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Sijia Kong

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**THÈSE DE DOCTORAT**  
**DE L'UNIVERSITÉ PSL**

Préparée à Mines Paris-PSL

**Prediction-based control of dynamical systems with  
stochastic input delay**

**Contrôle basé sur la prédiction de systèmes dynamiques  
avec retard d'entrée stochastique**

Soutenue par

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# List of Acronyms and Notations

## Acronyms

<b>LTI</b>	Linear Time-Invariant
<b>ODE</b>	Ordinary Differential Equation
<b>PDE</b>	Partial Differential Equation
<b>SDDE</b>	Stochastic Delay Differential Equation
<b>LMI</b>	Linear Matrix Inequality

## Notations

$(\cdot)^T$	transpose of a matrix
$\mathbf{I}_n$	identity matrix of size $n$ , $n \in \mathbb{N}$
$\mathbf{diag}(a_1, \dots, a_n)$	diagonal square matrix of dimension $n$ with diagonal elements $a_1, \dots, a_n$
$ \cdot $	Euclidean norm of a vector in $\mathbb{R}^n$ and induced euclidean norm of a real matrix, $ A  = \sqrt{\max(\lambda(A^T A))}$
$\ \cdot\ $	spatial $\mathcal{L}_2$ -norm of a signal $v : x \in [0, 1] \times \mathbb{R} \rightarrow v(x) \in \mathbb{R}$ , $\ v\  = \sqrt{\int_0^1 v(x)^2 dx}$ ,
$\lambda(M)$	spectrum of a square matrix $M$
$\min(\lambda(M))$	minimum eigenvalue of a real square matrix $M$
$\max(\lambda(M))$	maximum eigenvalue of a real square matrix $M$
$e_i$	$i^{\text{th}}$ unit vector in $\mathbb{R}^r$ , $r \in \mathbb{N}_+$ and $i \in \{1, \dots, r\}$ ,
$\mathbb{E}(x)$	expectation of a random variable $x$
$\mathbb{E}_{[s, x_0]}(x(t))$	conditional expectation of $x(t)$ at the instant $t$ knowing that $x(s) = x_0$ at the instant $s \leq t$
$\mathbf{Var}(x)$	variation of the random variable $x$
$\mathbf{Var}_{[s, x_0]}(x(t))$	conditional variation of $x(t)$ at the instant $t$ knowing that $x(s) = x_0$ at the instant $s \leq t$
$\sigma(x)$	standard deviation of the random variable $x$
$\sigma_{[s, x_0]}(x(t))$	conditional standard deviation of $x(t)$ at the instant $t$ knowing that $x(s) = x_0$ at the instant $s \leq t$

# Introduction (Version Français)

## 1. Contexte: retards stochastiques dans les systèmes dynamiques contrôlés

L'étude des systèmes à retard est un problème récurrent dû à l'omniprésence de retards dans les applications. Il est bien connu que les retards sont une cause potentielle d'instabilité dans les systèmes dynamiques et rendent la conception de contrôleurs difficile, car leur présence peut induire des comportements complexes [GN03]. Par exemple, de petits retards peuvent déstabiliser certains systèmes [HL01], tandis que des retards plus grands peuvent en stabiliser d'autres [BM75, Mac86]. D'autres cas où l'introduction volontaire de retards peut présenter des avantages pour le contrôle sont rapportés dans [Ric03], dont l'exemple des contrôleurs "deadbeat" [WNK96] et des résonateurs retardés [JO98], pour en citer quelques-uns.

Lorsque les retards sont supposés constants, des méthodes spectrales maintenant bien connues [MN07] peuvent permettre d'analyser la stabilité des systèmes à retard et les bifurcations d'équilibre et donc faciliter la conception de contrôleurs. Cependant, contrairement au cas de la dimension finie, l'équation caractéristique d'un système à retard admet un nombre infini de racines. Leur comportement et leur sensibilité peuvent donc être difficiles à étudier, et nécessitent donc des études poussées [MBN17, JGNB18, IGBN18]. Malheureusement, ces outils du domaine fréquentiel ne sont pas adaptés aux retards variant dans le temps.

Néanmoins, les retards temps-variants sont omniprésents dans les applications. Par exemple, en communication, la transmission de données s'accompagne toujours d'une différence entre l'instant d'émission et l'instant de réception d'un signal. Dans un système en réseau, ces informations peuvent être transmises via différents canaux, comme illustré sur la Figure 1, en fonction de l'état de congestion du réseau. Par conséquent, le temps de transit dépend de l'algorithme de routage et changera dans le temps. Avec l'essor des technologies d'information et de communication, la dernière décennie a été marquée par un énorme intérêt de la recherche pour de tels systèmes [CJO<sup>+</sup>11, HNX07]. Lorsque les données de capteur et de contrôle sont transmises via des canaux de communication numériques, les systèmes à données échantillonnées [ÅW13] sont un outil naturel pour la modélisation de système et la conception de contrôle [CHVdW<sup>+</sup>10]. Dans ce cas, les systèmes à échantillonnage périodique [HFO<sup>+</sup>17] se révèlent constituer un outil pratique pour capturer certaines caractéristiques des systèmes de contrôle en réseau, y compris les sauts d'échantillonnage, la perte de paquets ou les fluctuations causées par les interactions entre les algorithmes de contrôle et les protocoles temps réel d'ordonnancement. Alternativement, une autre façon courante d'expliquer le phénomène de commutation par paquets [HNX07] consiste à modéliser le retard comme une variable non déterministe [Gup11, Sch08].

En effet, les retards stochastiques sont communément utilisés pour modéliser les pertes de paquets et le réordonnancement de données, ce qui se produit fréquemment, par exemple, dans les systèmes de véhicules connectés [HCCDV13] : en plus du temps de réaction du conducteur, la communication sans fil de véhicule à véhicule [BK06, ZO17] utilisée pour communiquer avec les véhicules alentours peut engendrer de nombreux retards de communication et pertes de paquets [BPDV15, MQIO17]. Des retards stochastiques apparaissent également dans la modélisation des réseaux de régulation des



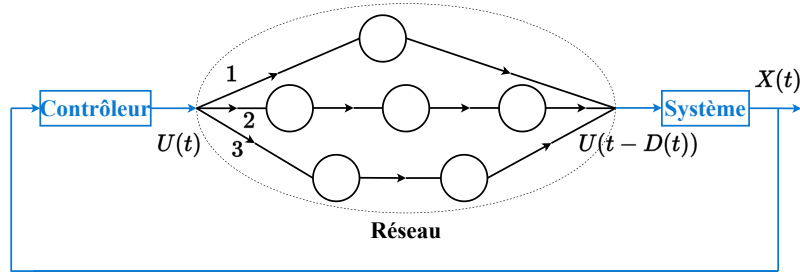


Figure 1: Le retard de communication induit par un réseau peut dépendre de l’algorithme de routine.

gènes [SO16], car le temps d’exécution des processus de transcription et de traduction est affecté par l’environnement cellulaire incertain [JLO<sup>+</sup>11, Lew03].

L’analyse de stabilité et la stabilisation des systèmes à temps discret à retards stochastiques ont été largement étudiées dans la littérature, par exemple dans [KÖC<sup>+</sup>94, NBW98, GQOM14]. Ces méthodes transforment généralement les retards discrets en systèmes dynamiques de dimension finie en étendant l’état du système, comme il est coutumier de procéder pour les systèmes à retard en temps discret. Dans cette formulation, le retard stochastique peut donc être exprimé comme un paramètre stochastique à temps discret, et la stabilité du système en boucle fermée est alors réécrite comme un problème de contrôle/estimation optimal, par exemple dans [Gup11, Sch08]. Le problème correspondant du contrôle à erreur quadratique moyenne minimale est obtenu en résolvant l’équation de Riccati correspondante, comme dans la théorie usuelle du filtrage de Kalman.

Cependant, les outils de stabilisation pour les processus en temps continu soumis à des retards stochastiques en temps continu sont encore peu nombreux dans la littérature. L’objectif principal de cette thèse est donc de proposer une méthodologie générique de stabilisation pour les systèmes dynamiques en temps continu avec des retards d’entrée stochastiques. Pour y parvenir, nous proposons de nous appuyer sur des contrôleurs par prédiction qui ont de forts mérites dans le cas d’un retard déterministe. Nous présentons donc tout d’abord cette technique de contrôle dans ce qui suit.

## 2. Contrôleur par prédiction: état de l’art de la stabilisation des systèmes à retard d’entrée

### 2.1. Le concept des contrôleurs par prédiction

Considérons un système mono-entrée linéaire invariant dans le temps (LTI) avec retard d’entrée constant

$$\dot{X}(t) = AX(t) + BU(t - D), \quad (1)$$

où  $X \in \mathbb{R}^n$ ,  $D$  est un retard d’entrée, et la paire de matrices  $(A, B)$  est commandable. Pour le système sans retard ( $D = 0$ ), une loi de commande assurant la stabilisation en boucle fermée est

$$U = KX(t), \quad (2)$$

où le gain de rétroaction  $K$  est tel que  $A + BK$  est Hurwitz. La dynamique en boucle fermée se réécrit alors sous la forme  $\dot{X} = (A + BK)X$  qui est exponentiellement stable. Le concept de contrôle par prédiction consiste à obtenir la même dynamique en boucle fermée que dans le cas sans retard en *compensant* le retard d’entrée. Pour ce faire, définissons la prédiction de l’état  $X$  du système sur un horizon de  $D$  unités de temps

$$P(t) = X(t + D), \quad (3)$$

et remplaçons-le simplement à la place de  $X(t)$  dans (2) pour définir

$$U = KP(t). \quad (4)$$

Cette loi de commande implique alors que  $U(t-D) = KX(t)$ , ce qui implique à son tour  $\dot{X} = (A+BK)X$  pour  $t \geq D$ . Le retard a ainsi disparu de la dynamique en boucle fermée, dont le comportement est alors plus facile à régler et à analyser.

Le principal problème restant est que l'état futur  $X(t+D)$  ne peut bien sûr pas être mesuré, car il est a priori non causal. Pourtant, une expression causale peut en être obtenue en appliquant la formule de variation de la constante à (1),

$$\begin{aligned} P(t) = X(t+D) &= e^{AD}X(t) + \int_t^{t+D} e^{A(t+D-s)}BU(s-D)ds \\ &= e^{AD}X(t) + \int_{t-D}^t e^{A(t-s)}BU(s)ds, \end{aligned} \quad (5)$$

Cette expression ne dépend plus alors que de l'état du système à l'instant  $t$  et des valeurs passées de l'entrée  $U$  sur la plage de temps  $(t-D, t)$ , qui sont connues à l'instant  $t$ . Ce contrôleur par prédiction est donc implémentable et s'écrit finalement sous la forme

$$U(t) = KP(t) = KX(t+D) = K \left[ e^{AD}X(t) + \int_{t-D}^t e^{A(t-s)}BU(s)ds \right]. \quad (6)$$

Ce contrôleur est particulièrement intéressant dans le cas de retards d'entrée non négligeables, tandis que la robustesse de la loi de rétroaction nominale sans retard peut être évoquée pour les petits retards d'entrée.

Cette technique a été présentée dans de nombreuses études depuis les années 1950. Au cours de cette période, le célèbre Prédicteur de Smith a tout d'abord été proposé [Smi57] en 1959, pour éliminer les retards dans les systèmes stables en boucle ouverte. La technique de rétroaction par prédicteur fut ensuite introduite dans le cadre de "l'allocation de spectre fini" par [May68, KP80, MO79, Wat86, WNKI92] et tant que "modèle réduit" dans [Art82], pour des systèmes linéaires temps-invariants à retard d'entrée, génériques. Dans ce manuscrit, nous faisons référence à cette technique sous la dénomination de "contrôle par prédiction".

Ce concept de contrôle par prédiction a depuis été utilisé dans de nombreux contextes (retards d'état et d'entrée, plusieurs retards d'entrées...). Nous en détaillons maintenant quelques extensions liées aux problèmes considérés dans ce manuscrit.

### 2.1.1. Système linéaire temps-invariant (LTI) avec retard d'entrée variable dans le temps

Même si l'article fondateur d'Artstein [Art82] pouvait traiter les retards variant dans le temps, ce n'est qu'avec [Nih91] qu'une conception explicite de prédicteur pour des retards d'entrée variant dans le temps a été obtenue.

Cette formulation a été proposée pour le système LTI

$$\dot{X}(t) = AX(t) + BU(t-D(t)), \quad (7)$$

où le retard d'entrée variable dans le temps  $D$  est supposé être une fonction bornée différentiable, telle que  $\dot{D}(t) < 1$  pour tout temps. Dans ce cas, on introduit la fonction de retard  $\phi(t) = t - D(t)$  et, comme précédemment, on souhaite calculer une prédiction telle que  $U(\phi(t)) = KX(t)$ . Ainsi, le contrôle par prédiction devrait écrire comme

$$\begin{aligned} U(t) = KX(\phi^{-1}(t)) &= K \left[ e^{A(\phi^{-1}(t)-t)}X(t) + \int_t^{\phi^{-1}(t)} e^{A(\phi^{-1}(t)-s)}BU(\phi(s))ds \right] \\ &= K \left[ e^{A(\phi^{-1}(t)-t)}X(t) + \int_{\phi(t)}^t e^{A(\phi^{-1}(t)-\phi^{-1}(s))}B \frac{U(s)}{\dot{\phi}(\phi^{-1}(s))}ds \right]. \end{aligned} \quad (8)$$

où nous avons appliqué la formule de variation de la constante à (7), comme précédemment.

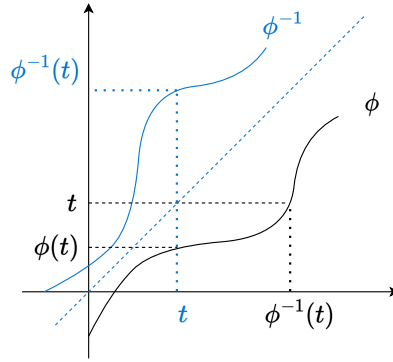


Figure 2: Fonction de retard  $\phi(t)$  et son inverse. Le calcul de  $\phi^{-1}(t)$  à un instant donné  $t$  nécessite la connaissance des valeurs futures de  $\phi$ .

Pour concevoir une telle loi de commande par prédiction, la fonction  $\phi(t) = t - D(t)$  doit être inversible. Ceci est garanti par le fait que  $\dot{D}(t) < 1$ , ce qui implique que  $\dot{\phi}(t) > 0$  et donc que la fonction  $\phi$  est croissante. Nous pouvons aussi noter que l’hypothèse  $\dot{D}(t) < 1$  est satisfaite pour une large classe de retards, parfois appelés retards hydrauliques [CP19] ou retards de transport [BPP16], et qui découlent du transport physique d’une matière (flux, etc).

Observons aussi que, si le retard d’entrée est constant, l’inverse de la fonction  $\phi$  est  $\phi^{-1}(t) = t + D$ , et nous pouvons également obtenir le contrôle par prédiction (6). Cependant, dans le cas général, l’horizon de prédiction n’est pas simplement égal à la valeur de retard actuelle, c’est-à-dire  $\phi^{-1}(t) \neq t + D(t)$ . En réalité, le calcul de cet inverse nécessite d’avoir une certaine connaissance des variations futures du retard, comme illustré sur la Figure 2, ce qui peut ne pas être le cas en pratique. C’est pourquoi certaines approches reposent sur une compensation inexacte du retard.

### Compensation robuste avec un contrôleur par prédiction.

Une approche alternative à la compensation de retard exacte fournie par (8) consiste à s’appuyer sur un horizon de prédiction approché, comme dans [BPCP13]

$$U(t) = K \left[ e^{AD(t)} X(t) + \int_{t-D(t)}^t e^{A(t-s)} BU(s) ds \right], \quad (9)$$

La logique de ce contrôleur est de supposer que le retard restera suffisamment proche de  $D(t)$  dans un avenir proche, et donc que (6) peut être utilisé directement avec la valeur de retard actuelle  $D(t)$  à la place de  $D$ . Cependant, en raison de la non-concordance entre les fonctions  $\phi^{-1}(t)$  et  $t + D(t)$ , cette loi de commande ne permet pas d’obtenir une compensation exacte du retard d’entrée. Cependant, le contrôleur par prédiction (9) peut encore stabiliser (7) à condition que les variations de retard soient suffisamment faibles, comme cela sera détaillé par la suite.

#### 2.1.2. Système non linéaire avec retard d’entrée constant

Dans [Krs09b], la première “méthode de prédiction” pour les systèmes à retard non linéaires avec retards d’entrée constants arbitrairement longs a été développées, et a ensuite été amenée à maturité dans [BLK13a]. Ces travaux considèrent la dynamique non linéaire mono-entrée

$$\dot{X}(t) = f(X(t), U(t - D)), \quad (10)$$

où  $X \in \mathbb{R}^n$ ,  $U \in \mathbb{R}$ ,  $f \in C^1(\mathbb{R}^n \times \mathbb{R}; \mathbb{R}^n)$  et  $D \geq 0$  est un retard d’entrée constant.

Le contrôle par prédiction correspondant repose alors sur deux éléments préliminaires :

- le système (10) est supposé ne pas exploser en temps fini, de sorte à ce que la loi de commande en boucle fermée puisse effectivement atteindre le système (au temps  $D > 0$ ) avant qu’il n’ait explosé et ne puisse plus avoir un quelconque effet sur le système dynamique ;

- l'existence d'une loi nominale de rétroaction dans le cas non-retardé  $\kappa : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}$  telle que la dynamique  $\dot{X}(t) = f(X(t), \kappa(X(t), t))$  est asymptotiquement stable.

Comme mentionné précédemment, cette loi de rétroaction nominale peut alors être utilisée pour compenser le retard d'entrée, en l'appliquant à une prédiction de l'état du système. Autrement dit, le contrôleur par prédiction  $U(t) = \kappa(P(t), t + D)$  où le prédicteur est défini implicitement par

$$P(t) = X(t + D) = X(t) + \int_{t-D}^t f(P(s), U(s)) ds, \quad (11)$$

permet d'obtenir la dynamique en boucle fermée  $\dot{X}(t) = f(X(t), \kappa(X(t), t))$  pour  $t \geq D$ , c'est-à-dire une dynamique asymptotiquement stable nominale non-retardée. L'équation (11) n'est rien d'autre que la forme intégrale de (10), avec la condition initiale  $P(t - D) = X(t)$ .

Ces éléments ont été étendus aux retards variant dans le temps dans [BLK13a], comme décrit dans les sections précédentes. Les cas bien plus complexes de retards dépendants de l'état, et certains types de retards dépendants des entrées, ont par exemple également été étudiés dans [BLK12, BPCP13, DBLK17].

## 2.2. Sensibilité au retard et défis liés à l'extension du contrôle par prédiction pour les retards stochastiques

Bien que le contrôleur par prédiction présente le grand avantage d'éliminer les retards en boucle fermée, améliorant ainsi les performances transitoires et simplifiant leur réglage, il présente également certains défauts. Le plus important est probablement que les techniques par prédiction sont sensibles aux incertitudes de retards (et dans une moindre mesure aux incertitudes portant sur les paramètres du système) [Pal80, YS87, Fen91, SS93]. Par conséquent, le choix de l'horizon de prédiction est une question cruciale.

De nombreux travaux ont étudié la robustesse des contrôleurs par prédiction à une erreur de retard. La plupart a été consacrée à l'obtention d'une limite supérieure d'erreur de retard admissible pour préserver la stabilité, à l'aide d'une analyse dans le domaine fréquentiel pour des retards constants [OR82, ALC00, MNL01, Nic01, Zho06].

Des propriétés similaires de robustesse aux erreurs de retard ont été obtenues récemment dans le domaine temporel pour des perturbations de retard plus complexes. Le plus important est probablement [BLK13b] qui fournit des résultats de compensation robuste pour une fonction de retard différentiable dans le temps dépendant potentiellement aussi de l'état du système. Il y est démontré que la compensation robuste est obtenue sous l'hypothèse que la plage de valeur du retard et son taux de variation sont suffisamment limités (et que la condition initiale du système permet à la variation du retard par rapport à l'état de rester petite). Un résultat similaire a été obtenu dans [KK13] mais avec une méthode de petit gain qui évite de limiter le taux de variation du retard, et des conditions suffisantes quantifiables, sous la forme d'inégalités linéaires matricielles portant sur l'amplitude des perturbations du retard, ont ensuite été récemment proposées dans [LPS19]. Les contrôleurs par prédiction détaillés dans ce manuscrit pour un retard d'entrée stochastique reposent sur ces propriétés de robustesse par rapport au retard pour un retard déterministe.

Dans le cas d'un retard stochastique, la compensation exacte du retard semble être un objectif déraisonnable. La première raison pour cela est que la fonction  $\phi$  introduite dans la section 2.1.1 n'est pas régulière et a donc peu de raisons d'être inversible. La seconde est que cette fonction inverse n'est en aucun cas causale et ne peut être prédite si la variation du retard est aléatoire. Enfin, du fait du caractère aléatoire des retards, une compensation exacte du retard, même si possible, conduirait à une loi de commande saccadée, qui serait alors impossible à appliquer en pratique en raison du temps de réponse limité d'un actionneur. Pour la même raison, même si la réalisation actuelle du retard  $D(t)$  est mesurée, l'appliquer comme horizon de prédiction dans (9) conduirait également à une loi

de commande saccadée, qui de toute façon ne serait pas nécessairement proche ou représentative des réalisations de retard encore à venir. Sur la base des propriétés de robustesse par rapport aux erreurs de retards susmentionnées, des stratégies de prédiction alternatives et adaptées sont envisagées dans ce manuscrit.

Pour obtenir des conditions suffisantes à la compensation robuste de retard stochastique, nous proposons d'utiliser le cadre des équations aux dérivées partielles et la reformulation du contrôleur par prédiction comme le contrôle frontière d'une équation de transport, tel qu'introduit dans [KS08].

## 2.3. Équation aux dérivées partielles (EDP) et interprétation du contrôle par prédiction sous l'angle du backstepping

### 2.3.1. Représentation du retard par une équation de transport

Considérons un signal d'entrée  $U$  retardé de  $D > 0$  unités de temps (constantes). Ce phénomène de retard est généralement représenté par l'état de dimension infinie  $U_t : s \in [-D, 0] \mapsto U(t + s)$ . Il peut être formulé de manière équivalente en définissant l'entrée distribuée  $v(x, t) = U(t + D(x - 1))$  pour  $x \in [0, 1]$  qui satisfait alors l'EDP de transport suivante

$$\begin{cases} Dv_t(x, t) = v_x(x, t) \\ v(1, t) = U(t), \end{cases} \quad (12)$$

où  $v_t$  et  $v_x$  représentent respectivement les dérivées partielles par rapport au temps et à l'espace. Cette dynamique du premier ordre est la plus simple des EDP hyperboliques, transportant la quantité  $U(t)$  de  $x = 1$  à  $x = 0$  avec une vitesse  $1/D$ , amenant à la condition de bord opposé  $v(0, t) = U(t - D)$ . Cette équivalence est illustrée sur la Figure 3.

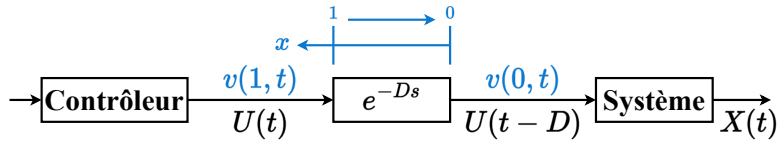


Figure 3: Actionneur distribué  $v(x, t)$  pour  $x \in [0, 1]$ .

De la sorte, le système à retard d'entrée (1) peut alors être réécrit comme la cascade de cette EDP de transport dans une équation différentielle ordinaire (EDO), c'est-à-dire comme

$$\begin{cases} \dot{X}(t) = AX(t) + Bv(0, t) \\ Dv_t(x, t) = v_x(x, t) \\ v(1, t) = 0. \end{cases} \quad (13)$$

Remarquons que cette réécriture permet d'obtenir une paramétrisation explicite du retard : alors qu'il s'agissait à l'origine d'un paramètre temporel de la loi de commande, le retard est désormais représenté comme un facteur bilinéaire dans (13). Nous reviendrons plus en détail sur cette caractéristique par la suite.

Cette équivalence entre les EDP hyperboliques et certaines classes de systèmes à retard est bien connue et rapportée par exemple dans [KM92, HL93, Ric03]. Pourtant, elle était rarement utilisée jusqu'à ce que le contrôle par prédiction (6) soit réinterprété comme le résultat d'une transformation backstepping de (13) dans [KS08], ce que nous allons maintenant détailler.

### 2.3.2. Transformation backstepping

Le concept du backstepping pour les EDP, généralisant tel que décrit dans [BK02] la technique de dimension finie éponyme, s'est avéré ces dernières années être une technique efficace et constructive pour

le contrôle aux frontières des EDP (voir [KS08]). Elle a été appliquée avec succès dans de nombreux contextes impliquant des EDP paraboliques [Liu03, SK04, SK05], des EDP hyperboliques de premier et second ordre [KSS06] ou encore l'équation de Navier-Stokes [CVK06] ou de Burgers [KMV08].

Pour les systèmes à retard, sa pertinence a été soulignée dans [KS08], où il est prouvé que la loi de contrôle par prédiction (6) peut en fait être considérée comme le résultat d'une transformation backstepping du système (13). En effet, la transformation de backstepping suivante de l'entrée distribuée  $v$

$$w(x, t) = v(x, t) - Ke^{ADx}X(t) - D_0 \int_0^x Ke^{AD(x-y)}Bv(y, t)dy, \quad (14)$$

avec la loi de commande (6) transforment (13) en le système cible suivant

$$\begin{cases} \dot{X}(t) = (A + BK)X(t) + Bw(0, t), \\ Dw_t(x, t) = w_x(x, t), \\ w(1, t) = 0. \end{cases} \quad (15)$$

où  $K$  est le gain de rétroaction de la loi de commande (6), choisi tel que  $A + BK$  soit Hurwitz. En raison de la condition de bord  $w(1, t) = 0$ , la variable de backstepping  $w$  converge alors vers zéro en temps fini, après  $D$  unités de temps, et le système en boucle fermée suit alors pour  $t \geq D$  un comportement nominal non-retardé (et exponentiellement stable). Par conséquent, le système dynamique (15) représente bien la compensation du retard d'entrée réalisée par le contrôle par prédiction (6).

À noter que la transformation (14) peut en fait être réécrite comme  $w(x, t) = v(x, t) - KX(t + Dx)$ , en utilisant la formule de variation de la constante pour (1) entre le temps  $t$  et  $t + Dx$ . Ainsi, la variable de backstepping  $w$  n'est rien d'autre que la superposition de deux phénomènes de transport : celui de la quantité future  $KX(t + D)$  et celui de l'entrée.

Alternativement, la stabilité exponentielle du système cible (15) peut être prouvée avec des arguments de Lyapunov, en introduisant la fonction de Lyapunov candidate

$$V(X, w) = X(t)^T P X(t) + bD \int_0^1 (1+x)w(x, t)^2 dx, \quad (16)$$

où  $P$  est la solution définie positive symétrique de l'équation de Lyapunov  $P(A + BK) + (A + BK)^T P = -Q$  pour une matrice définie positive donnée  $Q$  et  $b$  est une constante positive à déterminer. En effet, la dérivée de  $V$  le long des trajectoires de (15) s'écrit

$$\begin{aligned} \dot{V} &= -X(t)^T Q X(t) + 2X(t)^T P B w(0, t) + 2b \int_0^1 (1+x)w(x, t)w_x(x, t)dx \\ &= -X(t)^T Q X(t) + 2X(t)^T P B w(0, t) + 2bw(1, t)^2 - bw(0, t)^2 - b \int_0^1 w(x, t)^2 dx, \end{aligned} \quad (17)$$

où la dernière égalité est obtenue avec une intégration par parties. Alors, en appliquant l'inégalité de Young, nous avons

$$\begin{aligned} \dot{V} &\leq -\frac{\min \lambda(Q)}{2} |X(t)|^2 - \left( b - \frac{2|PB|^2}{\min \lambda(Q)} \right) w(0, t)^2 - b \int_0^1 w(x, t)^2 dx \\ &\leq -\min \left\{ \frac{\min \lambda(Q)}{2 \max \lambda(P)}, \frac{b}{D} \right\} V - \left( b - \frac{2|PB|^2}{\min \lambda(Q)} \right) w(0, t)^2. \end{aligned} \quad (18)$$

Par conséquent, en choisissant  $b \geq \frac{2|PB|^2}{\min \lambda(Q)}$ , la stabilité exponentielle du système cible s'ensuit, avec le taux de décroissance  $\eta = \min \left\{ \frac{\min \lambda(Q)}{2 \max \lambda(P)}, \frac{b}{D} \right\}$ .

Ces éléments ne constituent pas seulement des points de vue ou des preuves alternatives au concept de compensation de retard exposé dans la section 2.1. Ils comportent également deux caractéristiques qui peuvent se révéler utiles dans des contextes similaires :

- l'EDP de transport dans (13) introduit une paramétrisation explicite du retard, qui facilite l'analyse de la robustesse par rapport au retard et est également compatible avec le contrôle adaptatif (voir [IS12]) ;
- le système cible (15) peut être analysé avec la fonctionnelle de Lyapunov (16). Ainsi, cette représentation par une EDP de transport du retard, reformulée avec une transformation back-stepping, fournit un outil d'analyse de Lyapunov-Krasovskii, permettant également d'étudier la sensibilité au retard.

Par conséquent, ces outils ont été largement utilisés au cours de la dernière décennie pour obtenir des résultats de robustesse par rapport au retard pour des contrôleurs par prédiction, non seulement dans le cas d'un retard constant et d'une dynamique linéaire [BPK10] mais également pour des perturbations de retard complexes, dépendant du temps et de l'état, pour une dynamique non-linéaire [BLK13b, BLK13a]. De même, ce cadre a été récemment utilisé pour proposer une approche systématique de conception de contrôleurs adaptatifs au retard [Krs09a, ZK20].

De ce fait, ces outils se révèlent particulièrement intéressants dans le contexte des retards stochastiques étudiés dans ce manuscrit.

### 3. Modélisation des retards stochastiques et analyse de stabilité

Cette thèse étudie le problème des équations différentielles affectées par des retards aléatoires à temps continu.

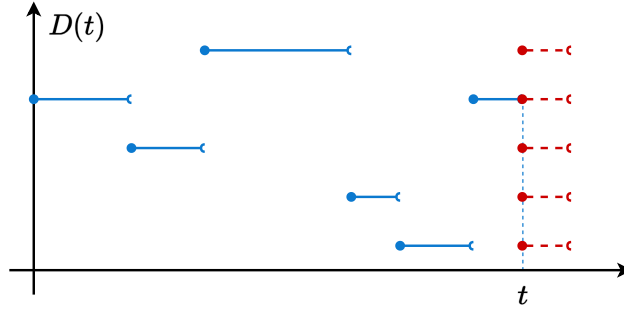
De nombreuses études ont abordé le cas d'un retard à temps discret, telles que [KÖC<sup>+</sup>94, NBW98, KMR03, GQOM14], ou d'équation différentielle stochastique à retard [HL93, IN64, KM13, Moh84, SZ18, CGMP20]. Parmi ces travaux, il convient de mentionner [CGMP20] qui a étudié le contrôle par prédiction d'une équation différentielle stochastique à retard linéaire, mais avec un retard d'entrée déterministe, même si variant dans le temps.

Néanmoins, seuls quelques travaux ont considéré le cas où le retard est une variable stochastique en temps continu. À notre connaissance, ces études sont [KM01a, KM01b, LM21, VM09b, VM09a, SSG<sup>+</sup>20]. Récemment, [VM09b, VM09a] a modélisé le retard sous la forme d'une fonction en dents de scie, dont les sauts se produisent à des instants qui sont les temps d'arrivée d'un processus de Poisson homogène. La dynamique considérée est donc en quelque sorte spécifique, et se rapproche plus à d'un système à données échantillonnées, mais stochastique. En conséquence, l'analyse de stabilité effectuée dans cette approche mélange la stabilité en moment et des approches de Lyapunov qui sont usuellement utilisées pour les systèmes à transmissions événementielles. Une autre approche récente similaire est celle de [SSG<sup>+</sup>20, SBO19] qui considère le retard comme un processus stochastique constant par morceaux de période constante connue, avec des valeurs en nombre fini et des probabilités de transition stationnaires. L'hypothèse selon laquelle les sauts de retard ne se produisent qu'à des instants fixes périodiques est également très similaire au cas spécifique des systèmes à données échantillonnées. Enfin, des études moins récentes de l'école russe [Kat67, KM00, KM01a] proposent de modéliser le retard comme un processus de Markov à nombre fini d'états. Cette approche permet de développer une analyse de stabilité générique. C'est l'une des raisons qui nous ont conduit à adopter ce modèle dans ce manuscrit, que nous allons maintenant détailler.

#### 3.1. Modélisation du retard avec un processus de Markov

Dans [KK60], le retard affectant un système dynamique est supposé être un processus de Markov avec un nombre fini d'états, c'est-à-dire,  $D(t) \in \{D_i, i \in \{1, \dots, r\}\}$  pour  $r \in \mathbb{N}$ .

Plus en détail, le retard  $D$  est modélisé par un processus stochastique  $\{D(t), t \geq 0\}$  d'un espace mesurable  $(\Omega, \mathcal{A})$  à  $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ , où  $\mathcal{B}(\mathbb{R})$  est la tribu de Borel de  $\mathbb{R}$ . Le retard est supposé être un

Figure 4: Exemple de réalisations de retard avec  $r = 5$  valeurs possibles.

processus de Markov, ce qui signifie qu'il satisfait la propriété de Markov,

$$\begin{aligned} Pr(D(t) = D_j | D(t_0) = D_i, \{D(u), 0 \leq u < t_0\}) &= Pr(D(t) = D_j | D(t_0) = D_i) \\ &\triangleq P_{ij}(t_0, t) \text{ for any } 0 \leq t_0 \leq t, \end{aligned} \quad (19)$$

c'est-à-dire que la réalisation du retard  $\{D(t) | t \geq t_0\}$ , étant donné l'état de retard initial  $D(t_0)$ , est indépendante des réalisations de retard passées  $\{D(u) | 0 \leq u < t_0\}$ . Les probabilités conditionnelles  $P_{ij} : (t_1, t_2) \in \mathbb{R}^2 \rightarrow [0, 1]$  introduites dans (19), qui quantifient la probabilité de passer de  $D_i$  au temps  $t_1$  à  $D_j$  au temps  $t_2$  ( $(i, j) \in \{1, \dots, r\}^2, t_2 \geq t_1 \geq 0$ ), sont appelées les probabilités de transition.

Les réalisations de ce processus de Markov sont supposées continues à droite, afin de pouvoir garantir l'existence de solutions pour les systèmes dynamiques en boucle ouverte ou fermée affectés par ce retard. Enfin, le retard ne peut prendre qu'un nombre fini de valeurs  $D_j$  que nous supposons, sans nuire à la généralité, ordonnées  $0 < D_1 < D_2 < \dots < D_r$  pour  $r \in \mathbb{N}$ . La Figure 4 montre un exemple de réalisation d'un tel retard.

Considérer un nombre fini de valeurs de retard est une pratique standard considérée dans de nombreuses références fournies précédemment, par exemple dans [KM01a, SBO19]. Dans le cas de systèmes en réseau, ces valeurs discrètes peuvent être considérées comme une mesure de l'état de congestion du réseau. De même, modéliser le retard par un processus de Markov découle du fait que la congestion d'un réseau dépend principalement de son état actuel ou de son dernier état connu.

Il est important de préciser que, sous ces hypothèses et si les probabilités de transition sont des fonctions régulières, il existe des fonctions positives  $\tau_{ij}, c_j = \sum_{k=1}^r \tau_{jk}$  ( $i, j \in \{1, \dots, r\}$ ) telles que  $\tau_{ii} = 0$  et que les probabilités de transition satisfont

$$\begin{aligned} \frac{\partial P_{ij}(s, t)}{\partial t} &= -c_j(t)P_{ij}(s, t) + \sum_{k=1}^r P_{ik}(s, t)\tau_{kj}(t), \quad s < t, \\ P_{ii}(s, s) &= 1 \quad \text{and} \quad , P_{ij}(s, s) = 0 \quad \text{for } i \neq j. \end{aligned} \quad (20)$$

Cette équation est connue sous le nom d'équation de Kolmogorov en avant (voir [Ros14, RH03]). Comme elle est fréquemment utilisée dans la suite de ce manuscrit, nous fournissons brièvement sa preuve, telle que proposée dans [RH03].

Considérons l'intervalle  $(0, t + \Delta t]$  pour  $t, \Delta t > 0$ . Divisons cet intervalle en deux parties  $(0, t]$  et  $(t, t + \Delta t]$ . Du fait de la propriété de Markov, la relation suivante (appelée équation de Chapman-Kolmogorov) peut être obtenue

$$\begin{aligned} P_{ij}(0, t + \Delta t) &= Pr(D(t + \Delta t) = D_j | D(0) = D_i) \\ &= \sum_{k=1}^r Pr(D(t + \Delta t) = D_j | D(0) = D_i, D(t) = D_k) Pr(D(t) = D_k | D(0) = D_i) \end{aligned} \quad (21)$$



$$\begin{aligned}
 &= \sum_{k=1}^r Pr(D(t + \Delta t) = D_j \mid D(t) = D_k) Pr(D(t) = D_k \mid D(0) = D_i) \\
 &= \sum_{k=1}^r P_{ik}(0, t) P_{kj}(t, t + \Delta t).
 \end{aligned}$$

où nous avons utilisé la définition de la probabilité conditionnelle et le fait que les événements  $\{D(t) = D_k\}$  pour  $k = 1, \dots, r$  sont indépendants. En soustrayant  $P_{ij}(0, t)$  des deux côtés de cette équation, on obtient

$$P_{ij}(0, t + \Delta t) - P_{ij}(0, t) = \sum_{k=1, k \neq j}^r P_{ik}(0, t) P_{kj}(t, t + \Delta t) - (1 - P_{jj}(t, t + \Delta t)) P_{ij}(0, t). \quad (22)$$

En divisant cette équation par  $\Delta t$  puis en prenant la limite pour  $\Delta t \rightarrow 0$ , on obtient donc

$$\begin{aligned}
 &\lim_{\Delta t \rightarrow 0} \frac{P_{ij}(0, t + \Delta t) - P_{ij}(0, t)}{\Delta t} \\
 &= \sum_{k=1, k \neq j}^r P_{ik}(0, t) \lim_{\Delta t \rightarrow 0} \frac{P_{kj}(t, t + \Delta t)}{\Delta t} - \lim_{\Delta t \rightarrow 0} \frac{1 - P_{jj}(t, t + \Delta t)}{\Delta t} P_{ij}(0, t) \\
 &= \sum_{k=1, k \neq j}^r P_{ik}(0, t) \underbrace{\lim_{\Delta t \rightarrow 0} \frac{P_{kj}(t, t + \Delta t) - P_{kj}(t, t)}{\Delta t}}_{\triangleq \tau_{kj}(t)} - \underbrace{\lim_{\Delta t \rightarrow 0} \frac{1 - P_{jj}(t, t + \Delta t)}{\Delta t}}_{\triangleq c_j(t)} P_{ij}(0, t),
 \end{aligned} \quad (23)$$

comme  $P_{ij}(t, t) = 0$  pour  $i \neq j$ . Il s'agit bien de l'équation de Kolmogorov en avant (20), définissant en outre  $\tau_{ii}(t) = 0$ . Notez que les limites ci-dessus sont bien définies sous l'hypothèse que les probabilités de transition sont des fonctions différentiables.

Enfin, du fait que  $\sum_{j=1}^r P_{ij}(0, t) = 1$ , on peut facilement déduire des définitions ci-dessus de  $\tau_{kj}$  et  $c_j$  que

$$\begin{aligned}
 c_j(t) - \sum_{k=1}^r \tau_{jk}(t) &= \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \left( 1 - P_{jj}(t, t + \Delta t) - \sum_{k=1, k \neq j}^r P_{jk}(t, t + \Delta t) \right) \\
 &= \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \underbrace{\left( 1 - \sum_{k=1}^r P_{jk}(t, t + \Delta t) \right)}_{=0},
 \end{aligned} \quad (24)$$

c'est-à-dire que  $c_j(t) = \sum_{k=1}^r \tau_{jk}(t)$  pour tout temps  $t$ .

Selon ces définitions, on observe que pour un temps d'échantillonnage  $\Delta t$  fixe et suffisamment petit,  $\tau_{ij}\Delta t \approx P_{ij}(t, t + \Delta t)$ . Ainsi,  $\tau_{ij}\Delta t$  peut être interprété comme la probabilité de transition de  $D_i$  à  $D_j$  sur l'intervalle  $[t, t + \Delta t)$ . De même,  $1 - c_j(t)\Delta t$  représente la probabilité de rester à  $D_j$  pendant l'intervalle de temps  $[t, t + \Delta t)$ .

Cette modélisation des retards permet par ailleurs de développer une analyse de stabilité générique.

### 3.2. Moyennisation Probabiliste de Retard

Modéliser le retard sous la forme de processus de Markov à nombre fini d'états permet alors de considérer chaque valeur de retard fixe séparément pour l'analyse de stabilité, en appliquant la technique connue sous le nom de Moyennisation Probabiliste de Retard [KM00], que nous allons maintenant présenter.

Considérons la dynamique

$$\dot{X}(t) = f(X(t), X(t - D(t))), \quad (25)$$

où  $X \in \mathbb{R}^n$  et  $D$  est un processus de Markov satisfaisant les propriétés précédentes. On peut montrer que  $\Psi = (X, D)$  est un processus de Markov en temps continu (nous l'admettons pour plus de clarté). Ainsi, considérons le générateur infinitésimal  $L$  [Dyn65], qui est défini, pour toute fonction régulière  $V : \Psi = (x, D) \in \mathbb{R}^n \times \mathbb{R} \mapsto V(\Psi) \in \mathbb{R}$ , comme

$$LV(\Psi(t)) = \lim_{s \rightarrow 0^+} \frac{\mathbb{E}_{[t, \Psi(t)]}(V(\Psi(t+s))) - V(\Psi(t))}{s}, \quad (26)$$

où  $\mathbb{E}_{[t, \Psi(t)]}(V(\Psi(t+s)))$  est l'espérance conditionnelle de  $V(\Psi(t+s))$ . Puisque  $D$  est un processus avec un nombre fini d'états  $D_j$  ( $j = 1, \dots, r$ ), et comme la probabilité d'être dans l'état  $D_j$  au temps  $t$  sachant que  $D(0) = D_i$  est  $P_{ij}(0, t)$ , alors

$$\mathbb{E}_{[0, \Psi(0)]}LV(\Psi(t)) = \sum_{j=1}^r P_{ij}(0, t) \mathbb{E}_{[0, \Psi(0)]}LV((X(t), D_j)). \quad (27)$$

De plus, par linéarité de l'espérance,

$$\begin{aligned} LV((X(t), D_j)) &= \lim_{s \rightarrow 0^+} \frac{1}{s} \left( \mathbb{E}_{[t, (X(t), D_j)]}(V((X(t+s), D(t+s)))) - V((X(t), D_j)) \right) \\ &= \lim_{s \rightarrow 0^+} \frac{1}{s} \left( \mathbb{E}_{[t, (X(t), D_j)]}(V((X(t+s), D(t+s)))) - \mathbb{E}_{[t, (X(t), D_j)]}(V((X(t+s), D_j))) \right) \\ &\quad + \lim_{s \rightarrow 0^+} \frac{1}{s} \left( \mathbb{E}_{[t, (X(t), D_j)]}(V((X(t+s), D_j))) - V((X(t), D_j)) \right). \end{aligned} \quad (28)$$

Considérons les deux termes de (28) séparément. Comme précédemment, puisque  $D$  est un processus avec un nombre fini d'états  $D_j$ , il s'ensuit que

$$\begin{aligned} &\lim_{s \rightarrow 0^+} \frac{1}{s} \left( \mathbb{E}_{[t, (X(t), D_j)]}(V((X(t+s), D(t+s)))) - \mathbb{E}_{[t, (X(t), D_j)]}(V((X(t+s), D_j))) \right) \quad (29) \\ &= \lim_{s \rightarrow 0^+} \frac{1}{s} \left( \sum_{k=1}^r P_{jk}(t, t+s) \mathbb{E}_{[t, (X(t), D_j)]}(V((X(t+s), D_k))) - \mathbb{E}_{[t, (X(t), D_j)]}(V((X(t+s), D_j))) \right) \\ &= \lim_{s \rightarrow 0^+} \left( \sum_{k=1, k \neq j}^r \frac{P_{jk}(t, t+s)}{s} \mathbb{E}_{[t, (X(t), D_j)]}(V((X(t+s), D_k))) \right. \\ &\quad \left. + \frac{P_{jj}(t, t+s) - 1}{s} \mathbb{E}_{[t, (X(t), D_j)]}(V((X(t+s), D_j))) \right) \\ &= \sum_{k=1, k \neq j}^r \tau_{jk}(t) V((X(t), D_k)) - c_j(t) V((X(t), D_j)), \end{aligned}$$

où la dernière ligne provient des définitions de  $\tau_{jk}$  et  $c_j$  dans (23). De plus, comme les réalisations de  $D$  sont continues à droite, il donc existe un  $\epsilon > 0$  suffisamment petit tel que  $D(t+s) = D(t) = D_j$  pour  $s \in [0, \epsilon)$ . Par conséquent, le deuxième terme de (28) vaut

$$\lim_{s \rightarrow 0^+} \frac{1}{s} \left( \mathbb{E}_{[t, (X(t), D_j)]}(V((X(t+s), D_j))) - V((X(t), D_j)) \right) = \frac{\partial V}{\partial X}(X(t), D_j) f(X(t), X(t - D_j)). \quad (30)$$

Ainsi, (27) peut être réécrit comme

$$\begin{aligned} \mathbb{E}_{[0, \Psi(0)]}LV(\Psi(t)) &= \mathbb{E}_{[0, \Psi(0)]} \left( \sum_{j=1}^r P_{ij}(0, t) \frac{\partial V}{\partial X}(X(t), D_j) f(X(t), X(t - D_j)) \right. \\ &\quad \left. + \sum_{j=1}^r P_{ij}(0, t) \left[ \sum_{k=1, k \neq j}^r \tau_{jk}(t) V((X(t), D_k)) - c_j(t) V((X(t), D_j)) \right] \right). \end{aligned} \quad (31)$$

En utilisant l'équation de Kolmogorov en avant (20), le dernier terme de (31) se simplifie comme

$$\begin{aligned}
 \sum_{j=1}^r P_{ij}(0, t) \left[ \sum_{k=1, k \neq j}^r \tau_{jk}(t) V((X(t), D_k)) - c_j(t) V((X(t), D_j)) \right] \\
 = \sum_{j=1}^r \left[ \sum_{k=1, k \neq j}^r \tau_{kj}(t) P_{ik}(0, t) - c_j P_{ij}(0, t) \right] V((X(t), D_j)) \\
 = \sum_{j=1}^r \frac{\partial P_{ij}}{\partial t}(0, t) V((X(t), D_j)). \tag{32}
 \end{aligned}$$

Finalement, nous obtenons ainsi

$$\begin{aligned}
 \mathbb{E}_{[0, \Psi(0)]} LV(\Psi(t)) = \mathbb{E}_{[0, \Psi(0)]} \left( \sum_{j=1}^r P_{ij}(0, t) \frac{\partial V}{\partial X}(X(t), D_j) f(X(t), X(t - D_j)) \right. \\
 \left. + \sum_{j=1}^r \frac{\partial P_{ij}}{\partial t}(0, t) V((X(t), D_j)) \right), \tag{33}
 \end{aligned}$$

ou encore

$$\begin{aligned}
 \mathbb{E}_{[0, \Psi(0)]} LV(\Psi(t)) = \mathbb{E}_{[0, \Psi(0)]} \left( \sum_{j=1}^r P_{ij}(0, t) \left[ \frac{\partial V}{\partial X}(X(t), D_j) f(X(t), X(t - D_j)) \right. \right. \\
 \left. \left. + \sum_{k=1}^r \tau_{jk}(t) (V((X(t), D_k)) - V((X(t), D_j))) \right] \right) \\
 \triangleq \mathbb{E}_{[0, \Psi(0)]} \left( \sum_{j=1}^r P_{ij}(0, t) L_j V(t) \right), \tag{34}
 \end{aligned}$$

qui est lié à l'espérance conditionnelle de  $V$  via la formule de Dynkin [Dyn65, Théorème 5.1, p. 133]

$$\mathbb{E}_{[0, \Psi(0)]} V(\Psi(t)) = V(\Psi(0)) + \int_0^t \mathbb{E}_{[0, \Psi(0)]} LV(\Psi(s)) ds. \tag{35}$$

Par conséquent, l'analyse de stabilité du système peut être effectuée en considérant chaque valeur de retard fixe séparément via le générateur de retard infinitésimal à retard fixé  $L_j V$ , puis en prenant la moyenne au sens de (33) ou (34). À noter que, comme souligné dans [KM00], la variation  $L_j V((X(t), D_j))$  impliquée dans (34) peut être comprise intuitivement par le fait que  $\Psi(t + s) - (X(t), D_j)$  est le résultat de deux facteurs. Le premier terme est déterminé par la dynamique à retard fixe  $D_j$  avec une probabilité proche de 1. Le second correspond aux "sauts" du processus  $D$  de l'état  $D_j$  à l'état  $D_k$ , qui se produit avec la probabilité  $\tau_{jk}(t)$ .

Pour illustrer la philosophie du concept de Moyennisation Probabiliste de Retard, prenons pour exemple la dynamique

$$\dot{X}(t) = -X(t) + D(t)X(t), \tag{36}$$

avec  $X \in \mathbb{R}$ . Même si  $D$  ne joue pas dans (36) le rôle d'un retard, mais d'un paramètre linéaire, l'étude de ce système illustre les principaux mécanismes en jeu.

Définissons la fonction de Lyapunov candidate  $V(X) = X^2$ . Considérons une valeur de retard fixe  $D_j$ . Comme  $V$  ne dépend pas de  $D$ , nous pouvons obtenir

$$L_j V(X(t)) = 2X(t)(-X(t) + D_j X(t)) = -2(1 - D_j)X(t)^2, \tag{37}$$

et la formule de Dynkin implique alors

$$\mathbb{E}_{[0, \Psi(0)]} V(\Psi(t)) = V(\Psi(0)) - \int_0^t 2 \left( 1 - \sum_{j=1}^r P_{ij}(0, s) D_j \right) \mathbb{E}_{[0, \Psi(0)]} V(\Psi(s)) ds, \quad (38)$$

d'où l'on conclut que le système est exponentiellement stable si l'espérance du retard  $\sum_{j=1}^r P_{ij}(0, t) D_j$  satisfait, par exemple, pour un  $\gamma$  positif donné,  $1 - \sum_{j=1}^r P_{ij}(0, t) D_j \geq \gamma > 0$ . Ceci est cohérent avec le fait qu'en raison de la linéarité de (36), l'espérance de l'état satisfait la dynamique de retard moyennée  $\dot{X}(t) = (1 - \sum_{j=1}^r P_{ij}(0, t) D_j) X(t)$ .

Ainsi, cette approche permet d'utiliser pour l'analyse de stabilité de Lyapunov une approche à retard constant facile à appréhender. Ceci a inspiré la méthode d'analyse de stabilité proposée dans ce manuscrit pour les systèmes à des retards d'entrée stochastiques.

#### 4. Méthodologie et contributions de cette thèse

Dans les sections précédentes, nous avons exposé les principaux ingrédients utilisés dans ce manuscrit pour concevoir un contrôle par prédiction en présence d'un retard d'entrée stochastique et analyser la stabilité de la boucle fermée correspondante, c'est-à-dire réécrire le retard d'entrée sous forme d'EDP de transport, appliquer une transformation de backstepping correspondant au contrôle par prédiction et enfin appliquer l'approche de Moyennisation Probabiliste de Retard au système cible généré. Comme nous le montrerons, ceci permet de proposer une procédure générique à la fois pour la conception du contrôle et l'analyse de stabilité de la boucle fermée correspondante. Plus précisément, en supposant que le retard soit un processus de Markov tel que présenté précédemment avec un nombre fini de valeurs, notre méthodologie repose sur les trois étapes suivantes :

- Représentation du phénomène de transport engendré par le retard, en introduisant une EDP de transport pour chaque valeur de retard potentielle ;
- Transformation de backstepping de la cascade des EDPs de transport dans une EDO ;
- Analyse de stabilité de Lyapunov du système cible correspondant, basée sur l'approche de Moyennisation Probabiliste de Retard.

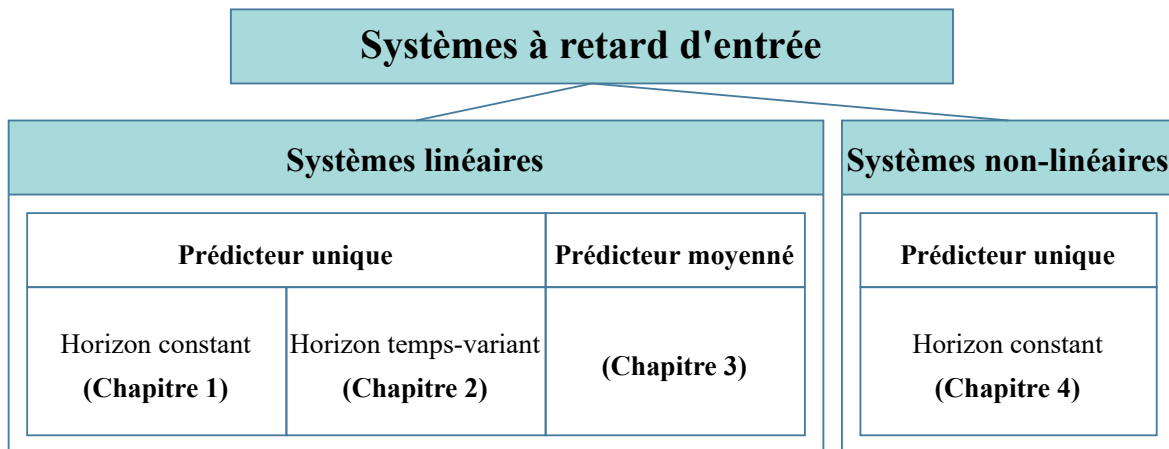


Figure 5: Présentation de l'organisation de cette thèse.

Une représentation générale du contenu de cette thèse est illustrée en Figure 5. Le manuscrit comporte 4 chapitres :

**Le Chapitre 1** considère le cas d'un système LTI soumis à un retard d'entrée aléatoire. Il étudie sa stabilisation par un contrôleur par prédiction à horizon constant. Tout d'abord, il est prouvé que l'horizon de prédiction doit être choisi suffisamment proche des valeurs prises par le retard, conformément à ce que l'intuition laisserait penser. Cette condition restrictive requiert alors que l'horizon de prédiction soit proche de chaque valeur potentielle de retard, et donc que les valeurs de retard soient toutes suffisamment proches.

Puis, en considérant une fonctionnelle de Lyapunov différente, nous parvenons à modifier cette condition déterministe suffisante sous une forme probabiliste qui tient compte de la distribution des retards. Cette deuxième condition généralise la première puisqu'elle requiert seulement que l'horizon de prédiction soit choisi suffisamment proche des valeurs de retard *en moyenne*.

**Le Chapitre 2** étend la conception du Chapitre 1 en considérant un horizon de prédiction variant dans le temps. Cela permet d'actualiser l'horizon de prédiction considéré au cours du temps et rend ainsi le contrôleur applicable à un plus large éventail de scénarios. Il est prouvé que, en plus d'exiger que la plage de prédiction reste suffisamment proche de la valeur de retard moyenne, tout comme dans le Chapitre 1, le taux de variation de l'horizon de prédiction doit également être limité.

**Le Chapitre 3** présente une conception alternative de contrôle par prédiction pour les systèmes LTI à retard d'entrée aléatoire. Contrairement aux approches basées sur un seul prédicteur des chapitres précédents, nous proposons dans ce chapitre de considérer plusieurs prédicteurs, un pour chaque valeur de retard potentielle, et d'en prendre la moyenne.

Nous illustrons numériquement sur plusieurs exemples que ce contrôleur surpasse le contrôleur précédent en termes de robustesse par rapport au retard, et fournissons quelques pistes d'explication sur les causes de cette différence de performances.

Enfin, **le Chapitre 4** étend le contrôle par prédicteur des Chapitres 1 et 2, mais dans un contexte non linéaire, ce qui complexifie substantiellement l'analyse de stabilité. Tout comme dans le Chapitre 1, nous considérons un contrôleur à horizon de prédiction constant et démontrons qu'il est possible de stabiliser le système, à condition que l'horizon de prédiction reste suffisamment proche des valeurs de retard.

Enfin, en conclusion, nous détaillons les perspectives de cette thèse.

## 5. Publications

Les travaux présentés dans cette thèse ont donné lieu aux publications suivantes :

- Journaux

1. SiJia Kong and Delphine Bresch-Pietri. "Prediction-based controller for linear systems with stochastic input delay." *Automatica* 138 (2022): 110149.
2. SiJia Kong and Delphine Bresch-Pietri. "Probabilistic Sufficient Conditions for Prediction-based Stabilization of Linear Systems with Random Input Delay." *IEEE Control Systems Letters* 6, 2270-2275, 2022. (With inclusion in the program of the 2022 American Control Conference)
3. Jean Auriol, SiJia Kong, Delphine Bresch-Pietri. "Explicit Prediction-Based Control for Linear Difference Equations With Distributed Delays", *IEEE Control Systems Letters* 6, 2864-2869, 2022 (With inclusion in the program of the 2022 Conference on Decision and Control)

- Conférences internationales

1. SiJia Kong and Delphine Bresch-Pietri. "Constant time horizon prediction-based control for linear systems with time-varying input delay." *IFAC-PapersOnLine* 53.2 (2020): 7665-7670.

- Chapitre d'ouvrages

1. SiJia Kong and Delphine Bresch-Pietri. "Prediction Control for Nonlinear Systems with Stochastic Input Delay." *Advances in Distributed Parameter Systems*. Springer, Cham, 2022. 27-44.



# Introduction (English Version)

## 1. Context: Stochastic delays in control systems

The study of delay systems is a problem of recurring interest due to the ubiquity of delays in applications. Time lags are well-known as a potential source of instability in dynamical systems and can make control design a challenging task by inducing complex behaviors [GN03]. For instance, small delays may destabilize some systems [HL01], while large ones may stabilize others [BM75, Mac86]. Other surprising cases where voluntary delay introductions can actually benefit the control are reported in [Ric03] and include deadbeat controllers [WNK96] and delayed resonators [JO98], to name a few.

When delays are assumed to be constant, well-established spectral methods [MN07] may allow for analyzing the stability of time-delay systems and bifurcations of equilibria and, thus, design controllers. Yet, contrary to the finite-dimensional case, the characteristic equation of a delay system admits an infinite number of roots, the behavior and sensitivity of which may be complicated to study and thus still require careful and advanced investigations [MBN17, JGNB18, IGBN18]. Unfortunately, these frequency-domain tools are not tailored to handle time-varying delays.

Nevertheless, time-varying delays are ubiquitous in applications. For instance, in communication, data transmission is always accompanied by a difference between the sending time and the delivery time of a signal. In a network system, this information can be transmitted through different channels as depicted in Figure 1b, depending on the congestion state of the network. Hence, the transmission time depends on the routine algorithm and will change over time. With the rise of information and communication technologies, the last decade has witnessed an enormous interest in the study of such systems [CJO<sup>+</sup>11, HNX07]. When sensor and control data are transmitted over a digital communication channel, sampled-data systems [ÅW13] are a natural tool for system modeling and control design [CHVdW<sup>+</sup>10]. In that context, systems with a periodic sampling [HFO<sup>+</sup>17] reveal a handy tool to capture some features of Networked Control Systems including sampling jitters, packet drop-outs or fluctuations due to the interactions between control algorithms and real-time scheduling protocols. Alternatively, another common approach for accounting for the phenomenon of packet-switching [HNX07] is to model the delay as a non-deterministic variable [Gup11, Sch08].

Indeed, stochastic delays have been commonly used to represent packet loss and reordering which often take place for connected vehicle systems for instance [HCCDV13]: in addition to the driver's reaction time, wireless vehicle-to-vehicle communication [BK06, ZO17] used to monitor vehicles

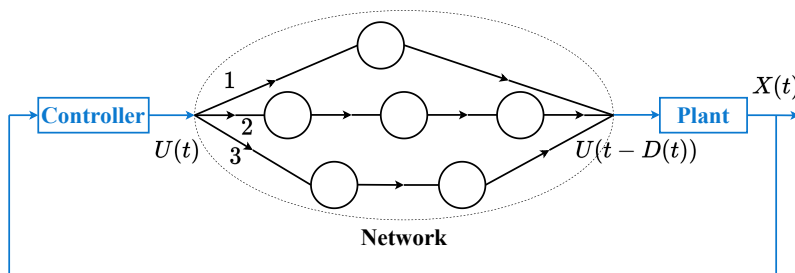


Figure 1b: Communication delay induced by a network may depend on the routine algorithm.



ahead when beyond the line of sight often introduces substantial communication delays and packet losses [BPDV15, MQIO17] while transmitting the remote vehicles information. Random delays also arise in the modeling of gene regulatory networks [SO16] as the execution times of transcription and translation processes are influenced by a noisy cell environment [JLO<sup>+</sup>11, Lew03].

The stability analysis and stabilization of discrete-time systems with stochastic delay have been investigated thoroughly in the literature, e.g. in [KÖC<sup>+</sup>94, NBW98, GQOM14]. These approaches typically recast the discrete-time delay as finite-dimensional dynamics by extending the system state, as usual for discrete-time delay systems. In this formulation, the random delay then acts as a random discrete-time parameter, and the stability of the closed-loop system is then rewritten as an optimal control/estimation problem as e.g. in [Gup11, Sch08]. The corresponding problem of minimum mean-squared error controller is obtained by solving the corresponding Riccati equation, as in the standard theory of Kalman filtering.

Yet, stabilization tools for continuous-time processes subject to continuous-time stochastic delays are lacking in general in the literature. Proposing a generic stabilization methodology for continuous-time dynamical systems subject to continuous-time stochastic input delay is the main objective of this dissertation. To achieve this goal, we propose to rely on prediction-based controllers which have strong merits in the deterministic delay case. Hence, we now present this control technique.

## 2. Prediction-based controller: state-of-art in stabilizing input-delay systems

### 2.1. The concept of prediction-based controllers

Let us consider a Linear Time-Invariant (LTI) Single-Input systems with constant input delay in the form of

$$\dot{X}(t) = AX(t) + BU(t - D), \quad (39)$$

in which  $X \in \mathbb{R}^n$ ,  $D$  is an input delay, and the matrix pair  $(A, B)$  is controllable. For the delay-free system ( $D = 0$ ), a control law ensuring closed-loop stabilization can be given as

$$U = KX(t), \quad (40)$$

in which the feedback gain  $K$  is such that  $A + BK$  is Hurwitz. The closed-loop dynamics then rewrites as  $\dot{X} = (A + BK)X$  which is exponentially stable. The concept of prediction-based control is to obtain the same closed-loop dynamics as in the delay-free case by *compensating* for the input delay. To do so, let us define the prediction of the system state  $X$  over a  $D$ -unit horizon as

$$P(t) = X(t + D), \quad (41)$$

and simply replace it in lieu of  $X(t)$  in (40) to define

$$U = KP(t). \quad (42)$$

This control law then implies that  $U(t - D) = KX(t)$  holds, which in turns implies  $\dot{X} = (A + BK)X$  for  $t \geq D$ . The delay effect has thus disappeared from the closed-loop dynamics, which is then easier to tune and analyze.

The main remaining problem is that the future system state  $X(t + D)$  of course cannot be measured as it is non-causal, a priori. Yet, an expression of it can be obtained by applying the variation of the constant formula to (39)

$$\begin{aligned} P(t) = X(t + D) &= e^{AD}X(t) + \int_t^{t+D} e^{A(t+D-s)}BU(s - D)ds \\ &= e^{AD}X(t) + \int_{t-D}^t e^{A(t-s)}BU(s)ds, \end{aligned} \quad (43)$$

This expression only depends on the system state at time  $t$  and past values of the input  $U$  on the time horizon  $(t - D, t)$ , which are known at time  $t$ . Therefore, it is implementable. The predictor feedback law finally writes as

$$U(t) = KP(t) = KX(t + D) = K \left[ e^{AD} X(t) + \int_{t-D}^t e^{A(t-s)} BU(s) ds \right]. \quad (44)$$

This design is of particular interest for non-negligible large-scale input delays, while the robustness of a nominal delay-free feedback law can be evoked for small ones.

This technique has been exposed in many studies since the 1950s. During this period, the well-known Smith Predictor was first proposed in 1959 [Smi57], to overcome the dead time delay in the open-loop stable systems. The predictor feedback technique was then introduced by [May68, KP80, MO79, Wat86, WNKI92] within the framework of ‘finite spectrum assignment’ and as the ‘reduction model’ in [Art82], to handle general time-invariant input-delay systems. In this manuscript, we refer to this technique under the name ‘prediction-based controller’.

This concept of prediction-based controller has been applied in several more general contexts (state and input delays, multiple input delays,...). We now present a few extensions which are relevant to the problems considered in this manuscript.

### 2.1.1. Linear time-invariant (LTI) system with time-varying input delay

Even if the seminal paper by Artstein [Art82] was able to cope with time-varying delay, the specific explicit prediction design for time-varying input delay was obtained only later by [Nih91]. This formulation was proposed for the LTI system

$$\dot{X}(t) = AX(t) + BU(t - D(t)), \quad (45)$$

in which the time-varying input delay  $D$  is supposed to be a bounded time-differentiable function, such that  $\dot{D}(t) < 1$  for all time. In this case, we introduce the delay function  $\phi(t) = t - D(t)$  and, as previously, wish to compute a prediction such that  $U(\phi(t)) = KX(t)$ . Thus, the predictor feedback control should write as

$$\begin{aligned} U(t) = KX(\phi^{-1}(t)) &= K \left[ e^{A(\phi^{-1}(t)-t)} X(t) + \int_t^{\phi^{-1}(t)} e^{A(\phi^{-1}(t)-s)} BU(\phi(s)) ds \right] \\ &= K \left[ e^{A(\phi^{-1}(t)-t)} X(t) + \int_{\phi(t)}^t e^{A(\phi^{-1}(t)-\phi^{-1}(s))} B \frac{U(s)}{\dot{\phi}(\phi^{-1}(s))} ds \right]. \end{aligned} \quad (46)$$

in which we applied the variation of constant formula to (45), as previously.

Notice that, to design such a prediction-based control law, the function  $\phi(t) = t - D(t)$  is required to be invertible. This is guaranteed by the fact that  $\dot{D}(t) < 1$  for all time, which implies that  $\dot{\phi}(t) > 0$  and thus that the function  $\phi$  is increasing. It is also worth observing that the assumption  $\dot{D}(t) < 1$  is satisfied for a large class of delays sometimes called hydraulic delays [CP19] or transport delays [BPP16], and which stem from the physical transportation of a material.

Let us also observe that, of course, if the input delay is constant, the inverse of the function  $\phi$  is  $\phi^{-1}(t) = t + D$ , and one recovers the prediction-based controller previously presented in (44). However, in the general case, the prediction horizon is not simply equal to the current delay value, that is,  $\phi^{-1}(t) \neq t + D(t)$ . In fact, computing this inverse requires to have some knowledge on the future delay variations, as illustrated in Figure 2b, which may not be the case in practice. This is why some approaches rely on an inexact delay compensation.

**Robust compensation with a prediction-based controller.** An alternative approach to the exact delay compensation provided by (46) is to rely on an approximate prediction horizon such as the

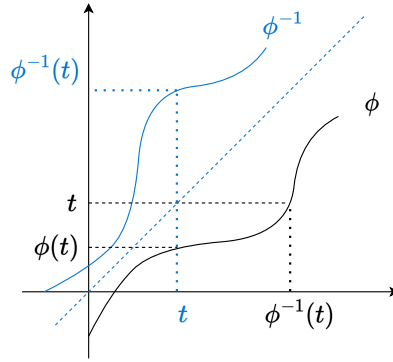


Figure 2b: Delay function  $\phi(t)$  and its inverse. The computation of  $\phi^{-1}(t)$  at a given time  $t$  requires the knowledge of future values of  $\phi$ .

following one proposed in [BPCP13]

$$U(t) = K \left[ e^{AD(t)} X(t) + \int_{t-D(t)}^t e^{A(t-s)} BU(s) ds \right], \quad (47)$$

The philosophy beyond this controller assumes that the delay will remain sufficiently close to  $D(t)$  in the near future and thus that (44) can directly be used with the current delay value  $D(t)$  instead of  $D$ . Yet, because there exists a mismatch between the functions  $\phi^{-1}(t)$  and  $t + D(t)$ , the exact compensation of the input delay is not achieved with this control law. However, under the condition that the delay variation is sufficiently small, the prediction-based controller (47) can still stabilize the delay system (45), as will be detailed in the sequel.

### 2.1.2. Nonlinear system with constant input delay

In [Krs09b], the first ‘predictor approach’ for nonlinear delay systems with arbitrary long constant input delays was developed, and was later brought to maturity in [BLK13a]. These works consider the nonlinear dynamics with a single input

$$\dot{X}(t) = f(X(t), U(t - D)), \quad (48)$$

where  $X \in \mathbb{R}^n$ ,  $U \in \mathbb{R}$ ,  $f \in C^1(\mathbb{R}^n \times \mathbb{R}; \mathbb{R}^n)$  and  $D \geq 0$  is a constant input delay.

The prediction-based control design then relies on two preliminary ingredients:

- the plant (48) is assumed to not escape in finite time, so that the closed-loop control law can actually reach the plant (at time  $D > 0$ ) before it explodes and could have any effect on the dynamics;
- the existence of a nominal delay-free feedback law  $\kappa : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}$  such that the dynamics  $\dot{X}(t) = f(X(t), \kappa(X(t), t))$  is asymptotically stable.

As previously, one can then rely on this nominal feedback law, fed with a prediction of the system state to compensate for the input delay. Namely, picking the prediction-based controller  $U(t) = \kappa(P(t), t + D)$  with the prediction defined implicitly as

$$P(t) = X(t + D) = X(t) + \int_{t-D}^t f(P(s), U(s)) ds, \quad (49)$$

yields the closed-loop dynamics  $\dot{X}(t) = f(X(t), \kappa(X(t), t))$  for  $t \geq D$ , that is, the nominal delay-free asymptotically stable dynamics. Notice that (49) is nothing more than the integral form of (48) with the initial condition  $P(t - D) = X(t)$ .

These elements were extended to time-varying delays in [BLK13a], as in the previous sections. The much more intricate cases of state-dependent delays, and some classes of input-dependent delays, were also investigated in [BLK12, BPCP13, DBLK17] for instance.

## 2.2. Delay sensitivity and challenges in extending prediction-based control for stochastic delays

Despite its strong merits, namely to eliminate the delay in closed-loop and thus to improve transient performances and to simplify its tuning, a prediction-based controller also exhibits some flaws. The most important one may be that prediction-based techniques suffer from being sensitive to delay mismatch (and, to a lesser extent, to plant parameters uncertainties) [Pal80, YS87, Fen91, SS93]. Hence, carefully designing the prediction horizon is a question of crucial importance.

Numerous works investigated the robustness of prediction-based controllers to such a mismatch. Most were devoted to the derivation of an upper-bound of admissible delay mismatch preserving stability, based on analysis in the frequency-domain for constant delays [OR82, ALC00, MNL01, Nic01, Zho06].

Similar delay-robustness properties were obtained recently in the time-domain for more complicated delay perturbations. The most important one is probably [BLK13b] which provides robust compensation results for a time-differentiable delay function potentially depending as well on the system state. Robust compensation is proved to hold under the assumptions that both the range of variation of the delay and its variation rate is sufficiently limited (and that the system initial condition allows the delay variation with respect to the state to remain small). A similar result was obtained in [KK13] but through a small-gain approach enabling to avoid restricting the delay variation rate, and sufficient quantifiable Linear Matrix Inequalities (LMI) conditions on the magnitude of the delay perturbations were then recently proposed in [LPS19]. The prediction-based designs detailed in this manuscript for a stochastic input delay ground on these delay robustness properties for a deterministic delay.

In the case of a stochastic delay, exact delay compensation seems like an unreasonable objective. The first reason for that is that the function  $\phi$  introduced in Section 2.1.1 is not smooth and has little reason to be invertible. The second one is that, in any case, this inverse function is not causal and, if the delay variation are random, cannot be predicted. Finally due to the random nature of the delay, exact delay compensation, even if possible, would result into a chattering control law, which would then be impossible to apply in practice, due to the limited response time of an actuator. For the same reason, even if the current delay realization  $D(t)$  is measured, applying it as horizon prediction in (47) would also result into a chattering control law, which in any case would not necessarily be close to or representative of the delay realizations yet to come. Grounding on the delay robustness properties previously mentioned, alternative and tailored prediction strategies are considered in this manuscript.

To obtain sufficient conditions for robust stochastic delay compensation, we propose to use the framework of Partial Differential Equations and the reformulation of the prediction-based controller as a specific boundary control of a transport equation, as introduced in [KS08].

## 2.3. A Partial Differential Equation (PDE) and backstepping viewpoint

### 2.3.1. Transport representation of the delay

Consider an input signal  $U$  delayed by  $D > 0$  (constant) units of time. This delay phenomenon is usually captured by the infinite-dimensional state  $U_t : s \in [-D, 0] \mapsto U(t+s)$ . It can be equivalently represented by defining the distributed input  $v(x, t) = U(t + D(x - 1))$  for  $x \in [0, 1]$  which then

satisfies the following transport PDE

$$\begin{cases} Dv_t(x, t) = v_x(x, t) \\ v(1, t) = U(t), \end{cases} \quad (50)$$

in which  $v_t$  and  $v_x$  denote the partial derivatives with respect to time and space, respectively. This first-order dynamics is the simplest of the hyperbolic PDEs, transport of the quantity  $U(t)$  from  $x = 1$  up to  $x = 0$  with a speed  $1/D$  resulting in the opposite boundary value  $v(0, t) = U(t - D)$ . This equivalence is illustrated in Figure 3b.

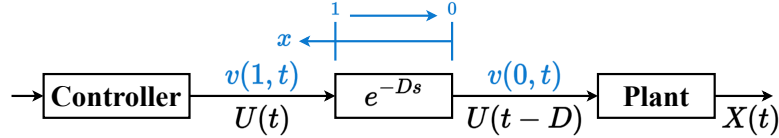


Figure 3b: Distributed actuator  $v(x, t)$  for  $x \in [0, 1]$ .

The input delayed dynamics (39) can then be rewritten as the cascade of this transport PDE into an Ordinary Differential Equation (ODE), that is, as

$$\begin{cases} \dot{X}(t) = AX(t) + Bv(0, t) \\ Dv_t(x, t) = v_x(x, t) \\ v(1, t) = 0. \end{cases} \quad (51)$$

Notice that this rewriting allows to obtain an explicit parameterization with respect to the delay: while it was originally a time argument of the control law, the delay now appears as a bilinear factor in (51). We will comment more on this feature in the sequel.

This equivalence between hyperbolic PDEs and some classes of delay systems is well-known and reported for instance in [KM92, HL93, Ric03]. Yet, it was seldom used before the prediction-based controller (44) was reinterpreted as the result of a backstepping transformation of (51) in [KS08]. We will now present this interpretation.

### 2.3.2. Backstepping transformation

The concept of backstepping for PDEs, generalizing the eponymous finite-dimensional technique as described in [BK02], has proved in recent years to be an efficient and constructive design technique for boundary control of PDEs (see [KS08]). It has been applied successfully in a large number of contexts, involving parabolic PDEs [Liu03, SK04, SK05], hyperbolic PDEs of first and second order [KSS06] or Navier-Stokes [CVK06] and Burgers [KMV08] equations, to cite a few examples.

For time-delay systems, its relevance was underlined in [KS08], which proved that the prediction-based control law (44) can actually be seen as the result of a backstepping transformation of system (51). In detail, the following backstepping transformation of the distributed input  $v$

$$w(x, t) = v(x, t) - Ke^{ADx}X(t) - D \int_0^x Ke^{AD(x-y)}Bv(y, t)dy, \quad (52)$$

along with the control law (44) transform (51) into the target system

$$\begin{cases} \dot{X}(t) = (A + BK)X(t) + Bw(0, t), \\ Dw_t(x, t) = w_x(x, t), \\ w(1, t) = 0. \end{cases} \quad (53)$$

in which  $K$  is the feedback gain of the control law (44), which is chosen such that  $A + BK$  is Hurwitz. Due to the boundary condition  $w(1, t) = 0$ , the backstepping variable  $w$  will then converge to zero

in finite-time, after  $D$  units of time, and the closed-loop system will then behave in a delay-free (exponentially stable) manner for  $t \geq D$ . Hence, the dynamics (53) indeed captures the input-delay compensation resulting from the prediction-based control law (44).

Notice that the transformation (52) can actually be rewritten as  $w(x, t) = v(x, t) - KX(t + Dx)$ , using the variation of constant formula for (39) between time  $t$  and  $t + Dx$ . Hence, in some sense, the backstepping variable  $w$  is nothing more than the superposition of two transport phenomena: the one of the future quantity  $KX(t + D)$  and the one of the input.

Alternatively to the above considerations, the exponential stability of the target system (53) can be proven with Lyapunov arguments, introducing the Lyapunov functional candidate

$$V(X, w) = X(t)^T P X(t) + bD \int_0^1 (1+x)w(x, t)^2 dx, \quad (54)$$

in which  $P$  is the symmetric positive definite solution of the equation  $P(A + BK) + (A + BK)^T P = -Q$  for a given positive definite matrix  $Q$  and  $b$  is a constant positive parameter to be designed. Indeed, evaluating the derivative of  $V$  along the trajectories of (53) yields

$$\begin{aligned} \dot{V} &= -X(t)^T Q X(t) + 2X(t)^T P B w(0, t) + 2b \int_0^1 (1+x)w(x, t)w_x(x, t) dx \\ &= -X(t)^T Q X(t) + 2X(t)^T P B w(0, t) + 2bw(1, t)^2 - bw(0, t)^2 - b \int_0^1 w(x, t)^2 dx, \end{aligned} \quad (55)$$

in which the last equality is obtained with an integration by parts. Then, applying Young's inequality, it follows that

$$\begin{aligned} \dot{V} &\leq -\frac{\min \lambda(Q)}{2} |X(t)|^2 - \left( b - \frac{2|PB|^2}{\min \lambda(Q)} \right) w(0, t)^2 - b \int_0^1 w(x, t)^2 dx \\ &\leq -\min \left\{ \frac{\min \lambda(Q)}{2 \max \lambda(P)}, \frac{b}{D} \right\} V - \left( b - \frac{2|PB|^2}{\min \lambda(Q)} \right) w(0, t)^2. \end{aligned} \quad (56)$$

Hence, choosing  $b \geq \frac{2|PB|^2}{\min \lambda(Q)}$ , the exponential stability of the target system follows, with the decay rate  $\eta = \min \left\{ \frac{\min \lambda(Q)}{2 \max \lambda(P)}, \frac{b}{D} \right\}$ .

These elements do not only constitute alternative viewpoints or proofs to the concept of delay compensation exposed in Section 2.1. They also present two features which can reveal useful in similar contexts:

- the transport PDE in (51) introduces an explicit delay parameterization, which eases delay-robustness analysis and is also compliant with adaptive control for instance (see [IS12]);
- the target system (53) can be analyzed with the Lyapunov functional (54). Hence, this transport PDE representation of the delay, reformulated with a backstepping transformation, equips the designer with a tool for Lyapunov-Krasovskii analysis, compliant as well with delay-sensitivity analysis.

Hence, these tools were widely used in the past decade to express delay-robustness results for prediction-based controllers, not only for constant delays and linear dynamics [BPK10] but also for the much more intricate case of time- and state-dependent delay perturbations and a nonlinear plant [BLK13b, BLK13a]. Likewise, this framework was recently used to propose a systematic approach to design delay-adaptive controllers [Krs09a, ZK20].

For these reasons, these tools reveal of particular interest in the context of stochastic delays studied in this manuscript.

### 3. Stochastic delay modeling and stability analysis

In this work, we investigate the problem of a differential equation affected by a stochastic continuous-time input delay.

Numerous work have investigated the question of a discrete-time delay as, e.g., in [KÖC<sup>+</sup>94, NBW98, KMR03, GQOM14] or of Stochastic Delay Differential Equations (SDDEs) in various contexts [HL93, IN64, KM13, Moh84, SZ18, CGMP20]. Among these studies, it is worth mentioning [CGMP20] which investigated prediction-based control for linear SDEs but with a deterministic time-varying input delay.

Yet, only a few works have considered the case where the delay is a continuous-time stochastic variable. Up to our knowledge, these few studies are [KM01a, KM01b, LM21, VM09b, VM09a, SSG<sup>+</sup>20]. Recently, [VM09b, VM09a] modeled the delay as a saw-tooth function with stochastic switches occurring at the arrival times of a homogeneous Poisson process. The dynamics at stake is hence somehow specific and resembles more a stochastically sampled-data one. Correspondingly, the stability analysis carried out in this approach mixes moment stability and Lyapunov approaches which are common for event-triggered systems. Another similar recent trend is the one of [SSG<sup>+</sup>20, SBO19] which considers the delay as a piece-wise constant process with known constant period, values in finite number and stationary transition probabilities. The assumption that the switches occur are only at periodic sampling time also strongly resembles the specific case of sampled-data systems. On the other hand, less recent studies by the Russian school [Kat67, KM00, KM01a] propose to model the delay as a Markov process with a finite number of states. This approach enables to develop a generic stability analysis. This is one of the reasons which lead us in this manuscript to adopt this modeling, which we will now detail.

#### 3.1. Delay modeling with a Markov process

In [KK60], the delay affecting a dynamical system is assumed to be a Markov process with a finite number of states, that is,  $D(t) \in \{D_i, i \in \{1, \dots, r\}\}$  for a given  $r \in \mathbb{N}$ .

In detail, the delay  $D$  is a stochastic process  $\{D(t), t \geq 0\}$  from a measurable space  $(\Omega, \mathcal{A})$  into  $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ , in which  $\mathcal{B}(\mathbb{R})$  is the Borel  $\sigma$ -algebra of  $\mathbb{R}$ . Assuming that the delay is a Markov process means that it satisfies the Markov property

$$\begin{aligned} Pr(D(t) = D_j | D(t_0) = D_i, \{D(u), 0 \leq u < t_0\}) &= Pr(D(t) = D_j | D(t_0) = D_i) \\ &\triangleq P_{ij}(t_0, t) \text{ for any } 0 \leq t_0 \leq t, \end{aligned} \quad (57)$$

that is, the delay realization  $\{D(t) | t \geq t_0\}$ , given the initial delay state  $D(t_0)$ , is independent of the past delay realizations  $\{D(u) | 0 \leq u < t_0\}$ . The conditional probabilities  $P_{ij} : (t_1, t_2) \in \mathbb{R}^2 \rightarrow [0, 1]$  introduced in (57) which quantify the probability to switch from  $D_i$  at time  $t_1$  to  $D_j$  at time  $t_2$  ( $(i, j) \in \{1, \dots, r\}^2, t_2 \geq t_1 \geq 0$ ) are called the transition probabilities.

The realizations of this Markov process are supposed to be right-continuous, in order to be able to guarantee existence of solutions to the open-loop or closed-loop dynamical systems affected by this delay. Finally, the delay can only take a finite number of values  $D_j$  which, without loss of generality, we assume to be ordered as  $0 < D_1 < D_2 < \dots < D_r$  for a given  $r \in \mathbb{N}$ . Figure 4b shows an example of realization of such a delay.

Considering a finite number of delay values is a standard practice considered in a large number of the aforementioned references, e.g., in [KM01a, SBO19]. In the case of network systems, these discrete values can be seen as a measure of the congestion state of the network. Similarly, modeling the delay as a Markov process stems from the fact that the congestion of a network mainly depends on its current or most recently known state.

It is important to specify that, under these assumptions and if the transition probabilities are smooth functions, there exist positive-valued functions  $\tau_{ij}, c_j = \sum_{k=1}^r \tau_{jk}$  ( $i, j \in \{1, \dots, r\}$ ) such that

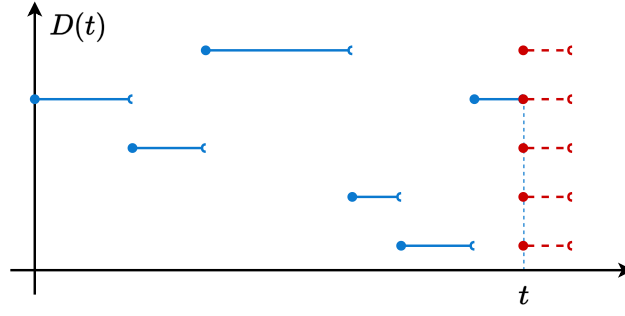


Figure 4b: Example of delay realizations with  $r = 5$  possible values.

$\tau_{ii} = 0$  and the transition probabilities satisfy

$$\begin{aligned} \frac{\partial P_{ij}(s, t)}{\partial t} &= -c_j(t)P_{ij}(s, t) + \sum_{k=1}^r P_{ik}(s, t)\tau_{kj}(t), \quad s < t, \\ P_{ii}(s, s) &= 1 \quad \text{and} \quad , P_{ij}(s, s) = 0 \quad \text{for} \quad i \neq j. \end{aligned} \quad (58)$$

This equation is well-known under the name of the Kolmogorov forward equation (see [Ros14, RH03]). As it is often used in the sequel of this manuscript, we briefly detail its proof, as provided e.g. in [RH03].

Consider the interval  $(0, t + \Delta t]$  for certain  $t, \Delta t > 0$ . Let us split this interval into two parts  $(0, t]$  and  $(t, t + \Delta t]$ . From the Markov property, one obtains the following relation (known as the Chapman-Kolmogorov equation)

$$\begin{aligned} P_{ij}(0, t + \Delta t) &= Pr(D(t + \Delta t) = D_j \mid D(0) = D_i) \\ &= \sum_{k=1}^r Pr(D(t + \Delta t) = D_j \mid D(0) = D_i, D(t) = D_k) Pr(D(t) = D_k \mid D(0) = D_i) \\ &= \sum_{k=1}^r Pr(D(t + \Delta t) = D_j \mid D(t) = D_k) Pr(D(t) = D_k \mid D(0) = D_i) \\ &= \sum_{k=1}^r P_{ik}(0, t) P_{kj}(t, t + \Delta t). \end{aligned} \quad (59)$$

where we used the definition of a condition probability and the fact that the events  $\{D(t) = D_k\}$  for  $k = 1, \dots, r$  are independent. Subtracting  $P_{ij}(0, t)$  from both sides of this equation yields

$$P_{ij}(0, t + \Delta t) - P_{ij}(0, t) = \sum_{k=1, k \neq j}^r P_{ik}(0, t) P_{kj}(t, t + \Delta t) - (1 - P_{jj}(t, t + \Delta t)) P_{ij}(0, t). \quad (60)$$

Dividing the equation by  $\Delta t$  and taking the limit as  $\Delta t \rightarrow 0$  gives

$$\begin{aligned} \lim_{\Delta t \rightarrow 0} \frac{P_{ij}(0, t + \Delta t) - P_{ij}(0, t)}{\Delta t} & \\ &= \sum_{k=1, k \neq j}^r P_{ik}(0, t) \lim_{\Delta t \rightarrow 0} \frac{P_{kj}(t, t + \Delta t)}{\Delta t} - \lim_{\Delta t \rightarrow 0} \frac{1 - P_{jj}(t, t + \Delta t)}{\Delta t} P_{ij}(0, t) \\ &= \sum_{k=1, k \neq j}^r P_{ik}(0, t) \underbrace{\lim_{\Delta t \rightarrow 0} \frac{P_{kj}(t, t + \Delta t) - P_{kj}(t, t)}{\Delta t}}_{\triangleq \tau_{kj}(t)} - \underbrace{\lim_{\Delta t \rightarrow 0} \frac{1 - P_{jj}(t, t + \Delta t)}{\Delta t}}_{\triangleq c_j(t)} P_{ij}(0, t), \end{aligned} \quad (61)$$



as  $P_{ij}(t, t) = 0$  for  $i \neq j$ . This is indeed the forward Kolmogorov equation (58), defining in addition  $\tau_{ii}(t) = 0$ . Notice that the limits above are well-defined if the transition probabilities are differentiable functions.

Finally, from the fact that  $\sum_{j=1}^r P_{ij}(0, t) = 1$ , one easily obtains from the above definitions of  $\tau_{kj}$  and  $c_j$  that

$$\begin{aligned} c_j(t) - \sum_{k=1}^r \tau_{jk}(t) &= \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \left( 1 - P_{jj}(t, t + \Delta t) - \sum_{k=1, k \neq j}^r P_{jk}(t, t + \Delta t) \right) \\ &= \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \underbrace{\left( 1 - \sum_{k=1}^r P_{jk}(t, t + \Delta t) \right)}_{=0}, \end{aligned} \quad (62)$$

that is, that  $c_j(t) = \sum_{k=1}^r \tau_{jk}(t)$  for all time  $t$ .

Observe that, according to these definitions and considerations, for a fixed and small enough sampling time  $\Delta t$ ,  $\tau_{ij}\Delta t \approx P_{ij}(t, t + \Delta t)$ . Hence,  $\tau_{ij}\Delta t$  can be interpreted as the probability of transition from  $D_i$  to  $D_j$  on the interval  $[t, t + \Delta t)$ . Similarly,  $1 - c_j(t)\Delta t$  represents the probability of staying at  $D_j$  during the time interval  $[t, t + \Delta t)$ .

This delay modeling then enables to develop a generic stability analysis.

### 3.2. Probabilistic delay averaging

Modeling the delay as Markov process with a finite number of state then allows to consider each fixed-delay value separately for stability analysis, applying the so-called technique of Probabilistic Delay Averaging [KM00], which we will expose now.

Consider the dynamics

$$\dot{X}(t) = f(X(t), X(t - D(t))), \quad (63)$$

in which  $X \in \mathbb{R}^n$  and  $D$  is a Markov process satisfying the previous properties. It can be proven that  $\Psi = (X, D)$  is a continuous-time Markov process (which we will admit, for the sake of clarity of the exposition). Hence, it is possible to consider its infinitesimal generator  $L$  [Dyn65], which, for any smooth function  $V : \Psi = (x, D) \in \mathbb{R}^n \times \mathbb{R} \mapsto V(\Psi) \in \mathbb{R}$ , is defined as

$$LV(\Psi(t)) = \lim_{s \rightarrow 0^+} \frac{\mathbb{E}_{[t, \Psi(t)]}(V(\Psi(t+s))) - V(\Psi(t))}{s}, \quad (64)$$

for  $t \in \mathbb{R}_+$ , and where  $\mathbb{E}_{[t, \Psi(t)]}(V(\Psi(t+s)))$  is the conditional expectation of  $V(\Psi(t+s))$ . As  $D$  is a process that has a finite number of states  $D_j$  ( $j = 1, \dots, r$ ), and as the probability of being in the state  $D_j$  at time  $t \geq 0$  starting from  $D(0) = D_i$  is  $P_{ij}(0, t)$ , it follows that

$$\mathbb{E}_{[0, \Psi(0)]}LV(\Psi(t)) = \sum_{j=1}^r P_{ij}(0, t) \mathbb{E}_{[0, \Psi(0)]}LV((X(t), D_j)). \quad (65)$$

Furthermore, due to the linearity of the expected value,

$$\begin{aligned} LV((X(t), D_j)) &= \lim_{s \rightarrow 0^+} \frac{1}{s} \left( \mathbb{E}_{[t, (X(t), D_j)]}(V((X(t+s), D(t+s)))) - V((X(t), D_j)) \right) \\ &= \lim_{s \rightarrow 0^+} \frac{1}{s} \left( \mathbb{E}_{[t, (X(t), D_j)]}(V((X(t+s), D(t+s)))) - \mathbb{E}_{[t, (X(t), D_j)]}(V((X(t+s), D_j))) \right) \\ &\quad + \lim_{s \rightarrow 0^+} \frac{1}{s} \left( \mathbb{E}_{[t, (X(t), D_j)]}(V((X(t+s), D_j))) - V((X(t), D_j)) \right). \end{aligned} \quad (66)$$

Let us consider separately the two terms in (66). As previously, as  $D$  is a process that has a finite number of states  $D_j$ , it follows that

$$\begin{aligned}
 & \lim_{s \rightarrow 0^+} \frac{1}{s} \left( \mathbb{E}_{[t, (X(t), D_j)]} (V((X(t+s), D(t+s)))) - \mathbb{E}_{[t, (X(t), D_j)]} (V((X(t+s), D_j))) \right) \quad (67) \\
 &= \lim_{s \rightarrow 0^+} \frac{1}{s} \left( \sum_{k=1}^r P_{jk}(t, t+s) \mathbb{E}_{[t, (X(t), D_j)]} (V((X(t+s), D_k))) - \mathbb{E}_{[t, (X(t), D_j)]} (V((X(t+s), D_j))) \right) \\
 &= \lim_{s \rightarrow 0^+} \left( \sum_{k=1, k \neq j}^r \frac{P_{jk}(t, t+s)}{s} \mathbb{E}_{[t, (X(t), D_j)]} (V((X(t+s), D_k))) \right. \\
 &\quad \left. + \frac{P_{jj}(t, t+s) - 1}{s} \mathbb{E}_{[t, (X(t), D_j)]} (V((X(t+s), D_j))) \right) \\
 &= \sum_{k=1, k \neq j}^r \tau_{jk}(t) V((X(t), D_k)) - c_j(t) V((X(t), D_j)),
 \end{aligned}$$

where the last line follows from the definitions of  $\tau_{jk}$  and  $c_j$  in (61). Furthermore, as the realizations of  $D$  are right-continuous, there exists  $\epsilon > 0$  small enough such that  $D(t+s) = D(t) = D_j$  for  $s \in [0, \epsilon)$  and, hence, the second term in (66) reads

$$\lim_{s \rightarrow 0^+} \frac{1}{s} \left( \mathbb{E}_{[t, (X(t), D_j)]} (V((X(t+s), D_j))) - V((X(t), D_j)) \right) = \frac{\partial V}{\partial X}(X(t), D_j) f(X(t), X(t - D_j)). \quad (68)$$

Hence, (65) rewrites as

$$\begin{aligned}
 \mathbb{E}_{[0, \Psi(0)]} LV(\Psi(t)) &= \mathbb{E}_{[0, \Psi(0)]} \left( \sum_{j=1}^r P_{ij}(0, t) \frac{\partial V}{\partial X}(X(t), D_j) f(X(t), X(t - D_j)) \right. \\
 &\quad \left. + \sum_{j=1}^r P_{ij}(0, t) \left[ \sum_{k=1, k \neq j}^r \tau_{jk}(t) V((X(t), D_k)) - c_j(t) V((X(t), D_j)) \right] \right). \quad (69)
 \end{aligned}$$

Using the forward Kolmogorov equation (20), the last term in (69) simplifies as

$$\begin{aligned}
 & \sum_{j=1}^r P_{ij}(0, t) \left[ \sum_{k=1, k \neq j}^r \tau_{jk}(t) V((X(t), D_k)) - c_j(t) V((X(t), D_j)) \right] \\
 &= \sum_{j=1}^r \left[ \sum_{k=1, k \neq j}^r \tau_{kj}(t) P_{ik}(0, t) - c_j P_{ij}(0, t) \right] V((X(t), D_j)) \\
 &= \sum_{j=1}^r \frac{\partial P_{ij}}{\partial t}(0, t) V((X(t), D_j)). \quad (70)
 \end{aligned}$$

Finally, one thus obtains

$$\begin{aligned}
 \mathbb{E}_{[0, \Psi(0)]} LV(\Psi(t)) &= \mathbb{E}_{[0, \Psi(0)]} \left( \sum_{j=1}^r P_{ij}(0, t) \frac{\partial V}{\partial X}(X(t), D_j) f(X(t), X(t - D_j)) \right. \\
 &\quad \left. + \sum_{j=1}^r \frac{\partial P_{ij}}{\partial t}(0, t) V((X(t), D_j)) \right), \quad (71)
 \end{aligned}$$

or, alternatively,

$$\begin{aligned} \mathbb{E}_{[0, \Psi(0)]} LV(\Psi(t)) &= \mathbb{E}_{[0, \Psi(0)]} \left( \sum_{j=1}^r P_{ij}(0, t) \left[ \frac{\partial V}{\partial X}(X(t), D_j) f(X(t), X(t - D_j)) \right. \right. \\ &\quad \left. \left. + \sum_{k=1}^r \tau_{jk}(t) (V((X(t), D_k)) - V((X(t), D_j))) \right] \right) \\ &\triangleq \mathbb{E}_{[0, \Psi(0)]} \left( \sum_{j=1}^r P_{ij}(0, t) L_j V(t) \right), \end{aligned} \quad (72)$$

which is related to the conditional expectation of  $V$  via Dynkin's formula [Dyn65, Theorem 5.1, p. 133]

$$\mathbb{E}_{[0, \Psi(0)]} V(\Psi(t)) = V(\Psi(0)) + \int_0^t \mathbb{E}_{[0, \Psi(0)]} LV(\Psi(s)) ds. \quad (73)$$

Consequently, the stability analysis of the system can be carried out by considering each fixed value of the delay separately through the fixed-delay infinitesimal generator  $L_j V$ , and then by averaging in the sense of (71) or (72). Notice that, as underlined in [KM00], the variation  $L_j V((X(t), D_j))$  involved in (72) can intuitively be understood as the fact that the increment  $\Psi(t + s) - (X(t), D_j)$  is the result of two contributing factors. The first one is determined, with a probability close to 1, by the dynamics with a fixed delay  $D_j$ . The second one corresponds to the "jumps" of the process  $D$  from a state  $D_j$  to a state  $D_k$  which take place with the probability  $\tau_{jk}(t)$ .

To illustrate the philosophy behind the concept of Probabilistic Delay Averaging, let us consider as an example the following dynamics

$$\dot{X}(t) = -X(t) + D(t)X(t), \quad (74)$$

with  $X \in \mathbb{R}$ . Even though  $D$  does not play the role of a delay in (74), but of a random parameter appearing linearly, the study of this system illustrates the main mechanisms at stake.

Let us define the Lyapunov functional candidate  $V(X) = X^2$ . Let us consider a fixed-delay value  $D_j$ . As  $V$  does not depend on  $D$ , one obtains that

$$L_j V(X(t)) = 2X(t)(-X(t) + D_j X(t)) = -2(1 - D_j)X(t)^2, \quad (75)$$

and Dynkin's formula then implies

$$\mathbb{E}_{[0, \Psi(0)]} V(\Psi(t)) = V(\Psi(0)) - \int_0^t 2 \left( 1 - \sum_{j=1}^r P_{ij}(0, s) D_j \right) \mathbb{E}_{[0, \Psi(0)]} V(\Psi(s)) ds, \quad (76)$$

from which one concludes that the system is exponentially stable if the delay expected value  $\sum_{j=1}^r P_{ij}(0, t) D_j$  is such that  $1 - \sum_{j=1}^r P_{ij}(0, t) D_j \geq \gamma > 0$  for a given positive  $\gamma$  for instance. This is in accordance here with the fact that, due to the linearity of (74), the state expected value satisfies the averaged-delay dynamics  $\dot{X}(t) = (1 - \sum_{j=1}^r P_{ij}(0, t) D_j) X(t)$ .

Consequently, this approach allows one to carry out a simple easy to use constant-delay approach for Lyapunov stability analysis. This inspired the core of the methodology for stability analysis proposed in this manuscript for stochastic input-delay systems.

## 4. Methodology and contributions of this thesis

In the previous sections, we exposed the main ingredients that are used in this manuscript for designing prediction-based controller in the presence of stochastic input-delay and analyzing the resulting closed-loop stability, namely rewriting the input delay as a transport PDE, applying a backstepping transformation corresponding to the prediction-based controller and finally applying the Probabilistic Delay Averaging approach to the resulting target system. As will appear, this enables to propose a generic procedure both for control design and resulting closed-loop stability analysis. In detail, assuming that the delay is the Markov process with a finite number of values previously presented, our methodology relies on the following three steps:

- Transport representation of the delay, in which a transport PDE is introduced for each potential delay value;
- Backstepping transformation of the resulting cascade of transport PDEs into an ODE;
- Lyapunov stability analysis of the corresponding target system, based on the Probabilistic Delay Averaging approach.

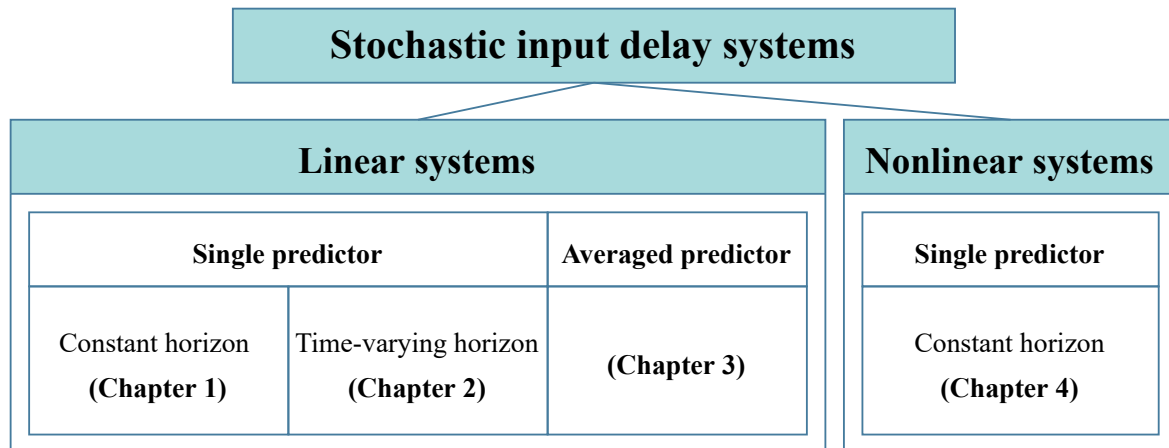


Figure 5b: The overview of the organization of this thesis.

The overview of this thesis is pictured in Figure 5b. It consists of 4 chapters:

**Chapter 1** considers the case of a LTI dynamics affected by a random input delay. It investigates its stabilization with a constant horizon predictor controller. First, it proves, in tune with the common-sense intuition, that the prediction horizon should be chosen in the vicinity of the delay values. This conservative condition demands that the prediction horizon is close to every potential delay value and thus, in turns, that the delay values are all sufficiently close.

Then, by considering a different Lyapunov functional, this sufficient deterministic condition is relaxed into a probabilistic form which takes the delay distribution into account. This second condition generalizes the first one as it only requires the prediction horizon to be chosen close enough to the delay values *in average*.

**Chapter 2** extends the design of Chapter 1 by considering a time-varying prediction horizon. This allows to update the prediction horizon under consideration with time and thus to tackle a much wider range of contexts. It is proven that, in addition to requiring as in Chapter 1 the prediction horizon to remain close enough to the delay values in average, its variation rate should be limited within a certain range.

**Chapter 3** puts forwards an alternative design of prediction-based control laws for LTI systems with random input delay. Contrary to the design approaches raised in the previous chapters, which are

based on a single predictor, we propose in this chapter to consider several predictors for each potential delay values and to average them.

We illustrate numerically on a few examples that this controller outperforms the previous one, in terms of delay robustness, and provide some leads on the reasons for this performance gap.

**Chapter 4** finally extends the single-predictor design of Chapter 1 and 2 but in a nonlinear context, which considerably complexifies the analysis. As in Chapter 1, a constant prediction horizon controller is considered and proven to allow for stabilization provided that the prediction horizon to remain close enough to the delay values.

Finally, as a conclusion, we give the perspectives to this work.

## 5. Publications

The works presented in this thesis has given rise to the following publications:

- Journals

1. SiJia Kong and Delphine Bresch-Pietri. "Prediction-based controller for linear systems with stochastic input delay." *Automatica* 138 (2022): 110149.
2. SiJia Kong and Delphine Bresch-Pietri. "Probabilistic Sufficient Conditions for Prediction-based Stabilization of Linear Systems with Random Input Delay." *IEEE Control Systems Letters* 6, 2270-2275, 2022. (With inclusion in the programm of the 2022 American Control Conference)
3. Jean Auriol, SiJia Kong, Delphine Bresch-Pietri. "Explicit Prediction-Based Control for Linear Difference Equations With Distributed Delays", *IEEE Control Systems Letters* 6, 2864-2869, 2022 (With inclusion in the programm of the 2022 Conference on Decision and Control)

- International Conferences

1. SiJia Kong and Delphine Bresch-Pietri. "Constant time horizon prediction-based control for linear systems with time-varying input delay." *IFAC-PapersOnLine* 53.2 (2020): 7665-7670.

- Book Chapter

1. SiJia Kong and Delphine Bresch-Pietri. "Prediction Control for Nonlinear Systems with Stochastic Input Delay." *Advances in Distributed Parameter Systems*. Springer, Cham, 2022. 27-44.

## Chapter 1

# Constant time-horizon prediction-based controller

This chapter is adapted from the paper *Prediction-based controller for linear systems with stochastic input delay*, published in *Automatica* in April 2022, and the paper *Probabilistic Sufficient Conditions for Prediction-Based Stabilization of Linear Systems With Random Input Delay*, published in the *IEEE Control Systems Letters*, in January 2022.

**Résumé** Ce chapitre s'intéresse à la stabilisation d'un système linéaire avec retard d'entrée aléatoire, à l'aide d'un contrôleur par prédiction. En modélisant le retard comme un processus de Markov à nombre d'états fini, nous prouvons tout d'abord qu'une prédiction à horizon constant permet une compensation robuste du retard, à condition que cet horizon de prédiction soit suffisamment proche des valeurs du retard. Dans un second temps, nous généralisons cette condition en montrant qu'il suffit qu'elle soit respectée *en moyenne*, dans le sens probabiliste du terme. Des résultats de simulation soulignent la pertinence pratique de cette condition.

**Abstract** This chapter focuses on the prediction-based stabilization of a linear system subject to a random input delay. Modeling the delay as a finite-state Markov process, it first proves that a constant time-horizon prediction enables robust compensation of the delay, provided the horizon prediction is sufficiently close to the delay values. It then relaxes this condition and proves that the horizon prediction is only required to be sufficiently close to the delay values *in average*. Simulation results emphasize the practical relevance of this condition.

## 1.1 Problem statement and controller design

This chapter focuses on the prediction-based control of the following LTI dynamics subject to a random input delay

$$\dot{X}(t) = AX(t) + BU(t - D(t)), \quad (1.1)$$

in which  $A$  and  $B$  are constant dynamics matrices such that the pair  $(A, B)$  is controllable and the  $\mathbb{R}^n$ -valued random variable  $X$  and  $U \in \mathbb{R}$  are the state and control input, respectively.

The random delay affecting the input is assumed to be a Markov process with a finite number of states, that is,  $D(t) \in \{D_i, i \in \{1, \dots, r\}\}$  for a given  $r \in \mathbb{N}$ . Without loss of generality, we assume in the sequel that these values are ordered as  $0 < \underline{D} \leq D_1 < D_2 < \dots < D_r \leq \bar{D}$  in which  $\underline{D}$  and  $\bar{D}$  are known positive bounds. In addition, we denote  $P_{ij} : \mathbb{R}^2 \rightarrow [0, 1]$  the transition probability functions, which quantify the probability to switch from  $D_i$  at time  $t_1$  to  $D_j$  at time  $t_2$  ( $((i, j) \in \{1, \dots, r\}^2, t_2 \geq t_1 \geq 0)$ ).

Finally, we assume that the realizations of this Markov process are right-continuous, in order to be able to guarantee existence of solutions to the open-loop or closed-loop systems.

Our control objective is to stabilize the dynamics (1.1) with a prediction-based controller, aiming at compensating, at least robustly, for the input delay. Indeed, due to the random nature of the delay, exact delay compensation, even if possible, would result into a chattering control law, which would then be impossible to apply in practice, due to the limited response time of an actuator. For this reason, we focus our attention on robust stabilization and, as a first step, we propose in this chapter to use the following prediction-based controller with constant time horizon  $D_0$

$$U(t) = K \left[ e^{AD_0} X(t) + \int_{t-D_0}^t e^{A(t-s)} BU(s) ds \right], \quad (1.2)$$

in which  $K$  is a feedback gain such that  $A + BK$  is Hurwitz, and  $D_0 \in [\underline{D}, \bar{D}]$  is constant. Notice that this prediction-based controller is nothing more than the usual prediction-based controller (44) applied in the constant delay case and which would compensate for the delay in closed-loop, would  $D_1 = \dots = D_r = D_0$ .

We will study in the following how to choose the prediction horizon and under which conditions one can guarantee that this controller will achieve stabilization of the dynamical systems (1.1). Before doing so, we first formulate that the closed-loop system admits a unique solution, which then enables us to analyze its behavior with the Probabilistic Delay Averaging approach (see Section 3.2 in the Introduction).

## 1.2 Preliminary result: well-posedness of the closed-loop system

Following [KK60], by a weak solution to the closed-loop system (1.1) and (1.2), we refer to a  $\mathbb{R}^n \times \mathcal{L}_2([- \bar{D}, 0], \mathbb{R}) \times \mathbb{R}$ -valued random variable  $(X(X_0, t), U_t(U_0, \cdot), D(t))$ , the realizations of which satisfy an integral form of (1.1) and (1.2), that is,

$$X(t) = X(0) + \int_0^t (AX(s) + BU(s - D(s))) ds, \quad (1.3)$$

and (1.2) for  $t \geq 0$ .

**Lemma 1.1** *For every initial condition  $(X_0, U_0) \in \mathbb{R}^n \times \mathcal{L}_2([- \bar{D}, 0], \mathbb{R})$ , the closed-loop system consisting of (1.1) and the control law (1.2) has a unique weak solution, which is such that*

$$X(t) = e^{At} X(0) + \int_0^t e^{A(t-s)} BU(s - D(s)) ds, \quad t \geq 0. \quad (1.4)$$

*Proof:* We first focus on the existence of a solution. Notice that, for the function  $X$  defined in (1.4), performing an integration by parts,

$$\begin{aligned} & X(0) + \int_0^t (AX(s) + BU(s - D(s))) ds \\ &= X(0) + \int_0^t BU(s - D(s)) ds + \int_0^t Ae^{As} X(0) ds + \int_0^t \left( \int_0^s Ae^{A(s-\xi)} ds \right) BU(\xi - D(\xi)) d\xi \quad (1.5) \\ &= e^{At} X(0) + \int_0^t e^{A(t-\xi)} BU(\xi - D(\xi)) d\xi = X(t), \end{aligned}$$

which does correspond to an integral form of (1.1). Hence,  $X$  defined in (1.4) is a weak solution to (1.1) if it is well-defined.

Let us denote  $\delta$  the Markov process taking values in  $\mathbb{R}^r$  and defined as  $\delta(t) = e_i$ , the  $i^{\text{th}}$  unit vector in  $\mathbb{R}^r$  where  $i$  is the index of the delay realization at time  $t$ . In other words,  $\delta(t) = e_i$  if and only if  $D(t) = D_i$ . Observe that the delayed input then rewrites as  $U(t - D(t)) = \delta(t) (U(t - D_1) \dots U(t - D_r))^T$ . As the realizations of  $D$ , and thus  $\delta$ , are right-continuous,  $\delta(t)$  is a bounded almost-everywhere continuous function ([CZ01, Exercise 4, p. 7]). It is thus and also square-integrable. As  $U_0 \in \mathcal{L}_2([-D, 0], \mathbb{R})$ , it follows from Cauchy-Schwarz's inequality, that  $t \rightarrow U(t - D(t))$  is integrable on the interval  $[0, \underline{D}]$ . Consequently, the integral in (1.4) is well-defined for  $t \in [0, \underline{D}]$ , and  $X$  is bounded on the interval  $[0, \underline{D}]$ .

Then, as  $U_0 \in \mathcal{L}_2([-D, 0], \mathbb{R})$ ,  $U$  defined in (1.2) remains bounded on the interval  $[0, \underline{D}]$  from the corresponding inverse Volterra integral equation [BPCE18, Yos60]. Consequently, by a straightforward iterative argument on time intervals of length  $\underline{D}$ , one can prove that both  $X$  and  $U$ , as defined through (1.4) and (1.2), remain bounded for positive times, and that, consequently, the integral in (1.4) is well-defined. Hence, (1.4) and (1.2) define a weak solution to the closed-loop system.

Secondly, we prove the uniqueness of this solution. Suppose that there exist two different solutions  $(X_1, U_1)$  and  $(X_2, U_2)$  for a given initial condition. It then holds

$$\begin{cases} (\dot{X}_1 - \dot{X}_2)(t) = A(X_1 - X_2) + B(U_1(t - D(t)) - U_2(t - D(t))) \\ (X_1 - X_2)(0) = 0, \end{cases} \quad (1.6)$$

with  $U_1 = U_2$  for  $t < 0$ , and for  $t \geq 0$

$$\begin{cases} U_1(t) = K \left( e^{AD_0} X_1(t) + \int_{t-D_0}^t e^{A(t-s)} BU_1(s) ds \right) \\ U_2(t) = K \left( e^{AD_0} X_2(t) + \int_{t-D_0}^t e^{A(t-s)} BU_2(s) ds \right). \end{cases} \quad (1.7)$$

For any delay realization, it thus holds that  $U_1(t - D(t)) = U_2(t - D(t))$  for  $t \in [0, \underline{D}]$ , which, in turns, gives  $X_1(t) = X_2(t)$  for  $t \in [0, \underline{D}]$ . Iterating on intervals of length  $\underline{D}$ , we then obtain  $X_1 = X_2$  and  $U_1 = U_2$  for  $t \in \mathbb{R}_+$ . ■

With Lemma 1.1, it follows that  $(X, U_t, \delta)$  defines a continuous-time Markov process. This is a prerequisite to apply the Probabilistic Delay Averaging approach, as detailed in Section 3.1 in the Introduction. With this Lemma, we are thus able to formulate the following robustness result.

## 1.3 Deterministic condition for closed-loop exponential stability

### 1.3.1 Main result

We prove in this section the following theorem, which states that the prediction-based controller (1.2) robustly compensate for the stochastic input delay, provided that the prediction horizon  $D_0$  is chosen sufficiently close to the delay values.



**Theorem 1.1** Consider the closed-loop system consisting of the system (1.1) and the control law (1.2). There exists a positive constant  $\varepsilon^*(K)$  such that, if

$$|D_0 - D_j| \leq \varepsilon^*(K), \quad j \in \{1, \dots, r\}, \quad (1.8)$$

there exist positive constants  $R$  and  $\gamma$  such that

$$\mathbb{E}_{[0, (X(0), U_0, D(0))]}(\Upsilon(t)) \leq R\Upsilon(0)e^{-\gamma t}, \quad (1.9)$$

with

$$\Upsilon(t) = |X(t)|^2 + \int_{t-\bar{D}-D_0}^t U(s)^2 ds. \quad (1.10)$$

Condition (1.8) guarantees that the prediction performed in (1.2) remains sufficiently accurate in the case of a stochastic delay. To be able to pick the prediction horizon  $D_0$  such that (1.8) is satisfied, the sequence of the random delay  $\mathbf{D} = (D_i)_{i \in \{1, \dots, r\}}$  should be limited in the vicinity of each other, namely in a ball of radius  $\varepsilon^*$ .

Note that this result is consistent with the delay-robustness results obtained in the deterministic delay case, namely [KBP20, BLK13b, KK13, CK16, LPS19] which all formulate robust compensation results requiring the range of variation of the delay (or of a perturbation affecting it) to be sufficiently limited. Hence, this theorem falls within this framework and extends it to the stochastic context.

We now provide the proof of this theorem in the following sections.

d

### 1.3.2 PDE representation of the delay and backstepping transformation

In order to prove Theorem 1.1, we follow the methodology proposed in the Introduction, and which consists in three steps.

#### 1.3.2.1 Step 1: Transport PDE cascading into an ODE $(X, \mathbf{v})$ and extended system $(X, \hat{\mathbf{v}}, \tilde{\mathbf{v}}, \mu)$

First, to represent the input delay, we define a distributed actuator vector as, for  $x \in [0, 1]$ ,  $\mathbf{v}(x, t) = (v_1(x, t) \ \cdots \ v_k(x, t) \ \cdots \ v_r(x, t))^T$  with  $v_k(x, t) = U(t + D_k(x - 1))$ . Each component of  $\mathbf{v}$  accounts for a different potential value of the stochastic delay. This enables to rewrite (1.1) as

$$\begin{cases} \dot{X}(t) = AX(t) + B\delta(t)^T \mathbf{v}(0, t) \\ \Lambda_D \mathbf{v}_t(x, t) = \mathbf{v}_x(x, t) \\ \mathbf{v}(1, t) = \mathbf{1}U(t), \end{cases} \quad (1.11)$$

in which  $\Lambda_D = \text{diag}(D_1, \dots, D_r)$ ,  $\mathbf{1}$  is a  $r$ -by-1 all-ones vector and  $\delta(t) \in \mathbb{R}^r$  is such that, if  $D(t) = D_j$ ,

$$\delta_i(t) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases} \quad (1.12)$$

Due to this definition,  $\delta$  is a Markov process with the same transition probabilities as the process  $D$ , but with the finite number of states ( $e_j$ ) instead of  $(D_j)$ . In the sequel,  $\delta$  and  $D$  will thus be equivalently used.

As illustrated in Figure 1.1, for each delay value  $D_j$  ( $j \in \{1, \dots, r\}$ ),  $v_j$  is used to represent the controller  $U$  in the time interval  $[t - D_j, t]$  with the variable  $x \in [0, 1]$ . There exist  $r$  different channels, such that for each channel, the control law enters as  $U(t)$  and propagates, respectively, with the speeds from  $\frac{1}{D_1}$  to  $\frac{1}{D_r}$ . It thus gives the outputs  $U(t - D_1)$  to  $U(t - D_r)$ , respectively. The random variable  $\delta$  then acts as a switching signal, selecting the channel and thus the delayed signal which

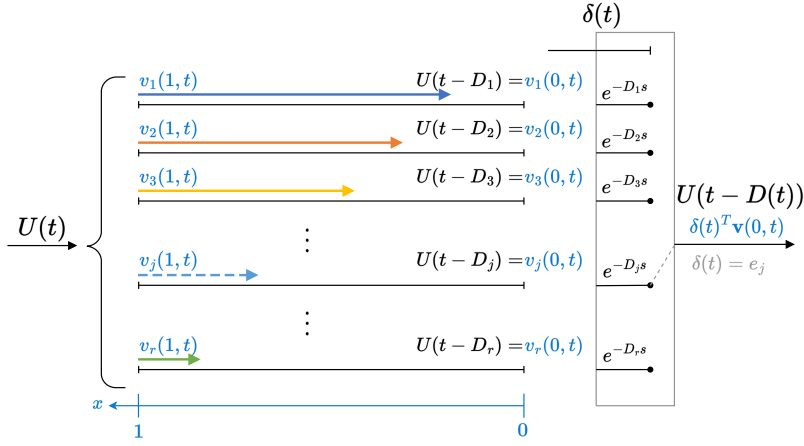


Figure 1.1: Schematic illustration of the distributed variables  $v_j$  and the role of the random process  $\delta$ , which acts as a selector.

enters the ODE. Note that the dynamics is reformulated in (1.11) as a dynamical system involving a random parameter, as studied in [KK60] or [CJV09] for instance. This allows us to use the technique of Probabilistic Delay Averaging to analyze the stability of the closed-loop, in the sequel.

We now provide a detailed proof of (1.11). As  $v_j(x, t) = U(t + D_j(x - 1))$ , it follows that  $U(t - D_j) = v_j(0, t)$ . Thus, if  $D(t) = D_j$ , with the definition of  $\mathbf{v}$ , the ODE can be rewritten as

$$\dot{X}(t) = AX(t) + BU(t - D_j) = AX(t) + Bv_j(0, t) = AX(t) + Be_j^T \mathbf{v}(0, t).$$

As  $\delta(t) = e_j$  if and only if  $D(t) = D_j$ . By definition of  $\delta$ , one then obtains  $\dot{X} = AX + B\delta^T \mathbf{v}(0, t)$ .

Besides, for  $j \in \{1, 2, \dots, r\}$ , as  $U(t + D_j(x - 1)) = v_j(x, t)$ , one easily obtains the following time- and space-derivatives

$$\begin{aligned} v_{jt} &= \dot{U}(t + D_j(x - 1)), \\ v_{jx} &= D_j \dot{U}(t + D_j(x - 1)). \end{aligned}$$

Thus, one gets that  $D_j v_{jt} = v_{jx}$ . With  $\Lambda_D = \text{diag}(D_1, \dots, D_r)$ , we then have  $\Lambda_D \mathbf{v}_t(x, t) = \mathbf{v}_x(x, t)$ .

Finally, from the definition of the actuator  $\mathbf{v}(x, t)$ , for  $j \in \{1, 2, \dots, r\}$ , we have  $v_j(1, t) = U(t)$ . Consequently, the boundary condition of  $\mathbf{v}$  at  $x = 1$  rewrites as

$$\mathbf{v}(1, t) = (v_1(1, t) \quad \dots \quad v_k(1, t) \quad \dots \quad v_r(1, t))^T = \mathbf{1}U(t).$$

**Extended system**  $(X, \hat{v}, \tilde{\mathbf{v}}, \mu)$ . In addition to the previous distributed variables, we consider  $\hat{v}(x, t)$  to represent the control input  $U(t)$  within the interval  $[t - D_0, t]$ , which is the time interval corresponding to the prediction-based control law (1.2). We also introduce the corresponding input estimation error  $\tilde{\mathbf{v}}(x, t)$ . They are defined as

$$\hat{v}(x, t) = U(t + D_0(x - 1)), \quad (1.13)$$

$$\tilde{\mathbf{v}}(x, t) = \mathbf{v}(x, t) - \mathbf{1}\hat{v}(x, t). \quad (1.14)$$

Then, the state  $(X(t), \hat{v}(x, t), \tilde{\mathbf{v}}(x, t))$  satisfies

$$\begin{cases} \dot{X}(t) = AX(t) + B\hat{v}(0, t) + B\delta(t)^T \tilde{\mathbf{v}}(0, t) \\ D_0 \hat{v}_t(x, t) = \hat{v}_x(x, t) \\ \hat{v}(1, t) = U(t) \\ \Lambda_D \tilde{\mathbf{v}}_t(x, t) = \tilde{\mathbf{v}}_x(x, t) - \Sigma_D \hat{v}_x(x, t) \\ \tilde{\mathbf{v}}(1, t) = \mathbf{0}, \end{cases} \quad (1.15)$$

in which  $\Sigma_D = (\frac{D_1-D_0}{D_0}, \dots, \frac{D_r-D_0}{D_0})^T$  and  $\mathbf{0}$  is a  $r$ -by-1 all-zeros vector. Notice that  $\hat{v}$  simply satisfies a transport PDE with velocity  $1/D_0$  while the dynamics of  $\tilde{v}$  is affected by the source term  $\Sigma_D \hat{v}_x(x, t)$  which accounts for the difference between the propagation speeds of  $\hat{v}$  and each component of  $\mathbf{v}$ .

Finally, to ease the stability analysis performed in the sequel, we also introduce the variable  $\mu(x, t) = U(t - D_0 + \bar{D}(x - 1))$  to describe the history of the input on a longer time window  $[t - \bar{D} - D_0, t - D_0]$ . Correspondingly, the extended dynamics of  $(X(t), \hat{v}(x, t), \tilde{v}(x, t), \mu(x, t))$  is

$$\begin{cases} \dot{X}(t) = AX(t) + B\hat{v}(0, t) + B\delta(t)^T \tilde{v}(0, t) \\ D_0 \hat{v}_t(x, t) = \hat{v}_x(x, t) \\ \hat{v}(1, t) = U(t) \\ \Lambda_D \tilde{v}_t(x, t) = \tilde{v}_x(x, t) - \Sigma_D \hat{v}_x(x, t) \\ \tilde{v}(1, t) = \mathbf{0} \\ \bar{D} \mu_t(x, t) = \mu_x(x, t) \\ \mu(1, t) = \hat{v}(0, t). \end{cases} \quad (1.16)$$

In other words,  $\hat{v}$  now cascades into the transport PDE satisfied by  $\mu$ .

### 1.3.2.2 Step 2: Target system $(X, w, \tilde{v}, \mu)$

Consider the following backstepping transformation of  $(X, \hat{v})$  corresponding to the prediction-based control law (1.2) (see [KS08])

$$w(x, t) = \hat{v}(x, t) - Ke^{AD_0x} X(t) - D_0 \int_0^x Ke^{AD_0(x-y)} B\hat{v}(y, t) dy. \quad (1.17)$$

**Lemma 1.2** *The backstepping transformation (1.17), jointly with the control law (1.2), transform the plant (1.16) into the target system  $(X(t), w(x, t), \tilde{v}(x, t), \mu(x, t))$*

$$\begin{cases} \dot{X}(t) = (A + BK)X(t) + B\delta(t)^T \tilde{v}(0, t) + Bw(0, t) \\ D_0 w_t(x, t) = w_x(x, t) - D_0 Ke^{AD_0x} B\delta(t)^T \tilde{v}(0, t) \\ w(1, t) = 0 \\ \Lambda_D \tilde{v}_t(x, t) = \tilde{v}_x(x, t) - \Sigma_D h(t + D_0(x - 1)) \\ \tilde{v}(1, t) = \mathbf{0} \\ \bar{D} \mu_t(x, t) = \mu_x(x, t) \\ \mu(1, t) = KX(t) + w(0, t), \end{cases} \quad (1.18)$$

in which,  $h$  is defined for  $t \geq 0$  as

$$\begin{aligned} h(t) = D_0 K \left[ (A + BK)e^{AD_0} X(t) + e^{AD_0} B\delta(t)^T \tilde{v}(0, t) + e^{AD_0} Bw(0, t) + D_0(A + BK) \right. \\ \left. \times \int_0^1 e^{AD_0(1-x)} B \left( w(x, t) + Ke^{(A+BK)D_0x} X(t) + \int_0^x KD_0 e^{(A+BK)D_0(x-y)} Bw(y, t) dy \right) dx \right]. \end{aligned} \quad (1.19)$$

*Proof:* The space-derivative of the backstepping transformation (1.17) can be written as

$$\begin{aligned} w_x(x, t) = \hat{v}_x(x, t) - KAD_0 e^{AD_0x} X(t) - D_0 KB\hat{v}(x, t) \\ - D_0 \int_0^x KAD_0 e^{AD_0(x-y)} B\hat{v}(y, t) dy. \end{aligned} \quad (1.20)$$

Besides, the time-derivative of (1.17) reads

$$\begin{aligned} w_t(x, t) = & \hat{v}_t(x, t) - D_0 \int_0^x K e^{AD_0(x-y)} B \hat{v}_t(y, t) dy \\ & - K e^{AD_0x} A X(t) - K e^{AD_0x} B \hat{v}(0, t) - K e^{AD_0x} B \delta(t)^T \tilde{v}(0, t). \end{aligned} \quad (1.21)$$

From (1.17), (1.20) and (1.21) with an integration by parts, we obtain the two equations with respect to  $w$  in (1.18). Finally, from the definition of  $\tilde{v}$  and  $\hat{v}$  in (1.13) and (1.14), one can observe that  $h(t + D_0(x - 1)) = \hat{v}_x(x, t) = D_0 \dot{U}(t + D_0(x - 1))$  which gives the desired expression of  $h(t)$  for  $t \geq 0$ , taking a time-derivative of (1.2) and using the inverse backstepping transformation of (1.17), which is

$$\hat{v}(x, t) = w(x, t) + K e^{(A+BK)D_0x} X(t) + \int_0^x K D_0 e^{(A+BK)D_0(x-y)} B w(y, t) dy. \quad (1.22)$$

■

Notice that, compared to (1.16), the main interest of the backstepping transformation (1.17) is that it allows to obtain the boundary condition  $w(1, t) = 0$  in (1.18). This considerably simplifies the Lyapunov analysis, which we are now ready to carry out.

### 1.3.3 Lyapunov Stability Analysis with Probabilistic Delay Averaging

Let us define the state of the target system (1.18) as  $\Psi = (X, w, \tilde{v}, \mu) \in \mathbb{R}^n \times \mathcal{L}_2([0, 1], \mathbb{R}) \times \mathcal{L}_2([0, 1], \mathbb{R}^r) \times \mathcal{L}_2([0, 1], \mathbb{R}) \triangleq \mathcal{D}_\Psi$ . From Lemma 1.1,  $(\Psi, \delta)$  defines a continuous-time Markov process. We can then consider its infinitesimal generator and use the Probabilistic Delay Averaging approach (see Section 3 of the Introduction).

With this aim in view, consider the following Lyapunov functional candidate

$$\begin{aligned} V(\Psi) = & X^T P X + b D_0 \int_0^1 (1+x) w(x)^2 dx + c \sum_{l=1}^r \int_0^1 (1+x) (e_l \cdot \mathbf{D})^T \tilde{v}(x)^2 dx \\ & + d \bar{D} \int_0^1 (1+x) \mu(x)^2 dx, \end{aligned} \quad (1.23)$$

with  $b, c, d > 0$ ,  $P$  the symmetric positive definite solution of the equation  $P(A+BK) + (A+BK)^T P = -Q$ , for a given symmetric positive definite matrix  $Q$ , and  $\mathbf{D} = (D_1 \dots D_r)_{r \in \mathbb{N}}^T$  and where  $\cdot$  denotes the Hadamard multiplication and the square in  $\tilde{v}(x)^2$  should be understood component-wise.

Recall the definition of the fixed-delay infinitesimal generator of the Markov process  $(\Psi, \delta)$ , which is obtained for the target system (1.18) by fixing  $\delta(t) = e_j$ , as

$$L_j V(\Psi) = \frac{dV}{d\Psi}(\Psi, e_j) f_j(\Psi), \quad (1.24)$$

in which  $f_j$  denotes the operator corresponding to the dynamics of the target system (1.18) with the fixed value  $\delta(t) = e_j$ , that is, for  $\Psi = (X, w, \tilde{v}, \mu)$ ,

$$f_j(\Psi)(x) = \begin{pmatrix} (A+BK)X + B e_j^T \tilde{v}(0) + B w(0) \\ \frac{1}{D_0} [w_x(x) - D_0 K e^{AD_0x} B e_j^T \tilde{v}(0)] \\ \Lambda_D^{-1} [\tilde{v}_x(x) - \Sigma_D h(\cdot + D_0(x-1))] \\ \frac{1}{D} \mu_x(x) \end{pmatrix}. \quad (1.25)$$

It is worth noticing that the expression in (1.24) has been simplified compared to the original one provided in (72), due to the fact that the expression of  $V$  in (1.23) does not depend explicitly on  $\delta$ .

For the sake of conciseness, in the sequel, we denote  $V(t)$ ,  $LV(t)$  and  $L_j V(t)$ , for short, instead of  $V(\Psi(t), \delta(t))$ ,  $LV(\Psi(t), \delta(t))$  and  $L_j V(\Psi(t))$  respectively.

### 1.3.3.1 Lyapunov analysis

Following the Probabilistic Delay Averaging approach, as a first step, one can focus on the derivative of the Lyapunov functional evaluated for a dynamic with a fixed delay, that is,  $L_j V$ . This allows to obtain the following intermediate result.

**Lemma 1.3** *Assume there exists a positive constant  $\varepsilon$  such that*

$$|D_0 - D_j| \leq \varepsilon, \quad j \in \{1, \dots, r\}. \quad (1.26)$$

*Then, there exist  $(b, c, d, \eta) \in (\mathbb{R}_+^*)^4$  which are independent of  $\varepsilon$  such that the Lyapunov functional  $V$  defined in (1.23) satisfies*

$$\sum_{j=1}^r P_{ij}(0, t) L_j V(t) \leq -(\eta - g(\varepsilon))V(t), \quad t \geq \bar{D}, \quad (1.27)$$

*with the function  $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  satisfying  $\lim_{\varepsilon \rightarrow 0} g(\varepsilon) = 0$ .*

The proof of this Lemma is detailed in the rest of this section, with a series of intermediate lemmas. First, taking a derivative of (1.23) and applying integrations by parts and Young's inequality, we obtain

$$\begin{aligned} \frac{dV}{d\Psi}(\Psi) f_j(\Psi) &= -X(t)^T Q X(t) + 2X(t)^T P B(w(0, t) + \tilde{v}_j(0, t)) \\ &\quad + 2b \int_0^1 (1+x)w(x, t)(w_x(x, t) - D_0 K e^{AD_0 x} B \tilde{v}_j(0, t)) dx \\ &\quad + 2c \sum_{l=1}^r \int_0^1 (1+x) \tilde{v}_l(x, t) \left( \tilde{v}_{lx}(x, t) + \left(1 - \frac{D_l}{D_0}\right) h(t + D_0(x-1)) \right) dx \\ &\quad + 2d \int_0^1 (1+x) \mu(x, t) \mu_x(x, t) dx \\ &\leq - \left( \frac{\min(\lambda(Q))}{2} - 4d|K|^2 \right) |X(t)|^2 - d \|\mu(t)\|^2 \\ &\quad - b(1 - 2D_0|K||B|e^{|A|D_0} \gamma_1) \|w(t)\|^2 - c \sum_{l=1}^r \left( 1 - \frac{2}{D_0} |D_0 - D_l| \gamma_2 \right) \|\tilde{v}_l(t)\|^2 \\ &\quad - \left( b - 4d - \frac{4|PB|^2}{\min(\lambda(Q))} \right) w(0, t)^2 \\ &\quad - \left( c - \frac{4|PB|^2}{\min(\lambda(Q))} - 2bD_0|K||B|e^{|A|D_0} \frac{1}{\gamma_1} \right) \tilde{v}_j(0, t)^2 - c \sum_{l \neq j} \tilde{v}_l(0, t)^2 \\ &\quad - d\mu(0, t)^2 + \frac{2c}{D_0} \sum_{l=1}^r |D_0 - D_l| \frac{1}{\gamma_2} \|h(t + D_0(\cdot - 1))\|^2, \end{aligned} \quad (1.28)$$

for any  $\gamma_1, \gamma_2 \geq 0$ .

If one wants to upper-bound (1.28) (or its average) with a negative quantity, a remaining problem is to bound the last term in (1.28). This yields the following lemma.

**Lemma 1.4** *Consider the function  $h$  defined in (1.19). There exists  $M_h > 0$  such that*

$$\|h(t + D_0(\cdot - 1))\|^2 \leq M_h V(t), \quad t \geq D_0. \quad (1.29)$$

*Proof:* First, from (1.19),  $h$  can be expressed as

$$h(t) = D_0 K \left[ e^{AD_0} AX(t) + e^{AD_0} BU(t - D(t)) + BU(t) - e^{AD_0} BU(t - D_0) + A \int_{t-D_0}^t e^{A(t-s)} BU(s) ds \right]. \quad (1.30)$$

Then, (1.30) gives

$$\begin{aligned} \|h(t + D_0(\cdot - 1))\|^2 &= \int_0^1 h(t + D_0(x - 1))^2 dx \\ &\leq 5|K|^2 D_0^2 \int_0^1 \left[ M_1 |X(t + D_0(x - 1))|^2 + M_2 |U(t + D_0(x - 1) - D(t + D_0(x - 1)))|^2 \right. \\ &\quad \left. + M_3 |U(t + D_0(x - 1))|^2 + M_4 |U(t + D_0(x - 2))|^2 \right. \\ &\quad \left. + |A|^2 \int_{t+D_0(x-2)}^{t+D_0(x-1)} e^{2|A|(t+D_0(x-1)-s)} |B|^2 |U(s)|^2 ds \right] dx, \end{aligned} \quad (1.31)$$

with positive constants defined as  $M_1 = e^{2|A|\bar{D}}|A|^2$ ,  $M_2 = e^{2|A|\bar{D}}|B|^2$ ,  $M_3 = |B|^2$ ,  $M_4 = e^{2|A|\bar{D}}|B|^2$ .

From the definition of the dynamics (1.1), it holds

$$\begin{aligned} |X(t + D_0(x - 1))| &= \left| e^{AD_0(x-1)} \left( X(t) - \int_{t+D_0(x-1)}^t e^{A(t-s)} BU(s - D(s)) ds \right) \right| \\ &\leq e^{|A|D_0} \left( |X(t)| + \int_{t+D_0(x-1)}^t e^{|A|(t-s)} |B| \sum_{j=1}^r |U(s - D_j)| ds \right) \\ &\leq e^{|A|D_0} |X(t)| + e^{2|A|D_0} |B| \sum_{j=1}^r \int_{t+D_0(x-1)}^t |U(s - D_j)| ds. \end{aligned} \quad (1.32)$$

Then, with (1.22), the equation (1.31) gives

$$\begin{aligned} \|h(t + D_0(\cdot - 1))\|^2 & \\ &\leq 5|K|^2 D_0^2 \left[ 2M_1 e^{2|A|D_0} |X(t)|^2 + 2M_1 r e^{4|A|D_0} |B|^2 (\|\mu(t)\|^2 + M_6 |X(t)|^2 + M_6 \|w(t)\|^2) \right. \\ &\quad \left. + M_2 r (\|\mu(t)\|^2 + M_6 |X(t)|^2 + M_6 \|w(t)\|^2) + M_3 M_6 |X(t)|^2 + M_3 M_6 \|w(t)\|^2 + M_4 \|\mu(t)\|^2 \right. \\ &\quad \left. + M_5 (\|\mu(t)\|^2 + M_6 |X(t)|^2 + M_6 \|w(t)\|^2) \right] \\ &\leq 5|K|^2 \bar{D}^2 [M_X |X(t)|^2 + M_w \|w(t)\|^2 + M_\mu \|\mu(t)\|^2], \end{aligned} \quad (1.33)$$

in which  $M_5 = |A|^2 e^{2|A|D_0} |B|^2$ ,  $M_6 = 3(1 + |K|^2 e^{2|A+BK|D_0} \max\{1, D_0^2 |B|^2\})$  and the positive constants  $(M_X, M_w, M_\mu)$  are defined as follows

$$\begin{cases} M_X = 2M_1 e^{2|A|\bar{D}} + M_6 (2M_1 r e^{4|A|\bar{D}} |B|^2 + M_2 r + M_3 + M_5), \\ M_w = M_6 (2M_1 r e^{4|A|\bar{D}} |B|^2 + M_2 r + M_3 + M_5), \\ M_\mu = 2M_1 r e^{4|A|\bar{D}} |B|^2 + M_2 r + M_4 + M_5. \end{cases} \quad (1.34)$$

Consequently, from the definition of Lyapunov functional  $V(t)$  in (1.23), it holds

$$\|h(t + D_0(\cdot - 1))\|^2 \leq 5(|K|\bar{D})^2 \max \left\{ \frac{M_X}{\min(\lambda(P))}, \frac{M_w}{b\underline{D}}, \frac{M_\mu}{d\underline{D}} \right\} V(t). \quad (1.35)$$

The lemma is then proved with the positive constant  $M_h = 5(|K|\bar{D})^2 \max \left\{ \frac{M_X}{\min(\lambda(P))}, \frac{M_w}{b\underline{D}}, \frac{M_\mu}{d\underline{D}} \right\}$ .

■

Therefore, applying (1.26) and Lemma 1.4, one gets for  $t \geq \bar{D}$ ,

$$\begin{aligned}
 \sum_{j=1}^r P_{ij}(0, t) \frac{dV}{d\Psi}(\Psi) f_j(\Psi) &\leq - \left( \frac{\min(\lambda(Q))}{2} - 4d|K|^2 \right) |X(t)|^2 - b \left( 1 - 2D_0|K||B|e^{|A|D_0}\gamma_1 \right) \|w(t)\|^2 \\
 &\quad - c \sum_{l=1}^r \left( 1 - \frac{2}{D_0}|D_0 - D_l|\gamma_2 \right) \|\tilde{v}_l(t)\|^2 - d\|\mu(t)\|^2 - \left( b - 4d - \frac{4|PB|^2}{\min(\lambda(Q))} \right) w(0, t)^2 \\
 &\quad - \left( c - \frac{4|PB|^2}{\min(\lambda(Q))} - 2bD_0|K||B|e^{|A|D_0}\frac{1}{\gamma_1} \right) \sum_{j=1}^r P_{ij}(0, t) \tilde{v}_j(0, t)^2 \\
 &\quad - c \sum_{j=1}^r P_{ij}(0, t) \sum_{l \neq j} \tilde{v}_l(0, t)^2 - d\mu(0, t)^2 + \frac{2cr\varepsilon}{D_0} \frac{1}{\gamma_2} M_h V(t), \tag{1.36}
 \end{aligned}$$

in which the positive constant  $M_h$  does not depend on  $\varepsilon$  and is defined in Lemma 1.4.

Observing that  $D_0 \in [\underline{D}, \bar{D}]$ , let us choose  $(b, c, d, \gamma_1, \gamma_2) \in (\mathbb{R}_+^*)^5$  as follows

- (a)  $d < \frac{\min(\lambda(Q))}{8|K|^2}$ ,
- (b)  $b \geq 4d + \frac{4|PB|^2}{\min(\lambda(Q))}$ ,
- (c)  $\gamma_1 < \frac{1}{2\bar{D}|K|e^{|A|\bar{D}}|B|}$ ,
- (d)  $\gamma_2 < \frac{1}{4} \min \left\{ \left( 1 - \frac{D_l}{\bar{D}} \right)^{-1}, \left( \frac{D_r}{\underline{D}} - 1 \right)^{-1} \right\}$ ,
- (e)  $c \geq \frac{4|PB|^2}{\min(\lambda(Q))} + 2b\bar{D}|K||B|e^{|A|\bar{D}}\frac{1}{\gamma_1}$ .

From (1.36), one then obtains (1.27) with

$$\eta = \min \left\{ \frac{\min(\lambda(Q)) - 8d|K|^2}{2 \max(\lambda(P))}, \frac{1 - 2\bar{D}|K||B|e^{|A|\bar{D}}\gamma_1}{2\bar{D}}, \frac{1}{4D_r}, \frac{1}{2\bar{D}} \right\} > 0, \tag{1.37}$$

$$g(\varepsilon) = \frac{2cr}{D_0} \frac{1}{\gamma_2} M_h \varepsilon. \tag{1.38}$$

Lemma 1.3 is then proved.

### 1.3.3.2 Proof of Theorem 1.1

Before concluding the proof of Theorem 1.1, we first formulate intermediate lemmas. The first one relates the functional (1.23) defined in terms of the variables of the target system with the functional (1.10) defined with respect to the original variables.

**Lemma 1.5** *The functionals  $V$  and  $Y$  defined respectively in (1.23) and (1.10) are equivalent, that is, there exist  $q_1, q_2 > 0$  such that  $q_1 V(t) \leq Y(t) \leq q_2 V(t)$  for  $t \geq 0$ .*

*Proof:* Firstly, we define another Lyapunov functional  $\Gamma(t)$  based on the state  $(X, v, \hat{v}, \mu)$

$$\Gamma(t) = |X(t)|^2 + \|v(t)\|^2 + \|\hat{v}(t)\|^2 + \|\mu(t)\|^2. \tag{1.39}$$

From the definition of distributed actuator  $\mu$ , in which  $\mu$  represents the controller  $U$  in the time interval  $[t - \bar{D} - D_0, t]$ , one gets

$$\Gamma(t) = |X(t)|^2 + \int_{t-D(t)}^t U(s)^2 ds + \int_{t-D_0}^t U(s)^2 ds + \int_{t-\bar{D}-D_0}^t U(s)^2 ds \tag{1.40}$$

and hence, we have

$$\Upsilon(t) \leq \Gamma(t) \leq 3\Upsilon(t). \quad (1.41)$$

From the backstepping transformation and its inverse, one deduces that there exist  $r_i, s_i > 0$  such that

$$\|v(t)\|^2 \leq 2\|\hat{v}(t)\|^2 + \|\tilde{v}(t)\|^2, \quad (1.42)$$

$$\|\hat{v}(t)\|^2 \leq r_1|X(t)|^2 + r_2\|w(t)\|^2, \quad (1.43)$$

$$\|\tilde{v}(t)\|^2 \leq 2\|v(t)\|^2 + 2\|\hat{v}(t)\|^2, \quad (1.44)$$

$$\|w(t)\|^2 \leq s_1|X(t)|^2 + s_2\|\hat{v}(t)\|^2. \quad (1.45)$$

Then, one can conclude that the functional  $\Gamma$  satisfies

$$\begin{aligned} \Gamma(t) &= |X(t)|^2 + \|v(t)\|^2 + \|\hat{v}(t)\|^2 + \|\mu(t)\|^2 \\ &\leq |X(t)|^2 + 3\|\hat{v}(t)\|^2 + 2\|\tilde{v}(t)\|^2 + \|\mu(t)\|^2 \leq \max\left\{\frac{1+3r_1}{\min(\lambda(P))}, \frac{3r_2}{bD_0}, \frac{2}{cD_r}, \frac{1}{d\bar{D}}\right\} V(t), \end{aligned} \quad (1.46)$$

and similarly, for the functional  $V$ ,

$$\begin{aligned} V(t) &\leq \max(\lambda(P))|X(t)|^2 + 2bD_0\|w(t)\|^2 + 2crD_r\|\tilde{v}(t)\|^2 + 2d\bar{D}\|\mu(t)\|^2 \\ &\leq (\max(\lambda(P)) + 2bD_0s_1)|X(t)|^2 + 4crD_r\|v(t)\|^2 + (2bD_0s_2 + 4crD_r)\|\hat{v}(t)\|^2 + 2d\bar{D}\|\mu(t)\|^2 \\ &\leq \max\{\max(\lambda(P)) + 2bD_0s_1, 4crD_r, 2bD_0s_2 + 4crD_r, 2d\bar{D}\}\Gamma(t). \end{aligned} \quad (1.47)$$

The desired result follows defining the constants  $q_1$  and  $q_2$  as,

$$\begin{cases} q_1 = (2 \max\{\max(\lambda(P)) + 2bD_0s_1, 4crD_r, 2bD_0s_2 + 4crD_r, 2d\bar{D}\})^{-1}, \\ q_2 = \max\left\{\frac{1+3r_1}{\min(\lambda(P))}, \frac{3r_2}{bD_0}, \frac{2}{cD_r}, \frac{1}{d\bar{D}}\right\}. \end{cases} \quad (1.48)$$

■ As Lemma 1.3 only concerns times greater than  $\underline{D}$ , we now formulate a second lemma which quantifies the stability of the system over the time interval  $[0, \bar{D}]$ .

**Lemma 1.6** *Consider the closed-loop system consisting of (1.1) and the controller (1.2). There exists a constant  $R_0$  such that the function  $\Upsilon$  defined in (1.10) satisfies*

$$\Upsilon(t) \leq R_0\Upsilon(0), \quad t \in [0, \bar{D}]. \quad (1.49)$$

*Proof:* Using (1.4) in Lemma 1.1, for  $t \in [0, \bar{D}]$ , and defining  $N_1 = 2e^{2|A|\bar{D}} \max\{1, |B|^2\bar{D}\}$ , it holds

$$\begin{aligned} |X(t)|^2 &\leq N_1 \left( |X(0)|^2 + \int_{-\bar{D}}^{t-\bar{D}} U(s)^2 ds \right) \\ &= N_1 \left( |X(0)|^2 + \int_{-\bar{D}}^{\min\{t-\bar{D}, 0\}} U_0(s)^2 ds + \int_{\min\{t-\bar{D}, 0\}}^{t-\bar{D}} U(s)^2 ds \right), \end{aligned} \quad (1.50)$$

with  $U(t) = U_0(t)$  for  $t \leq 0$ . Thus, by using Theorem 2 in [BPCE18], the prediction-based control law can be also written as

$$U(t) = K_D \left[ X(t) + \int_0^t \Phi_D(t, s) X(s) ds + \int_{-\bar{D}_0}^0 \Phi_0(t, s) U_0(s) ds \right], \quad (1.51)$$

with  $K_D = Ke^{A\bar{D}_0}$ , and  $\Phi_D$  and  $\Phi_0$  two continuous functions defined in [BPCE18]. Replacing (1.51) into (1.50), one obtains

$$\begin{aligned} |X(t)|^2 &\leq N_1 \left( |X(0)|^2 + \int_{-\bar{D}}^{\min\{t-\bar{D}, 0\}} U_0(s)^2 ds + 3K_D \int_{\min\{t-\bar{D}, 0\}}^{t-\bar{D}} |X(s)|^2 ds \right) \\ &+ 3K_D \int_{\min\{t-\bar{D}, 0\}}^{t-\bar{D}} \int_{-\bar{D}_0}^0 \Phi_0(s, \xi)^2 U_0(\xi)^2 d\xi ds + 3K_D \int_{\min\{t-\bar{D}, 0\}}^{t-\bar{D}} \int_0^s \Phi_D(s, \xi)^2 |X(\xi)|^2 d\xi ds. \end{aligned} \quad (1.52)$$



Thus, using again (1.50) and (1.51) and by a straightforward iteration on time intervals of length  $\underline{D}$ , we can get that there exist  $N_2 > 0$  and a continuous function  $\tilde{\Phi}_0$  such that

$$|X(t)|^2 \leq N_2 |X(0)|^2 + \int_{-\underline{D}}^0 \tilde{\Phi}_0(t, s) U_0(s)^2 ds. \quad (1.53)$$

Similarly, there exist a constant  $N_3 > 0$  and a continuous function  $\tilde{\tilde{\Phi}}_0$  such that

$$U(t)^2 \leq N_3 |X(0)|^2 + \int_{-\underline{D}}^0 \tilde{\tilde{\Phi}}_0(t, s) U_0(s)^2 ds. \quad (1.54)$$

Therefore, as from (1.10), it holds

$$\Upsilon(t) = |X(t)|^2 + \int_{t-\underline{D}-D_0}^0 U_0(s)^2 ds + \int_0^t U(s)^2 ds, \quad (1.55)$$

the conclusion follows from (1.53) and (1.54).  $\blacksquare$

We can now complete the last part of the proof of Theorem 1.1.

Firstly, as  $\lim_{\varepsilon \rightarrow 0} g(\varepsilon) = 0$ , there exists  $\varepsilon^* > 0$  such that  $\eta - g(\varepsilon) = \gamma > 0$  for  $\varepsilon < \varepsilon^*$ . Therefore, according to Dynkin's formula [Dyn65, Theorem 5.1, p. 132], from (1.27), one obtains for  $\varepsilon < \varepsilon^*$

$$\mathbb{E}_{[\underline{D}, (\Psi, D)(\underline{D})]}(e^{\gamma t} V(t)) - e^{\gamma \underline{D}} V(\underline{D}) = \mathbb{E}_{[\underline{D}, (\Psi, D)(\underline{D})]} \left( \int_{\underline{D}}^t [\gamma e^{\gamma s} V(s) + e^{\gamma s} L V(s)] ds \right) \leq 0. \quad (1.56)$$

from which one deduces that

$$\mathbb{E}_{[\underline{D}, (\Psi, D)(\underline{D})]}(V(t)) \leq V(\underline{D}) e^{-\gamma(t-\underline{D})}. \quad (1.57)$$

Hence, using standard conditional expectation properties, one gets that

$$\mathbb{E}_{[0, (\Psi, D)(0)]}(V(t)) \leq \mathbb{E}_{[0, (\Psi, D)(0)]}(V(\underline{D})) e^{-\gamma(t-\underline{D})}. \quad (1.58)$$

Using Lemma 1.5, it thus follows that  $\mathbb{E}_{[0, (\Upsilon(0), D(0))]}(\Upsilon(t)) \leq \frac{q_2}{q_1} \mathbb{E}_{[0, (\Upsilon(0), D(0))]}(\Upsilon(\underline{D})) e^{-\gamma(t-\underline{D})}$ . Finally, with Lemma 1.6, the function  $\Upsilon$  thus satisfies  $\mathbb{E}_{[0, (\Upsilon(0), D(0))]}(\Upsilon(t)) \leq \frac{q_2}{q_1} R_0 \Upsilon(0) e^{-\gamma(t-\underline{D})}$ , Theorem 1 is then proved with  $R = \frac{q_2}{q_1} R_0 e^{\gamma \underline{D}}$ .

## 1.4 Simulation results

To illustrate Theorem 1.1 and in particular the role played by the condition (1.8), we consider the following toy example

$$\dot{X}(t) = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} X(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} U(t - D(t)). \quad (1.59)$$

The control law (1.2) is applied with the feedback gain  $K = -[1 \ 2]$  resulting in conjugate closed-loop eigenvalues  $\lambda(A + BK) = \{-0.5000 + 1.3229i, -0.5000 - 1.3229i\}$ . The initial conditions are chosen as  $X(0) = [1 \ 0]^T$  and  $U(t) = 0$ , for  $t \leq 0$ . The integral in (1.2) is discretized using their zero-order hold approximations, in line with a suggestion in [MO79]. Finally, the simulations are carried out with a discrete-time solver in Matlab-Simulink and a sampling time  $\Delta t = 0.01$  s.

We consider 3 different delay values  $(D_1, D_2, D_3) = (0.1, 2.0, 2.1)$ . The initial transition probabilities are taken as<sup>1</sup>  $P_1(0, 0^+) = 0.02$ ,  $P_2(0, 0^+) = 0.69$  and  $P_3(0, 0^+) = 0.29$ , which means that

<sup>1</sup>The subscript  $i$  is omitted in this section, as the transition probabilities do not depend on the initial delay value. This is consistent with the fact that the expectation in (1.64) is conditioned by the initial delay value. Besides, to avoid a conflict between the initial condition in (58) and their discretized version used in simulation, we denote their initial conditions as  $P_j(0, 0^+)$ .

the delay values are initially concentrated in  $D_2$ , and then  $D_3$ . Notice that the delay margin of the closed-loop system (1.59) and (1.2) with constant delay  $D_0$  is  $\Delta D = 0.096$  (see [KBP20] for details on the computation of this quantity). Thus, the realizations of both  $D_1$  and  $D_3$  lead to a delay difference which is beyond the robustness margin of the closed-loop system, resulting in a challenging set-up.

To describe the evolution of the transition probability, we consider the forward Kolmogorov equation (58) (see the Introduction) with the transition rates

$$\tau(t) = (\{\tau_{ij}(t)\})_{1 \leq i, j \leq 3} = \tau^* \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & |\sin(kt)| \\ 0 & |\cos(kt)| & 0 \end{pmatrix}, \quad (1.60)$$

for constants  $\tau^* = 0.2$  and  $k = 0.1$ . In view of these coefficients, the delay values will fluctuate between  $D_2$  and  $D_3$ .

Firstly, we pick  $D_0 = 2$ . This results in a value  $\varepsilon = \max_{j=1,2,3} |D_0 - D_j| = 1.9$ . Figure 1.2 depicts the corresponding simulation results. In order to illustrate the mean-square feature of the stability result (1.9), we perform Monte-Carlo simulations, with 100 delay realizations. One can observe that the resulting mean value of the state, which approximates the expectation of Theorem 1.1, indeed converges to the origin.

On the other hand, the choice of a larger prediction horizon  $D_0 = 2.2$  (corresponding to the larger value  $\varepsilon = 2.1$ ) results into a diverging behavior pictured in Figure 1.3. This confirms that the choice of the prediction horizon  $D_0$  should be restricted to a range sufficiently close to the delay values to guarantee the stability of the dynamics.

Yet, the range of delay value is not the only factor to take into account. Another important one to consider is the probability distribution. Indeed, in the above example, even though the delay value  $D_1$  is quite far away from the other potential delay values and the prediction horizon  $D_0$  that we picked, it has a weak enough occurrence for the prediction horizon to be close in average to the delay realizations and to enable stabilization.

To illustrate this point, we consider a different probability distribution by now picking the transition rates as

$$\tau(t) = (\{\tau_{ij}(t)\})_{1 \leq i, j \leq 3} = \tau^* \begin{pmatrix} 0 & e^{-kt} & e^{-kt} \\ e^{-kt} & 0 & e^{-kt} \\ e^{-kt} & e^{-kt} & 0 \end{pmatrix}, \quad (1.61)$$

with constants  $\tau^* = 0.2$  and  $k = 0.1$ . This transition rate matrix  $\tau$  corresponds to the case of convergence to a uniform probability distribution (see Figure 1.4). We pick the horizon prediction as  $D_0 = 2$ , as previously. While this choice lead previously to a stable closed-loop behavior in Figure 1.2, due to the fact that  $D_0$  remains close to the most high-frequency delay realizations, it results in the present configuration into a diverging behavior pictured in Figure 1.4. Figure 1.5 illustrate this mechanism by picturing examples of delay realizations in both cases and the corresponding values of  $\mathbb{E}(|D_0 - D(t)|)$ . One can observe that this quantity has a much higher scale in the second case, which is likely to be the most determining factor influencing stability.

For this reason, we will now study how to take this feature into account in the stability analysis and, namely, how to relax the deterministic condition (1.8) of Theorem 1.1.

## 1.5 Probabilistic condition for closed-loop exponential stability

In this section, we aim at extending (1.8) to a probabilistic condition by taking the transition probabilities of the stochastic input delay into account. With this aim in view, we consider the dynamics of the transition probabilities  $P_{ij} : \mathbb{R}^2 \rightarrow [0, 1]$ . As detailed in the Introduction, under the previous assumptions made on the delay and assuming that the transition probabilities are differentiable

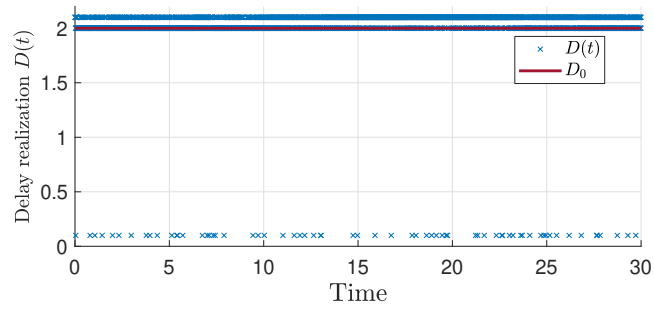
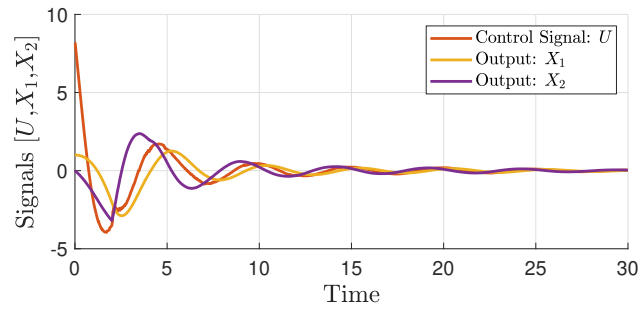
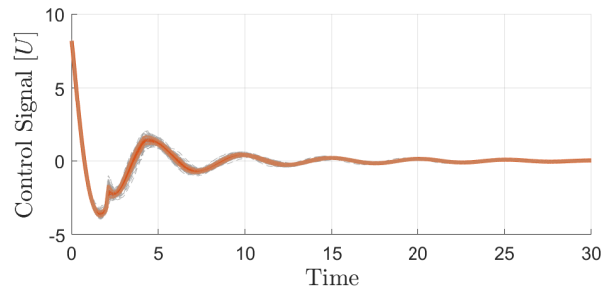
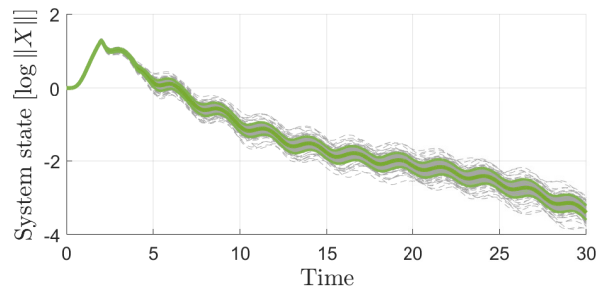
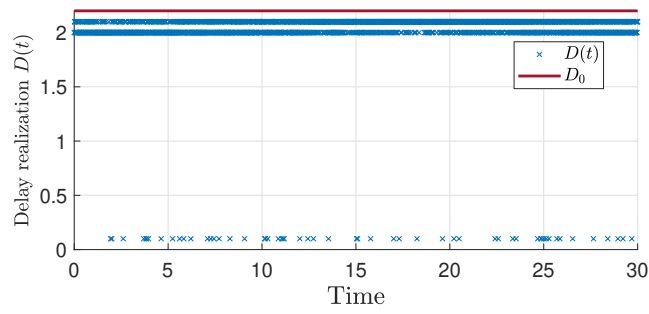
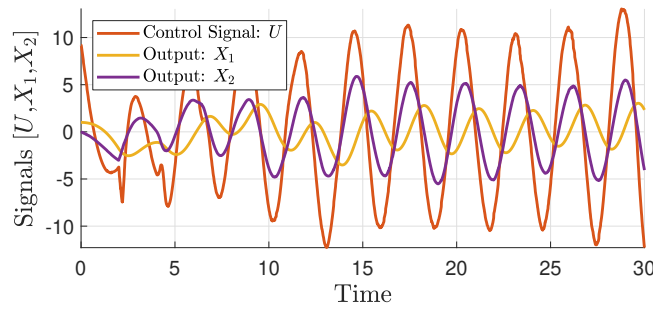

 (a) Example of a realization of the random delay  $D$ .

 (b) Example of a realization of the signals  $U$ ,  $x_1$  and  $x_2$  corresponding to the delay pictured in (a).

 (c) Monte Carlo simulation of the closed-loop input  $U$  (100 trials).

 (d) Monte Carlo simulation of the closed-loop input  $\log \|X\|$  (100 trials).

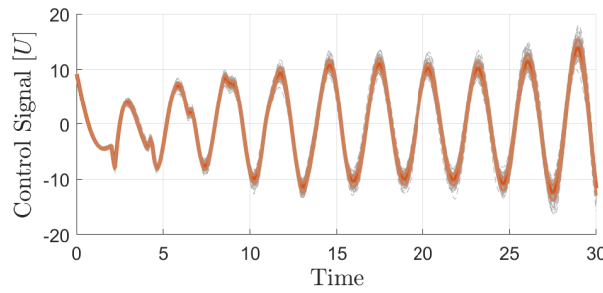
Figure 1.2: Simulation results of the closed-loop system (1.59) and (1.2) with the feedback gain  $K = -[1 \ 2]$  for  $\mathbf{D} = (0.1, 2.0, 2.1)^T$ ,  $X(0) = [1 \ 0]^T$  and  $U(t) = 0$  for  $t \leq 0$ . The prediction horizon is  $D_0 = 2.0$ . The transition probabilities follow the forward Kolmogorov equation with the transition rate defined in (1.60). (a) and (b) picture results corresponding to one delay realization. (c) and (d) present the results of 100 trials, in which the means and the standard deviations are highlighted by the coloured lines.



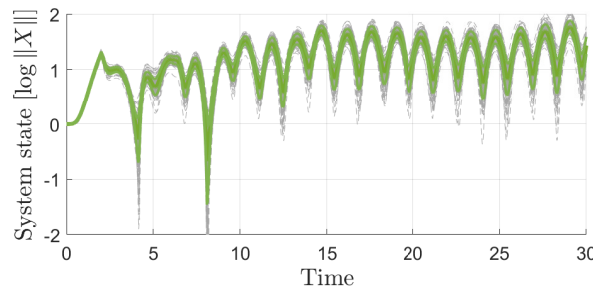
(a) Example of a realization of the random delay  $D$ .



(b) Example of a realization of the signals  $U$ ,  $x_1$  and  $x_2$  corresponding to the delay pictured in (a).

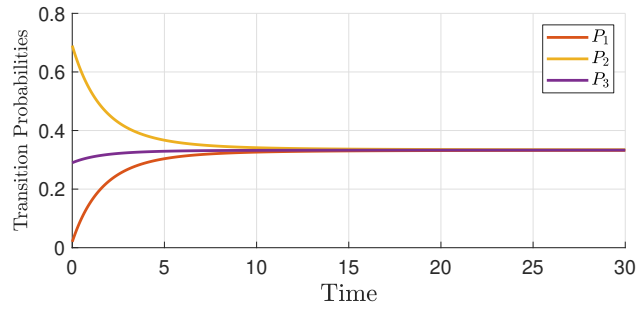


(c) Monte Carlo simulation of the closed-loop input  $U$  (100 trials).

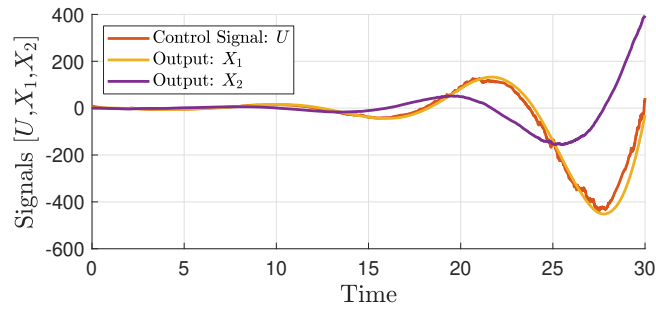


(d) Monte Carlo simulation of the closed-loop input  $\log \|X\|$  (100 trials).

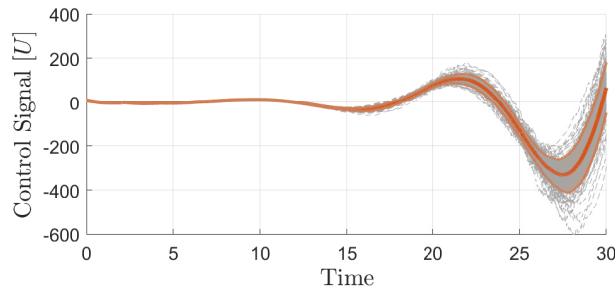
Figure 1.3: Simulation results of the closed-loop system (1.59) and (1.2) with the feedback gain  $K = -[1 \ 2]$  for  $\mathbf{D} = (0.1, 2.0, 2.1)^T$ ,  $X(0) = [1 \ 0]^T$  and  $U(t) = 0$  for  $t \leq 0$ . The prediction horizon is  $D_0 = 2.2$ . The transition probabilities follow the forward Kolmogorov equation with the transition rate defined in (1.60). (a) and (b) picture results corresponding to one delay realization. (c) and (d) present the results of 100 trials, in which the means and the standard deviations are highlighted by the coloured lines.



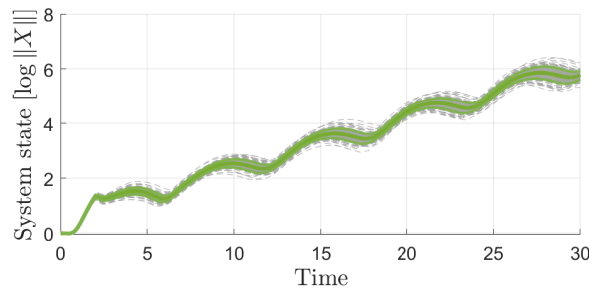
(a) Dynamic of the transition probabilities  $P_1$ ,  $P_2$  and  $P_3$ .



(b) Example of a realization of the signals  $U$ ,  $x_1$  and  $x_2$  corresponding to the delay pictured in Figure 1.5(b).

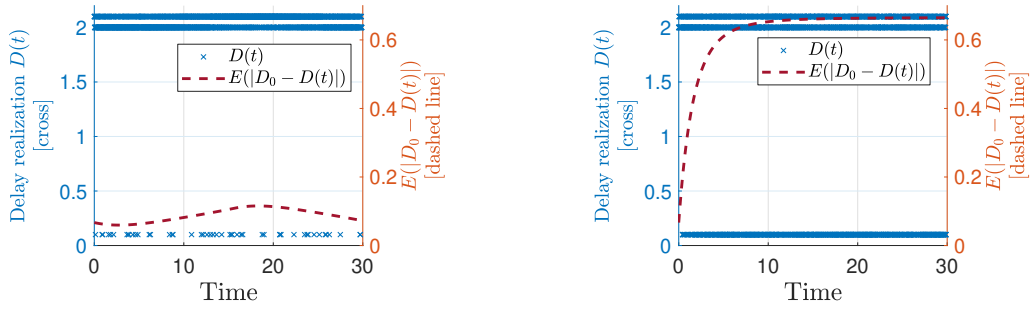


(c) Monte Carlo simulation of the closed-loop input  $U$  (100 trials).



(d) Monte Carlo simulation of the closed-loop input  $\log \|X\|$  (100 trials).

Figure 1.4: Simulation results of the closed-loop system (1.59) and (1.2) with the feedback gain  $K = -[1 \ 2]$  for  $\mathbf{D} = (0.1, 2.0, 2.1)^T$ ,  $X(0) = [1 \ 0]^T$  and  $U(t) = 0$  for  $t \leq 0$ . The prediction horizon is  $D_0 = 2.0$ . The transition probabilities follow the forward Kolmogorov equation with the transition rate defined in (1.61). (a) and (b) picture results corresponding to one delay realization. (c) and (d) present the results of 100 trials, in which the means and the standard deviations are highlighted by the coloured lines.



(a) Random delay  $D$  with transition rates  $\tau$  defined in (1.60). (b) Random delay  $D$  with transition rates  $\tau$  defined in (1.61).

Figure 1.5: Example of realizations of the random delay  $D$ , with the transition rate matrices defined in (1.60) and (1.61), respectively. The prediction horizon is picked as  $D_0 = 2.0$ , corresponding to the simulation results of Figures 1.2 and 1.4.

functions, they satisfy the following equation, called the forward Kolmogorov's equation,

$$\begin{aligned} \frac{\partial P_{ij}(s, t)}{\partial t} &= -c_j(t)P_{ij}(s, t) + \sum_{k=1}^r P_{ik}(s, t)\tau_{kj}(t), \text{ for } s < t, \\ P_{ii}(s, s) &= 1 \text{ and } P_{ij}(s, s) = 0 \text{ for } i \neq j, \end{aligned} \quad (1.62)$$

in which  $\tau_{ij}$  and  $c_j = \sum_{k=1}^r \tau_{jk}$  are positive-valued functions such that  $\tau_{ii}(t) = 0$ . We assume that the functions  $\tau_{ij}$  are upper-bounded by a certain constant  $\tau^* > 0$ .

### 1.5.1 Main Result

A probabilistic version of the previous theorem can be obtained as follows.

**Theorem 1.2** Consider the closed-loop system consisting of the system (1.1) and the control law (1.2). There exists a positive constant  $\varepsilon^*(K)$ , such that if, for all time  $t \geq 0$ ,

$$\mathbb{E}_{[0, D(0)]}(|D_0 - D(t)|) \leq \varepsilon^*(K), \quad (1.63)$$

then, the closed-loop system is mean-square exponentially stable, that is,

$$\mathbb{E}_{[0, (\Upsilon(0), D(0))]}(\Upsilon(t)) \leq R\Upsilon(0)e^{-\gamma t}, \quad (1.64)$$

for certain positive constants  $R$  and  $\gamma$  and with

$$\Upsilon(t) = |X(t)|^2 + \int_{t-3\bar{D}}^t U(s)^2 ds. \quad (1.65)$$

As Theorem 1.1, Theorem 1.2 requires the prediction horizon to be sufficiently close to the delay values, but *in average* only. This considerably generalizes the deterministic condition obtained in the previous section, by distinguishing among the delay distributions.

Besides, as for the previous proof, it is worth mentioning that an expression for the positive constant  $\varepsilon^*$  is provided<sup>2</sup> in the proof of Theorem 1.2 detailed in the sequel. However, this value is likely to be of very little practical use, due to the conservativeness of the Lyapunov analysis carried out. Nevertheless, thanks to this expression, one can observe that the positive constant  $\varepsilon^*$  depends on the feedback gain and that this dependence is likely to be considerable. Providing a quantitatively

<sup>2</sup>Namely, in Equation (1.94), involving itself various other parameters such as the intermediate constants chosen in (a)–(e) or introduced in Lemmas 1.4 and 1.9

meaningful bound and studying its relation with  $K$  in view of increasing the closed-loop robustness is out of the scope of the present work, but should be the focus of future works.

Finally, note that the interval of definition of the integral in (1.65) is longer than the one in (1.65) for a technical reason, namely, for an equivalent form of Lemma 1.4 to hold with the new form of Lyapunov functional used in the sequel.

We now provide the proof of this theorem in the following sections. This proof relies on the same steps as the ones previously introduced. For this reason, we will omit a large part of the elements previously introduced.

### 1.5.2 Transport PDEs and Backstepping transformation

As previously, we rewrite the input delay as a series of distributed inputs  $v_j$  cascaded into an ODE and introduce an additional variable  $\hat{v}$  corresponding to the prediction horizon of the controller. To this extended system  $(X, \hat{v}, \tilde{v})$ , the dynamics of which is given in (1.15), we add the distributed variable  $\mu$  as  $\mu(x, t) = U(t - D_0 + (3\bar{D} - D_0)(x - 1))$ , which represents the input history within the interval  $[t - 3\bar{D}, t - D_0]$  and satisfies the dynamics

$$\begin{cases} (3\bar{D} - D_0)\mu_t(x, t) = \mu_x(x, t) \\ \mu(1, t) = \hat{v}(0, t). \end{cases} \quad (1.66)$$

that is, again, a cascade of one transport PDE into a second one. This then leads to the following Lemma, the proof of which is similar to the one of Lemma 1.2.

**Lemma 1.7** *The backstepping transformation (1.17), jointly with the control law (1.2), transform the plant (1.15) and (1.66) into the target system  $(X, w, \tilde{v}, \mu)$*

$$\begin{cases} \dot{X}(t) = (A + BK)X(t) + B [\delta(t)^T \tilde{v}(0, t) + w(0, t)] \\ D_0 w_t(x, t) = w_x(x, t) - D_0 K e^{AD_0 x} B \delta(t)^T \tilde{v}(0, t) \\ w(1, t) = 0 \\ \Lambda_D \tilde{v}_t(x, t) = \tilde{v}_x(x, t) - \Sigma_D h(t + D_0(x - 1)) \\ \tilde{v}(1, t) = \mathbf{0} \\ (3\bar{D} - D_0)\mu_t(x, t) = \mu_x(x, t) \\ \mu(1, t) = KX(t) + w(0, t), \end{cases} \quad (1.67)$$

in which,  $h$  is defined in (1.19).

### 1.5.3 Lyapunov stability analysis

We now consider the Lyapunov functional candidate

$$\begin{aligned} V(\Psi, \delta) = & X^T P X + b D_0 \int_0^1 (1+x) w(x)^2 dx + c \int_0^1 (1+x) (\delta \cdot \mathbf{D})^T \tilde{v}(x)^2 dx \\ & + d(3\bar{D} - D_0) \int_0^1 (1+x) \mu(x)^2 dx, \end{aligned} \quad (1.68)$$

with  $b, c, d > 0$ ,  $P$  the symmetric positive definite solution of the equation  $P(A + BK) + (A + BK)^T P = -Q$ , for a given symmetric positive definite matrix  $Q$ , and  $\mathbf{D} = (D_1 \dots D_r)^T$  and where  $\cdot$  denotes the Hadamard multiplication and the square in  $\tilde{v}(x)^2$  should be understood component-wise. Note that, contrary to (1.23), the functional (1.68) explicitly depends on  $\delta$ . This should thus reflect the stochastic effect brought by the random input delay affecting the system.

As previously, we define the fixed-delay infinitesimal generator  $L_j$  of the Markov process  $(\Psi, \delta)$  as

$$L_j V(\Psi) = \frac{dV}{d\Psi}(\Psi, e_j) f_j(\Psi) + \sum_{l=1}^r (V_l(\Psi) - V_j(\Psi)) \tau_{jl}(t), \quad (1.69)$$

in which  $V_l(\Psi) = V(\psi, e_l)$  and  $f_j$  denotes the operator corresponding to the dynamics of the target system (1.67) with the fixed value  $\delta(t) = e_j$ , that is, for  $\Psi = (X, w, \tilde{\mathbf{v}}, \mu)$ ,

$$f_j(\Psi)(x) = \begin{pmatrix} (A + BK)X + Be_j^T \tilde{\mathbf{v}}(0) + Bw(0) \\ \frac{1}{D_0} [w_x(x) - D_0 K e^{AD_0 x} B e_j^T \tilde{\mathbf{v}}(0)] \\ \Lambda_D^{-1} [\tilde{\mathbf{v}}_x(x) - \Sigma_D h(\cdot + D_0(x-1))] \\ \mu_x(x)/(3\bar{D} - D_0) \end{pmatrix}. \quad (1.70)$$

Observe that this definition now involves the delay transition rates, and thus, in some sense, now explicitly accounts for the transition probabilities of the delay.

### 1.5.3.1 Lyapunov analysis

The use of the Probabilistic Delay Averaging approach allows one to formulate the following result.

**Lemma 1.8** *There exist  $(b, c, d) \in (\mathbb{R}_+^*)^3$  such that the Lyapunov functional  $V$  defined in (1.68) satisfies, for  $t \geq 2\bar{D}$ ,*

$$\sum_{j=1}^r P_{ij}(0, t) L_j V(t) \leq -(\eta - M \mathbb{E}_{[0, D_0(0)]}(|D_0 - D(t)|) - Ng(t)) V(t), \quad (1.71)$$

with  $\eta, M, N > 0$  positive constants and the function  $g$  defined as

$$g(t) \triangleq \sum_{j=1}^r |D_j - D_0|^2 \left( \frac{\partial P_{ij}(0, t)}{\partial t} + c_j(t) P_{ij}(0, t) \right), \quad (1.72)$$

with  $c_j$  defined in (1.62).

The proof of Lemma 1.8 is detailed in the remaining of this section.

For the first term in (1.69), from (1.70), applying integrations by parts and Young's inequality, one obtains

$$\begin{aligned} \frac{dV_j}{d\Psi}(\Psi) f_j(\Psi) &\leq - \left( \frac{\min(\lambda(Q))}{2} - 4d|K|^2 \right) |X(t)|^2 \\ &\quad - b(1 - 2D_0|K||B|e^{|A|D_0}\gamma_1) \|w(t)\|^2 - \left( b - 4d - \frac{4|PB|^2}{\min(\lambda(Q))} \right) w(0, t)^2 \\ &\quad - d\|\mu(t)\|^2 - \left( c - \frac{4|PB|^2}{\min(\lambda(Q))} - 2bD_0|K||B|e^{|A|D_0} \frac{1}{\gamma_1} \right) \tilde{\mathbf{v}}_j(0, t)^2 \\ &\quad - c \left( 1 - \frac{2}{D_0} |D_0 - D_j| \gamma_2 \right) \|\tilde{\mathbf{v}}_j(t)\|^2 + \frac{2c}{D_0 \gamma_2} |D_0 - D_j| \|h(t + D_0(\cdot - 1))\|^2, \end{aligned} \quad (1.73)$$

for any  $\gamma_1, \gamma_2 \geq 0$ . Observing that  $D_0 \in [\underline{D}, \bar{D}]$ , let us choose  $(b, c, d, \gamma_1, \gamma_2) \in (\mathbb{R}_+^*)^5$  as follows

- (a)  $d < \frac{\min(\lambda(Q))}{8|K|^2}$ ,
- (b)  $b \geq 4d + \frac{4|PB|^2}{\min(\lambda(Q))}$ ,



- (c)  $\gamma_1 < \frac{1}{2\bar{D}|K|e^{|\bar{A}|\bar{D}}|B|}$ ,
- (d)  $\gamma_2 < \frac{1}{4} \min \left\{ \frac{\bar{D}}{\bar{D}-D_1}, \frac{D}{D_r-\underline{D}} \right\}$ ,
- (e)  $c \geq \frac{4|PB|^2}{\min(\lambda(Q))} + 2b\bar{D}|K||B|e^{|\bar{A}|\bar{D}}\frac{1}{\gamma_1}$ ,

and define  $\eta_0 = \min\{\min(\lambda(Q))/2 - 4d|K|^2, b(1 - 2D_0|K||B|e^{|\bar{A}|\bar{D}_0}\gamma_1), d\}$ , which implies

$$\frac{dV_j}{d\Psi}(\Psi)f_j(\Psi) \leq -\eta_0 \left( |X(t)|^2 + \|w(t)\|^2 + \|\mu(t)\|^2 \right) + \frac{2c}{\gamma_2 D_0} |D_0 - D_j| \|h(t + D_0(\cdot - 1))\|^2. \quad (1.74)$$

Here, the last term remains to be bounded (which we will do by using Lemma 1.4, which still holds in the present context), but a negative term in  $\tilde{v}_{j_0}(t)$  is also missing (in which  $j_0$  is the index such that  $\delta(t)^T \tilde{\mathbf{v}}(t) = \tilde{v}_{j_0}(t)$ ). To create it, we rely on the following lemma.

**Lemma 1.9** *There exist  $N_X, N_w, N_\mu > 0$  such that, for all  $j \in \{1, \dots, r\}$  and  $t \geq \bar{D}$ ,*

$$\|\tilde{v}_j(t)\|^2 \leq N_X |X(t)|^2 + N_w \|w(t)\|^2 + N_\mu \|\mu(t)\|^2. \quad (1.75)$$

*Proof:* From the definition of the input estimation error (1.14), it follows that

$$\begin{aligned} \|\tilde{v}_j(t)\|^2 &= \int_0^1 (U(t + D_j(x - 1)) - U(t + D_0(x - 1)))^2 dx \\ &\leq \frac{4}{\underline{D}} \int_{t-D_0-\bar{D}}^t U(s)^2 ds \leq 4 \frac{\bar{D} + D_0}{\underline{D}} \left( \|\hat{v}(t)\|^2 + \|\mu(t)\|^2 \right). \end{aligned} \quad (1.76)$$

Besides, from (1.85), (1.75) follows with  $N_X = 4N_7(\bar{D} + D_0)/\underline{D}$ ,  $N_w = 4N_8(\bar{D} + D_0)/\underline{D}$  and  $N_\mu = 4(\bar{D} + D_0)/\underline{D}$ . ■

Using Lemma 1.4 and Lemma 1.9 for the index  $j_0 \in \{1, \dots, r\}$  such that  $e_{j_0} = \delta(t)$ , (1.74) finally rewrites as

$$\frac{dV_j}{d\Psi}(\Psi)f_j(\Psi) \leq -\eta V(t) + \frac{2cM_h}{\gamma_2 D_0} |D_0 - D_j| V(t), \quad (1.77)$$

with  $\eta = \frac{\eta_0}{2 \max\{\max(\lambda(P)), 2b\bar{D}, 2cD_r, \max\{N_X, N_w, N_\mu\}, 2d(3\bar{D}-\underline{D})\}}$ , in which  $N_X, N_w, N_\mu$  are defined in Lemma 1.9.

Let us now turn our attention to the second term in (1.73). By definition of the Lyapunov function (1.68), one obtains

$$\sum_{l=1}^r (V_l(\Psi) - V_j(\Psi)) \tau_{jl}(t) = \sum_{l=1}^r c \int_0^1 (1+x) \tau_{jl}(t) \left( D_l \tilde{v}_l(x, t)^2 - D_j \tilde{v}_j(x, t)^2 \right) dx, \quad (1.78)$$

in which, from the definition (1.14) of the input estimation error, one gets

$$\begin{aligned} D_l \tilde{v}_l(x, t)^2 - D_j \tilde{v}_j(x, t)^2 &= \left( \sqrt{D_l} \int_{t+D_0(x-1)}^{t+D_l(x-1)} \dot{U}(s) ds - \sqrt{D_j} \int_{t+D_0(x-1)}^{t+D_j(x-1)} \dot{U}(s) ds \right) \\ &\quad \times \left( \sqrt{D_l} \int_{t+D_0(x-1)}^{t+D_l(x-1)} \dot{U}(s) ds + \sqrt{D_j} \int_{t+D_0(x-1)}^{t+D_j(x-1)} \dot{U}(s) ds \right) \\ &\leq (1-x)^2 \left( |\sqrt{D_l} - \sqrt{D_j}| |D_j - D_0| + \sqrt{\bar{D}} |D_l - D_j| \right) \\ &\quad \times \sqrt{\bar{D}} (|D_l - D_0| + |D_j - D_0|) \max_{s \in [-\bar{D}, 0]} \dot{U}(t+s)^2. \end{aligned} \quad (1.79)$$

Then, one is left with obtaining a bound of the maximum value of the time-derivative of the input  $\dot{U}$  on the interval  $[t - \bar{D}, t]$ , which yields the following Lemma.

**Lemma 1.10** Consider the control law defined in (1.2), there exists a positive constant  $M_U$  such that

$$\max_{s \in [-\bar{D}, 0]} \dot{U}(t+s)^2 \leq M_U V(t), \quad t \geq 2\bar{D}. \quad (1.80)$$

*Proof:* The time-derivative of the control law (1.2) reads

$$\begin{aligned} \dot{U}(t+s) = & K \left[ e^{AD_0} (AX(t+s) + BU(t+s - D(t+s))) + BU(t+s) - e^{AD_0} BU(t+s - D_0) \right. \\ & \left. + A \int_{t+s-D_0}^{t+s} e^{A(t+s-\xi)} BU(\xi) d\xi \right]. \end{aligned} \quad (1.81)$$

for  $t+s \geq 0$  and  $s \in [-\bar{D}, 0]$ , and, hence, for  $t \geq \underline{D}$ . We will now bound every term in (1.81).

First, due to the form of the solution (1.4) to the dynamics (1.1), it holds

$$\begin{aligned} |X(t+s)| &= \left| e^{As} \left( X(t) - \int_{t+s}^t e^{A(t-\xi)} BU(\xi - D(\xi)) d\xi \right) \right| \\ &\leq e^{|A|\bar{D}} \left( |X(t)| + \int_{t+s}^t e^{|A|(t-\xi)} |B| \sum_{j=1}^r |U(\xi - D_j)| d\xi \right) \\ &\leq e^{|A|\bar{D}} \left( |X(t)| + e^{|A|\bar{D}} |B| r \int_{t-2\bar{D}}^t |U(\xi)| d\xi \right) \end{aligned} \quad (1.82)$$

Similarly, with the solution (1.4) and from the definition of the controller  $U$  in (1.2), we can bound the input as follows, for  $\ell \in [\max\{0, t - 2\bar{D}\}, t]$ ,

$$\begin{aligned} U(\ell) &= K \left[ e^{AD_0} X(\ell) + \int_{\ell-D_0}^{\ell} e^{A(\ell-\xi)} BU(\xi) d\xi \right] \\ &= K \left[ e^{AD_0} e^{-A(t-\ell)} \left( X(t) - \int_{\ell}^t e^{A(t-\xi)} BU(\xi - D(\xi)) d\xi \right) + \int_{\ell-D_0}^{\ell} e^{A(\ell-\xi)} BU(\xi) d\xi \right] \\ &\leq |K| \left[ e^{|A|(\bar{D}+t-\ell)} |X(t)| + e^{|A|(\bar{D}+2(t-\ell))} |B| \sum_{j=1}^r \int_{\ell}^t |U(\xi - D_j)| d\xi + e^{|A|\bar{D}} |B| \int_{t-\bar{D}}^t |U(\xi)| d\xi \right] \\ &\leq N_4 |X(t)| + N_5 \int_{t-3\bar{D}}^t |U(\xi)| d\xi \end{aligned} \quad (1.83)$$

with  $N_4 = |K|e^{2|A|D_0}$  and  $N_5 = |K||B|e^{2|A|D_0}(re^{|A|D_0} + 1)$ . This last inequality can be used to bound the pointwise input terms in (1.81), provided that  $t+s - D(t+s) \geq 0$ ,  $t+s \geq 0$  and  $t+s - D_0 \geq 0$  for all  $s \in [-\bar{D}, 0]$ , that is, that  $t \geq 2\bar{D}$ . Hence, for  $t \geq 2\bar{D}$ , using (1.82) and (1.83) in (1.81), one obtains

$$\begin{aligned} |\dot{U}(t+s)| &\leq |K| \left( |A| e^{|A|\bar{D}} \left( |X(t)| + e^{|A|\bar{D}} |B| r \int_{t-2\bar{D}}^t |U(\xi)| d\xi \right) \right. \\ &\quad \left. + |B| (1 + 2e^{|A|\bar{D}}) \left( N_4 |X(t)| + N_5 \int_{t-3\bar{D}}^t |U(\xi)| d\xi \right) + |A| e^{|A|\bar{D}} |B| \int_{t-2\bar{D}}^t |U(\xi)| d\xi \right) \\ &\leq \left( |K||A|e^{|A|\bar{D}} + |K||B|(1 + 2e^{|A|\bar{D}})N_4 \right) |X(t)| + \left( |K|e^{2|A|\bar{D}}|B|r + |K||B|(1 + 2e^{|A|\bar{D}})N_5 \right. \\ &\quad \left. + |K||B||A|e^{|A|D_0}N_5 \right) \int_{t-3\bar{D}}^t |U(\xi)| d\xi \\ &\leq N_6 |X(t)| + N_7 \|\hat{v}(t)\| + N_8 \|\mu(t)\| \end{aligned} \quad (1.84)$$

with  $N_6 = |K||A|e^{|A|\bar{D}} + |K||B|(1 + 2e^{|A|\bar{D}})N_4$ ,  $N_7 = (|K|e^{2|A|\bar{D}}|B|r + |K||B|(1 + 2e^{|A|\bar{D}})N_5 + |K||B||A|e^{|A|D_0}N_5)\bar{D}$  and  $N_8 = (|K|e^{2|A|\bar{D}}|B|r + |K||B|(1 + 2e^{|A|\bar{D}})N_5 + |K||B||A|e^{|A|D_0}N_5)3\bar{D}$ . Using Young's and Cauchy inequalities, the inverse of backstepping transformation (1.17) gives

$$\|\hat{v}(t)\|^2 \leq N_9 |X(t)|^2 + N_{10} \|w(t)\|^2, \quad (1.85)$$

with the positive constants  $N_9 = 3|K|^2 e^{2|A+BK|D_0}$  and  $N_{10} = 3(1 + |K|^2 D_0^2 e^{2|A+BK|D_0} |B|^2)$ . Applying (1.85) to (1.84), it thus holds

$$\left( \max_{s \in [-\bar{D}, 0]} \dot{U}(t+s) \right)^2 \leq C_X |X(t)|^2 + C_\mu \|\mu(t)\|^2 + C_w \|w(t)\|^2, \quad (1.86)$$

with constants defined as  $C_X = 3(N_6^2 + N_7^2 N_9)$ ,  $C_\mu = 3N_8^2$  and  $C_w = 3N_7^2 N_{10}$ .

The lemma is then proved with the positive constant  $M_2 = \max\{\frac{C_X}{\min(\lambda(P))}, \frac{C_w}{b\bar{D}}, \frac{C_\mu}{d\bar{D}}\}$ .  $\blacksquare$

Using Lemma 1.10 in (1.79), we then obtain

$$D_l \tilde{v}_l(x, t)^2 - D_j \tilde{v}_j(x, t)^2 \leq M_U M_3 \left( |D_j - D_0| + |D_l - D_j| |D_l - D_0| \right) V(t), \quad (1.87)$$

with  $M_3 = \max\left\{ (3\bar{D} - 2(\bar{D}\underline{D})^{1/2}) |\bar{D} - \underline{D}|, \bar{D} \right\}$ . Therefore, gathering (1.77), (1.78) and (1.87), one gets

$$L_j V(t) \leq -\eta V(t) + M_4 |D_0 - D_j| V(t) + N \sum_{l=1}^r \tau_{jl}(t) |D_l - D_j| |D_l - D_0| V(t), \quad (1.88)$$

with  $M_4 = 2c \left( \frac{M_h}{\gamma_2 D_0} + M_U M_3 r \tau^* \right)$  and  $N = 2c M_U M_3$ . Then, as  $\sum_{j=1}^r P_{ij}(0, t) |D_0 - D_j| = \mathbb{E}_{[0, D(0)]} (|D_0 - D(t)|)$ , the following inequality holds

$$\begin{aligned} \sum_{j=1}^r P_{ij}(0, t) L_j V(t) &\leq - \left( \eta - M_4 \mathbb{E}_{[0, D(0)]} (|D_0 - D(t)|) \right) V(t) \\ &\quad + N \sum_{j=1}^r P_{ij}(0, t) \sum_{l=1}^r \tau_{jl}(t) |D_l - D_j| |D_l - D_0| V(t). \end{aligned} \quad (1.89)$$

Hence, applying the triangle inequality and using (1.62), one finally gets

$$\begin{aligned} \sum_{j=1}^r P_{ij}(0, t) L_j V(t) &\leq - (\eta - M_4 \mathbb{E}_{[0, D(0)]} (|D(t) - D_0|)) V(t) + N \sum_{l=1}^r \sum_{j=1}^r P_{ij}(0, t) \tau_{jl}(t) |D_l - D_0|^2 V(t) \\ &\quad + N \tau^* \mathbb{E}_{[0, D(0)]} (|D(t) - D_0|) \sum_{l=1}^r |D_l - D_0| V(t) \\ &\leq - \left( \eta - (M_4 + N \tau^* r |\bar{D} - \underline{D}|) \mathbb{E}_{[0, D(0)]} (|D(t) - D_0|) \right) V(t) \\ &\quad + N \sum_{j=1}^r |D_j - D_0|^2 \left( \frac{\partial P_{ij}(0, t)}{\partial t} + c_j(t) P_{ij}(0, t) \right) V(t). \end{aligned} \quad (1.90)$$

Lemma 1.8 is then proved with  $M = M_4 + N \tau^* r |\bar{D} - \underline{D}|$ .

### 1.5.3.2 Conclusion of the stability analysis

With Lemma 1.8, we are now ready to conclude the proof of Theorem 1.2. Let us denote  $\gamma_0(t) = \eta - M \mathbb{E}_{[0, D(0)]} (|D(t) - D_0|) - Ng(t)$ , in which  $\eta$ ,  $M$ ,  $N$  and  $g$  are defined in Lemma 1.8 and introduce the functional  $Z$  as  $Z(t) = \exp\left(\int_{2\bar{D}}^t \gamma_0(s) ds\right) V(t)$ . Applying Lemma 1.8 and the definition of the infinitesimal generator  $L$  in (66), we obtain that

$$\mathbb{E}_{[2\bar{D}, (\Psi, D)(2\bar{D})]} (LZ(t)) = \exp\left(\int_{2\bar{D}}^t \gamma_0(s) ds\right) \mathbb{E}_{[2\bar{D}, (\Psi, D)(2\bar{D})]} \left( \gamma_0(t) V(t) + \sum_{j=1}^r P_{ij}(0, t) L_j V(t) \right) \leq 0.$$

Therefore, for  $t \geq 2\bar{D}$ , according to Dynkin's formula [Dyn65, Theorem 5.1, p. 132],

$$\mathbb{E}_{[2\bar{D},(\Psi,D)(2\bar{D})]}(Z(t)) - Z(2\bar{D}) = \mathbb{E}_{[2\bar{D},(\Psi,D)(2\bar{D})]} \left( \int_{2\bar{D}}^t LZ(s)ds \right) \leq 0, \quad (1.91)$$

from which, using standard conditional expectation properties, one deduces  $\mathbb{E}_{[0,(\Psi,D)(0)]}(Z(t)) \leq \mathbb{E}_{[0,(\Psi,D)(0)]}Z(2\bar{D})$ . In addition, due to the expression of  $g$  in (1.72), observe that

$$\int_0^t g(s)ds \leq (\bar{D} - \underline{D}) \left( \mathbb{E}_{[0,D(0)]}(|D_0 - D(t)|) + c^* \int_0^t \mathbb{E}_{[0,D(0)]}(|D_0 - D(s)|)ds \right), \quad (1.92)$$

as  $c_j = \sum_{k=1}^r \tau_{jk} \leq r\tau^* \triangleq c^*$ . Hence, it follows that

$$\begin{aligned} \mathbb{E}_{[0,(\Psi,D)(0)]}(Z(t)) &\geq \mathbb{E}_{[0,(\Psi,D)(0)]} \left( V(t) \exp(-N(\bar{D} - \underline{D})) \mathbb{E}_{[0,D(0)]}(|D_0 - D(t)|) \right. \\ &\quad \left. + \int_0^t (\eta - (M + Nc^*(\bar{D} - \underline{D}))) \mathbb{E}_{[0,D(0)]}(|D_0 - D(s)|) ds \right). \end{aligned} \quad (1.93)$$

Thus, if (1.63) holds with

$$\varepsilon^* \triangleq \frac{\eta}{2(M + Nc^*(\bar{D} - \underline{D}))}, \quad (1.94)$$

one obtains from (1.91) and (1.93)

$$\begin{aligned} \mathbb{E}_{[0,(\Psi,D)(0)]} \left( e^{-N(\bar{D} - \underline{D})\varepsilon^* + \frac{\eta}{2}t} V(t) \right) &\leq \mathbb{E}_{[0,(\Psi,D)(0)]}(Z(t)) \\ &\leq \mathbb{E}_{[0,(\Psi,D)(0)]}(Z(2\bar{D})) \leq e^{2\bar{D}\eta} \mathbb{E}_{[0,(\Psi,D)(0)]}(V(2\bar{D})), \end{aligned} \quad (1.95)$$

which implies, with  $\gamma = \frac{\eta}{2}$ ,

$$\mathbb{E}_{[0,(\Psi,D)(0)]}(V(t)) \leq \mathbb{E}_{[0,(\Psi,D)(0)]}(V(2\bar{D})) e^{2\bar{D}\eta + N(\bar{D} - \underline{D})\varepsilon^* - \gamma t}. \quad (1.96)$$

Similarly to the proof of Lemma 1.5, one can obtain that  $V$  defined in (1.68) and  $Y$  defined in (1.65) are equivalent, that is, there exist positive constants  $q_1$  and  $q_2$  such that for  $\forall t \geq 0$ ,  $q_1V(t) \leq Y(t) \leq q_2V(t)$  (see Lemma 1.6 for a proof of this fact in a similar case). It thus follows that  $\mathbb{E}_{[0,(\Psi,D)(0)]}(Y(t)) \leq \frac{q_2}{q_1} e^{2\bar{D}\eta + N(\bar{D} - \underline{D})\varepsilon^*} e^{-\gamma t}$ . In addition, as the dynamics (1.1) is linear, according to Lemma 1.6, there exists a constant  $R_0 > 0$  such that  $Y(t) \leq R_0Y(0)$ ,  $t \in [0, 2\bar{D}]$ . Consequently, (1.64) follows with  $R = R_0 e^{2\bar{D}\eta + N(\bar{D} - \underline{D})\varepsilon^*} q_2/q_1$ .

## 1.6 Conclusion and perspective

In this chapter, we investigated robust compensation of a random input delay by relying on a constant-horizon prediction. We first prove that, if this horizon is chosen close enough to all delay values, closed-loop stabilization is guaranteed. Then, we relaxed this condition and proved that it is sufficient that this condition holds *in average*. These studies were performed using a three-steps procedure consisting of:

- a transport PDE representation of the delay, with one input channel per delay value and a reformulation of the delay as a random process acting as an output selector;
- a backstepping transformation of a distributed variable accounting for the input over a past time window corresponding to the prediction horizon;
- an application of the Probabilistic Delay Averaging technique, to carry out the Lyapunov analysis.

This systematic procedure can be applied in a large number of contexts, as the following chapters of this manuscript will illustrate it.

However, a point of major importance to study in the future consists in quantifying the robustness margin  $\varepsilon^*$  and, in particular, its dependence on the feedback gain. Indeed, being able to tune the feedback gain to increase the robustness margin is an important practical question. Yet, the Lyapunov study provided above is very conservative and unlikely to be used to obtain meaningful robustness bounds. It is thus likely that an alternative viewpoint should complete the present work in the future.

Furthermore, adapting the prediction-horizon to the current delay distribution could also be an interesting design feature to explore, as it is likely to increase the closed-loop delay-robustness. This would allow to address the case of large variations of the delay, or at least its average. This is the problem under consideration in the following chapter.

## Chapter 2

# Time-varying horizon prediction-based controller

### Résumé

Ce chapitre poursuit l'étude d'un contrôle par prédiction pour système linéaire soumis à un retard d'entrée aléatoire, entamée au Chapitre 1, mais s'intéresse au développement d'un contrôleur à horizon de prédiction temps-variant. Nous prouvons que la stabilisation exponentielle en boucle ouverte est obtenue, pourvu que l'horizon de prédiction temps-variant reste suffisamment proche des valeurs du retard, mais également que le taux de variation de cet horizon de prédiction reste suffisamment faible.

**Abstract** This chapter studies further the prediction-based control of a linear system affected by a random input delay considered in Chapter 1 but investigates the design of a time-varying prediction horizon controller. We prove that closed-loop exponential stabilization is achieved, provided that the time-varying prediction horizon and the delay values are sufficiently close, but also that the variation rate of the prediction horizon remains sufficiently small.

## 2.1 Problem statement and controller design

This chapter wrestles with the same compensation problem as Chapter 1, which concentrates on the linear systems with a stochastic input delay as follows,

$$\dot{X}(t) = AX(t) + BU(t - D(t)), \quad (2.1)$$

in which  $A$  and  $B$  are constant dynamics matrices such that the pair  $(A, B)$  is controllable and the  $\mathbb{R}^n$ -valued random variable  $X$  and  $U \in \mathbb{R}$  are the state and control input, respectively.

The stochastic input delay  $D$  is assumed to be a Markov process with a finite number of states  $D(t) \in \{D_i, i \in \{1, \dots, r\}\}$  ( $r \in \mathbb{N}, t \geq 0$ ), and the potential delay values are ordered as  $0 < D_1 < D_2 < \dots < D_r$ . We denote  $P_{ij}(t_1, t_2)$  the transition probability of the delay. Assuming that these functions are differentiable, they satisfy (see the Introduction) the following Kolmogorov equation

$$\begin{aligned} \frac{\partial P_{ij}(s, t)}{\partial t} &= -c_j(t)P_{ij}(s, t) + \sum_{k=1}^r P_{ik}(s, t)\tau_{kj}(t), \text{ for } s < t, \\ P_{ii}(s, s) &= 1 \text{ and } P_{ij}(s, s) = 0 \text{ for } i \neq j, \end{aligned} \quad (2.2)$$

in which  $\tau_{ij}$  and  $c_j = \sum_{k=1}^r \tau_{jk}$  are positive-valued functions such that  $\tau_{ii}(t) = 0$ . We assume in the following that the functions  $\tau_{ij}$  are upper-bounded by a certain constant  $\tau^* > 0$ . Besides, we assume that the realizations of this Markov process are right-continuous, in order to assess the well-posedness of the system.

In this chapter, to better compensate for the random input delay and account for the potential variations of the delay probability distribution, we propose to use a prediction-based controller with a time-varying prediction horizon, as

$$U(t) = K \left[ e^{AD_0(t)} X(t) + \int_{t-D_0(t)}^t e^{A(t-s)} BU(s) ds \right], \quad (2.3)$$

in which  $K$  is a feedback gain such that  $A + BK$  is Hurwitz and  $D_0 : \mathbb{R}_+ \rightarrow [\underline{D}, \overline{D}]$  is a function to be designed and such that  $0 < \underline{D} \leq D_0(t) \leq \overline{D}$  for all  $t \geq 0$ . As this controller is directly inspired by the constant delay case, namely by replacing the constant delay  $D$  in (6) with the time-varying variable  $D_0(t)$ , it is likely that this controller will only hold for sufficiently small prediction horizon variations. This is the question we investigate in this chapter.

## 2.2 Sufficient condition for closed-loop exponential stability

### 2.2.1 Main result

We consider that the prediction horizon function  $D_0$  satisfies the following assumption.

**Assumption 2.1**  $D_0$  is a continuously differentiable function, and there exist constants  $\Delta_0^*$  and  $\Delta_1^*$  such that

$$\Delta_0^* \leq \dot{D}_0(t) \leq \Delta_1^* < 1. \quad (2.4)$$

Under this condition, we are then able to formulate the following result.

**Theorem 2.1** Consider the closed-loop system consisting of the plant (2.1) and of the controller (2.3) with a prediction horizon  $D_0 : \mathbb{R}_+ \rightarrow [\underline{D}, \overline{D}]$  satisfying Assumption 2.1. There exist positive constants  $\Delta^*(K)$  and  $\varepsilon^*(K)$ , such that if, for all time  $t \geq 0$ ,

$$\max\{|\Delta_0^*|, |\Delta_1^*|\} \leq \Delta^*(K), \quad (2.5)$$

$$\mathbb{E}_{[0, (\Upsilon, D)(0)]}(|D_0(t) - D(t)|^2) \leq \varepsilon^*(K), \quad (2.6)$$

then, the closed-loop system is mean-square exponentially stable, that is,

$$\mathbb{E}_{[0,(\Upsilon(0),D(0))]}(\Upsilon(t)) \leq R\Upsilon(0)e^{-\gamma t}, \quad (2.7)$$

for certain positive constants  $R$  and  $\gamma$  and with

$$\Upsilon(t) = |X(t)|^2 + \int_{t-3\bar{D}}^t U(s)^2 ds. \quad (2.8)$$

Theorem 2.1 requires the prediction horizon and the delay values to be sufficiently close in average, in the sense of (2.6). This is in accordance with the probabilistic condition obtained in Theorem 1.2, which requires a very similar property. Yet, with (2.5), Theorem 2.1 also requires the prediction horizon to vary sufficiently slowly. This is consistent with the constant-horizon feature of the prediction-based controller (1.2), as expected.

Obviously, the choice of the prediction horizon  $D_0(t)$  is a question of utmost importance, that we will address in Section 2.3. Let us just observe that a natural choice could be to pick the expected value of the delay, that is,  $D_0(t) = \mathbb{E}_{[0,D(0)]}(D(t))$ . Yet, this requires the knowledge of some statistical properties of the delay, which may not be always achievable. This is one of the reasons why we formulate the analysis in the general case, in the following.

The proof of Theorem 2.1 which we now detail follows the generic three-steps methodology proposed in Chapter 1.

## 2.2.2 Proof of Theorem 2.1

### 2.2.2.1 PDE representation

Corresponding to the random input delay  $D(t) \in \{D_1, \dots, D_r\}$ , we introduce the distributed actuator  $\mathbf{v}(x, t) = (v_1(x, t) \dots v_k(x, t) \dots v_r(x, t))^T$  with  $v_k(x, t) = U(t + D_k(x - 1))$ . System (2.1) then rewrites as

$$\begin{cases} \dot{X}(t) = AX(t) + B\delta(t)^T \mathbf{v}(0, t) \\ \Lambda_D \mathbf{v}_t(x, t) = \mathbf{v}_x(x, t) \\ \mathbf{v}(1, t) = \mathbf{1}U(t), \end{cases} \quad (2.9)$$

in which  $\Lambda_D = \text{diag}(D_1, \dots, D_r)$ ,  $\mathbf{1}$  is a  $r$ -by-1 all-ones vector and  $\delta(t) \in \mathbb{R}^r$  is such that, if  $D(t) = D_j$ ,  $\delta(t) = e_j$ , the  $j^{\text{th}}$ -unit vector.

Then, we introduce  $\hat{v}$  to represent the input  $U$  within the interval  $[t - D_0(t), t]$ , and the corresponding input error  $\tilde{\mathbf{v}}$ . Besides, let us consider  $\mu$  which describes the control input within the interval  $[t - 3\bar{D}, t - D_0(t)]$  as

$$\hat{v}(x, t) = U(t + D_0(t)(x - 1)), \quad (2.10)$$

$$\tilde{\mathbf{v}}(x, t) = \mathbf{v}(x, t) - \mathbf{1}\hat{v}(x, t), \quad (2.11)$$

$$\mu(x, t) = U(t - D_0(t) + (3\bar{D} - D_0(t))(x - 1)). \quad (2.12)$$

One can observe that these quantities are the same as the ones introduced in Chapter 1, but with the time-varying quantity  $D_0(t)$  in lieu of  $D_0$ . Again, this is consistent with the constant-horizon inspiration of the prediction-based controller (2.3).

Taking time- and space-derivatives of these quantities, one can easily obtain that the extended



state  $(X, \hat{v}, \tilde{v}, \mu)$  satisfies

$$\left\{ \begin{array}{l} \dot{X}(t) = AX(t) + B\hat{v}(0, t) + B\delta(t)^T \tilde{v}(0, t) \\ \hat{v}_t(x, t) = \pi_1(x, t)\hat{v}_x(x, t) \\ \hat{v}(1, t) = U(t) \\ \Lambda_D \tilde{v}_t(x, t) = \tilde{v}_x(x, t) - \rho(x, t)\hat{v}_x(x, t) \\ \tilde{v}(1, t) = \mathbf{0} \\ \mu_t(x, t) = \pi_2(x, t)\mu_x(x, t) \\ \mu(1, t) = \hat{v}(0, t), \end{array} \right. \quad (2.13)$$

with the distributed parameters  $\pi_1, \pi_2$  and  $\rho$  defined as

$$\begin{aligned} \pi_1(x, t) &= \frac{1 + \dot{D}_0(t)(x-1)}{D_0(t)}, \quad \pi_2(x, t) = \frac{1 - \dot{D}_0(t)x}{3\bar{D} - D_0(t)}, \\ \rho(x, t) &= (\rho_1 \quad \rho_2 \quad \cdots \quad \rho_r)^T \text{ with } \rho_j(x, t) = D_j \pi_1(x, t) - 1, \end{aligned}$$

in which, according to Assumption 2.1,  $\pi_1, \pi_2$  are positive-valued functions defined for  $x \in [0, 1]$  and  $t \geq 0$ .

### 2.2.2.2 Backstepping transformation

We now introduce the backstepping transformation

$$w(x, t) = \hat{v}(x, t) - Ke^{Ax D_0(t)} X(t) - KD_0(t) \int_0^x e^{A(x-y)D_0(t)} B\hat{v}(y, t) dy. \quad (2.14)$$

Again, these quantities are the same as the ones introduced in Chapter 1, but with the time-varying prediction horizon  $D_0(t)$  in lieu of  $D_0$ .

**Lemma 2.1** *The backstepping transformation (2.14), jointly with the control law (2.3) transform the plant (2.13) into the target system  $(X, w, \tilde{v}, \mu)$*

$$\left\{ \begin{array}{l} \dot{X}(t) = (A + BK)X(t) + B\delta(t)^T \tilde{v}(0, t) + Bw(0, t) \\ w_t(x, t) = \pi_1(x, t)w_x(x, t) - Ke^{Ax D_0(t)} B\delta(t)^T \tilde{v}(0, t) - \dot{D}_0(t)q(x, t) \\ w(1, t) = 0 \\ \Lambda_D \tilde{v}_t(x, t) = \tilde{v}_x(x, t) - \rho(x, t)h(t + D_0(t)(x-1)) \\ \tilde{v}(1, t) = \mathbf{0} \\ \mu_t(x, t) = \pi_2(x, t)\mu_x(x, t) \\ \mu(1, t) = KX(t) + w(0, t), \end{array} \right. \quad (2.15)$$

with the functions  $q$  and  $h$  defined as

$$q(x, t) = K(A + BK)e^{AD_0(t)x} X(t) + Ke^{AD_0(t)x} Bw(0, t), \quad (2.16)$$

$$\begin{aligned} h(t) &= D_0(t)K \left[ A\dot{D}_0(t)e^{AD_0(t)} X(t) + e^{AD_0(t)} (A + BK)X(t) + e^{AD_0(t)} B\delta(t)^T \tilde{v}(0, t) \right. \\ &\quad + BK e^{(A+BK)D_0(t)} X(t) + B \int_0^1 KD_0(t) e^{(A+BK)D_0(t)(1-y)} Bw(y, t) dy \\ &\quad + e^{AD_0(t)} Bw(0, t) - (1 - \dot{D}_0(t))e^{AD_0(t)} Bw(0, t) - (1 - \dot{D}_0(t))e^{AD_0(t)} BKX(t) \\ &\quad + \int_0^1 A e^{AD_0(t)(1-y)} B \left( w(y, t) + Ke^{(A+BK)D_0(t)y} X(t) \right. \\ &\quad \left. \left. + \int_0^y KD_0(t) e^{(A+BK)D_0(t)(y-z)} Bw(z, t) dz \right) dy \right]. \end{aligned} \quad (2.17)$$

*Proof:* Taking the time-derivative of (2.14), it follows that

$$\begin{aligned}
 w_t(x, t) &= \hat{v}_t(x, t) - Ke^{AD_0(t)x}AX(t) - Ke^{AD_0(t)x}B\hat{v}(0, t) - Ke^{AD_0(t)x}B\delta(t)^T\tilde{v}(0, t) \\
 &\quad - \dot{D}_0(t)xKAe^{AD_0(t)x}X(t) - \dot{D}_0(t)\int_0^x Ke^{AD_0(t)(x-y)}B\hat{v}(y, t)dy \\
 &\quad - D_0(t)\int_0^x KA(x-y)\dot{D}_0(t)e^{AD_0(t)(x-y)}B\hat{v}(y, t)dy \\
 &\quad - D_0(t)\int_0^x Ke^{AD_0(t)(x-y)}B\hat{v}_t(y, t)dy.
 \end{aligned} \tag{2.18}$$

By using (2.13) and an integration by parts, the term containing  $\hat{v}_t$  in (2.18) can be rewritten as

$$\begin{aligned}
 D_0(t)\int_0^x Ke^{AD_0(t)(x-y)}B\hat{v}_t(y, t)dy \\
 &= D_0(t)\int_0^x Ke^{AD_0(t)(x-y)}B\pi_1(y, t)\hat{v}_x(y, t)dy \\
 &= \int_0^x Ke^{AD_0(t)(x-y)}B(1 + \dot{D}_0(t)(y-1))\hat{v}_x(y, t)dy \\
 &= KB(1 + \dot{D}_0(t)(x-1))\hat{v}(x, t) - Ke^{AD_0(t)x}B(1 - \dot{D}_0(t))\hat{v}(0, t) \\
 &\quad + \int_0^x K(AD_0(t)(1 + \dot{D}_0(t)(y-1)) - \dot{D}_0(t))e^{AD_0(t)(x-y)}B\hat{v}(y, t)dy.
 \end{aligned} \tag{2.19}$$

Besides, the space-derivative of the backstepping transformation (2.14) is

$$\begin{aligned}
 w_x(x, t) &= \hat{v}_x(x, t) - KAD_0(t)e^{AD_0(t)x}X(t) - D_0(t)KB\hat{v}(x, t) \\
 &\quad - D_0(t)^2\int_0^x KAe^{AD_0(t)(x-y)}B\hat{v}(y, t)dy.
 \end{aligned} \tag{2.20}$$

Using (2.19), the difference between (2.18) and (2.20) then gives

$$\begin{aligned}
 D_0(t)w_t(x, t) - (1 + \dot{D}_0(t)(x-1))w_x(x, t) \\
 &= D_0(t)\left(\hat{v}_t(x, t) - Ke^{AD_0(t)x}AX(t) - Ke^{AD_0(t)x}B\hat{v}(0, t) - Ke^{AD_0(t)x}B\delta(t)^T\tilde{v}(0, t) \right. \\
 &\quad - \dot{D}_0(t)xKAe^{AD_0(t)x}X(t) - \dot{D}_0(t)\int_0^x Ke^{AD_0(t)(x-y)}B\hat{v}(y, t)dy \\
 &\quad - D_0(t)\int_0^x KA(x-y)\dot{D}_0(t)e^{AD_0(t)(x-y)}B\hat{v}(y, t)dy \\
 &\quad - KB(1 + \dot{D}_0(t)(x-1))\hat{v}(x, t) + Ke^{AD_0(t)x}B(1 - \dot{D}_0(t))\hat{v}(0, t) \\
 &\quad \left. - \int_0^x K(AD_0(t)(1 + \dot{D}_0(t)(y-1)) - \dot{D}_0(t))e^{AD_0(t)(x-y)}B\hat{v}(y, t)dy\right) \\
 &\quad - (1 + \dot{D}_0(t)(x-1))\left(\hat{v}_x(x, t) - KAD_0(t)e^{AD_0(t)x}X(t) - D_0(t)KB\hat{v}(x, t) \right. \\
 &\quad \left. - D_0(t)^2\int_0^x KAe^{AD_0(t)(x-y)}B\hat{v}(y, t)dy\right) \\
 &= -D_0(t)\dot{D}_0(t)KAe^{AD_0(t)x}X(t) - D_0(t)\dot{D}_0(t)Ke^{AD_0(t)x}B\hat{v}(0, t) \\
 &\quad - D_0(t)Ke^{AD_0(t)x}B\delta(t)^T\tilde{v}(0, t) \\
 &= -D_0(t)\dot{D}_0(t)K(A+BK)e^{AD_0(t)x}X(t) - D_0(t)\dot{D}_0(t)Ke^{AD_0(t)x}Bw(0, t) \\
 &\quad - D_0(t)Ke^{AD_0(t)x}B\delta(t)^T\tilde{v}(0, t).
 \end{aligned} \tag{2.21}$$

This leads to the second equation in (2.15) with the function  $q$  defined in (2.16).

Besides, from (2.13), we have  $h(t + D_0(x - 1)) = \hat{v}_x(x, t) = D_0(t)\dot{U}(t + D_0(t)(x - 1))$ . Then, with (2.23), (2.24) and (2.25), the function  $h$  can be represented as

$$\begin{aligned} h(t) &= D_0(t)\dot{U}(t) \\ &= D_0(t)K \left[ A\dot{D}_0(t)e^{AD_0(t)}X(t) + e^{AD_0(t)}(A + BK)X(t) + e^{AD_0(t)}B\delta(t)^T\tilde{\mathbf{v}}(0, t) \right. \\ &\quad \left. + BU(t) + e^{AD_0(t)}Bw(0, t) - (1 - \dot{D}_0(t))e^{AD_0(t)}BU(t - D_0(t)) \right. \\ &\quad \left. + \int_{t-D_0(t)}^t Ae^{A(t-s)}BU(s)ds \right]. \end{aligned} \quad (2.22)$$

Using the inverse backstepping transformation of (2.14), which is

$$\hat{v}(x, t) = w(x, t) + Ke^{(A+BK)D_0(t)x}X(t) + \int_0^x KD_0(t)e^{(A+BK)D_0(t)(x-y)}Bw(y, t)dy, \quad (2.23)$$

$U(t)$  and  $U(t - D_0(t))$  can be described in terms of the target system state  $(X, w, \tilde{\mathbf{v}}, \mu)$  as

$$U(t) = \hat{v}(1, t) = Ke^{(A+BK)D_0(t)}X(t) + \int_0^1 KD_0(t)e^{(A+BK)D_0(t)(1-y)}Bw(y, t)dy, \quad (2.24)$$

$$U(t - D_0(t)) = \hat{v}(0, t) = w(0, t) + KX(t). \quad (2.25)$$

Replacing in (2.22) the expressions of  $U(t)$  and  $U(t - D_0(t))$  given in (2.24) and (2.25) finally gives (2.17).

In addition, from the definition of the backstepping transformation (2.14) evaluated for  $x = 0$  and  $x = 1$ , one can also obtain the following relations

$$\begin{aligned} w(1, t) &= \hat{v}(1, t) - Ke^{AD_0(t)}X(t) - KD_0(t) \int_0^1 e^{A(1-y)D_0(t)}B\hat{v}(y, t)dy \\ &= \hat{v}(1, t) - U(t) = 0, \\ \mu(1, t) &= \hat{v}(0, t) = KX(t) + w(0, t), \\ \dot{X}(t) &= AX(t) + B\hat{v}(0, t) + B\delta(t)^T\tilde{\mathbf{v}}(0, t) \\ &= AX(t) + B(KX(t) + w(0, t)) + B\delta(t)^T\tilde{\mathbf{v}}(0, t). \end{aligned}$$

We finally obtain the target system represented by the distributed variables  $(X, w, \tilde{\mathbf{v}}, \mu)$ , and the lemma is then proved.  $\blacksquare$

Notice that the transport PDE of  $w$  in (2.15) involves two source terms, the first one accounts for the distributed input error at  $x = 0$  while the second one involves the variation rate of the time-varying prediction horizon. The latter should be sufficiently small to guarantee the stability of the target system, and gives rise to (2.5) in Theorem 2.1. Similarly, and as in the analysis carried out in Chapter 1 the source term appearing in the dynamics of  $\tilde{\mathbf{v}}$  is the one which leads to the second condition (2.6) in Theorem 2.1.

### 2.2.2.3 Step 3: Lyapunov stability analysis with probabilistic delay averaging

We consider the following Lyapunov functional candidate for the target system given in (2.15)

$$V(\Psi, \delta) = X^T P X + \alpha_1 \int_0^1 e^{\alpha_2 x} w(x)^2 dx + \beta_1 \int_0^1 e^{\beta_2 x} (\delta \cdot \mathbf{D})^T \tilde{\mathbf{v}}(x)^2 dx + \theta_1 \int_0^1 e^{\theta_2 x} \mu(x)^2 dx. \quad (2.26)$$

with  $\alpha_1, \alpha_2, \beta_1, \beta_2, \theta_1, \theta_2 > 0$ ,  $P$  the symmetric positive definite solution of the equation  $P(A + BK) + (A + BK)^T P = -Q$ , for a given symmetric positive definite matrix  $Q$ ,  $\mathbf{D} = (D_1 \dots D_r)^T$  and

where  $\cdot$  denotes the Hadamard multiplication and the square in  $\tilde{\mathbf{v}}(x)^2$  should be understood component-wise. This Lyapunov functional is very close to the one proposed in Chapter 1 in (1.68), but simply involves different kernels, which slightly simplifies the following Lyapunov computation.

As previously, we define the fixed-delay infinitesimal generator  $L_j$  of the Markov process  $(\Psi, \delta)$  as

$$L_j V(t) = \frac{dV}{d\Psi}(\Psi, e_j) f_j(\Psi, t) + \sum_{l=1}^r (V_l(\Psi) - V_j(\Psi)) \tau_{jl}(t), \quad (2.27)$$

in which  $V_l(\Psi) = V(\psi, e_l)$  and  $f_j$  denotes the operator corresponding to the dynamics of the target system (2.15) with the fixed value  $\delta(t) = e_j$ , that is, for  $\Psi = (X, w, \tilde{\mathbf{v}}, \mu)$ ,

$$f_j(\Psi, t)(x) = \begin{pmatrix} (A + BK)X + Be_j^T \tilde{\mathbf{v}}(0) + Bw(0) \\ \pi_1(x, t)w_x(x) - Ke^{AD_0(t)x} Be_j^T \tilde{\mathbf{v}}(0) - \dot{D}_0(t)q(x, t) \\ \Lambda_D^{-1} [\tilde{\mathbf{v}}_x(x) - \rho(x, t)h(t + D_0(t)(x - 1))] \\ \pi_2(x, t)\mu_x(x) \end{pmatrix}. \quad (2.28)$$

With all these ingredients, we are now ready to develop the Lyapunov analysis.

**Lemma 2.2** *There exist  $(\alpha_1, \alpha_2, \beta_1, \beta_2, \theta_1, \theta_2) \in (\mathbb{R}_+^*)^6$  and  $\Delta^* > 0$  such that, if*

$$\max\{|\Delta_0^*|, |\Delta_1^*|\} \leq \Delta^*, \quad (2.29)$$

*then, the Lyapunov functional  $V$  defined in (2.26) satisfies for  $t \geq 2\bar{D}$*

$$\sum_{j=1}^r P_{ij}(0, t) L_j V(t) \leq - \left( \eta - M_1 \mathbb{E}(|D(t) - D_0(t)|^2) - M_2 \mathbb{E}(|D(t) - D_0(t)|) - M_3 g(t) \right) V(t), \quad (2.30)$$

*for given positive constants  $\eta, M_1, M_2, M_3$ , and the function  $g$  defined as*

$$g(t) = \sum_{j=1}^r P_{ij}(0, t) \sum_{l=1}^r |D_l - D_0(t)|^2 \tau_{jl}(t).$$

We provide the proof of Lemma 2.2 in the remaining part of this section. Using integrations by parts, the first term in (2.27) gives

$$\begin{aligned} \frac{dV_j}{d\Psi}(\Psi) f_j(\Psi, t) &= -X(t)^T Q X(t) + 2X(t)^T P B (w(0, t) + \tilde{v}_j(0, t)) - \frac{1 - \dot{D}_0(t)}{D_0(t)} \alpha_1 w(0, t)^2 \\ &\quad - \alpha_1 \int_0^1 e^{\alpha_2 x} \frac{\alpha_2 (1 + \dot{D}_0(t)(x - 1)) + \dot{D}_0(t)}{D_0(t)} w(x, t)^2 dx \\ &\quad - 2\alpha_1 \int_0^1 e^{\alpha_2 x} w(x, t) K e^{AD_0(t)x} B (\dot{D}_0(t)w(0, t) + \tilde{v}_j(0, t)) dx \\ &\quad - 2\alpha_1 \dot{D}_0(t) \int_0^1 e^{\alpha_2 x} w(x, t) K e^{AD_0(t)x} (A + BK) X(t) dx - \beta_1 \tilde{v}_j(0, t)^2 \\ &\quad - \beta_1 \beta_2 \int_0^1 e^{\beta_2 x} \tilde{v}_j(x, t)^2 dx - 2\beta_1 \int_0^1 e^{\beta_2 x} \tilde{v}_j(x, t) \rho_j(x, t) h(t + D_0(t)(x - 1)) dx \\ &\quad + \theta_1 \frac{e^{\theta_2} (1 - \dot{D}_0(t))}{3\bar{D} - D_0(t)} (KX(t) + w(0, t))^2 - \frac{\theta_1}{3\bar{D} - D_0(t)} \mu(0, t)^2 \\ &\quad - \theta_1 \int_0^1 e^{\theta_2 x} \frac{\theta_2 (1 - \dot{D}_0(t)x) - \dot{D}_0(t)}{3\bar{D} - D_0(t)} \mu(x, t)^2 dx. \end{aligned} \quad (2.31)$$

To bound (2.31) with negative terms, we aim at finding the minimum value of  $\alpha_2(1 + \dot{D}_0(t)(x-1)) + \dot{D}_0(t)$  and  $\theta_2(1 - \dot{D}_0(t)x) - \dot{D}_0(t)$  for  $x \in [0, 1]$ . It holds  $\min_{x \in [0, 1]} (\alpha_2(1 + \dot{D}_0(t)(x-1)) + \dot{D}_0(t)) = \min\{\alpha_2(1 - \dot{D}_0(t)) + \dot{D}_0(t), \alpha_2 + \dot{D}_0(t)\}$ , as we know that this function is monotonic with respect to  $x$ . Hence, its minimum value will appear at one of its ends (when  $x = 0$  or  $x = 1$ , depending on the sign of  $\dot{D}_0(t)$ ). Furthermore, as these last functions are linear with respect to  $\dot{D}_0$ , with Assumption 2.1, it holds that  $\min_{x \in [0, 1]} (\alpha_2(1 + \dot{D}_0(t)(x-1)) + \dot{D}_0(t)) = \min\{\alpha_2(1 - \Delta_0^*) + \Delta_0^*, \alpha_2(1 - \Delta_1^*) + \Delta_1^*, \alpha_2 + \Delta_0^*\}$ . Similarly,  $\min_{x \in [0, 1]} (\theta_2(1 - \dot{D}_0(t)x) - \dot{D}_0(t)) \geq \min\{\theta_2 - \dot{D}_0(t), \theta_2(1 - \dot{D}_0(t)) - \dot{D}_0(t)\} \geq \min\{\theta_2 - \Delta_1^*, \theta_2(1 - \Delta_1^*) - \Delta_1^*\}$ .

Then, by applying Young's inequality and Assumption 2.1, we obtain the following inequality

$$\begin{aligned}
 \frac{dV_j}{d\Psi}(\Psi)f_j(\Psi, t) \leq & - \left( \frac{\min(\lambda(Q))}{4} - \frac{\theta_1 e^{\theta_2(1 - \Delta_0^*)}}{\bar{D}} |K|^2 \right) |X(t)|^2 \\
 & - \alpha_1 \left( \frac{\min\{\alpha_2(1 - \Delta_0^*) + \Delta_0^*, \alpha_2(1 - \Delta_1^*) + \Delta_1^*, \alpha_2 + \Delta_0^*\}}{\bar{D}} - |K|e^{|\Lambda|\bar{D}}|B|(\gamma_1 + \gamma_2)e^{\alpha_2} \right. \\
 & \left. - \frac{4\alpha_1 \max\{|\Delta_0^*|, |\Delta_1^*|\}^2 e^{2\alpha_2} |K|^2 e^{2|\Lambda|\bar{D}} |A + BK|^2}{\min(\lambda(Q))} \right) \|w(t)\|^2 - \beta_1 \beta_2 \|\tilde{v}_j(t)\|^2 \\
 & - \theta_1 \frac{\min\{\theta_2 - \Delta_1^*, \theta_2(1 - \Delta_1^*) - \Delta_1^*\}}{3\bar{D} - \underline{D}} \|\mu(t)\|^2 - \left( \alpha_1 \frac{1 - \Delta_1^*}{\bar{D}} \right. \\
 & \left. - \frac{4|PB|^2}{\min(\lambda(Q))} - \alpha_1 e^{\alpha_2} |K|e^{|\Lambda|\bar{D}}|B| \max\{|\Delta_0^*|, |\Delta_1^*|\}^2 \frac{1}{\gamma_1} - \frac{\theta_1 e^{\theta_2(1 - \Delta_0^*)}}{\bar{D}} \right) w(0, t)^2 \\
 & - \left( \beta_1 - \alpha_1 e^{\alpha_2} |K|e^{|\Lambda|\bar{D}}|B| \frac{1}{\gamma_2} - \frac{4|PB|^2}{\min(\lambda(Q))} \right) \tilde{v}_j(0, t)^2 \\
 & - 2\beta_1 \int_0^1 e^{\beta_2 x} \frac{D_j - D_0(t) + D_j \dot{D}_0(t)(x-1)}{D_0(t)} \tilde{v}_j(x, t) h(t + D_0(t)(x-1)) dx.
 \end{aligned} \tag{2.32}$$

The main problem left to consider now is to bound the last term of (2.32). To do so, we formulate two intermediate lemmas.

**Lemma 2.3** *Consider the control law defined in equation (2.3), there exists  $M_U > 0$  such that*

$$\left( \max_{s \in [-\bar{D}, 0]} \dot{U}(t+s) \right)^2 \leq M_U V(t), \quad t \geq 2\bar{D}. \tag{2.33}$$

*Proof:* The proof is similar to Lemma 1.10 with  $D_0$  a bounded function taking values in the interval  $[\underline{D}, \bar{D}]$ .

The time-derivative of control law (2.3) writes

$$\begin{aligned}
 \dot{U}(t+s) = & K \left[ A \dot{D}_0(t+s) e^{AD_0(t+s)} X(t+s) + e^{AD_0(t+s)} AX(t+s) \right. \\
 & + e^{AD_0(t+s)} BU(t+s - D(t+s)) + BU(t+s) \\
 & \left. - (1 - \dot{D}_0(t+s)) e^{AD_0(t+s)} BU(t+s - D_0(t+s)) + \int_{t+s-D_0(t+s)}^{t+s} A e^{A(t+s-\xi)} BU(\xi) d\xi \right].
 \end{aligned} \tag{2.34}$$

for  $t+s > 0$  and  $s \in [-\bar{D}, 0]$ . As we want to obtain the maximum value of  $\dot{U}$  within the interval  $[t - \bar{D}, t]$ , according to (2.34), it is necessary to bound  $U$  on the interval  $[t - 2\bar{D}, t]$ . When we refer to the definition of controller  $U$ , we find that we also need to bound  $X$  within the interval  $[t - 2\bar{D}, t]$ .

From the definition of the dynamics (2.1), for  $\theta \in [-2\bar{D}, 0]$ , it holds

$$\begin{aligned}
 |X(t+\theta)| &= \left| e^{A\theta} \left( X(t) - \int_{t+\theta}^t e^{A(t-\xi)} BU(\xi - D(\xi)) d\xi \right) \right| \\
 &\leq e^{|\Lambda|\theta} |X(t)| + e^{|\Lambda|\theta} \int_{t+\theta}^t e^{|\Lambda|(t-\xi)} |B| \sum_{j=1}^r |U(\xi - D_j)| d\xi \\
 &\leq e^{|\Lambda|\theta} |X(t)| + e^{2|\Lambda|\theta} |B| \sum_{j=1}^r \int_{t+\theta}^t |U(\xi - D_j)| d\xi.
 \end{aligned} \tag{2.35}$$

Then, with the definition of controller  $U$  in (2.3) and (2.35), we can thus bound  $U$  for  $l \in [\max\{0, t - 2\bar{D}\}, t]$  as follows

$$\begin{aligned}
 U(l) &= K \left[ e^{AD_0(l)} X(l) + \int_{l-D_0(l)}^l e^{A(l-\xi)} BU(\xi) d\xi \right] \\
 &= K \left[ e^{AD_0(l)} e^{A(l-t)} \left( X(t) - \int_l^t e^{A(t-\xi)} BU(\xi - D(\xi)) d\xi \right) + \int_{l-D_0(l)}^l e^{A(l-\xi)} BU(\xi) d\xi \right] \\
 &\leq |K| \left[ e^{3|\Lambda|\bar{D}} X(t) + e^{5|\Lambda|\bar{D}} |B| \sum_{j=1}^r \int_s^t |U(\xi - D_j)| d\xi + e^{|\Lambda|\bar{D}} |B| \int_{t-3\bar{D}}^t |U(\xi)| d\xi \right] \\
 &\leq |K| \left[ e^{3|\Lambda|\bar{D}} X(t) + (re^{5|\Lambda|\bar{D}} + e^{|\Lambda|\bar{D}}) |B| \int_{t-3\bar{D}}^t |U(\xi)| d\xi \right] \leq N_1 |X(t)| + N_2 \int_{t-3\bar{D}}^t |U(\xi)| d\xi,
 \end{aligned} \tag{2.36}$$

with  $N_1 = |K|e^{3|\Lambda|\bar{D}}$  and  $N_2 = |K||B|(re^{5|\Lambda|\bar{D}} + e^{|\Lambda|\bar{D}})$ . This last inequality can be used to bound the pointwise input terms in (2.34), provided that  $t+s-D(t+s) \geq 0$ ,  $t+s \geq 0$  and  $t+s-D_0(t+s) \geq 0$  for all  $s \in [-\bar{D}, 0]$ , that is, that  $t \geq 2\bar{D}$ . Hence, for  $t \geq 2\bar{D}$ , using (2.35), (2.36) and the target system (2.15), equation (2.34) gives

$$\begin{aligned}
 |\dot{U}(t+s)| &= |K| \left[ |A| |\dot{D}_0(t+s) + 1| e^{|\Lambda|D_0(t+s)} |X(t+s)| + e^{|\Lambda|D_0(t+s)} |B| |U(t+s-D(t+s))| \right. \\
 &\quad \left. + |B| |U(t+s)| + (1 - \dot{D}_0(t+s)) e^{|\Lambda|D_0(t+s)} |B| |U(t+s-D_0(t+s))| \right. \\
 &\quad \left. + \int_{t+s-D_0(t+s)}^{t+s} |A| e^{|\Lambda|(t+s-l)} |B| |U(l)| dl \right] \\
 &\leq |K| |A| e^{|\Lambda|\bar{D}} \max\{|1 + \Delta_0^*|, |1 + \Delta_1^*|\} \left( e^{|\Lambda|\bar{D}} |X(t)| + e^{2|\Lambda|\bar{D}} |B| \sum_{j=1}^r \int_{t-\bar{D}}^t |U(\xi - D_j)| d\xi \right) \\
 &\quad + N_3 \left( N_1 |X(t)| + N_2 \int_{t-3\bar{D}}^t |U(\xi)| d\xi \right) \\
 &= \left( |K| |A| e^{2|\Lambda|\bar{D}} \max\{|1 + \Delta_0^*|, |1 + \Delta_1^*|\} + N_3 N_1 \right) |X(t)| \\
 &\quad + \left( |K| |A| e^{3|\Lambda|\bar{D}} \max\{|1 + \Delta_0^*|, |1 + \Delta_1^*|\} |B| r + N_3 N_2 \right) (3\bar{D} - \underline{D}) \int_0^1 |\mu(x, t)| dx \\
 &\quad + \left( |K| |A| e^{3|\Lambda|\bar{D}} \max\{|1 + \Delta_0^*|, |1 + \Delta_1^*|\} |B| r + N_3 N_2 \right) \bar{D} \int_0^1 |\hat{v}(x, t)| dx \\
 &= N_4 |X(t)| + N_5 (3\bar{D} - \underline{D}) \int_0^1 |\mu(x, t)| dx + N_5 \bar{D} \int_0^1 |\hat{v}(x, t)| dx,
 \end{aligned} \tag{2.37}$$

with  $N_3 = |K| \left( e^{|\Lambda|\bar{D}} + 1 + e^{|\Lambda|\bar{D}} (1 - \Delta_0^*) + |A| e^{|\Lambda|\bar{D}} \right) |B|$ ,  $N_4 = |K| |A| e^{2|\Lambda|\bar{D}} \max\{|1 + \Delta_0^*|, |1 + \Delta_1^*|\} + N_3 N_1$  and  $N_5 = |K| |A| e^{3|\Lambda|\bar{D}} \max\{|1 + \Delta_0^*|, |1 + \Delta_1^*|\} |B| r + N_3 N_2$ .

Here we should limit  $\hat{v}$  by the target system variables  $(X, w)$ . Using Young's and Cauchy-Schwarz inequalities, the inverse of backstepping transformation given in (2.23) gives

$$\|\hat{v}(t)\|^2 \leq N_6|X(t)|^2 + N_7\|w(t)\|^2, \quad (2.38)$$

with the positive constants  $N_6 = 3|K|^2 e^{2|A+BK|\bar{D}}$  and  $N_7 = 3\left(1 + |K|^2 e^{2|A+BK|\bar{D}}|B|^2 \bar{D}^2\right)$ .

Applying (2.38) to (2.37), thus it holds

$$\left(\max_{s \in [-\bar{D}, 0]} \dot{U}(t+s)\right)^2 \leq C_X|X(t)|^2 + C_\mu\|\mu(t)\|^2 + C_w\|w(t)\|^2, \quad (2.39)$$

with constants  $C_X = 3N_4^2 + 3(N_5\bar{D})^2 N_6$ ,  $C_\mu = 3N_5^2(3\bar{D} - \underline{D})^2$  and  $C_w = 3(N_5\bar{D})^2 N_7$ .

The lemma is then proved with the positive constant  $M_U = \max\left\{\frac{C_X}{\min(\lambda(P))}, \frac{C_w}{\alpha_1}, \frac{C_\mu}{\theta_1}\right\}$ . ■

Lemma 2.3 can then be used to obtain the following Lemma about the distributed variable  $\tilde{v}_j$  and the function  $h$ .

**Lemma 2.4** Consider the function  $h$  defined in (2.17) and the variable  $\tilde{v}_j$  defined in (2.11). There exists a constant  $M_h$  such that the following inequalities are satisfied,

$$\|h(t + D_0(t)(\cdot - 1))\|^2 \leq M_h V(t), \quad t \geq 2\bar{D}, \quad (2.40)$$

$$|\tilde{v}_j(x, t)| \leq |D_j - D_0(t)|(M_U V(t))^{1/2}, \quad t \geq 2\bar{D}, \quad j \in \{1, \dots, r\}, \quad x \in [0, 1]. \quad (2.41)$$

in which  $M_U$  is the constant introduced in Lemma 2.3.

*Proof:* First,  $h$  defined in (2.17) can be expressed as  $h(t + D_0(t)(x - 1)) = \hat{v}_x(x, t)$ . According to the definition of the distributed actuator  $\hat{v}$  in (2.10), it then holds that  $h(t) = D_0(t)\dot{U}(t)$ . By replacing the variable  $t$  with  $t + D_0(t)(x - 1)$  for  $x \in [0, 1]$ , we obtain

$$\begin{aligned} \|h(t + D_0(t)(\cdot - 1))\|^2 &= \int_0^1 h(t + D_0(t)(x - 1))^2 dx \\ &= D_0(t) \int_0^1 \dot{U}(t + D_0(t)(x - 1))^2 dx, \end{aligned} \quad (2.42)$$

Consequently, with Lemma 2.3, and as the prediction horizon  $D_0$  is limited within the range  $[\underline{D}, \bar{D}]$ , it holds

$$\|h(t + D_0(t)(\cdot - 1))\|^2 \leq \bar{D} \left(\max_{s \in [-\bar{D}, 0]} \dot{U}(t+s)\right)^2 \leq \bar{D} M_U V(t). \quad (2.43)$$

Secondly, from (2.11) and the mean-value theorem, there exists  $\xi \in [t + \max\{D_0(t), D_j\}(x - 1), t + \min\{D_0(t), D_j\}(x - 1)]$  such that  $\tilde{v}_j(x, t)$  can be expressed as

$$\tilde{v}_j(x, t) = U(t + D_j(x - 1)) - U(t + D_0(t)(x - 1)) = (D_j - D_0(t))(x - 1)\dot{U}(\xi). \quad (2.44)$$

Then, by applying Lemma 2.3, (2.44) gives

$$|\tilde{v}_j(x, t)| \leq |D_j - D_0(t)| \left| \max_{s \in [-\bar{D}, 0]} \dot{U}(t+s) \right| \leq |D_j - D_0(t)|(M_U V(t))^{1/2}. \quad (2.45)$$

The Lemma is proved with the constant  $M_h = \bar{D} M_U$ . ■

We can now solve the problem of analyzing the last term in (2.32) with the above lemmas. Let us apply Lemma 2.4 and Cauchy-Schwarz's and Young's inequalities to the last term of (2.32). Then, we obtain

$$\begin{aligned}
 & -2\beta_1 \int_0^1 e^{\beta_2 x} \frac{D_j - D_0(t) + D_j \dot{D}_0(t)(x-1)}{D_0(t)} \tilde{v}_j(x, t) h(t + D_0(t)(x-1)) dx \\
 & \leq 2\beta_1 e^{\beta_2} \int_0^1 \frac{|D_j - D_0(t)|}{D_0(t)} |\tilde{v}_j(x, t)| |h(t + D_0(t)(x-1))| dx \\
 & \quad + 2\beta_1 e^{\beta_2} \int_0^1 \frac{D_j |\dot{D}_0(t)(x-1)|}{D_0(t)} |\tilde{v}_j(x, t)| |h(t + D_0(t)(x-1))| dx \\
 & \leq 2\beta_1 e^{\beta_2} \frac{|D_j - D_0(t)|}{\underline{D}} \int_0^1 |\tilde{v}_j(x, t)| |h(t + D_0(t)(x-1))| dx \\
 & \quad + 2\beta_1 e^{\beta_2} \frac{D_r \max\{|\Delta_0^*|, |\Delta_1^*|\}}{\underline{D}} \int_0^1 |\tilde{v}_j(x, t)| |h(t + D_0(t)(x-1))| dx \\
 & \leq \beta_1 \left( \gamma_3 \|\tilde{v}_j(t)\|^2 + \frac{e^{2\beta_2} |D_j - D_0(t)|^2}{\gamma_3 \underline{D}^2} M_h V \right) + 2\beta_1 e^{\beta_2} \frac{D_r \max\{|\Delta_0^*|, |\Delta_1^*|\}}{\underline{D}} |D_j - D_0(t)| \sqrt{M_U} \sqrt{M_h} V \\
 & = \beta_1 \gamma_3 \|\tilde{v}_j(t)\|^2 + \beta_1 \left( \frac{e^{2\beta_2} |D_j - D_0(t)|^2}{\gamma_3 \underline{D}^2} M_h + \frac{2e^{\beta_2} D_r \max\{|\Delta_0^*|, |\Delta_1^*|\}}{\underline{D}} |D_j - D_0(t)| \sqrt{M_U} \sqrt{M_h} \right) V.
 \end{aligned} \tag{2.46}$$

If we now use the fact  $D_0(t) \in [\underline{D}, \bar{D}]$  and (2.4) in Assumption 2.1, (2.32) then gives

$$\begin{aligned}
 \frac{dV_j}{d\Psi}(\Psi) f_j(\Psi, t) & \leq - \left( \frac{\min(\lambda(Q))}{4} - \frac{\theta_1 e^{\theta_2} (1 - \Delta_0^*)}{\bar{D}} |K|^2 \right) |X(t)|^2 \\
 & - \alpha_1 \left( \frac{\min\{\alpha_2(1 - \Delta_0^*) + \Delta_0^*, \alpha_2(1 - \Delta_1^*) + \Delta_1^*, \alpha_2 + \Delta_0^*\}}{\bar{D}} - |K| e^{|A|\bar{D}} |B| (\gamma_1 + \gamma_2) e^{\alpha_2} \right. \\
 & \quad \left. - \frac{4\alpha_1 \max\{|\Delta_0^*|, |\Delta_1^*|\}^2 e^{2\alpha_2} |K|^2 e^{2|A|\bar{D}} |A + BK|^2}{\min(\lambda(Q))} \right) \|w(t)\|^2 \\
 & - \beta_1 (\beta_2 - \gamma_3) \|\tilde{v}_j(t)\|^2 - \theta_1 \frac{\min\{\theta_2 - \Delta_1^*, \theta_2(1 - \Delta_1^*) - \Delta_1^*\}}{3\bar{D} - \underline{D}} \|\mu(t)\|^2 \\
 & - \left( \alpha_1 \frac{1 - \Delta_1^*}{\bar{D}} - \frac{4|PB|^2}{\min(\lambda(Q))} - \frac{\alpha_1 e^{\alpha_2} |K| e^{|A|\bar{D}} |B| \max\{|\Delta_0^*|, |\Delta_1^*|\}^2}{\gamma_1} \right. \\
 & \quad \left. - \frac{\theta_1 e^{\theta_2} (1 - \Delta_0^*)}{\bar{D}} \right) w(0, t)^2 - \left( \beta_1 - \frac{4|PB|^2}{\min(\lambda(Q))} - \frac{\alpha_1 e^{\alpha_2} |K| e^{|A|\bar{D}} |B|}{\gamma_2} \right) \tilde{v}_j(0, t)^2 \\
 & + \beta_1 e^{\beta_2} \left( (D_j - D_0(t))^2 \frac{e^{\beta_2}}{\gamma_3 \underline{D}^2} M_h + \frac{2D_r \max\{|\Delta_0^*|, |\Delta_1^*|\}}{\underline{D}} |D_j - D_0(t)| \sqrt{M_U} \sqrt{M_h} \right) V.
 \end{aligned} \tag{2.47}$$

To guarantee that the first terms in (2.47) are negative, we choose the constants  $\alpha_2$ ,  $\theta_2$  and  $\gamma_3$  as arbitrary but positive and the rest of the constants as follows

- |                                                                                |                                                                                                                         |
|--------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------|
| (a) $\gamma_1 < \frac{\alpha_2}{2\bar{D} K e^{ A \bar{D}} B } e^{-\alpha_2}$ , | (d) $\alpha_1 \geq \frac{4\bar{D} PB ^2}{\min(\lambda(Q))} + \theta_1 e^{\theta_2}$ ,                                   |
| (b) $\gamma_2 < \frac{\alpha_2}{2\bar{D} K e^{ A \bar{D}} B } e^{-\alpha_2}$ , | (e) $\beta_1 \geq \frac{4 PB ^2}{\min(\lambda(Q))} + \alpha_1 e^{\alpha_2}  K  e^{ A \bar{D}}  B  \frac{1}{\gamma_2}$ , |
| (c) $\theta_1 < \frac{\bar{D} \min(\lambda(Q))}{4e^{\theta_2}  K ^2}$ ,        | (f) $\beta_2 \geq \gamma_3$ .                                                                                           |

Now, assume that the condition (2.29) holds for  $\Delta^*(K)$  such that the following conditions are satisfied for  $\Delta \leq \Delta^*(K)$



$$\begin{aligned}
 \text{(g)} \quad & \alpha_2(1 - \Delta) - \Delta > 0, & \text{(j)} \quad & \gamma_1 < \frac{\min\{\alpha_2(1-\Delta)-\Delta, \alpha_2-\Delta\}}{2\bar{D}|K|e^{|\Lambda|\bar{D}}|B|e^{\alpha_2}} - \frac{2\alpha_1\Delta^2 e^{2\alpha_2}|K|^2 e^{2|\Lambda|\bar{D}}|A+BK|^2}{\min(\lambda(Q))|K|e^{|\Lambda|\bar{D}}|B|e^{\alpha_2}}, \\
 \text{(h)} \quad & \theta_2 > \max\left\{\Delta, \frac{\Delta}{1-\Delta}\right\}, & \text{(k)} \quad & \gamma_2 < \frac{\min\{\alpha_2(1-\Delta)-\Delta, \alpha_2-\Delta\}}{2\bar{D}|K|e^{|\Lambda|\bar{D}}|B|e^{\alpha_2}} - \frac{2\alpha_1\Delta^2 e^{2\alpha_2}|K|^2 e^{2|\Lambda|\bar{D}}|A+BK|^2}{\min(\lambda(Q))|K|e^{|\Lambda|\bar{D}}|B|e^{\alpha_2}}, \\
 \text{(i)} \quad & \theta_1 < \frac{\bar{D}\min(\lambda(Q))}{4e^{\theta_2(1+\Delta)}|K|^2}, & \text{(l)} \quad & \alpha_1 \geq \left(\frac{4|PB|^2}{\min(\lambda(Q))} + \theta_1 \frac{e^{\theta_2(1+\Delta)}}{\bar{D}}\right) \left(\frac{1-\Delta}{\bar{D}} - \frac{e^{\alpha_2}|K|e^{|\Lambda|\bar{D}}|B|\Delta^2}{\gamma_1}\right)^{-1}.
 \end{aligned}$$

Notice that , for  $\Delta = 0$ , these equations simplify as  $\alpha_2 > 0$ ,  $\theta_2 > 0$ ,  $\theta_1 < \frac{\bar{D}\min(\lambda(Q))}{4e^{\theta_2}|K|^2}$ ,  $\gamma_1, \gamma_2 < \frac{\alpha_2}{2\bar{D}|K|e^{|\Lambda|\bar{D}}|B|}e^{-\alpha_2}$  and  $\alpha_1 \geq \frac{4\bar{D}|PB|^2}{\min(\lambda(Q))} + \theta_1 e^{\theta_2}$ . Hence, using (a)-(f) and by continuity, there exists  $\Delta^*(K) > 0$  small enough such that (g)-(l) are satisfied.

Then, under this condition, (2.47) can be simplified as

$$\begin{aligned}
 \frac{dV_j}{d\Psi}(\Psi)f_j(\Psi, t) &\leq -\eta V + \beta_1(D_j - D_0(t))^2 \frac{e^{2\beta_2}}{\gamma_3 \bar{D}^2} M_h V \\
 &+ 2\beta_1 e^{\beta_2} \frac{2D_r \max\{|\Delta_0^*|, |\Delta_1^*|\}}{\underline{D}} |D_j - D_0(t)| \sqrt{M_U} \sqrt{M_h} V,
 \end{aligned} \tag{2.48}$$

with the positive constant  $\eta$  defined as

$$\begin{aligned}
 \eta = \min \left\{ \frac{\min(\lambda(Q))}{4 \max(\lambda(P))} - \frac{\theta_1 e^{\theta_2(1+\Delta^*)}|K|^2}{\bar{D} \max(\lambda(P))}, \frac{\beta_2 - \gamma_3}{e^{\beta_2}}, \frac{\min\{\theta_2 - \Delta_1^*, \theta_2(1 - \Delta_1^*) - \Delta_1^*\}}{(3\bar{D} - \underline{D})e^{\theta_2}}, \right. \\
 \left. \frac{\min\{\alpha_2(1 - \Delta) - \Delta, \alpha_2 - \Delta\}}{\bar{D}e^{\alpha_2}} - |K|e^{|\Lambda|\bar{D}}|B|(\gamma_1 + \gamma_2) - \frac{4\alpha_1(\Delta^*)^2 e^{\alpha_2}|K|^2 e^{2|\Lambda|\bar{D}}|A+BK|^2}{\min(\lambda(Q))} \right\}.
 \end{aligned} \tag{2.49}$$

On the other hand, the second term in (2.27) can be rewritten as

$$\sum_{l=1}^r (V_l(\Psi) - V_j(\Psi))\tau_{jl}(t) = \sum_{l=1}^r \beta_l \int_0^1 e^{\beta_2 x} \tau_{jl}(t) \left( D_l \tilde{v}_l(x, t)^2 - D_j \tilde{v}_j(x, t)^2 \right) dx, \tag{2.50}$$

in which, from the definition (2.11) of the input estimation error,

$$\begin{aligned}
 D_l \tilde{v}_l(x, t)^2 - D_j \tilde{v}_j(x, t)^2 &= \left( \sqrt{D_l} \int_{t+D_0(t)(x-1)}^{t+D_l(x-1)} \dot{U}(s) ds - \sqrt{D_j} \int_{t+D_0(t)(x-1)}^{t+D_j(x-1)} \dot{U}(s) ds \right) \\
 &\times \left( \sqrt{D_l} \int_{t+D_0(t)(x-1)}^{t+D_l(x-1)} \dot{U}(s) ds + \sqrt{D_j} \int_{t+D_0(t)(x-1)}^{t+D_j(x-1)} \dot{U}(s) ds \right) \\
 &\leq (1-x)^2 \left( |\sqrt{D_l} - \sqrt{D_j}| |D_j - D_0(t)| + \sqrt{D_r} |D_l - D_j| \right) \\
 &\times \sqrt{D_r} \left( |D_l - D_0(t)| + |D_j - D_0(t)| \right) \max_{s \in [-\bar{D}, 0]} \dot{U}(t+s)^2 \\
 &\leq \sqrt{D_r} \left( \left( \sqrt{D_r} - \sqrt{D_1} \right) |D_j - D_0(t)| + \sqrt{D_r} |D_l - D_j| \right) \\
 &\times \left( |D_l - D_0(t)| + |D_j - D_0(t)| \right) \max_{s \in [-\bar{D}, 0]} \dot{U}(t+s)^2.
 \end{aligned} \tag{2.51}$$

Then, applying Lemma 2.3 which bounds the maximum of  $\dot{U}$  over the time interval  $[t - \bar{D}, 0]$ , (2.51) rewrites as

$$\begin{aligned}
 D_l \tilde{v}_l(x, t)^2 - D_j \tilde{v}_j(x, t)^2 &\leq \left( \left( 2D_r - \sqrt{D_1 D_r} \right) |D_j - D_0(t)| |D_l - D_j| \right. \\
 &+ \left( 2D_r - \sqrt{D_1 D_r} \right) |D_j - D_0(t)|^2 + D_r |D_l - D_0(t)|^2 \\
 &+ D_r |D_j - D_0(t)| |D_l - D_0(t)| \Big) M_U V \\
 &\leq \left( \left( 3D_r - \sqrt{D_1 D_r} \right) \left( \bar{D} - \underline{D} \right) |D_j - D_0(t)| \right. \\
 &+ \left. \left( 2D_r - \sqrt{D_1 D_r} \right) \left( D_j - D_0(t) \right)^2 + D_r |D_l - D_0(t)|^2 \right) M_U V.
 \end{aligned} \tag{2.52}$$

Therefore, gathering (2.47), (2.50) and (2.52), one gets

$$\begin{aligned}
 L_j V(t) &\leq -\eta V + \beta_1 (D_j - D_0(t))^2 \frac{e^{2\beta_2}}{\gamma_3 \underline{D}^2} M_h V + 2\beta_1 e^{\beta_2} \frac{2D_r \Delta^*}{\underline{D}} |D_j - D_0(t)| \sqrt{M_U} \sqrt{M_h} V \\
 &\quad + \frac{\beta_1}{\beta_2} (e^{\beta_2} - 1) \sum_{l=1}^r \tau_{jl}(t) \left( (3D_r - \sqrt{D_1 D_r}) (\bar{D} - \underline{D}) |D_j - D_0(t)| \right. \\
 &\quad \left. + (2D_r - \sqrt{D_1 D_r}) (D_j - D_0(t))^2 + D_r |D_l - D_0(t)|^2 \right) M_U V \\
 &= -\eta V + (D_j - D_0(t))^2 M_1 V + |D_j - D_0(t)| M_2 V + \sum_{l=1}^r |D_l - D_0(t)|^2 \tau_{jl}(t) M_3 V,
 \end{aligned} \tag{2.53}$$

with  $M_1 = \beta_1 \frac{e^{2\beta_2}}{\gamma_3 \underline{D}^2} M_h + r(2D_r - \sqrt{D_1 D_r}) \frac{\beta_1}{\beta_2} (e^{\beta_2} - 1) \tau^* M_U$ ,  $M_2 = 2\beta_1 e^{\beta_2} \frac{2D_r \Delta^*}{\underline{D}} \sqrt{M_U} \sqrt{M_h} + \frac{\beta_1}{\beta_2} (e^{\beta_2} - 1) r \tau^* (3D_r - \sqrt{D_1 D_r}) (\bar{D} - \underline{D}) M_U$  and  $M_3 = \frac{\beta_1}{\beta_2} (e^{\beta_2} - 1) D_r M_U$ . Hence, the following inequality holds

$$\begin{aligned}
 \sum_{j=1}^r P_{ij}(0, t) L_j V(t) &\leq - \left( \eta - M_1 \mathbb{E}_{[0, D(0)]} (|D(t) - D_0(t)|^2) - M_2 \mathbb{E}_{[0, D(0)]} (|D(t) - D_0(t)|) \right) V(t) \\
 &\quad + M_3 \sum_{j=1}^r P_{ij}(0, t) \sum_{l=1}^r |D_l - D_0(t)|^2 \tau_{jl}(t) V.
 \end{aligned} \tag{2.54}$$

which is the one given in Lemma 2.2.

#### 2.2.2.4 Conclusion of the stability analysis

We are now ready to conclude the proof of Theorem 2.1. Let us denote  $\gamma_0(t) = \eta - M_1 \mathbb{E}_{[0, D(0)]} (|D(t) - D_0(t)|^2) - M_2 \mathbb{E}_{[0, D(0)]} (|D(t) - D_0(t)|) - M_3 g(t)$ , and introduce  $Z(t) = \exp\left(\int_{2\bar{D}}^t \gamma_0(s) ds\right) V(t)$ . Applying Lemma 2.2, we obtain  $LZ(t) = \gamma_0(t) Z(t) + \exp\left(\int_{2\bar{D}}^t \gamma_0(s) ds\right) LV(t)$ . Then, its expectation satisfies  $\mathbb{E}_{[2\bar{D}, (\Psi, D)(2\bar{D})]} (LZ(t)) = \gamma_0(t) \mathbb{E}_{[2\bar{D}, (\Psi, D)(2\bar{D})]} (Z(t)) + \exp\left(\int_{2\bar{D}}^t \gamma_0(s) ds\right) \sum_{j=1}^r P_{ij}(0, t) L_j V(t) \leq 0$ . Therefore, for  $t \geq 2\bar{D}$ , according to Dynkin's formula [Dyn65, Theorem 5.1, p. 132],

$$\mathbb{E}_{[2\bar{D}, (\Psi, D)(2\bar{D})]} (Z(t)) - Z(2\bar{D}) = \mathbb{E}_{[2\bar{D}, (\Psi, D)(2\bar{D})]} \left( \int_{2\bar{D}}^t LZ(s) ds \right) \leq 0, \tag{2.55}$$

from which, using standard conditional expectation properties, one deduces that  $\mathbb{E}_{[0, (\Psi, D)(0)]} (Z(t)) \leq \mathbb{E}_{[0, (\Psi, D)(0)]} (Z(2\bar{D}))$  for  $t \geq 2\bar{D}$ . In addition, from the Kolmogorov's equation (2.2), we observe that

$$\begin{aligned}
 \int_0^t g(s) ds &= \int_0^t \sum_{j=1}^r |D_j - D_0(s)|^2 \sum_{k=1}^r P_{ik}(0, s) \tau_{kj}(s) ds \\
 &= \int_0^t \sum_{j=1}^r |D_j - D_0(s)|^2 \left[ \frac{\partial P_{ij}(0, s)}{\partial t} + c_j(s) P_{ij}(0, s) \right] ds \\
 &\leq c^* \int_0^t \mathbb{E}_{[0, D(0)]} (|D(s) - D_0(s)|^2) ds + \mathbb{E}_{[0, D(0)]} (|D(t) - D_0(t)|^2), \\
 &\quad + 2 \int_0^t \mathbb{E} (|D(s) - D_0(s)|) \Delta^* ds,
 \end{aligned} \tag{2.56}$$

using an integration by parts and the facts that  $c_j = \sum_{k=1}^r \tau_{jk} \leq r\tau^* \triangleq c^*$  and  $|\dot{D}_0(t)| \leq \Delta^*$ . Hence, it follows that

$$\begin{aligned} \mathbb{E}_{[0,(\Psi,D)(0)]}(Z(t)) &\geq \mathbb{E}_{[0,(\Psi,D)(0)]}V(t) \exp\left(-M_3\mathbb{E}_{[0,(\Psi,D)(0)]}(|D(t) - D_0(t)|^2)\right) \\ &\quad + \int_0^t (\eta - (M_1 + M_3c^*)\mathbb{E}_{[0,(\Psi,D)(0)]}(|D(s) - D_0(s)|^2) \\ &\quad - (M_2 + 2M_3\Delta^*)\mathbb{E}_{[0,(\Psi,D)(0)]}(|D(s) - D_0(s)|)ds). \end{aligned} \quad (2.57)$$

Besides, it holds

$$\begin{aligned} \mathbb{E}_{[0,(\Psi,D)(0)]}(|D(t) - D_0(t)|^2) &= \left(\sum_{j=1}^r P_{ij}(0,t)|D(t) - D_0(t)|\right)^2 \\ &\leq r \sum_{j=1}^r P_{ij}(0,t)|D(t) - D_0(t)|^2 = r\mathbb{E}_{[0,(\Psi,D)(0)]}(|D(t) - D_0(t)|^2). \end{aligned} \quad (2.58)$$

Hence, if (2.6) holds with the constant  $\varepsilon^*$  defined as

$$\varepsilon^* \triangleq \left(\frac{\sqrt{r(M_2 + 2M_3\Delta^*)^2 + 2(M_1 + M_3c^*)\eta} - \sqrt{r}(M_2 + 2M_3\Delta^*)}{2(M_1 + M_3c^*)}\right)^2. \quad (2.59)$$

One obtains from (2.55) and (2.57)

$$\begin{aligned} \mathbb{E}_{[0,(\Psi,D)(0)]}\left(e^{-M_3\varepsilon^* + \frac{\eta}{2}t}V(t)\right) &\leq \mathbb{E}_{[0,(\Psi,D)(0)]}(Z(t)) \\ &\leq \mathbb{E}_{[0,(\Psi,D)(0)]}(Z(2\bar{D})) \leq e^{2\bar{D}\eta}\mathbb{E}_{[0,(\Psi,D)(0)]}(V(2\bar{D})), \end{aligned} \quad (2.60)$$

which implies, with  $\gamma = \frac{\eta}{2}$ ,

$$\mathbb{E}_{[0,(\Psi,D)(0)]}(V(t)) \leq \mathbb{E}_{[0,(\Psi,D)(0)]}(V(2\bar{D}))e^{2\bar{D}\eta + M_3\varepsilon^* - \gamma t}. \quad (2.61)$$

The proof can thus be concluded as in Chapter 1.

### 2.3 Possible choices of time-varying prediction horizon

The choice of the prediction horizon  $D_0(t)$  is a question of utmost importance. The most intuitive option may be to pick the expected value of the delay, that is,  $D_0(t) = \mathbb{E}_{[0,D(0)]}(D(t))$ , as it allows a priori to get a good estimation of the delay realizations in average. Furthermore, in this case, due to the assumptions made on the transition probabilities,  $D_0$  does satisfy Assumption 2.1. For this choice of prediction horizon, one can then reformulate Theorem 2.1 as follows.

**Corollary 2.1 (Corollary of Theorem 2.1)** *Consider the closed-loop system consisting of the system (2.1) and the control law (2.3) with  $D_0(t) = \mathbb{E}_{[0,D(0)]}(D(t))$ . There exist positive constants  $\Delta^*$  and  $\varepsilon^*(K)$  such that, if*

$$\max\{|\Delta_0^*|, |\Delta_1^*|\} \leq \Delta^*(K), \quad (2.62)$$

$$\sigma_{[0,D(0)]}(D) \leq \varepsilon^*(K), \quad (2.63)$$

*then, the closed-loop system is mean-square exponentially stable, that is, there exist positive constants  $R$  and  $\gamma$  such that*

$$\mathbb{E}_{[0,(\Upsilon,D)(0)]}(\Upsilon(t)) \leq R\Upsilon(0)e^{-\gamma t}, \quad t \geq 0, \quad (2.64)$$

*with*

$$\Upsilon(t) = |X(t)|^2 + \int_{t-3D_r}^t U(s)^2 ds. \quad (2.65)$$

This corollary explicitly involves the standard deviation of the delay, which reflects the degree of dispersion of the delay realizations around the expected value and requires it to be small enough to guarantee stabilization.

To achieve such a controller, it is necessary to have access to the current transition probabilities, or at least to data exchange statistics. For instance, if the plant and the controller are equipped with synchronized internal clocks, exchanged data could be time-stamped and the current delay realization may be known at all time. Yet, estimation of the expected value of the delay through this technique will lead to a slight time lag, as variations of the expected value will not be instantaneously captured by the measurement of a few delay realizations. This will translate into an estimation error of the expected value. With a triangle inequality, condition (2.63) can then be reformulated in this context as,

$$\sigma_{[0,D(0)]}(D(t)) + |D_0(t) - \mathbb{E}_{[0,D(0)]}(D(t))| \leq \varepsilon^*(K), \quad (2.66)$$

that is, both the delay standard deviation and the expected value estimation error are sufficiently small.

Other more sophisticated delay pattern can be picked. For instance, it turns out that delay overestimation sometimes turns out in practice to yield better transient performances than in the reverse situation.

## 2.4 Simulation results

In this section, we provide some simulation results to illustrate the previous results. We consider the same toy example as in Chapter 1, that is,

$$\dot{X}(t) = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} X(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} U(t - D(t)). \quad (2.67)$$

The control law (2.3) is applied with the feedback gain  $K = -[1 \ 2]$ . Here, we choose the prediction horizon as  $D_0(t) = \mathbb{E}_{[0,D(0)]}(D(t))$ . The initial conditions are chosen as  $X(0) = [1 \ 0]^T$  and  $U(t) = 0$ , for  $t \leq 0$ . Simulations are carried out with a fixed-step solver in Matlab-Simulink and a sampling time  $\Delta t = 0.01$  s. The delay values are  $(D_1, D_2, D_3) = (0.1, 2.0, 2.1)$  with initial transition probabilities  $P_1(0, 0^+) = 0.02$ ,  $P_2(0, 0^+) = 0.69$ ,  $P_3(0, 0^+) = 0.29$  and the transition rates  $\tau_{ij}$  defined as

$$\tau(t) = (\{\tau_{ij}(t)\})_{1 \leq i, j \leq 3} = \tau^* \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & |\sin(kt)| \\ 0 & |\cos(kt)| & 0 \end{pmatrix}, \quad (2.68)$$

An example of a corresponding time evolution is pictured in Figure 2.1, along with the variations of the expected value of the delay. Monte Carlo simulations were also carried out with 100 trials. Compared to the results of Chapter 1, one can observe similar though slightly improved transient performances. This improvement is likely to increase with higher rates of variations of the transition probabilities.

## 2.5 Conclusion and perspective

This chapter investigated the problem of exponential stabilization of LTI systems affected by a stochastic input delay, with a prediction-based controller with a time-varying prediction horizon. It proved that stabilization could be achieved provided that the rate of variation of the prediction horizon remains sufficiently small and provided that the delay realization remains sufficiently close to the prediction horizon, in average. We also proposed to use the expected value of the delay as prediction horizon, which then translates into the conditions that both the variation rate of the delay expected value and

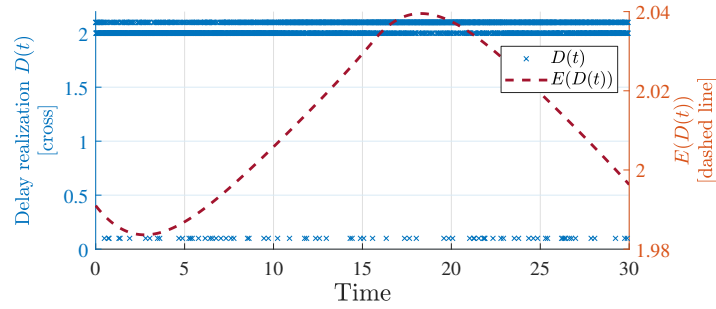
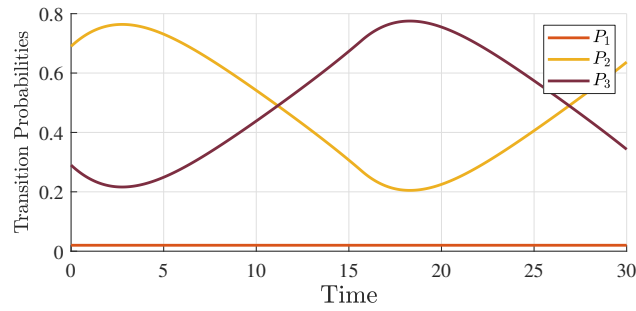
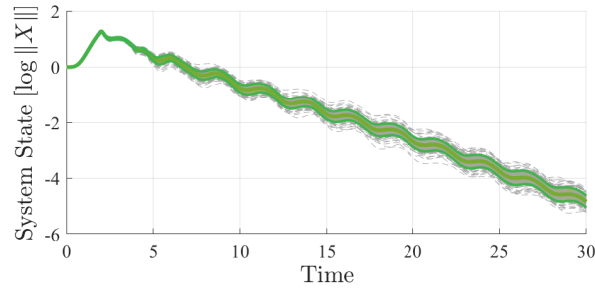
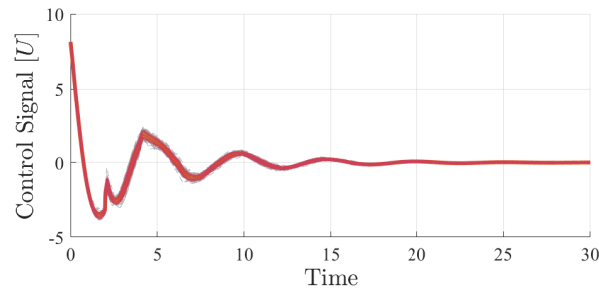

 (a) Example of a realization of the random delay  $D$ .

 (b) Dynamic of transition probabilities  $P_1$ ,  $P_2$  and  $P_3$ .

 (c) Monte Carlo results of the system state  $\log \|X\|$ .

 (d) Monte Carlo results of the control signal  $U$ .

Figure 2.1: Simulation results of the closed-loop system (2.67) and (2.3) with the feedback gain  $K = -[1 \ 2]$  for  $\mathbf{D} = (0.1, 2.0, 2.1)^T$ ,  $X(0) = [1 \ 0]^T$  and  $U(t) = 0$  for  $t \leq 0$ . The prediction horizon is  $D_0(t) = \mathbb{E}(D(t))$ . The transition probabilities follow the forward Kolmogorov equation with the transition rate defined in (2.68). (a) and (b) picture results corresponding to one delay realization. (c) and (d) present the results of 100 trials, in which the means and the standard deviations are highlighted by the colored lines.

of the standard deviation remain sufficiently small.

A potential interesting path to explore to improve the previous controller could be to rely on the prediction of the future delay value, in line with the standard exact compensation technique for a time-varying deterministic delay function, as presented in Section 2.1.1 in the Introduction. Namely, defining  $\phi_E(t) = t - \mathbb{E}_{[0, D(0)]}(D(t))$ , and assuming that this function is invertible, one would then pick the following controller

$$U(t) = K\hat{P}(t), \quad (2.69)$$

$$\begin{aligned} \hat{P}(t) &= e^{A(\phi_E^{-1}(t)-t)} X(t) + \int_t^{\phi_E^{-1}(t)} e^{A(\phi_E^{-1}(t)-s)} BU(\phi_E(s)) ds \\ &= e^{A(\phi_E^{-1}(t)-t)} X(t) + \int_{\phi_E(t)}^t e^{A(\phi_E^{-1}(t)-\phi_E^{-1}(s))} BU(s) \frac{ds}{\dot{\phi}_E(\phi_E^{-1}(s))}, \end{aligned} \quad (2.70)$$

in which  $\hat{P}$  is obtained with the variation of constant formula, integrating the dynamics  $\dot{X}(t) = AX(t) + BU(\phi_E(t))$  with the initial condition  $X(t)$  up to time  $\phi_E^{-1}(t)$ . Namely, this strategy aims at anticipating the future value of the delay expected value. Correspondingly, it should allow to better take into account the future variations of the delay distribution and, in detail, to get rid of the condition (2.5) which limits its range of variation. Yet, we were not able to obtain such a result yet, which is thus left for future works.

Notice however that determining this inverse  $\phi_E^{-1}(t)$  may be difficult to do in a causal way in practice. Indeed, one would need to be able to predict the future expected value of the delay, and thus its transition probabilities, which requires to integrate forward the Kolmogorov equation. Even if one may anticipate their general trend, it is very unlikely that these future values will be known perfectly. Thus, such a design should also involve a type of robustness property to prediction errors.

Another very important and interesting path to explore could be to adapt the prediction horizon, with as little information on the delay distribution as possible (namely, without the knowledge of the expected value of the delay, or without the knowledge of the delay values). This may be achieved with a delay-adaptive controller, but such a design remains to be entirely explored.



## Chapter 3

# Alternative averaged-prediction-based controller

**Résumé** Ce chapitre expose une conception alternative de contrôle par prédiction pour système linéaire à retard d'entrée aléatoire, basé sur des prédictions multiples. Chaque prédiction est calculée sur un horizon de temps différent, correspondant à l'une des valeurs potentielles du retard, et la moyenne de ses prédictions est alors utilisée dans la loi de contrôle. Des simulations numériques illustrent que ce nouveau contrôleur par prédiction engendre une compensation du retard plus robuste qu'une stratégie basée sur une unique prédiction telle que considérée dans les chapitres précédents. Ceci nous amène à l'hypothèse qu'il est préférable de prédire et de moyenner plutôt que de moyenner et de prédire. Nous expliquons cette différence par l'analyse de la dynamique moyennée, dans les deux cas de figure, et prouvons que la stabilisation exponentielle en boucle fermée est obtenue, à condition que l'écart type du retard soit suffisamment faible.

**Abstract** This chapter proposes an alternative prediction-based controller for a linear system with random input delay, building on multiple predictions. Each prediction is computed on a time horizon corresponding to a potential delay value, and their average is then used in the control law. Numerical simulations illustrate that the averaged-prediction controller leads to a more robust delay compensation than the control design based on a single prediction, as investigated in the previous chapter. This leads us to the conclusion that predicting and averaging results into better performances than averaging and predicting. We explain this difference by analyzing the average dynamics in both cases and prove that closed-loop exponential stabilization is achieved, provided that the standard deviation of the delay distribution is small enough.



### 3.1 A new controller design approach: averaged prediction

In this chapter, we propose to investigate an alternative prediction-based approach, compared to both Chapters 1 and 2. Again, we consider the following LTI system subject to a random input delay

$$\dot{X}(t) = AX(t) + BU(t - D(t)), \quad (3.1)$$

with a controllable constant matrices pair  $(A, B)$ , the system state  $X \in \mathbb{R}^n$ , the control input  $U \in \mathbb{R}$  and the stochastic input delay  $D$  modeled as a Markov process with a finite number of values, which are assumed to be ordered as  $0 < D_1 < D_2 < \dots < D_r$ . In addition, as previously, we assume that the realizations of this Markov process are right-continuous. To ease the exposition, *in the following of this chapter, we consider that the transition probabilities<sup>1</sup>  $P_{ij}$  are constant.*

In the previous chapters, we proposed to compensate for the random input delay by grounding on a state prediction. The essential question at stake was then the choice of the prediction horizon. A natural choice is the expected value of the delay. Denoting<sup>2</sup> this expectation as  $\mathbb{E}(D)$ , this leads to the following control law

$$U_{D_{avg}}(t) = K \left[ e^{A\mathbb{E}(D)} X(t) + \int_{t-\mathbb{E}(D)}^t e^{A(t-s)} BU(s) ds \right], \quad (3.2)$$

in which  $K$  is a feedback gain such that  $A + BK$  is Hurwitz. This design then led to the sufficient conditions given in Corollary 2.1 and which are simplified here as the second condition (2.63) only, due to the fact that the transition probabilities are assumed to be constant.

An alternative (maybe less natural) choice is to ground the control law on the predictors resulting from each possible delay value

$$\hat{P}_j(t) = e^{AD_j} X(t) + \int_{t-D_j}^t e^{A(t-s)} BU(s) ds, \quad (3.3)$$

and to then consider their average value as

$$U_{P_{avg}}(t) = K \sum_{j=1}^r P_{ij} \hat{P}_j(t), \quad (3.4)$$

in which, again, the feedback gain  $K$  is such that  $A + BK$  is Hurwitz.

Notice that this design relies on the same ingredients as before, but in reverse order: first, predicting in (3.3), and then averaging in (3.4). Yet, in this configuration, one needs to compute  $r$  different predictions  $\hat{P}_j$ , which may seem quite demanding from a computational point of view. Furthermore, compared to the previous controller design approaches, this may require storing the history of the input on a much longer time window when  $D_r \gg \mathbb{E}(D)$ . On the other hand, in the case where the delay values all have the same scale, the memory requirements should be similar to the ones of the control law (3.2) and the number of additional computations should be reduced (as numerous computations are the same for the different prediction horizons). In other words, in this case, the price to pay to compute  $r$  predictions instead of one is relatively small compared to the one of choosing a prediction-based controller over a memoryless one.

<sup>1</sup>Recall that these functions  $P_{ij} : \mathbb{R}^2 \rightarrow [0, 1]$  quantify the probability to switch from  $D_i$  at time  $t_1$  to  $D_j$  at time  $t_2$  ( $(i, j) \in \{1, \dots, r\}^2, t_2 \geq t_1 \geq 0$ ).

<sup>2</sup>In this chapter, for the sake of clarity and simplicity,  $\mathbb{E}_{[0, D(0)]}(D)$  and  $\sigma_{[0, D(0)]}(D)$ , which are always conditional on  $[0, D(0)]$ , are denoted as  $\mathbb{E}(D)$  and  $\sigma(D)$ , respectively. Notice that we omit the time variable as these values are assumed to be constant in this chapter.

### 3.1.1 Numerical comparison

To compare the merits of the two prediction-based control laws, in this section, we consider two toy examples.

#### 3.1.1.1 One-dimensional system

We first begin with a one-dimensional unstable LTI system defined as follows

$$\dot{X}(t) = X(t) + U(t - D(t)), \quad (3.5)$$

with  $r = 3$  possible input delay values  $(D_1, D_2, D_3) = (1.7, 2.0, 2.3)$ . We choose a fairly small sampling time  $\Delta t = 0.001$  s to avoid any accuracy issues potentially arising for high gain values. The integrals in (3.2) and (3.4) are discretized using their zero-order hold approximations, in line with a suggestion in [MO79].

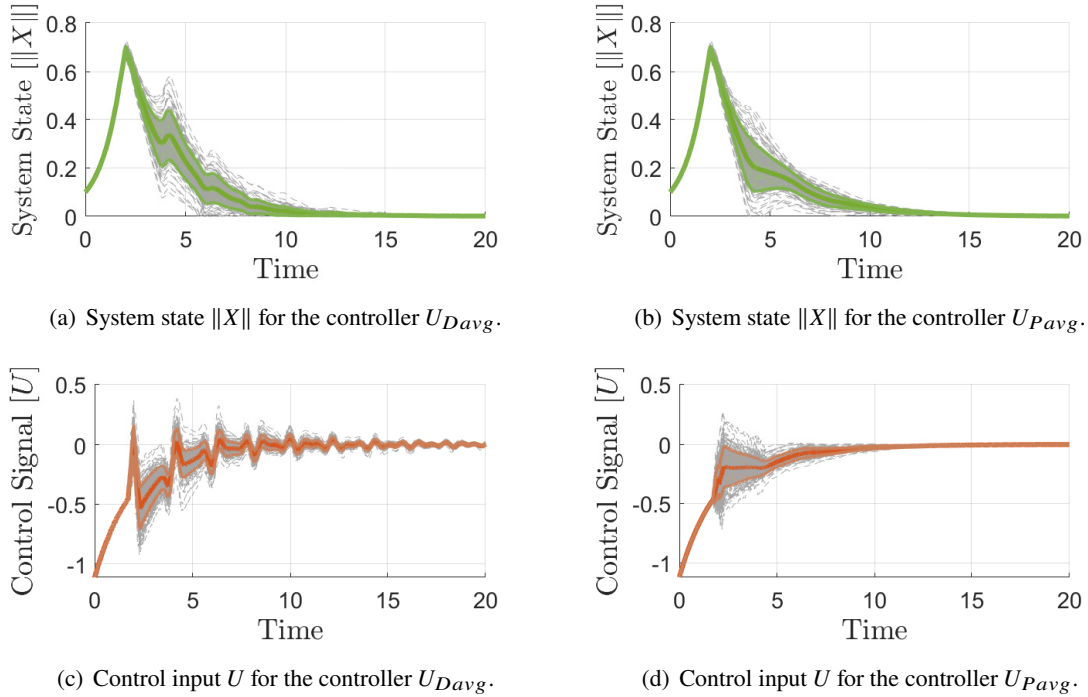


Figure 3.1: Simulation results of the closed-loop systems consisting of (3.5) and, respectively, the controller  $U_{D_{avg}}$  defined in (3.2) (left column) and the controller  $U_{P_{avg}}$  defined in (3.4) (right column). The feedback gain is picked as  $K = -1.5$ , the potential delay values are  $\mathbf{D} = (1.7, 2.0, 2.3)^T$  and the initial conditions are  $X(0) = 0.1$  and  $U(t) = 0$  for  $t \leq 0$ . The transition probabilities are constant and such that  $(P_{i1} \ P_{i2} \ P_{i3}) = (0.125 \ 0.75 \ 0.125)$ . (a) and (b) present the system state evolution, while (c) and (d) present the one of the control input, for 100 trials. The means and the standard deviations are highlighted by the colored lines.

The transition probabilities are taken as  $(P_{i1} \ P_{i2} \ P_{i3}) = (0.125 \ 0.75 \ 0.125)$  resulting in the delay expectation  $\mathbb{E}(D) = 2$  and a standard deviation  $\sigma(D) = 0.15$ . To ensure the Hurwitz matrix  $A + BK$ , with  $A = B = 1$ , the control law (3.2) and (3.4) are applied with the feedback gain  $K = -1.5$ . Notice that the delay margin of the corresponding closed-loop system with the constant input delay  $D = 2$  is  $\Delta D = 0.091$  (see [MNL01] for details on the computation of this quantity). As such, the realizations of both  $D_1$  and  $D_3$  lead to a delay difference which is three times the robustness margin of the closed-loop system and the standard deviation of the delay is also beyond this margin. Despite

this challenging set-up, both controllers achieve closed-loop stabilization, with, as can be observed in Figure 3.3, similar enough performances.

To compare further these transient performances, we picked three different feedback gains  $K = -1.5$ ,  $K = -2.0$  and  $K = -3.0$ . Corresponding average trajectories are depicted in Figure 3.2. These three feedback gain values yield closed-loop stabilization for the controller with averaged prediction (3.4), with a time response decaying (in average) with the magnitude of the feedback gain. Yet, one can observe that the obtainable performances are limited with the prediction with averaged delay (3.2), as the higher value of the feedback gain results in an unstable behavior. Hence, the averaged prediction (3.4) seems to present a larger robustness margin.

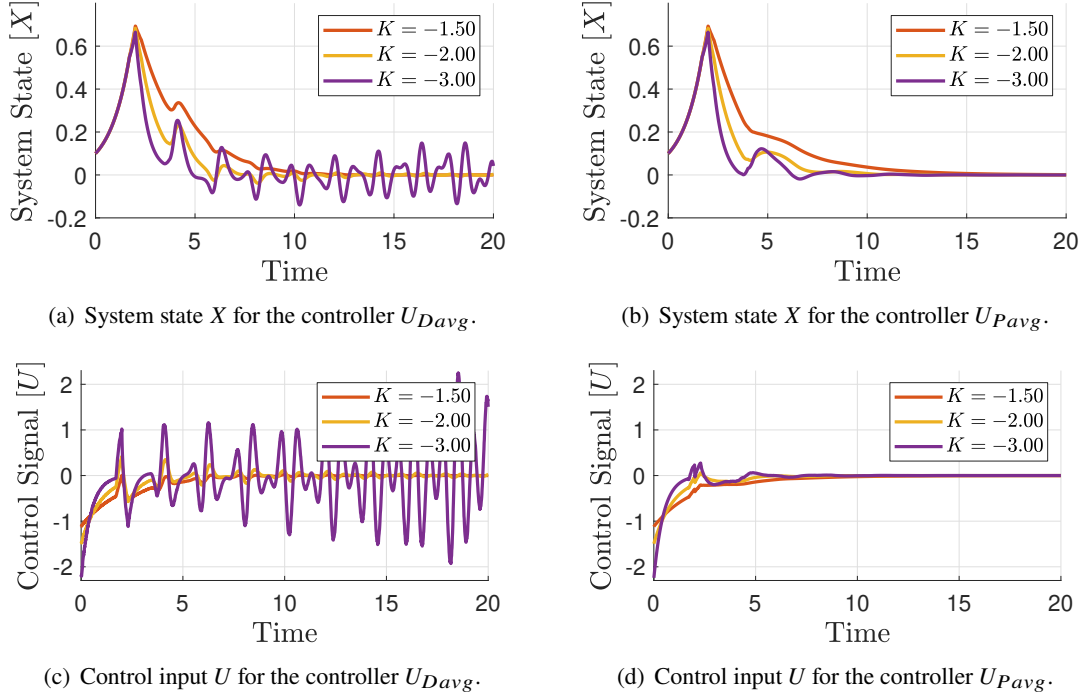


Figure 3.2: Simulation results of the closed-loop systems consisting of (3.5) and, respectively, the controller  $U_{Davg}$  defined in (3.2) (left column) and the controller  $U_{Pavg}$  defined in (3.4) (right column). Different feedback gain values are considered:  $K = -1.5$ ,  $K = -2$  and  $K = -3$ . The potential delay values are  $\mathbf{D} = (1.7, 2.0, 2.3)^T$  and the initial conditions are  $X(0) = 0.1$  and  $U(t) = 0$  for  $t \leq 0$ . The transition probabilities are constant and such that  $(P_{i1} \ P_{i2} \ P_{i3}) = (0.125 \ 0.75 \ 0.125)$ . (a) and (b) present the mean of the system state over 100 different trials, while (c) and (d) present the one of the control input.

This systematic comparison was carried out by considering various stabilizing feedback gains and transition probabilities distributions, and observing when closed-loop stabilization was obtained. The (quite arbitrary) criterion selected to assess that stabilization is achieved is that the 95% response time of the closed-loop system is smaller than 3 times the nominal time constant for the smallest feedback gain ( $|K| = 1.01$ ). In detail, we selected the constant transition probabilities as  $(P_{i1} \ P_{i2} \ P_{i3}) = (a \ 1 - 2a \ a)$  for a varying positive parameter  $a$ , resulting, for the sake of comparison, in the same constant delay expectation  $\mathbb{E}(D) = 2$ , but a standard deviation  $\sigma(D)$  varying linearly with  $a$ . Corresponding results are pictured in Figure 3.3.

One can observe that the stability region corresponding to the controller based on the average prediction is much larger than the one obtained with the controller based on the average delay.

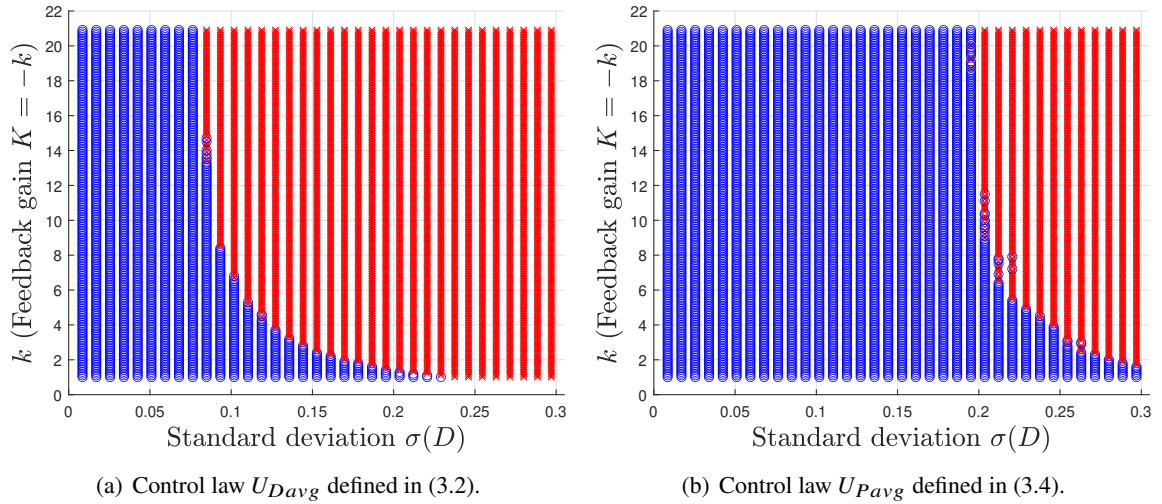


Figure 3.3: Stability regions obtained for the closed-loop systems consisting of (3.5) and, respectively, the two prediction-based control laws (3.2) (left) and (3.4) (right). The initial conditions are picked as  $X(0) = 0.1$  and  $U(t) = 0$  for  $t < 0$  and the delay values as  $\mathbf{D} = (1.7, 2.0, 2.3)^T$ . The transition probabilities are chosen as  $(P_{i1} \ P_{i2} \ P_{i3}) = (a \ 1 - 2a \ a)$  for a varying  $a > 0$ . For each couple,  $(K, a)$ , one delay realization was considered. Asymptotic stability of the resulting closed-loop system is depicted with a blue circle, and its absence with a red cross.

### 3.1.1.2 Two-dimensional system

To consider a more complex case, let us compare the performances obtained with these two controllers on the same second-order LTI system as the one of Chapters 1 and 2

$$\dot{X}(t) = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} X(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} U(t - D(t)), \quad (3.6)$$

with  $r = 3$  potential input delay values  $(D_1, D_2, D_3) = (0.1, 2.0, 2.1)$ . The transition probabilities are taken as  $(P_{i1} \ P_{i2} \ P_{i3}) = (0.02 \ 0.69 \ 0.29)$ .

We picked two different feedback gains:  $K = -[1 \ 2]$  and  $K = -[5 \ 6.5]$ , which corresponds to closed-loop eigenvalues  $\lambda(A + BK) = [-0.5000 + 1.3229i \ -0.5000 - 1.3229i]$  and  $\lambda(A + BK) = [-1.5 \ -4]$ , respectively. Corresponding simulation results are pictured in Figure 3.4 and illustrate that, as previously, the stability region corresponding to the controller based on the average prediction is larger than the one obtained with the controller based on the average delay, and that the first one thus yields better transient performances.

In a nutshell, both simulation cases indicate that predicting and averaging results into better performances than averaging and predicting. It turns out that the analysis of the average state dynamics can shed some light on these surprisingly dissimilar performances.

## 3.2 Some elements of explanation of the difference between the two controller designs

To better understand the different closed-loop behaviors resulting, respectively, from the two controllers (3.2) and (3.4), we first investigate further the dynamics of the ‘averaged system’.

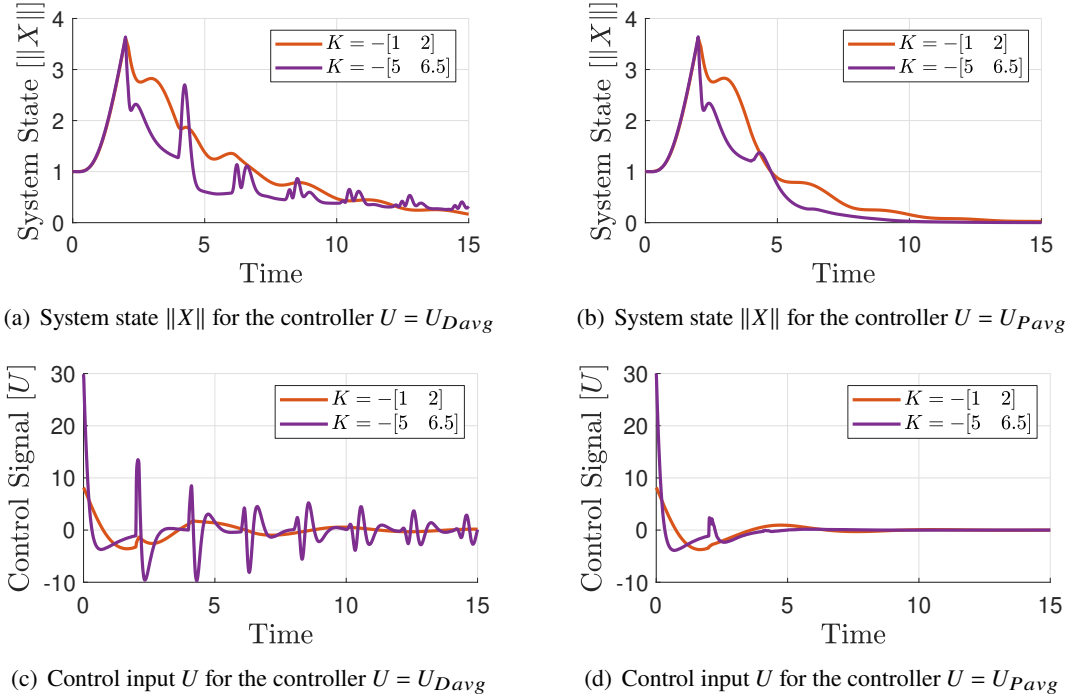


Figure 3.4: Simulation results of the closed-loop systems consisting of (3.6) and, respectively, the controller  $U_{Davg}$  defined in (3.2) (left column) and the controller  $U_{Pavg}$  defined in (3.4) (right column). Different feedback gain values are considered:  $K = -\begin{bmatrix} 1 & 2 \\ 5 & 6.5 \end{bmatrix}$ , resulting in the closed-loop eigenvalues  $\lambda(A + BK) = [-0.5000 + 1.3229i \quad -0.5000 - 1.3229i]$  and  $\lambda(A + BK) = [-1.5 \quad -4]$ , respectively. The potential delay values are  $\mathbf{D} = (0.1, 2.0, 2.1)^T$  and the initial conditions are  $X(0) = [1 \quad 0]^T$  and  $U(t) = 0$  for  $t \leq 0$ . The transition probabilities are constant and such that  $(P_{i1} \quad P_{i2} \quad P_{i3}) = (0.02 \quad 0.69 \quad 0.29)$ . (a) and (b) present the mean of the system state over 100 different trials, while (c) and (d) present the one of the control input.

### 3.2.1 Dynamics of the averaged system

In this subsection only, we consider that the transition probabilities are time-varying functions, contrary to the rest of the chapter.

**Lemma 3.1** *Consider the dynamics (3.1) with  $U$  any essentially bounded function. Then,  $(X, D)$  defines a continuous-time Markov process and the average state  $\bar{X} = \mathbb{E}_{[0, (X(0), U_{[0]}, D(0))]}(X)$  satisfies the following averaged dynamics*

$$\dot{\bar{X}}(t) = A\bar{X}(t) + \sum_{j=1}^r BP_{ij}(0, t)U(t - D_j). \quad (3.7)$$

Furthermore, if  $U$  is defined as either the prediction-based controller (3.2) or (3.4),  $(X, U_{[t]}, D)$  defines a continuous-time Markov process and the average state  $\bar{X} = \mathbb{E}_{[0, (X(0), U_{[0]}, D(0))]}(X)$  and the average input  $\bar{U} = \mathbb{E}_{[0, (X(0), U_{[0]}, D(0))]}(U)$  are related through the averaged dynamics

$$\dot{\bar{X}}(t) = A\bar{X}(t) + \sum_{j=1}^r BP_{ij}(0, t)\bar{U}(t - D_j). \quad (3.8)$$

*Proof:* Let us prove the closed-loop case of the prediction-based controller (3.2). According to Theorem 1.1,  $(X, U_{[t]}, D)$  defines a continuous-time Markov process. Let us then apply its

corresponding component-wise infinitesimal generator to the system state  $X$  as follows

$$LX(t) = (LX_l(t))_{1 \leq l \leq n} \triangleq \left( \limsup_{\Delta t \rightarrow 0^+} \frac{1}{\Delta t} \left( \mathbb{E}_{[t, (X(t), U_{[t], D(t)})]} (X_l(t + \Delta t)) - X_l(t) \right) \right)_{1 \leq l \leq n}.$$

Then, as the Probabilistic Delay Averaging technique presented in Introduction, as the process  $D$  takes only a finite number of values, it follows that

$$\begin{aligned} \mathbb{E}_{[0, (X(0), U_{[0], D(0)})]} (LX_l(t)) &= \sum_{j=1}^r P_{ij}(0, t) \mathbb{E}_{[0, (X(0), U_{[0], D(0)})]} (L_j X_l(t)) \\ &= \sum_{j=1}^r P_{ij}(0, t) \mathbb{E}_{[0, (X(0), U_{[0], D(0)})]} ((AX(t) + BU(t - D_j))_l) \end{aligned} \quad (3.9)$$

in which the last equality follows from the fact that  $X_l$  does not depend explicitly on the delay  $D$ . Hence, due to the linearity of the expected value, one gets that

$$\begin{aligned} &\mathbb{E}_{[0, (X(0), U_{[0], D(0)})]} (LX(t)) \quad (3.10) \\ &= \sum_{j=1}^r P_{ij}(0, t) \left( A \mathbb{E}_{[0, (X(0), U_{[0], D(0)})]} (X(t)) + B \mathbb{E}_{[0, (X(0), U_{[0], D(0)})]} (U(t - D_j)) \right) \\ &= A \bar{X}(t) + \sum_{j=1}^r P_{ij}(0, t) B \bar{U}(t - D_j), \end{aligned}$$

as  $\sum_{j=1}^r P_{ij}(0, t) = 1$ . Applying Dynkin's formula [Dyn12] to each component in (3.9), one finally concludes that

$$\begin{aligned} \bar{X}(t) &= \mathbb{E}_{[0, (X(0), U_{[0], D(0)})]} (X(t)) = X(0) + \int_0^t \mathbb{E}_{[0, (X(0), U_{[0], D(0)})]} (LX(s)) ds \quad (3.11) \\ &= X(0) + \int_0^t \left[ A \bar{X}(s) + \sum_{j=1}^r B P_{ij}(0, t) \bar{U}(s - D_j) \right] ds. \end{aligned}$$

The two other cases can be obtained similarly. ■

Lemma 3.1 is an important element in the understanding of the mechanisms of the dynamics at stake: according to it, the averaged system state behaves as if each input channel, delayed by a corresponding value  $D_j$ , was affecting the dynamics, with a weight  $P_{ij}(0, t)$ . Hence, the averaged system is controlled by the virtual input  $W(t) = \sum_{j=1}^r P_{ij}(0, t) \bar{U}(t - D_j)$ .

Figure 3.5 illustrates the veracity of this lemma, by comparing the state of the averaged dynamics (3.7) with the mean of  $N$  realizations of the original dynamics (3.1). In detail, we considered the two-dimensional stable dynamics

$$\dot{X}(t) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} X(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} U(t - D(t)), \quad (3.12)$$

with  $r = 3$  potential delay values  $(D_1, D_2, D_3) = (0.1, 2.0, 2.1)$  and the constant transition probabilities  $(P_{i1}, P_{i2}, P_{i3}) = (0.02, 0.69, 0.29)$ . The initial conditions are taken as  $X(0) = [1 \ 0]^T$  and  $U(t) = 0$ , for  $t \leq 0$ . The open-loop input  $U$  is a random process, following a normal distribution  $\mathcal{N}(0, 10)$ . Increasing the number of realizations from  $N = 100$  to  $N = 500$  emphasizes the fit between the two curves representing respectively the state of the averaged dynamics (3.7)  $\bar{X}$  with the mean of  $N$  realizations of the original dynamics  $\frac{1}{N} \sum_{i=1}^N X$ , in accordance with Lemma 3.1.

Let us now analyze the effect of the two previously presented control laws on this averaged dynamics. This will provide some possible explanations on the different closed-loop performances. In the rest of the chapter, we go back to the case where the transition probabilities are constant, to simplify the exposition.

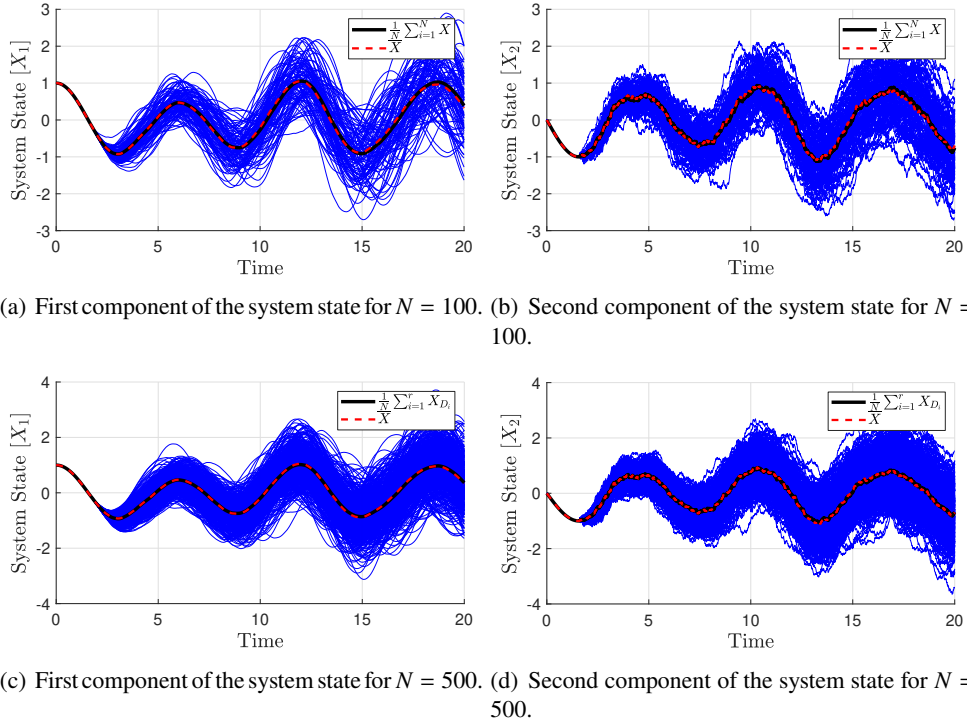


Figure 3.5: Comparison between the averaged system (3.7) and the mean of  $N$  realizations (top:  $N = 100$ , bottom:  $N = 500$ ) of the original system (3.1), for the dynamics (3.12). The open-loop input  $U$  is a random process, following a normal distribution  $\mathcal{N}(0, 10)$ . The delay values are  $(D_1, D_2, D_3) = (0.1, 2.0, 2.1)$  and the constant transition probabilities  $(P_{i1}, P_{i2}, P_{i3}) = (0.02, 0.69, 0.29)$ . The initial conditions are chosen as  $X(0) = [1 \ 0]^T$  and  $U(t) = 0$ , for  $t \leq 0$ . The red line represents the average of Monte Carlo simulations of  $N$  trials, represented with blue lines, while the black line represents the averaged system state  $\bar{X}$ .

### 3.2.2 Closed-loop averaged systems

Let us replace the two prediction-based control laws (3.2) and (3.4) into the averaged dynamics (3.8).

#### 3.2.2.1 With the averaged-delay prediction-based control law $U_{D_{avg}}$ of (3.2)

First, by linearity of the expected value, the averaged version of the control law (2.3) is

$$\bar{U}(t) = K \left[ e^{A\mathbb{E}(D)} \bar{X}(t) + \int_{t-\mathbb{E}(D)}^t B e^{A(t-s)} \bar{U}(s) ds \right]. \quad (3.13)$$

in which, using the fact that  $\sum_{k=1}^r P_{ik} = 1$  and integrating (3.8) on the  $r$  time intervals  $[t - D_k, t]$  ( $k = 1, \dots, r$ ),

$$\bar{X}(t) = \sum_{k=1}^r P_{ik} \bar{X}(t) = \sum_{k=1}^r P_{ik} \left[ e^{AD_k} \bar{X}(t - D_k) + \int_{t-D_k}^t B \sum_{j=1}^r P_{ij} e^{A(t-s)} \bar{U}(s - D_j) ds \right]. \quad (3.14)$$

Plugging this averaged controller into the averaged system (3.8), and with the help of (3.14), one gets

$$\begin{aligned} \dot{\bar{X}}(t) = & (A + BK) \bar{X}(t) + BK \sum_{j=1}^r P_{ij} \left( \left( e^{A\mathbb{E}(D)} - e^{AD_j} \right) \bar{X}(t - D_j) \right. \\ & \left. + \sum_{k=1}^r P_{ik} \int_{t-\mathbb{E}(D)-D_j}^{t-D_k-D_j} e^{A(t-D_j-s)} B \bar{U}(s) ds \right). \end{aligned} \quad (3.15)$$

This closed-loop average dynamics cannot be further simplified a priori. Hence, as could be expected, the input delay is only partially compensated. Besides, the last two terms in (3.15) could reasonably be interpreted as non-commutation error terms. We now turn our attention to the second control law.

### 3.2.2.2 With the averaged prediction control law $U_{Pavg}$ of (3.4)

Similarly, the averaged version of the control law (3.4) is

$$\bar{U}(t) = K \sum_{j=1}^r P_{ij} \left[ e^{AD_j} \bar{X}(t) + \int_{t-D_j}^t B e^{A(t-s)} \bar{U}(s) ds \right]. \quad (3.16)$$

Then, the control law based on the average of the predictions leads to the closed-loop dynamics

$$\begin{aligned} \dot{\bar{X}}(t) &= (A + BK) \bar{X} \\ &+ BK \left( \sum_{j=1}^r P_{ij} \sum_{k=1}^r P_{ik} \left[ e^{AD_k} \bar{X}(t - D_j) + \int_{t-D_k-D_j}^{t-D_j} e^{A(t-D_j-s)} B \bar{U}(s) ds \right] \right. \\ &\left. - \sum_{k=1}^r P_{ik} \left[ e^{AD_k} \bar{X}(t - D_k) + \int_{t-D_k}^t B \sum_{j=1}^r P_{ij} e^{A(t-s)} \bar{U}(s - D_j) ds \right] \right) \\ &= (A + BK) \bar{X} + BK \sum_{j,k=1}^r P_{ij} P_{ik} \left( e^{AD_k} \bar{X}(t - D_j) + \int_{t-D_k-D_j}^{t-D_j} e^{A(t-D_j-s)} B \bar{U}(s) ds \right. \\ &\left. - e^{AD_k} \bar{X}(t - D_k) - \int_{t-D_k-D_j}^{t-D_j} B e^{A(t-D_j-s)} \bar{U}(s) ds \right) \\ &= (A + BK) \bar{X}(t) + BK \sum_{j=1}^r P_{ij} \left( \sum_{k=1}^r P_{ik} e^{AD_k} - e^{AD_j} \right) \bar{X}(t - D_j). \end{aligned} \quad (3.17)$$

Again, the input-delay is only partially compensated, leaving delayed terms which may be interpreted as a type of covariance of the delayed averaged state. Yet, compared to (3.15), one can observe that (3.17) does not involve a distributed delay error term, which has been removed by the input averaging feature of the control law (3.4), corresponding somehow to the one present in (3.8). This remarkable difference is likely to explain the performance gap observed in simulations for the two control laws.

To validate numerically this analysis, consider the characteristic equations corresponding to (3.15) and (3.17) respectively. For the one-dimensional toy example studied previously in Section 3.1.1.1, one can easily compute the corresponding characteristic roots (with the QPmR routine for instance [VZ14]) for each value of the couple  $(K, \sigma)$  (or  $(K, a)$ ) and thus determine the stability regions. These are reported in Figure 3.6 and perfectly match the boundary of the stability region previously obtained. It confirms the key role played by the average dynamics (3.8) when analyzing the closed-loop impact of the control law.

The formal comparison of the closed-loop dynamics (3.15) and (3.17) to prove rigorously the benefit of using (3.4) over (3.2) remains to be carried out in future works. Yet, it is worth observing that, alternatively to the methodology for stability analysis proposed in the two previous chapters, one can directly rely on these averaged dynamics to establish sufficient conditions for closed-loop stability.



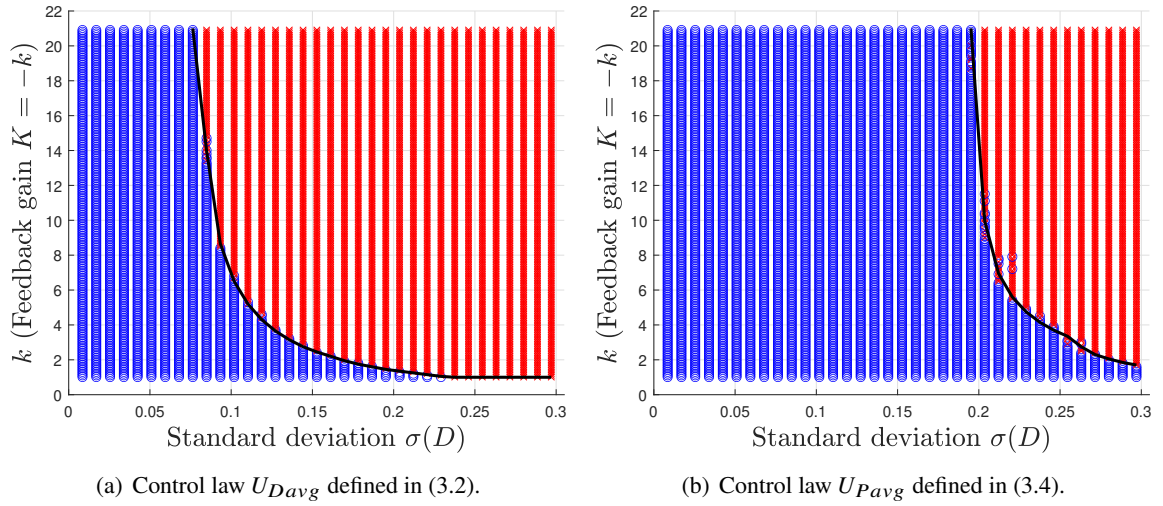


Figure 3.6: Stability regions obtained for the closed-loop systems consisting of (3.5) and, respectively, the two prediction-based control laws (3.2) (left) and (3.4) (right). The initial conditions are picked as  $X(0) = 0.1$  and  $U(t) = 0$  for  $t < 0$  and the delay values as  $\mathbf{D} = (1.7, 2.0, 2.3)^T$ . The transition probabilities are chosen as  $(P_{i1} \ P_{i2} \ P_{i3}) = (a \ 1 - 2a \ a)$  for a varying  $a > 0$ . For each couple,  $(K, \sigma)$  or similarly  $(K, a)$ , one delay realization was considered. Asymptotic stability of the resulting closed-loop system is depicted with a blue circle, and its absence with a red cross. Black lines depict the boundaries of the stability regions of the closed-loop average systems (3.15) and (3.17), which were obtained by computing the corresponding characteristic roots.

### 3.3 Stabilization result with an averaged-prediction controller based on the closed-loop averaged system

As the closed-loop dynamics corresponding to the averaged-prediction controller (3.17) can be compactly written under the following LTI pointwise-delay differential equation form

$$\dot{\bar{X}}(t) = (A + BK)\bar{X}(t) + \sum_{j=1}^r \bar{A}_j \bar{X}(t - D_j), \quad (3.18)$$

in which  $\bar{A}_j = BKP_{ij} (\sum_{k=1}^r P_{ik} e^{AD_k} - e^{AD_j})$  for  $j \in \{1, \dots, r\}$ , one can then analyze its stability with standard Lyapunov tools [NVDD98], as the following theorem illustrates it.

**Theorem 3.1** Consider the closed-loop system consisting of the system (3.1) and the control law (3.4). If there exist positive definite matrices  $P, S_j > 0$  such that

$$\bar{Q} = \begin{bmatrix} P(A + BK) + (A + BK)^T P + \sum_{j=1}^r S_j & \Theta \\ \Theta^T & -\Sigma_S \end{bmatrix} < 0. \quad (3.19)$$

with  $\Theta = [P\bar{A}_1 \ \dots \ P\bar{A}_r]$  and  $\Sigma_S = \text{diag}(S_1, \dots, S_r)$ , then the closed-loop system is asymptotically stable. Besides, the Linear Matrix Inequality (3.19) holds if the delay standard deviation  $\sigma_{[0, D(0)]}(D)$  is sufficiently small.

*Proof:* We consider the following Lyapunov functional candidate (see [Ric03, NVDD98, GCK03])

$$V(\bar{X}_t) = \bar{X}(t)^T P \bar{X}(t) + \sum_{j=1}^r \int_{t-D_j}^t \bar{X}(s)^T S_j \bar{X}(s) ds, \quad (3.20)$$

in which  $P, S_j > 0$  are positive definite matrices. The derivative of the Lyapunov functional thus yields

$$\begin{aligned} \dot{V}(\bar{X}_t) &= 2\bar{X}(t)^T P \left( (A + BK)\bar{X}(t) + \sum_{j=1}^r \bar{A}_{D_j} \bar{X}(t - D_j) \right) \\ &\quad + \sum_{j=1}^r \left( \bar{X}(t)^T S_j \bar{X}(t) - \bar{X}(t - D_j)^T S_j \bar{X}(t - D_j) \right) \\ &= \bar{\Psi}(t)^T \bar{Q} \bar{\Psi}(t), \end{aligned} \quad (3.21)$$

with  $\bar{\Psi}(t) = [\bar{X}(t) \ \bar{X}(t - D_1) \ \dots \ \bar{X}(t - D_r)]^T$  and  $\bar{Q}$  defined in (3.19). Then, in view of Lyapunov-Krasovskii Theorem, the asymptotic stability of the closed-loop system (3.18) is then reduced to the matrix inequality (3.19).

Furthermore, using the Schur complement, the constraint (3.19) can be transformed as

$$P(A + BK) + (A + BK)^T P + \sum_{j=1}^r S_j + \Theta \Sigma_S^{-1} \Theta^T < 0, \quad (3.22)$$

in which

$$\Theta \Sigma_S^{-1} \Theta^T = \sum_{j=1}^r P \bar{A}_j S_j^{-1} \bar{A}_j^T P^T. \quad (3.23)$$

Let us observe that, as  $|M|^2 = \max \lambda(M^T M)$  is the maximal spectral value of a given  $n \times m$ -matrix  $M$ , it follows from the spectral theorem that  $M^T M \leq |M|^2 I_m$  and also that

$$M_1^T M_2^T M_2 M_1 \leq |M_2|^2 M_1^T M_1, \quad (3.24)$$

for matrices  $M_1$  and  $M_2$  of appropriate dimensions. Hence, denoting  $S_j^{1/2}$  the square root of the positive definite matrix  $S_j$ , it follows that

$$\bar{A}_j S_j^{-1} \bar{A}_j^T \leq |S_j^{-1/2}|^2 \bar{A}_j \bar{A}_j^T = |S_j^{-1}| \bar{A}_j \bar{A}_j^T. \quad (3.25)$$

Besides, as

$$\bar{A}_j \bar{A}_j^T = P_{ij}^2 B K \left( \sum_{k=1}^r P_{ik} e^{AD_k} - e^{AD_j} \right) \left( \sum_{k=1}^r P_{ik} e^{A^T D_k} - e^{A^T D_j} \right) K^T B^T, \quad (3.26)$$

one then obtains that

$$\bar{A}_j S_j^{-1} \bar{A}_j^T \leq |S_j^{-1}| P_{ij}^2 \left| \sum_{k=1}^r P_{ik} e^{A^T D_k} - e^{A^T D_j} \right|^2 B K K^T B^T. \quad (3.27)$$

Applying the mean-value theorem to the function  $D \in \mathbb{R} \mapsto e^{A^T D}$  on the interval  $[\min \{D_k, D_j\}, \max \{D_k, D_j\}]$ , one then gets that

$$\begin{aligned} \sum_{k=1}^r P_{ik} e^{A^T D_k} - e^{A^T D_j} &= \sum_{k=1}^r P_{ik} (e^{A^T D_k} - e^{A^T D_j}) \\ &= \sum_{k=1}^r P_{ik} A^T e^{A^T \xi_{ik}} (\max \{D_k, D_j\} - \min \{D_k, D_j\}), \end{aligned} \quad (3.28)$$

in which  $\xi_{ik}$  is a given scalar belonging to the interval  $[\min \{D_k, D_j\}, \max \{D_k, D_j\}]$ . Consequently,

$$\left| \sum_{k=1}^r P_{ik} e^{A^T D_k} - e^{A^T D_j} \right| \leq |A^T| e^{|A^T| D_r} \sum_{k=1}^r P_{ik} |D_k - D_j|, \quad (3.29)$$

and, thus, from (3.23) and (3.27),

$$\Theta \Sigma_S^{-1} \Theta^T \leq |A^T|^2 e^{2|A^T| D_r} \max_{j=1, \dots, r} |S_j^{-1}| \left( \sum_{j=1}^r P_{ij}^2 \left[ \sum_{k=1}^r P_{ik} |D_k - D_j| \right]^2 \right) P B K K^T B^T P^T. \quad (3.30)$$

As, applying Young's inequality, it holds

$$\begin{aligned} \sum_{j=1}^r P_{ij}^2 \left[ \sum_{k=1}^r P_{ik} |D_k - D_j| \right]^2 &\leq \sum_{j=1}^r P_{ij} \left[ \sum_{k=1}^r P_{ik} |D_k - D_j| \right]^2 \\ &\leq r \sum_{j=1}^r P_{ij} \sum_{k=1}^r P_{ik}^2 (D_k - D_j)^2 \\ &\leq r \sum_{j=1}^r P_{ij} \sum_{k=1}^r P_{ik} (D_k - D_j)^2 = 2r \sigma_{[0, D(0)]}(D)^2, \end{aligned} \quad (3.31)$$

one concludes that

$$\Theta \Sigma_S^{-1} \Theta^T \leq \sigma_{[0, D(0)]}(D)^2 2r |A^T|^2 e^{2|A^T| D_r} \max_{j=1, \dots, r} |S_j^{-1}| P B K K^T B^T P^T. \quad (3.32)$$

Consequently, for any positive definite matrices  $S_j$  and with  $P$  the solution to the Lyapunov equation  $P(A + BK) + (A + BK)^T P = -I_n - \sum_{j=1}^r S_j$ , the matrix inequality (3.19) is satisfied for  $\sigma_{[0, D(0)]}(D)$  small enough, namely, such that

$$\sigma_{[0, D(0)]}(D)^2 \leq \frac{1}{2r |A^T|^2 e^{2|A^T| D_r} \max_{j=1, \dots, r} (|S_j^{-1}|) \max(\lambda(P B K K^T B^T P^T))}. \quad (3.33)$$

Theorem 3.1 is only an example of the stability conditions which may be obtained from the averaged systems (3.17) or (3.18). Less conservative conditions are likely to be obtained with more sophisticated Lyapunov functions provided in [NVDD98, GCK03, Bri14], for instance. ■

For the sake of illustration, let us again consider the one-dimensional toy example studied previously in Section 3.1.1.1. For each value of the couple  $(K, \sigma)$ , one can verify whether the LMI (3.19) is satisfied or not, using for instance the YALMIP toolbox [Lof04]. Figure 3.7 compares the estimated stability results obtained by this LMI with the actual stability region computed by solving the corresponding characteristic roots of (3.17). One can observe that a reasonably large portion of the stability region is captured by the LMI (3.19). Yet, its general trend may have little interest in practice as it does not cover low-gain stability, which is likely to be used to counteract high values of delay standard deviations.

### 3.4 Conclusion and perspective

This chapter investigates prediction-based control design for a linear system subjected to a random input delay, modeled as a continuous-time Markov process, with a finite number of values and constant transition probabilities. Two different prediction-based control laws are considered and compared: (i) the first one uses a single state prediction with the expectation of the delay as prediction horizon, as proposed in Chapter 2; (ii) the second one builds on multiple predictions, each one computed on a

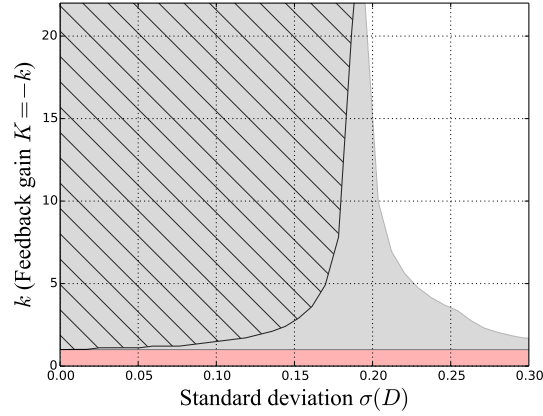


Figure 3.7: Stability regions obtained for the closed-loop systems consisting of (3.5) and the average-prediction control law (3.4). The stability obtained with the LMI (3.19) is pictured by the black-dashed region. The grey region presents the actual stability region, obtained by computing the corresponding characteristic roots. The region in red represents values of the feedback gain which do not guarantee that  $A + BK$  is Hurwitz. The initial conditions are picked as  $X(0) = 0.1$  and  $U(t) = 0$  for  $t < 0$  and the delay values as  $\mathbf{D} = (1.7, 2.0, 2.3)^T$ . The transition probabilities are chosen as  $(P_{i1} P_{i2} P_{i3}) = (a \ 1 - 2a \ a)$  for a varying  $a > 0$ .

time horizon corresponding to a potential delay value, which are then averaged. In the second case, we prove that closed-loop exponential stabilization is achieved under a condition quantitatively similar to the one obtained for the averaged-delay prediction-based controller, namely, provided that the standard deviation of the delay distribution is small enough.

Yet, numerical simulations carried out for both controllers illustrate that the averaged-prediction controller leads to a more robust delay compensation. This leads to the assumption that predicting and averaging results into better performances than averaging and predicting. We explain this difference by obtaining the averaged dynamics, and the corresponding averaged closed-loop differential equations in both cases. We also illustrated how, for a linear time invariant system, this closed-loop dynamics can be used to provide sufficient stabilization conditions.

Nevertheless, the formal comparison of the respective closed-loop behaviors remains to be carried out in the future. Such an analysis may ground on the fact that the closed-loop averaged system (3.15) exhibits a distributed input term. A potential path to follow could be, thanks to the inverse Artstein transformation [BPCE18], to rewrite this quantity as a memory term on the state  $X$ , that is  $\int_0^t \Phi(t, s)X(s)ds$  for a certain kernel  $\Phi$ , and study how the addition of this term affects the closed-loop eigenvalues. Another possibility could be to rely on spectral considerations, but this study would then difficulty extend to the case of time-varying transition probabilities.

The extension of this multi-predictor technique to nonlinear dynamics is also an interesting, but much more complex path to explore. Indeed, even if, in all likelihood, the concept of averaged prediction can be fairly straightforwardly extended to nonlinear plants, one would then be unable to rely on an explicit form of the averaged dynamics, as the proof of Lemma 3.1 relies on the linearity of both the expected value and the dynamics. Thus, one cannot perform stability analysis with the averaged system, as proposed in this chapter.

This is why, in the following chapter, we propose to use the general methodology already applied in Chapters 1 and 2, in the nonlinear context.



## Chapter 4

# Extension to nonlinear stochastic input delay systems

This chapter is adapted from the book chapter *Prediction Control for Nonlinear Systems with Stochastic Input Delay*, published in "Advances in Distributed Parameter Systems", Vol in the series "Advances in Delays and Dynamics" (eds) J. Auriol, J. Deutscher, G. Mazanti and G. Valmorbida, 2021.

**Résumé** Dans ce chapitre, nous étendons nos travaux précédents au cas d'un système non linéaire soumis à un retard d'entrée aléatoire, en considérant une prédiction à horizon constant et une loi de contrôle correspondant au cas nominal non-retardé. Nous prouvons que la stabilisation en boucle fermée peut être obtenue sous réserve que la condition initiale soit restreinte, de sorte à ne pas sortir du bassin d'attraction correspondant à la loi de contrôle nominale avant que le contrôle en boucle fermée n'atteigne le système, mais également sous la condition que l'horizon de prédiction soit suffisamment proche des valeurs du retard. Ce résultat généralise donc le Théorème 1.1 au cas non-linéaire.

**Abstract** In this chapter, we extend our previous works to the case of a nonlinear system subjected to a random input delay, by considering a constant horizon prediction-based controller feeding a nominal control law exponentially stabilizing the delay-free dynamics. We prove that closed-loop stabilization can be obtained, provided that the system initial condition is restricted to guarantee that the system state does not leave the basin of attraction of the nominal control law before the closed-loop controller kicks in, but also provided that the prediction horizon remains sufficiently close to the delay values. Hence, this result generalizes Theorem 1.1 to the nonlinear case.

## 4.1 Problem statement and control design

This chapter focuses on the prediction-based control of the following autonomous nonlinear dynamics,

$$\dot{X}(t) = f(X(t), U(t - D)), \quad (4.1)$$

in which the  $\mathbb{R}^n$ -valued random variable  $X$  and  $U \in \mathbb{R}$  are the state and control input, respectively.  $f$  is a nonlinear function of class  $C^1$  such that  $f(0, 0) = 0$ .

The random delay affecting the input is assumed to be a Markov process with a finite number of states, that is,  $D(t) \in \{D_i, i \in \{1, \dots, r\}\}$  for a given  $r \in \mathbb{N}$ . Without loss of generality, we assume in the sequel that these values are ordered as  $0 < \underline{D} \leq D_1 < D_2 < \dots < D_r \leq \overline{D}$  in which  $\underline{D}$  and  $\overline{D}$  are known positive bounds. In addition, we denote  $P_{ij} : \mathbb{R}^2 \rightarrow [0, 1]$  the transition probability functions, which quantify the probability to switch from  $D_i$  at time  $t_1$  to  $D_j$  at time  $t_2$  ( $(i, j) \in \{1, \dots, r\}^2$ ,  $t_2 \geq t_1 \geq 0$ ).

Finally, we assume that the realizations of this Markov process are right-continuous, in order to be able to guarantee the existence of solutions to the open-loop or closed-loop systems.

The control objective is to stabilize plant (4.1) following a prediction-based approach, despite the random nature of the input delay. As in the previous chapter, this will be achieved by a robust delay compensation approach, extending the constant-horizon prediction design proposed in Chapter 1. Before detailing our control design, we first further characterize the delay-free characteristics of the plant under consideration.

**Assumption 4.1** *There exists a feedback law  $\kappa$  of class  $C^1$  such that, the plant*

$$\dot{X} = f(X, \kappa(X)), \quad (4.2)$$

*is exponentially stable with basin of attraction  $\mathcal{A} \subset \mathbb{R}^n$ , i.e., there exists a positive definite function  $W : \mathbb{R}^n \rightarrow \mathbb{R}_+$  such that, for all compact  $C \subset \mathcal{A}$ , there exist positive constants  $\alpha$ ,  $C_1$ ,  $C_2$  and  $C_3$  such that, for all  $X \in C$ ,*

$$\frac{dW}{dX}(X) f(X, \kappa(X)) \leq -\alpha |X|^2, \quad (4.3)$$

$$C_1 |X|^2 \leq W(X) \leq C_2 |X|^2, \quad (4.4)$$

$$\left| \frac{dW}{dX}(X) \right| \leq C_3 |X|. \quad (4.5)$$

Assuming exponential stability of the delay-free closed-loop dynamics in lieu of asymptotic stability is not restrictive, in the sense that some dynamical systems for which the origin is only asymptotically stable can have a zero-delay robustness property. This prohibits us from using a robust compensation controller as the one we aim at designing.

We now detail two additional technical assumptions.

**Assumption 4.2** *The function  $f(X, U)$  is globally Lipschitz, i.e., there exists  $C_L > 0$  such that for  $\forall (U_1, U_2) \in \mathbb{R}^2$  and  $\forall (X_1, X_2) \in (\mathbb{R}^n)^2$*

$$|f(X_1, U_1) - f(X_2, U_2)| \leq C_L |(X_1, U_1) - (X_2, U_2)|. \quad (4.6)$$

**Assumption 4.3** *The feedback law  $\kappa$  defined in Assumption 4.1 is such that  $\kappa(0) = 0$  and there exists a positive constant  $C_4$  such that*

$$\left| \frac{d\kappa}{dX} \right| \leq C_4, X \in C, \text{ for all compact } C \subset \mathcal{A}. \quad (4.7)$$

Contrary to Assumption 4.1, these last two assumptions are more restrictive and more technical. They aim at simplifying the stability analysis detailed in the following, but should be relaxed in future works.

Note that Assumption 4.2 jointly with the fact that  $f(0, 0) = 0$  guarantees that the plant does not escape in finite time, which is a necessary condition for stabilization to be achieved for any delay values. Therefore, we do not need to assume strong forward completeness here, as usually done for prediction-based control for nonlinear dynamics [BLK13a], but, in all likelihood, future works will have to replace Assumption 4.2 by this property.

In line with the nonlinear prediction-based control strategy for deterministic delays presented in Section 2.1.2 of the Introduction and the constant-horizon prediction design proposed in Chapter 1 for a linear system with a random input delay, we define the controller as

$$U(t) = \kappa(\hat{P}(t)), \quad (4.8)$$

in which the predictor  $\hat{P}$  is defined for  $\theta \in [t - D_0, t]$  as

$$\hat{P}(\theta, t) = X(t) + \int_{t-D_0}^{\theta} f(\hat{P}(s, t), U(s)) ds, \quad t - D_0 \leq \theta \leq t. \quad (4.9)$$

Notice that this prediction-based controller is nothing more than the usual prediction-based controller (44) applied in the constant delay case and which would compensate for the delay in closed-loop, would  $D_1 = \dots = D_r = D_0$ .

We study in the following how to choose the prediction horizon and under which conditions one can guarantee that this controller will achieve stabilization of the dynamical systems (4.1). As could be expected, the obtained condition is the direct extension of the one obtained in Chapter 1 in the context of a linear dynamics.

## 4.2 Sufficient condition for closed-loop exponential stability

In the nonlinear context under consideration, we can formulate a twin result to Theorem 1.1.

**Theorem 4.1** *Consider the closed-loop system consisting of the system (4.1) satisfying Assumptions 4.1–4.3 and the control law (4.8). Let  $C$  be a compact set of  $\mathcal{A}$  containing the origin. Then, there exist positive constants  $\varepsilon^*$  and  $\rho^*$  such that, if*

$$|D_0 - D_j| \leq \varepsilon^*, \quad j \in \{1, \dots, r\}, \quad (4.10)$$

and if

$$Y(0) \leq \rho^*, \quad (4.11)$$

there exist positive constants  $R$  and  $\gamma$  (independent of the initial condition) such that

$$\mathbb{E}_{[0, Y(0)]}(Y(t)) \leq RY(0)e^{-\gamma t}, \quad (4.12)$$

with

$$Y(t) = |X(t)|^2 + \int_{t-\bar{D}-D_0}^t U(s)^2 ds. \quad (4.13)$$

As for a linear dynamic, condition (4.10) requires the sequence of the random delay  $(D_i)_{1 \leq i \leq r}$  to be limited in the vicinity  $\varepsilon^*$  of the prediction horizon  $D_0$ , and thus to be as well in a vicinity  $2\varepsilon^*$  of each other. The main difference with Theorem 1.1 is the condition (4.11) which restricts the size of the initial condition. This is due to the fact that, during the first  $\bar{D}$  units of time, the closed-loop control input may not reach the plant, due to the presence of the input delay. Hence, if the initial input



is too large, it may steer the system state outside of the basin of attraction, thus prohibiting stabilizing it with the feedback law  $\kappa$ . To avoid such a phenomenon, one then has to restrict the initial condition.

The condition (4.10) is of course quite restrictive and should be relaxed to a probabilistic one, as the one obtained in Theorem 1.2. In that sense, Theorem 4.1 only constitutes a preliminary step in view of the stabilization of a broad class of contexts for nonlinear systems. We will detail these points further in the conclusion section of this chapter, but we now turn our attention to the proof of this theorem, which is obtained following the generic three-steps methodology proposed in Chapter 1.

### 4.3 PDE Representation of the Delay and Backstepping Transformation

The first ingredient we use to analyze the closed-loop behavior is to represent the random delay with a distributed actuator vector as, for  $x \in [0, 1]$ ,  $\mathbf{v}(x, t) = (v_1(x, t) \ \cdots \ v_k(x, t) \ \cdots \ v_r(x, t))^T$  with  $v_k(x, t) = U(t + D_k(x - 1))$ . In addition, we introduce  $\hat{v}(x, t)$  to represent the control input  $U(t)$  within the interval  $[t - D_0, t]$ , and the corresponding input estimation error  $\tilde{\mathbf{v}}(x, t)$  defined as

$$\begin{cases} \hat{v}(x, t) = U(t + D_0(x - 1)) \\ \tilde{\mathbf{v}}(x, t) = \mathbf{v}(x, t) - \mathbf{1}\hat{v}(x, t). \end{cases} \quad (4.14)$$

Then, we define  $\mu(x, t) = U(t - D_0 + \bar{D}(x - 1))$  to represent the controller within the interval  $[t - D_0 - \bar{D}, t - D_0]$ . This enables to rewrite the ODE with random input delay (4.1) as the extended state  $(X, \hat{v}, \tilde{\mathbf{v}}, \mu)$ ,

$$\begin{cases} \dot{X}(t) = f(X(t), \hat{v}(0, t) + \delta(t)^T \tilde{\mathbf{v}}(0, t)) \\ D_0 \hat{v}_t(x, t) = \hat{v}_x(x, t) \\ \hat{v}(1, t) = U(t) \\ \Lambda_D \tilde{\mathbf{v}}_t(x, t) = \tilde{\mathbf{v}}_x - \Sigma_D \hat{v}_x \\ \tilde{\mathbf{v}}(1, t) = \mathbf{0} \\ \bar{D} \mu_t(x, t) = \mu_x(x, t) \\ \mu(1, t) = \hat{v}(0, t), \end{cases} \quad (4.15)$$

in which  $\Sigma_D = \left( \frac{D_1 - D_0}{D_0} \ \cdots \ \frac{D_r - D_0}{D_0} \right)^T$ ,  $\mathbf{0}$  is a  $r$ -by-1 all-zeros vector, and  $\Lambda_D = \text{diag}(D_1, \dots, D_r)$ ,  $\mathbf{1}$  is a  $r$ -by-1 all-ones vector and  $\delta \in \mathbb{R}^r$  is such that, if  $D(t) = D_j$ , then  $\delta(t) = e_j$ , the  $j^{\text{th}}$ -unit vector. Hence,  $\delta$  is a random process with the same transition probabilities as the stochastic process  $D$ , but with the finite number of states ( $e_i$ ) instead of ( $D_i$ ). The random variable  $\delta$  then acts as a switching signal, selecting the channel and thus the delayed signal which enters the ODE.

The second ingredient that we use is the following backstepping transformation (see [BLK13a])

$$w(x, t) = \hat{v}(x, t) - \kappa(\hat{p}(x, t)), \quad 0 \leq x \leq 1, \quad (4.16)$$

in which the distributed predictor  $\hat{p}$  is defined as

$$\hat{p}(x, t) = X(t) + D_0 \int_0^x f(\hat{p}(y, t), \hat{v}(y, t)) dy, \quad 0 \leq x \leq 1. \quad (4.17)$$

Notice that this backstepping transformation is similar to the one used in the deterministic nonlinear context but applied here to the estimated distributed input  $\hat{v}$ .

**Lemma 4.1** *The backstepping transformation (4.16), jointly with the control law (4.8)–(4.9), transform the plant (4.15) into the target system  $(X, w, \tilde{v}, \mu)$*

$$\begin{cases} \dot{X}(t) = f(X(t), \kappa(X(t)) + w(0, t) + \delta(t)^T \tilde{v}(0, t)) \\ D_0 w_t(x, t) = w_x(x, t) + r_w(x, t) \\ w(1, t) = 0 \\ \Lambda_D \tilde{v}_t(x, t) = \tilde{v}_x - \Sigma_D h(t + D_0(x - 1)) \\ \tilde{v}(1, t) = \mathbf{0} \\ \bar{D} \mu_t(x, t) = \mu_x(x, t) \\ \mu(1, t) = \kappa(X(t)) + w(0, t), \end{cases} \quad (4.18)$$

in which  $h$  is defined for  $t \geq 0$  as

$$h(t) = D_0 \frac{d\kappa}{dX}(\hat{p}(1, t)) f(\hat{p}(1, t), w(0, t) + \kappa(X(t)) + \delta(t)^T \tilde{v}(0, t)), \quad (4.19)$$

and  $r_w$  is defined for  $x \in [0, 1]$  as  $r_w(x, t) = -D_0 \frac{d\kappa}{dX}(\hat{p}(x, t)) \Phi(x, 0, t) \tilde{f}(t)$ , with

$$\tilde{f}(t) = f(X(t), \kappa(X(t)) + w(0, t) + \delta(t)^T \tilde{v}(0, t)) - f(X(t), \kappa(X(t)) + w(0, t)). \quad (4.20)$$

Besides,  $\Phi$  is the state-transition matrix associated with the equation

$$r_x(x, t) = D_0 \frac{\partial f}{\partial X}(\hat{p}(x, t), w(x, t) + \kappa(\hat{p}(x, t))) r(x, t), \quad (4.21)$$

with  $\hat{p}(x, t)$  defined in (4.17) a function of  $X(t)$  and  $w(y, t)$  for  $y \in [0, x]$  and  $x \in [0, 1]$ .

*Proof:* Define  $q(x, t) = D_0 \hat{p}_t(x, t) - \hat{p}_x(x, t)$ . Taking time- and space-derivatives of  $\hat{p}$  defined in (4.17), one obtains

$$\begin{aligned} q(x, t) &= D_0 \left[ f(X(t), \hat{v}(0, t) + \delta(t)^T \tilde{v}(0, t)) + \int_0^x \frac{\partial f}{\partial X}(\hat{p}(y, t), \hat{v}(y, t)) D_0 \hat{p}_t dy \right. \\ &\quad + \int_0^x \frac{\partial f}{\partial U}(\hat{p}(y, t), \hat{v}(y, t)) \hat{v}_x dy - f(X(t), \hat{v}(0, t)) \\ &\quad \left. - \int_0^x \frac{\partial f}{\partial X}(\hat{p}(y, t), \hat{v}(y, t)) \hat{p}_x dy - \int_0^x \frac{\partial f}{\partial U}(\hat{p}(y, t), \hat{v}(y, t)) \hat{v}_x dy \right] \\ &= D_0 \int_0^x \frac{\partial f}{\partial X}(\hat{p}(y, t), \hat{v}(y, t)) q(y, t) dy + D_0 \tilde{f}(t), \end{aligned} \quad (4.22)$$

with  $\tilde{f}(t)$  defined in (4.20). This integral equation can be rewritten under the differential form

$$\begin{cases} q_x(x, t) = D_0 \frac{\partial f}{\partial X}(\hat{p}(x, t), \hat{v}(x, t)) q(x, t) \\ q(0, t) = D_0 \tilde{f}(t). \end{cases} \quad (4.23)$$

Thus, introducing the state-transition matrix  $\Phi$  associated with (4.23), it follows that

$$q(x, t) = D_0 \Phi(x, 0, t) \tilde{f}(t). \quad (4.24)$$

As the time- and space- derivative of the backstepping transformation (4.16) are given as

$$w_t(x, t) = \hat{v}_t(x, t) - \frac{d\kappa}{dX}(\hat{p}(x, t)) \hat{p}_t(x, t), \quad (4.25)$$

$$w_x(x, t) = \hat{v}_x(x, t) - \frac{d\kappa}{dX}(\hat{p}(x, t)) \hat{p}_x(x, t), \quad (4.26)$$

the dynamics of the backstepping variable  $w$  then follows.

In addition, the expression of  $h(t)$  for  $t \geq 0$  is obtained by taking a time derivative of (4.8)

$$h(t + D_0(x - 1)) = \hat{v}_x(x, t) = D_0 \dot{U}(t + D_0(x - 1)) = D_0 \frac{dk}{dX}(\hat{p}(1, t)) \dot{\hat{P}}(t). \quad (4.27)$$

■

We are now ready to analyze the closed-loop system stabilization.

#### 4.4 Stability analysis of the closed-loop system

Before proceeding with the Lyapunov analysis based on our third ingredient, the Probabilistic Delay Averaging approach, we first formulate a well-posedness result for this system, in order to guarantee that it defines a Markov process.

With this aim in view, let us define the state of the target system (4.18) as  $\Psi = (X, w, \tilde{\mathbf{v}}, \mu) \in \mathbb{R}^n \times \mathcal{L}_\infty([0, 1], \mathbb{R}) \times \mathcal{L}_\infty([0, 1], \mathbb{R}^r \times \mathcal{L}_\infty([0, 1], \mathbb{R})) \triangleq \mathcal{D}_\Psi$ . Following [KK60], by a weak solution to the closed-loop system (4.1) and (4.8)–(4.9), we refer to a  $\mathbb{R}^n \times \mathcal{L}_\infty([-D, 0], \mathbb{R}) \times \mathbb{R}$ -valued random variable  $(X(X_0, t), U_t(U_0, \cdot), D(t))$ , the realizations of which satisfy an integral form of (4.1) and (4.8)–(4.9). Similarly, by a weak solution to (4.18), we refer to a  $\mathcal{D}_\Psi \times \mathbb{R}$ -valued random variable  $(X(X_0, t), w(w_0, \cdot, t), \tilde{\mathbf{v}}(\tilde{\mathbf{v}}_0, \cdot, t), \mu(\mu_0, \cdot, t), D(t))$ , the realizations of which are a weak solution of (4.18), that is, in the standard weak solution sense of [CZ12] for the transport PDEs and under an integral form for the ODE.

**Lemma 4.2** *For every initial condition  $(X_0, U_0) \in \mathbb{R}^n \times \mathcal{L}_\infty([-D, 0], \mathbb{R})$ , the closed-loop system consisting of the plant (4.1) satisfying Assumptions 4.1–4.2 and the control law defined in (4.8)–(4.9) has a unique weak solution.*

*Consequently, for each initial condition in  $\mathcal{D}_\Psi$ , the target system (4.18) also has a unique weak solution.*

*Proof:* We follow an induction argument. Let us start by observing that, as  $D(t) \geq \underline{D}$ ,  $t \in [0, \underline{D}] \rightarrow U(t - D(t))$  is essentially bounded as  $U_0 \in \mathcal{L}_\infty([-D, 0], \mathbb{R})$ . Hence, according to [Hal69, Theorem I.5.1, Theorem I.5.2] and as the plant is forward complete due to Assumption 4.2 and the fact that  $f(0, 0) = 0$ , there exists a unique solution for  $X$  and this solution is defined on  $[0, \underline{D}]$ . Besides, taking a time-derivative of (4.9) for  $\theta = t$ , one obtains the equivalent differential equation

$$\begin{aligned} \dot{\hat{P}}(t, t) &= f(\hat{P}(t, t), U(t)) - f(X(t), U(t - D_0)) + f(X(t), U(t - D(t))) \\ &= f(\hat{P}(t, t), \kappa(\hat{P}(t, t))) - f(X(t), U(t - D_0)) + f(X(t), U(t - D(t))) \\ &\triangleq f(\hat{P}(t, t), \kappa(\hat{P}(t, t))) + G(t). \end{aligned} \quad (4.28)$$

Note that  $X$  is bounded on  $[0, \underline{D}]$ , according to the previous considerations, and  $t \rightarrow U(t - D_0)$  and  $t \rightarrow U(t - D(t))$  are essentially bounded as  $U_0 \in \mathcal{L}_\infty([-D, 0], \mathbb{R})$ . Hence, the function  $G$  is essentially bounded on  $[0, \underline{D}]$  with the fact that the function  $f$  is continuous. With Assumption 4.1, the positive definite function  $W$  satisfies

$$\dot{W}(\hat{P}) = \frac{dW}{dX}(\hat{P})f(\hat{P}, \kappa(\hat{P})) + \frac{dW}{dX}(\hat{P})G(t) \leq -\frac{\alpha}{2}|\hat{P}|^2 + \frac{C_3}{\alpha}|G(t)|^2 \leq -\frac{\alpha}{2C_1}W(\hat{P}) + \frac{C_3}{\alpha}|G(t)|^2. \quad (4.29)$$

From the derivative of function  $W$ , it holds

$$W(\hat{P}(t)) \leq e^{-\frac{\alpha}{2C_1}t} W(\hat{P}(0)) + \frac{C_3}{\alpha} \int_0^t e^{-\frac{\alpha}{2C_1}(t-s)} |G(s)|^2 ds. \quad (4.30)$$

Therefore,  $\hat{P}$  is well-defined on  $t \in [0, \underline{D}]$ , and so is  $U$  as  $\kappa$  is a continuous mapping according to Assumption 4.1. The result finally follows by a direct induction on time intervals of length  $\underline{D}$ . ■

From Lemma 4.2,  $(\Psi, \delta)$  does define a continuous-time Markov process, and we can therefore rely on the Probabilistic Delay Averaging approach to perform the stability analysis.

We consider the following Lyapunov functional candidate

$$\begin{aligned} V(\Psi) = & W(X) + bD_0 \int_0^1 (1+x)w(x)^2 dx + c \sum_{l=1}^r \int_0^1 (1+x)(e_l \cdot \mathbf{D})^T \tilde{\mathbf{v}}(x)^2 dx \\ & + d\bar{D} \int_0^1 (1+x)\mu(x)^2 dx, \end{aligned} \quad (4.31)$$

with  $b, c, d > 0$ , and  $\mathbf{D} = (D_1 \dots D_r)^T$  and where  $\cdot$  denotes the Hadamard multiplication and the square in  $\tilde{\mathbf{v}}(x)^2$  should be understood component-wise. We define the fixed-delay infinitesimal generator  $L_j$  of the Markov process  $(\Psi, \delta)$  as

$$L_j V(\Psi) = \frac{dV}{d\Psi}(\Psi) F_j(\Psi), \quad (4.32)$$

in which  $F_j$  denotes the operator corresponding to the dynamics of the target system (4.18) with the fixed value  $\delta(t) = e_j$ , that is, for  $\Psi = (X, w, \tilde{\mathbf{v}}, \mu)$ ,

$$F_j(\Psi)(x) = \begin{pmatrix} f(X(t), \kappa(X(t)) + e_j^T \tilde{\mathbf{v}}(0) + w(0)) \\ \frac{1}{D_0} [w_x(x) + r_w(x)] \\ \Lambda_D^{-1} [\tilde{\mathbf{v}}_x(x) - \Sigma_D h(\cdot + D_0(x-1))] \\ \frac{1}{D} \mu_x(x) \end{pmatrix}. \quad (4.33)$$

With these elements, we can now formulate the following result.

#### 4.4.1 Lyapunov analysis

**Lemma 4.3** *Consider the closed-loop system consisting of (4.1) and the control law (4.8)–(4.9) satisfying Assumptions 4.1–4.3. Let  $C$  be a compact set of  $\mathcal{A}$  containing the origin. Assume there exists a positive constant  $\varepsilon$  such that*

$$|D_0 - D_j| \leq \varepsilon, \quad j \in \{1, \dots, r\}, \quad (4.34)$$

and that  $X(t) \in C$  for a certain  $t \geq \bar{D}$ , then, there exist  $b, c, d \in \mathbb{R}_+^{*3}$  which are independent of  $\varepsilon$  such that the Lyapunov functional  $V$  defined in (4.31) satisfies

$$\sum_{j=1}^r P_{ij}(0, t) L_j V(t) \leq -(\eta - g(\varepsilon))V(t), \quad (4.35)$$

with  $\eta > 0$  independent of  $\varepsilon$  and the function  $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  satisfying  $\lim_{\varepsilon \rightarrow 0} g(\varepsilon) = 0$ .

The remainder of this subsection is dedicated to the proof of this lemma. Applying integrations by parts and Young's inequality, one obtains

$$\begin{aligned} \frac{dV}{d\Psi}(\Psi) F_j(\Psi) \leq & \frac{dW}{dX} f(X(t), \kappa(X(t)) + w(0, t) + \tilde{v}_j(0, t)) \\ & - b(1 - 2\gamma_1) \|w(t)\|^2 - bw(0, t)^2 + 2b \frac{1}{\gamma_1} \|r_w(x, t)\|^2 \\ & - c \sum_{l=1}^r \left( 1 - 2 \left| 1 - \frac{D_l}{D_0} \right| \gamma_2 \right) \|\tilde{v}_l(t)\|^2 - c \sum_{l=1}^r \tilde{v}_l(0, t)^2 \\ & + 2c \sum_{l=1}^r \left| 1 - \frac{D_j}{D_0} \right| \frac{1}{\gamma_2} \|h(t + D_0(\cdot - 1))\|^2 \\ & - d\mu(0, t)^2 - d\|\mu(t)\|^2 + 2d\mu(1, t)^2. \end{aligned} \quad (4.36)$$

From the definition of  $\tilde{f}$  in (4.20) with  $\delta(t) = e_j$  and Assumptions 4.2–4.3, it holds

$$\|r_w(x, t)\|^2 = D_0^2 C_4^2 e^{2D_0 C_L} C_L^2 \tilde{v}_j(0, t)^2. \quad (4.37)$$

Then, with Assumption 4.2, we obtain the following inequality

$$\begin{aligned} \frac{dV}{d\Psi}(\Psi) F_j(\Psi) &\leq \frac{dW}{dX} \left( f(X(t), \kappa(X(t)) + C_L |w(0, t) + \tilde{v}_j(0, t)|) \right) \\ &\quad - b(1 - 2\gamma_1) \|w(t)\|^2 - bw(0, t)^2 + 2b \frac{1}{\gamma_1} D_0^2 C_L^2 C_4^2 e^{2D_0 C_L} \tilde{v}_j(0, t)^2 \\ &\quad - c \sum_{l=1}^r \left( 1 - 2 \left| 1 - \frac{D_l}{D_0} \right| \gamma_2 \right) \|\tilde{v}_l(t)\|^2 - c \sum_{l=1}^r \tilde{v}_l(0, t)^2 \\ &\quad + 2c \sum_{l=1}^r \left| 1 - \frac{D_l}{D_0} \right| \frac{1}{\gamma_2} \|h(t + D_0(\cdot - 1))\|^2 \\ &\quad - d\mu(0, t)^2 - d\|\mu(t)\|^2 + 2d\mu(1, t)^2. \end{aligned} \quad (4.38)$$

To bound the term involving the function  $h$ , we formulate the following intermediate lemma.

**Lemma 4.4** *Consider the function  $h$  defined in (4.19). There exists  $M > 0$  such that*

$$\|h(t + D_0(\cdot - 1))\|^2 \leq MV(t), \quad t \geq D_0. \quad (4.39)$$

*Proof:* From the definition of the control law (4.8)–(4.9), (4.27) and Assumptions 4.2–4.3, then

$$\begin{aligned} |h(t)|^2 &= D_0^2 \left( \frac{d\kappa}{dX}(\hat{p}(1, t)) \hat{p}_t(1, t) \right)^2 \leq C_4^2 |f(\hat{p}(1, t), \hat{v}(1, t)) + \Phi(1, 0, t) \tilde{f}(t)|^2 \\ &\leq C_4^2 \left( |f(\hat{p}(1, t), \hat{v}(1, t))| + e^{D_0 C_L} |\tilde{f}(t)| \right)^2. \end{aligned} \quad (4.40)$$

Hence, using (4.20) and Assumption 4.2, one obtains

$$\begin{aligned} \|h(t + D_0(\cdot - 1))\|^2 &= \int_0^1 h(t + D_0(x - 1))^2 dx \\ &\leq 2C_4^2 \left( \int_0^1 |f(\hat{p}(1, t + D_0(x - 1)), \hat{v}(1, t + D_0(x - 1)))|^2 dx + e^{2D_0 C_L} \int_0^1 |\tilde{f}(t + D_0(x - 1))|^2 dx \right) \\ &\leq 2C_4^2 \left( C_L^2 \int_0^1 (|\hat{P}(t + D_0(x - 1))| + |U(t + D_0(x - 1))|)^2 dx + 2C_L^2 e^{2D_0 C_L} \int_0^1 U(t + D_0(x - 2))^2 dx \right. \\ &\quad \left. + 2C_L^2 e^{2D_0 C_L} \int_0^1 U(t + D_0(x - 1) - D(t + D_0(x - 1)))^2 dx \right). \end{aligned} \quad (4.41)$$

Besides, from (4.9) and Assumption 4.2, for  $\theta \in [t - D_0, t]$ ,

$$\begin{aligned} |\hat{P}(\theta)| &\leq |X(t)| + C_L \int_{t-D_0}^{\theta} (|\hat{P}(s)| + |U(s)|) ds \\ &\leq \left( |X(t)| + C_L \int_{t-D_0}^{\theta} |U(s)| ds \right) + C_L \int_{t-D_0}^{\theta} |\hat{P}(s)| ds. \end{aligned} \quad (4.42)$$

Using Grönwall's inequality, it follows that

$$\begin{aligned} |\hat{P}(\theta)| &\leq |X(t)| + C_L \int_{t-D_0}^{\theta} |U(s)| ds + C_L \int_{t-D_0}^{\theta} (|X(t)| + C_L \int_{t-D_0}^s |U(\xi)| d\xi) e^{C_L(\theta-s)} ds \\ &= e^{C_L(\theta-t+D_0)} |X(t)| + C_L \int_{t-D_0}^{\theta} e^{C_L(\theta-s)} |U(s)| ds. \end{aligned} \quad (4.43)$$

Replacing (4.43) into (4.41), one has

$$\begin{aligned} \|h(t + D_0(\cdot - 1))\|^2 &\leq 2C_L^2 D_0^2 C_4^2 \int_0^1 \left( e^{C_L D_0 x} |X(t)| + |U(t + D_0(x - 1))| \right. \\ &+ C_L \int_{t-D_0}^{t+D_0(x-1)} e^{C_L(t+D_0(x-1)-s)} |U(s)| ds \Big)^2 dx + 4C_L^2 D_0^2 C_4^2 e^{2D_0 C_L} \left( \int_0^1 U(t + D_0(x - 2))^2 dx \right. \\ &\left. + \int_0^1 U(t + D_0(x - 1) - D(t + D_0(x - 1)))^2 dx \right). \end{aligned} \quad (4.44)$$

Now, define the inverse of the backstepping transformation of (4.16), which is given as

$$\hat{v}(x, t) = w(x, t) + \kappa(\hat{\rho}(x, t)), \quad (4.45)$$

where  $\hat{\rho}$  satisfies

$$\hat{\rho}(x, t) = X(t) + D_0 \int_0^x f(\hat{\rho}(y, t), w(y, t) + \kappa(\hat{\rho}(y, t))) dy. \quad (4.46)$$

Similarly, using Assumption 4.2 and Grönwall's inequality, one can also limit  $\hat{\rho}(x, t)$  as

$$\hat{\rho}(x, t) \leq e^{D_0 C_L (1+C_4)x} |X(t)| + D_0 C_L \int_0^x e^{D_0 C_L (1+C_4)(x-y)} |w(y, t)| dy. \quad (4.47)$$

Then, from (4.45), one obtains

$$\|\hat{v}(t)\|^2 \leq 3C_4^2 e^{2D_0 C_L (1+C_4)} |X(t)|^2 + 3(C_4^2 D_0^2 C_L^2 e^{2D_0 C_L (1+C_4)} + 1) \|w(t)\|^2. \quad (4.48)$$

Hence, observing that

$$\begin{aligned} \int_0^1 U(t + D_0(x - 1) - D(t + D_0(x - 1)))^2 dx &\leq \sum_{j=1}^r \int_0^1 U(t + D_0(x - 1) - D_j)^2 dx \\ &\leq 2r \left( \|\hat{v}(t)\|^2 + \|\mu(t)\|^2 \right), \end{aligned} \quad (4.49)$$

and using (4.48) in (4.44), it follows that

$$\|h(t + D_0(\cdot - 1))\|^2 \leq M_X |X(t)|^2 + M_w \|w(t)\|^2 + M_\mu \|\mu(t)\|^2, \quad (4.50)$$

in which  $(M_X, M_w, M_\mu)$  are positive constants defined as follows

$$\begin{cases} M_X = 6C_L^2 \bar{D}^2 C_4^2 (e^{2C_L \bar{D}} + C_4^2 e^{2C_L \bar{D}(1+C_4)} (3 + 3C_L^2 e^{2C_L \bar{D}} + 2(1 + 2r)e^{2C_L \bar{D}})), \\ M_w = 6C_L^2 \bar{D}^2 C_4^2 (C_4^2 \bar{D}^2 C_L^2 e^{2\bar{D} C_L (1+C_4)} + 1) (3 + 3C_L^2 e^{2C_L \bar{D}} + 2(1 + 2r)e^{2C_L \bar{D}}), \\ M_\mu = 4(1 + 2r) C_L^2 \bar{D}^2 C_4^2 e^{2C_L \bar{D}}. \end{cases} \quad (4.51)$$

Consequently, from the definition of the Lyapunov functional  $V$  in (4.31), one finally gets the desired result with  $M = \bar{D}^2 \max \left\{ \frac{M_X}{C_1}, \frac{M_w}{b\bar{D}}, \frac{M_\mu}{d\bar{D}} \right\}$ . ■

With this lemma, applying Assumptions 4.2–4.3 and Young's inequality, it follows from (4.38) that

$$\begin{aligned} \sum_{j=1}^r P_{ij}(0, t) \frac{dV}{d\Psi}(\Psi) F_j(\Psi) &\leq - \left( \frac{\alpha}{2} - 4dC_4^2 \right) |X(t)|^2 - b(1 - 2\gamma_1) \|w(t)\|^2 \\ &- c \sum_{l=1}^r \left( 1 - 2 \left| 1 - \frac{D_l}{D_0} \right| \gamma_2 \right) \|\tilde{v}_l(t)\|^2 - d \|\mu(t)\|^2 - \left( b - 4d - \frac{C_3^2 C_L^2}{\alpha} \right) w(0, t)^2 \\ &- \left( c - 2b \frac{1}{\gamma_1} D_0^2 e^{2D_0 C_L} C_4^2 C_L^2 - \frac{C_3^2 C_L^2}{\alpha} \right) \sum_{j=1}^r P_{ij}(0, t) \tilde{v}_j(0, t)^2 \\ &- c \sum_{j=1}^r P_{ij}(0, t) \sum_{l \neq j} \tilde{v}_l(0, t)^2 - d\mu(0, t)^2 + \frac{2c r \varepsilon}{D_0} \frac{1}{\gamma_2} MV(t), \end{aligned} \quad (4.52)$$

in which  $M$  is a positive constant which does not depend on  $\varepsilon$  and is defined in Lemma 4.4.

Observing that  $D_0 \in [\underline{D}, \overline{D}]$ , let us choose  $(b, c, d, \gamma_1, \gamma_2) \in (\mathbb{R}_+^*)^5$  as follows

- (a)  $d < \frac{\alpha}{8C_4^2}$ ,
- (b)  $b \geq 4d + \frac{C_3^2 C_L^2}{\alpha}$ ,
- (c)  $\gamma_1 \leq \frac{1}{4}$ ,
- (d)  $\gamma_2 \leq \frac{1}{4} \min \left\{ \left(1 - \frac{D_1}{D}\right)^{-1}, \left(\frac{D_r}{D} - 1\right)^{-1} \right\}$ ,
- (e)  $c \geq 2b \frac{1}{\gamma_1} \overline{D}^2 e^{2\overline{D}C_L} C_4^2 C_L^2 + \frac{C_3^2 C_L^2}{\alpha}$ .

From (4.52), one then obtains (4.35) with the well-defined positive constant  $\eta = \min \left\{ \frac{\alpha - 8dC_4^2}{2C_2}, \frac{1}{4\overline{D}} \right\}$ , and the function  $g(\varepsilon) = \frac{2c\varepsilon}{D_0} \frac{1}{\gamma_2} M$ , which satisfies  $\lim_{\varepsilon \rightarrow 0} g(\varepsilon) = 0$ .

#### 4.4.2 Conclusion of the proof of Theorem 1

Firstly, as  $\lim_{\varepsilon \rightarrow 0} g(\varepsilon) = 0$ , there exists  $\varepsilon^* > 0$  such that  $\eta - g(\varepsilon) = \gamma > 0$  for  $\varepsilon < \varepsilon^*$ . Therefore, according to Dynkin's formula [Dyn65, Theorem 5.1, p. 132], assuming temporarily that  $X(t) \in C$  for all  $t \geq \overline{D}$ , from (4.35), one obtains for  $\varepsilon < \varepsilon^*$

$$\mathbb{E}_{[0, \Phi(0)]}(V(t)) - \mathbb{E}_{[0, \Phi(0)]}(V(\overline{D})) \leq \mathbb{E}_{[0, \Phi(0)]} \left[ \int_{\overline{D}}^t -\gamma V(s) ds \right]. \quad (4.53)$$

Consequently, applying Grönwall's inequality to (4.53) (see [KM01a])

$$\mathbb{E}_{[0, \Phi(0)]}(V(t)) \leq \mathbb{E}_{[0, \Phi(0)]}(V(\overline{D})) e^{-\gamma(t-\overline{D})}. \quad (4.54)$$

Similarly to Lemma 1.5, by applying Young's and Grönwall's inequalities to the backstepping transformation (4.16) and its inverse (4.45), there exist positive constants  $q_1$  and  $q_2$  such that for  $\forall t \geq 0$ ,  $q_1 V(t) \leq Y(t) \leq q_2 V(t)$ . It thus follows that  $\mathbb{E}_{[0, Y(0)]}(Y(t)) \leq \frac{q_2}{q_1} Y(\overline{D}) e^{-\gamma(t-\overline{D})}$ . To relate  $Y(\overline{D})$  to  $Y(0)$ , we formulate the following lemma.

**Lemma 4.5** *Consider the dynamics (4.1) satisfying Assumptions 4.1–4.3. There exists a constant  $R_0$  such that the function  $Y$  defined in (4.13) satisfies*

$$Y(t) \leq R_0 Y(0), \quad t \in [0, \overline{D}]. \quad (4.55)$$

*Proof:* Firstly, with Assumption 4.2, notice that

$$\begin{aligned} |X(t)| &\leq e^{C_L t} |X(0)| + C_L \int_0^t e^{C_L(t-s)} |U(s - D(s))| ds \\ &\leq e^{C_L t} |X(0)| + C_L \sum_{j=1}^r \int_0^t e^{C_L(t-s)} |U(s - D_j)| ds. \end{aligned} \quad (4.56)$$

For  $t \in [0, \overline{D}]$ , with  $U(t) = U_0(t)$  for  $t \leq 0$ , (4.56) then gives

$$|X(t)|^2 \leq N_1 |X(0)|^2 + N_2 \int_{-\overline{D}}^{\min\{t-\underline{D}, 0\}} U_0(s)^2 ds + N_2 \int_{\min\{t-\underline{D}, 0\}}^{t-\underline{D}} U(s)^2 ds, \quad (4.57)$$

with  $N_1 = 2e^{2C_L\bar{D}}$  and  $N_2 = rC_L^2N_1$ . Besides, using Assumption 4.3 and (4.43), the prediction-based control law satisfies the inequality

$$|U(t)| = |\kappa(\hat{P}(t))| \leq C_4 \left( e^{C_LD_0}|X(t)| + C_L \int_{t-D_0}^t e^{C_L(t-s)}|U(s)|ds \right), \quad (4.58)$$

from which one obtains

$$U(t)^2 \leq N_3|X(t)|^2 + N_4 \int_{t-D_0}^t U(s)^2 ds, \quad (4.59)$$

with  $N_3 = 2C_4^2e^{2C_LD_0}$  and  $N_4 = C_L^2N_3$ . Replacing (4.59) into (4.57) and using again (4.57), one obtains

$$\begin{aligned} |X(t)|^2 &\leq N_1|X(0)|^2 + N_2 \int_{-\bar{D}}^{\min\{t-\underline{D},0\}} U_0(s)^2 ds + N_2N_3 \int_{\min\{t-\underline{D},0\}}^{t-\underline{D}} X(s)^2 ds \\ &\quad + N_2N_4D_0 \int_{\min\{t-\underline{D},0\}-D_0}^{t-\underline{D}} U(s)^2 ds \leq N_5|X(0)|^2 + N_6 \int_{-\bar{D}}^{t-\underline{D}} U(s)^2 ds, \end{aligned} \quad (4.60)$$

with positive constants  $N_5 = N_1(1 + N_2N_3(\bar{D} - \underline{D}))$  and  $N_6 = N_2(1 + N_2N_3(\bar{D} - \underline{D}) + N_4D_0)$ . Hence, with (4.59), it follows that

$$U(t)^2 \leq N_3N_5|X(0)|^2 + (N_3N_6 + N_4) \int_{-\bar{D}}^t U(s)^2 ds, \quad (4.61)$$

and, applying Grönwall's lemma, one obtains

$$U(t)^2 \leq N_7|X(0)|^2 + N_8 \int_{-\bar{D}}^t U_0(s)^2 ds, \quad (4.62)$$

with  $N_7 = N_3N_5e^{(N_3N_6+N_4)\bar{D}}$  and  $N_8 = (N_3N_6 + N_4)e^{(N_3N_6+N_4)\bar{D}}$ . Consequently, it also holds

$$X(t)^2 \leq N_9|X(0)|^2 + N_{10} \int_{-\bar{D}}^t U_0(s)^2 ds, \quad (4.63)$$

with  $N_9 = N_5 + N_6(\bar{D} - \underline{D})N_7$  and  $N_{10} = N_6(1 + (\bar{D} - \underline{D})N_8)$ . Therefore, as from (4.13), it holds

$$\Upsilon(t) = |X(t)|^2 + \int_{t-\bar{D}-D_0}^0 U_0(s)^2 ds + \int_0^t U(s)^2 ds, \quad (4.64)$$

the conclusion follows from (4.62) and (4.60).  $\blacksquare$

With Lemma 4.5, we can now conclude the proof of Theorem 4.1. Indeed, the function  $\Upsilon$  thus satisfies  $\mathbb{E}_{[0,\Upsilon(0)]}(\Upsilon(t)) \leq \frac{q_2}{q_1}R_0\Upsilon(0)e^{-\gamma(t-\bar{D})}$ , if  $X(t) \in C$  for all  $t \geq \bar{D}$ . Finally, let us denote  $\sqrt{\rho_0^*}$  such that  $B(0, \sqrt{\rho_0^*}) \subset C$  and observe that  $|X(t)|^2 \leq \Upsilon(t)$  for all  $t \geq 0$ . Consequently, a sufficient condition for  $X(t)$  belonging to  $C$  is  $\Upsilon(t) \leq \rho_0^*$  and the result follows with  $\rho^* = \frac{q_1}{q_2R_0}\rho_0^*$  and  $R = \frac{q_2}{q_1}R_0e^{\gamma\bar{D}}$ .

## 4.5 Simulation results

To illustrate Theorem 4.1 and in particular the role played by the condition (4.10), we consider the following toy example with the system state  $X = [X_1 \quad X_2]^T$

$$\begin{cases} \dot{X}_1(t) = -X_1(t) + 2\sin(X_2(t))(1 + X_1(t)) \\ \dot{X}_2(t) = X_1(t)X_2(t) + \frac{2X_2(t)^2}{1 + X_2(t)^2} + U(t - D(t)). \end{cases} \quad (4.65)$$



The control law  $\kappa$  satisfying Assumption 4.1 is chosen as

$$\kappa(X) = -X_1(t)X_2(t) - X_2(t). \quad (4.66)$$

Indeed, for  $D \equiv 0$ , this results into the closed-loop system

$$\begin{cases} \dot{X}_1(t) = -X_1(t) + 2\sin(X_2(t))(1 + X_1(t)) \\ \dot{X}_2(t) = -X_2(t) + \frac{2X_2(t)^2}{1 + X_2(t)^2}, \end{cases} \quad (4.67)$$

which can be proven to be exponentially stable by computing its Jacobian at the origin, which has eigenvalue  $\lambda = -1$  of multiplicity two. Furthermore, one can prove that its basin of attraction is  $\mathcal{A} = \mathbb{R} \times ]-\infty, 1[$  by considering the Lyapunov function  $W_0 = X_1^2 + aX_2^2$  for a convenient positive constant  $a$ .

The initial conditions are chosen as  $X(0) = [0.08 \ 0.08]^T$ ,  $P(0) = [0 \ 0]^T$  and  $U(t) = 0$ , for  $t \leq 0$ . The integral in (4.8) is discretized using a its zero-order hold approximation, in line with a suggestion in [MO79]. Finally, the simulations are carried out with a discrete-time solver in Matlab-Simulink, with a sampling time  $\Delta t = 0.01$  s.

We consider the case of 3 different delay values  $(D_1, D_2, D_3) = (2.5, 2.75, 4.5)$  with the initial transition probabilities  $(P_1(0, 0^+), P_2(0, 0^+), P_3(0, 0^+)) = (0.29, 0.69, 0.02)$ . We also consider the Kolmogorov's equation (58) to describe the variation of the transition probabilities with the transition rates  $\tau_{ij}$  chosen as

$$\tau(t) = (\{\tau_{ij}(t)\})_{1 \leq i, j \leq 3} = \tau^* \begin{pmatrix} 0 & |\cos(kt)| & 0 \\ |\sin(kt)| & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (4.68)$$

with constants  $\tau^* = 0.2$  and  $k = 0.1$ . The delay value will therefore oscillate between  $D_2$  and  $D_3$  (see Figure 4.1).

We first pick the prediction horizon as  $D_0 = 2.75$ , resulting in an asymptotically stable closed-loop behavior depicted in Figure 4.1. The main reason for this is that the prediction horizon  $D_0$  remain sufficiently close to the two delays which are most often realized, namely,  $D_1 = 2.5$  and  $D_2 = 2.75$ , while  $D_3 = 4.5$  has a weak probability of occurrence. Yet, as in the linear case, the probability distribution of the delay plays an important role, as can be observed by considering a different transition rate as

$$\tau(t) = (\{\tau_{ij}(t)\})_{1 \leq i, j \leq 3} = \tau^* \begin{pmatrix} 0 & e^{-kt} & e^{-kt} \\ e^{-kt} & 0 & e^{-kt} \\ e^{-kt} & e^{-kt} & 0 \end{pmatrix}, \quad (4.69)$$

with constants  $\tau^* = 0.2$  and  $k = 0.1$ , resulting into a diverging behavior pictured in Figure 4.2.

Hence, extending condition (4.10) to a probabilistic form such as in Theorem 1.2 is an important path to follow in future works.

Besides, it is worth underlying that the toy example (4.65) does not satisfy neither Assumption 4.2, nor Assumption 4.3. Indeed, on the one hand, for  $(X_a, U)$  and  $(X_b, U)$  with  $X_a = [X_{a1} \ X_{a2}]^T$ ,  $X_b = [X_{b1} \ X_{b2}]^T$  and  $X_{a1} = X_{b1}$ , one gets

$$\begin{aligned} f_1(X_a, U) - f_1(X_b, U) &= -X_{a1} + 2\sin(X_{a2})(1 + X_{a1}) + X_{b1} - 2\sin(X_{b2})(1 + X_{b1}) \\ &= 2(\sin(X_{a2}) - \sin(X_{b2}))(1 + X_{a1}), \end{aligned} \quad (4.70)$$

$$f_2(X_a, U) - f_2(X_b, U) = X_{a1}(X_{a2} - X_{b2}) + \frac{2X_{a2}^2}{1 + X_{a2}^2} - \frac{2X_{b2}^2}{1 + X_{b2}^2}. \quad (4.71)$$

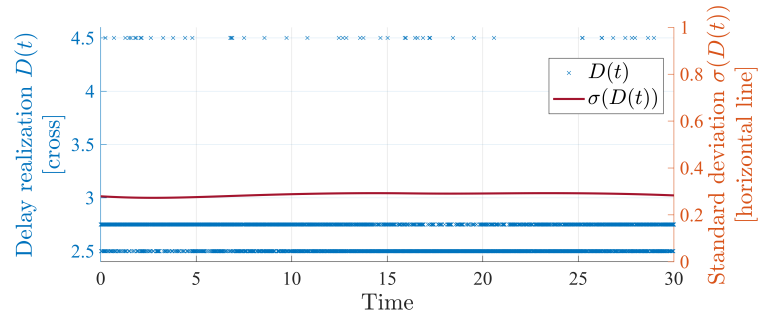
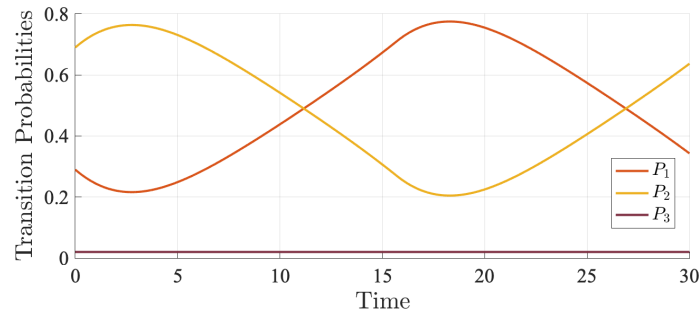
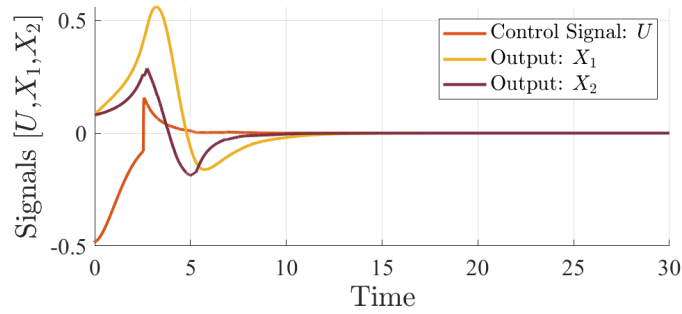
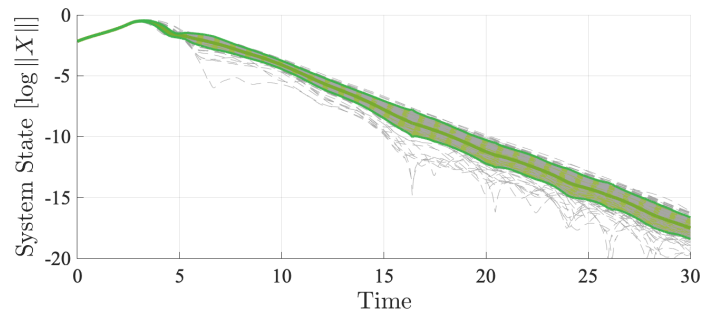

 (a) Example of a realization of the stochastic delay  $D$ .

 (b) Dynamic of the transition probabilities  $P_1$ ,  $P_2$  and  $P_3$ .

 (c) Example of a realization of the signals  $U$ ,  $X_1$  and  $X_2$  corresponding to the delay pictured in (a).

 (d) Monte Carlo simulation of  $\log \|x\|$  (100 trials).

Figure 4.1: Simulation results of the closed-loop system (4.65) and (4.66) for  $\mathbf{D} = (2.5, 2.75, 4.5)^T$ ,  $X(0) = [0.08 \ 0.08]^T$  and  $U(t) = 0$  for  $t \leq 0$ . The prediction horizon is  $D_0 = 2.75$ . The transition probabilities follow the forward Kolmogorov equation with the transition rate defined in (4.68). (a), (b) and (c) picture results corresponding to one delay realization. (d) and (e) present the results of 100 trials, in which the means and the standard deviations are highlighted by the coloured lines.

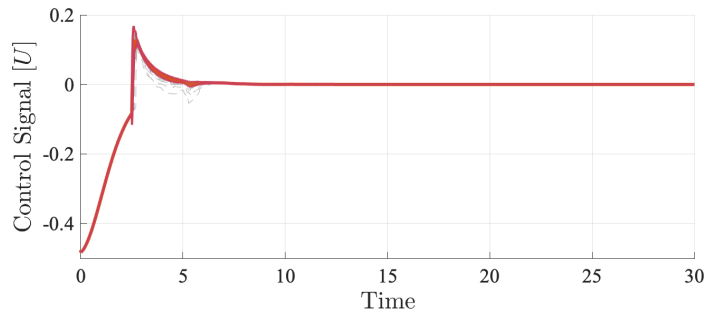

 (e) Monte Carlo simulation of the closed-loop input  $U$  (100 trials).

Figure 4.1: Continued.

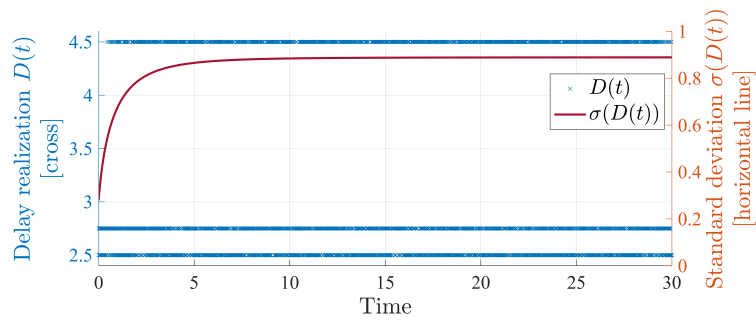
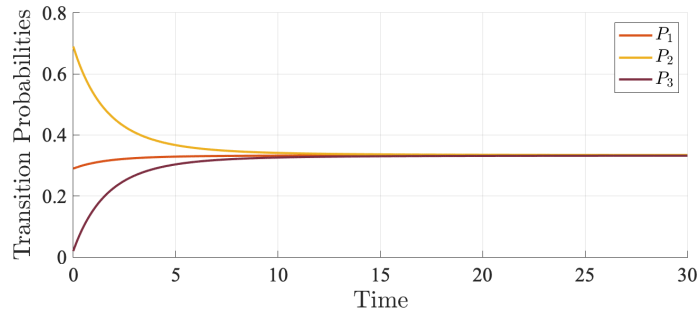
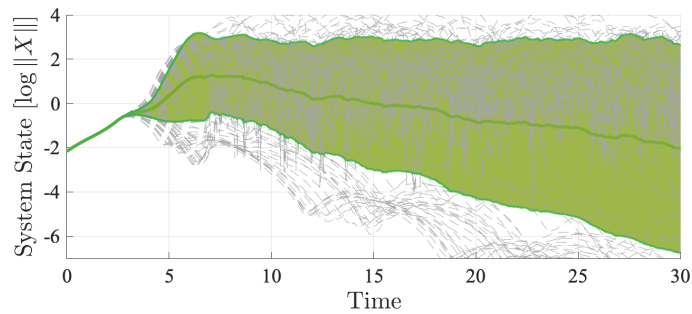

 (a) Example of a realization of the stochastic delay  $D$ .

 (b) Dynamic of the transition probabilities  $P_1$ ,  $P_2$  and  $P_3$ .

 (c) Monte Carlo simulation of  $\log \|x\|$  (100 trials).

Figure 4.2: Simulation results of the closed-loop system (4.65) and (4.66) for  $\mathbf{D} = (2.5, 2.75, 4.5)^T$ ,  $X(0) = [0.08 \ 0.08]^T$  and  $U(t) = 0$  for  $t \leq 0$ . The prediction horizon is  $D_0 = 2.75$ . The transition probabilities follow the forward Kolmogorov equation with the transition rate defined in (4.69). (a) and (b) picture results corresponding to one delay realization. (c) presents the results of 100 trials, in which the means and the standard deviations are highlighted by the coloured lines.

Hence,  $|f(X_a, U) - f(X_b, U)|$  cannot be bounded under the form  $C_L \|(X_a, 0) - (X_b, 0)\| = C_L |X_{a2} - X_{b2}|$  with a uniform constant  $C_L$ , due to the terms in  $X_{a1}$  appearing in both (4.70)–(4.71). Besides, on the other hand,

$$\frac{\partial \kappa}{\partial X}(X) = \begin{bmatrix} -X_2 \\ -X_1 - 1 \end{bmatrix}, \quad (4.72)$$

which is not uniformly bounded for compacts of  $\mathcal{A}$ . Hence, this confirms that Assumptions 4.2 and Assumption 4.3 are unnecessary, and considerably limit the class of nonlinear systems under consideration. This is why future works should start first by modifying the present analysis to get rid of these two conditions, before trying to distinguish between the probability distributions of the delay.

## 4.6 Conclusion and perspective

This chapter makes a preliminary exploration of robust compensation of a stochastic input delay for nonlinear systems. It proposes to use a constant horizon prediction, to feed a control law exponentially stabilizing the delay-free plant. We proved that the resulting closed-loop system is exponentially stable, provided that the prediction horizon is close enough to the delay values and that the initial condition of the system is limited, to avoid the system state to leave the basin of attraction of the nominal feedback law before the closed-loop control kicks in. This result was obtained following the general methodology already applied in Chapters 1 and 2.

This preliminary study raises challenging and exciting new problems to handle. The first important one would be to modify the present analysis to avoid relying on the uniform bound of Assumptions 4.2 and 4.3. In the deterministic case, this could be resolved by obtaining semi-global results, relying on the fact that both the vector field and the nominal feedback laws are locally Lipschitz functions, which is a reasonable assumption as it is required to guarantee well-posedness of the considered dynamics. Yet, in the stochastic case, this technique reveals difficult to apply. Indeed, the Probabilistic Delay Averaging technique aims at obtaining stability results with respect to the mean value, which is difficult to handle with both nonlinearities and semi-global Lyapunov techniques. The previous point might be resolved by employing a Lyapunov functional which does not depend explicitly on the delay, as the one proposed in this chapter. Yet, this would prohibit to obtain stability conditions bearing on the probability distribution of the delay.

Other questions include the possibility to apply the designs proposed in the previous chapters for linear systems in the nonlinear context, namely, to use a time-varying prediction horizon or, more importantly, an average prediction approach. Indeed, this last technique has shown numerically interesting performances for linear systems, which may transpose to the nonlinear case.



# Conclusion and Perspectives

## Contributions

In this dissertation, we considered the problem of closed-loop stabilization of dynamical systems affected by a random input delay, by the means of robust compensation with prediction-based controllers. We proposed a generic procedure both for control design and stability analysis of the resulting closed-loop system. In detail, modeling the delay as a Markov process with a finite number of values, our methodology relies on the following three steps:

- Transport representation of the delay, in which a transport PDE is introduced for each potential delay value;
- Backstepping transformation of the resulting cascade of transport PDEs into an ODE;
- Lyapunov stability analysis of the corresponding target system, based on the Probabilistic Delay Averaging approach.

We applied this generic methodology successfully in numerous contexts, throughout this manuscript.

In **Chapter 1**, we considered the stabilization of an LTI system affected by a random input delay with a constant horizon predictor. We first proved that, if this horizon is chosen close enough to all delay values, closed-loop stabilization is guaranteed. Then, we relaxed this condition and proved that it is sufficient that this condition holds *in average*.

**Chapter 2** generalized this design by considering a time-varying prediction horizon. It proved that stabilization could be achieved provided that the rate of variation of the prediction horizon remains sufficiently small and provided that the delay realization remains sufficiently close to the prediction horizon, in average. We also proposed to use the expected value of the delay as prediction horizon, which then translates into the conditions that both the variation rate of the delay expected value and of the standard deviation remain sufficiently small.

Alternatively, **Chapter 3** investigates a new control design paradigm, based on multiple predictions, each one computed on a time horizon corresponding to a potential delay value, which are then averaged. We prove that closed-loop exponential stabilization is achieved under a condition quantitatively similar to the one obtained for the single-prediction design. Yet, numerical simulations

Table 1: Controller designs proposed in this manuscript.

Chapter	Dynamics	Controller $U$	Prediction type	Eq.
Ch. 1	LTI	$K\hat{X}(t + D_0)$	Constant horizon $D_0$	(1.2)
Ch. 2	LTI	$K\hat{X}(t + D_0(t))$	Time-varying horizon $D_0(t)$	(2.3)
Ch. 3	LTI	$K \sum_{j=1}^r P_{ij}(0, t) \hat{X}(t + D_j)$	Averaged prediction	(3.4)
Ch. 4	Nonlinear	$\kappa(\hat{X}(t + D_0))$	Constant horizon $D_0$	(4.8)

(The notation  $\hat{X}$  is used to underline the fact that the state prediction is not exact.)

carried out for both controllers illustrate that the averaged-prediction controller leads to a more robust delay compensation. We explained this difference by obtaining the averaged dynamics, and the corresponding averaged closed-loop differential equations in both cases.

Finally, **Chapter 4** applies the constant horizon prediction-based controller proposed in Chapter 1 to a nonlinear dynamics, grounding on a control law exponentially stabilizing the delay-free plant. We proved that the resulting closed-loop system is exponentially stable, provided that the prediction horizon is close enough to the delay values and that the initial condition of the system is limited, to avoid the system state to leave the basin of attraction of the nominal feedback law before the closed-loop control kicks in.

## Perspectives

To conclude this thesis, let us sketch several possible future directions of research.

As mentioned in the manuscript, a few supplements to the works presented in this thesis should be addressed in the future.

First, a point of major importance to study in the future consists in quantifying the obtained robustness margins denoted  $\varepsilon^*$  and, in particular, their dependence on the feedback gain. Indeed, being able to tune the feedback gain to increase the robustness margin is an important practical question. Yet, the Lyapunov study provided above is very conservative and unlikely to be used to obtain meaningful robustness bounds. The closed-loop dynamics of the averaged system obtained in Chapter 3 might reveal useful to reach this goal.

A second interesting path to explore is to improve the time-varying horizon controller by incorporating an (estimated) prediction of the future delay value, in line with the standard exact compensation technique for a time-varying deterministic delay function, as presented in Section 2.1.1 in the Introduction. As this strategy aims at anticipating the future value of the delay (expected) value, it should allow to better take into account the future variations of the delay distribution and, in detail, to get rid of the condition (2.5) of Theorem 2.1 which limits its range of variation. Notice however that determining this delay prediction in a causal way is very likely to result into an inaccurate prediction horizon. Thus, such a design should also involve a type of robustness property to prediction errors.

Besides, the problem of robust compensation of random delay for nonlinear systems addressed in Chapter 4 only constitutes a preliminary exploration, which raises challenging questions to address. The first important one would be to modify the present analysis to avoid assuming that the plant under consideration is globally Lipschitz, which is a very restrictive condition. Another generalization of great significance would be to obtain a stabilization condition taking into account the probability distribution of the delay. Other questions include the possibility to apply the designs proposed in Chapters 1–3 in the nonlinear context, namely, to use a time-varying prediction horizon or, more importantly, an average prediction approach.

More generally, the robust compensation methodology for random delay presented in this thesis opens new paths to explore. The first one concerns extensions to more general contexts such as multi-input systems with distinct (random) delays [TKO16], systems involving (random) state delays, distributed delays, or even Stochastic Delay Differential Equations as recently investigated in [CGMP20].

Second, it would be essential to generalize the proposed methodology to handle an infinite number of delays. This may require a complete change of framework, but would widely broaden the range of applications.

Another very interesting problem to tackle is the adaptation of the prediction horizon, with as little information on the delay distribution as possible (namely, without the knowledge of the expected value of the delay, or without the knowledge of the delay values). This may be achieved with a

delay-adaptive controller, but such a design remains to be entirely explored.

Finally, we believe the analysis methodology proposed in this thesis could pave the way towards the analysis of more complex cascades of PDEs and ODEs, of various types, randomly connected.





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## RÉSUMÉ

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Cette thèse aborde le contrôle en boucle fermée d'un système dynamique en temps continu, affecté par un retard d'entrée aléatoire. Les retards stochastiques sont en effet fréquemment employés pour représenter les pertes de paquets ou les réordonnements qui peuvent se produire sur des systèmes en réseaux, par exemple. Néanmoins, peu d'outils de stabilisation existent dans la littérature pour des dynamiques continues avec retard stochastiques en temps continu. Cette thèse répond donc à ce besoin et propose une méthodologie générique de stabilisation à l'aide de contrôleurs par prédiction, outil dont les mérites ne sont plus à prouver dans le cas d'un retard déterministe d'entrée. En modélisant le retard d'entrée comme un processus de Markov à nombre d'états fini, nous étudions différents contrôleurs par prédiction et, d'une manière générale, prouvons que la stabilité boucle fermée en moyenne quadratique peut être obtenue, sous réserve que les valeurs des retards satisfasse une certaine condition de robustesse. Notre méthodologie d'analyse repose sur la reformulation du retard d'entrée par une Équation aux Dérivées Partielles de transport, puis sur l'application d'une transformation de backstepping correspondant au contrôleur par prédiction et enfin sur l'application de la technique de la Moyennisation Probabiliste de Retard au système cible correspondant. Ceci permet de présenter une procédure générique pour, à la fois, le développement de contrôleur et l'analyse de stabilité du système en boucle fermée correspondant, et ce, dans grande variété de contextes.

## MOTS CLÉS

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Systèmes à retard, systèmes à paramètres distribués, retard stochastique, backstepping, stabilité de Lyapunov.

## ABSTRACT

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This thesis investigates the closed-loop control of continuous-time dynamical systems affected by a random input delay. Stochastic delays are indeed commonly used to represent packet loss and reordering which often take place in network systems, for instance. Yet, stabilization tools for continuous-time processes subject to continuous-time stochastic delays are lacking in general in the literature. This thesis thus proposes a generic stabilization methodology relying on prediction-based controllers, which have strong merits in the deterministic delay case. Modeling the input delay as a Markov process with a finite number of states, we study different prediction-based controllers and, generally speaking, prove that mean-square closed-loop stability can be obtained, provided that the delay values satisfy a robustness condition. The analysis methodology we propose consists in rewriting the input delay as a transport Partial Differential Equation, applying a backstepping transformation corresponding to the prediction-based controller and finally applying the Probabilistic Delay Averaging approach to the resulting target system. This enables to present a generic procedure both for control design and stability analysis of the resulting closed-loop system, in a large variety of contexts.

## KEYWORDS

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Time-delay systems, stochastic delay, distributed parameter systems, backstepping, Lyapunov stability.