, the insurer needs to buy more bonds to satisfy the target w_t^b. We note $\mathbf{c}_t^{swap} = (c_{swap}(t,i))_{\in\{1,\dots,n\}}$ and have $\bar{B}(t,n,\mathbf{c}_t^{swap}) = 1$ from (2.2.3). We then define

$$\delta_t^b = w_t^b MV_t - \widehat{MV}_t^b \geq 0, \tag{2.2.13}$$

so that $w_t^b MV_t = \widehat{MV}_t^b + \delta_t^b \bar{B}(t,n,\mathbf{c}_t^{swap})$. The insurer will then

- buy $\frac{\delta^b}{n}$ at par bonds for each time to maturity $i \in \{1,\dots,n-1\}$ with coupon $c_{swap}^i(t)$,
- buy $\frac{\delta^b+\phi_{t-1}^b}{n}$ at par bonds with time to maturity n and coupon $c_{swap}^n(t)$.

Let us recall now that holding $\alpha > 0$ bonds with coupon c and $\alpha' \geq 0$ bonds with coupon c' and the same payment schedule is equivalent to hold $\alpha + \alpha'$ bonds with coupon $\frac{\alpha c+\alpha' c}{\alpha+\alpha'}$. Therefore, after the bond reallocation, the insurance company holds for each $i \in \{1,\dots,n\}$, $(\delta^b + \phi_{t-1}^b)/n$ bonds with time to maturity i and coupon

$$c_t^i = \mathbf{1}_{i\leq n-1} \frac{\phi_{t-1}^b c_{t-1}^{i+1} + \delta^b c_{swap}^i(t)}{\phi_{t-1}^b + \delta^b} + \mathbf{1}_{i=n} c_{swap}^n(t). \tag{2.2.14}$$

We can therefore write the market value of the bond portfolio as

$$MV_t^b = \phi_{t_3}^b \bar{B}(t,n,\mathbf{c}_t), \text{ with } \phi_{t_3}^b = \delta^b + \phi_{t-1}^b, \tag{2.2.15}$$

and in particular we have $\phi_{t_3}^b \geq \phi_{t-1}^b$.

We now have to update the book value of bonds. Since there are only purchases, the book value of the bought bonds is their market value. We thus set

$$\begin{aligned} BV_{t_3}^b &= BV_{t_1}^b + \frac{\delta^b}{n}\sum_{i=1}^{n-1} B(t,t+i,c_{swap}^i(t)) + \frac{\delta^b + \phi_{t-1}^b}{n} B(t,t+n,c_{swap}^n(t)) \\ &= BV_{t_1}^b + \delta^b + \frac{\phi_{t-1}^b}{n}. \end{aligned}$$

Sale of bonds ($w_t^b MV_t < \widehat{MV}_t^b$)

When $w_t^b MV_t < \widehat{MV}_t^b$, the insurer still has to buy bonds with time to maturity n, but he has to sell the other bonds to get an equally weighted bond portfolio. Thus, he has to find a position such that

$$w_t^b MV_t = \phi_{t_3}^b \left(\frac{1}{n}\sum_{i=1}^{n-1} B(t,t+i,c_{t-1}^{i+1}) + \frac{1}{n} B(t,t+n,c_{swap}^n(t)) \right).$$

Note that we necessarily have $\phi_{t_3}^b < \phi_{t-1}^b$, since the right-hand side corresponds to $\widehat{MV}_t^b$ for $\phi_{t_3}^b = \phi_{t-1}^b$. This gives

$$\phi_{t_3}^b = \frac{w_t^b MV_t}{\frac{1}{n}\sum_{i=1}^{n-1} B(t,t+i,c_{t-1}^{i+1}) + \frac{1}{n} B(t,t+n,c_{swap}^n(t))} > 0,$$

and the market value of the bond portfolio can be written as

$$MV_t^b = \phi_{t_3}^b \bar{B}(t, n, \mathbf{c}_t), \text{ with } c_t^i = \mathbf{1}_{i \leq n-1} c_{t-1}^{i+1} + \mathbf{1}_{i=n} c_{swap}^n(t) \text{ for } i = 1, \dots, n.$$

Let $\Delta\phi_t^b = \phi_{t_3}^b - \phi_{t-1}^b < 0$. The insurer has thus to buy $\frac{\phi_{t_3}^b}{n}$ at par bonds with time to maturity n and to sell, for each time to maturity $i \in \{1, \dots, n-1\}$, $\frac{\Delta\phi_t^b}{n}$ bonds with coupon c_{t-1}^{i+1}.

We now have to update the book values. We use the same approximation method as for the equity to evaluate the book value of the sold bonds. We thus set

$$BV_{t_3}^b = BV_{t_1}^b \left(1 + \frac{\Delta\phi_t^b}{\phi_{t-1}^b}\right) + \frac{\phi_{t_3}^b}{n} B(t, t+n, c_{swap}^n(t)) = BV_{t_1}^b \left(1 + \frac{\Delta\phi_t^b}{\phi_{t-1}^b}\right) + \frac{\phi_{t_3}^b}{n},$$

and the capital gain or loss on bond products is then given by

$$CGL_t^b = -\Delta\phi_t^b \left(\frac{1}{n} \sum_{i=1}^{n-1} B(t, t+i, c_{t-1}^{i+1}) - \frac{BV_{t_1}^b}{\phi_{t-1}^b}\right)$$

In the two following formulas, we sum up the book value update and the capital gain and loss on bonds in both selling and buying cases

$$BV_{t_3}^b = BV_{t_1}^b \left(1 - \frac{(\Delta\phi_t^b)^-}{\phi_{t-1}^b}\right) + \frac{n-1}{n} (\Delta\phi_t^b)^+ + \frac{\phi_{t_3}^b}{n}, \tag{2.2.16}$$

$$CGL_t^b = (\Delta\phi_t^b)^- \left(\frac{1}{n} \sum_{i=1}^{n-1} B(t, t+i, c_{t-1}^{i+1}) - \frac{BV_{t_1}^b}{\phi_{t-1}^b}\right), \tag{2.2.17}$$

since $\Delta\phi_t^b = \delta_b$ and $\phi_{t_3}^b = \phi_{t-1}^b + \delta_b$ in the buying case.

The capital gain or loss on equity CGL_t^s is directly taken into account for the profit sharing mechanism. Instead, the capital gain and loss on bonds CGL_t^b is handled separately in the French legislation and supply the capitalization reserve. Precisely, the capitalization reserve at time t is defined by

$$CR_t = \left(CR_{t-1} + CGL_t^b\right)^+. \tag{2.2.18}$$

If $CR_{t-1} + CGL_t^b < 0$, this quantity reduces the insurer's return of the period. Since the capitalization reserve is managed with a separate accounting, only

$$\Delta CR_t = CR_t - CR_{t-1} \tag{2.2.19}$$

appears in the balance sheet at Step 3, see Table 2.4.

Step 4: determination of the crediting rate

In order to determine the policyholder's earning rate $r_{ph}(t)$ on the period $(t-1, t)$, we propose a management decision that follows the regulatory constraints and is a reasonable trade-off between policyholders' and shareholders' interests. Most of the existing ALM model use a crediting rate that has been proposed by Grosen and Jørgensen [GL00] which is the minimal regulatory rate. Here, we propose a more sophisticated model for the crediting rate that we believe to be closer to practice. It involves a

Assets	Liabilities
$BV_{t_3}^s$	MR_{t_2}
$BV_{t_3}^b$	PSR_{t-1}
CGL_t^s	ΔCR_t
CGL_t^b	
G_t	$\widetilde{FI}_t + CGL_t^s - \left(CR_{t-1} + CGL_t^b\right)^-$

Table 2.4 – Book value balance sheet after step 3.

competitor rate and a control of the Latent Gain or Loss (LGP) and the Profit Sharing Reserve.

The existence of LGL, sometimes also called hidden reserve, results from the difference between market and book values. Formally they can be realized by selling and buying instantly the same amount of assets, but in practice this is just an account entry. It is a variable that the insurer can use as a control to determine the crediting rate for policyholders. In what follows we consider only LGL for stock assets since CGL on bond are constrained by the capitalization reserve and cannot be redistributed to policyholders nor shareholders. Let $MV_t^s = w_t^s MV_t$ the current market value of equity assets. The range of latent gain or loss can be described by the following interval

$$[-(MV_t^s - BV_{t_3}^s)^-, (MV_t^s - BV_{t_3}^s)^+].$$

There is a latent gain if $MV_t^s > BV_{t_3}^s$, and a latent loss if $MV_t^s < BV_{t_3}^s$. We define the latent gain or loss function by

$$LGL_t^s(\alpha) = -(1-\alpha)(MV_t^s - BV_{t_3}^s)^- + \alpha(MV_t^s - BV_{t_3}^s)^+, \quad \alpha \in [0,1]. \tag{2.2.20}$$

This function determines the amount of hidden reserve to distribute. Let us note that $LGL_t^s(\alpha)$ is non-decreasing with respect to α. The control $\alpha \in [0,1]$ models the fraction of LGL to register on the balance-sheet. The choice $\alpha = 1$ amounts to take all the gain or zero loss. The control $\alpha = 0$ takes all the loss or zero gain.

Remark 2.1. *We have $CGL_t^s LGL_t^s(\alpha) \geq 0$, i.e. CGL_t^s and $LGL_t^s(\alpha)$ have the same sign. If $\Delta\phi_t^s \geq 0$, this is true since $CGL_t^s = 0$. If $\Delta\phi_t^s < 0$, one has to notice that we have*

$$\frac{BV_{t-1}^s}{\phi_{t-1}^s} = \frac{BV_{t_3}^s}{\phi_{t_3}^s}$$

from (2.2.10). Therefore $S_t - \frac{BV_{t-1}^s}{\phi_{t-1}^s}$ is equal to $S_t - \frac{BV_{t_3}^s}{\phi_{t_3}^s} = (MV_t^s - BV_{t_3}^s)/\phi_{t_3}^s$, and these quantities have the same sign.

Another control for the insurer is the proportion $\rho \in (0,1]$ of profit sharing reserve to distribute. For simplicity, we will assume here that

$$\rho \in \{\bar{\rho}, 1\},$$

where $\bar{\rho} \in (0,1]$ is fixed. The insurer has then two possible choices: to use all the profit sharing reserve ($\rho = 1$) or to use only a part of it. Let us note that in our model, taking $\bar{\rho} = 1$ amounts to have no profit sharing reserve.

Remark 2.2. *Since $(1/2)^8 \approx 0.004$, we take in our experiments $\bar{\rho} = 1/2$ to be in line with the French legislation that requires to redistribute all the profit within 8 years. This is also the choice made by Berdin and Grundl [BG15] (see Equation (22) therein) who work under the German rule.*

Due to the participation rate, the minimal crediting rate depends on α and ρ. In case of latent gain ($MV_t^s \geq BV_{t_3}^s$), we note

$$TD_t(\alpha,\rho) = \widetilde{FI}_t - \left(CR_{t-1} + CGL_t^b\right)^- + \rho\left(PSR_{t-1} + CGL_t^s + LGL_t^s(\alpha)\right) \quad (2.2.21)$$

which is the amount that has to be redistributed to policyholders according to the participation rate π_{pr}. The first term corresponds to the coupon payment (2.2.8). The second term is the loss on the bonds that exceeds the capital reserve. The third term corresponds to the aggregated gains on equity and is nonnegative by Remark 2.1.

In case of a latent loss ($MV_t^s \geq BV_{t_3}^s$), we define

$$TD_t(\alpha,\rho) = \widetilde{FI}_t - \left(CR_{t-1} + CGL_t^b\right)^- + \rho PSR_{t-1} + CGL_t^s + LGL_t^s(\alpha) \quad (2.2.22)$$

the amount to be redistributed with the participation rate. From Remark 2.1, CGL_t^s and $LGL_t^s(\alpha)$ are nonpositive. Contrary to the gains, the insurer does not smooth the losses with a factor ρ.

We now sum up the amount to distribute (2.2.21) by the following formula that covers both capital gain or loss cases:

$$\begin{aligned} TD_t(\alpha,\rho) = \widetilde{FI}_t - \left(CR_{t-1} + CGL_t^b\right)^- &+ \rho\left(PSR_{t-1} + CGL_t^s + LGL_t^s(\alpha)\right) \\ &- (1-\rho)\left(CGL_t^s + LGL_t^s(\alpha)\right)^- . \end{aligned} \quad (2.2.23)$$

We have the following straightforward but important property.

Lemma 2.2.1. *The function $(\alpha,\rho) \in [0,1]^2 \mapsto TD_t(\alpha,\rho)$ is continuous and nondecreasing with respect to α and ρ. It is constant with respect to α if $S_t = BV_{t-1}^s/\phi_{t-1}^s$, otherwise it is increasing and affine with respect to α.*

We are now able to define the minimal distribution of returns that the insurer has to give to the (remaining) policyholders. The minimum guaranteed amount $R_t^G(\alpha,\rho)$ is defined by

$$R_t^G(\alpha,\rho) = \max\left\{\underline{R}_t^G, \pi_{pr} TD_t(\alpha,\rho)\right\}, \text{ with } \underline{R}_t^G = r^G(MR_{t_2} + PSR_{t-1}). \quad (2.2.24)$$

Note that the part of the profit sharing reserve has to be credited exactly as the mathematical reserve, since this reserve belongs to policyholders. Here, $\underline{R}_t^G$ is the minimum regulatory amount corresponding to the minimum rate r^G. Note that, in practice, the minimum regulatory rate is the maximum of r^G and of 60% of a technical rate called "taux moyen des emprunts d'Etat" that is an average of French sovereign bond rates, see for example paragraph A.6.2 of [BKP16]. Here, we assume for simplicity that r^G remains above this technical rate.

Beyond the minimum rate, the insurance company wants to credit at least the same rate as the other insurance companies in order to keep its policyholders. In fact, the surrender proportion p_t^e on $(t, t+1)$ usually depends on the difference between the crediting rate and the one of the other insurers. Thus, we assume that r_t^{comp} is a competitor rate. Typical choices can be

$$r_t^{comp} = r_t \text{ or } r_t^{comp} = \max(r_t, \eta r_{ph}(t-1)) \text{ with } \eta \in (0,1), \tag{2.2.25}$$

where $r_{ph}(t-1)$ is the crediting rate of the past period and r_t is the short interest rate. We define the target crediting amount by

$$R_t^\sigma(\alpha,\rho) = \max\left\{R_t^G(\alpha,\rho); R_t^{comp}\right\}, \text{ with } R_t^{comp} = r_t^{comp}(MR_{t_2} + PSR_{t-1}). \tag{2.2.26}$$

This is the amount that the insurance company tries to distribute if possible, which we discuss now.

We now determine α_t, ρ_t and the amount R_t^{ph} to be credited to policyholders. By Lemma 2.2.1, we know that we are in one of the following four distinct cases, going from the more to the less favorable case for the insurance company and the policyholder.

Case A: $\pi_{pr} TD_t(0,\bar\rho) \geq \max\left\{R_t^G; R_t^{comp}\right\}$.
This means that the target amount can be credited to policyholders without dissolving unrealized gains if any or by realizing all the latent losses. The insurer decides then to take

$$\alpha_t = 0, \;\; \rho_t = \bar\rho,$$

and credit the target amount $R_t^{ph} = R_t^\sigma(\alpha_t,\rho_t) = \pi_{pr} TD_t(\alpha_t,\rho_t)$ to policyholders.

Case B: $\pi_{pr} TD_t(1,\bar\rho) \geq \max\left\{R_t^G; R_t^{comp}\right\}$ and $\pi_{pr} TD_t(0,\bar\rho) < \max\left\{R_t^G; R_t^{comp}\right\}$.
This means that the target amount can be credited to policyholders, but the insurer has to realize some latent gain or cannot realize all the latent loss. We assume that the insurer decides to realize as little (resp. much) as possible latent gains (resp. losses). Note that by Lemma 2.2.1, the function $\alpha \mapsto TD_t(\alpha,\bar\rho)$ cannot be constant in Case B, and the insurer has to find the value α such that $\pi_{pr} TD_t(\alpha,\bar\rho) = \max\left\{R_t^G; R_t^{comp}\right\}$. Lemma 2.2.1 gives that the function is affine with respect to α, and therefore

$$\alpha_t = \frac{\frac{1}{\pi_{pr}} \max\left\{R^G; R^{comp}\right\} - TD_t(0,\bar\rho)}{TD_t(1,\bar\rho) - TD_t(0,\bar\rho)}.$$

The insurer also takes $\rho_t = \bar\rho$ and credits then $R_t^{ph} = R_t^\sigma(\alpha_t,\rho_t) = \pi_{pr} TD_t(\alpha_t,\rho_t)$ to policyholders.

Case C: $R_t^G \leq \pi_{pr} TD_t(1,\bar\rho) < \max\left\{R_t^G; R_t^{comp}\right\}$
The target amount cannot be reached with the available latent resources, but the minimal guaranteed rate can be reached. We assume then that the insurer makes its best effort on the latent gains or losses by taking

$$\alpha_t = 1 \text{ and } \rho_t = \bar\rho.$$

The amount $R_t^{ph} = \pi_{pr} TD_t(1,\bar\rho)$ is thus credited to policyholders.

Case D: $\pi_{pr} TD_t(1,\bar\rho) < R_t^G$.
In this case, the insurance company uses the whole profit sharing reserve and takes $\rho_t = 1$. It also takes $\alpha_t = 1$ and credits then $R_t^{ph} = \max(\pi_{pr} TD_t(1,1), R_t^G)$ to the policyholders.

Thus, in all cases, the minimum guaranteed amount $R_t^G(\alpha_t, \rho_t)$ is given to policyholders. We define then the crediting rate and update the mathematical and profit sharing reserves as follows:

$$r_{ph}(t) = \frac{R_t^{ph}}{MR_{t_2} + PSR_{t-1}}, \tag{2.2.27}$$

$$MR_t = MR_{t_2}\left(1 + r_{ph}(t)\right), \tag{2.2.28}$$

$$PSR_t = PSR_{t-1} r_{ph}(t) + (1-\rho)\left(PSR_{t-1} + (CGL_t^s + LGL_t^s(\alpha))^+\right). \tag{2.2.29}$$

The profit sharing reserve at time t is thus obtained as the proportion $1-\rho$ of the realized gains and of the updated profit sharing reserve. We also update the book value of stock assets to take into account the realized gain or loss:

$$BV_{t_4}^s = BV_{t_3}^s + LGL(\alpha_t).$$

The shareholder's margin comprises a percentage $1 - \pi_{pr}$ on the amount to be distributed $TD_t(\alpha_t, \rho_t)$ minus its contribution to bail out the company when the minimal amount cannot be met:

$$AM_t = (1-\pi_{pr})\, TD_t(\alpha_t, \rho_t) - (R_t^G(\alpha_t, \rho_t) - \pi_{pr} TD_t(\alpha_t, \rho_t))^+. \tag{2.2.30}$$

Note that this contribution is only needed in Case D when $R_t^G > \pi_{pr} TD_t(1,1)$.

Table 5 details the composition of the Book value Balance sheet after the crediting operation. As mentioned previously, the capitalization reserve is managed separately

Assets	Liabilities
$BV_{t_4}^s$ $BV_{t_3}^b$	MR_t PSR_t ΔCR_t AM_t

Table 2.5 – Book value balance sheet after Step 4.

from other technical reserves. While the mathematical provision and the profit sharing reserve are linked to the performance of the portfolio, regulatory constraints require to invest the capitalization reserve in sovereign bonds Here, we assume that it is invested in a one period zero-coupon bond. Since the capitalization reserve belongs to the equity capital of the insurance company, the interests coming from the capitalization reserve are given to shareholders. Their cash flow is then the sum of the accounting margin AM_t and the yield of the capitalization reserve:

$$P\&L_t = AM_t + CR_{t-1}\left(\frac{1}{P(t-1,t)} - 1\right). \tag{2.2.31}$$

Step 5: externalization of the shareholders' margin and of the capitalization reserve from the accounting

The margin AM_t that determines the accounting return on capital invested by shareholders on $(t-1,t)$ must be removed from the balance sheet. The same has to be done for the capitalization reserve movement ΔCR_t, since the capitalization reserve is handled separately. One then has to clear the amount $AM_t + \Delta CR_t$ from the balance sheet. If this amount were externalized in cash, one would have to calculate the capital gains made on it: the gain on equity assets should then be distributed to policyholders with the participation rate π_{pr} and the gain on bonds should modify the capitalization reserve. Thus, one would have to repeat the previous steps indefinitely. To avoid this difficulty, we assume if $AM_t + CR_t > 0$ that a fraction of the assets corresponding to the accounting value $AM_t + CR_t$ is removed. This amounts to fund the shareholders' margin and the capitalization reserve with a fraction of the portfolio instead of cash. If $AM_t + CR_t < 0$, we simply buy the quantity of assets and bonds with weights w_t^s and w_t^b that corresponds to this book value.

This procedure leads to the following update of the stock book value

$$BV_t^s = BV_{t_4}^s \left(1 - \frac{(AM_t + \Delta CR_t)^+}{BV_{t_4}}\right) + w_t^s (AM_t + \Delta CR_t)^-,$$

where $BV_{t_4} = BV_{t_4}^s + BV_{t_3}^b$ since the bond book value is unchanged at Step 4. The corresponding position is

$$\phi_t^s = \phi_{t_3}^s \left(1 - \frac{(AM_t + \Delta CR_t)^+}{BV_{t_4}}\right) + \frac{w_t^s (AM_t + \Delta CR_t)^-}{S_t}$$

We do the same for the bonds and have

$$BV_t^b = BV_{t_3}^b \left(1 - \frac{(AM_t + \Delta CR_t)^+}{BV_{t_4}}\right) + w_t^b (AM_t + \Delta CR_t)^-$$

with the corresponding position

$$\phi_t^b = \phi_{t_4}^b \left(1 - \frac{(AM_t + \Delta CR_t)^+}{BV_{t_4}^G}\right) + \frac{w_t^b (AM_t + \Delta CR_t)^-}{\bar{B}(t, n, \mathbf{c}_t)}.$$

Note that for simplicity, we assume here that we can buy (when $AM_t + CR_t < 0$) bonds with time to maturity i and coupon c_t^i. It would have been possible to buy bonds at par, but this would require then to modify again the coupon rates accordingly, similarly to Step 3. Table 2.1 represents the balance sheet at the end of Step 5 and thus at the and of the whole reallocation procedure.

2.2.3 Closing of the strategy at time T

We now describe how the ALM portfolio is closed at time T. The insurance company starts with the implementation of Step 1 described in Section 2.2.2. Things change then since the insurer has to liquidate the portfolio. Since the insurer closes its portfolio, we consider the policyholders that exit on $(T-1, T)$ and the others that exit at time T in the same way. All the assets are sold and all the capital gains or losses are realized, and the profit sharing reserve is released. The capital gain or loss realized when liquidating the stock portfolio is given by:

$$CGL_T^s = \phi_{T-1}^s \left(S_T - \frac{BV_{T-1}^s}{\phi_{T-1}^s}\right).$$

We now focus on the equally weighted basket of bonds. Keeping in mind that the bond with shortest time to maturity has come due, the bond portfolio comprises bonds from maturity 1 to $n-1$, and the capital gain or loss is therefore given by:

$$CGL_T^b = \phi_{T-1}^b \left(\frac{1}{n} \sum_{i=1}^{n-1} B(T, T+i, c_{T-1}^{i+1}) \right) - \left(BV_{T-1}^b - \frac{\phi_{T-1}^b}{n} \right).$$

This quantity impacts the capitalization reserve level as follows

$$CR_T = (CR_{T-1} + CGL_T^b)^+.$$

The terminal bonus declaration is rather simple. The insurer must liquidate the profit sharing reserve since it belongs to its policyholders and comply with the minimum guaranteed rate of return r^G. Let us define

$$TD_T = FI_T - (CR_{T-1} + CGL_T^b)^- + PSR_{T-1} + CGL_T^s, \tag{2.2.32}$$

the amount to distribute to policyholders. The credited amount to policyholders is:

$$R_T^G = \max \left\{ \pi_{pr} TD_T, r^G (MR_{T-1} + PSR_{T-1}) \right\}. \tag{2.2.33}$$

Note that we do not consider for the final date a competitor rate since all the contracts terminate. We then define the crediting rate $r_{ph}(T) = \frac{R_T^G}{MR_{T-1}+PSR_{T-1}}$, the mathematical reserve $MR_T = MR_{T-1}(1 + r_{ph}(T))$ and $PSR_T = r_{ph}(T) PSR_{T-1}$, exactly as in Equations (2.2.27), (2.2.28) and (2.2.29).

We define then the final accounting margin of the shareholders by

$$AM_T = (1 - \pi_{pr}) TD_T - (R_T^G - \pi_{pr} TD_T)^+.$$

Since the capitalization reserve RC_T is a part of the equity of the insurance company, it is given to shareholders. The terminal shareholders P&L is then:

$$P\&L_T = AM_T + CR_{T-1} \left(\frac{1}{B(T-1, T)} - 1 \right) + CR_T. \tag{2.2.34}$$

At the maturity of the contract, the mathematical and profit sharing reserves MR_T must be paid to policyholders. The terminal cash outflow is thus

$$COF_T = MR_T + PSR_T. \tag{2.2.35}$$

2.2.4 Overall performance of the ALM

To assess the solvency situation, Solvency II regulation requires to value assets and liabilities on a market-consistent basis. It prescribes to use the Best Estimates Liabilities and the Basic Own-Funds (also called Net Asset Value) to value the liability of the company. We explicit these key quantities in our framework.

The Basic Own-Funds (BOF) corresponds to the market-consistent valuation of the equity capital of the firm. It is determined as the present value of future shareholders P&L cash flows under the pricing measure $\mathbb{Q}$. If we consider a short-rate model $(r_t, t \geq 0)$, the BOF is given as follows:

$$BOF_0 = \mathbb{E}^{\mathbb{Q}} \left[\sum_{t=1}^{T} e^{-\int_0^t r_s ds} P\&L_t \right]. \tag{2.2.36}$$

More generally, if $(\mathcal{F}_t, t \geq 0)$ is the filtration representing the market information, we can define the Basic Own-Funds at time $t \in \{0, \dots, T-1\}$ as

$$BOF_t = \mathbb{E}^{\mathbb{Q}} \left[\sum_{u=t+1}^{T} e^{-\int_t^u r_s ds} (P\&L_u) \middle| \mathcal{F}_t \right]$$

.

The Best Estimates Liability (BEL) represents the total debt of the insurer. It corresponds to the discounted sum (present-value) of future surrender cash outflows and terminal liability payment

$$BEL_0 = \mathbb{E}^{\mathbb{Q}} \left[\sum_{t=1}^{T} e^{-\int_0^t r_s ds} COF_t \right]. \tag{2.2.37}$$

We more generally define

$$BEL_t = \mathbb{E}^{\mathbb{Q}} \left[\sum_{u=t+1}^{T} e^{-\int_t^u r_s ds} COF_u \middle| \mathcal{F}_t \right] \text{ for } t \in \{0, \dots, T-1\}$$

Since we are considering a pool of policyholders running off, all these cash flows are generated from the initial Mathematical Reserve. We therefore get the so-called "no-leakage" condition:

$$MR_0 = BOF_0 + BEL_0.$$

More generally, we have

$$\forall t \in \{0, \dots, T-1\},\ MR_t + PSR_t + CR_t = BOF_t + BEL_t.$$

2.2.5 Solvency Capital Requirement of the ALM with the standard formula

To determine the Solvency Capital Requirement (SCR), the supervision authority provides a standard formula that consists in various stress tests for different type of risks. The risks are divided between modules and sub-modules and combined into a global SCR for market risk according to an aggregation formula. The detailed description of the SCR calculation can be found in the note written by the European Insurance and Occupational Pensions Authority (EIOPA), Section 2 of [EIOPA12]. Short descriptions of the standard formula can be found in the appendix of [BSS15] or in Subsection 3.2 of [Boo17]. In our model that takes into account equity and riskless bonds, we only have to consider the equity and interest rate modules, and we briefly explain how to use the standard formula in our framework.

Remark 2.3. *Besides equity and interest rate modules, there are many other SCR risk modules that take part into the SCR for market risk. Among them, the spread risk module that deals with credit risk is perhaps the most important one in practice. In principle, it is possible to extend our model and to add corporate bonds. Thus, the insurance company targets to invest the portfolio according to the nonnegative weights* w_t^s, w_t^b *and* w_t^{cb} *that describe respectively the proportion invested in stocks, sovereign bonds and corporate bonds. One has then to precise the portfolio of corporate bonds, and the difficulty is to keep the model tractable. A simple choice would be to consider a pool of* $N_{cb} \in \mathbb{N}^*$ *different companies with the same credit grade and to assume that* *the corporate bonds are uniformly invested among these companies, i.e. each corporate bond has a relative weight* $1/N_{cb}$*. Let denote by* $\mu > 0$ *their common default intensity and define* $B_\mu(t,n,c) = \sum_{i=1}^{n} ce^{-\mu i}P(t,t+i) + e^{-\mu n}P(t,t+n)$ *the price at time* t *of a corporate bond with coupon* c *and maturity* $t+n$*, and* $\bar{B}_\mu(t,n,\tilde{\mathbf{c}}_t) = \frac{1}{n}\sum_{i=1}^{n} B(t,i,\tilde{c}_t^i)$ *for* $\tilde{\mathbf{c}}_t = (\tilde{c}_t^i)_{i\in\{1,\dots,n\}}$*. The reallocation procedure described in Subsection 2.2.2 can then be adapted to have*

$$\phi_t^s = \frac{w_t^s MV_t}{S_t}, \phi_t^b = \frac{w_t^b MV_t}{\bar{B}(t,n,\mathbf{c}_t)}, \text{ and } \phi_t^{cb} = \frac{w_t^{cb} MV_t}{\bar{B}_\mu(t,n,\tilde{\mathbf{c}}_t)}$$

but it requires to model a few other issues:

- *The number of defaults between* $t-1$ *and* t*. If we assume independence, it follows a binomial distribution with parameters* N_{cb} *and* $1-e^{-\mu}$*.*
- *The recovery value of defaulted bonds. It can be taken equal to zero or to a deterministic proportion of the nominal value.*

Last, the standard formula for the SCR spread module can then be implemented by applying a stress on μ*. Clearly, the spread module is directly affected by the model chosen for the recovery value and for the dependence between defaults. This is left for future research. Since the description of the ALM model is already quite complex, we have made the choice for clarity to only consider equity and riskless bonds.*

Equity module

The SCR for equity risk SCR_{eq} is determined by the variation of the Basic Own-Funds BOF_0 after a negative shock on the equity asset class that occurs immediately after time 0, i.e. after the first asset allocation. The negative shock $s^{eq} \in (-1,0)$ assumes that the value of stock assets decreases with a certain percentage. The shock prescribed by the EIOPA may differ according to the type of equity, see [EIOPA12] p. 140 and Section 3 of [GM12]. Here, we recall that S should be seen as a weighted average of stocks (like indices) in which the insurance company invests. Thus, we assume that

$$S_{0+}^{shock} = S_0\left(1+s^{eq}\right),$$

where s^{eq} is the corresponding average of the shocks prescribed by the EIOPA. We note $BOF_0^{eq\_shock}$ the Basic Own-Funds calculated with this shock. The SCR for equity risk is then defined by:

$$SCR_{eq} = (BOF_0^{eq\_shock} - BOF_0)^- = (BOF_0 - BOF_0^{eq\_shock})^+ \tag{2.2.38}$$

Interest rate module

To estimate the solvency capital for the risk on interest rates, the EIOPA provides upward and downward shocks to the initial term structure. As for the equity, the shocks are assumed to occur immediately after the first allocation at time 0. Let us suppose that we observe at time 0 market prices of zero-coupon bonds $t \mapsto P^{mkt}(0,t)$ and we note $R^{mkt}(0,t) = -\frac{1}{t}\log(P^{mkt}(0,t))$ the corresponding yield curve. The shifted yield curves are then given by:

$$R^{up/down}(0,t) = (1+s_t^{up/down})R^{mkt}(0,t), \tag{2.2.39}$$

where s_t^{up} (resp. s_t^{down}) is the upward (resp. downward) shock to the yield with maturity t. These coefficients have been recommended in p. 137 of [EIOPA12] and implemented by the European Commission in the Articles 166 and 167 of the Delegated Regulation [Com15][1]. They are summarized in the Table 2.6 below.

t	1	2	3	4	5	6	7	8	9	10
s_t^{up}	70%	70%	64%	59%	55%	52%	49%	47%	44%	42%
s_t^{down}	−75%	−65%	-56%	-50%	-46%	-42%	39%	-36%	-33%	-31%

t	11	12	13	14	15	16	17	18	19	20
s_t^{up}	39%	37%	35%	34%	33%	31%	30%	29%	27%	26%
s_t^{down}	−30%	−29%	-28%	-27%	-28%	-28%	-28%	-28%	-29%	-29%

Table 2.6 – Stress factors of the standard formula given by the EIOPA [EIOPA12] in December 2012

For years $t \geq 90$, the regulator prescribes the shocks $s_t^{up} = 0.2$ and $s_t^{down} = -0.2$. Between $t_a = 20$ and $t_b = 90$ years, a the linear interpolation method has to be used to get the shocks:

$$\forall t \in [t_a, t_b],\ s_t^{up/down} = s_{t_a}^{up/down} + (s_{t_b}^{up/down} - s_{t_a}^{up/down})\frac{t - t_a}{t_b - t_a}. \tag{2.2.40}$$

The SCR for up and down shock are determined by the variation of the basic own-funds if the stressed yield-curve is used instead of the initial term-structure. Namely, we set $SCR_{up} = (BOF_0 - BOF_0^{up})^+$ and $SCR_{down} = (BOF_0 - BOF_0^{down})^+$. The SCR for the risk on interest rates is defined as the worst one of the two shocks:

$$SCR_{int} = \max(SCR_{up}, SCR_{down}). \tag{2.2.41}$$

Remark 2.4. *Let us note* $f^{mkt}(0,t) = -\log\left(\frac{P^{mkt}(0,t+1)}{P^{mkt}(0,t)}\right) = (t+1)R^{mkt}(0,t+1) - tR^{mkt}(0,t)$*, that can be seen as a forward rate on* $(t,t+1)$*. Let* $(s_t)_{t\in\mathbb{N}^*}$ *be prescribed (deterministic) shocks and* $R^{shock}(0,t) = R^{mkt}(0,t)(1+s_t)$*. Let* $f^{shock}(0,t) = (t+1)R^{shock}(0,t+1) - tR^{shock}(0,t)$ *the stressed forward rate. We then have*

$$f^{shock}(0,t) - f^{mkt}(0,t) = (t+1)s_{t+1}R^{mkt}(0,t+1) - ts_tR^{mkt}(0,t).$$

For large maturity t*, it is likely to have* $R^{mkt}(0,t+1) \approx R^{mkt}(0,t)$*, which gives*

$$f^{shock}(0,t) - f^{mkt}(0,t) \approx s_{t+1}R^{mkt}(0,t+1) + t(s_{t+1} - s_t)R^{mkt}(0,t).$$

Due to the multiplication by t*, even small variations of the stress factor (and also of* $s_{t+1} - s_t$*) may lead to important variations of the shocked forward rate. For example, if we assume for simplicity* $R^{mkt}(0,t) = r$ *for all* t*, the downward shock of Table 2.6 gives* $f^{down}(0,13) = r - \frac{27}{100}r + \frac{13}{100}r = \frac{86}{100}r$ *and* $f^{down}(0,14) = \frac{58}{100}r$*, leading to an important variation of the downward shocked forward rate between maturities* 13 *and* 14*. The same is observed with* $f^{down}(0,18) = r - \frac{28}{100}r + \frac{18}{100}r = \frac{90}{100}r$ *and* $f^{down}(0,19) = \frac{71}{100}r$*.*

[1] Besides, a minimal increase (resp. decrease) of 1% is assumed for $R^{up}(0,t)$ (resp. $R^{down}(0,t)$), see also Boonen [Boo17], p. 411.

This methodology has been set up when the interest rates were around 2% or 3%, but is no longer relevant for very low or even negative interest rates. If we suppose for simplicity that $R^{mkt}(0,t) = 0$, then the multiplicative stress rule (2.2.39) leaves rates unchanged and therefore leads to a null SCR. Also, for negative rates, the formula (2.2.39) inverts the sign of the stress: the upward stress factor leads to a decrease of the interest rate and conversely. To bypass this issue, the EIOPA has recently recommended in 2018 [EIOPA18] to add an additive factor, i.e. to replace (2.2.39) by

$$R^{up/down}(0,t) = (1 + s_t^{up/down})R^{mkt}(0,t) + b_t^{up/down}. \tag{2.2.42}$$

Between $t_a = 20$ and $t_b = 90$ years, the interpolation formula (2.2.40) is kept for s_t. An analogous formula is used for $b_t^{up/down}$:

$$\forall t \in [t_a, \tilde{t_b}],\; b_t^{up/down} = b_{t_a}^{up/down} + (b_{\tilde{t_b}}^{up/down} - b_{t_a}^{up/down})\frac{t - t_a}{\tilde{t_b} - t_a},$$

with $t_a = 20$, $\tilde{t_b} = 60$ years and $b_t^{up/down} = 0$ for $t \geq \tilde{t_b}$.

t	1	2	3	4	5	6	7	8	9	10
s_t^{up}	61%	53%	49%	46%	45%	41%	37%	34%	32%	30%
s_t^{down}	-58%	-51%	-44%	-40%	-40%	-38%	-37%	-38%	-39%	-40%
b_t^{up}	2.14%	1.86%	1.72%	1.61%	1.58%	1.44%	1.30%	1.19%	1.12%	1.05%
b_t^{down}	-1.16%	-0.99%	-0.83%	-0.74%	-0.71%	-0.67%	-0.63%	-0.62%	-0.61%	-0.61%
t	11	12	13	14	15	16	17	18	19	20
s_t^{up}	30%	30%	30%	29%	28%	28%	27%	26%	26%	25%
s_t^{down}	-41%	-42%	-43%	-44%	-45%	-47%	-48%	-49%	-49%	-50%
b_t^{up}	1.05%	1.05%	1.05%	1.02%	0.98%	0.98%	0.95%	0.91%	0.91%	0.88%
b_t^{down}	-0.60%	-0.60%	-0.59%	-0.58%	-0.57%	-0.56%	-0.55%	-0.54%	-0.52%	-0.50%

Table 2.7 – Stress factors of the standard formula given by the EIOPA [EIOPA18] in February 2018.

Aggregation Formula

The SCR for market risk is a combination between the equity and interest rate risk in our framework. It is defined as follows (see Articles 164 and 165 of [Com15]):

$$SCR_{mkt} = \sqrt{SCR_{eq}^2 + SCR_{int}^2 + 2\varepsilon SCR_{eq}SCR_{int}} \tag{2.2.43}$$

where the “correlation factor” $\varepsilon = 0$ if the interest rate exposure is due to the upward shock and $\varepsilon = \frac{1}{2}$ if it is due to the downward shock.

2.3 Asset Model

The insurance company invests policyholders’ deposits between two asset classes: riskless bonds and stocks. Therefore, we have to model the equity asset S_t and the interest rates, for which we choose a short-rate model $(r_t, t \geq 0)$. Since we are considering a portfolio on a long run with a low rebalancing frequency, we chose (as it is usually done for ALM, see e.g. [BG15], [BKP16] or [GGH+08]) simple models with a clear parametrization. Of course, more sophisticated models exist for hedging and pricing purposes for shorter time horizon, but they are not really suited for ALM purposes. We denote $(\Omega, \mathcal{F}, \mathbb{Q})$ a risk-neutral probability space and $(\mathcal{F}_t, t \geq 0)$ the filtration that represents market information. Let $(W_t, Z_t)_{t\geq 0}$ be a standard two-dimensional Brownian motion under $\mathbb{Q}$ and we set

$$Z_t^\gamma = \gamma W_t + \sqrt{1-\gamma^2} Z_t, \text{ for } \gamma \in [-1, 1]. \tag{2.3.1}$$

Since we will mainly focus in this paper on the SCR valuation with the standard formula that has to be made under a risk-neutral measure, we will not model the asset dynamics under the real world probability. However, modeling for both risk-neutral and real world probabilities is relevant for ALM to determine, for example, an optimal asset allocation under SCR constraints. This is however beyond the scope of the paper.

Before specifying the equity and interest rate dynamics, we first describe the surrender rate model for policyholders. We consider that the surrender rate is the sum of a component $\underline{p} \in (0, 1)$ quantifying structural surrenders and a market contingent surrender rate $DSR(\Delta_t)$, where $\Delta_t = r_{ph}(t) - r_t^{comp}$ is the spread between the crediting rate to policyholders $r_{ph}(t)$ defined in (2.2.27) and the competitor rate r_t^{comp} defined by (2.2.25). The function DSR is defined as follows

$$DSR(\Delta) = \begin{cases} DSR_{max} & \text{for } \Delta < \alpha, \\ DSR_{max}\dfrac{\beta - \Delta}{\beta - \alpha} & \text{for } \alpha \leq \Delta \leq \beta, \\ 0 & \text{for } \Delta > \beta, \end{cases} \tag{2.3.2}$$

where $DSR_{max} \in (0, 1 - \underline{p})$ is the maximum surrender rate, α the massive surrender threshold and β the triggering surrender threshold. Therefore, surrenders occur with a proportion p_t^e also called exit rate:

$$p_t^e = \underline{p} + DSR(\Delta_t). \tag{2.3.3}$$

We assume that the equity asset follows a Black-Scholes model:

$$S_t = S_0 \exp\left(\int_0^t r_s ds + \sigma_S W_t - \frac{\sigma_S^2}{2} t\right), \tag{2.3.4}$$

where $\sigma_S > 0$ is the volatility. Concerning interest rates, we consider a priori two different models: the shifted Vasicek model (called Vasicek++ later on, see Brigo and Mercurio [BM06] Sections 3.2.1 and 3.8) and the Hull and White model (see e.g. [BM06] Section 3.3), that allow for negative interest rates. The Vasicek++ model assumes that

$$r_t = x_t + \varphi(t), \tag{2.3.5}$$

$$x_t = x_0 + \int_0^t k(\theta - x_s) ds + \sigma_r Z_t^\gamma, \tag{2.3.6}$$

for some parameters $x_0, \theta \in \mathbb{R}$, $k, \sigma_r > 0$, and $\varphi : \mathbb{R}_+ \to \mathbb{R}$. The Hull and White model assumes that

$$r_t = r_0 + \int_0^t k(\vartheta(s) - r_s) ds + \sigma_r Z_t^\gamma, \tag{2.3.7}$$

with parameters $r_0 \in \mathbb{R}$, $k, \sigma_r > 0$, and $\vartheta : \mathbb{R}_+ \to \mathbb{R}$. Both models have very similar properties: these are Gaussian models with explicit zero-coupon bond prices and closed formula for caplets, and we refer to [BM06] for further details and for examples of calibration to market data. In fact, as noticed by Brigo and Mercurio ([BM06], p. 101) the two models are identical if we take

$$\vartheta(t) = \theta + \varphi(t) + \frac{1}{k}\varphi'(t). \quad (2.3.8)$$

Let us note however that the Vasicek++ parametrization offers a slightly larger class of dynamics: for any piecewise continuous function ϑ, we can find a piecewise C^1 function φ such that (2.3.8) holds. Instead, there is no ϑ satisfying (2.3.8) when φ is piecewise continuous and not differentiable. We just recall the zero-coupon bond prices at time t with maturity $T \geq t$ in the Vasicek++ model:

$$P^{V++}(r_t, t, T) = A(T-t)\exp\left(-\int_t^T \varphi(s)ds - (r_t - \varphi(t))g_k(T-t) - \theta(T-t-g_k(T-t))\right),$$
$$g_k(t) = \frac{1-e^{-kt}}{k}, \quad A(t) = \exp\left(\frac{\sigma_r^2}{2k^2}g_k(t) - \frac{\sigma_r^2}{4k}g_k^2(t)\right)$$

and in the Hull and White model:

$$P^{HW}(r_t, t, T) = A(T-t)\exp\left(-r_t g_k(T-t) - \int_t^T (1-e^{-k(T-s)})\vartheta(s)ds\right).$$

The methodology to calibrate these models to market data (as required by the regulation, see e.g. [VEKLP17] p.8) is the same. For each parameter set $(x_0, \theta, k, \sigma_r)$ (resp. (r_0, k, σ_r)), there exists a unique deterministic function φ (resp. ϑ) that perfectly fits the zero-coupon bond prices $P^{mkt}(0,t)$ observed in the market (or deduced from market data). Therefore, one tries to find the parameters that better fit the market data on options such as caplet or swaption prices, and then pick the corresponding function φ or ϑ. These models comply with the Solvency II regulation that imposes to fit the initial term structure of interest rates and to approximate well market prices of the options. To perform the perfect fit of the zero-coupon bond prices, one typically assumes some parametrization of the functions φ and ϑ. A typical choice is to assume these functions to be piecewise constant or piecewise linear. Once parameterizations are chosen for φ and ϑ, the Vasicek++ and Hull and White model may no longer be the same: the Vasicek++ model with piecewise constant φ is not a Hull and White model with piecewise constant ϑ, and conversely. In what follows, we argue that the parametrization of the Vasicek++ model is much more convenient for dealing with the standard formula.

To implement the standard formula described in Section 2.2.5, one has to re-calibrate the models to the stressed zero-coupon curve. Since the stressed factors given in Tables 2.6 and 2.7 are given on an annual basis, it is rather natural to consider piecewise constant shapes for $\varphi(t)$ and $\vartheta(t)$:

$$\varphi(t) = \sum_{i=0}^{\infty} \varphi_i \mathbb{1}_{t\in[i,i+1[}, \ \vartheta(t) = \sum_{i=0}^{\infty} \vartheta_i \mathbb{1}_{t\in[i,i+1[}.$$

We denote by φ^{mkt} (resp. ϑ^{mkt}) the function calibrated in the Vasicek++ (resp. Hull and White) model to $P^{mkt}(0,t)$ and φ^{shock} (resp. ϑ^{shock}) the function calibrated to $\exp\left(-t[(1+s_t)R^{mkt}(0,t)+b_t]\right)$. We keep the parameters $(x_0, \theta, k, \sigma_r)$ constant (i.e. as before the shock) for the Vasicek++ model, and the shock is then entirely absorbed by the shift function φ. For the Hull and White model, a first idea would be to keep the parameters (r_0, k, σ_r) constant and change the function ϑ. However, this leads to unreasonable large or low values of ϑ, as well as unrealistic oscillations, which are due to the mean reversion. Let us take for example the case of the upward shock. Since r_t starts from r_0, ϑ_0 has to be very high to reproduce the shock. Then, the value of r_1 is likely to be very high and one has to take ϑ_1 very low to fit the shock, and so on. To reduce these fluctuations, we take $(1 + s_1)r_0 + b_1$ as the initial short rate value after the shock and keep the parameters (k, σ_r) constant. This reduces the fluctuations, but we already observe in ϑ^{shock} in Figure 2.1 that they remain quite significant. From the zero-coupon bond price formulas, we easily get:

$$\begin{aligned}\exp\left(-t[s_t R^{mkt}(0,t) + b_t]\right) &= \exp\left(\int_0^t \varphi^{mkt}(s) - \varphi^{shock}(s)ds\right)\\ &= \exp\left(-(s_1 r_0 + b_1)g_k(t) + \int_0^t (1 - e^{-k(t-s)})(\vartheta^{mkt}(s) - \vartheta^{shock}(s))ds\right).\end{aligned}$$

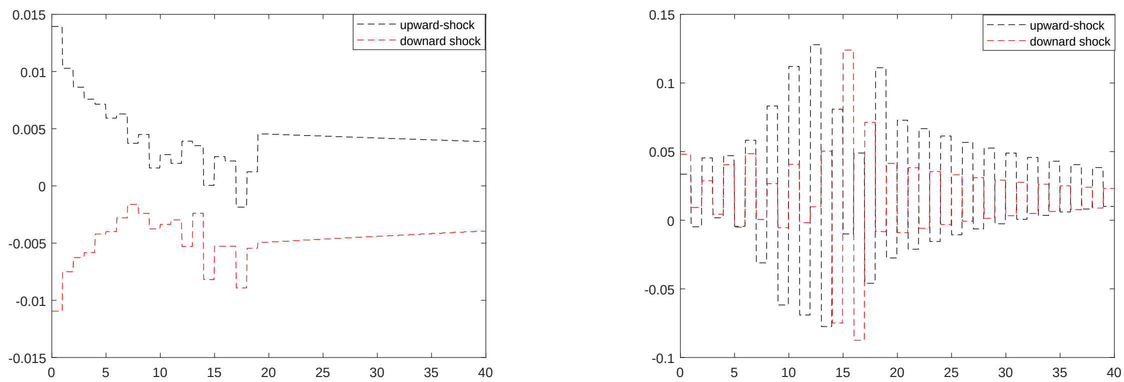

Figure 2.1 – Calibrated piecewise constant functions $t \mapsto \varphi^{shock}(t)$ (left) and $t \mapsto \vartheta^{shock}(t)$ (right) after the upward and downward shocks specified in Table 2.6 with no additive shock (i.e. $b_t = 0$), $r_0 = 0.02$ and $k = 0.2$.

Figure 2.1 illustrates the calibrated functions. We have considered the case where the zero-coupon bond prices $P^{mkt}(0,t)$ are given by a Vasicek model with $r_0 = \theta = 0.02$, $k = 0.2$ and $\sigma = 0.01$. We assume $r_t^{comp} = r_t$ and constant allocation targets $w_t^b = 0.95$ and $w_t^s = 0.05$. The first striking point is the oscillations of ϑ^{shock}, making the Hull and White model poorly realistic after the shock. Instead, the variations of φ^{shock} are much more reasonable. We still observe some unlikely moves between years 10 and years 20: as explained in Remark 2.4, this is due to small variations of the stress factors that are amplified by the maturity. Thus, the Vasicek++ model has much more meaning after the shock than the Hull and White model. We have plotted in Figure 2.2 the crediting rate to policyholders, as well as the empirical distribution of the cases A, B, C and D described in Step 4 that determines the crediting rate to policyholders. We observe significant oscillations of the mean crediting rate for the Hull and White model, that can be seen also from the important oscillations between the proportions of case A and case C. These oscillations come from the fluctuations of r_t, which is taken as the competitor rate. In contrast, the mean crediting rate and the distribution of the cases A, B, C and D is much more regular in the Vasicek++ model. We have also done the same analysis for the downward shocks: oscillations again appear in the Hull and White model, but they are less marked because of the minimum guaranteed rate.

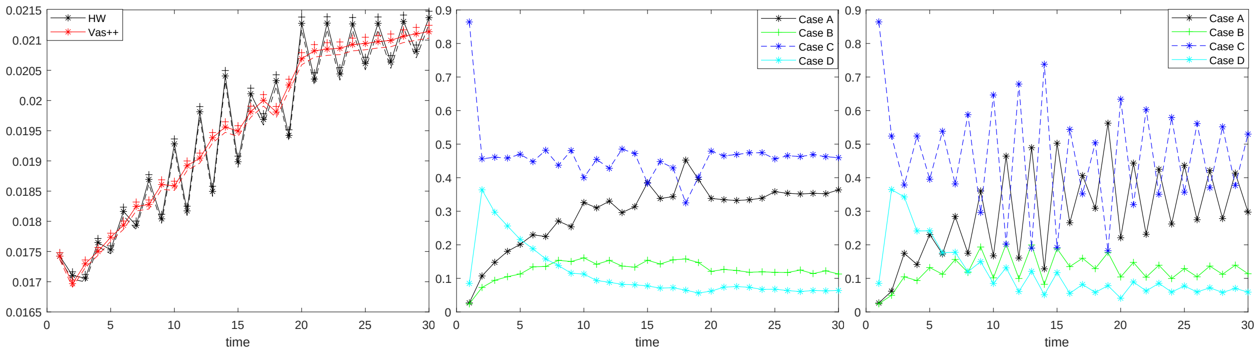

Figure 2.2 – Simulations after the upward shock on interest rates described in Table 2.6. Left: mean of the crediting rate $r_{ph}(t)$ defined in (2.2.27) with the 95% confidence interval. Middle (resp. Right): proportions of the cases A, B, C and D in the Vasicek++ (resp. Hull and White) model.

We now focus on SCR_{int} defined by (2.2.41). Figure 2.3 shows the value of the SCR in function of k when the central model is a Vasicek model with $r_0 = \theta = 0.02$, $\sigma = 0.01$ and k. We observe almost the same value for SCR_{down} when comparing the Vasicek++ model and the Hull and White model, but there is a significant difference for SCR_{up} in favor of the Vasicek++ model, which then affects then SCR_{int} when the upward shock has a greater contribution than the downward shock.

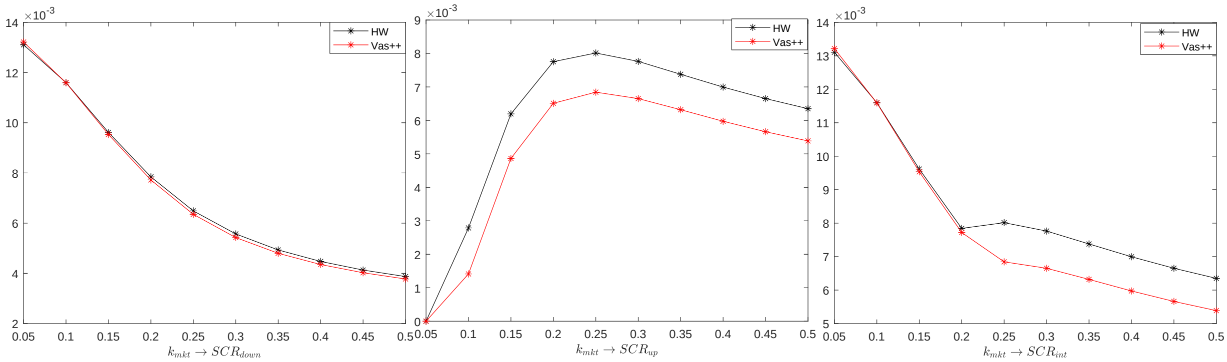

Figure 2.3 – SCR values with Vasicek++ and Hull and White models in function of k for SCR_{down} (left), SCR_{up} (middle) and SCR_{int} (right).

Last, we have plotted in Figure 2.4 the functions φ^{shock} and ϑ^{shock} after the upward and downward shocks with the new recommendations of the EIOPA given in Table 2.7. Here, the zero-coupon bond prices $P^{mkt}(0,t)$ are given by a Vasicek model with $r_0 = \theta = 0.005$, $k = 0.2$ and $\sigma = 0.01$. We observe even more oscillations with the Hull and White models, which makes this model irrelevant after the shocks. Surprisingly, for the Vasicek++ model, we notice that the shifted functions cross after 30 years: φ^{up} (resp. φ^{down}) becomes negative (resp. positive). Thus, the upward (resp. downward) shock on the risk-free interest rates leads to a downward (resp. upward) shock on the spot rate after approximately year 35, which is puzzling. This behavior is mostly due to the phasing out of the additive term that is less innocuous as one may think. In the simplest constant rate model, we observe that stopping the phasing out of b at time 60 has a significant effect.

This study tends to show that shifted models such as Vasicek++ or CIR++ have much more meaning after the shocks prescribed by the EIOPA than mean-reverting

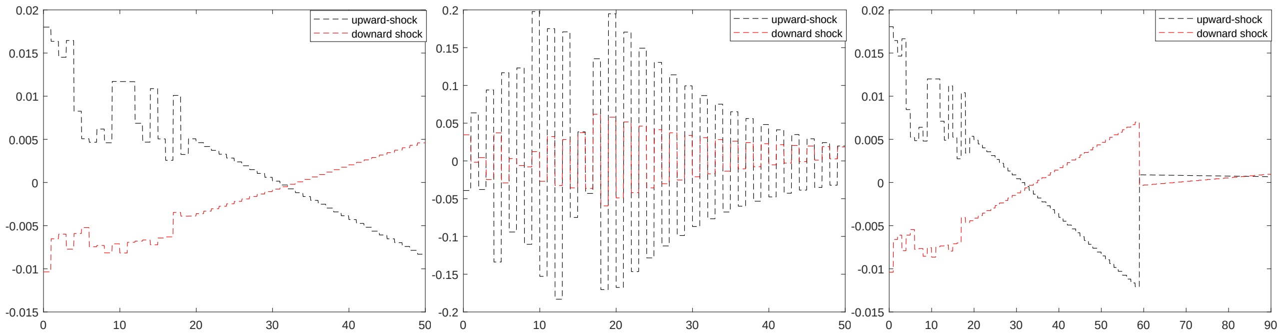

Figure 2.4 – Calibrated piecewise constant functions $t \mapsto \varphi^{shock}(t)$ (left) and $t \mapsto \vartheta^{shock}(t)$ (middle) after the upward and downward shocks specified in Table 2.7. Right: calibrated piecewise constant functions $t \mapsto \varphi^{shock}(t)$ for a constant rate model (Vasicek model with $r_0 = \theta = 0.005$ and $\sigma = 0$).

curve models such as Hull and White or Black and Karasinski models. In shifted models, the shock is translated in the shifted functions φ that directly affects r_t. Instead, for mean-reverting models, the variations of ϑ only impacts r_t (and thus $R(0,t)$) after some time (typically $1/k$), it is necessary to have strong variations of ϑ to follow the variations of $R(0,t)$ at each year t. This explains heuristically why oscillations are observed to fit the shocked curve. Thus, in our numerical experiments, we will work with the Vasicek++ model.

2.4 Numerical results

In this section, we provide numerical results for different model parameters. We compute the solvency capital requirement of the insurance company for an ALM portfolio over $T = 30$ years by using the standard formula. We study and discuss the impact of the shocks prescribed by the standard formula, and the corresponding values of the SCR modules. In our simulations, we sample exactly N paths $(r^i_t, \int_{t-1}^t r^i_s ds, S^i_t)_{t\in\{1,\dots,T\}}$, for $1 \le i \le N$, and use the same simulations for the central and shocked frameworks: this gives a fair comparison between different settings and models. More precisely, given x_t and W_t, we know from (2.3.6) and (2.3.1) that $\left(x_{t+\Delta}, \int_t^{t+\Delta} x_s ds, W_{t+\Delta}\right)$ follows a multivariate normal distribution with mean $(x_t e^{-k\Delta} + \theta(1-e^{-k\Delta}), (x_t-\theta)\frac{1-e^{-k\Delta}}{k} + \theta\Delta, W_t)$ and covariance

$$\begin{bmatrix} \sigma^2 \frac{1-e^{-2k\Delta}}{2k} & \frac{\sigma^2}{2}\left(\frac{1-e^{-k\Delta}}{k}\right)^2 & \sigma\gamma\frac{1-e^{-k\Delta}}{k} \\ \frac{\sigma^2}{2}\left(\frac{1-e^{-k\Delta}}{k}\right)^2 & \frac{\sigma^2}{k^2}\left(\Delta - \frac{1-e^{-k\Delta}}{k}\right) - \frac{1}{2k}\left(\frac{1-e^{-k\Delta}}{k}\right)^2 & \sigma\gamma\frac{k\Delta-1+e^{-k\Delta}}{k^2} \\ \sigma\gamma\frac{1-e^{-k\Delta}}{k} & \sigma\gamma\frac{k\Delta-1+e^{-k\Delta}}{k^2} & \Delta \end{bmatrix}.$$

From (2.3.5) and (2.3.4), we then obtain N independent paths $(r^i_t, \int_{t-1}^t r^i_s ds, S^i_t)_{t\in\{1,\dots,T\}}$, $1 \le i \le N$, and can calculate the BOF estimator

$$\frac{1}{N}\sum_{i=1}^N \sum_{t=1}^T e^{-\int_0^t r^i_s ds} P\&L^i_t \tag{2.4.1}$$

of (2.2.36) in the central and shocked frameworks. In the first subsection, we present general results with the shocks given by the present regulation in Europe [Com15] and with interest rates around 2%. Then, we analyze in the second subsection the importance of the cash flow matching in the ALM, and discuss its impact on the SCR. In the third subsection, we do a similar analysis with low interest rates around 0.5% using the last table of shocks recommended by the EIOPA (Table 2.7).

2.4.1 Analysis of the SCR with the standard formula

We work with the parameters describes in Tables 4.1 and 4.2: the interest rate model follows a Vasicek model mean-reverting around 2% while the minimum guaranteed rate is set at $r^G = 1.5\%$. Thus, this is a setting somehow well balanced, in the sense that all the cases A, B, C and D that determine the crediting rate occur with a significant proportion. This is confirmed by the empirical distribution plotted at the left in Figure 2.5. If r^G were higher (resp. lower) we would observe mostly cases C and D (resp. A and B).

To determine the constant allocation in stock and bond that we consider in our simulations, we have drawn in the right of Figure 2.5 the different SCR components as a function of w^s, as well as the global SCR given by formula (4.3.3). We use the shocks given by Table 2.6. In our simulations, we are looking for an allocation that makes the SCR on equity and the SCR on interest rates of the same order, since the aggregation formula (4.3.3) somehow encourages to diversify the risk components. This is achieved by $w^s = 0.05$. Note that in this case, this is also the allocation that minimizes the SCR. As one may expect, the SCR on equity is increasing with respect to w^s. The risk-neutral valuation dissuades from taking risk and is questionable in the life insurance context, as pointed out by Vedani et al. [VEKLP17], while it is required for the standard formula. The monotonicity is also observed for the downward (resp. upward) shock on the equity: the higher w^s, the less the insurance company has capital gains (resp. loss) from the downward shocks. Here, the curves of SCR_{up} and SCR_{down} cross also around $w^s = 0.05$.

Stock model	Short-rate model
$S_0 = 1$	$r_0 = \theta = 0.02$
$\sigma_S = 0.1$	$\sigma_r = 0.01$
$\gamma = 0$	$k = 0.2$

Table 2.8 – Market-model parameters

Management Parameters	Liability Parameters
Allocation in stock $w^s = 0.05$	Lapse triggering threshold $\beta = -0.01$
Allocation in bond $w^b = 0.95$	Massive lapse triggering threshold $\alpha = -0.05$
Participation rate $\pi_{pr} = 0.9$	Maximum lapse dynamic lapse rate $DSR_{max} = 0.3$
Minimum guaranteed rate $r^G = 0.015$	Static lapse rate $\underline{p} = 0.05$
Competitor rate $r_t^{comp} = r_t$	
Smoothing coefficient of the PSR: $\bar{\rho} = 0.5$	
Bond portfolio maximal maturity $n = 20$	

Table 2.9 – Liability and management parameters

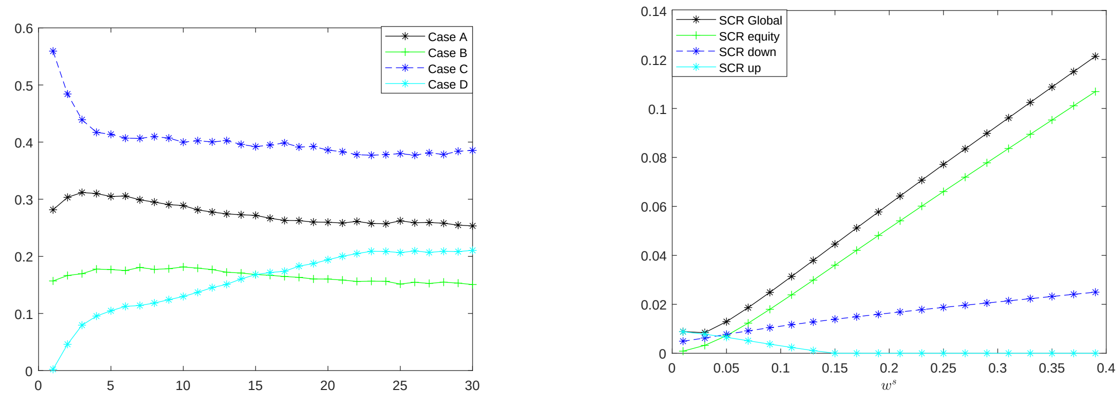

Figure 2.5 – Left: Empirical distribution of the cases A, B, C, and D determining the crediting rate, in function of the time t. Right: the SCR modules in function of the constant allocation weight in equity w^s.

We now analyze the shocks. In Figure 2.6, we have plotted the empirical means of the crediting rate and of the exit rate after the equity shock $s^{eq} = -0.39$. We have also indicated with a plus sign (resp. a dotted line) the upper (resp. lower) bound of the 95% confidence interval. As one may expect, the equity shock gives an important loss resulting in a lower crediting rate and thus a higher exit rate. Nonetheless, the effects on these rates are moderate due to the guaranteed rate: on average, the maximal difference between the competitor rate r_t is about 0.5%, and therefore only few scenarios are at some time above the surrender triggering threshold β. The shocks on the interest rate, illustrated in Figure 2.7 mix different effects. The downward shock gives an important gain at the beginning, but in the long run it makes it harder for the insurance company to credit the minimal guaranteed rate. This is known as the reinvestment risk in the literature. This fact is confirmed by the plot of the mean value of the average coupon rate $\frac{1}{n}\sum_{i=1}^{n} c_t^i$, that is even slightly below $r^G = 1.5\%$ after 20 years. This plot of the average coupon rate also illustrates the rolling mechanism described in Equation (2.2.14). Conversely, the upward shock gives an important initial loss, but in the long run it makes it much easier for the insurance company to credit the minimal guaranteed rate. Also, because of the initial loss, the insurer tends to credit at the beginning rather low rates to policyholders while the competitor rate r_t is high: this has an important effect on the surrender rate.

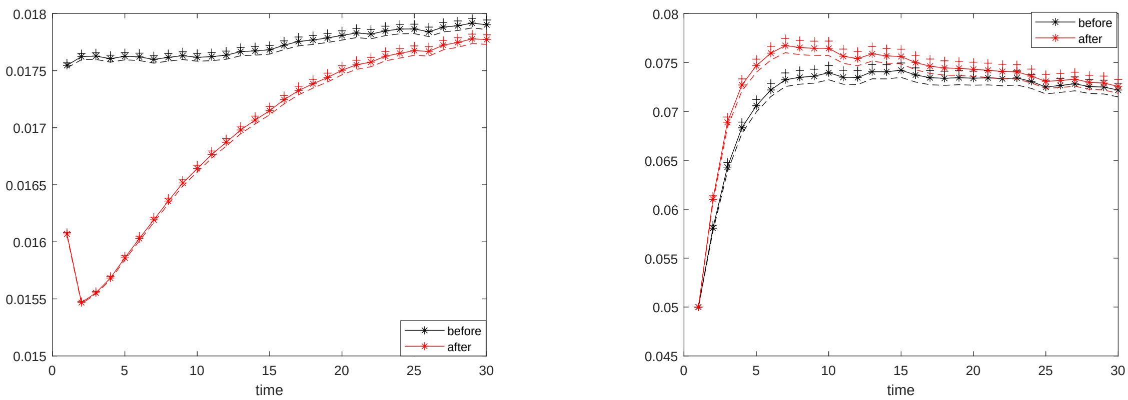

Figure 2.6 – Before and after the equity shock of 39%. Evolution of the mean crediting rate $\mathbb{E}[r_{ph}(t)]$ (left) and of the mean exit rate $\mathbb{E}[p_t^e]$ (right) in function of the time t.

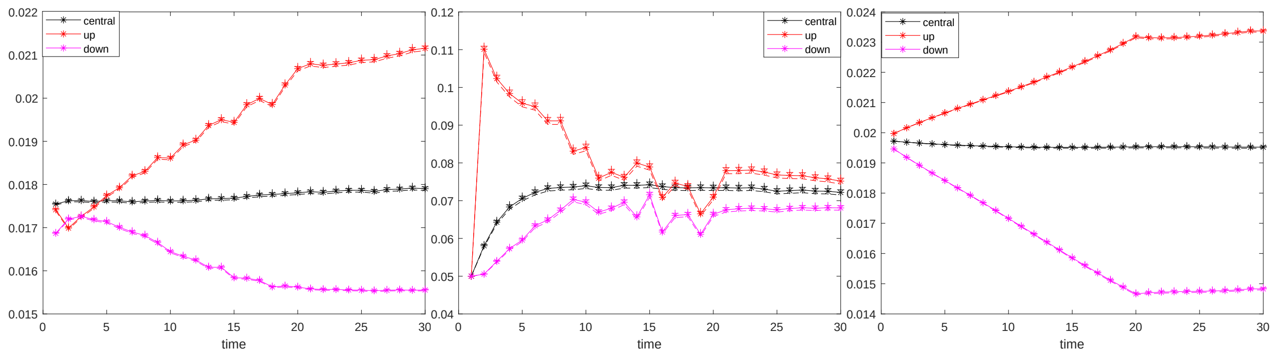

Figure 2.7 – Before and after the downward and upward shocks on interest rates. Evolution of the mean crediting rate $\mathbb{E}[r_{ph}(t)]$ (left), of the mean exit rate $\mathbb{E}[p^e_t]$ (middle) and of the average coupon in the Bond portfolio (right)in function of the time t.

Let us mention here that we have also run the same ALM strategy when the competitor rate is $r^{comp}_t = \max(r_t, 0.9r_{ph}(t-1))$. We have observed rather minor differences with the case $r^{comp}_t = r_t$. Therefore for the simplicity of the exposition, we have preferred to keep $r^{comp}_t = r_t$ in this numerical section. Before going fur-

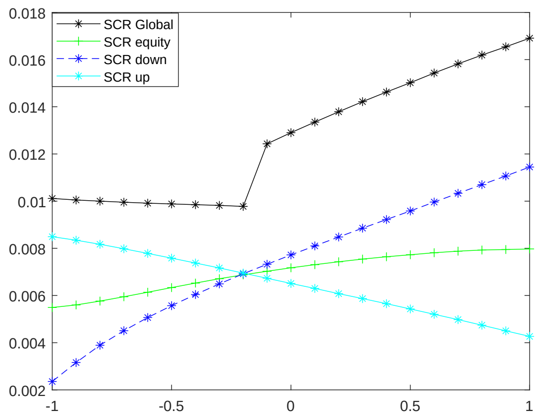

Figure 2.8 – Values of the different SCR modules as a function of γ, the correlation between bonds and stocks.

ther with the analysis of the importance of the cash flow matching, we have drawn the dependence of the different SCR modules as a function of γ that tunes the correlation between the equity and the interest rate. This is the kind of quantitative study an ALM model may help for. We observe that SCR_{eq} and SCR_{down} are decreasing and SCR_{up} is increasing with respect to γ. The aggregated SCR (4.3.3) is first slightly decreasing when $SCR_{down} < SCR_{up}$ and then increasing. We notice an important discontinuity at $SCR_{down} = SCR_{up}$ which is due to the ε coefficient in formula (4.3.3) that goes from 0 to 1/2 when SCR_{down} goes above SCR_{up}. In the range $[-0.5, 0.5]$ usually observed for the correlation between stocks and sovereign bonds (see e.g. Pericoli [Per18] or Rankin and Shah Idil [RI14]), we observe an important variation of 50% of SCR_{mkt}, half of which is contained by the discontinuity. Such a discontinuity in the SCR formula is unfair and may encourage the insurer to be at the edge of this discontinuity: a continuous formula for SCR_{mkt} such as $\max(\sqrt{SCR_{eq}^2 + SCR_{up}^2}, \sqrt{SCR_{eq}^2 + SCR_{down}^2 + SCR_{eq}SCR_{down}})$ would avoid this.

2.4.2 SCR for some dynamic allocation strategies

We analyze briefly in few simple examples how the SCR valuation with the standard formula may be impacted by considering dynamic strategies. Namely, we consider the four following strategies for $t \in \{0, 1, \dots, T\}$:

$$w_t^{S,0} = \frac{5}{100}, \quad w_t^{S,1} = \frac{t+5}{100}\mathbb{1}_{t\leq 5} + \frac{10}{100}\mathbb{1}_{t>5}, \quad w_t^{S,2} = \frac{10-t}{100}\mathbb{1}_{t\leq 5} + \frac{5}{100}\mathbb{1}_{t>5} \quad w_t^{S,3} = \frac{10}{100}. \tag{2.4.2}$$

Strategy 1 (resp. 2) starts from an initial 5% (resp. 10%) proportion of stocks that is gradually increased up to 10% (resp. decreased down to 5%) after 5 years. As a comparison, we consider Strategies 0 and 3 with respective constant allocations 5% and 10% in stocks. In Table 2.10, the mean values of the Basic-Own-Funds and of the different modules of the SCR are displayed. Figure 2.9 shows the empirical distributions (kernel density estimation with Gaussian kernel) of the Basic-Own-Funds variation after the shock on equity for the four strategies.

	BOF	SCR_{rate}	SCR_{eq}	SCR_{mkt}
Strategy 0	0.0209	0.0076	0.0072	0.0129
Strategy 1	0.0186	0.0089	0.0079	0.0146
Strategy 2	0.0199	0.0098	0.0221	0.0283
Strategy 3	0.0176	0.0109	0.0209	0.0280

Table 2.10 – Different values of the SCR modules of the strategies (2.4.2).

As one may expect, the main driver when using the standard formula is the initial allocation since it relies on different shocks at time $t = 0$. Thus, the SCR_{mkt} mean values and the distributions of $BOF_0^{eq\_shock} - BOF_0$ are rather close between Strategies 0 and 1 (resp. 2 and 3). This indicates that the standard formula mostly ignores the dynamic feature of the strategy and basically depends on the initial asset allocation.

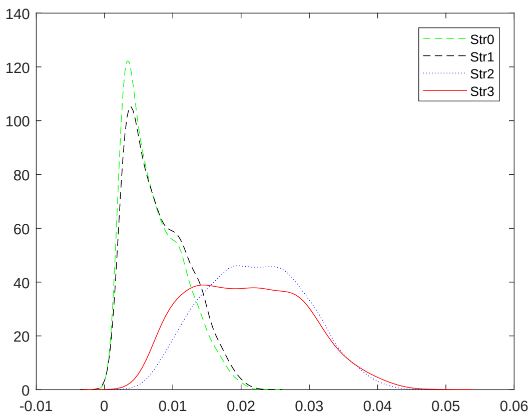

Figure 2.9 – Empirical distribution of $BOF_0^{eq\_shock} - BOF_0$ for the different allocation strategies (2.4.2).

2.4.3 Study of the cash flow matching

In this paragraph, we want to assess the relevance of an original feature of our ALM model: the cash flow matching between the bond assets and the liabilities. This feature reproduces a common practice of insurance companies. In order to have an idea of a good choice of n (the maximal maturity of the bond combination (2.2.3)), we have plotted on the left of Figure 2.10 the Basic Own-Funds defined in (2.2.36) as a function of n for the central and shocked settings. Thus, the difference between the curve of the central case and the shocked cases gives the values of SCR_{up} and SCR_{down}. We use here the same parameters as in Tables 4.1 and 4.2. We see that in the central case, the BOF is maximized around $n = 20$, but is anyway rather flat between $n = 15$ and $n = 30$. As one may expect, the BOF is increasing with respect to n in the downward shocked scenario: the more the insurer invests in long maturity bonds, the more he benefits from the decrease of the interest rates. For the upward shocks, two effects are mixed. On the one hand, the longer is the bond maturity, the greater is the loss due to the shock. On the other hand, the insurer has interest to match well the bond assets and the liabilities in order to keep as many policyholders as possible, since the high interest rates will be profitable in the long run. Thus, the higher BOF are obtained for $n = 7$ and $n = 8$. If the insurance company wants to have the minimal

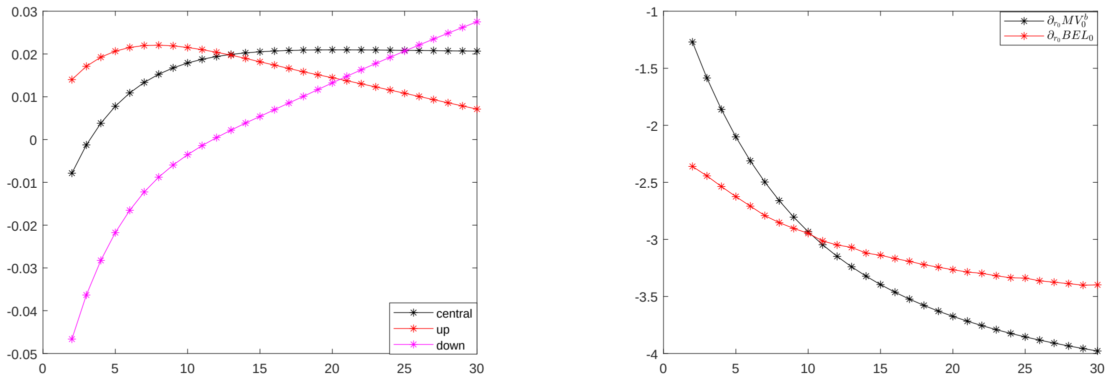

Figure 2.10 – Left: mean value of the Basic Own-Funds in function of n (defining the bond combination (2.2.3) in which bonds are invested) in the central framework and with the upward and downward shocks on interest rates. Right: Macaulay durations of the assets ($\partial_{r_0} MV_0^b$) and of the liabilities ($\partial_{r_0} BEL_0$) in function of n.

$SCR_{int} = \max(SCR_{up}, SCR_{down})$ with the standard formula, it has to choose n around where the curves cross. Thus, $n = 20$ appears to be the choice that minimizes SCR_{int}. Note that due to the discontinuity of formula (4.3.3), this is also a very good choice since we have also $SCR_{up} > SCR_{down}$: with $n = 21$, we would have $SCR_{up} < SCR_{down}$ and thus $\varepsilon = 1/2$, leading to a greater SCR_{mkt}. Thus, we can tune the value of n to satisfy $SCR_{up} > SCR_{down}$ and benefit from a better diversification coefficient of the standard formula. In comparison, we have also considered the Macaulay duration of the assets and of the liabilities after the initial allocation, i.e. $\partial_{r_0} MV_0^b$ and $\partial_{r_0} BEL_0$. They are plotted in the right of Figure 2.10, as a function of n. The curves cross around $n = 10$, which means that to be hedged against small variations of the interest rate at time 0, the insurance company should take $n = 10$. Note that from the graph on the left, this choice leads to a lower BOF in the central framework and to a much higher SCR with the standard formula. This demonstrates if it were necessary that hedging small variations is not the same as hedging shocks.

A proxy model. The discussion above already shows the importance of the choice of n. To go further, we would like now to compare with a simpler model where there is no cash flow matching. This proxy model works as follows. At time 0, the insurer invests in a single at-par coupon bearing bond with yield to maturity n_p and unitary market-value given by:

$$\bar{MV}_0^b = B(n_p, c_{swap}^{n_p}(0)) = 1$$

At each time $t \in \{1, \dots, T\}$, the insurance company re-balances it according to target weights. The available capital now is reinvested in a single bond with duration n_p. However, to approximate the full model, we do not consider that the company sells all of its current bond of maturity $n_p - 1$ to buy new bonds with longer term n_p: this would imply a tremendous realization of capital gains or losses, leading to important changes in book values. To deal with this issue and make a fair comparison with the original model, we propose the following approximation. Before reallocating his portfolio, the insurer adjusts his holding in bonds and compute $\phi_{t_2}^b$ such that

$$\phi_{t-1}^b B(n_p - 1, c_{t-1}^b) = \phi_{t_2}^b B(n_p, \tilde{c}_t^n),$$

where

$$\tilde{c}_t^n = \frac{1}{n} c_{swap}^n(t) + (1 - \frac{1}{n}) c_{t-1}^b,$$

and we assume that this procedure does not lead to a realization of CGL. The purpose of this approximation is to adjust the holding in bonds in order to leave the current market-value $\phi_{t-1}^b B(n_p - 1, c_{t-1}^b)$ of the bond portfolio unchanged while taking into account the reinvestment risk of the original model. In particular, the original model reinvests the nominal value at the swap rate, which justifies the term $\frac{1}{n} c_{swap}^n(t)$, and keeps a fraction $1 - \frac{1}{n}$ of bonds with unchanged coupons.

To determine the maturity n_p of the bond used in the proxy model, we choose the maturity n_p in order to keep approximately the same gain or loss between both models after the downward or upward shock:

$$\Delta MV_0 \approx \Delta MV_0^{proxy}(n_p) \tag{2.4.3}$$

After numerical investigation, the choice $n_p = \frac{n}{2}$ is satisfactory in practice. Then, to obtain exactly the same size of shocks in terms of loss or gain, we adjust then the position in bonds in the proxy model:

$$\phi_{0^+}^{b,proxy} MV_{0+}^{b,proxy} = \phi_{0+}^b MV_{0+}^b. \tag{2.4.4}$$

This adjustment is important to compare the models in a fair way: thus, the shocks induces the same initial loss or gain. Figure 2.11 illustrates the difference between the original and the proxy models on the Basic Own-Funds distribution in the central setting and after the interest rate shocks. Table 2.11 indicates the corresponding BOF mean values, and Table 2.12 the associated SCR values. We observe two effects. In all cases, we observe that the BOF distribution has less variance and is more peaked in the original model. This is expected: the basket of bonds allows a good cash flow matching between the nominal value of the expiring bonds and the surrendering policyholders. The second effect concerns the interest rate shocks. In both cases, the original model performs much better than the proxy model. This is again due to the cash flow matching. These shocks induce large latent gains or losses on bonds: with the proxy model, the insurer is forced to realize a part of it since he pays policyholders by

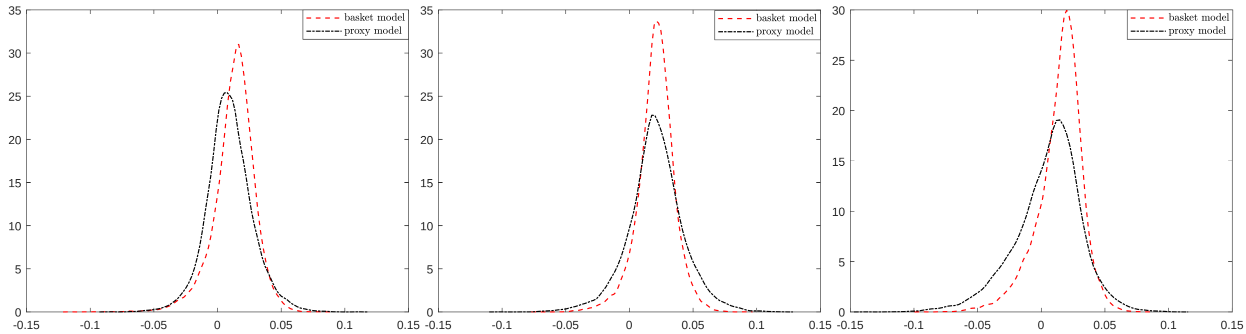

Figure 2.11 – Empirical BOF distributions with the proxy model and the original (Basket) model with a basket of bonds, after the downward shock (left), the upward shock (right) and in the central framework (middle).

selling a fraction of his portfolio. For the central setting and the shock on equity, there is no longer important latent gains or losses on bonds, and the mean BOF values are roughly the same between both models. Note that this is clear for the upward shock since realizing losses reduce the yearly P&L, but the interpretation for the downward shock is less obvious. Besides, on this second effect, we notice that the upward shock is even more expensive than the downward shock for the proxy model: from Table 2.11, the difference with the original basket model is equal to 0.0036 for the downward shock, versus 0.0092 for the upward shock. This difference is due to the massive rate of surrenders in the upward shock (see Figure 2.7), which is caused by the increase of the competitor rate r_t. Thus, these increased surrenders have again to be paid by selling a greater fraction of portfolio, leading to realize even more losses. This effect has already been noticed in the literature, see [KBG19].

	Basket	Proxy
Central	0.0208 [0.0206,0.02010]	0.0207 [0.0203,0.0210]
Equity shock	0.0136 [0.0134, 0.0139]	0.0134 [0.0130, 0.0137]
Downward shock	0.0130 [0.0128, 0.0133]	0.0094 [0.0091, 0.0096]
Upward shock	0.0145 [0.0142, 0.0147]	0.0053 [0.0049, 0.0056]

Table 2.11 – Mean value of the BOF in the original and the proxy models with 95% confidence interval, under the central and shocked settings.

Let us now comment quickly on the different SCR values in Table 2.12. An obvious remark is that the standard formula that relies on mean values is not sensitive to the BOF distributions and does not reward if they are more peaked with less variance. This is a clear weakness of the standard formula. Thus, the first effect described just above has no impact on the SCR, and for example the values of SCR_{eq} are the same under both models. In contrast, the second effect has some impact on the interest rate modules of the SCR, leading to some improvement of SCR_{mkt}. Note that the improvement is nonetheless tamed by the fact that in the aggregated formula (4.3.3), we use $\varepsilon = 0$ for the proxy model and $\varepsilon = 1/2$ for the original model with a basket of bonds.

	Basket	Proxy
SCR_{eq}	0.0072	0.0073
SCR_{down}	0.0078	0.0113
SCR_{up}	0.0063	0.0154
SCR_{mkt}	0.0119	0.0170

Table 2.12 – Different values of the SCR modules.

2.4.4 Impact of mortality risk on the cash flow matching

We investigate the impact of mortality on the cash flow matching strategy in bonds. More precisely, we are interested in the optimal choice of n that minimizes SCR_{mkt}, as in the previous section. For simplicity, we assume that all the policyholders have the same age x at time 0 and that we know for all $t \in \{0, \ldots, T-1\}$ the probability q_{x+t} that a policyholder alive at age $x+t$ dies before age $x+t+1$. Then, we assume that the overall exit rate of the portfolio is given by:

$$p_t^e = q_{x+t} + DSR(\Delta_t), \qquad (2.4.5)$$

which replaces Equation (2.3.3). We use otherwise the same parameters. We have run our simulations with the life table given in Table 2.13 and we consider three different cohorts: $x = 50$, $x = 60$ and $x = 70$. We refer to Subsection 2.4.3 for a discussion on the optimal choice of n to minimize SCR_{mkt}. Following this analysis, we observe from Figure 2.12 that the younger the cohort, the larger is the optimal maturity of investment in bonds. This is expected since the maturity of bonds has to match the surrenders as well as possible. The optimal values are $n = 37$ for $x = 50$, $n = 30$ for $x = 60$ and $n = 20$ for $x = 70$ (as for the example of Figure 2.10).

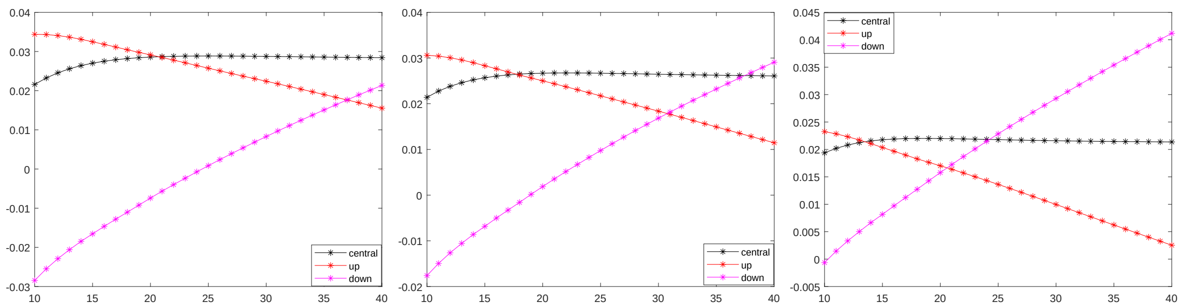

Figure 2.12 – Mean value of the Basic-Own-Funds in function of n in central the framework and with upward and downward shock on interest rates. Left: 50 years-old cohort at inception, Middle: 60 years-old cohort, Right: 70 years-old cohort.

This numerical example shows the ability of the model to deal with time varying structural surrenders, in particular to determine the optimal maturity of the investment of bonds for the SCR. Nonetheless, a possible improvement of our model would be to allow for a dynamic choice of n. Typically, one would like to decrease n (i.e. the maturity of bonds) slowly as long as the portfolio is aging in order to better match the cash flows. Changing n would also be relevant to hedge the dynamic surrenders. However, changing the value n along the time implies also movements of book values, which may be suboptimal if the changes are too frequent. The study of this trade-off between cash flow matching and book value movements is left for futher research.

Age	q_x	Age	q_x	Age	q_x	Age	q_x	Age	q_x
50	0.004740	60	0.006115	70	0.014742	80	0.049067	90	0.140516
51	0.004915	61	0.006422	71	0.016640	81	0.054883	91	0.155380
52	0.005065	62	0.006808	72	0.018813	82	0.061269	92	0.171816
53	0.005187	63	0.007286	73	0.021284	83	0.068274	93	0.190025
54	0.005279	64	0.007874	74	0.024077	84	0.075952	94	0.210235
55	0.005348	65	0.008587	75	0.027218	85	0.084366	95	0.232303
56	0.005430	66	0.009446	76	0.030731	86	0.093590	96	0.254792
57	0.005540	67	0.010470	77	0.034643	87	0.103710	97	0.277225
58	0.005685	68	0.011680	78	0.038982	88	0.114824	98	0.299489
59	0.005873	69	0.013096	79	0.043779	89	0.127049	99	0.321487

Table 2.13 – The probability q_x that a person aged exactly x dies before exact age $x + 1$, extracted from the S1PFL mortality table (All pensioners, Female, Lives) produced by the Institute of Actuaries and Faculty of Actuaries. It is available on the link https://www.actuaries.org.uk/learn-and-develop/continuous-mortality-investigation/cmi-mortality-and-morbidity-tables/s1-series-tables

2.4.5 Analysis in a low interest rate framework

In this paragraph, we investigate our model in a framework where interest rates are low (around 0.5%) to be closer to the current interest rates. This was the rate observed for the 10Y bonds issued by France (OAT) at the beginning of 2019. As explained in Section 2.2.5, the multiplicative shocks are no longer relevant in this context and we have applied the last recommendation of the EIOPA given in Table 2.7. The model parameters are specified in Tables 2.14 and 2.15. Note that we have considered here a higher structural surrender rate of 10%, and therefore a smaller value of n, the maximal maturity of the basket of bonds. Again, we have taken a constant allocation in equity w^s that is such that the SCR_{eq} and SCR_{int} are approximately of the same order.

Stock model	Short-rate model
$S_0 = 1$	$r_0 = \theta = 0.005$
$\sigma_S = 0.1$	$\sigma_r = 0.01$
$\gamma = 0$	$k = 0.2$

Table 2.14 – Market-model parameters in the low yield framework.

Management Parameters	Liability Parameters
Allocation in stock $w^s = 0.08$	Lapse triggering threshold $\beta = -0.01$
Allocation in bond $w^b = 0.92$	Massive lapse triggering threshold $\alpha = -0.05$
Participation rate $\pi_{pr} = 0.9$	Maximum lapse dynamic lapse rate $DSR_{max} = 0.3$
Minimum guaranteed rate $r^G = 0$	Static lapse rate $\underline{p} = 0.1$
Competitor rate $r_t^{comp} = r_t$	
Smoothing coefficient of the PSR: $\bar{\rho} = 0.5$	
Bond portfolio maximal maturity $n = 10$	

Table 2.15 – Liability and management parameters in the low yield framework.

We have plotted in Figure 2.13 the crediting rate, the surrending proportion and the average coupon rate. The behavior is roughly the same as the one observed in Figure 2.7 for higher interest rates, and we do not repeat the interpretation. We also observe that on the downward shock, we get negative average coupon values. The small differences between both cases can be explained by the change of method for the shocks. First, we notice in our examples that the additive term in the shocks makes the shocks stronger at the beginning. For example, the spread on the mean exit rate between the upward shock and the central framework is about 0.09 at year 2 instead of 0.05 in Figure 2.7 for a 2% interest rate. Of course, the multiplicative shock would have been stronger for a higher interest rate, but a simple calculation made on constant interest rates indicates that the 1Y shock obtained with zero interest rates and Table 2.7 (2.14%) is almost the same as the one obtained with 3% interest rates and Table 2.6 (2.1%). Another difference is that the crediting rates change of monotonicity after 10-15 years in the shocked frameworks: it is first increasing (resp. decreasing) and then decreasing (resp. increasing) for the upward (resp. downward) shock. This is mostly due to the fact that the shift functions (left of Figure 2.4) have opposite monotonicity after year 20, while they remain essentially parallel (left of Figure 2.1) with the shocks of Table 2.6. Since $n = 10$, this has an effect on the coupon rates from year 10 and on the competitor rate from year 20. Another interesting plot is the

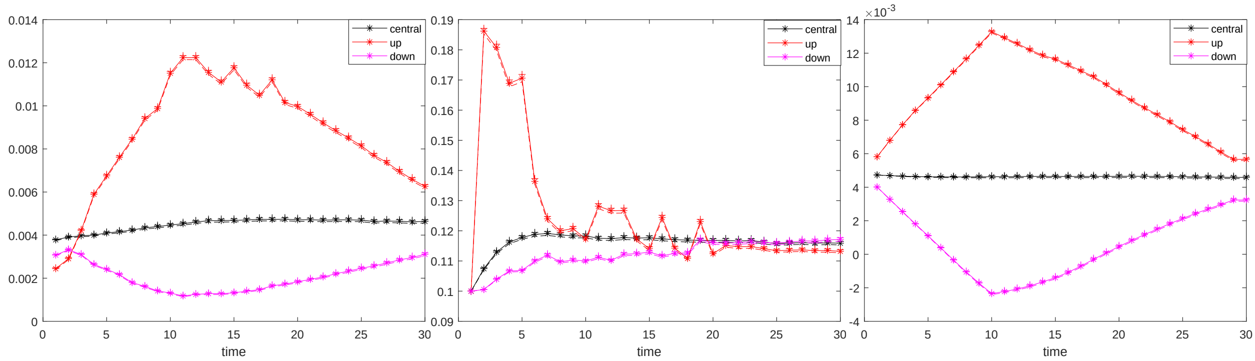

Figure 2.13 – Before and after the downward and upward shocks on interest rates. Evolution of the mean crediting rate $\mathbb{E}[r_{ph}(t)]$ (left), of the mean exit rate $\mathbb{E}[p^e_t]$ (middle) and of the average coupon in the Bond portfolio (right) in function of the time t.

calculation of the BOF in function of n, the maximum maturity of the basket of bonds, which is displayed in Figure 2.14. The behavior is very similar to the one observed on the left of Figure 2.10. Nonetheless, we see here that the SCR_{up} and SCR_{down} cross around $n = 12$, making this choice optimal for the minimization of the SCR_{int} and even SCR_{mkt} since we then have $SCR_{up} > SCR_{down}$. Thus, contrary to the previous case, the best choice of n to minimize SCR_{int} is not $1/\underline{p}$. More suprisingly, it does not also satisfy $n \leq 1/\underline{p}$, as one should take to have the nominal values of expiring bonds greater than the value of the surrending contracts. This shows anyway that our model can be a useful tool to determine the investment in different bond maturities.

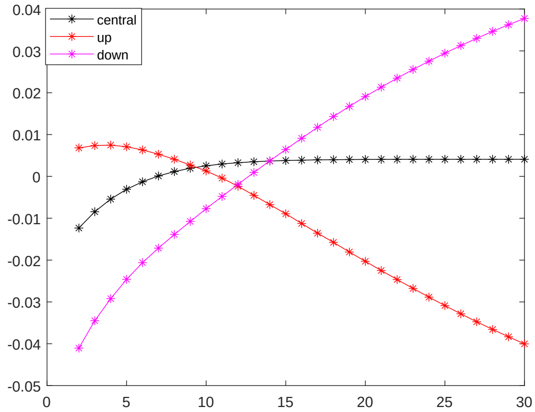

Figure 2.14 – Mean value of the Basic Own-Funds in function of n (defining the bond combination (2.2.3) in which bonds are invested) in the central framework and with the upward and downward shocks on interest rates.

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CHAPTER 3

FURTHER NUMERICAL INVESTIGATION ON THE ALM MODEL

Contents

In this chapter, we supplement the analysis of the ALM model and quantify the impact of regulatory decisions on the insurer's balance sheet. This study is motivated by the very recent consultation paper [EIOa] that EIOPA launch on the 2020 review of Solvency II. We investigates an extrapolation method of the yield curve (Smith Wilson) different from the methodology used in chapter 2 which is closer to practice. In this chapter only, we rely on the Smith Wilson method required by the EIOPA to construct the yield curve which is given as input of the ALM model. The aim of this part is not to discuss potential pitfall of the current regulation but rather to show that the model is flexible and can be adapted to the different methodology of computation of the SCR. Furthermore, we show that it can also deal with potential changes in the regulation that are currently considered by EIOPA, especially concerning the derivation of the interest rate risk-free curve. The core of this chapter is devoted to investigation of the impact of extrapolation methods of the term structure of interest-rate on the overall capital charge. It becomes a hot topic among insurers and contributes to the debate around market consistency and fair valuation approach in insurance, therefore having a model that is able to incorporate these potential changes is of primary importance, especially to measure the impact of these changes. Currently, the industry standard extrapolation procedure needs exogenous parameters that are provided by the regulator which may lead to significant impact on the insurers' balance sheet and the overall solvency capital. While insurance companies can at least partially hedge the interest-rate risk, they cannot hedge regulatory decision on external parameter, therefore analyzing the sensitivities to this parameter are of primary importance for life insurers. In SII, for the fair valuation of the insurer's balance sheet, discount factors extracted from market financial instrument (Government Bonds and Swaps) appears to be essential input for ALM models. For insurance companies where commitments toward policyholders spread over several decades, no market data are available and extrapolation methods are used to determine discount-factors for maturities where no liquid market-values are available. From [EIOb], The interest-rate curve used to discount future cash flows in SII must comprise a relevant interest-rate term structure extracted from liquid interest rate instrument and several adjustment (volatility and matching adjustments) are used to mitigate the effect of pro cyclicality induced by large financial shocks on the insurers' balance-sheet and act as shifts on the yield curve. The chapter proceed as follows. In the first section, we describe the procedure to derive the SCR quantile in the ALM model. In the next section, we describe the EIOPA's methodology to derive the risk-free yield curve. The section includes calibration result of the fitted yield curve using artificial market data. The final section of this chapter present numerical application and estimation outcome. SCR comparison between the standard formula and the quantile approach are summarized. Next, we assess the impact of regulatory parameters (UFR, LLP, convergence period) on the SCR.

3.1 Solvency Capital Requirement: the quantile formulation

In this paragraph, we describe the procedure to compute the SCR quantile in the ALM model. As pointed out in [CN+14], the definition of the SCR is provided only in a descriptive form (Article 101 of the SII directive) which allows for different mathematical interpretation. A general mathematical framework and several mathematical formulations are provided in their article. In our framework, we will follow the intuitive definition of Bauer et al.[BRS12]. Let us consider an insurance firm with current own-fund level BOF_t at time t. To ensure that the firm will remain solvent one-year ahead with some very high confidence level $1-\alpha$, the company may need to hold today an extra-amount $x^\star$ as a buffer against the risks faced by the firm. Assuming that the amount $x^\star$ is locked and invested in a bank saving account providing a yield $e^{\int_t^{t+1} r_s ds}$, the extra-amount should satisfy

$$x^\star = \inf\{x \in \mathbb{R},\ \mathbb{P}\left(BOF_{t+1} + xe^{\int_t^{t+1} r_s ds} \geq 0|\mathcal{F}_t\right) \geq 1-\alpha\} \tag{3.1.1}$$

Intuitively, the extra-amount to hold is the smallest amount that need to be added to the current own-funds in order to avoid the "bankruptcy event" with a $1-\alpha$ probability. The previous formula can be written as stated by the SII directive in term of one-year loss distribution

$$x^\star = \inf\{x \in \mathbb{R},\ \mathbb{P}\left(BOF_t - BOF_{t+1}e^{-\int_t^{t+1} r_s ds} \leq x + BOF_t|\mathcal{F}_t\right) \geq 1-\alpha\} \tag{3.1.2}$$

which implies that

$$BOF_t + x^\star = q_\alpha(L_{t+1}) \tag{3.1.3}$$

where $q_\alpha(L_{t+1})$ is the α-quantile of the $t+1$-*year loss distribution*

$$L_{t+1} = BOF_t - BOF_{t+1}e^{-\int_t^{t+1} r_s ds} \tag{3.1.4}$$

Hence

$$SCR_t = BOF_t + x^\star = q_\alpha(L_{t+1}) \tag{3.1.5}$$

A metrics that is monitored in the ALM, related to the SCR is the so-called the so-called *solvency ratio* as follows

$$SR_t = \frac{BOF_t}{SCR_t} \tag{3.1.6}$$

We should compute both of these quantities (SCR and Solvency ratio) in the ALM model in 3.3 using both definition of the SCR.

3.2 EIOPA's construction of the interest-rate curve

In this section, we review some key elements of the methodology proposed by the supervision authority (EIOPA) to construct a regulatory interest rate curve that will be used as input of the ALM model. For a comparison to the asset-side of the balance-sheet and the computation of the SCR, the liability of the insurance company should be valued market consistently. This value shall be derived using "a relevant risk-free interest rate structure" (Article 75 of Directive 2009/38/EC). EIOPA proposed to specify the curve using observed interest rate instrument comprising European investment grade bonds or swap contracts. However, the discounting approach require discount factors when no market data or not sufficiently liquid traded instrument are available. Therefore, an extrapolation procedure is needed to extend the interest-rate curve beyond the last observable point. In SII, the industry standard yield curve extrapolation method is a technique developed by Smith and Wilson (2001). The method uses the available market data to exactly fit bond or swap prices and extrapolate them for non observable maturities using a weighted average of the last observable point to a predetermined long-term equilibrium called *Ultimate Forward Rate* (UFR). The speed of convergence to this level is control by a mean reversion parameter. The method is based on additional exogenous parameter which are the Last-Liquid Point (LLP) and the convergence period after the LLP until the equilibrium is reached. In what follows, we review some key element of the method and describe in more detail the exogenous parameter given by EIOPA that plays a fundamental role in the fitting methodology. The investigation of these parameter will motivate our numerical study.

3.2.1 Ultimate Forward Rate (UFR)

In order to exclude "artificial volatility", further mitigate effect of pro-cyclicality, and deal with the lack of liquid instrument for long-term maturities, EIOPA introduce an exogenous fixed parameter, the *Ultimate Forward rate* (Delegated Act 2014/51/EU of the European Commission), which is supposed to reflect market expectation on interest-rate. It consists of a combination between historical estimates of the expected inflation and short term real rates. To be more specific, EIOPA suggests that the introduction of the UFR will make solvency ratio less susceptible to potential market disturbance by putting less weight on the liquid part of the yield curve and making the illiquid part of the yield curve less prone to shock on supply and demand. This parameter has prompted several discussion in the current low interest-rate environment, which lead EIOPA to reduce its current level. The applicable UFR for 2021 is fixed at 3.6%. In 2017, the UFR was set to 4.2%.

3.2.2 Last Liquid Point and Convergence Period

The Last-Liquid-Point (LLP) correspond to the last data point where interest rate market instruments are considered sufficiently liquid to enter in the construction of the interest-rate curve. An important point of critique is the position of the LLP which is currently set to 20 years. Market participants argue that the EU interest-rate market is still liquid beyond 20 years ([KOP12]), the amounts of government bonds with maturities 20 to 30 years is higher than maturities 10 to 20 years (Chief Risk Officer Forum 2010 [For10]). Recently, EIOPA published a consultation paper (2019/2020) which opens up the possibility to extend the Last Liquid Point for the Risk Free Rate to 30 years or even 50 years for the Euro curve. Once the LLP and the UFR are determined, a convergence period until the UFR is reached is set to 40 years by EIOPA. This convergence period is also subject to debate, and EIOPA indicates in [EIOa] that it would consider new values for the convergence periods in order to make it more consistent with market curves. Currently, EIOPA is exploring several solutions, among them an extension of the convergence period from 40 to 100 years in order to reduce the weight of the UFR and increase the weight of the liquid part of the yield curve.

3.2.3 The Smith-Wilson model

In this paragraph, we provide a brief description of the Smith-Wilson extrapolation method with a particular emphasize on the EIOPA's adaptation of the method provided in its technical published documentation. The reference document is "technical documentation of the methodology to derive EIOPA's risk free interest rate term structure" [EIOb]. Roughly speaking, the method aims at constructing a zero coupon curve $t \mapsto P(0,t)$ that *exactly fit* the market prices of all observable market instruments. The key property of the approach is that extrapolated forward interest will converge toward a long forward interest-rate (the UFR) that is a given input of the model. It uses the available market data to exactly fit the observed bond instrument and extrapolate them for non observable maturities using a weighted average of the last observable data point and the long-term equilibrium. Let us now introduce the general Smith-Wilson model framework. It is assumed that the market price of a certain number N of fixed income instrument is observed at time 0. For clarity of the presentation, we assume that the market data comprises only zero-coupon bonds. Let $t_1, \dots, t_N$ their observed maturities where t_N is the LLP in the EIOPA's terminology. The Smith-Wilson model model the discount factor function as

$$P(0,t) = e^{-\text{UFR}t} + \sum_{j=1}^{N} \xi_j K_j(t) \tag{3.2.1}$$

where $(\xi_j)_{j=1,\dots,N}$ are N parameters to be fitted and the function $t \mapsto K_j(t)$ are Kernel functions called *Wilson function* which are defined as

$$K_j(t) = W(t, t_j) \;\; t > 0 \tag{3.2.2}$$

Where $(u,v) \mapsto W(u,v)$ is a symmetric function defined as

$$W(u,v) = e^{-\text{UFR}(u+v)} \left(\alpha \min\{u,v\} - e^{-\alpha \max\{u,v\}} \text{sinh}(\alpha \min\{u,v\}) \right) \tag{3.2.3}$$

The Smith-Wilson function is a result of an optimization procedure related to exponential tension spline method. It corresponds to the solution to an optimization problem that seeks to interpolate the data under the constraint that a set of given market data points must be fitted exactly. The optimization procedure aims at making both the first (slope) and second order (the convexity) derivative small in order to minimize the integral of their squared value. The α parameter controls the speed of convergence toward the UFR. It also controls the smoothness of the curve. A higher α leads to faster convergence toward the UFR whereas lower α value will give more weight to market data (see Figure 3.1). Observe also that the Wilson function $t \mapsto K_j(t)$ converges to 0 when $t \to +\infty$.

Hence, the Smith-Wilson extrapolated price can be viewed as as a sum of a long-term discount factor (with UFR) and a "correction term" that tends to 0 as t$\to +\infty$.

Calibration to Market Data

From now on, we assume that the market price of N zero coupon bond $(P^{mkt}(0,t_j))_{j=1,\dots,N}$ with maturities $t_1, \dots, t_N$ are observed. The matching equation of the market price leads to the following linear system of equations

$$\begin{cases} P^{mkt}(0,t_1) = e^{-\text{UFR}t_1} + \sum_{j=1}^N \xi_j W(t_1,t_j) \\ \vdots \\ P^{mkt}(0,t_N) = e^{-\text{UFR}t_N} + \sum_{j=1}^N \xi_j W(t_N,t_j) \end{cases}$$

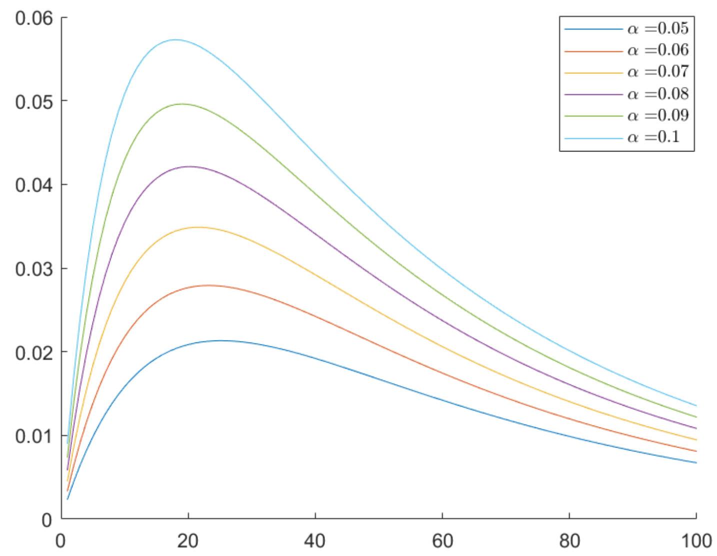

Figure 3.1 – Wilson function $v \mapsto W(u, v)$ for different speed α parameters and $u = 1$

which can be written in matrix form :

$$P^{mkt} = b_{UFR} + W_\alpha \xi \tag{3.2.4}$$

where:

$$\begin{aligned} P^{mkt} &= \left(P^{mkt}(0, t_1), \ldots, P^{mkt}(0, t_N)\right)^\top \\ b_{UFR} &= \left(e^{-UFRt_1}, \ldots, e^{-UFRt_N}\right)^\top \\ \xi &= (\xi_1, \ldots, \xi_N)^\top \end{aligned}$$

and

$$W = \begin{bmatrix} W_\alpha(t_1, t_1) & \cdots & W_\alpha(t_1, t_N) \\ \vdots & \ddots & \vdots \\ W_\alpha(t_N, t_1) & \cdots & W_\alpha(t_N, t_N) \end{bmatrix}$$

The solution of the linear system is given by

$$\widehat{\xi} = W^{-1}\left(P^{mkt} - b_{UFR}\right) \tag{3.2.5}$$

and the calibrated Smith Wilson curve is

$$\widehat{P}(0, t) = e^{-\mathrm{UFR}t} + \sum_{j=1}^{N} \widehat{\xi}_j K_j(t) \tag{3.2.6}$$

Using the relationship between zero coupon prices and yield/ Forward-rate, the fitted interest-rate term structure can be deduced from the map $t \mapsto \widehat{P}(0, t)$ as follows

$$\begin{aligned} \widehat{R}(0, t) &= \frac{-\log\left(\widehat{P}(0, t)\right)}{t} \\ \widehat{F}(0, t) &= -\partial_t \log\left(\widehat{P}(0, t)\right) \end{aligned}$$

Optimization of the speed parameter α

The convergence procedure toward the UFR, materialized by the α parameter is also regulated by EIOPA. In this section we describe the EIOPA's approach described in the technical document ([EIOb] p.46) to derive α. It is determine as the smallest value $\widehat{\alpha}$, which however must be above a lower bound 0.05 in order to ensure convergence at the convergence point $T_\infty = \max\{LLP + 40 \text{ years}; 60 \text{ years}\}$ in the sense that the gap function between extrapolated forward prices $t \mapsto \widehat{F}(0,t)$ and the UFR do not exceed 0.01%

$$\widehat{\alpha} = \inf\{\alpha \geq 0.05,\ |UFR - \widehat{F}(0, T_\infty)| \leq 0.01\%\} \tag{3.2.7}$$

Let us mention that the optimization can be trouble some as pointed out in [LL16] as the gap function can have singularities.

3.3 Numerical Application

We now illustrate the application of the calibrated Smith-Wilson yield curve using the EIOPA's implementation instruction in the ALM model. Once the risk-free term structure is derived, it is given as an input of the ALM model. In particular, the short rate model is calibrated on this EIOPA yield curve (shifts stress factor in the Vasicek++ model are calibrated using the methodology described 2.3). *In all this section, we use the ALM model described in chapter 2 with parameters given in Table 2.6, 4.1 and 4.2. The confidence interval for the quantile estimates is fixed at 95%.* In order to assess the impact of the UFR on long-term liabilities, we projected the ALM portfolio up to $T = 65$ years. Firstly, we present our calibration result and discuss the impact of changes in parameters that are currently discussed on the shape of the yield-curve. Next, we show that our SCR estimation using the standard formula based on shocks and the quantile approach. Then we discuss how changes in regulatory parameters (UFR level, position of the last-liquid point, convergence period) affect the technical provisions (Best Estimate BEL_0, Basic-Own-Funds BOF_0, SCR) in our ALM framework.

3.3.1 Smith-Wilson Model Calibration Result

The market data considered for our numerical experiment correspond to artificial zero-coupon bond $t \mapsto P^{mkt}(0,t)$ that were generated with the parameter provided in 4.1 up to the LLP which is set to 20 years in the base case situation. Based on these data, we are left with a $N = 20$ dimensional random vector $P^{mkt} = \left(P^{mkt}(0,t_1), \ldots, P^{mkt}(0,t_N)\right)^T$. The UFR considered in the numerical application is equal to 4.2%, which was the UFR specified by EIOPA until 2018. The next step is to specify the velocity of convergence α. In the present illustration, we used the optimization procedure 3.2.7 which lead to a value $\widehat{\alpha} = 0.1304$ with our market data. Finally, the calibration of the model involves finding the vector $\widehat{\xi} \in \mathbb{R}^N$ as solution of 3.2.5. Figure 3.2 plots the calibrated Smith-Wilson yield curve $t \mapsto R(0,t)$ and Figure 3.3 displays the forward-rate curve $t \mapsto F(0,t)$ from 1 to 120 years. Observed that at $T_\infty = 60$ years, the forward-rate has converged toward the UFR. In Figure 3.2 we have also displayed the central and shocked yield curve using the EIOPA stress factor provided in 2.6. We next displays in Figure 3.4 the plot representing the fitted SW yield curve for varying UFR values to highlight the sensitivity w.r.t the UFR. We see that lower UFR values implies zero-coupon bond yield that are lowered in the extrapolated zone. We shall quantify the impact of these lowered yield on the SCR in next sections.

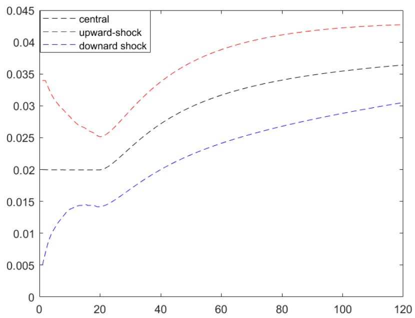

Figure 3.2 – Calibrated SW yield curve $t \mapsto R(0,t)$ in central and shocked framework

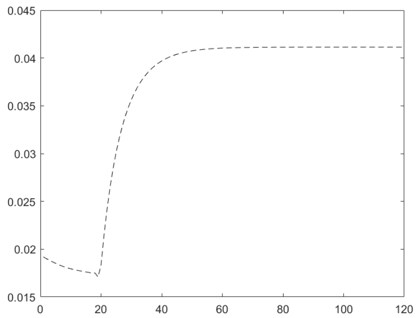

Figure 3.3 – Calibrated SW curve Forward-rate $t \mapsto F(0,t)$ with $T_\infty = 60$ years

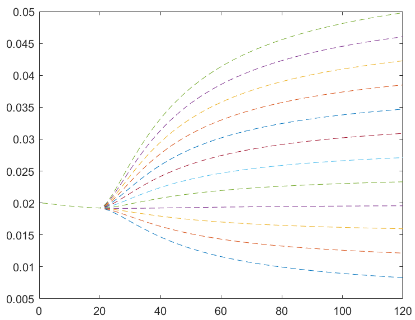

Figure 3.4 – Calibrated SW yield curve $t \mapsto R(0,t)$ for different UFR values

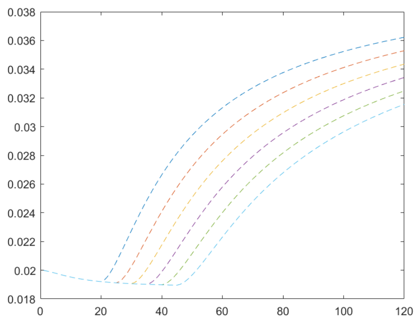

Figure 3.5 – Calibrated SW yield curve $t \mapsto R(0,t)$ for different LLP values

3.3.2 Comparison of the SCR formulas

In this paragraph, we analyze the SCR estimation obtained using the standard formula based on the aggregation of marginal shocks and the quantile formulation. To estimate the SCR with the standard formula, we use the estimator 2.4.1 with $N = 5000$ scenarios. In order to reduce the variance of the aggregated SCR, we use the same scenarios to derive the the shocked version of the Basic Own funds. The overall SCR_{mkt} is obtained using the aggregation formula in 4.3.3. For the quantile approach, we fix a computational budget of 10^6 simulations. We followed the optimal allocation provided in Broadie et al. [BDM11], Gordy and Juneja [GJ10] to obtain $M = 10^4$ primary scenarios and $K = 100$ inner simulations. The ALM market value balance sheet items (BEL_0, BOF_0) and the SCR using both approaches are summarized in Table 3.1.

BOF_0	BEL_0	SCR_0^{std}	SCR_0^{qtl}	$\frac{BOF_0}{SCR_{mkt}}$	$\frac{BOF_0}{SCR_0^{qtl}}$	$\frac{SCR_0^{std}-SCR_0^{qtl}}{SCR_0^{qtl}}$
0.0295	0.9695	0.0109	0.0103	2.7040	2.8760	5.83%

Table 3.1 – ALM model output using the EIOPA yield risk-free yield curve

We observe that both SCR are rather close with this setting, however, the standard formula requires about 6% more capital than the quantile approach. Hence the solvency ratio obtained with the quantile SCR are higher than the one obtain with the Standard formula. At the scale of an insurance company, the difference is actually quite high. In addition, observe that our setting, the EIOPA curve leads to a very favorable situation for the company which explains this high solvency ratio (higher than 270%). The use of UFR equal to 4.2% means that bond yield will keep increasing which is then translated in the shift factors. As time goes by, the minimum guaranteed rate is easier to fund which implies higher shareholders' P&L and lower SCR. Even if the SCR quantile value is lower than the standard formula in our setting, let us mention that sometimes, it can be profitable to use the Standard formula in a partial internal model. An obvious case is if the current interest rate level is low, using the multiplicative shock proposed by the EIOPA [EIOPA18], would lead to lower values of SCR than the quantile approach. To overcome this issue additive shocks on the yield curve have been proposed by EIOPA in [EIOPA18].

One year loss distribution

To complete our numerical investigation on the SCR quantile, we displayed the one-year loss distribution function $L_1 = BOF_0 - BOF_1 e^{-\int_0^1 r_s ds}$ using Nested Monte-Carlo simulations. From Figure 3.6, we see that the distribution is fat-tail and asymmetric. The average number of observation above the quantile is estimated to 0.0141. As suggested in Boonen [Boo17], Expected Shortfall is more appropriate in this case to take into account extreme events.

3.3.3 Impact of the UFR level on the SII balance-sheet

In this section, we describe the impact of the UFR level on SCR modules and Solvency Ratios. In Figure 3.8, the solvency ratio SR_0 is an increasing function of the UFR level. From Figure 3.9, we see that for low UFR values the main contribution to the

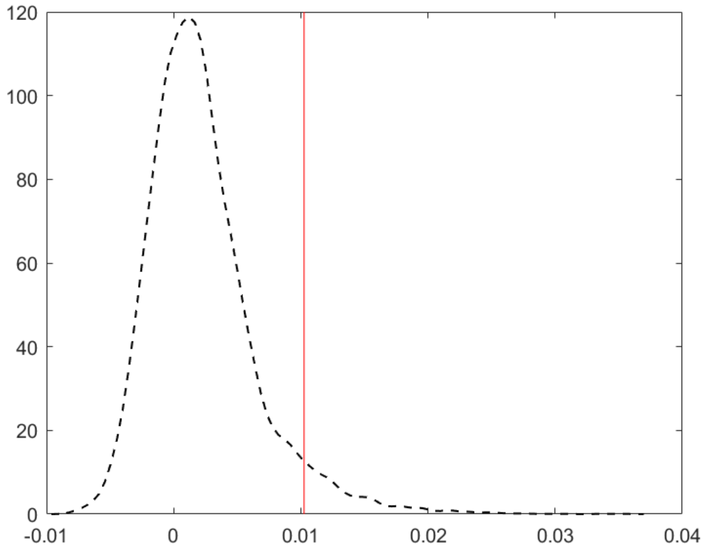

Figure 3.6 – one year loss distribution $L_1 = BOF_0 - BOF_1 e^{-\int_0^1 r_s ds}$, and SCR quantile (SCR_α value in red)

Solvency ratios comes from the SCR component. When the UFR is too low, bond yield in the EIOPA yield curve will get closer to the minimum guaranteed rate $r_G = 1.5\%$ which becomes more expensive to fund. As another side effect, the competition rate which is given by the short-rate that is calibrated on the EIOPA UFR curve also diminishes. As a consequence policyholders are less prone to surrender their contract and a larger proportion of the policyholder will remain in the portfolio. We also observe a discontinuity in the SCR function when the UFR level reaches a certain threshold. As already pointed out in 2.4.1, this is because of the ε discontinuity introduced in the Standard formula. In Figure 3.9, the discontinuity occurs when the upward shock is the worst shock in the SCR_{int} component. Another interesting point is that for large values of the UFR, the aggregated SCR_0^{std} does not vary too much whereas the level of basic-own funds BOF_0 keep increasing. Therefore, the main contributor in the improved Solvency Ratios comes from higher BOF_0 values when the UFR level is large enough. To summarize, up to a certain threshold, SCR values decrease faster than BOF_0, after that quite the opposite occurs, BOF_0 values keep increasing and SCR values do not change very much. From the balance sheet condition, since $MV_0 = BEL_0 + BOF_0$, increasing BOF_0 leads to lower BEL_0 values (see 3.8). Hence BEL_0 will decrease when the UFR increase, because the target-rate which is given by the short-rate calibrated on the EIOPA curve is easier to fund.

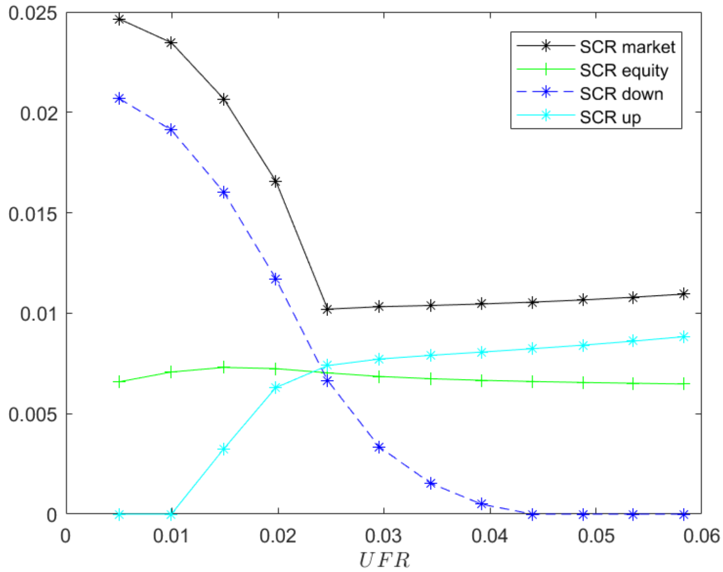

Figure 3.7 – SCR modules in function of the UFR level

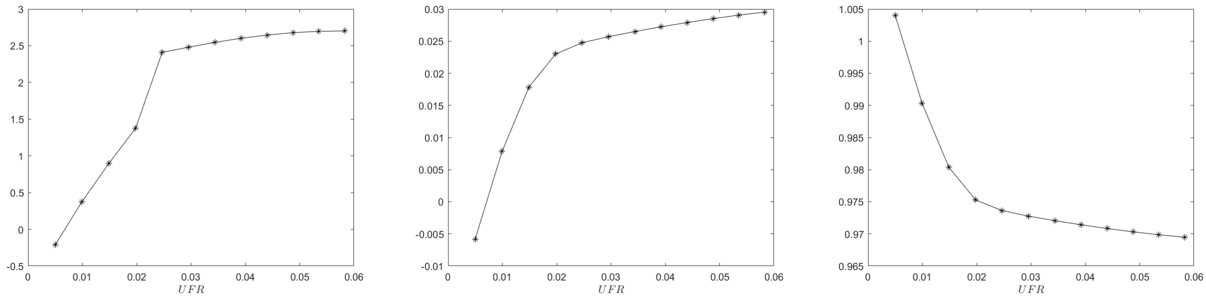

Figure 3.8 – Left: SR_0, Middle: BOF_0, Right: BEL_0 in function of the UFR level

SCR values for different maturities

We now study how the SCR values is sensitive w.r.t the UFR level when different portfolio maturity T are considered. We considered three portfolios that matures respectively in 30 and 65 years. We shall denote the respective portfolios P_{30} and P_{65}. In any case, we work with the parameters described in Table 4.1 and 4.2. The portfolio P_{30} is associated to the oldest age cohort while P_{65} represent a portfolio of liability associated to a young age cohort. The aim of this section is to quantify the exposure of the insurance company for long-term and shorter term liabilities. In Tables 3.2, 3.3 and 3.4 we see that the effect of the UFR level is smaller for portfolio P_{30}. In particular, the Best Estimate for P_{30} is practically unaffected by the level of the UFR. The reason is that the UFR will only affect the discounted cash flows for maturity 20 years or longer. For portfolio P_{65}, the activation period of the UFR is the longest (20 years to 65 years) which result in a higher impact on the balance sheet.

UFR	$T = 30$	$T = 65$
2%	0.0208	0.0228
4%	0.0209	0.0272
6%	0.0215	0.0296

Table 3.2 – Estimate of BOF_0 in function of the portfolio maturity T and the UFR

UFR	$T = 30$	$T = 65$
2%	0.9780	0.9755
4%	0.9782	0.9714
6%	0.09779	0.9694

Table 3.3 – Estimate of BEL_0 in function of the portfolio maturity T and the UFR

UFR	$T = 30$	$T = 65$
2%	0.0126	0.0166
4%	0.0095	0.0105
6%	0.0090	0.0110

Table 3.4 – Estimate of SCR_{mkt} in function of the portfolio maturity T and the UFR

3.3.4 Impact of the LLP on the SII balance sheet

We next present a pair of plots that provide insight of the effect of the Last-Liquid-Point on SII technical reserves. In our numerical setting, increasing the LLP will increase the effect of the liquid part of the yield curve which comprises bond yield around 2%. Consequently, the period where bond yield increases toward the UFR level $UFR = 4.2\%$ is reduce when the LLP increase. This result in deteriorated solvency situations for large values of the LLP since the company benefits from increasing bond in a shorter period. It result in lower BOF_0 and SCR_{mkt} values.

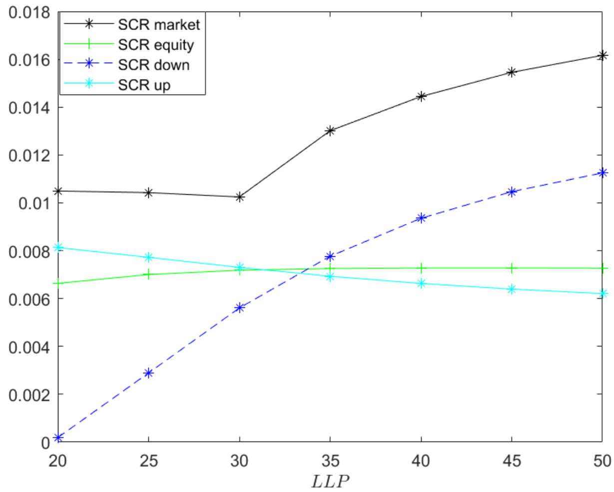

Figure 3.9 – SCR modules in function of the LLP

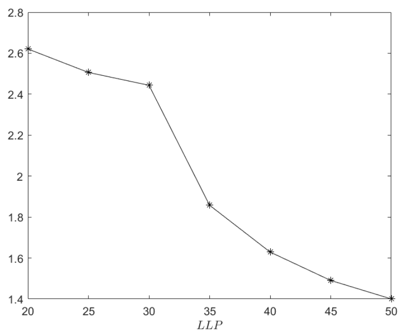

Figure 3.10 – $SR_0 = \frac{BOF_0}{SCR_0^{std}}$ in function of LLP

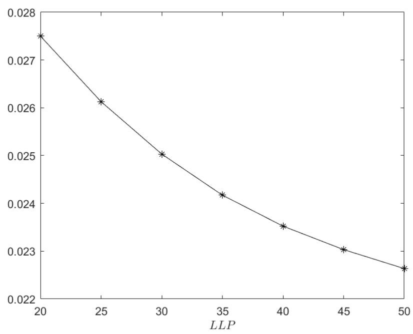

Figure 3.11 – BOF_0 in function of LLP

Part II

Numerical Methods for Nested Simulation: Application in ALM model

CHAPTER 4

MLMC FOR THE COMPUTATION OF FUTURE SCR

Contents

This chapter is an article written with A.Alfonsi and J.Arturo Infante Acevedo [ACIA20b] submitted for publication.

Abstract. This paper studies the multilevel Monte-Carlo estimator for the expectation of a maximum of conditional expectations. This problem arises naturally when considering many stress tests and appears in the calculation of the interest rate module of the standard formula for the SCR. We obtain theoretical convergence results that complements the recent work of Giles and Goda [GG19] and gives some additional tractability through a parameter that somehow describes regularity properties around the maximum. We then apply the MLMC estimator to the calculation of the SCR at future dates with the standard formula for an ALM savings business on life insurance. We compare it with estimators obtained with Least Square Monte-Carlo or Neural Networks. We find that the MLMC estimator is computationally more efficient and has the main advantage to avoid regression issues, which is particularly significant in the context of projection of a balance sheet by an insurer due to the path dependency. Last, we discuss the potentiality of this numerical method and analyse in particular the effect of the portfolio allocation on the SCR at future dates.

4.1 Introduction

Solvency II is a regulatory framework introduced in Europe in the period post-financial crisis of 2008. Solvency II establishes the requirements to be met to exercise the insurance or reinsurance activity in Europe and aims to protect policyholders and to give stability in the financial sector of the European Union.

One of the advantages of the Solvency II directive is that the computation required to evaluate the Solvency Required Capital (SCR) considers the specific risks borne by the insurers in comparison to the previous rules where the need of own funds ignored, for example, part of risks embedded in the asset side of the balance sheet. Indeed, under Solvency II the evaluation of the SCR amounts either to use the standard formula by applying shocks to each asset class or to calculate a quantile of the conditional law of the profits and losses for variations of the initial state of the market, given a portfolio of contracts.

Today, the SCR indicator is one of the most important Key Performance Indicator used by companies to monitor the activity. In particular, the so-called Solvency II ratio computed as the ratio between the "Eligible Own Fund" (EOF) and the SCR measures the solvency capacity of the insurers and it is followed by analysts to evaluate them in financial markets. Nevertheless, it is important to remark that the SCR corresponds to the amount of required capital in a 1 year horizon. Then, at time t to have an idea of the total amount of required capital during the life of a product or over the duration of the business, it is not only necessary to compute the SCR_t but also to estimate the SCR in future dates (SCR_{t+1}, SCR_{t+2}, etc).

The aim of this work is to deal with the problem of computing SCR at future dates which has several practical applications to real problems that arise in the insurance industry. One of the first applications that should be cited comes from the regulatory side and is called ORSA (Own Risk and Solvency Assessment) process which aims to evaluate from a prospective point of view the solvency needs related to the specific risk profile of the insurance companies. In order to do that, the computations of the SCR at future dates is necessary to ensure that the insurer is able to integrate the regulatory constraints in terms of solvency during the strategic plan horizon.

Other important applications appear when the notion of cost of capital is concerned. In general, the cost of capital refers to the desired return on the immobilized capital during the life of a product or over the duration of the business and can be written under the following form:

$$\textit{Targeted return} \times \textit{Immobilized Capital}$$

where *Targeted return* represents an expected return, for example, from a shareholder standpoint. It is possible to exhibit a relation between the Immobilized Capital and the SCR computed at futures dates by considering the simple following reasoning. At $t = 0$, the SCR_0 is the amount the shareholder needs to immobilize at inception in order to pay its liabilities in unfavorable cases. At $t = 1$, the insurance company pays the expected return to the shareholder on SCR_0 and lends from him the amount SCR_1 to continue to exercise the insurance activity. By repeating this mechanism at each time step until the end of the business or the product maturity (time T), the total immobilized capital at $t = 0$ corresponds to

$$\begin{aligned}\textit{Immobilized Capital}_0 &= SCR_0 + PV(\sum_{t=1}^{T} SCR_t - \textit{Targeted return} \times SCR_{t-1}) \\ &= (1 - \textit{Targeted return})PV(\sum_{t=1}^{T} SCR_{t-1}),\end{aligned}$$

where $PV(x)$ stands for the present value of x and given that $SCR_T = 0$ (it is assumed that the company closes its business at the end of the year T and then there is no need of capital between T and $T + 1$). Thus, among the applications related to the cost of capital, one can mention:

(i) Applications for ALM (Asset Liability Management) when a Strategic Asset Allocation needs to be computed for a given portfolio: to evaluate the optimality of an asset allocation, a criterion based on the sum of the present values of shareholder margins minus the amount of cost of capital generated by the asset allocation is usually studied. The idea of this approach is to analyse if the future gains generated by the portfolio meet the shareholder's expectations in terms of cost of capital.

(ii) Applications for pricing, when evaluating if future margins pay the return expected by the shareholders. Before launching a new product, the insurers evaluate the profitability of that product and then compare expected future shareholder margins with the need of capital generated by the new business.

Finally, the computation of future SCR can be used as a tool for studying the solvency of the company under different economic scenarios. For example, the current low interest rates environment yields in several questions on the solvency of the insurance companies and on the sustainability of the Savings business. In addition, the SCR computation at future dates allows to better understand the pattern of cash-flows generated by a product during the lifetime of the business. In particular, the approach based on shocks employed in this work as, for example, the shocks on the market conditions is useful to study the evolution of the balance sheet and the policyholder behavior under those shocked conditions.

In general, today, the computations where SCR at futures dates are needed are based on rough estimations from the initial SCR which can ignore the evolution of the risk profile of the insurer and may lead to bad decisions impacting the business.

Thus, the goal of the present work is to develop numerical methods for the calculation of the SCR required in the future. We focus here on the calculation of SCR with the standard formula, which is fully described by the documents of the European Insurance and Occupational Pensions Authority (EIOPA) [EIOPA12, EIOPA18]. Basically, this standard formula consists in applying different shocks on the different market sectors: the impact on the portfolio of each shock is evaluated in a risk-neutral world, and the SCR is then evaluated by using an aggregation formula from these impacts.

Many works in the literature deals with the numerical computation of the SCR so that we cannot be exhaustive. Devineau and Loisel [DL09], Bauer et al. [BRS12] have investigated numerical methods based on nested simulations. Bauer et al. [BMBR09], Krah et al. [KNK18] and Floryszczak et al. [FCM16] have used Least Square Monte-Carlo (regress now) methods for the risk while Pelsser and Schweizer [PS16], Cambou and Filipović have developed the replicating portfolio (or regress later) approach. Recently, Cheredito et al. [CEW20] and Fernandez-Arjona and Filipović [FAF20] have proposed to use neural networks to approximate the conditional expectation. Up to our knowledge, there are however no dedicated study on the use of multilevel Monte-Carlo estimators for the calculation of the SCR with practical application in an insurance context. This paper fills this gap. Besides, most of the paper deals with the quantile formulation of the SCR and focus on the calculation of the current value of the SCR (there are few exceptions such as Vedani and Devineau [VD12]). Here, we consider instead the calculation of the SCR with the standard formula at future dates. Last, most of the literature either use simple Markovian underlying models or consider instead models from insurance companies that are black boxes, which makes difficult the reproducibility of the results. Here, we are in between and make our experiments on a synthetic ALM model that we recently developed and fully presented in [ACIA20a] which takes into account many path-dependent features of the ALM for life insurance.

We now describe the formal mathematical framework and consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let X and Y be two random variables such that X takes values in a general measurable space $(G, \mathcal{G})$ and Y takes values in $\mathbb{R}^P$, $P \in \mathbb{N}^*$. We make the following assumptions:

(A.1) Y is square integrable $\mathbb{R}^P$-valued random variable,

(A.2) $\phi : G \to \mathbb{R}$ is a measurable real-valued function ϕ such that $\phi(X)$ is square integrable.

For the financial application that we consider in this paper, X generally represents the market information up to some time, and therefore may be the realization of asset paths. For example, we may take $G = \mathcal{C}([0,t], \mathbb{R}^d)$ if we consider a market with $d \in \mathbb{N}^*$ continuous assets up to time $t > 0$. However, under some Markovian assumption, the market information at time t may simply be sum up by the current value of the assets, in which case we may take $G = \mathbb{R}^d$. We are interested in the problem of computing nested expectations of the form :

$$I = \mathbb{E}\left[h\left(\mathbb{E}\left[Y^1|X\right], \ldots, \mathbb{E}\left[Y^P|X\right]\right)\phi(X)\right], \tag{4.1.1}$$

where $h : \mathbb{R}^P \to \mathbb{R}$ is a measurable function with sublinear growth (i.e. $\exists C > 0, \forall x \in \mathbb{R}^P, |h(x)| \leq C(1 + |x|)$), which ensures by Assumptions (A.1) and (A.2) that I is well defined. In Formula (4.1.1), $\mathbb{E}[Y^i|X]$ typically represents the expected loss at time t if one implements the shock (or stress test) number $i \in \{1, \ldots, P\}$ immediately after time t. The function h describes the aggregation of the shocks in terms of own funds while the function ϕ weights the different events up to t.

The calculation of I is usually made by using a nested Monte-Carlo method: one simulates J independent samples of X called primary scenarios and then, for each primary scenarios, one simulates independently K independent samples of Y to approximate the conditional expectations involved in (4.1.1) by the corresponding empirical mean. This method has been investigated by Gordy and Juneja [GJ10] and Broadie et. al [BDM11] to calculate the probability of large losses and the Value-at-Risk. Under mild assumptions, the optimal tuning to approximate I with a precision of $\varepsilon > 0$ is to take $J = O(\varepsilon^{-2})$ primary scenarios and $K = O(\varepsilon^{-1})$ secondary scenarios. Thus, the overall complexity is in $O(\varepsilon^{-3})$. The multilevel Monte-Carlo method (MLMC) developed by Giles [Gil08] has been applied to the calculation of nested expectations by Haji-Ali [HA12], Bujok et al. [BHR15] and Giorgi et al. [GLP20]. Under some regularity assumptions on h, they show that the antithetic MLMC estimator achieves a precision $\varepsilon > 0$ with a computational cost of $O(\varepsilon^{-2})$. Under additional regularity assumptions on h or on the probability density function of (X, Y), Giorgi et al. [GLP20] have applied the Richardson-Romberg Multilevel method developed by Lemaire and Pagès [LP17] that improves the convergence of the MLMC estimator.

In this work, we focus on the case where h is the maximum function that is sublinear, but does not satisfy the standard regularity assumptions. We are thus interested in the problem of computing nested expectations of the form

$$I = \mathbb{E}\left[\max\left\{\mathbb{E}\left[Y^1|X\right], \ldots, \mathbb{E}\left[Y^P|X\right]\right\}\phi(X)\right]. \tag{4.1.2}$$

Such kind of expectation arises in the standard formula for the calculation of the interest rate module of the SCR. More generally, the problem of computing (4.1.2) occurs when one has to determine the worst of a set of P shocks (or stress tests) on a portfolio of securities at some future time t called risk horizon. For the interest rate module of the SCR, one rather has to compute $\mathbb{E}\left[\max\left\{\mathbb{E}\left[Y^1|X\right], \ldots, \mathbb{E}\left[Y^P|X\right], 0\right\}\phi(X)\right]$, which amounts to add a zero coordinate to Y. When the function ϕ is nonnegative and such that $\mathbb{E}[\phi(X)] = 1$, $\phi(X)$ can be seen as a change of probability on the different events up to time t. The function $\phi(X)$ does not add any technical difficulty in our study, but it enables us to perform the evolution up to time t under the real probability and the evaluation of the losses under the risk-neutral probability, as it is recommended by Solvency II. Studies of MLMC estimators for nested expectations for irregular functions h with applications to risk management have recently been made by Giles and Haji-Ali [GHA19], Bourgey et al. [BDMGZ20] and Giorgi et al. [GLP20]. In a very recent work, Giles and Goda [GG19] have studied precisely the problem of computing (4.1.2) with the MLMC method.

The contribution of this paper is twofold. First, we provide an original mathematical analysis of the MLMC estimator for the calculation of (4.1.2) that completes the result obtained by Giles and Goda [GG19]. Our analysis relies on different arguments and the required assumptions are therefore also different. In particular, Giles and Goda make some technical assumptions to control the probability of two elements being close to the maximum. These assumptions are replaced in our analysis by an integrability assumption involving a parameter $\eta \in (0, 1)$ that gives some additional flexibility in the application of the MLMC estimator. Our second contribution is to apply this method to an ALM model for life insurance that takes into account the main characteristic of the business: book values, profit-sharing mechanism, minimum guaranteed rate, etc.

Thus, the model is truly path-dependent so that the conditional expectation at time t really involves the past dynamics making the use of regression techniques more delicate. One of the main advantage of the MLMC estimator is to calculate directly I and skip the question of regression. The second main advantage is that it provides an estimator with accuracy ε and with a computational cost in $O(\varepsilon^{-2})$: it is thus asymptotically as efficient as a Monte-Carlo method for plain expectations. In our numerical study, we compare the estimation of I with MLMC, Least Square Monte-Carlo (LSMC) estimator and the use of Neural Networks (NN), and demonstrate the main advantages of the MLMC estimator.

The paper is organized as follows. Section 4.2 presents the mathematical results on the estimation of I with nested Monte-Carlo and MLMC. Technical proofs are postponed to Appendix A.1. Section 4.3 then deals with the application to ALM. Subsections 4.3.1 and 4.3.3 present the ALM model for life insurance business that we developed in [ACIA20a] while Subsection 4.3.2 recalls the calculation of the SCR with the standard formula. Subsection 4.3.4 compares the numerical performance of the MLMC estimator with estimators obtained with LSMC or NN. Last, Subsection 4.3.5 shows the interest of analysing the SCR at future dates, exhibiting some interesting properties such as the dependence of the SCR on the portfolio allocation or on the market risk premia.

4.2 Mathematical analysis of Monte-Carlo estimators of I

4.2.1 Nested Monte-Carlo estimator

In order to compute I defined by (4.1.2), the classical approach is to approximate the inner and outer expectation using Monte-Carlo estimators. The procedure consists in generating an i.i.d sample $(X_1, \dots, X_J)$ of X called outer (or primary) scenarios. Then, conditionally on X_i, we sample $(Y_{i,1}, \dots, Y_{i,K})$ called inner (or secondary scenarios) following the conditional law of Y given $X = X_i$ and approximate the conditional expectation $\mathbb{E}\left[Y^p | X = X_i\right]$, for $p \in \{1, \dots, P\}$, by

$$\widehat{E}^p_{i,K} = \frac{1}{K} \sum_{k=1}^{K} Y^p_{i,k} \tag{4.2.1}$$

The outer expectation is then approximated using the standard MC estimator :

$$\widehat{I}_{J,K} = \frac{1}{J} \sum_{j=1}^{J} \max\left\{\widehat{E}^1_{j,K}, \dots, \widehat{E}^P_{j,K}\right\} \phi(X^j) \tag{4.2.2}$$

This estimator has been studied for example by Gordy and Juneja [GJ10] in the context of portfolio risk measurement. The nested simulation procedure introduces two level of error since we combine the estimates from the outer and inner levels of simulation to compute I. In a standard way, we analyse the Mean-Square Error (MSE) of the estimate $\text{MSE}(\widehat{I}_{J,K}) = \mathbb{E}\left[|\widehat{I}_{J,K} - I|^2\right]$ and use the bias-variance decomposition:

$$\text{MSE}(\widehat{I}_{J,K}) = \text{bias}^2(\widehat{I}_{J,K}) + \text{Var}(\widehat{I}_{J,K}),$$

where $\text{bias}(\widehat{I}_{J,K}) = \mathbb{E}[\widehat{I}_{J,K}] - I$.

Notation.

- *We set* $E_X^p := \mathbb{E}[Y^p|X]$ *for* $p \in \{1,\dots,P\}$ *and* $M_X^p = \max\{\mathbb{E}[Y^1|X],\dots,\mathbb{E}[Y^p|X]\}$.
- *Let* $K \in \mathbb{N}^*$ *and* $Y_1,\dots,Y_K$ *be an i.i.d. sample following the conditional law of* Y *given* X. *Then, we set*

$$\forall p = 1,\dots,P, \quad \widehat{E}_K^p = \frac{1}{K}\sum_{k=1}^K Y_k^p \text{ and } \widehat{M}_K^p = \max\{\widehat{E}_K^1,\dots,\widehat{E}_K^p\}. \tag{4.2.3}$$

- *Besides, when* K *is even, we define*

$$\forall p = 1,\dots,P, \quad \widehat{E}_{K/2}^{p,\prime} = \frac{2}{K}\sum_{k=K/2+1}^K Y_k^p \text{ and } \widehat{M}_{K/2}^{p,\prime} = \max\{\widehat{E}_{K/2}^{1,\prime},\dots,\widehat{E}_{K/2}^{p,\prime}\}. \tag{4.2.4}$$

From the LLN, we have $\widehat{E}_K^p \to E_X^p$ and $\widehat{M}_K^p \to M_X^p$ almost surely as $K \to +\infty$. The next theorem analyses the MSE of the nested estimator and provides estimates that will be then useful for the analysis of the MLMC estimator.

Theorem 4.1. *Let* $P \geq 2$ *and* $\eta \in (0,1]$. *Let* X, Y *be random variables defined on a probability space* $(\Omega,\mathcal{F},\mathbb{P})$. *We assume that (A.1) and (A.2) hold, and we define, for* $p \in \{1,\dots,P\}$, $\sigma_p(X) = \sqrt{\mathrm{Var}(Y^p|X)}$, $\Sigma_p^{1+\eta}(X) = \sum_{i=1}^p \sigma_i^{1+\eta}(X)$ *and*

$$C_p(X) = 2^\eta \Sigma_p^{1+\eta}(X)\sum_{p'=2}^p \frac{1}{|E_X^{p'} - M_X^{p'-1}|^\eta}.$$

Assume that the following condition holds:

(i) $\forall\ p = 2,\dots,P,\ \mathbb{P}\left(M_X^{p-1} = E_X^p\right) = 0$,

(ii) $\Sigma^2 = \mathbb{E}[\Sigma_P^2(X)\phi^2(X)] < \infty$ *and* $C = \mathbb{E}[C_P(X)|\phi(X)|] < \infty$.

Then, we have

$$\left|\mathbb{E}\left(\left(\widehat{M}_K^P - M_X^P\right)\phi(X)\right)\right| \leq \frac{C}{K^{\frac{1+\eta}{2}}} \quad \text{and } \mathbb{E}\left(\left(\widehat{M}_K^P - M_X^P\right)^2\phi^2(X)\right) \leq \frac{\Sigma^2}{K}. \tag{4.2.5}$$

Besides, if $V = \mathrm{Var}(M_X^P\phi(X)) < \infty$, *we get*

$$MSE(\widehat{I}_{J,K}) \leq \frac{C^2}{K^{1+\eta}} + \frac{2V}{J} + \frac{2\Sigma^2}{JK}. \tag{4.2.6}$$

With this upper bound, taking $K = O(\varepsilon^{-\frac{2}{1+\eta}})$ *and* $J = O(\varepsilon^{-2})$ *is an asymptotically optimal choice to get* $MSE(\widehat{I}_{J,K}) = O(\varepsilon^2)$ *while minimizing the computation cost* JK.

Remark 4.2. *Let us note that the assumptions (i) and* $C < \infty$ *of Theorem 4.1 are only needed to improve the upper bound on the bias. If it does not hold, we still have*

$$\left|\mathbb{E}\left(\left(\widehat{M}_K^P - M_X^P\right)\phi(X)\right)\right| \leq \mathbb{E}\left(\left|\left(\widehat{M}_K^P - M_X^P\right)\phi(X)\right|\right) \leq \frac{\Sigma}{\sqrt{K}}.$$

Note that the speed in $O(K^{-1/2})$ *is optimal. Consider the example where* $Y = (Y^1, Y^2)$ *and, given* X, Y^1 *and* Y^2 *are independent normal distribution with unit variance and the same mean* $m(X)$. *Then,* $M_X^2 = m(X)$ *and, given* X, $\widehat{M}_K^2 - M_X^2$ *has the same law as* $\frac{1}{\sqrt{K}}\max(G^1,G^2)$ *where* G^1 *and* G^2 *are independent standard normal variables.*

Remark 4.3. *For practical applications such as the standard formula for the SCR interest rate module, one usually considers the positive part of the maximum. This amounts to add the coordinate $Y^{P+1} = 0$ in our framework. Thus, if we assume in addition that $\mathbb{P}(M_X^P = 0) = 0$ and $\tilde{C} = \mathbb{E}\left[\left(C_P(X) + \frac{2^\eta \Sigma_P^{1+\eta}(X)}{M_X^P}\right)|\phi(X)|\right] < \infty$, then*

$$\left|\mathbb{E}\left(\left((\widehat{M}_K^P)^+ - (M_X^P)^+\right)\phi(X)\right)\right| \le \frac{\tilde{C}}{K^{\frac{1+\eta}{2}}} \quad \text{and } \mathbb{E}\left(\left((\widehat{M}_K^P)^+ - (M_X^P)^+\right)^2 \phi^2(X)\right) \le \frac{\Sigma^2}{K}.$$

Remark 4.4. *Let us assume for simplicity that $\phi \equiv 1$ and there exists $\underline{\sigma}, \overline{\sigma} \in \mathbb{R}_+^*$ such that for all $p \in \{1, \dots, P\}$,*

$$\underline{\sigma} \le \sigma_p(X) \le \overline{\sigma}, \text{ a.s.}$$

Then, the integrability condition (ii) of Theorem 4.1 is equivalent to have $\mathbb{E}[|E_X^p - M_X^{p-1}|^{-\eta}] < \infty$ for all $p \in \{2, \dots, P\}$. Suppose now that $E_X^p - M_X^{p-1}$ admits a probability density $f_p(x)$ that is continuous and does not vanish at 0. Then, the integrability condition near 0 gives

$$\int_{-\varepsilon}^{\varepsilon} |x|^{-\eta} f_p(x)dx < \infty \iff \eta < 1.$$

This indicates that, in a quite general framework, condition (ii) of Theorem 4.1 is not satisfied for $\eta = 1$ but may be satisfied for any $0 < \eta < 1$.

The proof of Theorem 4.1 is a consequence of the next lemma, whose proof is postponed to Appendix A.1.2. The analysis is rather standard, but the difficulty is to handle in the bias analysis the irregularity of the maximum when two (or more) arguments equal. This is why we need Assumption (i) and the finiteness of C in Assumption (ii). These assumptions are different from Assumptions 2 and 3 that are used by Giles and Goda [GG19] in a similar context. With their assumptions, they obtain a bias in $O(1/K^{1-\delta})$ for any arbitrary $0 < \delta < 1$. Here, we directly see the link between the integrability assumption and the bias in $O(1/K^{\frac{1+\eta}{2}})$. Besides, let us note that we do need to assume the boundedness of any moments of $Y^p\phi(X)$, $p \in \{1, \dots, P\}$ (Assumption 1 of [GG19]) since we are using a different approach that does not make use of the Burkholder-Davis-Gundy inequality.

Lemma 4.5. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space and $\eta \in (0, 1]$. Besides the random variable X, we consider real valued random variables $\widehat{\theta}_K^i$ and functions $\varphi_i(X)$, with $i \in \{1, 2\}$ that satisfy the following conditions :*

(i) $\widehat{\theta}_K^i \xrightarrow[K\to\infty]{} \varphi_i(X)$ *a.s*

(ii) There are nonnegative measurable functions C_i and σ_i^2 such that for all $K \in \mathbb{N}^*$:

$$\left|\mathbb{E}\left[\widehat{\theta}_K^i - \varphi_i(X)|X\right]\right| \le \frac{C_i(X)}{K^{\frac{1+\eta}{2}}}, \tag{4.2.7}$$

$$\mathbb{E}\left[\left|\widehat{\theta}_K^i - \varphi_i(X)\right|^2 |X\right] \le \frac{\sigma_i^2(X)}{K}. \tag{4.2.8}$$

(iii) Setting $\varphi_{21}(X) = \varphi_2(X) - \varphi_1(X)$, we have $\mathbb{P}(|\varphi_{12}(X)| = 0) = 0$.

Then, we have with

$$C(X) = \mathbb{1}_{\varphi_{21}(X)<0} C_1(X) + \mathbb{1}_{\varphi_{21}(X)>0} C_2(X) + 2^\eta \frac{\sigma_1^{1+\eta}(X) + \sigma_2^{1+\eta}(X)}{|\varphi_{21}(X)|}, \tag{4.2.9}$$

$$\sigma^2(X) = \sigma_1^2(X) + \sigma_2^2(X), \tag{4.2.10}$$

the following estimates:

$$\left|\mathbb{E}\left[\max\{\widehat{\theta}_K^1, \widehat{\theta}_K^2\} - \max\{\varphi_1(X), \varphi_2(X)\}|X\right]\right| \leq \frac{C(X)}{K^{\frac{1+\eta}{2}}}, \tag{4.2.11}$$

$$\mathbb{E}\left[\left|\max\{\widehat{\theta}_K^1, \widehat{\theta}_K^2\} - \max\{\varphi_1(X), \varphi_2(X)\}\right|^2 |X\right] \leq \frac{\sigma^2(X)}{K}. \tag{4.2.12}$$

Proof of Theorem 4.1. We first prove by induction on $P \geq 2$ that

$$\left|\mathbb{E}\left(\widehat{M}_K^P - M_X^P \middle| X\right)\right| \leq \frac{C_P(X)}{K^{\frac{1+\eta}{2}}} \text{ and } \mathbb{E}\left(\left(\widehat{M}_K^P - M_X^P\right)^2 \middle| X\right) \leq \frac{\Sigma_P^2(X)}{K}. \tag{4.2.13}$$

We apply Lemma 4.5 noticing that $\mathbb{E}[\widehat{E}_K^p - E_X^p|X] = 0$ and $\mathbb{E}[(\widehat{E}_K^p - E_X^p)^2|X] = \frac{\sigma_p^2(X)}{K}$, for $p \in \{1, \dots, P\}$. First, this gives the result for $P = 2$. Second, with the induction hypothesis for P, Lemma 4.5 gives that (4.2.13) is satisfied for $P+1$ with

$$C_{P+1}(X) = C_P(X) + 2^\eta \frac{\Sigma_P^{1+\eta}(X) + \sigma_{P+1}^{1+\eta}(X)}{|\widehat{E}_K^{P+1} - M_X^P|^\eta} \text{ and } \Sigma_{P+1}^2(X) = \Sigma_P^2(X) + \sigma_{P+1}^2(X),$$

which gives the claim.

Since $\text{bias}(\widehat{I}_{J,K}) = \mathbb{E}\left[\left(\widehat{M}_K^P - M_X^P\right)\phi(X)\right]$, we get $|\text{bias}(\widehat{I}_{J,K})| \leq \frac{\mathbb{E}[C_P(X)|\phi(X)|]}{K^{\frac{1+\eta}{2}}} = \frac{C}{K}$. Similarly, we have

$$\begin{aligned}
\text{Var}(\widehat{I}_{J,K}) = \frac{1}{J}\text{Var}[\widehat{M}_K^P \phi(X)] &\leq \frac{2}{J}\text{Var}\left[\left(\widehat{M}_K^P - M_X^P\right)\phi(X)\right] + \frac{2}{J}\text{Var}[M_X^P \phi(X)] \\
&\leq \frac{2}{J}\mathbb{E}\left[\left(\widehat{M}_K^P - M_X^P\right)^2 \phi^2(X)\right] + \frac{2}{J}\text{Var}[M_X^P \phi(X)],
\end{aligned}$$

which leads to (4.2.6).

Last, we notice that for $c_1, c_2 > 0$, the minimization of JK given $\frac{c_1}{K^{1+\eta}} + \frac{c_2}{J} = \varepsilon^2$ leads to $J = \frac{c_2}{(1+\eta)c_1} K^{1+\eta}$ and thus $K = O(\varepsilon^{-\frac{2}{1+\eta}})$ and $J = O(\varepsilon^{-2})$. Since $\frac{1}{JK} \leq \frac{1}{J}$ and $\frac{1}{JK} = O(\varepsilon^{2+\frac{2}{1+\eta}})$ is negligible with respect to $\frac{1}{K^2}$ and $\frac{1}{J}$, this choice is asymptotically optimal: it gives $MSE(\widehat{I}_{J,K}) = O(\varepsilon^2)$ with a computational cost in $O(\varepsilon^{-3-\frac{1-\eta}{1+\eta}})$. □

4.2.2 The Multilevel Monte-Carlo estimator

We now present the Multilevel Monte-Carlo (MLMC) estimator of I. We consider $L \in \mathbb{N}$ that represents the number of levels. Let $J_0, \dots, J_L \in \mathbb{N}^*$ and $K_0, \dots, K_L \in \mathbb{N}^*$ be such that

$$\forall l \in \{1, \dots, L\}, \; K_l = K_0 2^l. \tag{4.2.14}$$

For each level $l \in \{0, \dots, L\}$, we consider $(X_{l,j}, 1 \leq j \leq J_l)$ i.i.d. random variables having the same distribution as X, and random variables $(Y_{l,j,k}, 1 \leq j \leq J_l, 1 \leq k \leq K_l)$

that are independent given $(X_{l,j}, 1 \leq j \leq J_l)$ and such that $Y_{l,j,k}$ follows the distribution of Y given $X = X_{l,j}$. These random variables are assumed to be independent between levels, i.e. $(X_{l,j}, Y_{l,j,k}, 1 \leq j \leq J_l, 1 \leq k \leq K_l)_{l\in 0,\ldots,L}$ are independent. Then, we define for $l \in \{0, \ldots, L\}$ and $p \in \{1, \ldots, P\}$:

$$\widehat{E}^p_{l,j,K} = \frac{1}{K} \sum_{k=1}^{K} Y^p_{l,j,k}, K \in \{1, \ldots, K_l\} \tag{4.2.15}$$

$$\widehat{M}^p_{l,j,K} = \max(\widehat{E}^1_{l,j,K}, \ldots, \widehat{E}^p_{l,j,K}) \tag{4.2.16}$$

Then, the MLMC estimator of I is defined by

$$\widehat{I}^{MLMC} = \frac{1}{J_0} \sum_{j=1}^{J_0} \widehat{M}^P_{0,j,K_0} \phi(X_{0,j}) + \sum_{l=1}^{L} \frac{1}{J_l} \sum_{j=1}^{J_l} (\widehat{M}^P_{l,j,K_l} - \widehat{M}^P_{l,j,K_{l-1}}) \phi(X_{l,j}). \tag{4.2.17}$$

Let us assume that the assumptions of Theorem 4.1 hold. We have bias$(\widehat{I}^{MLMC}) = \mathbb{E}\left[\left(\widehat{M}^P_{K_L} - M^P_X\right)\phi(X)\right] = O(K_L^{-\frac{1+\eta}{2}}) = O(2^{-\frac{1+\eta}{2}L})$. Besides, we have

$$\begin{aligned}\mathrm{Var}\left(\left(\widehat{M}^P_{K_l} - \widehat{M}^P_{K_{l-1}}\right)\phi(X)\right) &\leq 2\mathrm{Var}\left(\left(\widehat{M}^P_{K_l} - M^P_X\right)\phi(X)\right) + 2\mathrm{Var}\left(\left(\widehat{M}^P_{K_{l-1}} - M^P_X\right)\phi(X)\right) \\ &= O(K_l^{-1}) = O(2^{-l})\end{aligned}$$

and the computational cost of $(\widehat{M}^P_{l,j,K_l} - \widehat{M}^P_{l,j,K_{l-1}})\phi(X_{l,j})$ is $O(K_l) = O(2^l)$. We can thus apply Theorem 1 [Gil15], which leads to the following result.

Proposition 4.6. *Let us assume that the assumptions of Theorem 4.1 hold for some $\eta \in (0, 1]$. Then, by taking when $\varepsilon \to 0$*

$$L = \left\lceil \frac{2}{1+\eta} \frac{|\log(\varepsilon)|}{\log(2)} \right\rceil, J_0 = 2^{\left\lceil \frac{2|\log(\varepsilon)|+|\log(|\log(\varepsilon)|)|}{\log(2)} \right\rceil} = O(\varepsilon^{-2}|\log(\varepsilon)|) \text{ and } J_l = J_0 2^{-l}, l \in \{1, \ldots L\}, \tag{4.2.18}$$

we have $MSE(\widehat{I}^{MLMC}) = \mathbb{E}[(\widehat{I}^{MLMC} - I)^2] = O(\varepsilon^2)$ with a computational cost in $O(\varepsilon^{-2}\log^2(\varepsilon))$.

If only the assumption $\Sigma^2 < \infty$ of Theorem 4.1 holds, the same conclusion holds by taking $\eta = 0$ in (4.2.18).

Proof. We just check that the parameters achieve the claim. From the bias-variance decomposition, we get by using Theorem 4.1, (4.2.14) and (4.2.18) that there is a positive constant C such that

$$MSE(\widehat{I}^{MLMC}) \leq C\left(\frac{1}{K_L^{1+\eta}} + \frac{1}{J_0} + \sum_{l=1}^{L} \frac{1}{J_l K_l}\right) = C\left(\frac{2^{-(1+\eta)L}}{K_0^{1+\eta}} + \frac{1}{J_0} + \frac{L}{J_0 K_0}\right).$$

The choice of L gives $2^{-(1+\eta)L} \leq \varepsilon^2$ and the choice of J_0 then gives $\frac{L}{J_0} = O(\varepsilon^2)$. Last the computational cost is given by $\sum_{l=0}^{L} J_l K_l = L J_0 K_0 = O(\varepsilon^{-2}\log^2(\varepsilon))$. In the case where we only know $\Sigma^2 < \infty$, only the second statement of Equation (4.2.5) holds, and we get

$$\left|\mathbb{E}\left[\left(\widehat{M}^P_{K_L} - M^P_X\right)\phi(X)\right]\right| \leq \frac{\Sigma}{\sqrt{K_L}} = \frac{\Sigma}{\sqrt{K_0}} 2^{-L/2},$$

which gives the second claim with the same arguments. □

Remark 4.7. *Let us note that the analysis of the computational cost gives that it is asymptotically bounded by $C\varepsilon^2\log^2(\varepsilon)$ for some constant $C > 0$, but it does not analyse precisely this constant. Nonetheless, since this cost is LJ_0K_0, this constant can be chosen to be proportional to the number of levels.*

Thus, the analysis of the bias given by Theorem 4.1 under the integrability assumption $\mathbb{E}[C_P(X)|\phi(X)|] < \infty$ enables to reduce the number of levels and then to reduce this constant.

It is however possible to construct a better estimator using the MLMC antithetic estimator

$$\widehat{I}_A^{MLMC} = \frac{1}{J_0}\sum_{j=1}^{J_0}\widehat{M}^P_{0,j,K_0}\phi(X_{0,j}) + \sum_{l=1}^{L}\frac{1}{J_l}\sum_{j=1}^{J_l}\left(\widehat{M}^P_{l,j,K_l} - \frac{\widehat{M}^P_{l,j,K_{l-1}} + \widehat{M}^{P,\prime}_{l,j,K_{l-1}}}{2}\right)\phi(X_{l,j}), \tag{4.2.19}$$

where we set for $p \in \{1,\dots,P\}$,

$$\widehat{E}^{p,\prime}_{l,j,K_{l-1}} = \frac{1}{K_{l-1}}\sum_{k=K_{l-1}+1}^{K_l} Y^p_{l,j,k} \text{ and } \widehat{M}^{p,\prime}_{l,j,K_{l-1}} = \max(\widehat{E}^{1,\prime}_{l,j,K_{l-1}},\dots,\widehat{E}^{p,\prime}_{l,j,K_{l-1}}). \tag{4.2.20}$$

This is a rather natural idea to reduce the variance contribution of each level, see Section 9.1 of [Gil15]. However, the irregularity of the maximum function makes the analysis of the variance more delicate as if it were a smooth function. Giles and Goda [GG19] give an analysis of the variance that require again the boundedness of any moments of $Y^p\phi(X)$, $p \in \{1,\dots,P\}$ (Assumption 1 of [GG19]) and assumptions to control the probability that another component is close to the maximum (Assumptions 2 and 3 of [GG19]). Here, our proof relies on a different argument and only requires a moment condition that quantifies in a different way the probability that two or more arguments in the maximum are close to the maximum. Details are in the Appendix (see Proposition A.7).

Remark 4.8. *For the calculation of (4.1.1) with a general function h, the antithetic MLMC estimator is defined by*

$$\frac{1}{J_0}\sum_{j=1}^{J_0} h(\widehat{E}^1_{0,j,K_0},\dots,\widehat{E}^P_{0,j,K_0})\phi(X_{0,j})$$
$$+\sum_{l=1}^{L}\frac{1}{J_l}\sum_{j=1}^{J_l}\left(h(\widehat{E}^1_{l,j,K_l},\dots,\widehat{E}^P_{l,j,K_l}) - \frac{h(\widehat{E}^1_{l,j,K_{l-1}},\dots,\widehat{E}^P_{l,j,K_{l-1}}) + h(\widehat{E}^{1,\prime}_{l,j,K_{l-1}},\dots,\widehat{E}^{P,\prime}_{l,j,K_{l-1}})}{2}\right)\phi(X_{l,j}).$$

In particular, it is possible to estimate by MLMC the value of (4.1.1) for different functions h with the same simulations.

Theorem 4.9. *Let $\eta \in (0,1]$. We assume that the assumptions of Theorem 4.1 hold and besides that*

$$\forall p \in \{2,\dots,P\},\ \mathbb{E}\left[\frac{D_{2+\eta}(X)}{|E^p_X - M^{p-1}_X|^\eta}\phi^2(X)\right] < \infty,$$

where $D^p_{2+\eta}(X) = \mathbb{E}[|Y^p - \mathbb{E}[Y^p|X]|^{2+\eta}|X]$. Then, by taking when $\varepsilon \to 0$

$$L = \left\lceil \frac{2}{1+\eta}\frac{|\log(\varepsilon)|}{\log(2)}\right\rceil, J_0 = 2^{\left\lceil\frac{2|\log(\varepsilon)|}{\log(2)}\right\rceil} = O(\varepsilon^{-2}) \text{ and } J_l = \lceil J_0 2^{-(1+\frac{\eta}{4})l}\rceil, l \in \{1,\dots L\}, \tag{4.2.21}$$

we have $MSE(\widehat{I}_A^{MLMC}) = \mathbb{E}[(\widehat{I}_A^{MLMC} - I)^2] = O(\varepsilon^2)$ *with a computational cost in* $O(\varepsilon^{-2})$.

Proof. We have $\text{bias}(\widehat{I}_A^{MLMC}) = \text{bias}(\widehat{I}^{MLMC}) = O(2^{-\frac{1+\eta}{2}L})$. By Proposition A.7, the variance of each level satisfies

$$\text{Var}\left(\left(\widehat{M}^P_{K_l} - \frac{\widehat{M}^P_{K_{l-1}} + \widehat{M}^{P,\prime}_{K_{l-1}}}{2}\right)\phi(X)\right) = O(K_l^{-\left(1+\frac{\eta}{2}\right)}) = O(2^{-l\left(1+\frac{\eta}{2}\right)}),$$

and the computational cost of $\left(\widehat{M}^P_{K_l} - \frac{\widehat{M}^P_{K_{l-1}} + \widehat{M}^{P,\prime}_{K_{l-1}}}{2}\right)\phi(X)$ is in $O(K_l) = O(2^l)$. We are thus in the framework of Theorem 1 of [Gil15], and we just check that the choice of parameters (4.2.21) gives the claim. By using the bias variance decomposition, we have

$$MSE(\widehat{I}_A^{MLMC}) \leq C\left(2^{-(1+\eta)L} + \frac{1}{J_0} + \sum_{l=1}^{L}\frac{1}{J_l K_l^{1+\frac{\eta}{2}}}\right) \leq C\left(\varepsilon^2 + \varepsilon^2\sum_{l=0}^{L}2^{-\frac{\eta}{4}l}\right).$$

Since $\sum_{l=0}^{L}2^{-\frac{\eta}{4}l} \leq \sum_{l=0}^{\infty}2^{-\frac{\eta}{4}l} = \frac{1}{1-2^{-\frac{\eta}{4}}}$, we indeed have $MSE(\widehat{I}_A^{MLMC}) = O(\varepsilon^2)$. Observing that for $\varepsilon \in \mathbb{R}_+^*$ small enough, we have $J_0 2^{-\left(1+\frac{\eta}{4}\right)L} \geq 1$ and thus $J_l \leq 2J_0 \times 2^{-\left(1+\frac{\eta}{4}\right)l}$ for $l \in \{0, \dots, L\}$, we can upper bound the computational cost as follows

$$\sum_{l=0}^{L}J_l K_l \leq 2J_0K_0\sum_{l=0}^{L}2^{-\frac{\eta}{4}l} \leq \frac{4K_0\varepsilon^{-2}}{1-2^{-\frac{\eta}{4}}}. \qquad \square$$

Remark 4.10. *We can easily extend Theorem 4.9 if we assume that the assumption of Theorem 4.1 is true for some* $\eta_1 \in (0,1]$ *and that*

$$\forall p \in \{2,\dots,P\},\ \mathbb{E}\left[\frac{D^p_{2+\eta_2}(X)}{|E^p_X - M^{p-1}_X|^{\eta_2}}\phi^2(X)\right] < \infty,$$

for some $\eta_2 > 0$. *If we then take*

$$L = \left\lceil \frac{2}{1+\eta_1}\frac{|\log(\varepsilon)|}{\log(2)}\right\rceil, J_0 = 2^{\left\lceil \frac{2|\log(\varepsilon)|}{\log(2)}\right\rceil} = O(\varepsilon^{-2}) \text{ and } J_l = \lceil J_0 2^{-\left(1+\frac{\eta_2}{4}\right)l}\rceil, l \in \{1,\dots L\},$$

we get in the same way that $MSE(\widehat{I}_A^{MLMC}) = O(\varepsilon^2)$ *with a computational cost in* $O(\varepsilon^{-2})$. *However, roughly speaking, the integrability assumption of Theorem 4.1 for the bias deals with the integrability of* $\frac{1}{|E^p_X - M^{p-1}_X|^{\eta_1}}$ *when* $|E^p_X - M^{p-1}_X|$ *is close to* 0, *similarly as the assumption for the variance estimate. Thus, it is rather natural to consider* $\eta_1 = \eta_2$, *and we state Theorem 4.9 in this case for sake of simplicity.*

4.2.3 Least-Square Monte Carlo techniques for Nested Expectations

In this paragraph, we aim at presenting briefly the classical technique of regression in our context, i.e. for the calculation of I. For simplicity, we only consider here regressors that are indicator functions.

Let $N_r \in \mathbb{N}^*$ be the number of regressors. We consider $B_1, \dots, B_{N_r} \in \mathcal{G}$ disjoint measurable sets of the space where X takes values, and we define for $n \in \{1, \dots, N_r\}$ and $p \in \{1, \dots, P\}$,

$$\alpha_n^p = \mathbb{E}[Y^p | X \in B_n] = \frac{\mathbb{E}[Y^p \mathbb{1}_{X \in B_n}]}{\mathbb{P}(X \in B_n)} \quad \text{(with the convention } 0/0 = 0).$$

Then we have

$$\forall p \in \{1, \dots, P\},\ \mathbb{E}\left[\left(Y^p - \sum_{n=1}^{N_r} \alpha_n^p \mathbb{1}_{X \in B_n}\right)^2\right] = \min_{\alpha_1, \dots, \alpha_{N_r} \in \mathbb{R}} \mathbb{E}\left[\left(Y^p - \sum_{n=1}^{N_r} \alpha_n \mathbb{1}_{X \in B_n}\right)^2\right],$$

i.e. $\sum_{n=1}^{N_r} \alpha_n^p \mathbb{1}_{X \in B_n}$ is the L^2 projection of Y^p on $\{\sum_{n=1}^{N_r} \alpha_n \mathbb{1}_{X \in B_n} : \alpha_1, \dots, \alpha_{N_r} \in \mathbb{R}\}$. It is a natural proxy of E_X^p, which is the L^2 projection on the larger space of $\sigma(X)$-measurable random variables. We then define $\gamma_n^P = \max_{p=1,\dots,P} \alpha_n^p$, so that $\sum_{n=1}^{N_r} \gamma_n^P \mathbb{1}_{X \in B_n}$ approximates M_X^P.

Let us consider $(X_j, Y_j)_{1 \le j \le J}$ an i.i.d. sample following the distribution of (X, Y). We define

$$\widehat{\alpha}_{n,J}^p = \frac{\sum_{j=1}^J Y_j^p \mathbb{1}_{X_j \in B_n}}{\sum_{j=1}^J \mathbb{1}_{X_j \in B_n}} \quad \text{(with the same convention } 0/0 = 0)$$

and have similarly

$$\forall p \in \{1, \dots, P\},\ \frac{1}{J}\sum_{j=1}^J \left(Y_j^p - \sum_{n=1}^{N_r} \widehat{\alpha}_{n,J}^p \mathbb{1}_{X_j \in B_n}\right)^2 = \min_{\alpha_1, \dots, \alpha_{N_r} \in \mathbb{R}} \frac{1}{J}\sum_{j=1}^J \left(Y_j^p - \sum_{n=1}^{N_r} \alpha_n \mathbb{1}_{X_j \in B_n}\right)^2.$$

We define $\widehat{\gamma}_{n,J}^P = \max_{p=1,\dots,P} \widehat{\alpha}_{n,J}^p$, so that $\sum_{n=1}^{N_r} \widehat{\gamma}_{n,J}^P \mathbb{1}_{X_j \in B_n}$ approximates $M_{X_j}^P$. Thus, we define the Least Square Monte-Carlo estimator of I by

$$\widehat{I}^{LSMC} = \frac{1}{J}\sum_{j=1}^J \phi(X_j) \sum_{n=1}^{N_r} \widehat{\gamma}_{n,J}^P \mathbb{1}_{X_j \in B_n}. \tag{4.2.22}$$

We are interested in estimating the MSE of this estimator. The next proposition gives a framework to analyse it, which is useful to determine asymptotically the number of regressors and the number of Monte-Carlo samples to reach a given precision $\varepsilon > 0$.

Proposition 4.11. *For $p = 1, \dots, P$, we set $\sigma_p(x) = \mathrm{Var}(Y^p | X = x)$, and assume that there exists $\overline{\sigma}, \overline{\phi} \in \mathbb{R}_+^*$ such that for all $x \in G$, $\sigma_p^2(x) \le \overline{\sigma}^2$ and $|\phi(x)| \le \overline{\phi}$. Then, we have*

$$\mathbb{E}[(\widehat{I}^{LSMC} - I)^2] \le 2\overline{\phi}^2 \left(\frac{\overline{\sigma}^2 N_r P + \mathbb{E}[(M_X^P)^2]}{J} + \sum_{p=1}^P \mathbb{E}\left[\left(E_X^p - \sum_{n=1}^{N_r} \alpha_n^p \mathbb{1}_{X \in B_n}\right)^2\right]\right).$$

We now suppose in addition that:

1. *$G = [0,1]^d$, $N_r = n_r^d$ and for any $n \in \{1, \dots, N_r\}$,*

$$B_n = \left[\frac{i_1}{n_r}, \frac{i_1 + 1}{n_r}\right) \times \dots \times \left[\frac{i_d}{n_r}, \frac{i_d + 1}{n_r}\right).$$

where $i_1, \dots, i_d \in \{0, \dots, n_r - 1\}$ are the unique integers determined by the decomposition in base n_r of $n - 1$, i.e. $n - 1 = \sum_{k=1}^d i_k n_r^{k-1}$,

2. *for any $p \in \{1, \dots, P\}$, the function $G \ni x \mapsto E_x^p = \mathbb{E}[Y^p | X = x]$ is Lipschitz continuous with constant L for the $\|\|_\infty$ norm on $\mathbb{R}^d$.*

Then, we have

$$\mathbb{E}[(\widehat{I}^{LSMC} - I)^2] \leq 2\overline{\phi}^2 \left(\frac{\sigma^2 N_r P}{J} + \frac{PL^2}{N_r^{2/d}} \right).$$

With this upper bound, taking $J \sim c\varepsilon^{-d-2}$ and $N_r \sim c'\varepsilon^{-d}$ for some constants $c, c' > 0$ is an asymptotic optimal choice to have $\mathbb{E}[(\widehat{I}^{LSMC} - I)^2] = O(\varepsilon^2)$, with an overall computational cost in $O(\varepsilon^{-d-2})$.

In comparison with the MLMC estimator, it is worth to notice that $\widehat{I}^{LSMC}$ suffers from the curse of dimensionality. The larger is the dimension of G (the space where X takes values), the more it requires computational effort. As we will see, for the problem of the calculation of the SCR for ALM management, this is particularly detrimental.

Proof. From the definition of $\widehat{I}^{LSMC}$ (4.2.22), we get by using $(a+b)^2 \leq 2a^2 + 2b^2$ and Jensen's inequality:

$$\begin{aligned}
\mathbb{E}[(\widehat{I}^{LSMC} - I)^2] &= \mathbb{E}\left[\left(\widehat{I}^{LSMC} - \frac{1}{J}\sum_{j=1}^J \phi(X_j) M_{X_j}^P + \frac{1}{J}\sum_{j=1}^J \phi(X_j) M_{X_j}^P - I \right)^2 \right] \\
&\leq \mathbb{E}\left[\frac{2}{J}\sum_{j=1}^J \phi^2(X_j) \left(\sum_{n=1}^{N_r} \widehat{\gamma}_{n,J}^P \mathbb{1}_{X_j \in B_n} - M_{X_j}^P \right)^2 \right] + 2\frac{\mathrm{Var}(\phi(X) M_X^P)}{J} \\
&\leq 2\overline{\phi}^2 \left(\mathbb{E}\left[\frac{1}{J}\sum_{j=1}^J \left(\sum_{n=1}^{N_r} \widehat{\gamma}_{n,J}^P \mathbb{1}_{X_j \in B_n} - M_{X_j}^P \right)^2 \right] + \frac{\mathbb{E}[(M_X^P)^2]}{J} \right).
\end{aligned}$$

Now, Theorem 8.2.4 [Gob16] gives

$$\mathbb{E}\left[\frac{1}{J}\sum_{j=1}^J \left(E_{X_j}^p - \sum_{n=1}^{N_r} \widehat{\alpha}_{n,J}^p \mathbb{1}_{X_j \in B_n} \right)^2 \right] \leq \frac{\sigma^2 N_r}{J} + \mathbb{E}\left[\left(E_X^p - \sum_{n=1}^{N_r} \alpha_n^p \mathbb{1}_{X \in B_n} \right)^2 \right].$$

We recall that $\widehat{\gamma}_{n,J}^P = \max_{p=1,\dots,P} \widehat{\alpha}_{n,J}^p$ and observe that for $X_j \in B_n$, we have

$$(M_{X_j}^P - \max_{p=1,\dots,P} \widehat{\alpha}_{n,J}^p)^2 \leq \max_{p=1,\dots,P} (E_{X_j}^p - \widehat{\alpha}_{n,J}^p)^2 \leq \sum_{p=1}^P (E_{X_j}^p - \widehat{\alpha}_{n,J}^p)^2$$

since $|\max_{p=1,\dots,P} a_p - \max_{p=1,\dots,P} b_p| \leq \max_{p=1,\dots,P} |a_p - b_p|$ for any $a, b \in \mathbb{R}^P$. This gives the first upper bound.

We now consider the case $G = [0,1]^d$ with the related assumptions. Then, for $X \in B_n$, we have for any p

$$\begin{aligned}
|E_X^p - \alpha_n^p| &= \left| E_X^p - \int_{x \in B_n} E_x^p \mathbb{P}(X \in dx | X \in B_n) \right| \\
&\leq \int_{x \in B_n} |E_X^p - E_x^p| \mathbb{P}(X \in dx | X \in B_n) \leq \frac{L}{n_r} = \frac{L}{N_r^{1/d}},
\end{aligned}$$

since $\|X - x\|_\infty \leq \frac{1}{n_r}$ for $X, x \in B_n$. This gives the second bound. To have this upper bound smaller than $C\varepsilon^2$ for some constant $C > 0$, one must at least have $N_r \geq c_1 \varepsilon^{-d}$

and $J \geq c_2 N_r \varepsilon^{-2}$ for some constants $c_1, c_2 > 0$, which leads to take $N_r \sim c' \varepsilon^{-d}$ and $J \sim \varepsilon^{-d-2}$.

Last, we observe that the computational cost to find n such that $x \in B_n$ is constant since $i_k = \lfloor n_r x_k \rfloor$ and $n = 1 + \sum_{k=1}^{d} i_k n_r^{k-1}$. Therefore, computing all the $2N_r$ sums $\sum_{j=1}^{J} Y_j^p \mathbb{1}_{X_j \in B_n}$ and $\sum_{j=1}^{J} \mathbb{1}_{X_j \in B_n}$ that define can be achieved with a computational cost of $O(J)$, and the calculation of (4.2.22) costs similarly $O(J)$. Since $J \sim c\varepsilon^{-d-2}$, we get the claim. □

4.2.4 Numerical results on a toy example: the Butterfly Call Option with the Black-Scholes model

The goal of this section is to illustrate the theoretical results on a simple case where the conditional expectations are known explicitly. Thus, we consider an asset following the Black-Scholes model:

$$S_t = S_0 \exp\left(\sigma W_t - \frac{\sigma^2}{2} t\right), \ t \geq 0,$$

where W is a standard Brownian motion and $\sigma > 0$ is the volatility. We consider a butterfly option with payoff at time $T > 0$:

$$\psi(S_T) = (S_T - K_1)^+ + (S_T - K_2)^+ - 2\left(S_T - \frac{K_1 + K_2}{2}\right)^+,$$

where $0 < K_1 < K_2$. The price of this butterfly option at time $t \in [0, T]$ is given by

$$\begin{aligned}\mathbb{E}[\psi(S_T)|S_t] &= \text{Call}^{\text{BS}}(T-t, S_t, K_1) + \text{Call}^{\text{BS}}(T-t, S_t, K_2) - 2\text{Call}^{\text{BS}}(T-t, S_t, K_1 + K_2) \\ &=: \text{Butterfly}(T-t, S_t)\end{aligned}$$

with $\text{Call}^{\text{BS}}(t, s, K) = s\mathcal{N}(\frac{1}{\sigma\sqrt{t}} \ln(s/K) + \frac{\sigma}{2}\sqrt{t}) - K\mathcal{N}(\frac{1}{\sigma\sqrt{t}} \ln(s/K) - \frac{\sigma}{2}\sqrt{t})$, where $\mathcal{N}$ is the cumulative distribution function of the standard normal distribution.

Now, we consider multiplicative upward and downward shocks $s^{up/down}$ that occur instantaneously at time t, we want to compute the worst loss between these shocks when it is positive. Since the Black-Scholes model is multiplicative with respect to the spot value, this shocks amounts to multiply the asset by $(1 \pm s^{up/down})$. Hence, setting $X = S_t$, $Y^1 = (\psi(S_T) - \psi((1+s^{up})S_T))$ and $Y^2 = \left(\psi(S_T) - \psi((1+s^{down})S_T)\right)$ we want to compute the following quantity :

$$I = \mathbb{E}\left[\max\left\{\mathbb{E}[Y^1|X], \mathbb{E}[Y^2|X], 0\right\}\right].$$

We are thus indeed in our general framework with $P = 3$ and $Y^3 = 0$ and $\phi(x) = 1$, and we have

$$M_X^3 = \max\left\{\text{Butterfly}(T-t, (1+s^{up})X), \text{Butterfly}(T-t, (1+s^{down})X), 0\right\}.$$

Since X follows a log-normal distribution, the exact value of I can be thus obtained by numerical integration

Numerical values. In all our numerical experiments, we consider the initial price $S_0 = 100$, the volatility $\sigma = 0.3$, the strikes $K_1 = S_0 + a$ and $K_2 = S_0 - a$ with $a = 50$, the option maturity $T = 2$ years and perform the shocks at $t = 1$ year. In our tests, we take $s^{up} = 0.2$ and $s^{down} = -0.2$.

Figure 4.1 illustrates the bias $\mathbb{E}[\widehat{M}_K^3 - M_X^3]$ in function of K with a log-scale. The expectation is approximated by the Nested Monte-Carlo with $J = 10^4$ to get a negligible statistical error. As a comparison, the function $K \mapsto 1/K$ is drawn, and we observe that two curves are quite parallel, which indicates that the bias behaves asymptotically like c/K. Also, we have drawn in Figure 4.2 the variance of $\widehat{M}_{K_l}^3 - \frac{\widehat{M}_{K_{l-1}}^3 + \widehat{M}_{K_{l-1}}^{3,\prime}}{2}$ in function of K_l, and we observe a behaviour in $K_l^{-3/2}$. Thus, it is reasonable to apply then the Multilevel method with $\eta = 1$ to determine the parameters in Equation (4.2.21). We have drawn in Figure 4.3 the RMSE in function of the computational cost (defined by $\sum_{l=0}^{L} J_l K_l$) for different values of $\eta \leq 1$. We observe a behaviour in ε^{-2}, which is in line with Theorem 4.9. The RMSE is calculated empirically by running many times the Multilevel method.

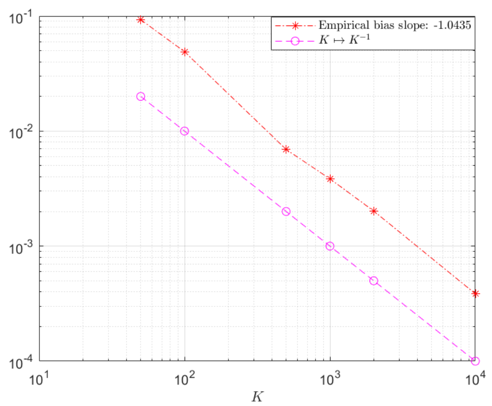

Figure 4.1 – Bias behaviour of the nested estimator

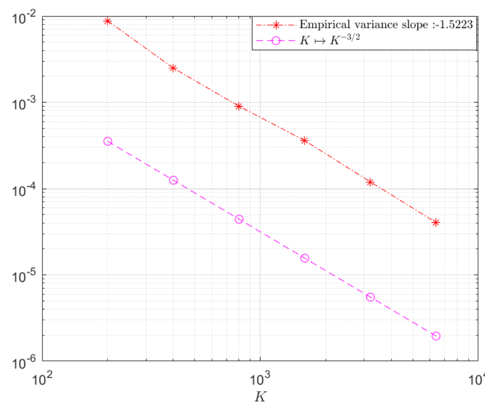

Figure 4.2 – Var $\left(\widehat{M}_{K_l}^3 - \frac{\widehat{M}_{K_{l-1}}^3 + \widehat{M}_{K_{l-1}}^{3,\prime}}{2}\right)$ in function of K_l

We now present the implementation of LSMC estimator. We note that $X' = \frac{1}{\sigma}\log(X) + \sigma/2$ is a standard normal distribution and therefore takes with a probability greater than 99% its values in $[-3, 3]$. We notice that $\mathbb{E}[Y^p|X] = \mathbb{E}[Y^p|X']$ and take the following regressors

$$\mathbb{1}_{X' \in [-3+6j/N_r, -3+6(j+1)/N_r]}, \quad j = 0, \ldots N_r - 1.$$

Up to a translation, we are thus in the framework of Proposition 4.11. In Figure 4.4, we have plotted the RMSE as a function of the number of samples J, which is also the computational cost of the method. The behaviour is in line with the theoretical result given by Proposition 4.11.

We already see on this one-dimensional example that the MLMC estimator has some benefit in terms of convergence with respect to the LSMC estimator. As we will see in the next section for the SCR estimation, this benefit is much more important when X takes values in a high-dimensional space.

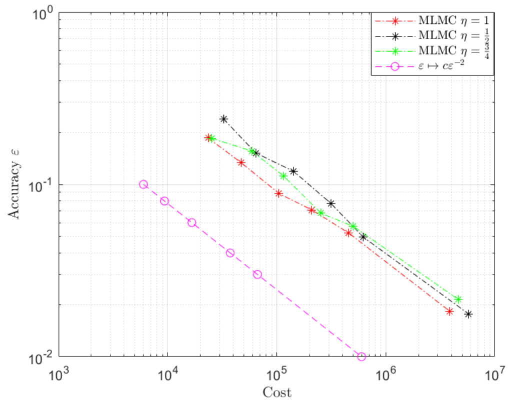

Figure 4.3 – Empirical RMSE of the MLMC antithetic estimator as a function of the cost (log-scale) for different values of η.

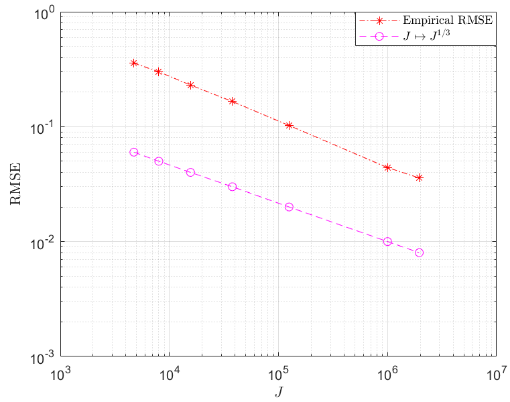

Figure 4.4 – Empirical RMSE of the LSMC estimator as a function of the number of samples J (log-scale)

4.3 Calculation of the SCR with the Standard Formula in an ALM model

In this section, we want to illustrate to compare the MLMC and LSMC methods on a more realistic example for the application in insurance. Namely, we consider the case of Asset Liability Management (ALM) for life insurance contracts and are interested with the calculation of the SCR with the standard formula after t years. This example is of practical interest and the conditional expectations that are at stake are typically high-dimensional. In fact, the process that determines the strategy is really path-dependent and involves book values, market values, crediting rates, etc. Here, we will use the recent ALM model that we have developed in [ACIA20a], and we focus on the interest rate module of the SCR with the standard formula.

4.3.1 The ALM model in a nutshell

In this section, we briefly present the ALM model developed in [ACIA20a] and refer to this paper for the full details. We consider an insurance company that handles a life insurance, namely a General Account guaranteed with participation contracts. We consider a runoff portfolio with an initial Mathematical Reserve MR_0 corresponding to policyholders' deposit. The Capitalization Reserve (a buffer for capital gains on bonds imposed by the French legislation) and the Profit Sharing Reserve (a buffer for capital gains on stocks to smooth the crediting rate) are empty at time 0, i.e. $CR_0 = 0$ and $PSR_0 = 0$. At time 0, the insurance company invests MR_0 in two asset classes, stocks and riskless bonds, with respective weights $w_0^S \in [0,1]$ and $w_0^b = 1 - w_0^S$. Thus the initial Market Value and Book Value in stock (resp. in bonds) is given by

$$MV_0^S = BV_0^S = w_0^S MR_0 \text{ (resp. } MV_0^b = BV_0^b = w_0^b MR_0).$$

During all the ALM strategy, the insurance company invests in an equity asset $(S_t)_{t\geq 0}$ that may be a stock index or more generally an average of stocks with weights corresponding to the investment of the insurance company. It therefore has $\phi_0^S = MV_0^S/\phi_0^S$ equity assets at time 0. The insurance company also invests in bonds, and we assume that this investment is made with an equally weighted portfolio of bonds with maturities $1, \dots, n$. We introduce some notation to precise this: we denote $P(t, t+i)$ the price of a Zero-Coupon bond at time t with maturity $t+i$, $B(t, n, c) = \sum_{i=1}^n cP(t, t+i) + P(t, t+n)$ the price at time t of a bond with constant coupon $c \in \mathbb{R}$, unit nominal value and maturity n and for $\mathbf{c} = (c^i)_{i\in\{1,\dots,n\}} \in \mathbb{R}^n$ we denote by

$$\bar{B}(t, n, \mathbf{c}) = \frac{1}{n} \sum_{i=1}^{n} B(t, i, c^i)$$

the value of an equally weighted portfolio of bonds with coupons $\mathbf{c}$. During all the ALM strategy, we assume that bonds are bought at par with the swap rate $c_{swap}(t, n) = \frac{1-P(t,t+n)}{\sum_{i=1}^n P(t,t+i)}$. We set $\mathbf{c}_0 = (c_{swap}(0, i))_{i\in\{1,\dots,n\}}$ and have $\bar{B}(0, n, \mathbf{c}_0) = 1$. At time 0, the insurance company has then $\phi_0^b = MV_0^b = MV_0^b / \bar{B}(0, n, \mathbf{c}_0)$ assets $\bar{B}(0, n, \mathbf{c}_0)$.

We assume that the portfolio is handled up to time $T \in \mathbb{N}^*$ and that it is static on each period $(t-1, t)$, $t \in \{1, \dots, T\}$. At each time t, it is reallocated in such a way to have at the end of the reallocation

$$\phi_t^S = \frac{w_t^S MV_t}{S_t}, \ \phi_t^b = \frac{w_t^b MV_t}{\bar{B}(t, n, \mathbf{c}_t)}$$

quantities of equity assets and bonds, where MV_t denotes the market value of the portfolio at time t and $w_t^S = 1 - w_t^b \in [0, 1]$ is the target weight decided for ALM strategy. The coupons $\mathbf{c}_t \in \mathbb{R}^n$ are determined by the reallocation procedure that we describe now and takes into account the specificities of life insurance contracts. We decompose this reallocation in five steps:

1. *Calculation of the cash inflows and book value movements related to the bonds.* Since the portfolio composition is unchanged on $(t-1, t)$, the insurer receives $\frac{\phi_{t-1}^b}{n}\left(1 + \sum_{i=1}^n c_{t-1}^i\right)$ corresponding to the nominal value of the expiring bonds and the coupons. The value of the matured bonds $\frac{\phi_{t-1}^b}{n}$ is removed from the book value of bonds BV_t^b.

2. *Payment of the policyholders that exit their contract.* The proportion of policyholders that exit on $[t-1, t]$ is denoted by p_{t-1}^e. It is modelled as the sum of a deterministic part related to the relevant life table and of a dynamic part modelling surrenders $DSR(\Delta_{t-1}) = DSR_{max} \mathbb{1}_{\Delta_t \le \alpha} + DSR_{max} \frac{\beta - \Delta_t}{\beta - \alpha} \mathbb{1}_{\alpha < \Delta_t < \beta}$, where Δ_{t-1} is the difference between the crediting rate to policyholders $r_{ph}(t-1)$ and a competitor rate r_{t-1}^{comp}. We assume that policyholders exit uniformly on $[t-1, t]$, and the amount to pay is thus $p_{t-1}^e MR_{t-1}(1 + r_G/2)$, where r_G is the minimum guaranteed rate. This means that they are remunerated with this rate on the last period.

3. *Reallocation step.* At this step, the market value of the portfolio is given by

$$MV_t = G_t + \phi_{t-1}^S S_t + \frac{\phi_{t-1}^b}{n} \sum_{i=1}^{n-1} B(t, i, c_{t-1}^{i+1}),$$

where G_t is the liquidity gap that corresponds to the difference between the cash inflows and outflows of the two first steps. The second term $\phi_{t-1}^S S_t$ represents the market value of equity assets, and the last term the market value of bond assets.

Note that a bond at time $t-1$ with maturity $i+1$ and coupon c_{t-1}^{i+1} becomes at time t a bond with maturity i with the same coupon.

The portfolio is reallocated with the prescribed weights $w_t^S \in [0,1]$ and $w_t^b = 1 - w_t^S$ given by the ALM strategy. The amount of equity assets to hold is thus given by $\phi_t^S = w_t^S MV_t / S_t$. If this quantity is greater than ϕ_{t-1}^S, there is a purchase of $\phi_t^S - \phi_{t-1}^S$ equity assets which increases the book value BV_t^S by $(\phi_t^S - \phi_{t-1}^S)S_t$. If this quantity is lower than ϕ_{t-1}^S, there is a sell of $\phi_{t-1}^S - \phi_t^S$ equity assets which decreases the book value BV_t^S by the factor ϕ_t^S/ϕ_{t-1}^S. This generates capital gain or loss on stocks that is registered, since capital gain has to be redistributed to policyholders with a participation rate $\pi_{pr} \in [0,1]$.

The reallocation in bonds follows the same principles but is more involved. At the end of this step, the portfolio in bonds is made with ϕ_t^b combinations of bonds $\bar{B}(t, n, \mathbf{c}_t)$. Since the bonds are bought at par, there is a precise relation between $\mathbf{c}_t$, $\mathbf{c}_{t-1}$ and the swap rates at time t. According to the French legislation rules, the capital gain or loss on bonds is stored in the Capitalization Reserve and is separated from the ALM portfolio. Details can be found in [ACIA20a].

4. *Determination of the crediting rate.* This step determines the policyholders' earning rate $r_{ph}(t)$ on the period $(t-1, t)$. Due to regulatory constraint, it has to be greater than the minimum guaranteed rate r_G and the amount distributed to policyholders has to be greater than the proportion π_{pr} of the gains (participation rate). Besides the insurance company compares $r_{ph}(t)$ with a competitor rate r_t^{comp} (typically the market short rate) and tries at best to have $r_{ph}(t) \geq r_t^{comp}$ to avoid dynamic surrenders. We call "target rate", the maximum rate given by these three constraints.

   The amount to distribute is typically made with the coupons, the capital gain or loss on stocks and possibly dividends. To smooth these gains along the years, the insurance company uses a Profit Sharing Reserve. In addition, the insurance company may also want to realize a part of latent gain or loss on stocks. In the model developed in [ACIA20a], the amount to distribute depends on all these quantities. We distinguish four cases (from the best to the worst).

   (A) The target rate can be distributed without using latent gain or by realizing all the latent loss on stocks.

   (B) The target rate can be distributed by using latent gain or without realizing all the latent loss on stocks. The proportion of gain or loss is determined accordingly.

   (C) The target rate cannot be reached with the available amount, but the minimum guaranteed rate can be distributed. The insurance company then uses all the latent gain in order to serve the best possible rate.

   (D) The minimum guaranteed rate cannot be reached with the available amount. Then, the insurance company clears out the Profit Sharing Reserve and credit the policyholders with the lowest rate above r_G that also satisfies participation rate constraints.

   Once $r_{ph}(t)$ is determined, the Mathematical Reserve of the remaining policyholders is updated accordingly: $MR_t = MR_{t-1}(1 - p_{t-1}^e)(1 + r_{ph}(t))$. The Profit

Sharing Reserve and the book value of stocks are also modified according to the case. The shareholder's margin can be calculated as well as the profit and loss $P\&L_t$ generated on the period $(t-1,t)$, which is defined as the sum of the shareholder's margin and the interest generated by the Capitalization Reserve. Again, all the details can be found in [ACIA20a].

5. *Externalization of the Capitalization Reserve and of Shareholders' margin from the accounting.* This last step is a technical accounting operation that slightly change the quantities of assets and the book values, while keeping unchanged the target weights w_t^S and w_t^b.

The last step at the final time T follows the same lines: instead of being reallocated, the portfolio is cleared and policyholders get back the remaining Mathematical Reserve MR_T.

4.3.2 The Solvency Capital Requirement with the standard formula

We now present the main lines of the SCR calculation with the standard formula as indicated by the EIOPA [EIOPA12, EIOPA18]. Let us denote by $(\mathcal{F}_t, t \geq 0)$ the filtration representing the market information at time $t \geq 0$ and $\mathbb{Q}$ the pricing measure. We consider a short-rate model $(r_t, t \geq 0)$ for interest rates and define at time $t \in \{0, \ldots, T-1\}$ the Basic Own Funds by

$$BOF_t = \mathbb{E}^{\mathbb{Q}}\left[\sum_{u=t+1}^{T} e^{-\int_t^u r_s ds} P\&L_u \middle| \mathcal{F}_t\right],$$

i.e. the expected value of the discounted future profits and losses. The principle of the standard formula is to apply shocks on each asset class (equity, interest rate, etc.) and evaluate the variation of Basic Own Funds. Then, the SCR on market risk is obtained by using a given formula that aggregates all risk modules. In this paper, we focus on the interest rate module, where upward and downward shocks are prescribed by the regulator. The methodology to apply these shocks is described in Section 2.5 of [ACIA20a]. We have used in our simulations the shocks specified in [EIOPA12]. At time t, the SCR value of the interest module is then defined by

$$SCR_t^{int} = \max\{BOF_t - BOF_t^{\text{upward shock}}, BOF_t - BOF_t^{\text{downward shock}}, 0\},$$

where shocks are applied at time t on the interest-rate curve. We also set

$$SCR_t^{up} = \max\{BOF_t - BOF_t^{\text{upward shock}}, 0\} \text{ and } SCR_t^{down} = \max\{BOF_t - BOF_t^{\text{downward shock}}, 0\},$$

so that $SCR_t^{int} = \max(SCR_t^{down}, SCR_t^{up})$. At time $t = 0$, SCR_0^{int} is a number that can be calculated by Monte-Carlo. This has been investigated in [ACIA20a]. However, for the ALM strategy, it may be useful to have quantitative insights on the evolution of the SCR along the time to asses the cost of capital. Thus, in this paper we are interested with the valuation of

$$I = \mathbb{E}^{\mathbb{P}}[\max\{BOF_t - BOF_t^{\text{upward shock}}, BOF_t - BOF_t^{\text{downward shock}}, 0\}], \quad (4.3.1)$$

the average value under the historical (or real) probability of the SCR at time t. If we denote by $\frac{d\mathbb{P}}{d\mathbb{Q}}\Big|_{\mathcal{F}_t}$ the change of probability, we have

$$I = \mathbb{E}^{\mathbb{Q}}\left[\frac{d\mathbb{P}}{d\mathbb{Q}}\Big|_{\mathcal{F}_t} \max\{BOF_t - BOF_t^{\text{upward shock}}, BOF_t - BOF_t^{\text{downward shock}}, 0\}\right], \quad (4.3.2)$$

and we are precisely in the framework of Section 4.2 if X denotes a random variable that represents all the market information up to time t (i.e. $\sigma(X) = \mathcal{F}_t$). The equity module of the SCR is similarly defined by

$$SCR_t^{eq} = \max\{BOF_t - BOF_t^{\text{equity shock}}, 0\},$$

where the equity shock amounts to a strong decrease of S immediately after t. Usually, the maximum with zero is useless since the shock is always negative. Last the standard formula that defines the SCR on market risk as follows (see Articles 164 and 165 of [Com15]):

$$SCR_t^{mkt} = \sqrt{(SCR_t^{eq})_t^2 + (SCR^{int})_t^2 + 2\varepsilon SCR_t^{eq} SCR_t^{int}} \quad (4.3.3)$$

where $\varepsilon = 0$ if the interest-rate exposure is due to the upward-shock on interest rates and $\varepsilon = \frac{1}{2}$ if it is due to the downward shock of the interest rate module. Thus, the expected value of the SCR is given by:

$$\mathbb{E}^{\mathbb{P}}[SCR_t^{mkt}] = \mathbb{E}^{\mathbb{Q}}\left[\frac{d\mathbb{P}}{d\mathbb{Q}}\Big|_{\mathcal{F}_t} \sqrt{(SCR_t^{eq})_t^2 + (SCR^{int})_t^2 + 2\varepsilon SCR_t^{eq} SCR_t^{int}}\right].$$

4.3.3 The stock and short-rate models

We consider $(W_t, Z_t)_{t\geq 0}$ a standard two-dimensional Brownian motion under $\mathbb{Q}$. Following [ACIA20a], we assume that the equity assets follows a Black-Scholes model and that the short interest rate follows a Vasicek++ (or Hull and White) model:

$$\frac{dS_t}{S_t} = r_t dt + \sigma_S dW_t \quad (4.3.4)$$

$$r_t = x_t + \varphi(t), \text{ with } dx_t = k(\theta - x_t)dt + \sigma_r(\gamma dW_t + \sqrt{1-\gamma^2}dZ_t), \quad (4.3.5)$$

where $\gamma \in [-1, 1]$ tunes the dependence between equity and interest rates. We assume $k, \theta, \sigma_S, \sigma_r > 0$. As explained in [ACIA20a], the shift function $\varphi : \mathbb{R}_+ \to \mathbb{R}$ is particularly convenient to implement the shocks prescribed by the EIOPA. Mainly, shocks amounts to modify the shift, leaving the dynamics of x unchanged, which makes easy to calculate the ALM strategies in both normal and shocked cases on each sample.

A new feature with respect to [ACIA20a] is that we now also consider the dynamics under the real probability $\mathbb{P}$. We assume here for simplicity the following basic change of probability

$$\frac{d\mathbb{P}}{d\mathbb{Q}}\Big|_{\mathcal{F}_t} = \exp\left(\lambda^W W_t + \lambda^Z Z_t - \frac{1}{2}((\lambda^W)^2 + (\lambda^Z)^2 + 2\gamma\lambda^W\lambda^Z)t\right) =: L_t, \quad (4.3.6)$$

with $\lambda^W, \lambda^Z \in \mathbb{R}$. By the Cameron-Martin theorem, $dW_t^{\mathbb{P}} = dW_t - \lambda_t^W dt$ and $dZ_t^{\mathbb{P}} = dZ_t - \lambda^Z dt$ are independent Brownian motions under $\mathbb{P}$. We then have the following dynamics under $\mathbb{P}$:

$$\frac{dS_t}{S_t} = (r_t + \lambda^W \sigma_S)dt + \sigma_S dW_t^{\mathbb{P}}$$
$$r_t = x_t + \varphi(t), \text{ with } dx_t = k\left(\theta + \sigma_r \frac{\gamma\lambda^W + \sqrt{1-\gamma^2}\lambda^Z}{k} - x_t\right)dt + \sigma_r(\gamma dW_t^{\mathbb{P}} + \sqrt{1-\gamma^2}dZ_t^{\mathbb{P}}),$$

To run this asset model with the ALM model described in Subsection 4.3.1, we have to be able to sample S_t, r_t and the change of probability L_t at each time $t \in \mathbb{N}$. It is possible to do it exactly by using the following recurrence formula

$$S_t = S_{t-1}\exp\left(\int_{t-1}^t x_u du + \int_{t-1}^t \varphi(u)du + \sigma_S(W_t - W_{t-1}) - \frac{\sigma_S^2}{2}\right),$$
$$x_t = x_{t-1}e^{-k} + \theta(1-e^{-k}) + \sigma_r \int_{t-1}^t e^{-k(t-u)}(\gamma dW_u + \sqrt{1-\gamma^2}dZ_u),$$
$$L_t = L_{t-1}\exp\left(\lambda^W(W_t - W_{t-1}) + \lambda^Z(Z_t - Z_{t-1}) - \frac{1}{2}((\lambda^W)^2 + (\lambda^Z)^2 + 2\gamma\lambda^W\lambda^Z)\right),$$

and

$$\int_{t-1}^t x_u du = \frac{1}{k}(x_{t-1} - x_t) + \theta + \frac{\sigma_r}{k}[\gamma(W_t - W_{t-1}) + \sqrt{1-\gamma^2}(Z_t - Z_{t-1})].$$

The law of $(W_t - W_{t-1}, Z_t - Z_{t-1}, \int_{t-1}^t e^{-k(t-u)}(\gamma dW_u + \sqrt{1-\gamma^2}dZ_u))$ is a centered Normal distribution with covariance

$$\begin{bmatrix} 1 & 0 & \gamma\frac{1-e^{-k}}{k} \\ 0 & 1 & \sqrt{1-\gamma^2}\frac{1-e^{-k}}{k} \\ \gamma\frac{1-e^{-k}}{k} & \sqrt{1-\gamma^2}\frac{1-e^{-k}}{k} & \frac{1-e^{-2k}}{2k} \end{bmatrix}.$$

This is the same law as $\left(G_1, G_2, \gamma\frac{1-e^{-k}}{k}G_1 + \sqrt{1-\gamma^2}\frac{1-e^{-k}}{k}G_2 + \sqrt{\frac{1-e^{-2k}}{2k} - \left(\frac{1-e^{-k}}{k}\right)^2}G_3\right)$, where G_1, G_2, G_3 are independent standard Normal variables. Once this triplet is sampled exactly, we can calculate easily (S_t, x_t, L_t) using the formulas above.

4.3.4 Numerical experiments I: comparison between methods to calculate $\mathbb{E}[SCR_t^{int}]$

We now present numerical results on the calculation of I defined by (4.3.1). We use the following parameters for the ALM model and for the asset model. They are summarized in Tables 4.1 and 4.2. Unless specified, we also consider $\mathbb{P} = \mathbb{Q}$, i.e. that the real and risk-neutral probability are the same. We will discuss however later on the impact of this change of probability for the SCR.

Stock model	Short-rate model
$S_0 = 1$	$r_0 = \theta = 0.02$
$\sigma_S = 0.1$	$\sigma_r = 0.01$
$\gamma = 0$	$k = 0.2$

Table 4.1 – Market-model parameters

Management Parameters	Liability Parameters
Target allocation in stock $w_t^s = 0.05$	Dynamic surrenders triggering thresholds $\beta = -0.01$ and $\alpha = -0.05$
Target allocation in bond $w_t^b = 0.95$	Maximum lapse dynamic surrender rate $DSR_{max} = 0.3$
Participation rate $\pi_{pr} = 0.9$	Deterministic constant exit rate $\underline{p} = 0.05$
Minimum guaranteed rate $r^G = 0.015$	Time horizon: $T = 30$ years
Competitor rate $r_t^{comp} = r_t$	
Smoothing coefficient of the PSR: $\bar{\rho} = 0.5$	
Bond portfolio maximal maturity $n = 20$	
Projection Horizon $T = 30$	

Table 4.2 – Liability and management parameters

The implementation of the MLMC antithetic estimator is easily made by using (4.2.21). Instead, the implementation of the LSMC raises some issues. The main one is to choose the regressors. In fact, the ALM model presented in Subsection is truly path-dependent, and one needs to know $(r_{t'}, S_{t'})_{t'\in\{1,\dots,t\}}$ to determine the book values, the different reserves and the Bond portfolio at time t. Thus, SCR_t^{int} depends on all the past before t. For $t = 1$ the dimension of the regression space is equal to 2 and the choice of the regressors r_1 and S_1 is obvious. When t gets larger, this is no longer the case and in view of the theoretical complexity result of Proposition 4.11 one cannot afford to use all the $2t$ regressors. It is then important to select few regressors. We explain now the procedure that we have used.

Selection of the regressors for the LSMC estimator

In Table 4.3 we have listed 12 relevant risk-factors for the insurance company. We will select the most relevant ones for the SCR interest rate module by using a forward selection procedure.

Attribute	Risk-factor description
$X_t^1 = S_t$	Equity asset value
$X_t^2 = r_t$	Short rate
$X_t^3 = \phi_t^S$	Position in Stock
$X_t^4 = \phi_t^b$	Position in bonds
$X_t^5 = BV_t^b$	Book value of bonds
$X_t^6 = BV_t^S$	Book value of equity assets
$X_t^7 = MR_t$	Mathematical Reserve
$X_t^8 = PSR_t$	Profit sharing reserve
$X_t^9 = CR_t$	Capitalization Reserve
$X_t^{10} = MV_t$	Portfolio market value
$X_t^{11} = \phi_t^b \bar{B}(t, n, \mathbf{c_t})$	Market value of bonds
$X_t^{12} = \phi_t^S S_t$	Market value of equity assets

Table 4.3 – Non exhaustive list of risk factors

To do so, we sample J_v scenarios up to time t of the ALM model. This produces in particular J_v samples of $(X_t^{1,j}, \dots, X_t^{12,j})_{j=1,\dots,J_v}$. Then, we approximate for each scenario the value of the interest rate module of the SCR, $SCR_t^{int,j}$, by using a Nested Monte-Carlo with K of secondary scenarios. We note $\widehat{SCR}_t^{Nested,j}$ these approximations (we drop for readability the superscript "int" in Paragraph 4.3.4). In our numerical application, we have taken $J_v = 2000$ validation scenarios and $K = 10^4$ inner scenarios. Let $\widehat{SCR} : \mathbb{R}^{12} \to \mathbb{R}$ be a function approximating the SCR from the values of X. We now consider the empirical RMSE, i.e.

$$\sqrt{\frac{1}{J_v}\sum_{i=1}^{J_v}(\widehat{SCR}_t^{Nested,j} - \widehat{SCR}(X_t^j))^2}$$

as a criterion to assess the accuracy of the regression function $\widehat{SCR}$.

We start by selecting the first variable. Up to a linear rescaling of the sample we may assume without loss of generality that all the variables are in $[0,1]$. We consider the 12 possible regressor functions for $l \in \{1,\dots,12\}$,

$$\sum_{i=0}^{n_r-1} \hat{\alpha}_i^l \mathbb{1}_{X_t^l \in [i/n_r,(i+1)/n_r)}, \text{ with } \hat{\alpha}_i^l = \frac{\sum_{j=1}^{J_v} \widehat{SCR}_t^{Nested,j} \mathbb{1}_{X_t^{l,j} \in [i/n_r,(i+1)/n_r)}}{\sum_{j=1}^{J_v} \mathbb{1}_{X_t^{l,j} \in [i/n_r,(i+1)/n_r)}},$$

and select $l_1^* \in \{1,\dots,12\}$ that achieves the lowest RMSE. Once l_1 is selected, we consider the following 11 regressor functions for $l \in \{1,\dots,12\} \setminus \{l_1\}$:

$$\sum_{i_1,i_2=0}^{n_r-1} \hat{\alpha}_{i_1,i_2}^{l_1,l} \mathbb{1}_{X_t^{l_1} \in [i_1/n_r,(i_1+1)/n_r)} \mathbb{1}_{X_t^l \in [i_2/n_r,(i_2+1)/n_r)},$$

$$\text{with } \hat{\alpha}_{i_1,i_2}^{l_1,l} = \frac{\sum_{j=1}^{J_v} \widehat{SCR}_t^{Nested,j} \mathbb{1}_{X_t^{l_1,j} \in [i_1/n_r,(i_1+1)/n_r)} \mathbb{1}_{X_t^{l,j} \in [i_2/n_r,(i_2+1)/n_r)}}{\sum_{j=1}^{J_v} \mathbb{1}_{X_t^{l_1,j} \in [i_1/n_r,(i_1+1)/n_r)} \mathbb{1}_{X_t^{l,j} \in [i_2/n_r,(i_2+1)/n_r)}}.$$

We then select the regressor $l_2 \in \{1,\dots,12\} \setminus \{l_1\}$ that gives the smallest RMSE. We then proceed similarly to select the next variables. We have run this selection for $t = 10$ with $n_r = 5$. Table 4.4 shows the result of this algorithm and indicate the Book values of bonds as the more significant variable to approximate the SCR module on interest rates. We notice that the RMSE is significantly reduced by using the second variable. In contrast, the third variable moderately improves the criterion. Since the number of variables is also a limitation then for the use of the LSMC estimator, we do not go further in the selection procedure. Figure 4.5 illustrates the approximation of the values of $\widehat{SCR}_t^{Nested,j}$ by the regression function with the two first regressors.

	Best attribute	$RMSE$
First variable	BV_t^b ($l_1 = 5$)	0.8739
Second variable	r_t ($l_2 = 2$)	0.5292
Third variable	S_t ($l_3 = 1$)	0.5084

Table 4.4 – Result of the Forward selection procedure for SCR_t^{int} with $t = 10$.

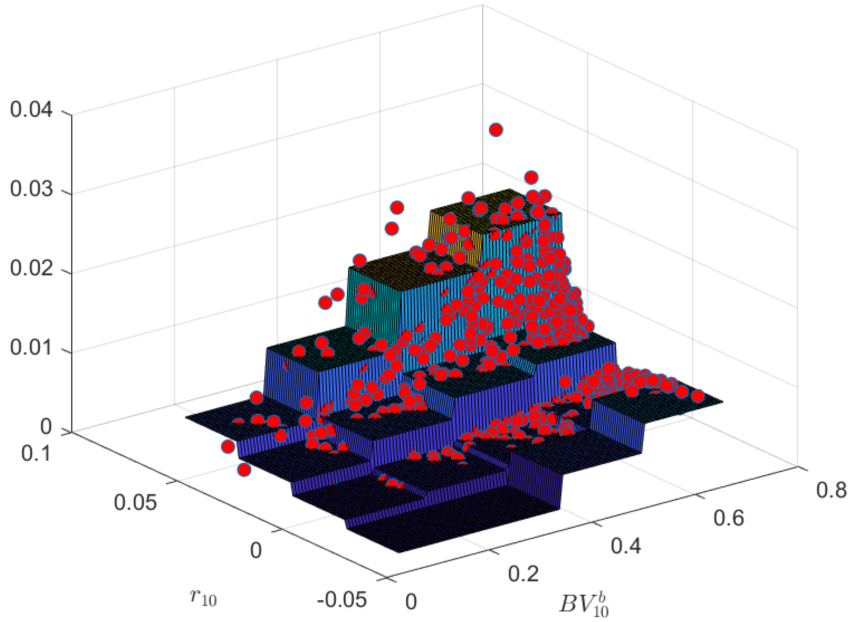

Figure 4.5 – Plot of the points $\widehat{SCR}_t^{Nested,j}$ and of the estimated regression function with the two first selected regressors.

Comparison between MLMC and LSMC estimators

We focus on the calculation of $\mathbb{E}[SCR_t^{int}]$ with $t = 10$ years. We now compare and test numerically the MLMC antithetic estimator $\widehat{I}_A^{MLMC}$ defined by (4.2.19) with the LSMC estimator $\widehat{I}^{LSMC}$ defined by (4.2.22) using the local cube basis and the regressors selected by the procedure described in Paragraph 4.3.4.

In Figure 4.6, we have drawn the Root Mean Square Error of the estimator $\widehat{I}_A^{MLMC}$ and of the estimators $\widehat{I}^{LSMC}$ using the first, the two first and the three first selected regressors. In order to derive the RMSE of the different estimators, as no closed formulas is available in this framework, we rely on a full nested Monte-Carlo procedure based on a fixed simulation budget of $\Gamma = 10^8$ sample paths to approximate the true value of I. The allocation between primary and secondary scenarios correspond to $M \approx \Gamma^{\frac{2}{3}}$ primary samples and $K \approx \Gamma^{\frac{1}{3}}$ inner scenarios, as prescribed by Theorem 4.1 for $\eta = 1$. To compute the RMSE of the different estimators, we produce $N_{batch} = 10$ independent simulations $(\widehat{I}_j^*)_{j=1,\dots,N_{batch}}$ and indicate the empirical RMSE $\sqrt{\frac{1}{N_{batch}} \sum_{j=1}^{N_{batch}} |\widehat{I}_j^* - I|^2}$. We plot the empirical RMSE's of the different estimators as a function J (with $J := \sum_{l=0}^{L} J_l K_l$ for the MLMC estimator). This represents the number of samples, as well as the computational cost (in log-scale) that is in $O(J)$ for both estimators.

Concerning the LSMC estimators, we notice that the estimators with two regressors does much better with the estimator with one regressor. Instead, the interest of using a third regressor is tiny. We also observe that the RMSE does not really decrease after 10^4 samples on our example. This is due to the regression error: since we approximate SCR_t^{int} by a function of two or three variables, there is no way to go beyond a certain level of precision. This is particularly noticeable for $t \geq 10$ years: the projection of the balance sheet through the ALM model is truly non-Markovian and the history up to

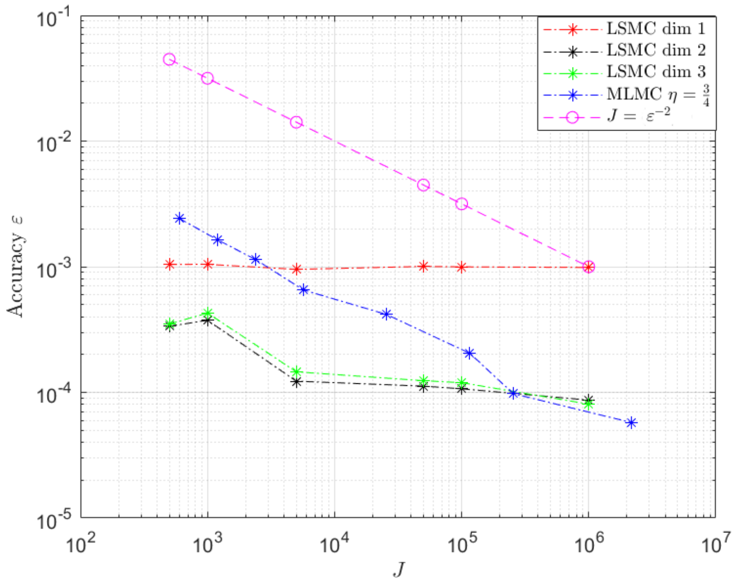

Figure 4.6 – Empirical RMSE's of the LSMC and MLMC estimators in function of the computational effort J (with $J := \sum_{l=0}^{L} J_l K_l$ for the MLMC estimator). The computational time needed for the forward selection used by the LSMC estimator is not taken into account in this plot.

time t cannot be summed up by two or three variables. In comparison, the convergence of the MLMC antithetic estimator is in line with Theorem 4.9 and is asymptotically more accurate than the LSMC estimator. Besides, the MLMC estimator avoids the step of selecting regressors that requires computational time and may be determinant for the accuracy of the LSMC estimator. Last, we notice that for a same level of precision, the computational time required by the MLMC is slightly smaller than the one required by the LSMC estimator. More precisely, the computational time needed for $J = 2 \times 10^5$ (where the three estimators have quite the same accuracy) are 9950 seconds for $\widehat{I}_A^{MLMC}$, 11230 seconds for $\widehat{I}^{LSMC}$ with two regressors ($n_r = 23$) and 12650 seconds for $\widehat{I}^{LSMC}$ with three regressors ($n_r = 13$).

We now make a comment on the choice of the parameter η for the MLMC antithetic estimator. We recall that η is, roughly speaking, related to the probability that two (or more) arguments of the maximum function are close to the maximum, see Theorems 4.1 and 4.9. Heuristically, the smaller is this probability, the larger can be η, which then reduces the number of levels and then the computational cost. In Figure 4.7, we have plotted the convergence of the MLMC antithetic estimator for $\eta \in \{1/2, 3/4, 1\}$ in function of the theoretical computational cost $\sum_{l=0}^{L} J_l K_l$. Basically, the three estimators converge, but the one obtained with $\eta = 1$ does not seem to be asymptotically in $O(\varepsilon^{-2})$ while the two others are in line with the theoretical convergence in $O(\varepsilon^{-2})$. This shows the interest of the parameter η in a practical application and explains why we have chosen to take $\eta = 3/4$ is our experiments in Figure 4.6.

Comparison between LSMC and the use of Neural Network

In Paragraph 4.3.4, we have noticed that a significant drawback of the LSMC estimator with respect to the MLMC antithetic estimator is that it requires first to select regres-

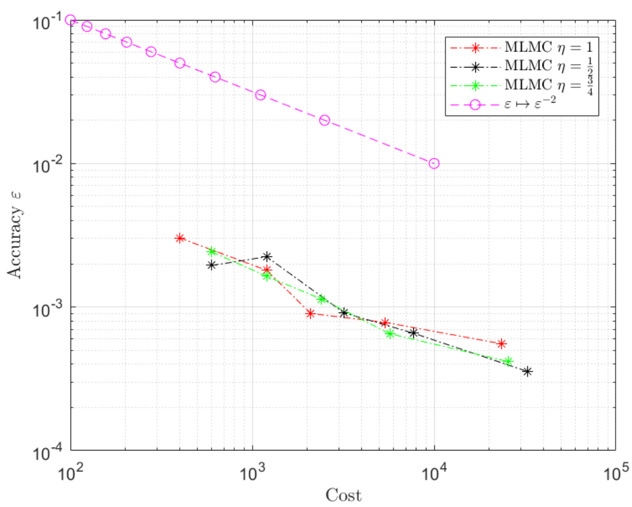

Figure 4.7 – Convergence of the MLMC estimator for different values of η in function of the computational cost $\sum_{l=0}^{L} J_l K_l$.

sors. Beyond the computational time needed by this selection, there is a significant regression error. A natural idea to skip the selection step in the regression is to use

Input Feature
X_t^1 Bond Book-Value BV_t^b
X_t^2 Stock Book-value BV_t^s
X_t^3 Position in bond ϕ_t^b
X_t^4 Position in stock ϕ_t^s
X_t^5 Profit-Sharing Reserve level PSR_t
X_t^6 Mathematical Reserve level MR_t
X_t^7 Capitalization Reserve level CR_t
X_t^8 Spread crediting rate/competing rate Δ_t
X_t^9 Stock price S_t
X_t^{10} Interest-rate r_t

Table 4.5 – Input feature of the Neural Network Algorithm

Neural Networks (NN). We have implemented a feedforward neural network with one hidden-layer. The hidden-layer is made with 10 or 50

neurons. The activation function used is the sigmoid function. To train the network, we generate J number of primary outer scenarios $(X_{t,j})_{1\leq j\leq J}$ for which only one inner simulation is performed to obtain Z_j that represents the maximal variation of the discounted P&L due to the shocks . Then, one minimizes

$$\frac{1}{J}\sum_{j=1}^{J}(Z_j - \mathrm{NN}(X_{t,j}))^2, \tag{4.3.7}$$

where NN is the function generated by the neural network, so that it approximates the desired conditional expectation. The input features of the network have not been pre-processed: the optimization above has to be enough for detecting the relevant variables for the approximation of the conditional expectation. Once the NN has been obtained, we then estimate $\mathbb{E}[SCR_t^{int}]$ simply by the empirical mean $\hat{I}^{NN} := \frac{1}{J}\sum_{j=1}^{J} \mathrm{NN}(X_{t,j})$. Since we only use the NN on the training sample, the standard problem of overfitting is not an issue for our application. We compare the RMSE of this estimator with the RMSE obtained with the LSMC method. The aim of our procedure is to assess if a Neural Network feeding with a whole range of input features is able to select relevant attributes and to compare with the LSMC method with well-chosen features.

J	LSMC dim 1	LSMC dim 2	LSMC dim 3	NN: 10 neurons	NN: 50 neurons
500	1.0e-3	3.36e-4	3.50e-4	7.075e-4	7.56e-4
10^3	1.0e-3	3.75e-4	4.28e-4	6.46e-4	3.017e-4
5×10^3	9.52e-4	1.23e-4	1.46e-4	1.63e-4	1.8153e-4
5×10^4	1.0e-3	1.12e-4	1.24e-4	6.60e-5	7.29e-5
10^5	9.97e-4	1.068e-4	1.19e-4	6.32e-5	6.92e-5
10^6	9.86e-4	8.67e-5	8.06e-5	4.22e-5	4.50e-5

Table 4.6 – RMSE of $\mathbb{E}[SCR_t^{int}]$ for $t = 10$ given by the Neural Network (one hidden layer with the indicated number of neurons) and the LSMC in function of J

Table 4.6 indicates the RMSE of the estimator with the different methods. First, we notice that there is no need on our example to consider many neurons: a simple layer with 10 neurons in enough and do as well as the NN with 50 neurons in terms of RMSE. We notice also that the estimator given by the NN is slightly better than the one obtained with the LSMC with two or three regressors when the training sample gets large. However, the use of neural networks present serious drawbacks. First, it requires to store all the samples to achieve the minimization of (4.3.7) while the LSMC (and also MLMC) estimator only uses once each sample. Second, the time needed by the minimization (indicated in Table 4.7) is important, making at the end this method less competitive than MLMC. Note that one could be then tempted to train the NN on a smaller size of samples and then use it for large J: one would then face the problem of overfitting, which we want to avoid.

J	500	10^3	5×10^3	5×10^4	10^5
Time (s)	18.8	30.4	50.5	273.1	1693

Table 4.7 – Time required in seconds for the optimization of (4.3.7) for a neural network with 10 neurons.

To sum up, the use of NN can indeed be useful to reduce the approximation error observed by using LSMC estimators. However, it both demands memory and computational time, making the gain with respect to the LSMC not obvious. The MLMC estimator presents the clear advantage to avoid this issue of function approximation, and to avoid any storage of data.

4.3.5 Numerical experiments II: some insights on the ALM

We now present some applications of the MLMC antithetic estimator for the ALM. One of the major issue in ALM is to determine the optimal asset allocation between the different asset class backed to the insurance portfolio. For that reason, it is crucial to evaluate precisely the amount of SCR required by the strategy. We are interested in calculating $\mathbb{E}[SCR_t^{mkt}]$, $\mathbb{E}[SCR_t^{eq}]$ and $\mathbb{E}[SCR_t^{int}]$, see Subsection 4.3.2 for the definition of these modules. Note that at time $t > 0$, the SCR_t^{eq} and SCR_t^{int} are random variables so that we cannot use the aggregation formula to get directly $\mathbb{E}[SCR_t^{mkt}]$ from $\mathbb{E}[SCR_t^{eq}]$ and $\mathbb{E}[SCR_t^{int}]$. Note that by using the MLMC Antithetic estimator, it possible and easy to calculate at the same time these expectations: see Remark 4.8 for the general expression of these estimators that we use for different functions h. At each level l, one simulates J_l primary scenarios up to time t. Then, one simulates for each primary scenario K_l secondary scenarios, on which we perform four different evolutions: the first one without any shock, the second one with the equity shock, the third one with the upward shock on interest rates and the last one with the downward shock on interest rates. Then, one computes the corresponding empirical means related to the calculation of SCR_t^{eq} and SCR_t^{int} and use the aggregation formula (4.3.3) for SCR_t^{mkt}. Let us note that the discontinuity induced by the coefficient ε may in principle deteriorate the MLMC estimation. However, we have thus run the MLMC estimator with a regularization of this coefficient and noticed a tiny impact of the regularization . This can be heuristically understood from Figure 4.11: the activation of ε may occur on a wide range (perhaps the whole range) of values of SCR^{int}, which smooths the phenomenon.

Figures 4.8, 4.9 and 4.10 illustrate respectively the different values of the SCR modules $\mathbb{E}[SCR_t^{mod}]$ for $mod \in \{mkt, eq, int, up, down\}$ in function of the constant allocation weight w^S in equity for $t = 0$, $t = 10$ and $t = 20$ years (at $t = 0$, we can remove the expectation). As one may expect, these values globally decrease with respect to the time since we are considering a run-off portfolio with an exit rate greater than 5%. We notice several interesting points.

- At time $t = 10$, $\mathbb{E}[SCR_t^{int}]$ is significantly larger than $\max(\mathbb{E}[SCR_t^{up}], \mathbb{E}[SCR_t^{down}])$, which shows that deterministic proxy values of SCR modules may induce errors. We no longer observe this phenomenon at time $t = 20$ because the greater shock is always given by the upward shock and we have then $SCR_{20}^{int} \approx SCR_{20}^{down}$ (the green and blue curves coincide). This is explained in the next point.

- The main effects of the shocks on the interest rate are the following. The upward (resp. downward) shock leads to an immediate decrease (resp. increase) of the portfolio market value, but on the long run higher (resp lower) rates gives a better (resp. worse) profitability. Here, we are considering a run-off portfolio with final maturity $T = 30$. Thus, as t increases, the effect on the long run of this shocks get less important making the immediate effect on market value dominant. Thus, at $t = 20$ the downward shock is harmless while the upward shock gets painful. This explains why we observe $SCR_{20}^{int} \approx SCR_{20}^{down}$.

- The aggregation formula (4.3.3) somehow encourages to have SCR^{int} and SCR^{eq} of the same order: if $SCR^{int} >> SCR^{eq}$ it is possible to invest more in equity to have a better average return with a moderate increase of SCR^{mkt}, and if $SCR^{int} << SCR^{eq}$, one should reduce the investment in equity to reduce SCR^{mkt}. Therefore, it is interesting to look at the allocation that is such that $\mathbb{E}[SCR_t^{int}] = \mathbb{E}[SCR_t^{eq}]$. We see from Figures 4.8, 4.9 and 4.10 that the corresponding weight w^S evolves slightly. We get $w^S \approx 0.05$ for $t = 0$, $w^S \approx 0.06$ for $t = 10$ and $w^S \approx 0.05$ for $t = 20$, which is still a relative variation of 20%. This shows that a better evaluation of the SCR along the time may lead to significant adjustments on the investment strategy.

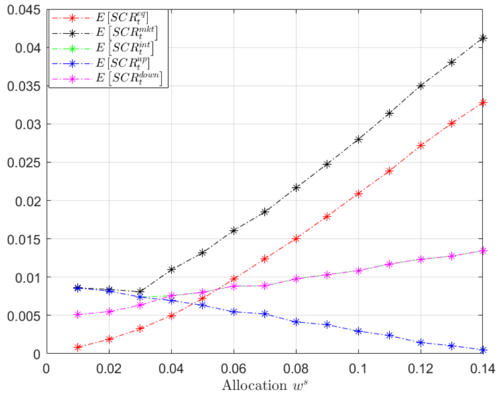

Figure 4.8 – Values of the SCR modules in function of the constant allocation weight w^S in equity for $t = 0$.

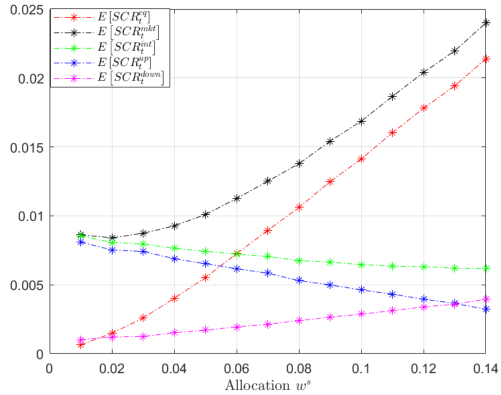

Figure 4.9 – Expected values of the SCR modules in function of the constant allocation weight w^S in equity for $t = 10$.

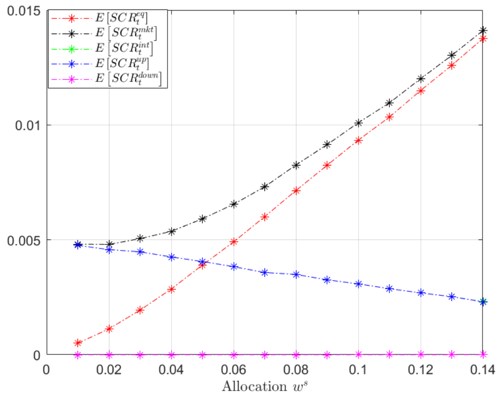

Figure 4.10 – Expected values of the SCR modules in function of the constant allocation weight w^S in equity for $t = 20$.

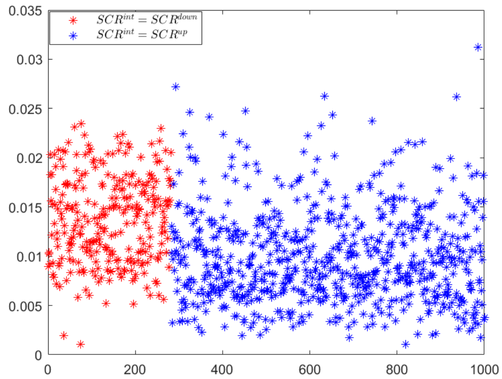

Figure 4.11 – Sample of 1000 approximated values of SCR_{10}^{int} given by a nested estimator: in red (resp. blue) are the values obtained when the downward shock is greater (resp. smaller) than the upward shock on interest rates.

Besides the calculation of the SCR, we can also use of the MLMC Antithetic estimator to calculate the sensitivity of the SCR with respect to some variations of parameters or market prices. These sensitivities are interesting information for management and they can also be useful to calculate quickly the a proxy value of the SCR. For example, suppose that we have computed the value of $\mathbb{E}[SCR_{10}^{mkt}]$ and, after few days, we want to estimate the new value of $\mathbb{E}[SCR_{10}^{mkt}]$ by taking into account the small variation on the equity and on the interest rates. Then, we can do this easily if one has computed the values of the sensitivities $\frac{\mathbb{E}[SCR_{10}^{mkt}(r_0+\delta r_0)-SCR_{10}^{mkt}(r_0)]}{\delta r_0}$ and $\frac{\mathbb{E}[SCR_{10}^{mkt}(S_0+\delta S_0)-SCR_{10}^{mkt}(S_0)]}{\delta S_0}$, where implicitly all values are kept constant but respectively r_0 and S_0. Note that these sensitivities can be computed with the MLMC antithetic estimator with the same samples that are needed for the estimation of SCR^{mkt}. Table (4.8) indicates the sensitivities obtained with our parameters with $\delta r_0 = 0.001$ and $\delta S_0 = 0.01$.

Last, the MLMC estimation is a tool for example to analyse how the SCR depends on the risk premia of stocks and interest rates. If the evaluation of the SCR has to be

$\frac{\mathbb{E}[SCR_{10}^{mkt}(S_0+\delta S_0)-SCR_{10}^{mkt}(S_0)]}{\delta S_0}$	$\frac{\mathbb{E}[SCR_{10}^{mkt}(r_0+\delta r_0)-SCR_{10}^{mkt}(r_0)]}{\delta r_0}$
0.0228	-0.0845

Table 4.8 – Sensitivities of the SCR_{10}^{mkt} with $w^S = 0.05$.

performed for regulatory reasons under a risk neutral framework, it is more relevant for ALM to calculate $\mathbb{E}^{\mathbb{P}}[SCR_t^{mod}]$ under the real probability, which corresponds to the average value of the own funds that will be necessary at time t. From equations (4.3.2) (generalized to any other SCR module) and (4.3.6), it is possible to see the impact of the risk premia λ^W and λ^Z on each SCR module. In Figures 4.12 and 4.13, we have indicated the more remarkable ones: the dependence of $\mathbb{E}^{\mathbb{P}}[SCR_t^{int}]$ on λ^Z and of $\mathbb{E}^{\mathbb{P}}[SCR_t^{eq}]$ on λ^W. We notice that the larger is λ^Z, the larger is $\mathbb{E}^{\mathbb{P}}[SCR_t^{int}]$. This can be understood as follows. A higher λ^Z leads to a higher mean reverting level for the short rate r. Thus, under the real probability measure (on time $[0,t]$), bonds are better remunerated and at time t the amount of savings (mathematical reserve) is higher. Since the evaluation of SCR_t^{int} is risk neutral, λ^Z has then no incidence on this evaluation. Thus, we observe a higher value of $\mathbb{E}^{\mathbb{P}}[SCR_t^{int}]$ simply because the mathematical reserve at time t is higher because of better returns. The same interpretation holds for $\mathbb{E}^{\mathbb{P}}[SCR_t^{eq}]$: the higher is λ^W, the higher is the amount of savings at time t and therefore the higher is $\mathbb{E}^{\mathbb{P}}[SCR_t^{eq}]$.

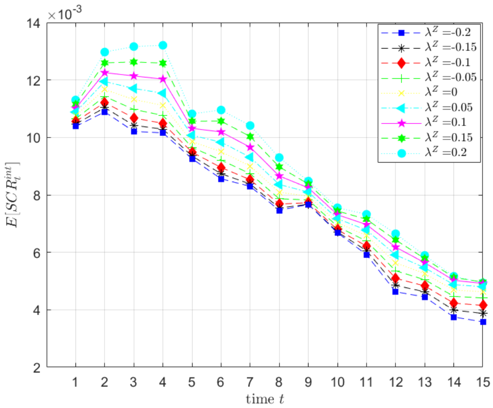

Figure 4.12 – Estimated values of $\mathbb{E}^{\mathbb{P}}[SCR_t^{int}]$ in function of t for different risk premia λ^Z.

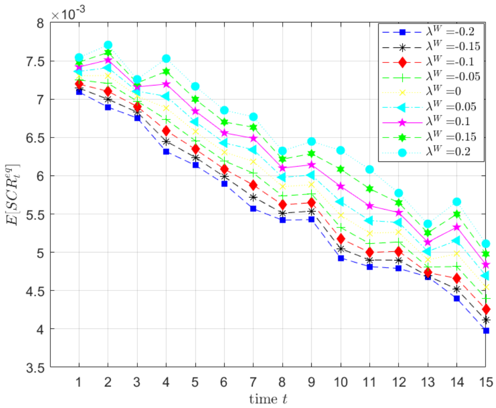

Figure 4.13 – Estimated values of $\mathbb{E}^{\mathbb{P}}[SCR_t^{eq}]$ in function of t for different risk premia λ^W.

APPENDIX A

TECHNICAL PROOFS

A.1 Technical proofs for Theorems 4.1 and 4.9

A.1.1 Preliminary results

In this section, we gather elementary but useful results for the analysis of the nested and the multilevel Monte-Carlo estimators.

Proposition A.1. *Let $g_0(u) = \max\{u, 0\}$ and, for $\varepsilon > 0$, $g_\varepsilon(u) = \frac{u^2}{2\varepsilon}\mathbb{1}_{u\in[0,\varepsilon]} + (u - \frac{\varepsilon}{2})\mathbb{1}_{u>\varepsilon}$. The function g_ε is C^1 and piecewise C^2 with*

$$g'_\varepsilon(u) = \frac{u}{\varepsilon}\mathbb{1}_{u\in[0,\varepsilon]} + \mathbb{1}_{u>\varepsilon},\ \ g''_\varepsilon(u) = \frac{1}{\varepsilon}\mathbb{1}_{u\in[0,\varepsilon]}$$

Moreover, g_ε is 1-Lipschitz and we have

$$g_\varepsilon \le g_0 \le g_\varepsilon + \frac{\varepsilon}{2}.$$

In addition, for any $\theta, \widehat{\theta}$ such that $\theta \le \widehat{\theta} \in \mathbb{R}$ we have:

$$\forall a \in \mathbb{R}, 0 \le \int_\theta^{\widehat{\theta}} g''_\varepsilon(t+a)dt \le \int_{\mathbb{R}} g''_\varepsilon(t)dt = 1. \tag{A.1.1}$$

Finally, the following asymptotic properties holds : $\forall\ u \in \mathbb{R}\ g_\varepsilon(u) \xrightarrow[\varepsilon\to 0]{} g_0(u)$, $\forall\ u \in \mathbb{R}\ g'_\varepsilon(u) \xrightarrow[\varepsilon\to 0]{} \mathbb{1}_{u>0}$ and $\int_{\mathbb{R}} g''_\varepsilon(u)\varphi(u)du \xrightarrow[\varepsilon\to 0]{} \varphi(0)$ for any function $\varphi : \mathbb{R} \to \mathbb{R}$ that is right-continuous at 0.

Lemma A.2. *Let $\theta, \widehat{\theta} \in \mathbb{R}$ and $\{a_\varepsilon\}$ an arbitrary function that converges to a as $\varepsilon \to 0$, then:*

$$\limsup_{\varepsilon\to 0} \left|\int_\theta^{\widehat{\theta}} g''_\varepsilon(t+a_\varepsilon)dt\right| \le \left|\mathbb{1}_{\theta\le -a\le\widehat{\theta}} - \mathbb{1}_{\widehat{\theta}\le -a\le\theta}\right| \tag{A.1.2}$$

Proof of Lemma A.2. Without loss of generality, we assume that $\theta < \widehat{\theta}$. First we have that :

$$\int_\theta^{\widehat{\theta}} g''_\varepsilon(t+a_\varepsilon)dt = \frac{1}{\varepsilon}\int_{A_\varepsilon} 1dt$$

where $A_\varepsilon = [\theta, \widehat{\theta}] \cap [-a_\varepsilon, \varepsilon - a_\varepsilon]$. Hence, if $-a < \theta$ or $\widehat{\theta} < -a$, it exists $\varepsilon_0 > 0$ such that $\forall \varepsilon \in [0, \varepsilon_0]$, $A_\varepsilon = \emptyset$. In this case:

$$\limsup_{\varepsilon \to 0} \left| \int_\theta^{\widehat{\theta}} g''_\varepsilon(t + a_\varepsilon) dt \right| = 0$$

Otherwise (i.e if $\theta \leq -a \leq \widehat{\theta}$) we always have: $A_\varepsilon \subset [-a_\varepsilon, \varepsilon - a_\varepsilon]$. Therefore, we have $\limsup_{\varepsilon \to 0} \left| \int_\theta^{\widehat{\theta}} g''_\varepsilon(t + a_\varepsilon) dt \right| \leq 1$ by (A.1.1). Thus we obtain

$$\limsup_{\varepsilon \to 0} \left| \int_\theta^{\widehat{\theta}} g''_\varepsilon(t + a_\varepsilon) dt \right| \leq \mathbb{1}_{\{\theta \leq -a \leq \widehat{\theta}\}}. \square$$

Lemma A.3. *Let $\theta, \widehat{\theta} \in \mathbb{R}$. Then, we have*

$$\left| \mathbb{1}_{\theta \leq 0 \leq \widehat{\theta}} - \mathbb{1}_{\widehat{\theta} \leq 0 \leq \theta} \right| \leq \mathbb{1}_{\theta \widehat{\theta} \leq 0}, \tag{A.1.3}$$

$$\mathbb{1}_{\theta \widehat{\theta} \leq 0} \leq \frac{|\widehat{\theta} - \theta|}{|\theta|} \text{ for } \theta \neq 0. \tag{A.1.4}$$

Proof. Inequality (A.1.3) is an equality when $\theta \neq 0$ or $\theta = 0, \widehat{\theta} \neq 0$ and an obvious inequality when $\theta = \widehat{\theta} = 0$. The right hand side of (A.1.4) is nonnegative. When $\theta < 0$ and $\widehat{\theta} \geq 0$ (resp. $\theta > 0$ and $\widehat{\theta} \leq 0$) , we have $|\widehat{\theta} - \theta| = \widehat{\theta} - \theta \geq -\theta = |\theta|$ (resp. $|\widehat{\theta} - \theta| = -\widehat{\theta} + \theta \geq \theta = |\theta|$). □

Lemma A.4. *Let $(Z_k)_{k \in \mathbb{N}^*}$ be an i.i.d. sequence of square integrable real valued random variables. Let $\mu = \mathbb{E}[Z_1]$ and $\sigma = \sqrt{\mathrm{Var}[Z_1]}$. For $\gamma \in [1, 2]$, we define $D_\gamma = \sigma^\gamma$ and for $\gamma > 2$, $D_\gamma = \mathbb{E}[|Z_1 - \mu|^\gamma] \in [0, +\infty]$. Then, we have*

$$\mathbb{E}\left[\left| \frac{1}{K} \sum_{k=1}^K Z_k - \mu \right|^\gamma \right] \leq C_\gamma \frac{D_\gamma}{K^{\gamma/2}},$$

with $C_\gamma = 1$ for $\gamma \in [1, 2]$ and $C_\gamma = (2\sqrt{\gamma - 1})^\gamma$ for $\gamma > 2$.

Proof. For $\gamma \in [1, 2]$, We have from Jensen inequality

$$\mathbb{E}\left[\left| \frac{1}{K} \sum_{k=1}^K Z_k - \mu \right|^\gamma \right] \leq \mathbb{E}\left[\left| \frac{1}{K} \sum_{k=1}^K Z_k - \mu \right|^2 \right]^{\gamma/2} = \sigma^\gamma / K^{\gamma/2}$$

. For $\gamma > 2$, this result is stated in Corollary 2.5 [GHJvW18] □

A.1.2 Nested Monte-Carlo estimator

Proof of Lemma 4.5. Let $\bar{b}(X) = \mathbb{E}\left[\max\{\widehat{\theta}_K^1, \widehat{\theta}_K^2\} - \max\{\varphi_1(X), \varphi_2(X)\} | X \right]$. Since $\max\{a, b\} = a + g_0(b - a)$ for $a, b \in \mathbb{R}$, we get

$$\bar{b}(X) = \mathbb{E}\left[\widehat{\theta}_K^1 - \varphi_1(X) + g_0(\widehat{\theta}_K^{21}) - g_0(\varphi_{21}(X)) | X \right] \tag{A.1.5}$$

Now, we observe that

$$\bar{b}(X) = \lim_{\varepsilon \to 0} \bar{b}_\varepsilon(X) \text{ with } \bar{b}_\varepsilon(X) = \mathbb{E}\left[\widehat{\theta}_K^1 - \varphi_1(X) + g_\varepsilon(\widehat{\theta}_K^{21}) - g_\varepsilon(\varphi_{21}(X)) | X \right]. \tag{A.1.6}$$

Since g_ε is 1-Lipschitz, we have

$$|g_\varepsilon(\widehat{\theta}^{21}_K) - g_\varepsilon(\varphi_{21}(X))| \leq |\widehat{\theta}^{21}_K - \varphi_{21}(X)| \leq |\widehat{\theta}^{1}_K - \varphi_1(X)| + |\widehat{\theta}^{2}_K - \varphi_2(X)|.$$

Then, we get (A.1.6) by using the integrability assumption (ii) and Lebesgue's dominated convergence theorem. Since g_ε is C^1 and piecewise C^2, we can make a Taylor expansion to obtain:

$$\bar{b}_\varepsilon(X) = \mathbb{E}\left[\widehat{\theta}^{1}_K - \varphi_1(X) + g'_\varepsilon(\varphi_{21}(X))\left(\widehat{\theta}^{21}_K - \varphi_{21}(X)\right) + \int_{\varphi_{21}(X)}^{\widehat{\theta}^{21}_K} \left(\widehat{\theta}^{21}_K - t\right) g''_\varepsilon(t)dt|X\right]$$

Then, since $g'_\varepsilon(\varphi_{21}(X))$ is $\sigma(X)$-measurable, using Proposition A.1 we get

$$\begin{aligned}\lim_{\varepsilon\to 0}\mathbb{E}\left[\widehat{\theta}^{1}_K - \varphi_1(X) + g'_\varepsilon(\varphi_{21}(X))\left(\widehat{\theta}^{21}_K - \varphi_{21}(X)\right)|X\right] &= \mathbb{E}\left[\widehat{\theta}^{1}_K - \varphi_1(X) + \mathbb{1}_{\varphi_{21}(X)>0}\left(\widehat{\theta}^{21}_K - \varphi_{21}(X)\right)|X\right]\\ &= \mathbb{E}\left[\mathbb{1}_{\varphi_{21}(X)\leq 0}\left(\widehat{\theta}^{1}_K - \varphi_1(X)\right) + \mathbb{1}_{\varphi_{21}(X)>0}\left(\widehat{\theta}^{2}_K - \varphi_2(X)\right)|X\right]\end{aligned}$$

Using condition (ii) we get :

$$\left|\mathbb{E}\left[\mathbb{1}_{\varphi_{21}(X)\leq 0}\left(\widehat{\theta}^{1}_K - \varphi_1(X)\right) + \mathbb{1}_{\varphi_{21}(X)>0}\left(\widehat{\theta}^{2}_K - \varphi_2(X)\right)|X\right]\right| \leq \frac{\mathbb{1}_{\varphi_{21}(X)\leq 0}C_1(X) + \mathbb{1}_{\varphi_{21}(X)>0}C_2(X)}{K^{\frac{1+\eta}{2}}} \tag{A.1.7}$$

Now we focus on the remainder in the Taylor decomposition. Using Lemma A.2 and the dominated convergence theorem and then Lemma A.3 with $\varphi_{21}(X) \neq 0$ a.s. (Assumption (iii)), we get

$$\begin{aligned}\lim_{\varepsilon\to 0}\left|\mathbb{E}\left[\int_{\varphi_{21}(X)}^{\widehat{\theta}^{21}_K}\left(\widehat{\theta}^{21}_K - t\right) g''_\varepsilon(t)dt|X\right]\right| &\leq \mathbb{E}\left[|\widehat{\theta}^{21}_K - \varphi_{21}(X)|\limsup_{\varepsilon\to 0}\left|\int_{\varphi_{21}(X)}^{\widehat{\theta}^{21}_K} g''_\varepsilon(t)dt\right||X\right]\\ &\leq \mathbb{E}\left[|\widehat{\theta}^{21}_K - \varphi_{21}(X)||\mathbb{1}_{\varphi_{21}(X)\leq 0\leq \widehat{\theta}^{21}_K} - \mathbb{1}_{\widehat{\theta}^{21}_K\leq 0\leq \varphi_{21}(X)}||X\right]\\ &\leq \mathbb{E}\left[|\widehat{\theta}^{21}_K - \varphi_{21}(X)|\mathbb{1}_{\widehat{\theta}^{21}_K\varphi_{21}(X)\leq 0}|X\right]\\ &\leq \mathbb{E}\left[\frac{|\widehat{\theta}^{21}_K - \varphi_{21}(X)|^{1+\eta}}{|\varphi_{21}(X)|^\eta}|X\right] = \frac{\mathbb{E}\left[|\widehat{\theta}^{21}_K - \varphi_{21}(X)|^{1+\eta}|X\right]}{|\varphi_{21}(X)|^\eta}\end{aligned}$$

Now, we use the norm inequality $\mathbb{E}\left[|\widehat{\theta}^{21}_K - \varphi_{21}(X)|^{1+\eta}|X\right]^{\frac{1}{1+\eta}} \leq \mathbb{E}\left[|\widehat{\theta}^{1}_K - \varphi_1(X)|^{1+\eta}|X\right]^{\frac{1}{1+\eta}} + \mathbb{E}\left[|\widehat{\theta}^{2}_K - \varphi_2(X)|^{1+\eta}|X\right]^{\frac{1}{1+\eta}}$ and the convexity of $x \mapsto x^{1+\eta}$ to get

$$\begin{aligned}\mathbb{E}\left[|\widehat{\theta}^{21}_K - \varphi_{21}(X)|^{1+\eta}|X\right] &\leq 2^\eta\left(\mathbb{E}\left[|\widehat{\theta}^{1}_K - \varphi_1(X)|^{1+\eta}|X\right] + \mathbb{E}\left[|\widehat{\theta}^{2}_K - \varphi_2(X)|^{1+\eta}|X\right]\right)\\ &\leq 2^\eta\frac{\sigma_1^{1+\eta}(X) + \sigma_2^{1+\eta}(X)}{K^{\frac{1+\eta}{2}}}\end{aligned} \tag{A.1.8}$$

With (A.1.6), equations (A.1.7) and (A.1.8) give the bias estimate (4.2.11).

We now focus on the variance. The proof is straightforward using condition (ii) and the inequality $|\max\{a_1, a_2\} - \max\{b_1, b_2\}| \leq \max\{|a_1 - b_1|, |a_2 - b_2|\}$:

$$\begin{aligned}\mathbb{E}\left[\left|\max\{\widehat{\theta}^{1}_K, \widehat{\theta}^{2}_K\} - \max\{\varphi_1(X), \varphi_2(X)\}\right|^2|X\right] &\leq \mathbb{E}\left[\max\{|\widehat{\theta}^{1}_K - \varphi_1(X)|^2, |\widehat{\theta}^{2}_K - \varphi_2(X)|^2\}|X\right]\\ &\leq \mathbb{E}\left[|\widehat{\theta}^{1}_K - \varphi_1(X)|^2|X\right] + \mathbb{E}\left[|\widehat{\theta}^{2}_K - \varphi_2(X)|^2|X\right]\\ &\leq \frac{\sigma_1^2(X) + \sigma_2^2(X)}{K} = \frac{\sigma^2(X)}{K}.\end{aligned}$$

□

A.1.3 Antithetic MLMC estimator

In this section we prepare the proof of Theorem 4.9 and start with a useful preliminary lemma.

Lemma A.5. *Let $p \geq 2$ and $K \in 2\mathbb{N}^*$. With the notation introduced in (4.2.3) and (4.2.4), the following property holds:*

$$\mathbb{E}\left[\left|\widehat{M}_K^p - \frac{\widehat{M}_{K/2}^p + \widehat{M}_{K/2}^{p,\prime}}{2}\right|^2 |X\right] \leq 2\mathbb{E}\left[\left|\widehat{M}_K^{p-1} - \frac{\widehat{M}_{K/2}^{p-1} + \widehat{M}_{K/2}^{p-1,\prime}}{2}\right|^2 |X\right] + 2\mathbb{E}\left[h^2\left(\widehat{M}_K^{p-1} - \widehat{E}_{K/2}^p, \widehat{M}_{K/2}^{p-1,\prime} - \widehat{E}_{K/2}^{p,\prime}\right)|X\right],$$

where $h(x,y) = \left(\frac{x+y}{2}\right)^+ - \frac{(x)^+ + (y)^+}{2}$.

Proof of Lemma A.5. Observing that $\forall a, b \in \mathbb{R}$, $\max\{a,b\} = a + (b-a)^+$, we deduce that

$$\begin{aligned}
\widehat{M}_K^p - \frac{\widehat{M}_{K/2}^p + \widehat{M}_{K/2}^{p,\prime}}{2} &= \max\{\widehat{E}_K^p, \widehat{M}_K^{p-1}\} - \frac{\max\{\widehat{E}_{K/2}^p, \widehat{M}_{K/2}^{p-1}\} + \max\{\widehat{E}_{K/2}^{p,\prime}, \widehat{M}_{K/2}^{p-1,\prime}\}}{2} \\
&= \widehat{E}_K^p + (\widehat{M}_K^{p-1} - \widehat{E}_K^p)^+ - \frac{\widehat{E}_{K/2}^p + (\widehat{M}_{K/2}^{p-1} - \widehat{E}_{K/2}^p)^+ + \widehat{E}_{K/2}^{p,\prime} + (\widehat{M}_{K/2}^{p-1,\prime} - \widehat{E}_{K/2}^{p,\prime})^+}{2} \\
&= (\widehat{M}_K^{p-1} - \widehat{E}_K^p)^+ - \frac{(\widehat{M}_{K/2}^{p-1} - \widehat{E}_{K/2}^p)^+ + (\widehat{M}_{K/2}^{p-1,\prime} - \widehat{E}_{K/2}^{p,\prime})^+}{2} \\
&= (\widehat{M}_K^{p-1} - \widehat{E}_K^p)^+ - \left(\frac{\widehat{M}_{K/2}^{p-1} + \widehat{M}_{K/2}^{p-1,\prime}}{2} - \widehat{E}_K^p\right)^+ \\
&\quad + h\left(\widehat{M}_K^{p-1} - \widehat{E}_{K/2}^p, \widehat{M}_{K/2}^{p-1,\prime} - \widehat{E}_{K/2}^{p,\prime}\right),
\end{aligned}$$

using that $\widehat{E}_K^p = \frac{\widehat{E}_{K/2}^p + \widehat{E}_{K/2}^{p,\prime}}{2}$ for the third and fourth equality. We conclude using that $(a+b)^2 \leq 2(a^2+b^2)$. □

The following lemmas gives a bound on h

Lemma A.6. *Let $h(x,y) = \left(\frac{x+y}{2}\right)^+ - \frac{(x)^+ + (y)^+}{2}$ for $x, y \in \mathbb{R}$. Then, we have $h(x,y) = -\frac{|x| \wedge |y|}{2}\mathbb{1}_{xy \leq 0}$.*

Proof. By distinguishing the cases as follows, we get the claim:

$$h(x,y) = \begin{cases} y/2 & \text{if } x+y \geq 0, x > 0, y < 0, \\ x/2 & \text{if } x+y \geq 0, x < 0, y > 0, \\ -x/2 & \text{if } x+y < 0, x > 0, y < 0, \\ -y/2 & \text{if } x+y < 0, x < 0, y > 0, \\ 0 & \text{otherwise.} \end{cases}$$ □

Proposition A.7. *We use the notation introduced in (4.2.3) and (4.2.4). Let $\eta > 0$ and $D_{2+\eta}^p(X) = \mathbb{E}[|Y^p - \mathbb{E}[Y^p|X]|^{2+\eta}|X]$. We assume that $\mathbb{P}(E_X^p = M_X^{p-1}) = 0$ for all $p \in \{2, \dots, P\}$ and*

$$\forall p \in \{2, \dots, P\},\ \mathbb{E}\left[\frac{D_{2+\eta}^p(X)}{|E_X^p - M_X^{p-1}|^\eta}\phi^2(X)\right] < \infty.$$

Then, there exist a constant $C \in \mathbb{R}_+^*$ such that

$$\mathrm{Var}\left[\left(\widehat{M}_K^P - \frac{\widehat{M}_{K/2}^P + \widehat{M}_{K/2}^{p,\prime}}{2}\right)\phi(X)\right] \leq \frac{C}{K^{1+\eta/2}}.$$

Proof. Let us define for $p = 1, \dots, P$,

$$U_K^p = \mathbb{E}\left[\left|\widehat{M}_K^p - \frac{\widehat{M}_{K/2}^p + \widehat{M}_{K/2}^{p,\prime}}{2}\right|^2 \middle| X\right]$$

$$\varepsilon_K^p = \mathbb{E}\left[h^2\left(\widehat{M}_K^{p-1} - \widehat{E}_{K/2}^p, \widehat{M}_{K/2}^{p-1,\prime} - \widehat{E}_{K/2}^{p,\prime},\right)|X\right].$$

We notice that $U_K^1 = 0$, and Lemma A.5 gives $U_K^p \leq 2(U_K^{p-1} + \varepsilon_K^p)$ for $p = 2, \dots, P$. A straightforward induction leads to

$$U_K^P \leq \sum_{p=2}^P 2^{P+1-p}\varepsilon_K^p. \tag{A.1.9}$$

The variance being smaller than the expectation of the square, we get by using the tower property of the conditional expectation

$$\mathrm{Var}\left[\left(\widehat{M}_K^P - \frac{\widehat{M}_{K/2}^P + \widehat{M}_{K/2}^{p,\prime}}{2}\right)\phi(X)\right] \leq \sum_{p=2}^P 2^{P+1-p}\mathbb{E}[\varepsilon_K^p\phi^2(X)]. \tag{A.1.10}$$

For $p = 2, \dots, P$, we define the following random variables

$$H_X^p = M_X^{p-1} - E_X^p,\ \widehat{H}_{K/2}^p = \widehat{M}_{K/2}^{p-1} - \widehat{E}_{K/2}^p,\ \widehat{H}_{K/2}^{p,\prime} = \widehat{M}_{K/2}^{p-1,\prime} - \widehat{E}_{K/2}^{p,\prime}.$$

We now use Lemma A.6 and the equality $\mathbb{1}_{\widehat{H}_{K/2}^p\widehat{H}_{K/2}^{p,\prime}<0} = \mathbb{1}_{\widehat{H}_{K/2}^p\widehat{H}_{K/2}^{p,\prime}<0}\mathbb{1}_{\widehat{H}_{K/2}^pH_X^p<0} + \mathbb{1}_{\widehat{H}_{K/2}^p\widehat{H}_{K/2}^{p,\prime}<0}\mathbb{1}_{\widehat{H}_{K/2}^{p,\prime}H_X^p<0}$ that is true a.s. since $\mathbb{P}(H_X^p = 0) = 0$ to get

$$\begin{aligned}
\mathbb{E}[\varepsilon_K^p\phi^2(X)] &= \frac{1}{4}\mathbb{E}\left[\left(\min\left(|\widehat{H}_{K/2}^p|, |\widehat{H}_{K/2}^{p,\prime}|\right)^2\right)\phi^2(X)\mathbb{1}_{\widehat{H}_{K/2}^p\widehat{H}_{K/2}^{p,\prime}<0}\mathbb{1}_{\widehat{H}_{K/2}^pH_X^p<0}\right] \\
&\quad + \frac{1}{4}\mathbb{E}\left[\left(\min\left(|\widehat{H}_{K/2}^p|, |\widehat{H}_{K/2}^{p,\prime}|\right)^2\right)\phi^2(X)\mathbb{1}_{\widehat{H}_{K/2}^p\widehat{H}_{K/2}^{p,\prime}<0}\mathbb{1}_{\widehat{H}_{K/2}^{p,\prime}H_X^p<0}\right] \\
&\leq \frac{1}{4}\left(\mathbb{E}\left[|\widehat{H}_{K/2}^p|^2\phi^2(X)\mathbb{1}_{\widehat{H}_{K/2}^pH_X^p<0}\right] + \mathbb{E}\left[|\widehat{H}_{K/2}^{p,\prime}|^2\phi^2(X)\mathbb{1}_{\widehat{H}_{K/2}^{p,\prime}H_X^p<0}\right]\right) \\
&= \frac{1}{2}\mathbb{E}\left[|\widehat{H}_{K/2}^p|^2\phi^2(X)\mathbb{1}_{\widehat{H}_{K/2}^pH_X^p<0}\right],
\end{aligned}$$

since $\widehat{H}_{K/2}^p$ and $\widehat{H}_{K/2}^{p,\prime}$ have the same law given X. Now, we use that $|\widehat{H}_{K/2}^p| \leq |\widehat{H}_{K/2}^p - H_X^p|$ on $\{\widehat{H}_{K/2}^pH_X^p < 0\}$ and Lemma A.3 gives $\mathbb{1}_{\widehat{H}_{K/2}^pH_X^p<0} \leq \frac{|\widehat{H}_{K/2}^p - H_X^p|^\eta}{|H_X^p|^\eta}$ for $\eta > 0$. This leads to

$$\mathbb{E}[\varepsilon_K^p\phi^2(X)] \leq \frac{1}{2}\mathbb{E}\left[\frac{|\widehat{H}_{K/2}^p - H_X^p|^{2+\eta}}{|H_X^p|^\eta}\phi^2(X)\right].$$

We now use Lemma A.4 and get $\mathbb{E}[|\widehat{H}_{K/2}^p - H_X^p|^{2+\eta}|X] \leq C_{2+\eta}\frac{D_{2+\eta}^p(X)}{(K/2)^{1+\eta/2}}$, and therefore

$$\mathbb{E}[\varepsilon_K^p\phi^2(X)] \leq 2^{\eta/2}C_{2+\eta}\mathbb{E}\left[\frac{D_{2+\eta}^p(X)}{|H_X^p|^\eta}\phi^2(X)\right].$$

Using this bound in (A.1.10), we get the claim with $C = 2^{\eta/2}C_{2+\eta}\sum_{p=2}^P 2^{P+1-p}\mathbb{E}[\frac{D_{2+\eta}^p(X)}{|H_X^p|^\eta}\phi^2(X)]$. □

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