



Topological insulators and artificial crystals for Hydro-Elastic Waves

Federigo Ceraudo

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DE L'UNIVERSITÉ PSL

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Laboratoire de Physique et Mécanique des Milieux Hétérogènes

Topological insulators and artificial crystals for Hydro-Elastic Waves

Isolants topologiques et cristaux artificiels pour les ondes hydro-élastiques

Soutenue par

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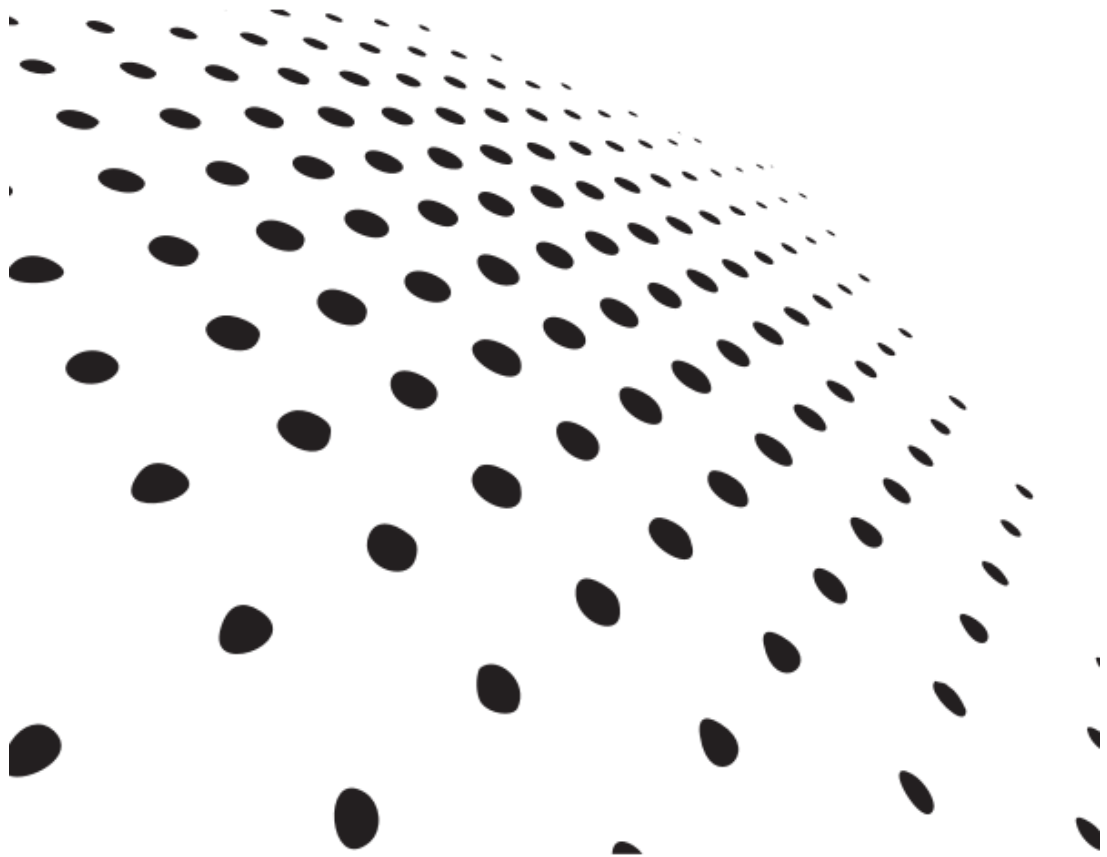
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Topological insulators and artificial crystals for Hydro-Elastic Waves

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Résumé

Titre: Isolants topologiques et cristaux artificiels pour les ondes Hydro-élastiques

Cette thèse porte sur le comportement des Ondes Hydro-Élastiques (HEWs), des ondes à la surface de l'eau qui interagissent avec une membrane élastique, en utilisant les idées et méthodologies qui dérivent de la topologie et de la physique de la matière condensée comme le *Quantum Hall Effect* (QHE). Les phases topologiques sont étudiées dans plusieurs domaines physiques et dans la majorité des cas, afin d'observer les propriétés liées aux isolants topologiques, il est nécessaire de construire un système où la symétrie par renversement du temps (*T-symmetry*) est brisée. En se concentrant sur les systèmes avec des ondes à la surface de l'eau, ce n'est pas trivial de briser la *T-symmetry*, dans ce cas ils sont nécessaires des flux d'eau rotationnelles qui comportent une forte injection d'énergie dans le système. Une autre approche pour faire face à ce défi expérimental est de mimer le *Quantum Spin Hall Effect* (QSHE), dans ce cas la *T-symmetry* est préservée. Nous proposons d'étudier les phases topologiques en utilisant une nouvelle approche basée sur les HEWs ou l'interaction entre une membrane de silicone et des ondes à la surface d'un liquide peut conduire à des guides onde robustes et immunes au *back-scattering*. Cette étude est basée sur des expériences en laboratoire qui permettent de reconstruire le champ de hauteur précisément. D'abord nous caractérisons la membrane de silicone en mesurant son module de Young, la relation de dispersion et l'atténuation, en montrant que nous sommes capables de modifier sa propagation en changeant les propriétés géométriques de la membrane. Nous étudions le comportement d'une perforation triangulaire sur la membrane, qui peut être vue comme une cavité 2D, en montrant ses modes de résonance. Nous utilisons un pattern triangulaire de perforations triangulaires, ce que l'on appelle un cristal artificiel, pour mimer la structure de bande d'un vrai cristal: les gaps de type Bragg et d'hybridation sont présentés expérimentalement. Nous démontrons qu'une inversion de bande se manifeste quand deux patterns avec des orientations différentes sont mis ensemble en créant une interface qui support un mode de bord topologique.

Mots clés: mécanique des fluides, ondes, hydroélasticité, cristaux artificiels, topologie, isolants topologiques

Abstract

Title: Topological insulators and artificial crystals for Hydro-Elastic Waves

This thesis concerns the study of Hydro-Elastic Waves (HEWs): waves at the surface of a liquid interacting with an elastic floating membrane, merging concepts and methodologies from topology typically and recently applied to condensed-matter physics. At this point, the study of topological phases involves many classical physical systems where artificial crystals (*i.e.* sonic or photonic crystals) are employed to observe new transport properties mainly characterized by robustness against defects and suppression of the back-scattering (spin-locked transport). We propose to mimic the Quantum Spin Hall effect and Quantum Valley Hall effect describing the topological phases using a new approach based on HEWs where the interaction between a silicone membrane spatially structured and water waves can lead to unidirectional robust waveguides immune to back-scattering. The study is based on a laboratory experiment which makes it possible to measure the wave field precisely via the synthetic Schlieren method. We first characterize the silicone membrane measuring its Young modulus, the dispersion relation and the attenuation of a HEW, showing that we are able to modify its propagation by changing the geometrical properties of the membrane. Then, before analysing the collective response of a spatially arranged structure, we describe the behaviour of a single triangular perforation on the membrane demonstrating that it acts like a 2D cavity; its resonant modes are obtained via modal decomposition and play a crucial role. We manufacture multiple triangular patterns of triangular perforations, the so-called artificial crystal for HEWs, to mimic the band structure of a solid-state one: Bragg and hybridization gaps are experimentally shown. We demonstrate that a band inversion takes place when two different patterns with a particular rotational direction are put together (also called domain wall). The latter, in other words, is a straight interface that supports a topological edge mode that leads to a new robust pseudospin-dependent transport. At last, we experimentally prove the existence of a topological edge mode using an interface different from a straight line, a V-shape, where the attenuation losses due to the corner along the path are largely suppressed.

Keywords: fluid mechanics, waves, hydroelasticity, artificial crystals, topology, topological insulators

Table of contents

Acknowledgements	iii
Résumé	v
Abstract.....	vii
Table of contents.....	ix
Introduction	13
Chapter 1: Hydro-Elastic Waves and experimental setup.....	23
1.1 Introduction.....	23
1.2 Waves in floating elastic ice sheets: mathematical formulation	23
1.3 Dispersion relation of a HEW: properties and length scales.....	26
1.4 Experimental setup	28
1.4.1 Fast checkerboard demodulation	31
1.4.2 Wave source generator.....	32
1.4.3 CCD sampling and folding	35
1.5 Main experimental results concerning HEWs obtained by Lucie Domino.....	36
1.6 Conclusion	40
Chapter 2: From structured media to topology.....	43
2.1 Introduction.....	43
2.2 Bragg scattering.....	44
2.3 Bloch-Floquet's formalism and reciprocal lattice	46
2.4 The 1D "bi-atomic" periodic structure: local resonances	49
2.5 2D periodic structure: properties of the triangular lattice	52
2.6 Topology applied to Condensed Matter physics.....	54
2.6.1 Topological phase transition	54

Table of contents

2.6.2	Topological invariant: Chern number	55
2.6.3	Chern number and symmetries	57
2.6.4	Topological insulator: classical equivalent.....	58
2.7	Conclusion	61
Chapter 3: Hydro-Elastic Waves: characterization.....		63
3.1	Introduction.....	63
3.2	Experimental dispersion relation of a HEW	63
3.3	Attenuation of a HEW.....	69
3.4	Singular triangular cavity.....	74
3.5	Conclusion	86
Chapter 4: Artificial crystals for Hydro-Elastic Waves: Valley transport		89
4.1	Introduction.....	89
4.2	Band structure of a periodic media: analysis	90
4.3	Artificial monotype crystal for Hydro-Elastic Waves: the triangular lattice	93
4.4	Experimental results for a monotype crystal for HEW	96
4.5	Valley transport: main concepts	101
4.6	Experimental results for the Valley Hall effect for HEW	102
4.7	Conclusion	109
Chapter 5: Topological insulator: hexagonal crystal		111
5.1	Introduction.....	111
5.2	Hexagonal crystal for HEW	112
5.3	Experimental results with a hexagonal crystal.....	115
5.4	Observation of a topological edge mode via Fourier analysis	120
5.5	Conclusion	130
Chapter 6: Conclusions and future perspectives		133
6.1	Conclusions.....	133

Table of contents

6.2	Future perspectives	135
6.2.1	Pattern to obtain a local excitation of the hexagonal structure.....	135
6.2.2	Hexagonal crystal made by circular resonators for HEW	137
6.2.3	Kekulé lattice for HEW	138
6.2.4	Zig-zag paths for HEW	140
6.2.5	On a larger scale: IFREMER, Brest	142
References		147

Introduction

Hydroelasticity combines the deformations of elastic bodies to hydrodynamic excitations. One of the most studied examples of hydroelasticity is the propagation of Hydro-Elastic Waves (HEW) into and within sea ice fields [1], [2]. An elastic plate above a fluid [3] can be used as a model to describe the motion of an ice plate, as done in the work of D. Kheysin who investigated it in detail [4], [5], [6], [7]. This model allows to reproduce such a physical context at a laboratory scale using, for example, a tank filled with water covered by an elastic floating structure as previously done by L. Deike [8].

Important results in this field come from the PhD thesis of L. Domino who characterized HEW, demonstrated a way to control their propagation by tuning the thickness of the elastic sheet and made multiple artificial crystals showing their band structure [9]. Extending the work of L. Domino, in the present experimental thesis we explore how a structured media based on HEW can be a new platform to explore topological phenomena making a bridge between two fields: Fluid Mechanics and Topology.

“Topology is a branch of mathematics that studies properties of spaces that are invariant under any continuous deformation” [10] (traditionally explained with a funny image involving a doughnut and a mug Fig. I-1). It has been recently applied to condensed matter Physics and led to interesting transport properties explained with a different perspective: the topological order. Referring to the doughnut’s transformation into a mug there is no difference, in term of topological order, between them; in fact, both objects have same genus which is defined as the number of holes (usually indicated as $g = 1$). Considering an object with the same shape of the number 8, this time its genus is $g = 2$ like it has two holes. Topology offers us another way to order matter, we can for example classify objects using the genus. So far, it could look just like a geometrical property, far from a physical problem but having a deeper look it actually applies very well to Condensed Matter Physics. A very famous example, explained with Topology is: the Quantum Hall effect (QHE).

Introduction

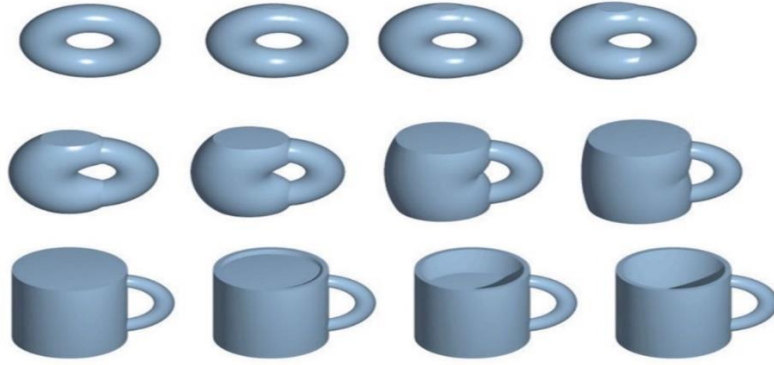


Figure I-1: Topological equivalence of a doughnut and a mug. Deforming one into the other one, we can observe that both have just one hole (the deformation does not include breaking or joined parts). Both have the same topological invariant known as its genus (equal to 1 in this case) [11].

The latter is the first example of the new topological phases of matter. In fact, the state responsible for this effect does not break any symmetries so, two different phases possess the same symmetry [12]. In particular, at a fixed magnetic field, the electron population distribution in the quantized orbits results in a quantization of the electrical resistance. This quantization is originated and explained with Topology. Von Klitzing discovered it in 1980 [13], received a Nobel prize in 1985 and constructed a new way to classify matter based on a new paradigm involving the concept of topological order. Starting from the QHE it is possible to realise the so-called Quantum Hall insulator which has the same behaviour of an insulator excepting for its surface which exhibits conducting states (when a periodic magnetic flux is applied or in other words breaking the time reversal symmetry or T-symmetry).

The study of topological phases, does not concern only Quantum systems, in fact it has been successfully applied to a variety of classical physical systems, such as photonic [14], [15], sonic crystals [16], [17], [18], [19] and water waves systems [20], [21]. Typically, all these are 2D examples so, the equivalent conducting state lays on a line. A relevant example in acoustics involving circulating fluids is the work Xu Ni et al [16]. They realised a 2D network of acoustic resonators, Fig. I-2, where the T-symmetry is broken by a circulating air flow which plays the role of a magnetic field. Like it is a 2D system, the goal is to observe a topological mode along an axis of an insulator material. Each site of this artificial sonic crystal can be seen as an atom or meta-atom linked by three channels.

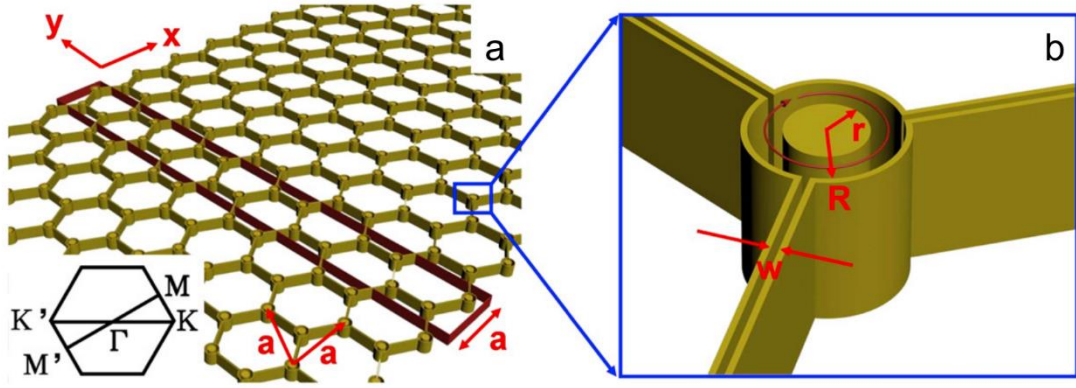


Figure I-2: a) Schematic of a 2D sonic crystal made by meta-atoms linked to each other by three channels. b) Meta-atom view showing the circulating airflow which plays the role of a “magnetic field”. Adapted from [16].

Its geometrical construction with the circulating airflow allows the propagation of a topological mode over a specific path, the interface between the sonic crystal and a rigid boundary.

Each field presents different challenges to realize the QHE that requires breaking of the T-symmetry. Focusing on water wave systems the role of the magnetic field can be replaced with rotational water flows but it experimentally implies many difficulties linked to non-linear effects and to the control of the large amount of energy demanded by the rotational water flows.

Another approach to face this experimental challenge is to mimic the Quantum Spin Hall Effect (QSHE) instead, realizing a topological insulator (TI) [22], [23] (in this case the T-symmetry is preserved). In the Quantum realm, a topological insulator is a material presenting an insulating behaviour excepting for its surface where there exist conducting states thanks to the spin-orbital coupling. This interesting property comes from its geometrical construction and does not need an external magnetic field to break the T-symmetry. Moreover, such states present robustness to bending, cavity and disorder. In water wave systems a relevant study has been conducted by Shiqiao Wu *et al* [24]. They created an artificial crystal, following a triangular pattern, on the bottom of a tank filled with water Fig. (I-3 a). The unit cell, Fig. (I-3 b), is made by a circular ring with specific values for R and r .

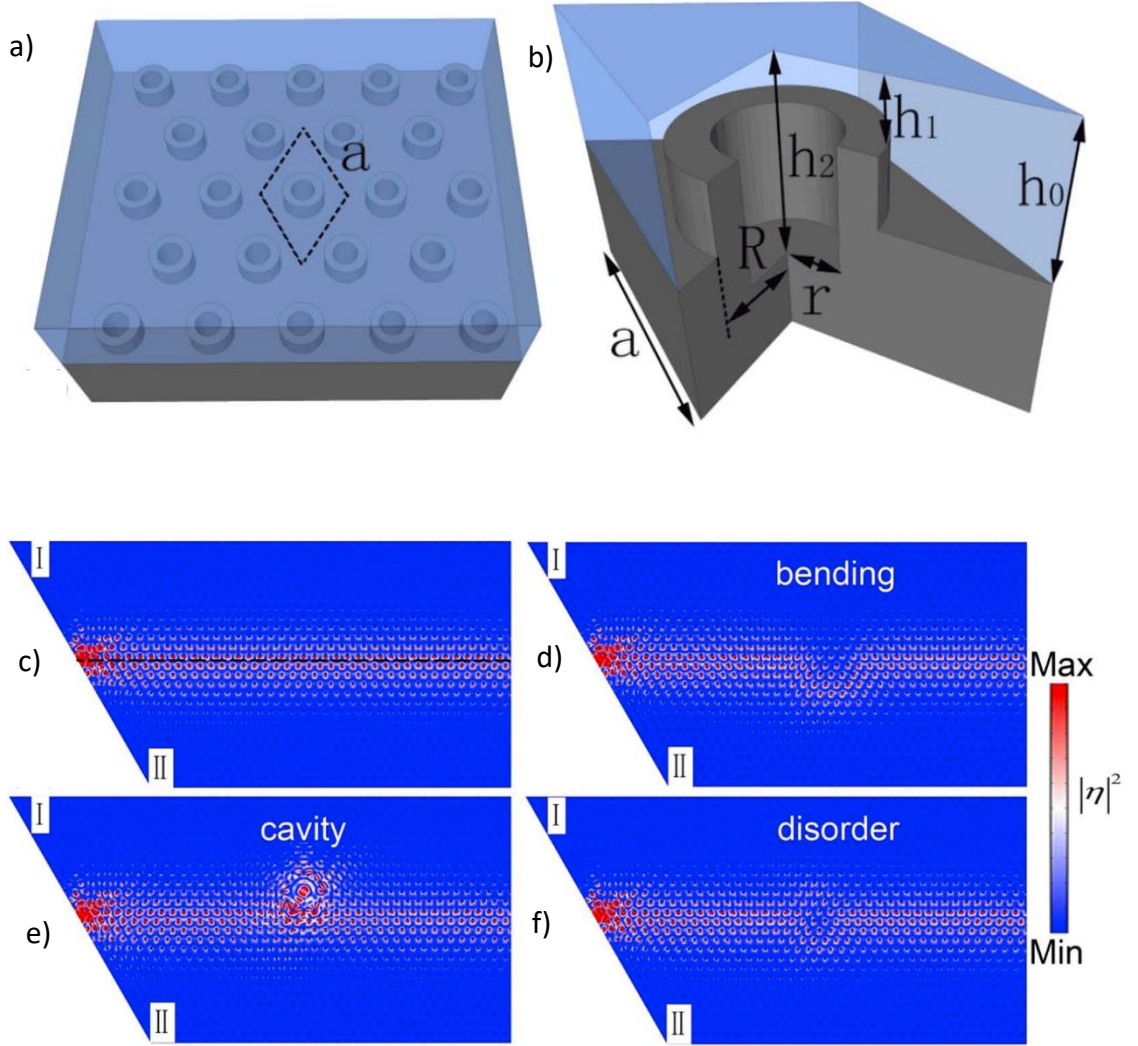


Figure I-3: a) Schematic of a 2D crystal for water waves placed on the bottom of a tank. b) Geometry of each site composing the crystal or unit cell. c) Topological edge mode through an interface made by two patterns on the bottom of a tank filled with water. This mode shows robustness against bending (d), cavity (e) and disorder (f). Red and blue indicate the square modulus of water wave height. Adapted from [24].

Then, they created two different patterns varying R and r , called I and II. Pattern I is placed next to pattern II forming an interface where they verified the presence of a topological edge mode via simulations, Fig. (I-3 c) which is robust to bending (Fig. I-3 d), cavity (Fig. I-3 e) and disorder (Fig. I-3 f). Red and blue indicate the square modulus of water wave height.

We will see how a topological edge mode can be observed in a HEW system. From a physical perspective it is fascinating how an exotic concept typical of Quantum Physics has a classic equivalent that can lead to a new kind of robust waveguide immune to

Introduction

back-scattering in water waves systems. An example of it at big scale is represented by the equatorial Yanai and Kelvin waves which exist because of the breaking of the T symmetry by Earth's rotation, Fig. (I-4). Topology plays a crucial role even in Earth's climate system [25].

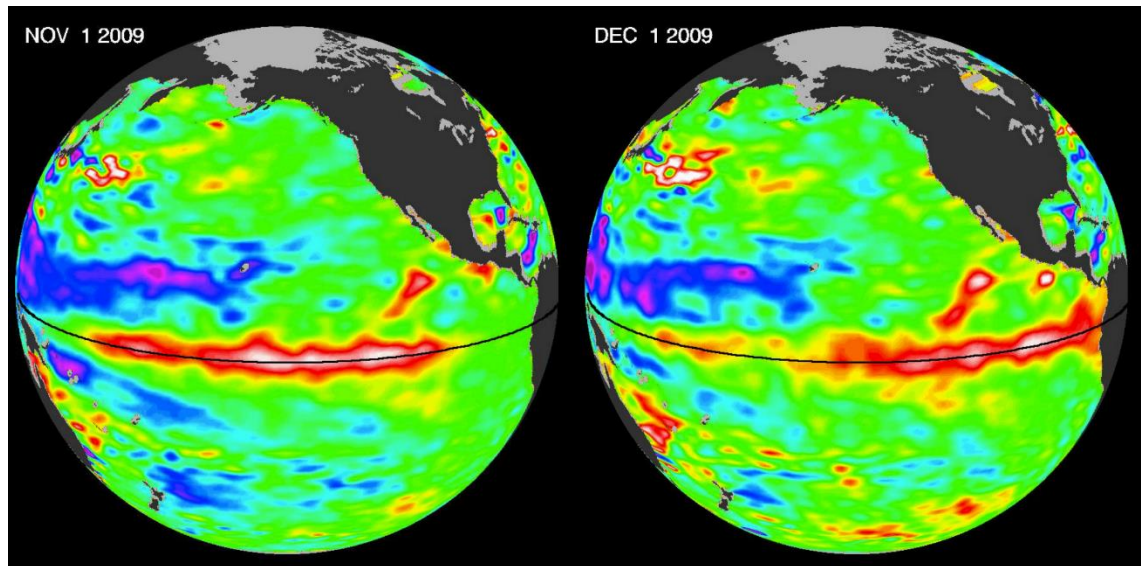


Figure I-4: Propagation of a temperature's anomaly in the Pacific Ocean during the climate pattern called El Niño [26], now explained as a topological wave [25].

Organization of the manuscript

The main body of this thesis consists of six chapters organized as follows:

Chapter 1

In this chapter we present the fundamental equations to obtain the dispersion relation of a Hydro-Elastic Wave (HEW). This is the starting point to understand how this kind of waves propagates and under which conditions they are studied (deep water condition and thin elastic plate). We discuss the three limiting regimes and the length scales associated, highlighting the flexural regime which is a peculiarity of this kind of waves. We illustrate the experimental setup, the technique to reconstruct the height field of a HEW in our physical environment and its limit caused by ray crossing.

Chapter 2

The second chapter is devoted to the propagation of waves in a structured media. The easiest 1D example is the Bragg's mirror which allows to understand how the dispersion relation changes when waves propagate in a structured media, showing band gaps due to the spatial arrangement of the lattice considered and by the local resonances. The formalism of Bloch is presented to explain the Bragg's mirror in a different way and to introduce the properties of the triangular lattice (the 2D structured media employed in the experimental part). We present an introduction to Topology which can be usefully applied to classical physical systems, illustrating the concepts of topological invariant and topological phase transition, leading to new interesting transport properties. We focus on the Quantum Spin Hall effect that can have a classical equivalent based on a triangular lattice which naturally sustains a four-fold degeneracy necessary condition to construct a Topological Insulator.

Chapter 3

This chapter is devoted to the characterization of the silicone membrane used in our experiments. It is composed mainly by three sections where we study: the Young's modulus, the attenuation length and the modes of a triangular cavity. We start by the

Introduction

measurement of the Young's modulus E (static value) of the silicone membrane via an extensometer. This value allows us to plot the theoretical dispersion relation of a HEW (E is contained in the flexural term of the dispersion relation). We verify it experimentally comparing the theoretical curve with the experimental one reconstructed via Fourier transform. The result of this comparison is that we need to use the dynamic value of E instead of the static one because the silicone membrane is a polymer which implies an increased value of E when it is excited by a source wave generator (like in our experiments). The second measure concerns the attenuation length of a HEW in order to show that this is a suitable kind of wave for experiments to study the interactions between an incident wave and structured media. The third part is based on the characterization of a triangular perforation on the silicone membrane. We demonstrate that it acts like a triangular cavity and the good agreement between the experimental modes and theoretical ones.

Chapter 4

In this chapter we present artificial crystals for HEW. We start by the analysis of the band diagram following the critical points in order to understand how to construct a crystal to support a topological edge mode. We characterize a triangular lattice for HEWs made by triangular resonators (monotype artificial crystal) with its dispersion relation, showing band gaps due to the Bragg scattering and the local resonances. Then, we introduce the Valley transport to create an artificial crystal where a topological mode can propagate without breaking the time reversal symmetry. We experimentally verify the propagation of an edge mode through a crystal made by two patterns of triangular resonators in a triangular lattice with different rotations, in other words, we propose a classical equivalent of the QSH effect for HEW (also called Topological Insulator).

Chapter 5

In this chapter we present a triangular lattice with a hexagonal shape in order to demonstrate that it is possible for a HEW to follow a particular path. The band

Introduction

inversion is achieved with a different geometry for the unit cell of the triangular lattice compared to the previous chapter. The focus, in this case, is on the control of the wave's propagation through a structure where two interfaces allow or not the propagation of an edge mode, creating a path different from a straight line. Moreover, these interfaces support a bidirectional mode. The experimental results are presented.

Chapter 6

In the end we summarize all results obtained. Main conclusions from the presented work are formed and possibilities of further research opened by this study are offered.

Chapter 1: Hydro-Elastic Waves and experimental setup

1.1 Introduction

In this chapter we present the theory to study HEWs using the model for waves in floating elastic ice plates and we will see how we can adapt it to a simple laboratory experiment based on a tank filled with water covered by an elastic sheet. We see how to write the dispersion relation of a HEW and the related possible regimes. We describe the experimental setup, in particular, the sampling techniques used to reconstruct the height field. The last part is devoted to the experimental results previously obtained during the PhD thesis of Lucie Domino [9] which is an important start to understand the goal of this research.

1.2 Waves in floating elastic ice sheets: mathematical formulation

We start with the mathematical model, proposed in [27], for an elastic plate that floats on the water surface.

For analytical purposes we assume a thin elastic homogeneous plate of infinite extent and a fluid domain of finite depth H . We assume that waves have an amplitude much smaller than the wavelength and the problem is linearized in amplitude. The (x,y) -plane is here the horizontal plane and the z -axis is the vertical direction (Fig. 1.1). The elastic plate has a uniform thickness d and flexural rigidity (or flexural modulus) $D = EI$ [28] where E is the Young's modulus, $I = d^3/(12(1 - \nu^2))$ and ν the Poisson's ratio.

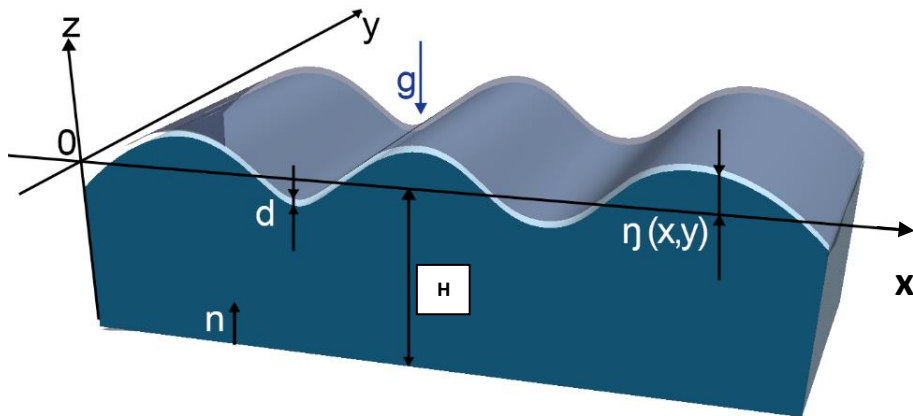


Figure 1.1: Diagram of a floating structure to model an ice plate.

In the approximation of small amplitude water waves and large enough frequencies so that viscous effects are confined to the fluid domain boundaries, there exists a velocity potential $\phi(x,y,z,t)$ that satisfies the three-dimensional Laplace equation [27]:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \quad \text{in the fluid domain,} \quad (1.1)$$

with the rigid and impermeable bed condition:

$$\frac{\partial \phi}{\partial z} = 0 \quad \text{on } z=-H \quad (1.2)$$

The linearized kinematic boundary condition on the plate covered surface (considering on $z=0$ the rest interface) is [2], [29]:

$$\frac{\partial}{\partial t} \eta(x, y, t) = \frac{\partial \phi}{\partial z} \quad \text{on } z=0 \quad (1.3)$$

where $\nabla = (\partial/\partial x, \partial/\partial y)$ and $\eta(x, y, t)$ is the vertical deflection of the plate. In the presence of in-plane compressive forces N acting on the floating elastic plate along the (x,y) -plane, the elastic plate deflection satisfies the thin plate equation [30]:

$$\left(D \nabla^4 + N \nabla^2 + \rho_p d \frac{\partial^2}{\partial t^2} \right) \eta(x, y, t) = -p_s(x, y, t) \quad \text{on } z=0 \quad (1.4)$$

with p_s being the external pressure acting on the floating plate and ρ_p the density of the plate. Assuming that the atmospheric pressure is constant and equal to zero, p_s reduces to the hydrodynamic pressure $p(x, y, t)$. Bernoulli's equation yields the linearized hydrodynamic pressure $p(x, y, t)$ on the plate-covered mean surface as follow:

$$p(x, y, t) = -\rho \frac{\partial}{\partial t} \phi + \rho g \eta \quad \text{on } z=0 \quad (1.5)$$

where ρ is the density of homogeneous fluid and g is the gravity acceleration. Combining eq. (1.4) and eq. (1.5) we obtain:

$$\left(D \nabla^4 + N \nabla^2 + \rho_p d \frac{\partial^2}{\partial t^2} \right) \eta(x, y, t) = \rho \frac{\partial}{\partial t} \phi - \rho g \eta \quad \text{on } z=0$$

we can write it as:

$$\left(D\nabla^4 + N\nabla^2 + \rho_p d \frac{\partial^2}{\partial t^2} + \rho g\right) \eta(x, y, t) = \rho \frac{\partial}{\partial t} \phi \quad \text{on } z=0 \quad (1.6)$$

Using eq. (1.1), eq. (1.3), eq. (1.6), the linearized plate-covered boundary condition on the mean surface in the presence of uniform flow and compressive force is obtained as:

$$\left(\frac{D}{\rho} \frac{\partial^4}{\partial z^4} - \frac{N}{\rho} \frac{\partial^2}{\partial z^2} + \frac{\rho_p d}{\rho} \frac{\partial^2}{\partial t^2} + g\right) \frac{\partial \phi}{\partial z} = \frac{\partial^2}{\partial t^2} \phi \quad \text{on } z=0 \quad (1.7)$$

seeking propagating wave solutions, the vertical displacement of the elastic plate can be written in the following form:

$$\eta(x, y, t) = A \cos(\mathbf{k} \cdot \mathbf{r} - \omega t) \quad (1.8)$$

where A is the amplitude of the surface displacement, $\mathbf{k} = (k_x, k_y)$ is the wave vector, $\mathbf{r} = (x, y)$ is the vector position and ω the angular frequency.

The velocity potential in water in the case of finite depth writes:

$$\phi(x, y, z, t) = -A \frac{\left(\frac{D}{\rho} k^4 - \frac{N}{\rho} k^2 - \frac{\rho_p d}{\rho} \omega^2 + g\right) \cosh(k(h-y))}{\omega \cosh(kh)} \sin(\mathbf{k} \cdot \mathbf{r} - \omega t) \quad (1.9)$$

where the wavenumber k satisfies:

$$\omega^2 = \left(\frac{D}{\rho} k^4 + \frac{N}{\rho} k^2 - \frac{\rho_p d}{\rho} \omega^2 + g\right) k \tanh(kh) \quad (1.10)$$

We can simplify eq. (1.10) assuming that the wavelength of the waves $\lambda \gg d$ of the elastic plate which implies that we can neglect the solid acceleration term $\gamma \omega^2$ where $\gamma = \rho_p d / \rho$. In other words, the elastic restoring force is much stronger than the force due to the inertia of the solid.

The dispersion relation becomes:

$$\omega^2 = \left(\frac{D}{\rho} k^4 + \frac{N}{\rho} k^2 + g\right) k \tanh(kh) \quad (1.11)$$

N , in our case, reduces to the mechanical tension T so we can rewrite Eq. (1.11) as:

$$\omega^2 = \left(\frac{D}{\rho} k^4 + \frac{T}{\rho} k^2 + g\right) k \tanh(kh) \quad (1.12)$$

This is the dispersion relation of a Hydro-Elastic Wave.

1.3 Dispersion relation of a HEW: properties and length scales

Looking at the right side of Eq. (1.12) we have three terms that correspond to bending, a tension term basically analogous to a capillary term [31] and a gravity term. All three terms are multiplied by $\tanh(kh)$ which can be approximated to kh in case of shallow-water ($\lambda \gg h$) reducing Eq. (1.12) to:

$$\omega^2 = \left(\frac{D}{\rho} k^6 + \frac{T}{\rho} k^4 + gk^2 \right) h \quad (1.13)$$

In case of deep water ($\lambda \ll h$) the term $\tanh(kh) \cong 1$ so Eq. (1.12) becomes:

$$\boxed{\omega^2 = \frac{D}{\rho} k^5 + \frac{T}{\rho} k^3 + gk} \quad (1.14)$$

This is the equation that we are going to use in all the following chapters. There are three important length scales associated with eq. (1.14); we can observe that balancing the first and second term gives the transition between bending and tension waves:

$$\lambda_{DT} = 2\pi \sqrt{\frac{D}{T}}$$

Balancing the second term and third one we obtain the length scale associated to the transition between tension and gravity waves:

$$\lambda_{gT} = 2\pi \sqrt{\frac{T}{g\rho}}$$

Balancing the first and third term we have the transition between bending and gravity waves which is the most relevant length scale for ice floes [32]:

$$\lambda_{Dg} = 2\pi \sqrt[4]{\frac{D}{g\rho}}$$

At short wavelegths the bending term of eq. (1.14) is dominant, this is the bending regime which allows us to modify the dispersion relation by changing the geometrical properties of the elastic plate as its thickness.

The dispersion relation eq. (1.14) is schematically represented in logarithmic axes frequency-wavelength in Fig. (1.2). For a more direct physical legibility we use the wavelength instead of k ($\omega/2\pi \rightarrow f, 2\pi/k \rightarrow \lambda$). It shows the bending, tension and gravity regimes with their corresponding slopes.

Compared to classical capillary-gravity surface waves, the dispersion relation of a HEW includes a bending term, which becomes dominant at short enough wavelengths ($\lambda < \lambda_{TD}$). In this regime, changing the properties of the elastic plate allows us to modify the speed of propagation.

Considering just the bending term (bending limit) we can write:

$$\omega = \left(\frac{Dk^5}{\rho} \right)^{\frac{1}{2}}$$

for the phase speed we have then:

$$v_\phi = \frac{\omega}{k} = \left(\frac{D}{\rho} \right)^{\frac{1}{2}} k^{\frac{3}{2}} \sim E^{\frac{1}{2}} \sim d^{\frac{3}{2}}$$

So, the membrane characteristics, as its thickness, have an influence on the phase speed.

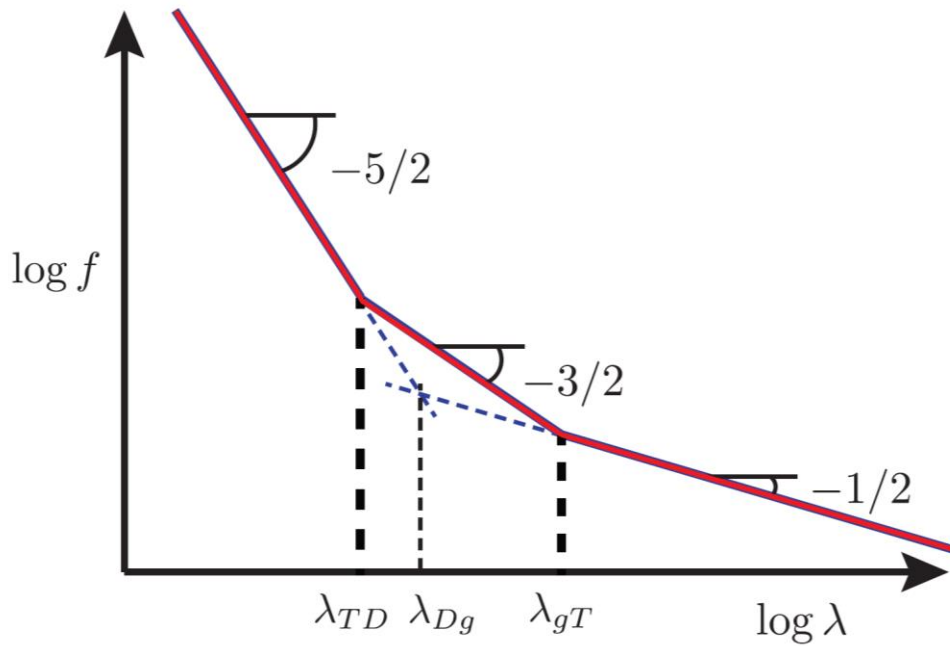


Figure 1.2: Schematic dispersion relation (f, λ) of a Hydro-Elastic Wave in logarithmic axes showing the bending, tension and gravity regimes, adapted from [9].

1.4 Experimental setup

The experimental setup is mainly composed by a glass tank (80 cm x 40 cm x 20 cm) filled by water (up to 17 cm high) and completely covered by an elastic floating sheet of silicone elastomer (thickness $d=0.5$ mm). The elastic sheet is optically transparent and characterized by the following values: Young's modulus $E = 1,6$ MPa, density $\rho=970$ kg/m³ and Poissons's ratio $\nu=0.5$. Perforations can be made in the elastic sheet to alter the propagation of waves. All patterns are regular perforations made by a cutter laser (Epilog laser Helix), programmed via PostScript [33], directly on all samples of elastic sheets.

The vibration exciter, controlled by a wave generator (working frequencies from 10 Hz to 120 Hz), is coupled with a straight paddle and it is located on the short side of the tank. A CCD camera (Basler, 2048 px x 2048 px, frame rate ~ 150 Hz), placed 2 m above the tank and a led panel with a printed 2D periodic pattern (checkerboard) just under it complete the experimental setup (see Fig. 1.3 a-b).

The reconstruction of the surface height is based on the synthetic Schlieren method [34] replacing the random dot pattern with a 2D periodic pattern [35] (black and white squares of side of 1 mm). A linear relation exists between the surface elevation gradient ∇h and the displacement field of the refracted image of a checkerboard if three approximations are taken into account (Fig. 1.3 c-d) [34]:

- Paraxial approximation, verified when $d_{CCD} \gg F_{field}$ (distance d_{CCD} between the CCD and the dot pattern and F_{field} field size)
- Weak slope approximation, it means that the angle between the vertical vector $\hat{z}$ and the normal vector to the interface $\hat{n}$ is small, $|\nabla h|^2 \ll 1$ so $\nabla h = \hat{z} - \hat{n}$
- Weak amplitude approximation, verified when the amplitude of the surface displacement is small compared to the mean height to measure

Ray crossing (caustics) is the main limitation of this technique (Fig. 1.4 a), it is caused by the surface curvature and/or surface-pattern distance h_0 . It is possible to calculate a critical surface-pattern distance h_c , which is proportional to λ^2 , to avoid ray crossing. Ray crossings are avoided if:

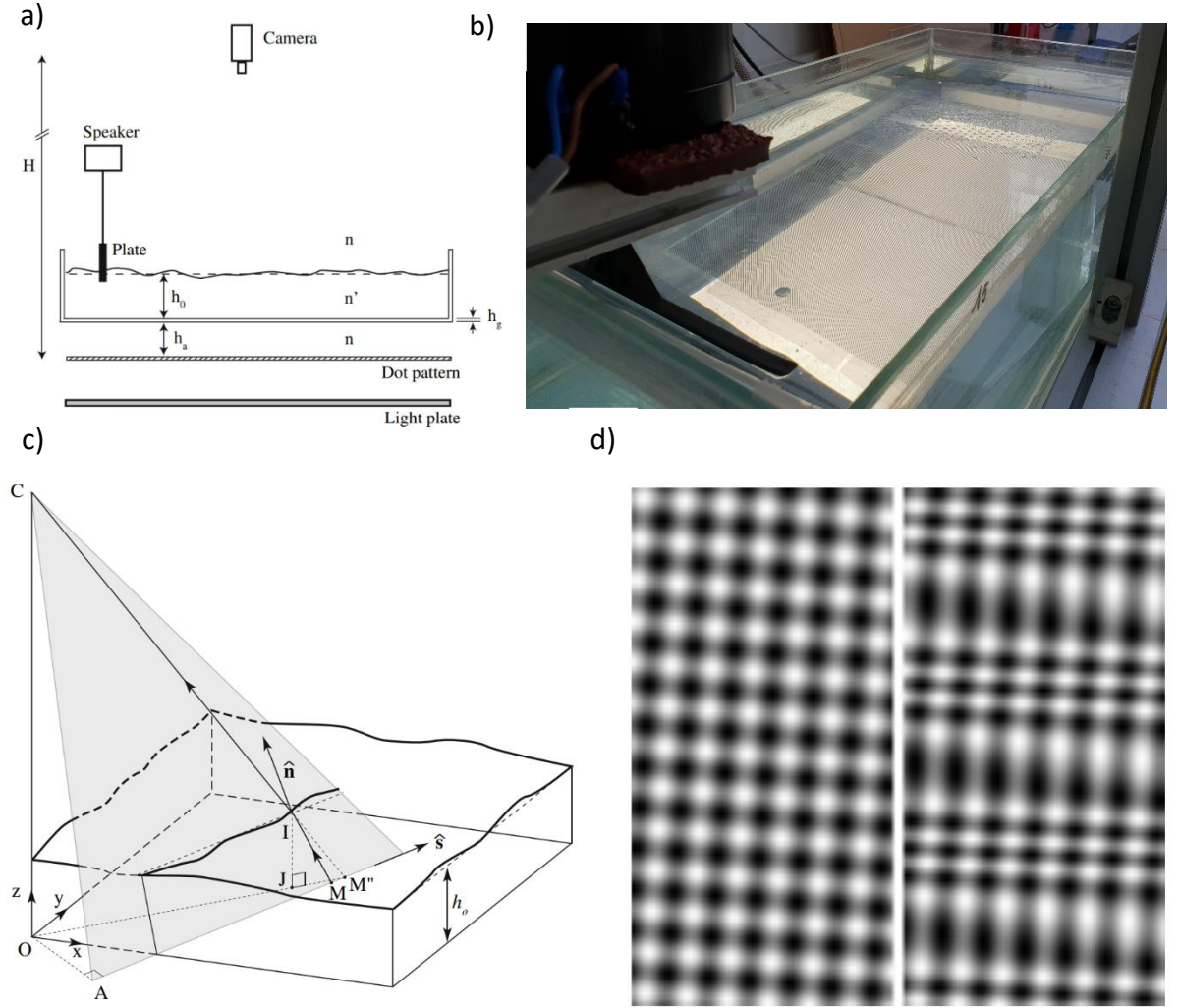


Figure 1.3: a) Experimental setup scheme [34]. b) View of the experimental setup, tank size 80 cm x 40 cm. c) 3D schematics considering a deformed interface by an incoming wave [34]. d) Checkerboard undistorted on the left side and distorted by the source wave, adapted from [35].

$$h_0 < h_c = \frac{\lambda^2}{4\pi^2 \alpha \eta_0},$$

$$\alpha = 1 - \frac{n}{n'}$$

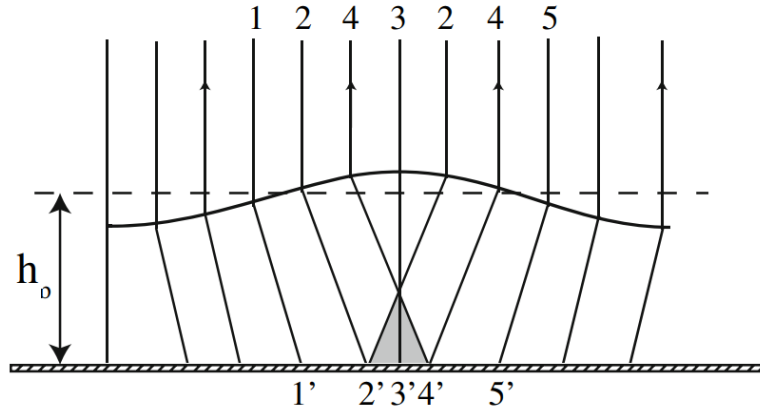
where n, n' are the refraction indices as shown in Fig. (1.3 a), $\alpha \cong 0,24$ for an air-water interface, λ is the wavelength and η_0 is the wave amplitude. Here is a table summarizing the typical values used for the experiments of this thesis:

Frequency f [Hz]	Wavelength λ [cm]	Wave amplitude η_0 [μ m]	α
10÷120	2 ÷ 8	10÷250	0,24

Table 1.1: typical ranges for frequency, wavelength and wave amplitude used in the experiments.

We plot the expression for the critical distance h_c against the wavelength in order to set the right surface-pattern distance value for our experiments. It is possible to see in Fig. (1.4 b) that, considering a wave amplitude $\eta_0 = 250 \mu m$, the surface-pattern distance to avoid ray crossings for all wavelengths in the range of table 1.1 is 17 cm. Consequently, we will use $h_0 = 16,9$ cm for all experiments.

a)



b)

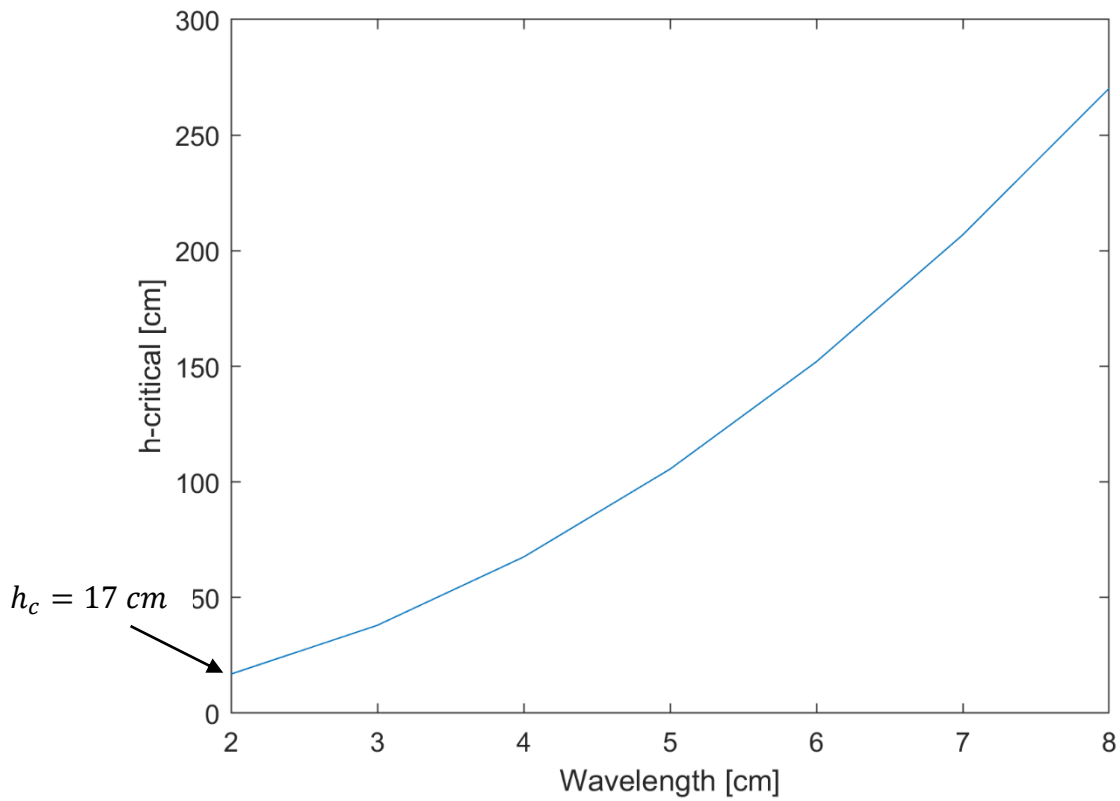


Figure 1.4: a) Ray crossing explanation: in 2' and 4' multiple rays compromised the reconstruction technique [34]. b) h critical for the typical waveleghts used in this thesis and wave amplitude $\eta_0 = 250 \mu m$.

1.4.1 Fast checkerboard demodulation

The free-surface synthetic Schlieren is an optical method that allow us to reconstruct the height field of the surface interfacing two fluids (typically water/air) where the reference image is a random dot pattern expressly designed to work with the digital image correlation (DIC) algorithm [34].

Instead of a random dot pattern it is possible to use a 2D periodic pattern in order to implement the Fast Checkerboard Demodulation (FCD) method which offers an accurate result like the DIC algorithm but in a significantly faster way [35]. The 2D periodic pattern, having a high spatial frequency $\mathbf{k}_0$, acts like a high frequency carrier wave involved by a phase modulation in analogy with the frequency modulation (FM) radio transmission [36]. In crystallography a general 2D periodic pattern I_0 (background pattern) can be written employing harmonic series:

$$I_0(\mathbf{r}) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} a_{m,n} e^{i(m\mathbf{k}_1+n\mathbf{k}_2)\cdot\mathbf{r}} \quad (1.15)$$

where $\mathbf{k}_1$ and $\mathbf{k}_2$ are the reciprocal lattice vectors and $\mathbf{k}_1 \times \mathbf{k}_2 \neq 0$, $\mathbf{r}$ is the pixel coordinates (x, y) . The distorted image $I(\mathbf{r})$ writes:

$$I(\mathbf{r}) = I_0(\mathbf{r} - \mathbf{u}(\mathbf{r})) \quad (1.16)$$

where $\mathbf{u}(\mathbf{r})$ is the vector displacement field. Combining eq. 1.15 and 1.16 we obtain for the distorted image:

$$I(\mathbf{r}) = I_0(\mathbf{r} - \mathbf{u}(\mathbf{r})) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} a_{m,n} e^{i(m\mathbf{k}_1+n\mathbf{k}_2)\cdot(\mathbf{r}-\mathbf{u}(\mathbf{r}))} \quad (1.17)$$

Eq. (1.18) underline the role of $\mathbf{u}(\mathbf{r})$: it modulates the phase of each harmonic within the series. Now it is possible to single out complex signals from the sums of eq. (1.15) and eq. (1.17) by Fourier domain filtering [37], [38], [39], [40]:

$$f_0 = a_0 e^{j\mathbf{k}_0 \cdot \mathbf{r}}$$

$$f = a_0 e^{j\mathbf{k}_0 \cdot (\mathbf{r} - \delta \mathbf{r})}$$

Once the extraction of the signals f_0 and f is done we can obtain $\mathbf{u}(\mathbf{r})$:

$$\mathbf{k}_0 \cdot \mathbf{u}(\mathbf{r}) = \text{Im}(\ln(f \cdot f_0^*))$$

So far, we presented the key step of this technique, we suggest referring to [35] for further details.

1.4.2 Wave source generator

The wave source generator consists in two parts, the first one is the vibration exciter (Brüel and Kjær 4809), the second one is a 3D object, created in the laboratory with a 3D printer in order to obtain plane waves or circular waves, fixed on the screw of the vibration exciter. For the plane waves we have a straight paddle made by a plastic material to respect the weight limit supported by the vibration exciter (see Fig. 1.5 c). A challenge about straight paddles is to avoid transverse fluctuations that would cause distortions in the height field reconstruction. That comes from the fact that the paddle impacts the membrane just on a side of it because of its flexibility. A good solution tested in the laboratory is to insert two tilted lateral supports between the straight paddle and the screw fixation (like a triangle, see Fig. 1.5 c); they give the stability to the paddle to correctly excite the membrane. We constructed it via OpenSCAD [41] verifying the weight limit (see Fig. 1.5 a) of the vibration exciter. We made, using the same strategy, a support to obtain a circular wave source with the advantage of local excitation of the membrane. It has an L shape to excite a particular area of the membrane and reconstruct the height field around it. Without the straight paddle the field of view would be partially occupied by the body of the vibration exciter so the reconstruction of the height field around the source would be compromised. However, a limit of the circular wave source is the noise made by the multiple reflections against the tank's walls. In fact, after a delay of approximately 2 seconds it is possible to observe in the tank the overlap of the main circular wave with all secondary waves. This noise can hide the effect that we want to observe and describe, that is why we used the circular wave source just to test our patterns (regular perforations on the membrane made via laser cutter, Fig. 1.6 d).

The vibration exciter works in the frequency range 10 Hz-20 KHz and has not a flat frequency response over the whole frequency range. Its frequency response is influenced by the support's weight which is more important for the low frequencies (10÷50 Hz). The first step is to characterize the wave source response in order to be able to set the same wave amplitude in the tank for each frequency in the working range (10÷120 Hz). We first put a silicone membrane covering the whole tank's

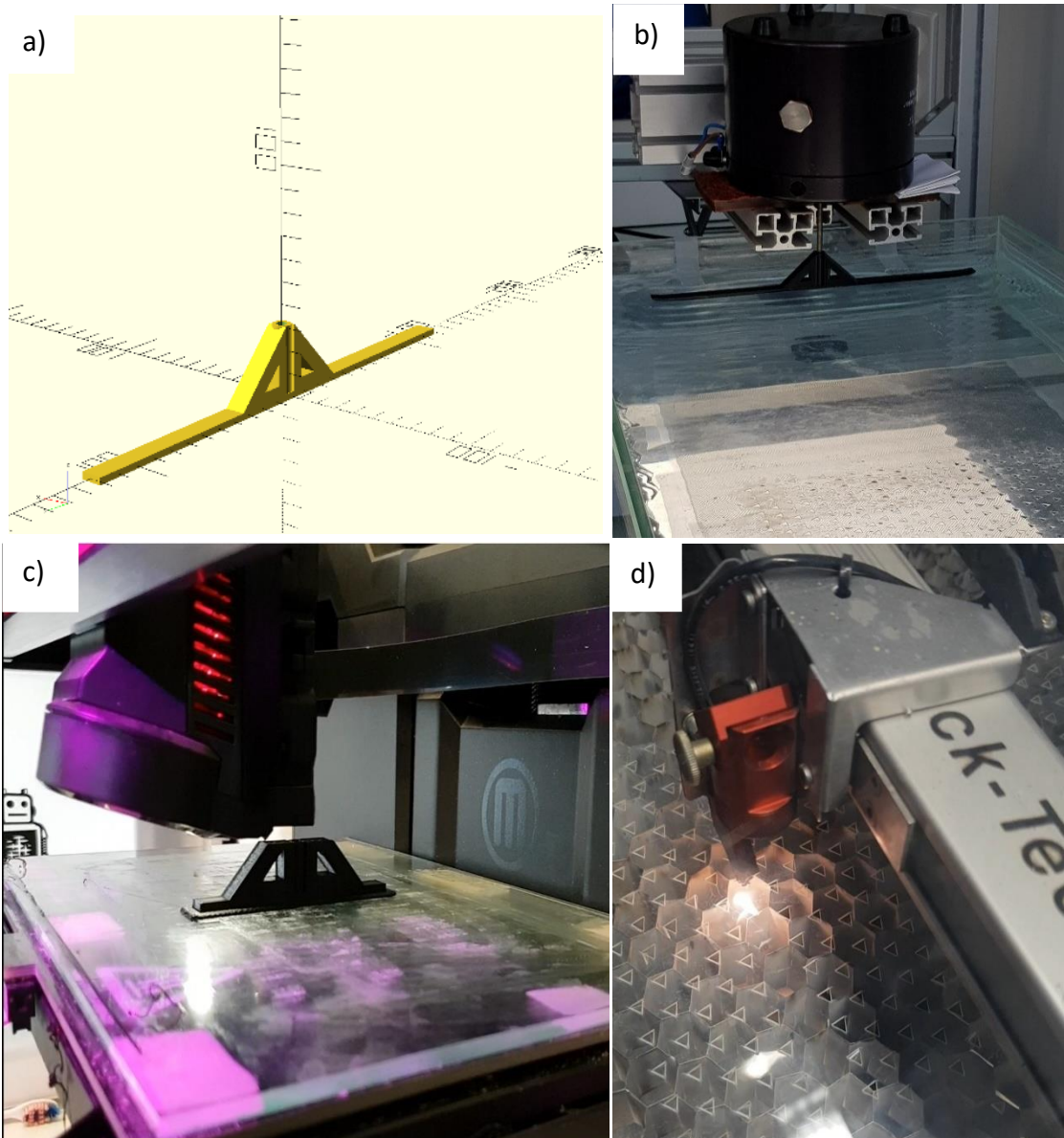


Figure 1.5: a) OpenSCAD project to make a straight paddle. b) Vibration exciter coupled with a 3D object to obtain plane waves. c) 3D printing of the lateral supports to stabilise the paddle. d) Cutter laser making regular perforations on the silicone membrane.

surface, then we attach the straight paddle to the vibration exciter and we place it in contact with the membrane (as shown in Fig 1.1 a-b). It is important to precise that the paddle softly touches the elastic sheet because we want to avoid water on the membrane and distortions in the reconstruction of height field. The vibration exciter is connected to an amplifier and a wave generator that provides a sine wave. Once the vibration exciter is on, we wait for 2 seconds to have a stationary wave in the tank. After this delay we start recording 33 frames (CCD fps=244,8 Hz) for each frequency in the

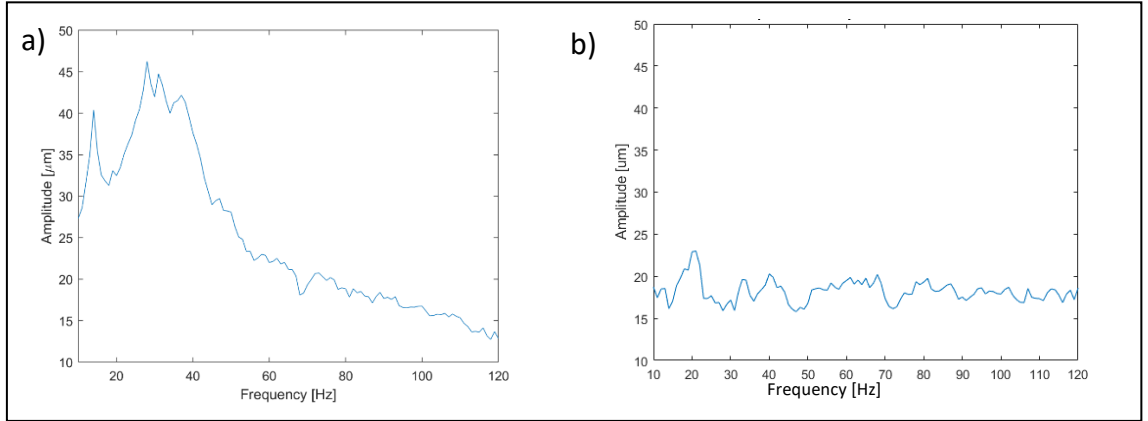


Figure 1.6: a) Frequency response of the wave source generator without calibration. b) Frequency response of the wave source generator after calibration.

range 10-120 Hz. The area recorded is in the middle of the membrane at about 23 cm from the paddle.

The reconstruction of the surface height based on the synthetic Schlieren method allows us to measure the amplitude of the incoming wave in the area selected and we obtain the graph in Fig. (1.6 a); as measured for the lowest frequencies the amplitude profile is very irregular. The amplitude profile can be corrected by applying a different amplification for each frequency in the working range from the wave generator. For example, from Fig. (1.6 a) we can observe that for $f = 30 \text{ Hz}$ the amplitude is 3 times compared to $f = 120 \text{ Hz}$; we need then an amplification factor of $1/3$ to obtain the same amplitude. Following this strategy, we made an array of correction factors for the amplification and the result is in Fig. (1.6 b) where it is shown the amplitude profile after calibration. Now, the vibration exciter has an acceptable flat frequency response so we are able to inject in the tank a wave with a particular amplitude and frequency. This is important to us for experimental reasons: generally, the effects that we want to study depends on a particular frequency that we do not know from the beginning so we measure and reconstruct the height field for a range of frequency. If we keep the same amplification for the source generator in order to have signal for the high frequencies (*i.e.* 120 Hz), for the low frequencies the excitation is so important that water goes over the membrane disturbing the whole measure. On the contrary, if we reduce the amplification to avoid water over the membrane then for the high frequencies there will be no excitation of the membrane. This calibration allows us to do our measurements in the same conditions for low and high frequencies.

1.4.3 CCD sampling and folding

The CCD used to record the images of the distorted checker-board has a typical acquisition rate f_{ps} between 100 and 150 Hz (choosing a smaller frame the f_{ps} can be increased). Considering that we work with a wave generator in a frequency range between 10 and 150 Hz, the Nyquist condition is not fulfilled (the minimum sample rate has to be equal or superior to the twice highest frequency of the signal that we want to reconstruct). This is, apparently, a case of undersampling, we would need a $f_{ps} > 300 \text{ Hz}$. The solution is based on the folding of the spectrum caused by the violation of the Nyquist condition and on the fact that we know the frequency of the sinusoidal signal made by the wave generator which is selected by us. In fact, the peak on the spectrum diagram associated with a signal with a frequency superior to $f_{ps}/2$ will appear in a position resulting from a mirror symmetry about $f_{ps}/2$. Referring to Fig. (1.5 a), let us consider an example to clarify the solution adopted in our experiments, where $f_{ps} = 108 \text{ Hz}$ and a signal with $f_{s2} = 83 \text{ Hz}$ (blue line); as the Nyquist condition is not respected, we obtain an artificial signal with f_{s1} (red line) which corresponds, in the frequency domain, to the spectral folding (Fig. 1.5 b). Because of that a peak will appear in a position mirrored about $f_{ps}/2 = 54 \text{ Hz}$. To calculate where to expect the folded peak we first calculate the frequency distance between $f_{ps}/2$ and f_{s2} obtaining $\delta = f_{s2} - f_{ps}/2 = 29 \text{ Hz}$. Then, $f_{s1} = f_{ps}/2 - \delta = 54 - 29 = 25 \text{ Hz}$. In this way the sampling rate will not cause any problem even when the Nyquist condition is not fulfilled.

Now, we would like to explain why we use this technique and its advantage. As we will see in the next chapters devoted to the experimental results, our goal is to verify if we can observe an edge mode propagating through a specific interface made on a silicone membrane. The input frequency (from the wave generator) is the same as the expected mode that we want to study, or in other words the response (output) of our system does not modify the input frequency. This means that we do not need to reconstruct an unknown signal, for which the accuracy in the frequency domain would be relevant. In our case, the f_{ps} can be low then resulting in an important advantage in terms of computational and memory cost. In fact, the higher the f_{ps} the more computational time and memory are needed. Just to give an idea a simple measure with our low f_{ps} needs about 500 Gb. With a higher f_{ps} (i.e. $f_{ps} = 300 \text{ Hz}$) we

would easily need more than 1 Tb per measure and a high computational cost which would not provide us any extra information about what we want to demonstrate.

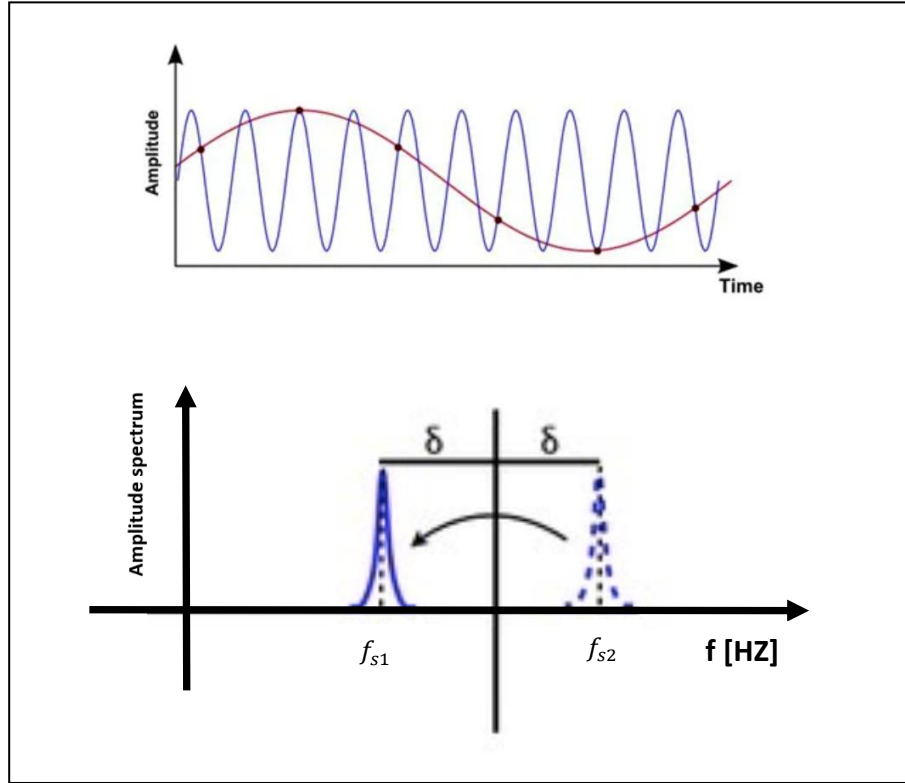


Figure 1.7: a) Real signal at f_{s2} in the time domain (blue line) with the undersampled reconstruction (red line) [42]. b) Spectral folding showing a folded peak. It is the result of a mirror symmetry around $f_{ps}/2$.

1.5 Main experimental results concerning HEWs obtained by Lucie Domino

Here, we want to summarise a part of the important results obtained by Lucie Domino during her PhD thesis that can be helpful for a better understanding of our research.

Dispersion relation

Lucie worked with silicone membranes of different thicknesses and she characterized each of them plotting the experimental and theoretical dispersion relation (Fig. 1.8 a). She confirmed the existence of a bending dominated regime (typically above 100 Hz) with a phase speed dependent on the membrane thickness. As seen at the end of paragraph (1.3), the membrane's thickness modifies the dispersion relation but has also an important impact for the experimental side. In fact, when a membrane is too thin (*i.e.* 300 μm) it is very complicated to manipulate it. For example, it is hard to

place it on the water surface. On the other hand, for a too thick membrane (*i.e.* $800\ \mu\text{m}$) the frequency range is higher compared to the previous example which bring us to face more important attenuation losses which depends on the frequency as we will see in detail in Chapter 3.

She found then, the good balance to study HEWs which is $500\ \mu\text{m}$. This is the thickness that we will always use during this thesis. Referring to Fig. (1.8 a) we can observe that almost all points of the dispersion relation for the membrane with thickness $t = 500\ \mu\text{m}$ are entirely below the prediction curve. This observation made us repeating the experiment in order to explain this shift between the theoretical and experimental dispersion relation. In Chapter 3 we will fully explain it.

Artificial crystals

Lucie made many artificial crystals for HEWs, in other words, she studied the interactions of a HEW with a specific pattern on the membrane.

What do we call pattern?

At the beginning she started by putting small cylinders, of the same material, on the membrane following the geometry of a 2D square lattice (this is a pattern) as shown on Fig. (1.8 b). The idea was to see how a HEW interacts with an artificial crystal which is an area of the membrane where the thickness is modulated following the square lattice configuration.

From the experimental view it was hard to observe any interaction. She had then the great idea of making perforations on the membrane so she removed parts of the silicone membrane instead of adding the same material.

This is an important start for us, in fact, all artificial crystals that we will analyse during this thesis will involve structures of regular perforations on the membrane. The artificial crystals are presented in Chapter 4.

Circular cavities

Before studying the behaviour of a pattern of perforations, she studied the interaction between a HEW and one single perforation. She was able to demonstrate that a single circular perforation acts like a resonator (or cavity) showing its modes. In particular,

she showed the first four modes, Fig. (1.8 c), comparing the theoretical and experimental modes. In our case, we will use the same idea of perforating the membrane but this time the shape will be an equilateral triangle. The reason of this choice is based on the fact that we want to study phenomena which require an anisotropic response from the resonator. A circular perforation is not then the right shape for our purpose. More details about the role of an anisotropic resonator are provided in Chapter 2 but here we can anticipate that like we want to study edge modes propagating through an interface we need to make a crystal presenting an interface.

What do we call an interface?

An interface is the border between two patterns one next to the other one. If we consider a pattern of triangular resonators, we can establish a rotation angle for a pattern and a different angle for the other pattern. Putting together the two patterns we will have an interface between them because at the border there is a difference in the spatial arrangement.

This is something that we could not apply to a pattern of circular resonators, as known a circle possesses a rotational symmetry so it is not possible to create two different patterns this way.

The characterization of a single triangular perforation is presented in Chapter 3.

Rotational experiment

Lucie wanted to study topological effects breaking the time reversal symmetry. In order to do so, she studied HEWs with a rotating tank (Fig. 1.8 d); the rotation implies the breaking of the T symmetry. Unfortunately, this approach has many experimental limits including a noise/signal too high, deformations of the membrane on the border and non-linear effects. This limit is well known by all the recent works involving topological modes in classical systems.

Lucia's work was precious to think about another approach. There exists a strategy to observe topological effects without breaking the T symmetry. We will see in the Chapter 2 (paragraph 2.6-2.7) how we can bypass this limit.

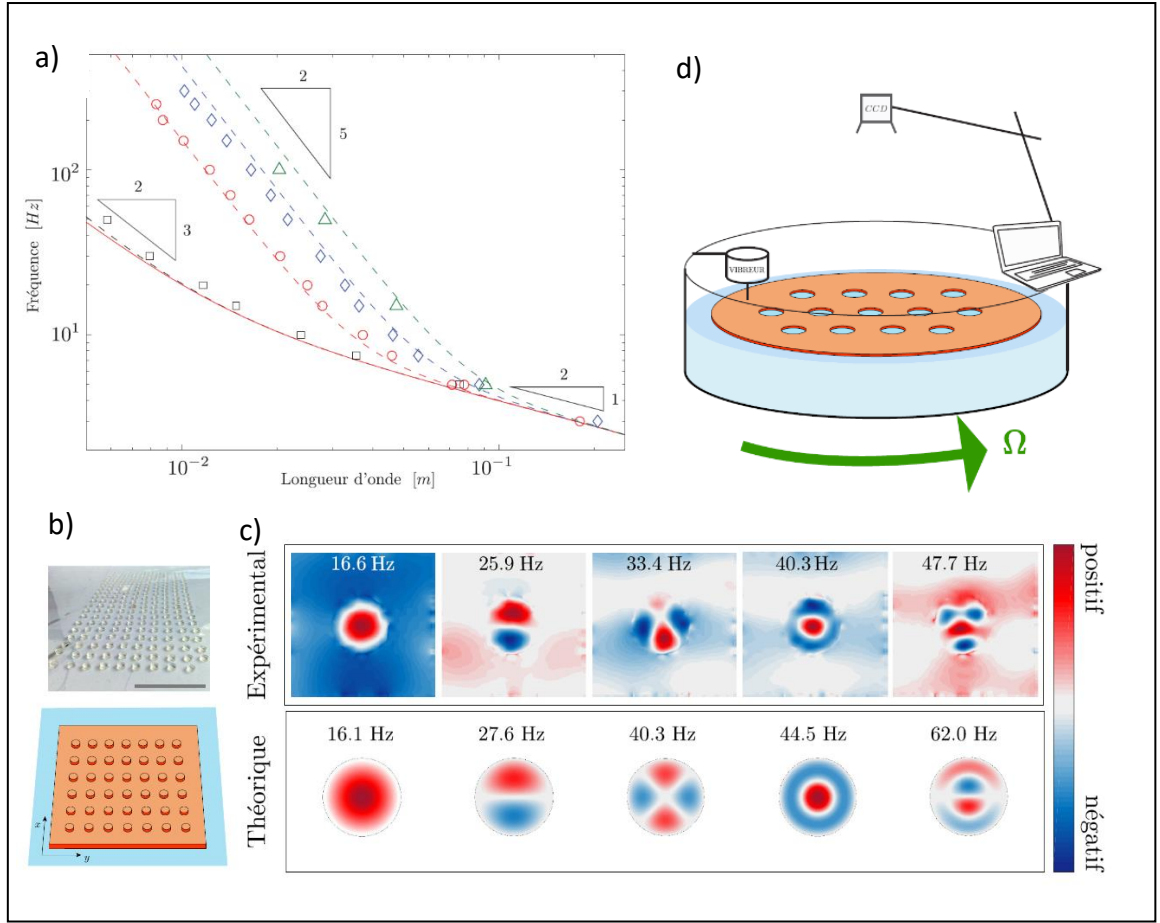


Figure 1.8: a) Dispersion relation for four cases: open water squared dots, membrane of thickness $t=300 \mu\text{m}$ red circles, $t=500 \mu\text{m}$ blue dots, $t=800 \mu\text{m}$ green triangular dots. b) Artificial crystal for HEW (square lattice). c) Experimental (up) and theoretical (down) modes for a circular perforation on a silicone membrane. d) Experimental setup to study an artificial crystal breaking the time reversal symmetry with a rotational tank. Adapted from [9].

Experimental differences

The experimental setup has been improved compared to the previous version adopted by Lucie Domino.

In particular the wave source generator is placed on an external support in order to avoid any vibration to the tank or to the CCD. In the previous version, it was placed right next to the tank causing noise.

The checker board is placed right next to the bottom of the tank, in order to avoid a layer of air which would deviate the light's path and it is fixed on the tank preventing distortions generated by a not flat checkerboard. We remind that the latter is a printed transparent sheet.

The last improvement concerns straight paddle connected to the wave source generator, previously presented, made via a 3D printer. This allows us to have plane waves instead of circular waves that were approximated to plane waves putting the source far from the area recorder by the CCD.

That is not just an approximation but it is also a source of noise because of the waves reflected in all directions in the tank resulting in a mode noisy and difficult reconstruction of the height field. In Fig. (1.9 a) we can observe the height field reconstruction with the previous source (circular wave source) and in Fig. (1.9 b) the actual version (plane waves), example at $f = 110 \text{ Hz}$.

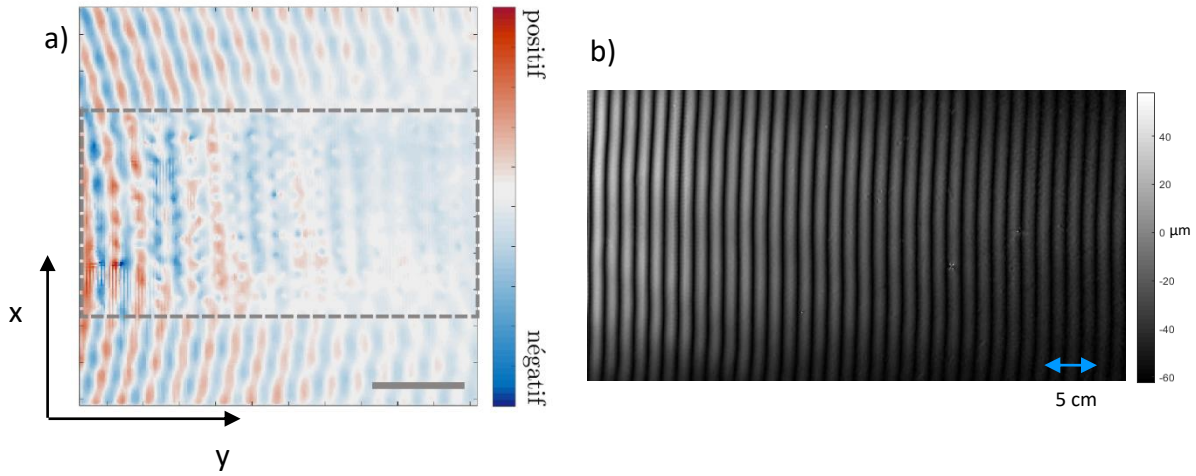


Figure 1.9: a) Previous version: height field reconstruction where a circular wave source is employed ($f=60 \text{ Hz}$); far from the source which is placed on the left side of the image the wave can be approximated as a plane wave, adapted from [9]. b) Actual version: height field reconstruction employing a straight paddle connected to the wave source ($f=110 \text{ Hz}$); in this case we do not need to be far from the source.

1.6 Conclusion

In this chapter we presented the fundamental equations to obtain the dispersion relation of a Hydro-Elastic Wave. This is the starting point to understand how this kind of waves propagates and under which conditions they are studied (deep water condition and thin elastic plate). We saw that the dispersion relation is composed by three terms: gravity, tension and bending term. The latter, is a peculiarity of HEW that allow us to easily modify the dispersion relation of a silicone membrane by changing, for example, its thickness. We illustrated the experimental setup and the technique, synthetic Schlieren method via fast checkerboard demodulation, to reconstruct the

height field of a HEW in our physical environment. Ray crossing, which is the main limit of this method is discussed. We explained how to bypass the Nyquist condition, providing the argumentations for the sample rate adopted. The last part is consecrated to the previous experimental results about HEWs obtained by Lucie Domino, which are an important start for this thesis.

Chapter 2: From structured media to topology

2.1 Introduction

In this chapter, we will first present the wave propagation in a one-dimensional periodic lattice showing how the dispersion relation is modified by a structured media with the gap opening process related to the Bragg scattering and local resonances. The gap opening process, as we will see, means that for a particular frequency range the wave propagation through such structure is not allowed. A famous example of 1D crystal presenting a band gap is the Bragg reflector and it will be explained also via the Bloch-Floquet's formalism which provides another possible interpretation of the problem.

Then, we will introduce the theoretical tools to understand how Topology can be applied in a classical system. Topology involves the geometry of the periodic media considered, that can be structured in a particular way leading to interesting transport phenomena. This is what we want to investigate, precisely, our goal is to demonstrate that we can observe a topological edge mode through a particular path. The latter is the interface of two 2D crystals one next to the other one which supports a propagating mode that belongs to the frequency range where the wave propagation is not allowed.

In order to understand the existence of this topological mode and this apparent contradiction, we will discuss first the properties of a 2D lattice, precisely the triangular lattice, defining the reciprocal lattice in the k -space, the Brillouin Zone, the critical points and the Irreducible Brillouin Zone.

Then, we will introduce the concepts of topological invariant and topological phase transition from which the existence of this mode depends. We will see that there are more ways to achieve a topological phase transition focusing on the symmetries of the problem. The first one, based on the Quantum Hall Effect, breaks the time reversal symmetry while the second one preserves it by imitating the Quantum Spin Hall effect (or equivalently realizing a Topological Insulator).

In other words, we will present all details to create a Topological insulator for HEWs.

2.2 Bragg scattering

Waves can be described by a traveling modulation of some physical quantity $f(x, t)$ periodic in time and space. Considering a single mode for simplicity, we can write it as:

$$f(x, t) = A \cos(\omega t - kx + \varphi)$$

where A is the amplitude, ω is the angular frequency, k is the wave number and φ the phase (see Fig. 2.1). Considering two waves with phases φ_1 and φ_2 , we can have several kind of interferences and two limit cases: destructive ($\Delta\varphi = \varphi_2 - \varphi_1 = \pi$) and constructive ($\Delta\varphi = 0$).

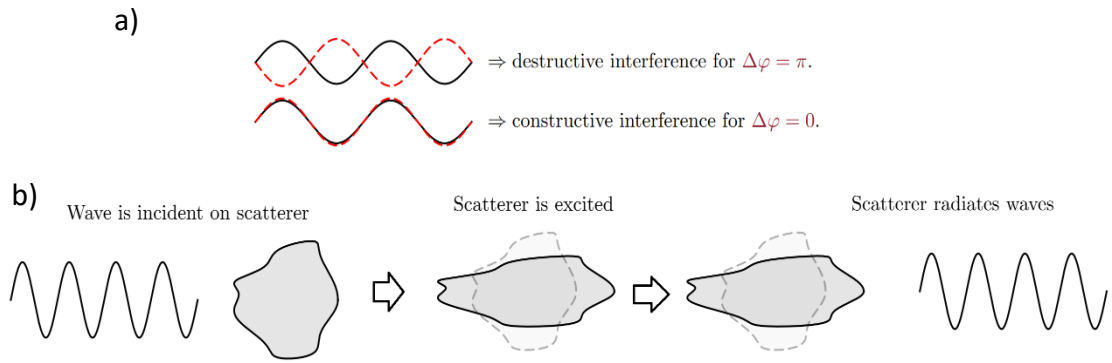


Figure 2.1: a) Destructive and constructive interference. b) The three phases of the interaction of an incident wave on a scatterer, it is first excited and then it radiates. Adapted from [43].

The phase difference $\Delta\varphi$ between the incident and radiated wave depends on the interaction between the wave and the scatterer and on the properties of the scatterer. Let's take two identical scatterers separated by a distance Δx (see Fig. 2.2), we can write the equations of the two radiated waves denoted by f_{r1} and f_{r2} :

$$f_{r1}(x, t) = \cos(\omega t - kx + \Delta\varphi_0)$$

$$f_{r2}(x, t) = \cos(\omega t - kx + \Delta\varphi_0 + k\Delta x)$$

We can observe that the destructive condition $k\Delta x = \pi$ can be written as:

$$\Delta x = \frac{\lambda}{2}$$

where λ denotes the wavelength.

If we extend this example to a periodic array of scatterers with a typical length-scale ΔL the destructive condition takes the name of Bragg condition:

$$\Delta L = \frac{\lambda}{2}$$

In other words when the Bragg condition is fulfilled waves cannot propagate through the array of scatterers (the phenomena is called Bragg scattering).

How is the Bragg condition represented in a (ω, k) diagram?

For example, let us consider the dispersion relation of electromagnetic waves in a vacuum is $\omega = ck$, where c is the speed of light in vacuum. When they interact with a 1D array of scatterers, like previously seen, the (ω, k) diagram presents a gap for the frequencies where the Bragg condition is met (see Fig. 2.3).

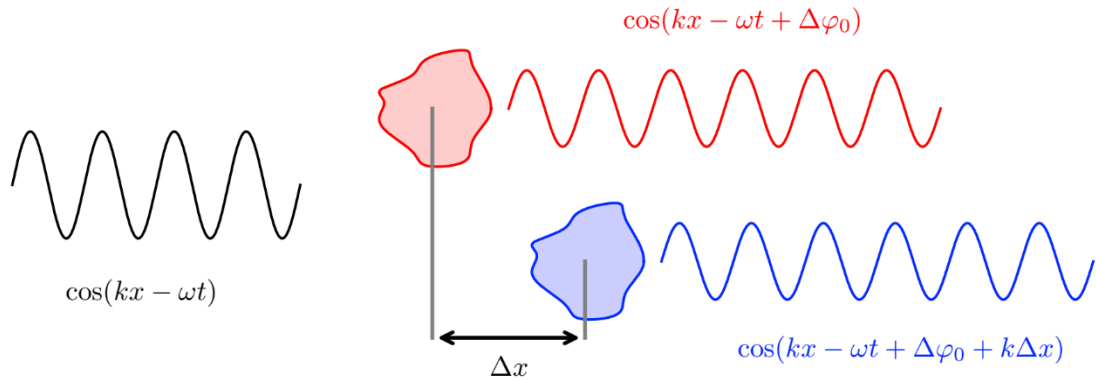


Figure 2.2: Representation of two radiated waves from two scatterers at a distance Δx , [43].

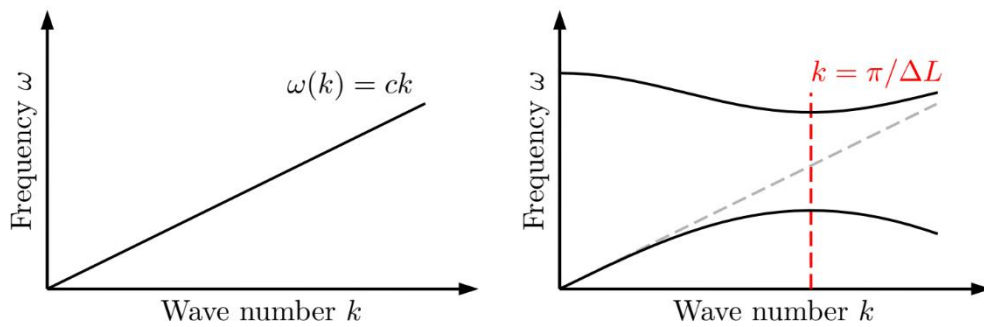


Figure 2.3: Dispersion relation of an electromagnetic wave in a vacuum on the left and in a 1D array of scatterers on the right, in this case a gaps open meaning that waves cannot propagate through the periodic structure, [43].

A relevant example in Optics, the easiest case of 1D photonic crystal, is the so-called Bragg mirror or Bragg reflector [44] (Fig. (2.4)) which is a structure made by an alternating sequence of layers of two materials with different refractive index, high (n_1) and low (n_2) indices, along one axis (the unit cell is composed by one layer of

each material). Lord Rayleigh is the first who explained the phase shift of the wave propagating in such a structured medium [45]. In particular, the thicknesses of the layers, t_1 and t_2 , are chosen in order to have the following expression for λ :

$$\lambda = 2m(t_1 + t_2) = 2mL$$

where m is an integer and $L = (t_1 + t_2)$ is the length of the unit cell.

As previously seen, in this case $\Delta\varphi = \pi$ so the waves interfere destructively and they are fully reflected (as a mirror). This kind of mirrors are designed therefore for a particular frequency and extensively used in laser cavities [46].

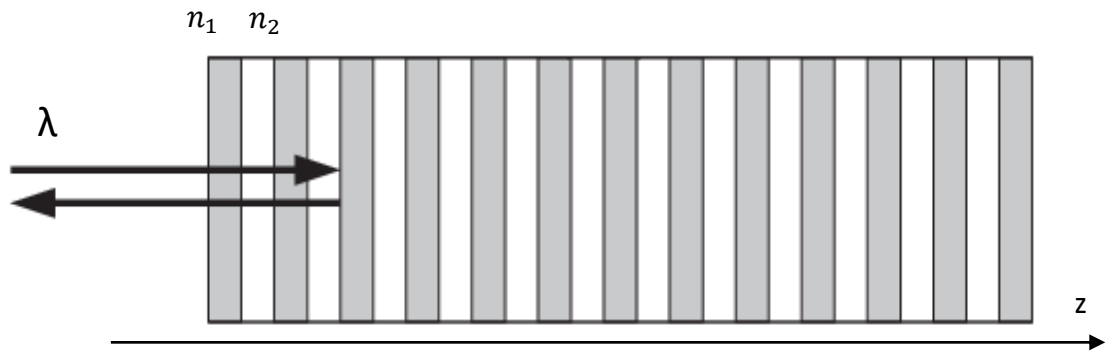


Figure 2.4: Bragg mirror or Bragg reflector: periodic structure made by an alternating sequence of layers of two materials with different refractive index (high (n_1) and low (n_2) indices) [47].

The Bragg mirror can have an equivalent explication using the Bloch-Floquet's formalism which gives us the opportunity to introduce some important tools to treat periodic media.

2.3 Bloch-Floquet's formalism and reciprocal lattice

We recall here some relevant concepts from solid state Physics, following [48], in order to introduce the Bloch formalism and the Bloch's theorem which give another possible explanation of the Bragg's mirror.

The vertices of any lattice structure can be seen as a collection of points (lattice points) associated with a set of basis vectors. The direct lattice is defined as the lattice point system together with the basis and it can be completely obtained by tessellating a suitable unit cell along the basis vectors $\mathbf{e}_i$ [48]. The unit cell is defined as the smallest

repeating unit having the full symmetry of the crystal structure. Let us define $\mathbf{r}_j$ the lattice points in a unit cell and $q(\mathbf{r}_j)$ the displacement of a lattice point within the unit cell. If a plane wave solution is admitted, $q(\mathbf{r}_j)$ can be expressed as:

$$q(\mathbf{r}_j) = q_j e^{(i\omega t - \mathbf{k} \cdot \mathbf{r}_j)}$$

where q_j is the amplitude, ω is the angular frequency and $\mathbf{k}$ is the wave vector of the plane wave. Considering the chosen unit cell and identifying any other cell via the integer pair (n_1, n_2) obtained by n_1 and n_2 translations along the basis vectors directions we can write the general expression for a point in the (n_1, n_2) cell:

$$\mathbf{r} = \mathbf{r}_j + n_1 \mathbf{e}_1 + n_2 \mathbf{e}_2$$

According to Bloch's theorem [49], the displacement $q(\mathbf{r})$ in any cell associated to the integer pair (n_1, n_2) in the direct lattice basis writes:

$$q(\mathbf{r}) = q(\mathbf{r}_j) e^{i\mathbf{k} \cdot (\mathbf{r} - \mathbf{r}_j)} = q(\mathbf{r}_j) e^{i(k_1 n_1 + k_2 n_2)}$$

where k_1 and k_2 are the components of the wave vector $\mathbf{k}$ along the basis vectors.

In other words, "Bloch's theorem (or Floquet's principle [50] when one-dimensional period structure is considered) states that for any periodic structure the change in complex wave without attenuation does not depend on the location of the unit cell within the structure and the wave propagation through the whole lattice can be studied in the frame of a single unit cell" [48].

The reciprocal lattice in the wave vector space (k -space) is defined by all points generated by the basis vectors of the direct lattice such that:

$$\mathbf{e}_i \cdot \mathbf{e}_j^* = \delta_{ij}$$

where $\mathbf{e}_i$ is the basis vectors of the direct lattice, $\mathbf{e}_j^*$ is the basis of the reciprocal lattice and δ_{ij} is the Kronecker delta function (considering a 2D lattice the subscripts i, j take the integer values 1 and 2) [48]. The wave vectors can be written using the reciprocal lattice base and like it is also periodic we can study only a restricted region, the so-called Brillouin Zones (BZ) which contains all the information [51]. "The first BZ is defined as a Wigner-Seitz or primate unit cell of the reciprocal lattice" [48], [52]. Bragg mirrors can be explained via the Bloch formalism that we have just introduced. We

consider waves with normal incidence as illustrated in Fig. 2.4 for which the incident and reflected fields components can be written as:

$$\Psi_i^{(I)}(z) = \Psi_i e^{j\beta_h z}$$

$$\Psi_i^{(R)}(z) = \alpha_i \Psi_i e^{-j\beta_h z}, \quad i = 1, 2$$

where β_h is the propagation constant in the homogeneous region with refractive index n_h and α_i is the reflection coefficient. The permittivity function is periodic along the z -axis and can be expressed via a Fourier series expansion as:

$$\varepsilon(z) = \sum_l \varepsilon_l e^{j l K z}$$

$$\varepsilon_l = \frac{1}{\Lambda} \int_{-\Lambda/2}^{\Lambda/2} \varepsilon(z) e^{-j n K z} dz$$

where K is the grating vector, Λ the grating period ($\Lambda = t_1 + t_2$) and ε_l the expansion coefficients. According to the Bloch-Floquet's theory, each field component in the periodic medium can be seen as the superposition of an infinite number of spatial harmonics characterized by the propagation constant k_{zn} [53]:

$$\Psi_i(z) = \sum_n \Psi_{in} e^{\pm j k_{zn} z}, \quad i = 1, 2$$

which is related to the fundamental harmonic:

$$k_{zn} = k_{z0} + nK$$

$$n = 0, \pm 1, \pm 2, \dots$$

In other words, the solution takes the form of an infinite number of plane waves modulated by a periodic function with the same periodicity of the 1D structure considered (accordingly to the Bloch's theorem). This justifies the periodicity of the dispersion relation which is commonly restricted to the range $[-\pi/\Lambda, \pi/\Lambda]$ where we have all the information, the so-called Brillouin Zone (BZ). The Brillouin Zone is an important concept that we will often use in the part devoted to Topology (paragraph 2.6).

Now, the Bragg reflector and the gap opening process is fully presented. We can pass to another kind of mechanism that creates band gaps: the local resonances.

2.4 The 1D “bi-atomic” periodic structure: local resonances

The Bragg mirror is a good example to illustrate how a gap opens in the (ω, k) diagram but it is not the only type of gap that can occur. In fact, local resonances can have a similar effect implying a frequency window where no waves can propagate. The easiest example to describe such effect is to consider the 1D bi-atomic periodic structure which consist in a chain of two types of masses (M_1 and M_2)-spring (μ stiffness) [54].

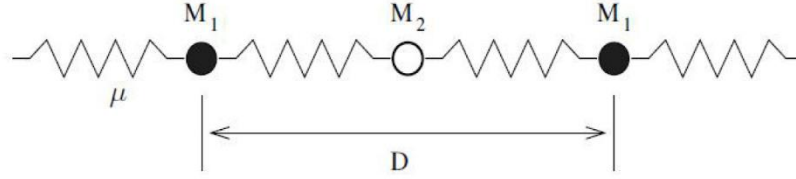


Figure 2.5: Scheme of the 1D bi-atomic periodic structure [54].

The equations of motion denoting u_n and v_n the displacement of M_1 and M_2 , in cell number n , respectively are:

$$M_1 \frac{\partial^2 u_n}{\partial t^2} - \mu(v_n + v_{n-1} - 2u_n) = 0 \quad (2.1)$$

$$M_2 \frac{\partial^2 v_n}{\partial t^2} - \mu(u_n + u_{n-1} - 2v_n) = 0 \quad (2.2)$$

According to the Bloch-Floquet theory the solution of Eq. (2.1) and Eq. (2.2) satisfies:

$$u_{s+n} = u_s e^{inK}$$

$$v_{s+n} = v_s e^{inK}$$

where K is the Bloch parameter and s, n integers. Considering a time harmonic solution, we have:

$$u_n = U e^{i(nKD - \omega t)} \quad (2.3)$$

$$v_n = V e^{i(nKD - \omega t)} \quad (2.4)$$

Inserting Eq. (2.3) and Eq. (2.4) into Eq. (2.1) and Eq. (2.2) we obtain a linear algebraic equation system:

$$(\omega^2 \mu^{-1} M_1 - 2)U + (1 + e^{-iKD})V = 0$$

$$(1 + e^{iKD})U + (\omega^2\mu^{-1}M_2 - 2)V = 0$$

Using the matrix form:

$$MX = 0$$

where

$$M = \begin{bmatrix} \omega^2\mu^{-1}M_1 - 2 & 1 + e^{-iKD} \\ 1 + e^{iKD} & \omega^2\mu^{-1}M_2 - 2 \end{bmatrix} \quad X = \begin{pmatrix} U \\ V \end{pmatrix}$$

This system has a non-trivial solution if $\text{Det}(M) = 0$ resulting in the dispersion relation:

$$\frac{M_1M_2}{\mu^2}\omega^4 - \frac{2(M_1+M_2)}{\mu}\omega^2 + 2(1 - \cos(KD)) = 0$$

with explicit solution:

$$\omega^2 = \mu \frac{M_1 + M_2 \pm \sqrt{(M_1+M_2)^2 - 2M_1M_2(1 - \cos(KD))}}{M_1M_2}$$

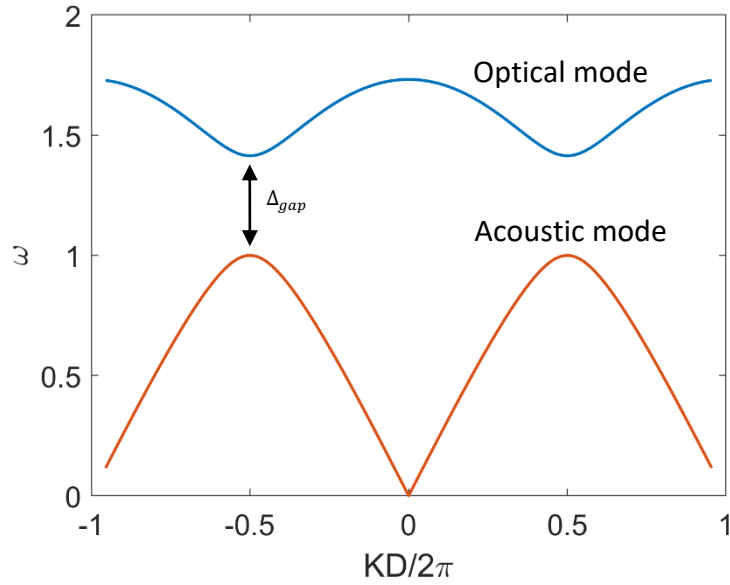


Figure 2.6: Dispersion relation of the 1D bi-atomic periodic structure (plotted using $M_1 = 1$, $M_2 = 2$ and $\mu=1$). The dispersion curve has two branches (optical and acoustic) and shows a band gap related to the local resonances meaning that no waves can propagate in that frequency range.

The plus sign result corresponds to the so-called optical mode while the other sign is called acoustic mode. The dispersion relation is shown in Fig. (2.6) where we can observe the separation between the two branches of the dispersion curve (optical and acoustic). This gap, also known as hybridization gap, is originated by the local resonance of the local oscillators which does not depend on the spacing of the periodic array like seen for the Bragg scattering but on the frequency of the local oscillator. The width Δ_{gap} of the band gap depends on the stiffness of the springs and on the masses considered. In particular, it can be calculated at the boundary of the BZ ($K = \pm \pi/D$) obtaining:

$$\Delta_{gap} = \sqrt{2\mu/M_1} - \sqrt{2\mu/M_2}$$

As we can see the hybridization gap arises from the difference between the resonant frequencies of the two masses and can be modified by changing the masses and stiffness values.

In order to better understand how the local resonances play a role within the context of the HEWs, we anticipate here that we will investigate the interaction between a triangular perforation on the silicone membrane and a HEW (Chapter 3: paragraph 3.4). The perforation, as we will see, acts as a resonator, or equivalently referring to the previous example as an oscillator, presenting resonant modes. At the resonance, the resonating process causes the decreasing of the amplitude transmitted resulting in a band gap in the dispersion relation. In other words, the energy of the incident wave is absorbed by the local resonator when the input frequency reaches the resonant frequency related to the triangular cavity. The width Δ_{gap} of the band gap employing cavities depends this time, on the frequency width associated to the resonance. We will see the frequency width of the resonant modes in detail in Chapter 3 (paragraph 3.4).

When more than one perforation, *i. e.* a lattice of perforations, is taken into account the effect of the amplitude decreasing is even more accentuated creating a large band gap in the dispersion relation. The latter will be shown in Chapter 4 where we present the artificial crystals for HEWs.

The gap opening process, due to Bragg scattering and local resonances, is now fully described. Before introducing Topology applied to a classical physical system, we still need the last important tools to treat periodic media which are presented in the next paragraph (2.5).

2.5 2D periodic structure: properties of the triangular lattice

So far, we considered just 1D periodic structures in order to explain how the wave propagation through such a media can be modified showing band gaps in the dispersion relation. It is time to extend our study to a 2D periodic structure employing the concepts related to the reciprocal lattice presented in the previous paragraph (2.3). Auguste Bravais studied and classified 2D and 3D lattices. For the 2D case we have the so-called Bravais lattice types which are five in total. One of them is the hexagonal lattice (also called triangular lattice) [55], as represented in Fig. (2.7) in the direct and reciprocal space, and it will be our focus because that is what we experimentally use in this thesis.

Let us see how to obtain the first BZ of a 2D triangular lattice which is the area of the dispersion relation containing all the information for a periodic media. Letting L be the lattice constant, we can express the basis vectors of the direct lattice as:

$$e_1 = L(1,0)$$

$$e_2 = L\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$$

and for the reciprocal lattice:

$$e_1^* = \frac{1}{L}\left(1, -\frac{1}{\sqrt{3}}\right)$$

$$e_2^* = \frac{1}{L}\left(0, \frac{2}{\sqrt{3}}\right)$$

The hexagonal lattice is preserved passing to the reciprocal space (Fig. 2.7), only its orientation is changed by $\pi/6$ with respect to the real-space lattice and primitive lattice vectors of length $4\pi/L\sqrt{3}$. Bisecting e_1^* and e_2^* and their equivalents, we obtain the first BZ. In the first BZ, there are three high-symmetry points, also called critical points, marked as Γ , M and K , positioned at $(0,0)$, $(\pi/L)(1, 1/\sqrt{3})$, and

$L(4\pi/3\sqrt{3}, 0)$ respectively [56] (and the area within these three points is called Irreducible Brillouin Zone (IBZ)). The IBZ can be seen as the first Brillouin zone reduced by all the symmetries.

These points are important because they are used to represent the dispersion relation of 2D periodic structures. Precisely, instead of having the wavevector on the x-axis and frequency on the y-axis like we usually expect for a dispersion relation, we have on the x-axis the three critical points (or the three high-symmetry points) marked as Γ , M and K .

This is almost a 3D representation, almost because it is still a 2D representation but with the evolution of the curve following the path $\Gamma \rightarrow K \rightarrow M$ (which is the Irreducible Brillouin Zone); refer to Fig. (2.7 c) to see where that path is in the 3D dispersion relation of graphene; the first part of the path, $\Gamma \rightarrow K$, is the direction k_x , then it continues to M and it closes on Γ . We will see in detail, in Chapter 4 (paragraph 4.2), a concrete example of 2D dispersion relation represented with the critical points. Now, that all elements related to the characterization of a periodic media are defined we can move to the concepts that belongs to Topology in order to see how the geometry of the problem can lead to interesting transport phenomena.

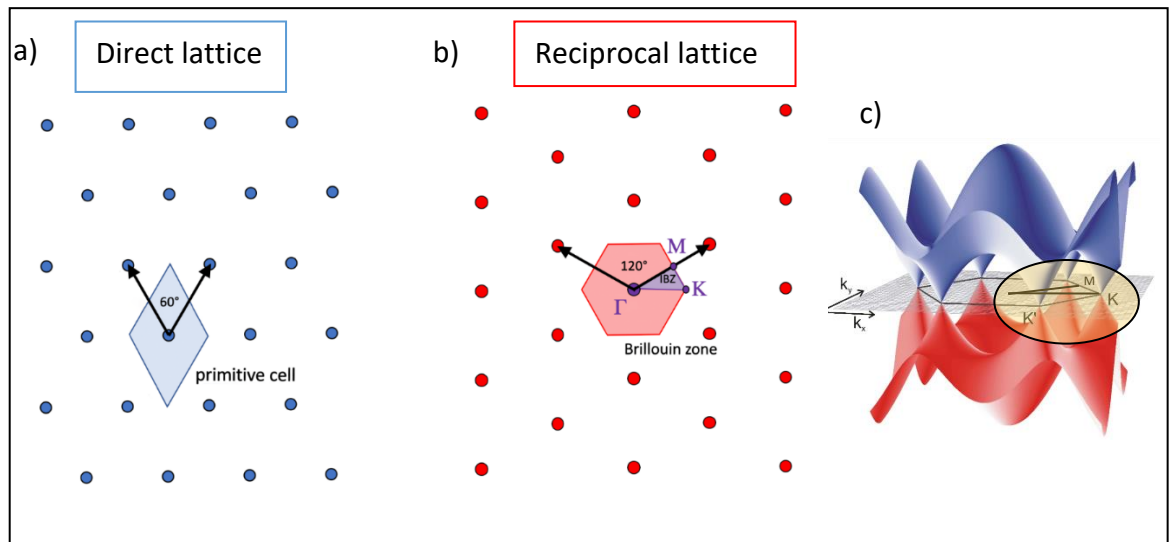


Figure 2.7: a) The triangular lattice in the direct space with the primivite vectors and primitive cell. b) The triangular lattice in the reciprocal space with its equivalent primitive vectors. The shaded red area shows the Brillouin zone while the purple one indicates the irreducible Brillouin zone (IBZ). Adapted from [57]. c) 3D dispersion relation of graphene showing the directions of the critical points (shadow yellow area). Adapted from [58].

2.6 Topology applied to Condensed Matter physics

We saw that the spatial arrangement plays an important role in term of wave propagation. In fact, this is observable in the dispersion relation which presents band gaps due to the Bragg scattering and local resonances depending on the geometrical parameters of our system. For the moment, we are able to know when the wave propagation is allowed like for a band-pass filter where the structured media behaves like an insulator in a specific frequency range. The next step is to use Topology to achieve interesting transport phenomena. Precisely, our goal is to demonstrate that we can observe a mode propagating through a path of an insulating material. In other words, we want to realise the so-called Topological Insulator TI or Quantum Spin Hall insulator (for HEWs in our case) which is globally an insulator presenting conducting states just on a line for the two-dimensional case (on a surface for the 3D case). A TI has then a double response depending on the area considered and its conducting states are a consequence of the TI's geometry.

Now, we present the main concepts to understand how to realise a TI.

2.6.1 Topological phase transition

Generally, in Physics a phase transition is related to a change of symmetry (symmetry breaking process). For example, considering the cooling process of a fluid into a crystalline solid, the continuous translation symmetry is broken. However, not all physical phenomena can be explained using this approach [59]. A very famous example that does not break any symmetry is the Quantum Hall Effect (or Quantum Hall insulator) originally observed in semiconductor heterostructures [60], [61]) and its explanations comes from a different phase transition, precisely a topological phase transition. The Quantum Hall insulator is an insulator in the bulk part presenting conducting states on the edge (in a 2D sample) when it is under a strong magnetic field. The conducting states are immune to back-scattering or dissipation and the associated Hall conductance σ_{xy} is quantized. This result cannot be explained with a symmetry breaking process like done before applying Topology to Condensed Matter Physics.

To better understand what a topological phase transition is, we need to define first a topological invariant which is a quantity that does not change (invariant) under

continuous deformations of the system. Let us consider three closed surfaces as shown in Fig. (2.8) [62]. All three are characterized by a topological invariant called genus (g) that represents the number of holes. Focusing on Fig. (2.8 a), we can see that a sphere or a spoon have the same genus $g = 0$ because both objects have no holes. For the torus and mug pair we have $g = 1$ (Fig. 2.8 b) and for the third example composed by a double torus and teapot $g = 2$.

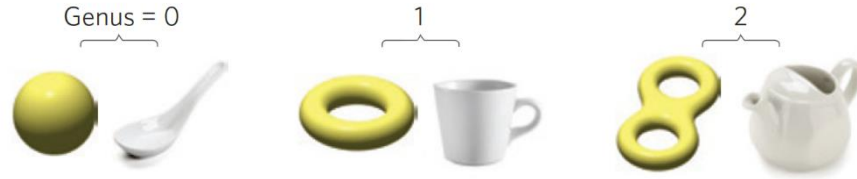


Figure 2.8: Three pairs of objects having the same topological invariant: the genus g ; $g=0$ for the pair sphere/spoon, $g=1$ for torus/mug and $g=2$ for double torus/teapot. Adapted from [62].

If we pass from the sphere to the torus, we have the so-called topological phase transition because its genus changes from 0 to 1. Interesting transport phenomena occur when we have a topological phase transition as modes immune to back-scattering. Now, that the concept of a topological phase transition is defined for a simple problem involving three different topologies, we can move to the topological invariant related to two-dimensional media spatially arranged and see how to construct a TI.

2.6.2 Topological invariant: Chern number

The topological invariant related to a band diagram is called Chern number (usually indicated with C) [64]. Physically, it is a quantity (integer $\mathbb{Z}$) that describes the global behavior of the wavefunctions on the bands and it is evaluated, considering the reciprocal space, over the Brillouin Zone which is a closed surface [62]. As we saw in paragraph (2.5), that is the part in the k -space containing all the information for a periodic media. The calculation of this topological invariant is not trivial and at the same time, it is not our purpose to calculate it because the interesting transport phenomena occur when there is a topological phase transition so, what we need is to create a periodic media where C changes like seen for the genus passing from a sphere

to a mug. Let us see with an example how we can obtain a topological phase transition. We start by the Bragg mirror presented in paragraph (2.2) that shows a band gap meaning that in a particular frequency range there is no propagation. The Bragg mirror is characterized by a zero Chern number and because of that it is often indicated as trivial mirror. If we put two Bragg mirrors one next to the other one, as done in Fig. (2.9 a) we do not have any topological transition because the topological invariant C is constant and equal to zero.

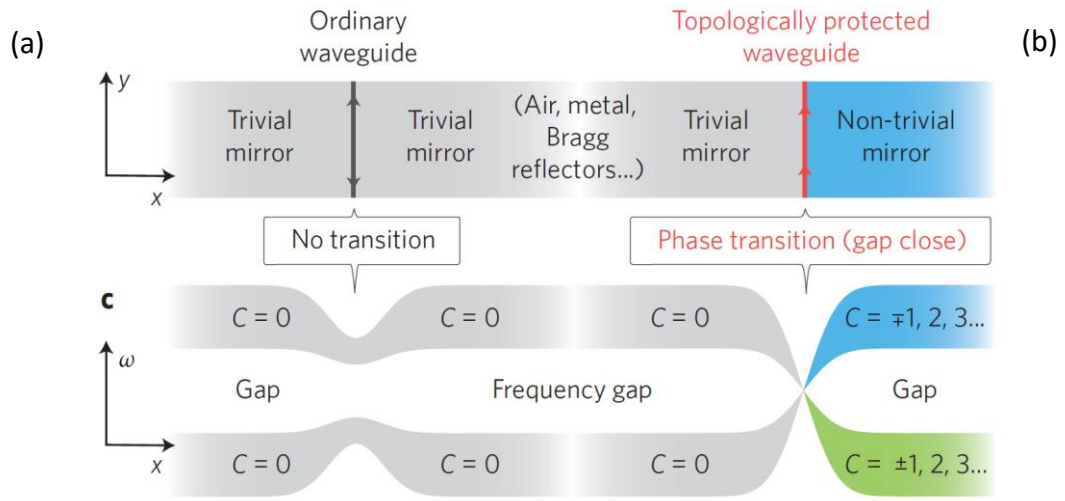


Figure 2.9: a) Ordinary waveguide obtained with trivials mirrors where no topological phase transitions occur; according to the band diagram the propagation is not allowed in a particular frequency range. b) Topological waveguide obtained combining a trivial mirror ($C=0$) with a non-trivial one ($C \neq 0$); in this case there is a topological phase transition confirmed by the variation of the topological invariant C and by the process of gap closing in the band diagram. This implies the existence of an edge mode that can propagate only in one direction as shown by the red arrow. Adapted from [62].

What we can obtain from such configuration is an ordinary waveguide for which the propagation is allowed or not according to the band diagram. Moreover, there is not a preferred direction for the wave propagation, when a mode can pass through the structured media it can propagate both ways as indicated by the double arrows on Fig. (2.9 a).

A completely different situation is represented in Fig. (2.9 b) where a trivial mirror ($C = 0$) is next to a non-trivial mirror ($C \neq 0$); in this case a topological phase transitions occurs like the topological invariant C changes on the interface between the two mirrors. In term of band diagram, it implies a gap closing process forming an

intersection in the middle of the band gap. That point represents a quadrupole degenerate state or also called 4-fold degeneracy. As confirmed by this example we do not need to calculate the Chern number, we just need to have a structured media where a topological phase transition occurs by choosing two different patterns, characterized by a different C , forming an interface that supports a topological edge mode. All that matters it is the variation of C and not the specific value assumed on each band.

2.6.3 Chern number and symmetries

The Chern number depends crucially on the symmetries of the problem, in fact, this topological invariant has a non-zero value when, a system breaks the time-reversal symmetry also called T -symmetry preserving the parity one (P -symmetry). We remind that the T -symmetry is defined as the symmetry under the reversal transformation of the temporal axis, in symbols:

$$T: t \rightarrow -t$$

Parity involves a different transformation, this time in the space domain, for which we have a sign change sign of the coordinate system, in symbols:

$$P: \begin{cases} x \\ y \\ z \end{cases} \rightarrow \begin{cases} -x \\ -y \\ -z \end{cases}$$

Breaking the T -symmetry, as the magnetic field does in a Quantum Hall insulator, could be a way to obtain a non-zero Chern number leading to a topological phase transition as seen so far. Referring to a concrete example with water waves this could be achieved employing rotating media however this is an approach that causes many experimental problems related to the control of its high rotational speed often required, viscous effects and non-linearities. Lucie Domino, during her thesis tried this approach but faced this kind of problems with a rotating tank.

Is there another way to obtain a topological phase transition that does not require the T -symmetry breaking? In other words, we need a time-reversal-invariant Quantum Hall insulator. This is possible and the idea is to mimic the Quantum Spin Hall Effect (QSHE) realizing a Topological Insulator as we will see in the next paragraph (2.6.4).

2.6.4 Topological insulator: classical equivalent

A Quantum Spin Hall insulator or also called Topological Insulator (TI) does not require the breaking of the time-reversal symmetry [65]. In fact, in such a system, there is no magnetic field that, as seen for Quantum Hall insulator, would break the T-symmetry. The latter is preserved because of the spin-orbit coupling which is a Quantum effect resulting from the interaction between the orbital angular momentum and the spin's particle. The spin-orbit coupling plays then the role of the magnetic field, in fact, the spin-up electrons will propagate in the opposite direction compared to spin-down ones even without a magnetic field so the moving electrons are guided by a spin-dependent force [66]. A Quantum Spin Hall insulator can be seen as two copies of a Quantum Hall system where, considering a 2D case, the bulk exhibits an insulating behaviour while the border presents conducting states. The latter, called edge states, because of the spin-orbit coupling, propagate in opposite directions guaranteeing a conducting behaviour on the edge of the insulator material without an external magnetic field.

Can the QSH effect have a classical equivalent? Or in other words, how the concept of spin-orbit coupling can be applied to a classical system?

Can we create a structure having an insulating behaviour in the bulk and a topological edge state for hydro-elastic waves?

The QSH effect can have a classical equivalent (in this case we will employ the terms pseudospin and pseudoatom as we are not in the Quantum domain). The full explication requires many notions, however here we will provide the main concepts to understand the logic chain conducting to the classical equivalent of the QSH effect. The main task is to realise a structure composed by two patterns with a different geometrical property that causes a topological phase transition. When a system exhibits a topological phase transition interesting transport phenomena occur as the aforementioned edge state for the QSH insulator and that is what we need.

The topological phase transition, as shown in Fig. (2.9 b) implies the process of gap closing in the band diagram or in other terms, the intersection point corresponds to a state which is four times degenerate. These are two ways to express the same concept.

Such a degenerate state exists for some lattices type, one of the most famous is the structure of graphene: the honeycomb lattice. Having a look of the literature, pseudospin-dependent Topological Insulators have been mostly realised with the honeycomb lattice [67], [68], however few studies involve a triangular lattice, which naturally sustains as well this four-fold degeneracy [69]. This is a result of crystal symmetry [70] and we take advantage of it to realise our crystal for hydro-elastic waves based on the triangular configuration. As many studies involve already the honey-comb lattice, we decided to choose the triangular lattice in order to contribute to this new domain where hydro-elastic waves meet Topology.

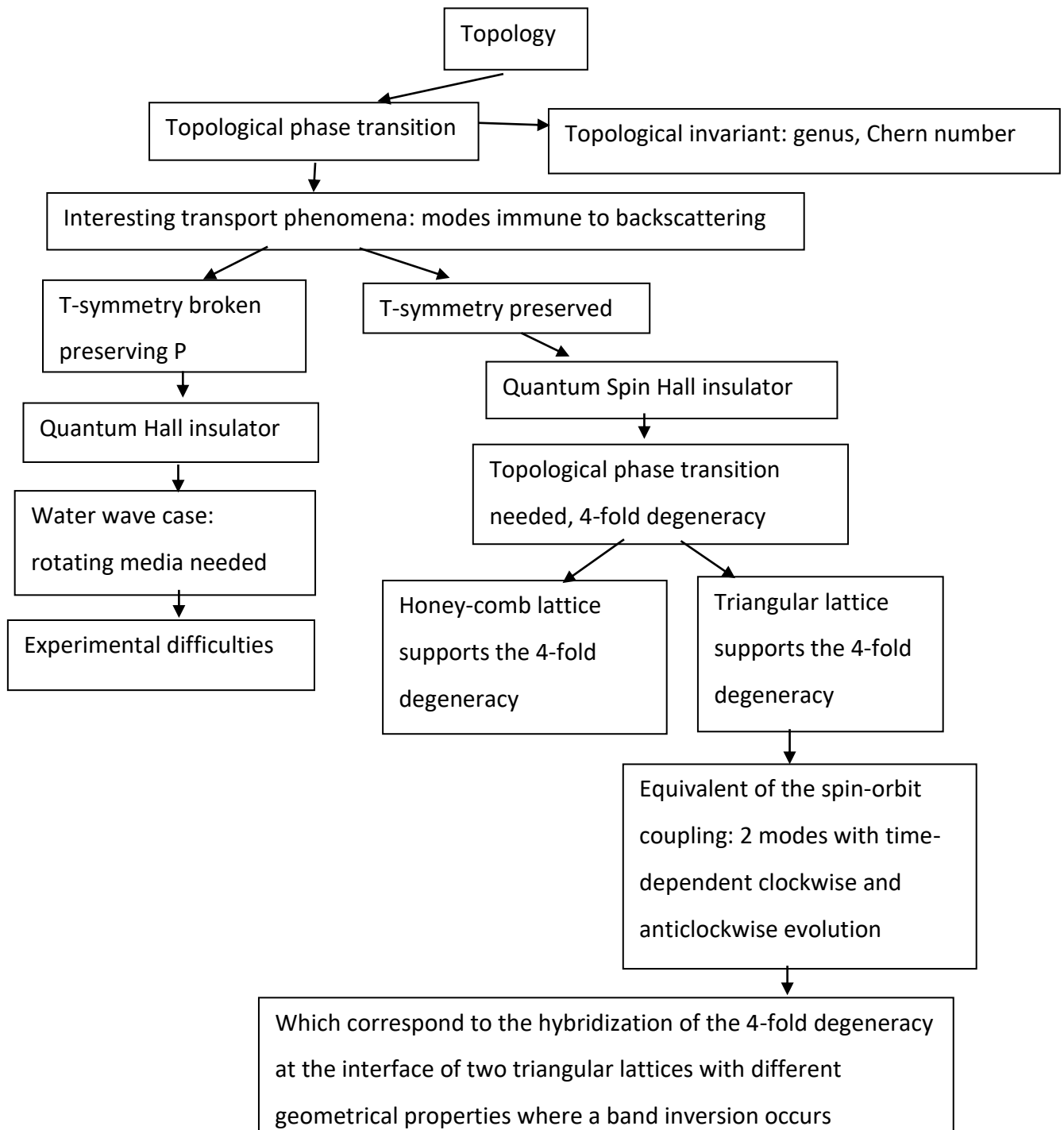
For the moment what we have is a structure that involves a topological phase transition, but as we need to preserve the time-reversal symmetry or in other words as we do not employ rotating media, we still have to find a classical equivalent for the spin-orbit coupling. What we need is a couple of modes with a time-dependent clockwise and anticlockwise evolution (also called helical states), imitating the spin-orbit coupling which implies, as we saw, the propagation of the spin-up electrons in the opposite direction compared to spin-down ones. These states are the closest representation of such coupling.

So, now the new question is: how can we create that couple of modes?

The four degenerate modes, resulting from the triangular lattice, play a crucial role. In fact, they are responsible of the realisation of that couple of modes needed to imitate the QSH effect. Precisely the 4-fold degeneracy corresponds to two particular pairs of modes called dipolar modes (also p state or orbital) and quadrupolar modes (also d states or orbital) imitating the p and d states of electrons. Now, focusing on a triangular lattice, by changing the geometrical properties of the pseudoatoms (like radius, size etc. of the unit cell) of the insulator, we can obtain a band diagram where the dipolar modes are in the lower band (lower frequencies) and the quadrupolar modes in the higher band (higher frequencies). Or, by changing again the geometrical properties such as radius, size of the unit cell, we can have the dipolar modes at higher frequencies and quadrupolar modes at low frequencies. In other words, the geometry tuning of the unit cell implies the flipping of the p and d modes (called band inversion). Putting together the two insulators, at the interface the two doublets p and d hybridize

[71], [72], forming the helical edge states (or conducting states) that translate the concept of spin-orbit coupling in a classical system. It is equivalent to say that when a band inversion occurs, we have a topological phase transition which means that the system considered can support an edge conducting state exhibiting an insulating behaviour in the bulk.

In the Chapter 4 we will show and discuss a real band diagram (not a sketch) to visualize all concepts presented so far, then we will finally show the crystal made for HEWs. We propose here a scheme of all steps involving Topology:



2.7 Conclusion

In this chapter we described how waves propagate through a structured media and how the dispersion relation can be modified by the spatial arrangement of the lattice considered and by the local resonances. Consequently, two types of gaps can be observed, type Bragg and hybridization. We started with the easiest example of 1D crystal, the Bragg mirror, providing the explanation based on the Bloch-Floquet's formalism. We illustrated the relevant properties of the 2D triangular lattice in direct and reciprocal space, showing how to obtain the Brillouin Zone, the area of the dispersion that contains all the information. Then, we saw how Topology can be applied to Physics leading to new interesting transport properties. In particular, we discussed the importance of the topological phase transition which explains the Quantum Hall effect. We presented different strategies based on the T-symmetry breaking in the first case, like for a Quantum Hall insulator and preserving the T-symmetry in the second case, like for a Quantum Spin Hall insulator. The latter is also called Topological Insulator and an equivalent for a classical system can be achieved employing a triangular lattice which naturally sustains a four-fold degeneracy at the BZ center, necessary condition for the QSH effect.

Chapter 3: Hydro-Elastic Waves: characterization

3.1 Introduction

In the previous chapter, we introduced the essential tools to understand how concepts coming from Topology can be applied to Physics. We want to focus now on the physical context of the Hydro-Elastic Waves of which we remind the definition: the wave propagating in a system composed by a liquid and a thin elastic sheet floating on its surface is a HEW. The interaction of ocean wave/sea-ice is a natural example of HEW (*i.e.* continuous ice like in central Arctic or individual ice floes [73] [74]). In this chapter we will see why this kind of wave is an excellent platform to observe various classes of topological phenomena starting by the attenuation and the dispersion relation of a HEW and incorporating the concept of cavity resonator.

3.2 Experimental dispersion relation of a HEW

As shown previously, the dispersion relation of an HEW in our system is:

$$\omega^2 = \left(\frac{D}{\rho} k^5 + \frac{T}{\rho} k^3 + gk \right) \tanh(kh_0) \quad (3.1)$$

where $\omega = 2\pi f$ is the angular frequency, $k = 2\pi/\lambda$ is the wavenumber, g is the gravitational acceleration, T is the mechanical tension in the elastic sheet, ρ is the fluid density, D is the flexural modulus of the elastic sheet and h_0 is the fluid depth [9], [75].

The previous relation is valid in the limit of small wave amplitude compared to the elastic sheet's thickness, in our case 500 μm .

Considering the deep water condition $\lambda \ll h_0$, eq. (3.1) reduces to:

$$\omega^2 = \frac{D}{\rho} k^5 + \frac{T}{\rho} k^3 + gk \quad (3.2)$$

In this experiment we use wavenumbers such that $k > k_{TD} = \sqrt{T/D}$ so the first term of eq. (3.2), distinctive of the HEW, is dominant:

$$\boxed{\omega^2 \sim \frac{D}{\rho} k^5} \quad (3.3)$$

As shown in eq. (3.3) we are able to modify the dispersion relation by changing the flexural modulus D which depends on properties of the elastic sheet used [28]:

$$D = Ed^3/12(1 - \nu)^2$$

where d is the membrane's thickness, E its Young's modulus and ν its Poisson's ratio. The elastic sheet is a thin (500 μm) silicone rubber film, translucent, (40° shore A hardness, $\nu=0,5$) provided by Silex [76]. We focus now on the Young's modulus, which is the measurement of a material's elasticity.

Using an extensometer (Instron 5865, see Fig. 3.1 a) it is possible to reconstruct the stress σ -strain ϵ relationship of a membrane's specimen via the related load-extension graph. In the elastic deformation region (known as Hooke's Law) E is simply the slope of the stress-strain graph. Fig. (3.1 c) shows the result of 4 tests of a 10 cm (L_0) high x 7 cm membrane with 500 μm of thickness.

Considering the expression of σ :

$$\sigma = \frac{F}{S} = E \epsilon = E \frac{\Delta L}{L_0}$$

$$E = \frac{F L_0}{S \Delta L}$$

The Young's modulus for our specimen is $1,6 \pm 0,1 \text{ MPa}$ and we will use this value in the dispersion relation eq. (3.2) (obtained via linear fit, see Fig. (3.1 c)).

We previously defined E like the slope of the stress-strain graph in the elastic deformation region, that is true for brittle materials like metals. Our membrane, being an elastomer, does not present a constant Young's modulus, but depends on the strain of the material. Rubber-like substances are generally treated with the Neo-Hookean model [77], [78], just for low strains we can assimilate their behavior to a linear one.

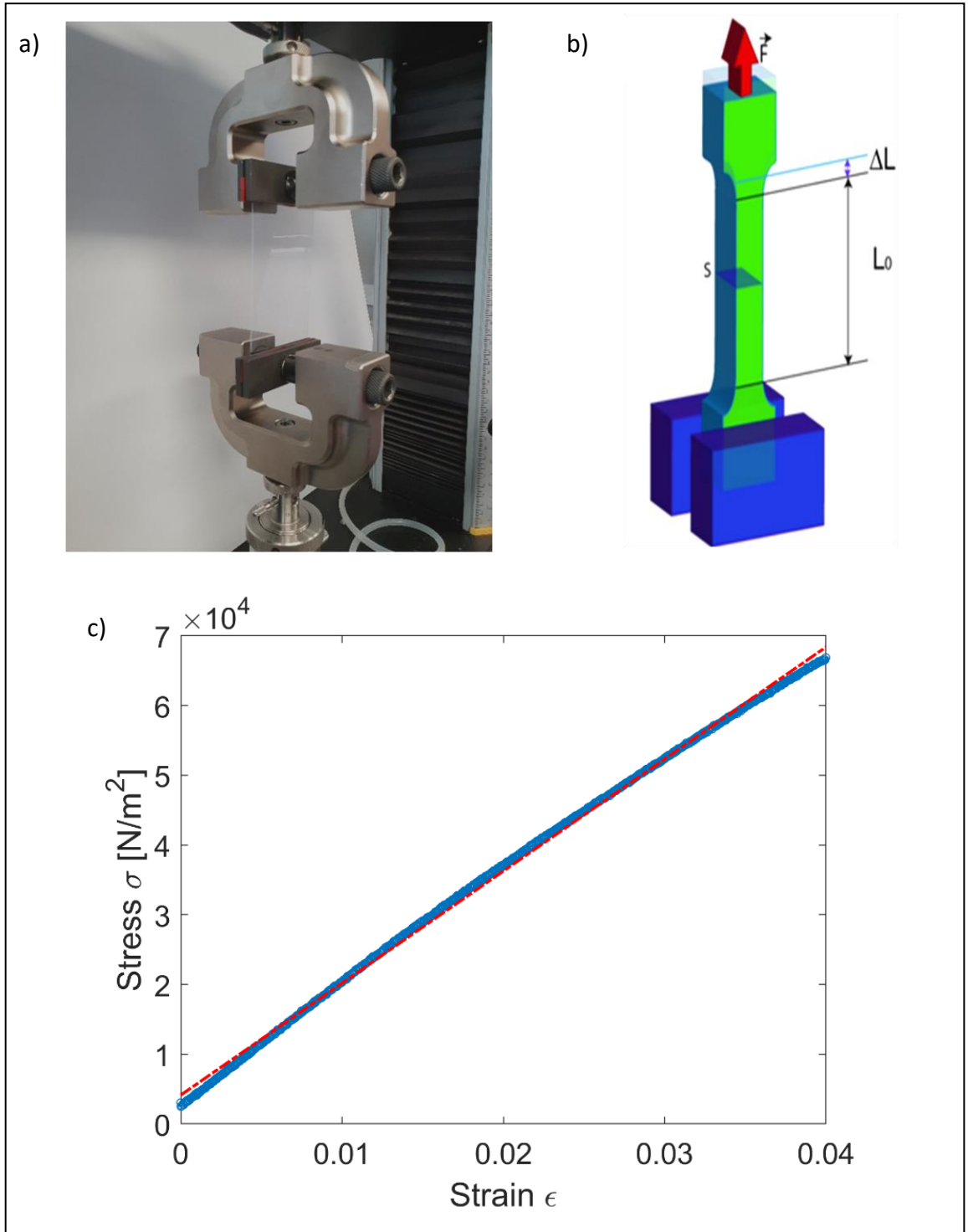


Figure 3.1: a) Extensometer Instron 5865 used to obtain the Load-extension diagram of our specimen. b) Scheme to calculate the Young's modulus, the green part corresponds to the specimen to test, adapted from [79]. c) Stress-strain diagram for the silicone membrane of thickness $d = 500 \mu\text{m}$ used in our experiments (blue dots), the red line is the linear fit.

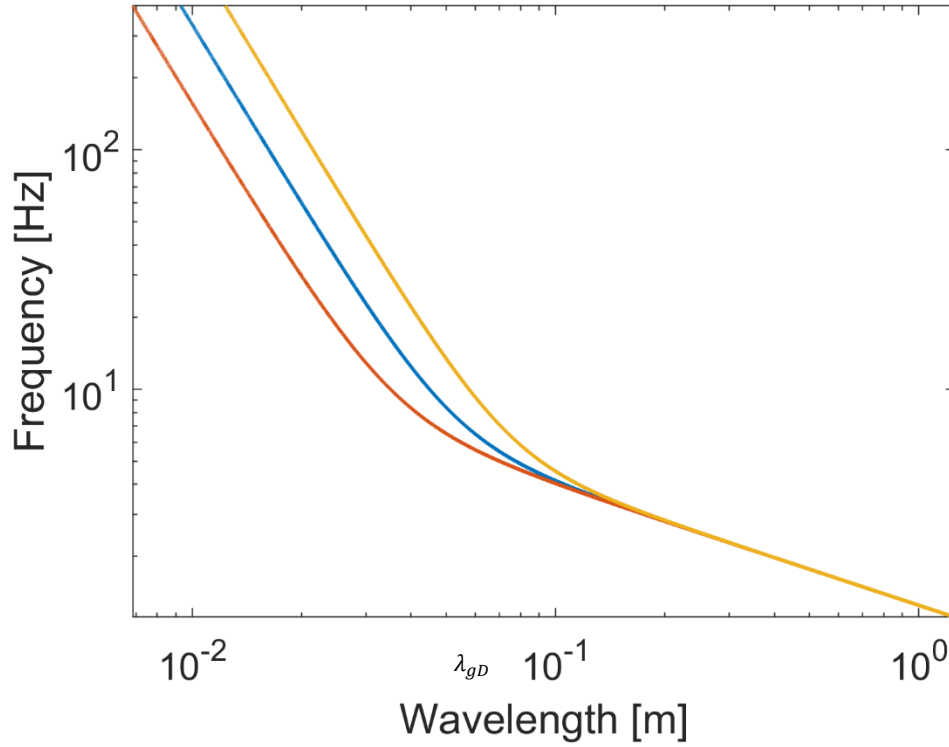


Figure 3.2: Dispersion relation of HEW for three different thickness values, red 300 μm , blue 500 μm and yellow 800 μm .

The theoretical dispersion relation of a HEW (Eq. 3.2) is shown on Fig. (3.2), for three different membrane's thicknesses ($d = 300, 500$ and $800 \mu\text{m}$, orange, blue and yellow respectively), adopting frequency and wavelength for the axis ($\omega/2\pi \rightarrow f, 2\pi/k \rightarrow \lambda$) for a more direct physical legibility. In the range of wavelengths where the flexural term is dominant ($1 - 10 \text{ cm}$) we can observe that the three dispersion relations are vertically shifted, it means that we can have a control over it just by changing d contained in the flexural modulus D (plotted using the following values: $g=9,81 \text{ m/s}^2$, $T=0,073 \text{ N/m}$, $E=1,6 \text{ MPa}$ and $\rho=1000 \text{ kg/m}^3$, $\nu=0,5$).

The same plot can be obtained experimentally: we proceed by filling up with water the glass tank up to 17 cm high; we put a membrane of the tank size on the water as illustrated on Fig. (1.3 b). The source generator is a straight paddle that generates plane waves in the frequency range 5-150 Hz. For each frequency we record 20 frames (CCD $\text{fps} = 89,0$) and we reconstruct the corresponding height field via the free-surface synthetic Schlieren (1.4.2). Now, we use a temporal Fourier transform to select only the frequency component equal to the frequency used by the source wave generator, in other words we filter the signal at the working frequency. We apply a

spatial Fourier transform on the filtered result and we obtain the distribution of energy in the k -space for a given temporal frequency. As expected the energy is localized in k -space corresponding to the propagation of a monochromatic plane wave. Repeating this for each frequency and piling all lines in the k -space we obtain the black lines in Fig. (3.4 a) which represents the experimental dispersion relation frequency-wave vector. In order to keep the same graphical representation for the dispersion relation, we reconstruct the diagram (f, λ) extracting k from the diagram (f, k) and converting it in wavelengths ($\lambda = 2\pi/k$). Fig. (3.4 b) shows the comparison between the theoretical dispersion relation of a HEW using the static value of E (green line) and the experimental results via Fourier transform (blue points). We can observe that for the low frequencies ($f < 10 \text{ Hz}$) there is a good agreement, then the two curves diverge. Having a deeper look on the behavior of elastomeric materials we discover that the Young's modulus depends on the frequency and varies then when it is excited by sin waves like in our case, implicating an important variation up to the 100% compared to the static case [80]. Applying a 100% variation to the static value we obtain a good agreement between the fit (red line) and our experimental data. The dynamical Young's modulus is two times the static one so, $E = 3,2 \text{ MPa}$ is the value that we will use for the rest of the thesis.

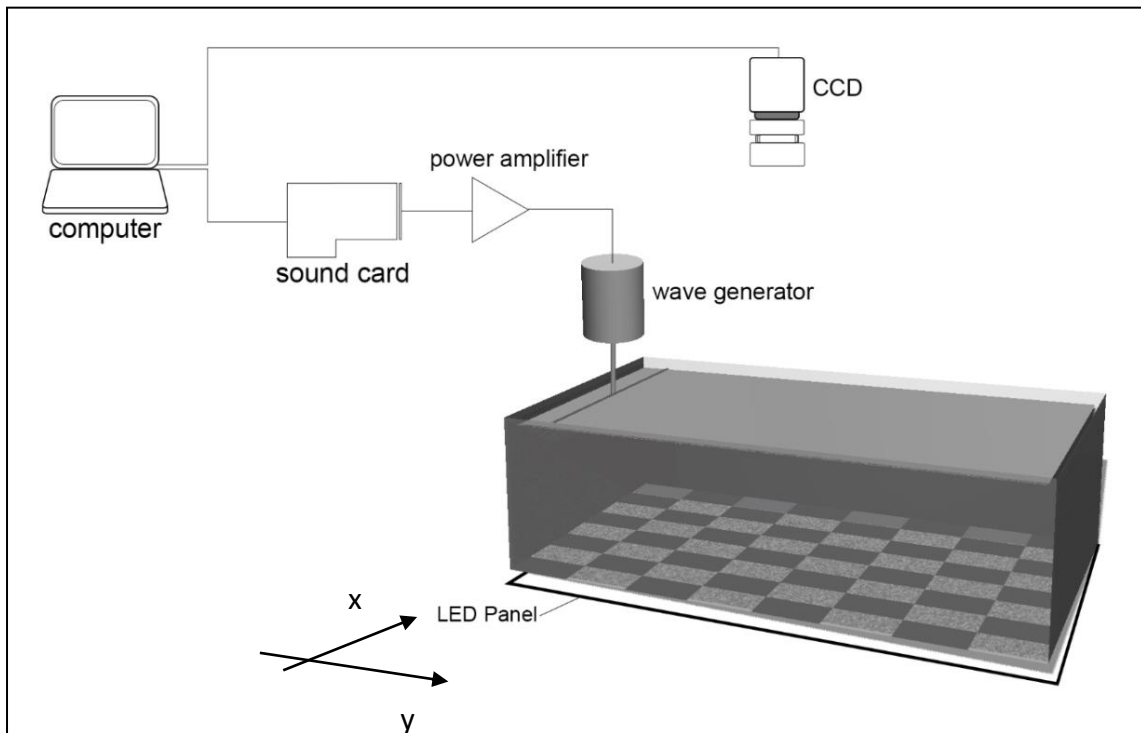


Figure 3.3: Scheme of the experimental setup to evaluate the dispersion relation of a HEW.

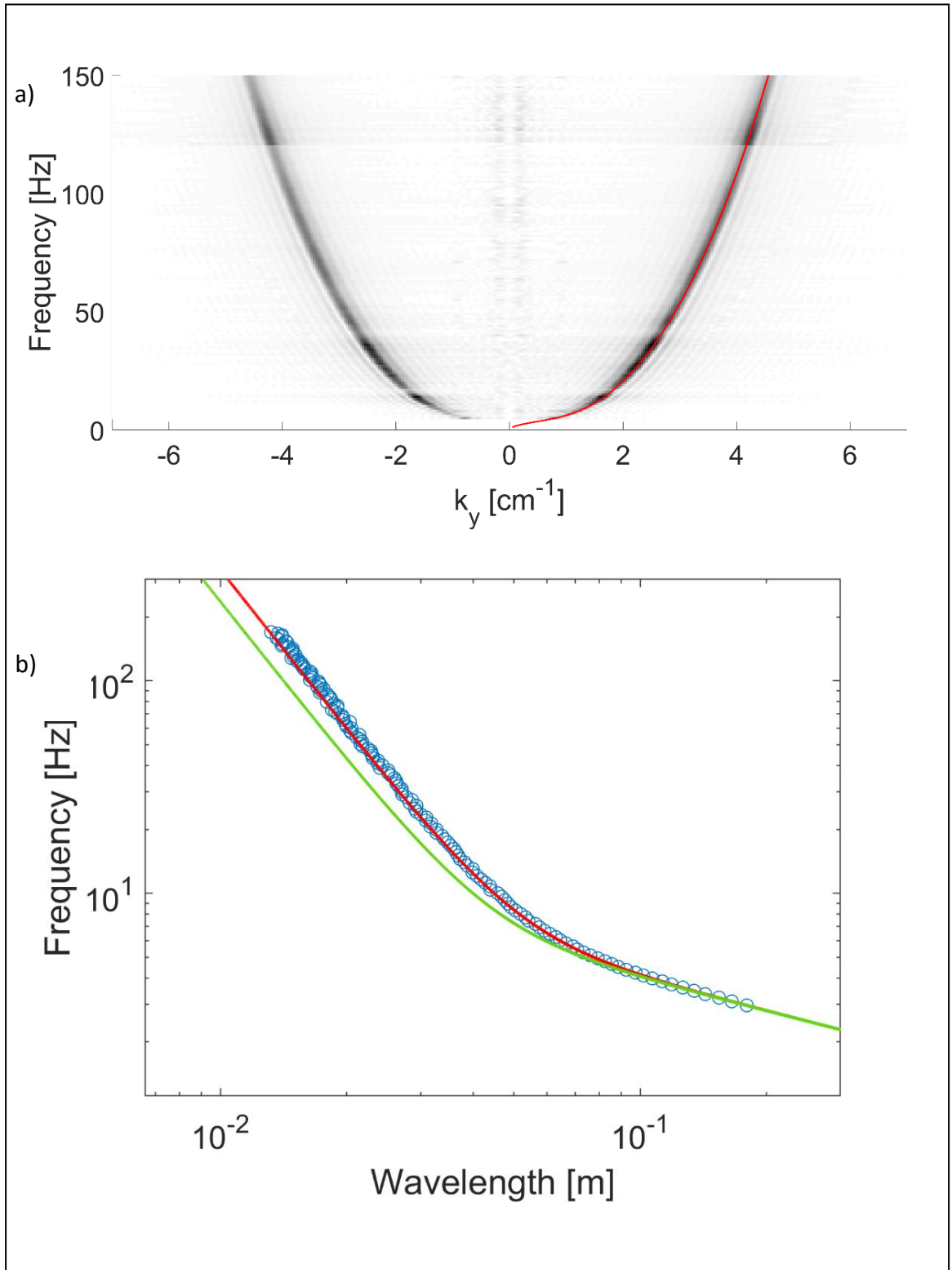


Figure 3.4: a) Theoretical dispersion relation (f,k) using the dynamical value for $E=3.2$ Mpa and the experimental result with via Fourier transform. B) Dispersion relation (f,λ) obtained via Fourier transform compared to the theoretical expression one using the static Young's modulus $E=1.6$ MPa (green line) and the dynamical one (red line).

3.3 Attenuation of a HEW

Attenuation plays a prominent role in water wave systems. Surface waves are characterized by a frequency-dependent attenuation length $L(f)$ defined from the expression of usual exponential function [81], [82]:

$$A(y, f) = A_0 e^{-\alpha y} = A_0 e^{-\frac{y}{L(f)}}$$

where A_0 is the maximum magnitude of the wave, α the attenuation coefficient and y the position along the propagation axis.

To experimentally study the attenuation of a HEW we use the same setup previously seen to obtain the dispersion relation (working frequency range 10-120 Hz). For each frequency we record 34 frames (CCD $fps = 104,6$) and we reconstruct the corresponding height field. The image obtained (see Fig. 3.5 a) can be seen like a matrix $M(x, y)$ where each column is represented by a damping sinusoidal function as:

$$I(y) = I_0 e^{-\alpha y} \sin(ky + \varphi)$$

where I_0 is the maximum magnitude of the wave and φ is a phase shift providing a proper beginning to the fitting curve [83]. We average the image along the x -axis and we obtain a vector containing the signal or wave's profile (see Fig. 3.5 b, example at $f=17$ Hz). We repeat the average operation over each frame, discarding noisy frames, and then we repeat the whole procedure for each frequency. We then determine the attenuation length, by fitting an exponential to the local maxima of the wave profile. The attenuation length for the frequency range 10-120 Hz is shown in Fig. (3.5 c).

We can observe that the high-frequency components dropped faster compared the low-frequency ones. The attenuation length for HEW is much bigger compared to the free surface classical water wave system ($L_{HEW} \gg L_{free}$). This allows us to study and make extended patterns on the silicone sheet which interact with the incoming wave without compromising attenuation losses [84] over a range of 10-20 cm.

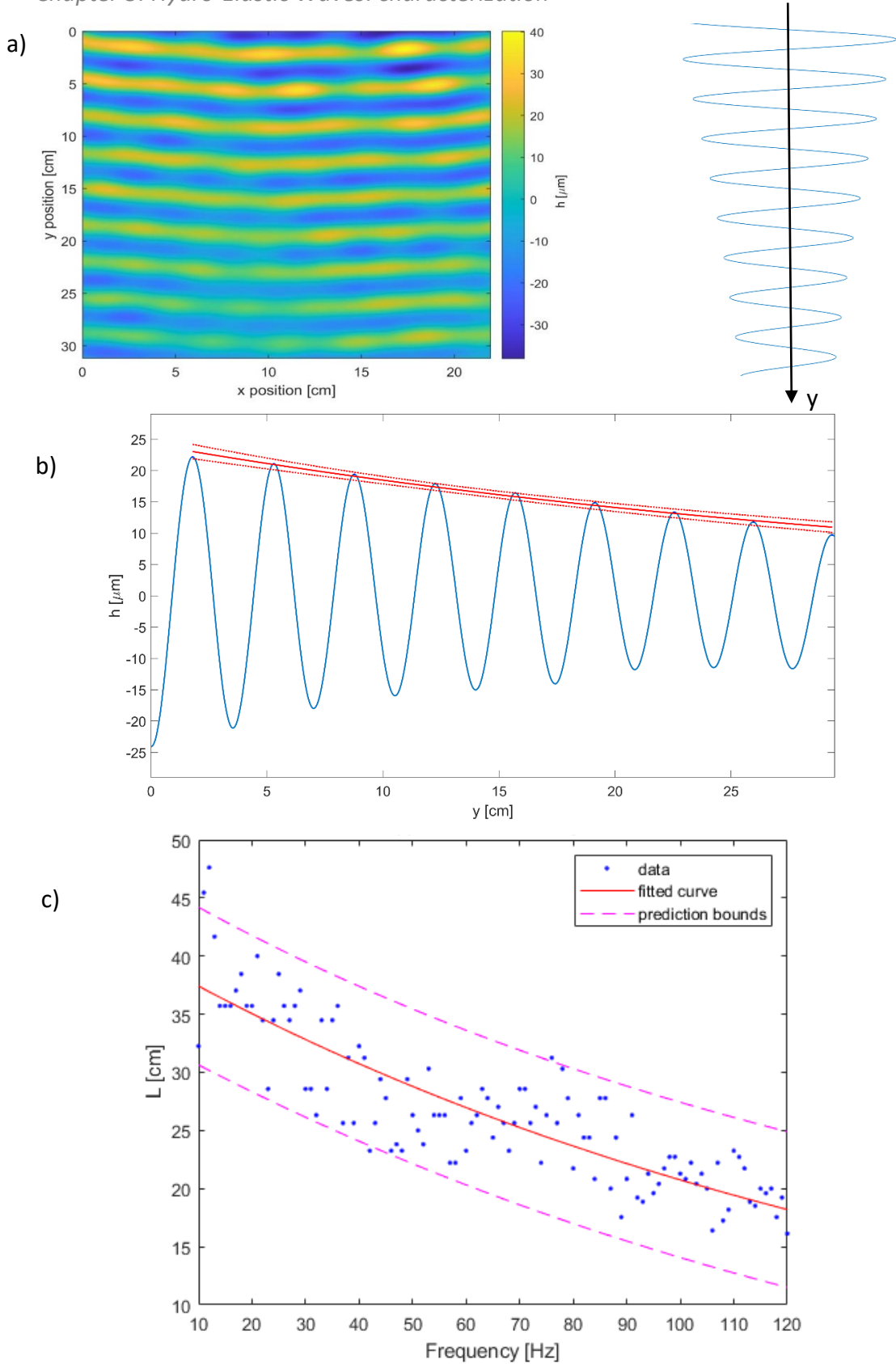


Figure 3.5: Example of height field reconstruction to evaluate the attenuation of a HEW. b) Attenuation profile for a wave at $f=17$ Hz. c) Attenuation length $L(f)$ for the frequency range 10-120 Hz with prediction bounds (fuchsia lines); the fit is performed with a second order polynomial equation $R^2 = 0,74$.

Another approach, based on a modulation technique is also possible as we recorded a film. Precisely, we apply a temporal Fourier transform using all frames over a vertical line in the center of the frame filtered at the input frequency. We call the result of the temporal Fourier transform $\hat{H}$. Then, we perform an exponential fit (Ae^{bx}) on $abs(\hat{H})$ in order to extract the attenuation length via the fit parameter b . In Fig. (3.6 a) we can see an example of the real part of $\hat{H}$ which is the result of the temporal Fourier transform applied to the sequence of frames filtered at $f = 57\text{Hz}$ (similarly to the previous method). Then, repeating this procedure for each frequency, as previously done, we plot $L(f)$ as shown in Fig. (3.6 b). Both techniques provide a consistent result with some differences: for the low frequencies ($f < 60\text{ Hz}$) the demodulation technique response results in a higher spread of data because of the reflected waves in

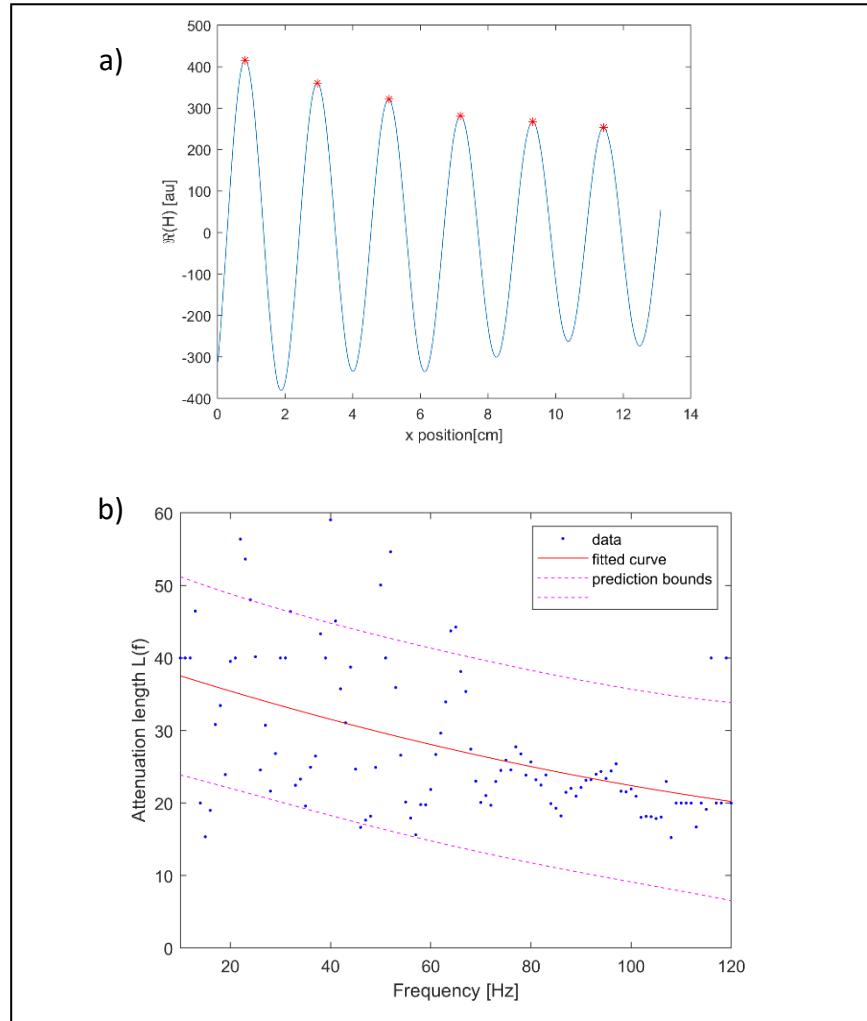


Figure 3.6: a) Real part of the result of the temporal Fourier transform over the whole sequence of frames, filtered at $f=57\text{ Hz}$. b) Attenuation length obtained via demodulation where the red line indicated the the fit performed with a power law equation.

the tank; in fact, as the attenuation length is higher for the low frequencies, the reflected waves disturb this attenuation measure. However, for the high frequencies, as the reflected waves are attenuated the response is less spread out compared to the previous method. Probably a bigger tank would improve this attenuation measure via the demodulation technique. For the moment we can observe that we have a decreasing behaviour but we cannot state about a precise function to describe $L(f)$, even though, the fit is performed with a power law equation (Cx^d : $d = 0,7 \pm 0,4$, $R^2 = 0,28$) inspired by the Stokes's law of sound attenuation for which $L(f) \sim f^{-2}$; the fit does not confirm such a behavior. More experiments are necessary to further investigate it.

In order to verify that the attenuation has a less significant impact to HEWs we can compare the characteristic time associated to a HEW with the open-water case. The characteristic time is usually defined to be the time in which a quantity, in this case the wave's amplitude, decreases by $1/e$. In order to write its expression, we can take advantage of what measured so far, the attenuation length:

$$\tau = \frac{L}{v_\phi}$$

where $v_\phi = \omega/k$ is the phase velocity of the wave.

Now we need to determine the phase velocity for a HEW. Considering the dispersion relation of a HEW, where the bending term is dominant (Eq. 3.3) we have:

$$\omega = \sqrt{\frac{D}{\rho}} k^{\frac{5}{2}}$$

So the phase velocity of a HEW writes:

$$v_\phi = \frac{\omega}{k} = \frac{1}{k} \left(\sqrt{\frac{D}{\rho}} k^{\frac{5}{2}} \right) = \sqrt{\frac{D}{\rho}} k^{\frac{3}{2}}$$

The expression of the characteristic time associated to a HEW is:

$$\tau = \frac{L}{v_\phi} = \frac{L}{\sqrt{\frac{D}{\rho}} k^{\frac{3}{2}}}$$

For the open-water case the characteristic time is [85]:

$$\tau_{water} \sim \frac{1}{2\nu_w k^2}$$

where ν_w is water dynamic viscosity at 20 °C.

For further details about the derivation of the characteristic time for water waves, all details are in the J. Lighthill's work [85]. Now that we have both expressions, we can plot the experimental data obtained from the attenuation measure previously seen. In Fig. 3.7 (plotted using the following values: $g=9,81 \text{ m/s}^2$, $T=0,073 \text{ N/m}$, $E=3,2 \text{ MPa}$ and $\rho=1000 \text{ kg/m}^3$, $\nu=0,5$), the red line indicates the characteristic time for a HEW which is always bigger than the open-water case (blue line). In other words, a HEW attenuates in a longer time compared to water waves. We can also observe that the attenuation losses, even for HEW, become too significant when $k > 4 \text{ cm}^{-1}$ or equivalently 110 Hz, that is why most of our experiments are conducted in the frequency range 10-120 Hz.

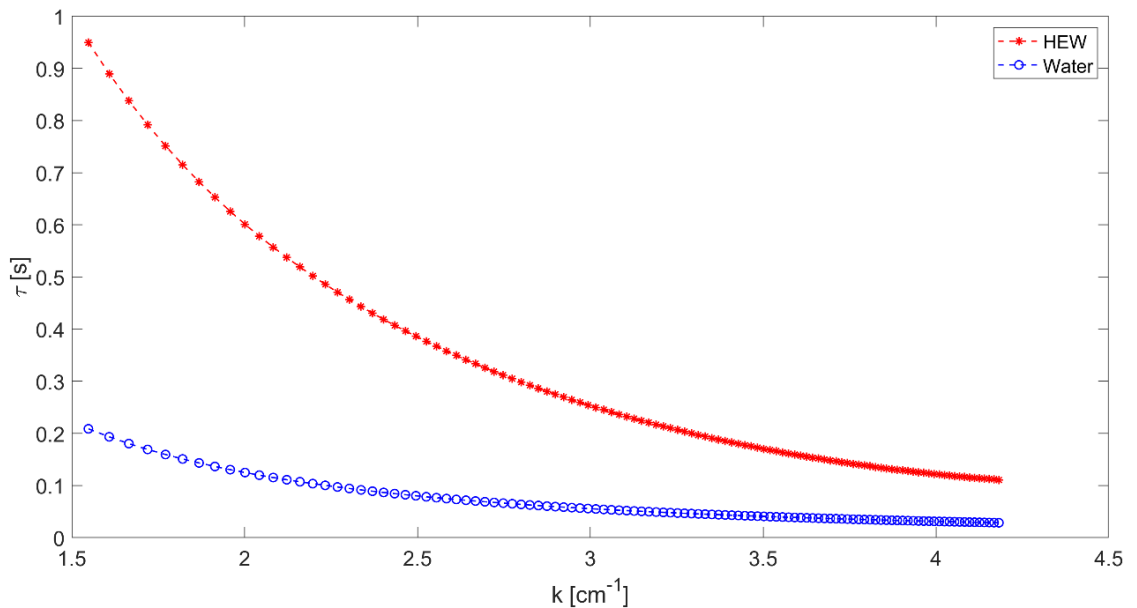


Figure 3.7: Characteristic time comparison between HEW (red line) and open water (blue line).

3.4 Singular triangular cavity

So far, we focused on the membrane's characterization and verified that we are able to modify the dispersion relation of HEW by changing, for example, its thickness. This allows us to have a control over the wave propagation which is less affected by the attenuation compared to the open water case.

The goal of this thesis is to study the interactions between artificial crystals, which are regular perforations on the membrane, and HEW. The high attenuation length, in the frequency range 10-120 Hz, assures as a suitable platform to study interactions phenomena that would be slightly detectable in a typical water wave system.

Our artificial crystals are made by perforations and each of them plays an important role that we are going to analyze in this paragraph. In particular, we will see that a simple perforation acts like a resonator leading to interesting properties when multiple resonators interact with a HEW. Then, in the next chapters (4 and 5) we will study the behavior of all resonators together, or in other words a 2D system with spatial structure.

It is worth mentioning that Lucie Domino studied also artificial crystals during her thesis [9], using a different shape, a circular hole, for her resonators (also called cavities). Because of its rotational symmetry the response of those cavities is isotropic.

In order to show, interesting transport properties for our crystals, it is important to consider a shape without the same rotational symmetry. The shape selected for our study is the equilateral triangle (among many other options like *i.e.* ellipses). The reason for that choice is that we need an anisotropic response of the cavity to create crystals, with interesting transport properties, that we will present in Chapter 4.

The first step before studying periodic patterns is to characterize the interaction between a plane HEW and a single triangular cavity. We anticipate resonance effects when the frequency of the incoming wave matches that of one of the eigenmodes of the cavity. The Dirichlet eigenproblem for an equilateral triangle was first solved by Lamé considering the vibrational modes of an elastic membrane [86], [87], [88].

The D'Alembert equation for the wave amplitude z (here the free surface height within the cavity) is:

$$\frac{\partial^2 z}{\partial t^2} = c^2 \Delta z \quad (3.1)$$

where c is the phase velocity of the wave. We consider Dirichlet boundary conditions taking $z = 0$ at the edge which is clearly an approximation for two reasons: first in our system the wave propagating through the membrane changes slightly the vertical position of the cavity's boundary, second the cavity's border is responsible for the energy transfer between the membrane and the free surface within the cavity. It is worth mentioning that the coupling between the incoming wave in the elastic membrane and the oscillating free surface within the cavity is due to the contact line at the edge of the cavity and due to the continuity of the liquid flow field below the surface. Here, we ignore the details of this coupling and analyse the possible eigenmodes of this cavity.

Lamé remarkably showed that its solution can be expressed with a combination of sines and cosines, here we present just the main steps of his arguments to find the solution of the Eq. (3.1).

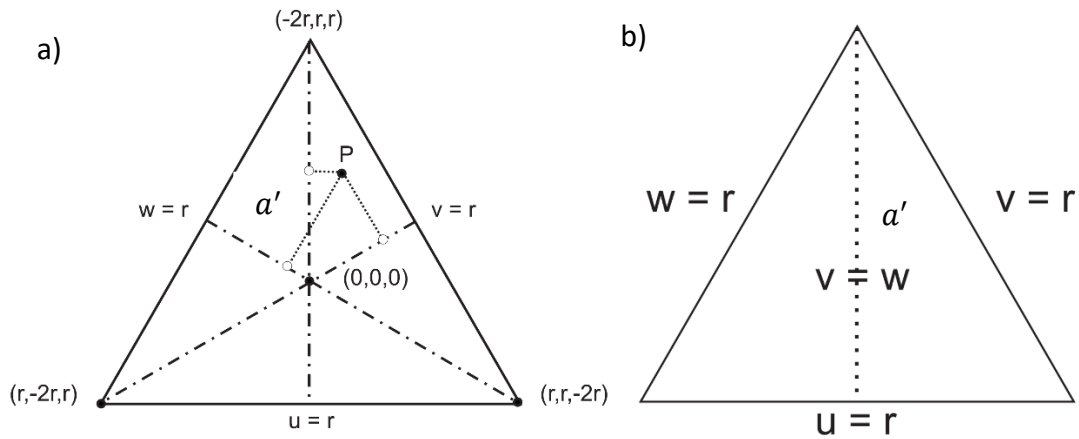


Figure 3.8: a) Triangular coordinate system. b) Modal line of symmetry/antisymmetry, adapted from [89].

Lamé first defined the triangular coordinates (u, v, w) of a point P , Fig. (3.8 a), by

$$u = r - y$$

$$v = \frac{\sqrt{3}}{2} \left(x - \frac{h}{2} \right) + \frac{1}{2} (y - r)$$

$$w = \frac{\sqrt{3}}{2} \left(\frac{h}{2} - x \right) + \frac{1}{2} (y - r)$$

where h is the side of the triangle and $r = h/2\sqrt{3}$ its inradius. The triangular coordinates satisfy the following relation:

$$u + v + w = 0$$

Its use simplifies the application of boundary conditions since the three sides of the triangle write $u = r, v = r$ and $w = r$.

The wave equation can be temporally and spatially separated leading to the spatial Helmoltz equation:

$$\Delta\psi(x, y) + k^2\psi(x, y) = 0, \quad (x, y) \in \tau;$$

where τ is the the equilateral triangle. For the Dirichlet boundary condition $\psi(x, y) = 0, (x, y) \in \partial\tau$.

Introducing the orthonormal coordinates (ξ, η) :

$$\xi = u, \eta = v - w$$

the Helmoltz equation can be written as

$$\frac{\partial^2 \psi}{\partial \xi^2} + 3 \frac{\partial^2 \psi}{\partial \eta^2} + k^2 \psi = 0$$

if we look for a separated solution as

$$f(\xi) \cdot g(\eta)$$

we obtain:

$$f'' + \alpha^2 f = 0;$$

$$g'' + \beta^2 g = 0;$$

$$\alpha^2 + 3\beta^2 = k^2$$

so we have spatially separated solutions of the form:

$$\psi(u, v, w) = f(u) \cdot g(v - w)$$

where f and g are trigonometric functions.

The next step is to write ψ decomposing it into symmetric and antisymmetric parts about the location $v = w$, Fig. (3.8 b)

$$\psi(u, v, w) = \psi_s(u, v, w) + \psi_a(u, v, w),$$

where

$$\psi_s(u, v, w) = \frac{\psi(u, v, w) + \psi(u, w, v)}{2}, \quad \psi_a(u, v, w) = \frac{\psi(u, v, w) - \psi(u, w, v)}{2}$$

denoting ψ_s the symmetric mode and ψ_a the antisymmetric mode.

Lamé demonstrated that $\psi(x, y)$ can be represented by trigonometric series (fundamental theorem) which has consequences about the determination of the domains that have trigonometric eigenfunctions. The equilateral triangle allows to construct such a family of eigenfunctions.

The last step is the constructions of the symmetric and antisymmetric mode, starting by the first one; ψ_s vanishes when $u = -2r$ and $u = r$ while being even as a function of $v - w$ so Lamé seeked for a solution as:

$$\sin \left[\frac{\pi l}{3r} (u + 2r) \right] \cdot \cos [\beta_1 (v - w)], \quad (3.2)$$

where l is an integer and $\left[\frac{\pi l}{3r} \right]^2 + 3\beta_1^2 = k^2$.

Only one term, or two terms with this form cannot satisfy the Dirichlet boundary condition along $v = r$ [89]. Nonetheless, the sum of three terms as Eq. (3.2) can:

$$\begin{aligned} & \sin \left[\frac{\pi l}{3r} (u + 2r) \right] \cdot \cos [\beta_1 (v - w)] + \sin \left[\frac{\pi l}{3r} (u + 2r) \right] \cdot \cos [\beta_2 (v - w)] \\ & + \sin \left[\frac{\pi l}{3r} (u + 2r) \right] \cdot \cos [\beta_3 (v - w)] \end{aligned}$$

with

$$\left[\frac{\pi l}{3r}\right]^2 + 3\beta_1^2 = \left[\frac{\pi l}{3r}\right]^2 + 3\beta_2^2 = \left[\frac{\pi l}{3r}\right]^2 + 3\beta_3^2 = k^2 \quad (3.3)$$

Using trigonometric identities and the relation $l + m + n = 0$ the symmetric mode is given by:

$$\begin{aligned} T_s^{m,n} = & \sin\left[\frac{\pi l}{3r}(u + 2r)\right] \cdot \cos\left[\frac{\pi(m - n)}{9r}(v - w)\right] \\ & + \sin\left[\frac{\pi m}{3r}(u + 2r)\right] \cdot \cos\left[\frac{\pi(n - l)}{9r}(v - w)\right] \\ & + \sin\left[\frac{\pi n}{3r}(u + 2r)\right] \cdot \cos\left[\frac{\pi(l - m)}{9r}(v - w)\right] \end{aligned} \quad (3.4)$$

Similar arguments are used for the construction of antisymmetric mode that we simply report here, refer to Lamé's work for further details [86], [87], [88]:

$$\begin{aligned} T_a^{m,n} = & \sin\left[\frac{\pi l}{3r}(u + 2r)\right] \cdot \sin\left[\frac{\pi(m - n)}{9r}(v - w)\right] \\ & + \sin\left[\frac{\pi m}{3r}(u + 2r)\right] \cdot \sin\left[\frac{\pi(n - l)}{9r}(v - w)\right] \\ & + \sin\left[\frac{\pi n}{3r}(u + 2r)\right] \cdot \sin\left[\frac{\pi(l - m)}{9r}(v - w)\right] \end{aligned} \quad (3.5)$$

In order to verify if this is an orthonormal set of eigenfunctions the two equations (3.4) and (3.5) can be transformed using trigonometric identities in an alternative form:

$$\begin{aligned} T_s^{m,n} = & \frac{1}{2} \left\{ \sin\left[\frac{2\pi}{9r}(lu + mv + nw + 3lr)\right] + \sin\left[\frac{2\pi}{9r}(nu + mv + lw + 3nr)\right] \right. \\ & + \sin\left[\frac{2\pi}{9r}(mu + nv + lw + 3mr)\right] + \sin\left[\frac{2\pi}{9r}(mu + lv + nw + 3mr)\right] \\ & \left. + \sin\left[\frac{2\pi}{9r}(nu + lv + mw + 3nr)\right] + \sin\left[\frac{2\pi}{9r}(lu + nv + mw + 3lr)\right] \right\} \end{aligned} \quad (3.6)$$

$$T_a^{m,n} = \frac{1}{2} \left\{ \cos \left[\frac{2\pi}{9r} (lu + mv + nw + 3lr) \right] - \cos \left[\frac{2\pi}{9r} (nu + mv + lw + 3nr) \right] \right. \\ \left. + \cos \left[\frac{2\pi}{9r} (mu + nv + lw + 3mr) \right] - \cos \left[\frac{2\pi}{9r} (mu + lv + nw + 3mr) \right] \right. \\ \left. - \cos \left[\frac{2\pi}{9r} (nu + lv + mw + 3nr) \right] - \cos \left[\frac{2\pi}{9r} (lu + nv + mw + 3lr) \right] \right\} \quad (3.7)$$

The direct integration of the equations (3.6) and (3.7) easily demonstrates, in this form, the orthogonality of this set by employing the inner product defined as $\langle f, g \rangle = \iint_{\tau} f g dA$. The completeness of such collection of eigenfunctions is demonstrated by the work of Pinsky [90] based on group-theoretical arguments.

The triangular coordinate system is the natural system to construct such a family of eigenfunctions, for practical reasons it is now easier to switch to the Cartesian coordinates for the following part. In Cartesian coordinates the solution of Eq. (3.2) writes [91]:

$$z_{m,n} = 2 \sin \frac{(m-n)\pi x}{a} \cos \frac{(m+n)\pi y}{a \tan \alpha} \\ - 2 \sin \frac{(2m+n)\pi x}{a} \cos \frac{n\pi y}{a \tan \alpha} + 2 \sin \frac{(2n+m)\pi x}{a} \cos \frac{m\pi y}{a \tan \alpha}$$

where m, n are integers with $m \neq n$ and $m > n$ (the gravest mode is given by $m=1$ and $n=0$), a' is the triangle's height and $\alpha = \frac{\pi}{6}$. The first simulated symmetric modes are represented on Fig. (3.11 e-f).

Lamé obtained that the frequency of oscillation of the mode is given by:

$$f_{m,n} = \frac{\omega}{2\pi} = \frac{c}{a'} \sqrt{q}, \quad q = m^2 + n^2 + mn \quad (3.8)$$

where c is the phase speed ω/k . This result $f_{m,n} \propto \sqrt{q}$ comes from the combination of Eq. (3.3), the previous relation $l + m + n = 0$ and the boundary conditions.

The triangular cavity is a perforation on the membrane so there is no bending term in the dispersion relation:

$$\omega^2 = \frac{\sigma k^3}{\rho} + gk$$

where $\rho = 1000 \text{ kg/m}^3$ and σ is the surface tension of water, namely $0,07 \text{ N/m}$.

If we assume that gravity is negligible, which is verified for the range of frequency starting from 20 Hz (the gravitational term dominates from 13Hz), the dispersion relation is:

$$\omega = \sqrt{\frac{\sigma k^3}{\rho}} \quad \text{or} \quad k = \left(\frac{\rho \omega^2}{\sigma}\right)^{\frac{1}{3}}$$

and a phase speed:

$$c(k) = \left(\frac{\sigma \omega}{\rho}\right)^{\frac{1}{3}} \quad (3.9)$$

The phase speed represents the connection between the eigenstructure of an equilateral triangular cavity and the physics of the hydro-elastic waves. Combining (3.8) and (3.9) we obtain:

$$f_{m,n} = \frac{\omega}{2\pi} = \left(\frac{\sigma \omega}{\rho}\right)^{\frac{1}{3}} \cdot \frac{1}{a'} \sqrt{m^2 + n^2 + mn}$$

$$f_{m,n} = \sqrt{\frac{2\pi\sigma}{\rho a'^3}} (m^2 + n^2 + mn)^{\frac{3}{4}} \quad (3.10)$$

The mathematical and physical problem is summarized by Eq. (3.10); The next step is to experimentally characterize our triangular cavity in order to show its first modes and to compare for each of them its predicted frequency of oscillation with the experimental one. Our setup is shown in Fig. (1.3 a) with a membrane perforated in the middle following an equilateral triangular shape (height $a' = 1,6 \text{ cm}$). This particular height comes from the fact that the reconstruction technique is based on a checker-board and to have enough squares (which play the role of pixels for a CCD) within the triangular shape we need a perforation as big as possible. However, there is a limit about its size because the frequency associated to the resonant modes depends on it. The good size to reconstruct the modes and to stay in the frequency range suitable for our experimental setup is a triangle with $a' = 1,6 \text{ cm}$.

We send plane waves (propagating perpendicularly to the base of the triangle) with a

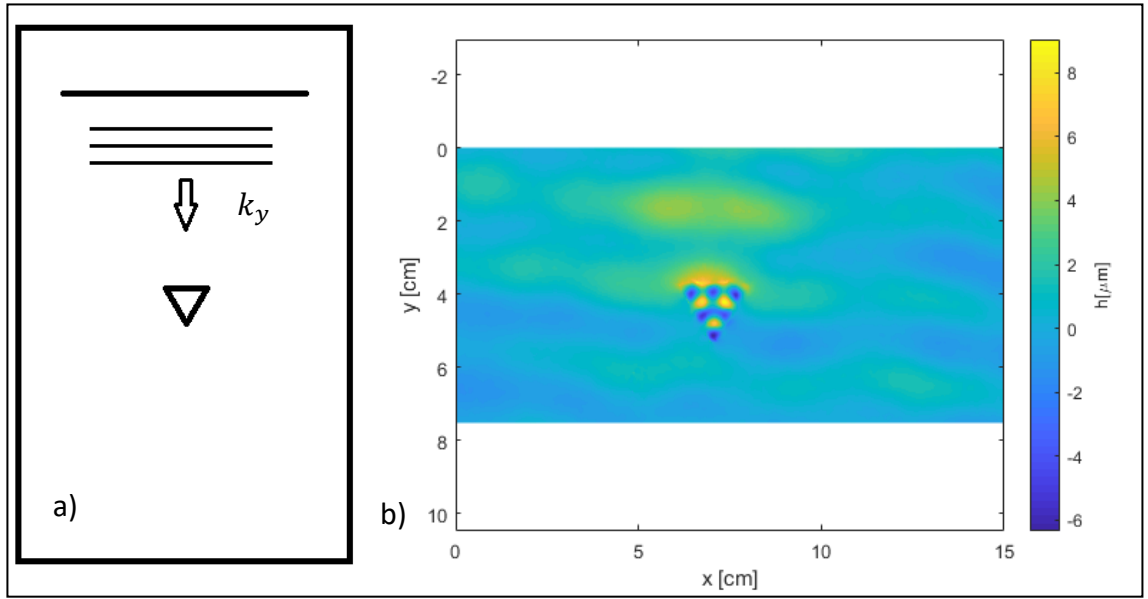


Figure 3.9: a) Schematic representation of the membrane used to analyse a triangular cavity excited by a plane wave generator typically tuned from 20 to 90 Hz. b) Height field reconstruction example with a triangular cavity at $f=48$ Hz.

frequency ranging between 20 and 90 Hz and we reconstruct the height field using the Synthetic Schlieren method (Chapter 1.4.2) centred in the triangle, see Fig. (3.9) for the orientation between the incident wave and the triangular cavity and for a typical height field reconstruction.

We reconstruct the height field for each frequency and we visually observe the cavity's behaviour, in particular the amplitude within the cavity which has a strong dependence on the frequency injected. We select the frequencies that maximize the cavity's response and we compare the reconstructed height field for those frequencies. We can see in Fig. (3.10 a) a typical example of a 3D height field reconstruction for $f=28$ Hz which has two important peculiarities, first the amplitude within the cavity is 2 times the amplitude of the incoming wave confirming the typical response of a resonant mode, second its shape has an excellent agreement with the theoretical one, Fig. (3.10 d). We apply this method for each frame which allows us to find three resonant symmetric modes ($f=28$ Hz, $f=48$ Hz and $f=73$ Hz) summarized in Fig. (3.11) where we compare the experimental modes, on the left side (a)-(c), with the theoretical ones obtained with Lamé's equation on the right side (d)-(f). This confirms that our triangular cavity behaves like a resonant cavity and we are able to excite it. So far this method shows an excellent agreement between the experimental and

simulated modes but we cannot quantify it by just a visual check. We propose to take advantage of the two important properties of our collection of eigenfunctions $\{T_s^{m,n}; T_a^{m,n}\}$: the orthonormality and completeness of the set.

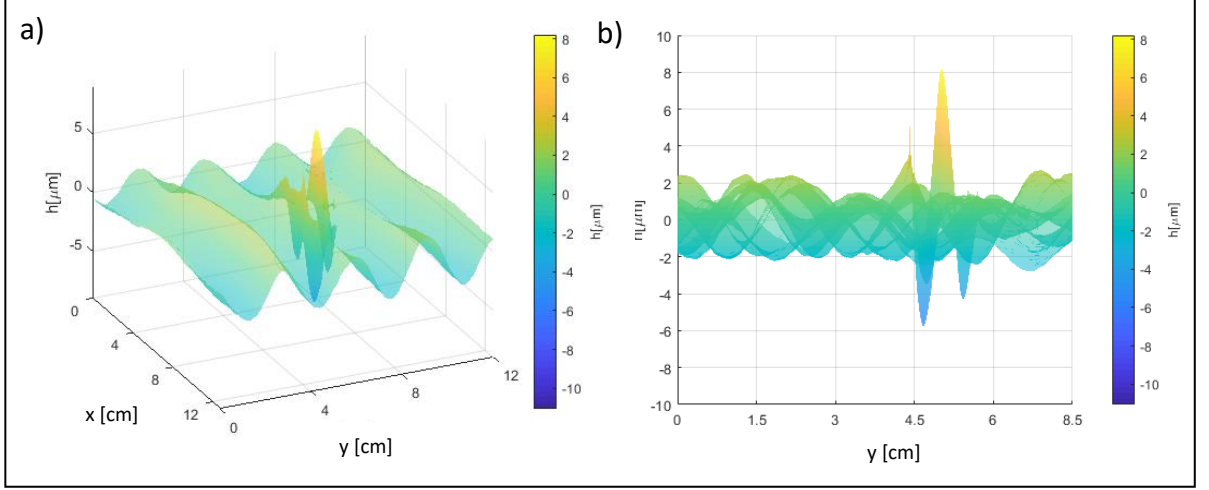


Figure 3.10: a) Height field reconstruction of a part of the membrane including the triangular cavity and incoming/outcoming wave generated by the plane wave generator for $f=28$ Hz. b) Height field reconstruction side view to highlight that the amplitude within the cavity is 2 times the amplitude of the incoming wave.

Each height field within the cavity can be projected on a basis made by the theoretical modes of the cavity as a result of its orthonormality and completeness [90], [92], [93]. We use just the first 10 modes obtained from $z_{m,n}$ in Cartesian coordinates as a basis that we call B . In the following table it is shown how $B = \{B_1 + \dots + B_{10}\}$ is constructed:

	B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_8	B_9	B_{10}
m,n	1,0	2,0	2,1	3,0	3,1	3,2	4,0	4,1	4,2	4,2

Table 3.1: Construction of the basis B .

We numerically verified that the 10 modes selected are dominant in the frequency range employed. We define a dot product as:

$$\langle z, B_i \rangle = \int_T z(x, y) B_i(x, y) dx dy \quad (3.11)$$

where z is the wave field measured within the triangle, B is the modal basis and $(x, y) \in T$ is the cavity's area. We compute Eq. (3.11) for each wave field measured in the frequency range from 20 to 90 Hz to obtain the coefficients α_i corresponding to

the projection on the basis B . The result of this operation can be represented in two ways:

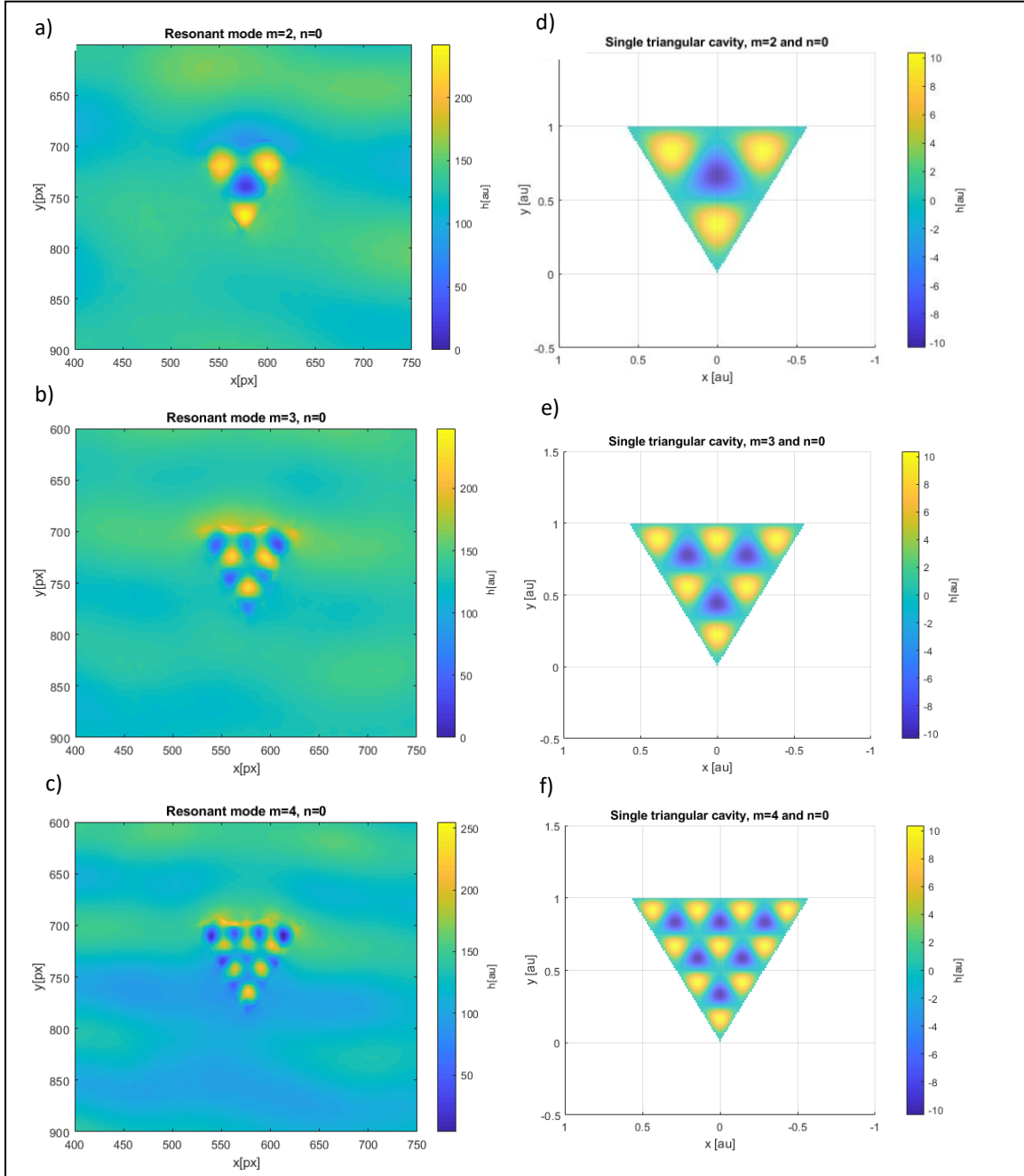


Figure 3.11: (a)-(c) Height field reconstruction inside the triangular cavity for the modes $(m,n)=(2,0)$ $f=28$ Hz (a), $(m,n)=(3,0)$ $f=48$ Hz (b) and $(m,n)=(4,0)$ $f=73$ Hz (c). (d)-(f) Theoretical prediction of the modes $(m,n)=(2,0)$ (d), $(m,n)=(3,0)$ (e) and $(m,n)=(4,0)$ (f).

1. We can see the distribution of the energy of a particular mode over the frequency range in order to find its maximum and its related frequency; we plot the following ratio:

$$\frac{|\alpha_m(f)|^2}{\sum_{n=1}^{10} |\alpha_n(f)|^2}$$

where α_m is the coefficient related to the modes $i=2,4$ and 7 .

Fig. (3.12 a-c) shows the energy ratio for the modes $(m,n)=(2,0)$, $(3,0)$ and $(4,0)$ and the corresponding frequencies are 28 Hz, 48 Hz and 73 Hz. This operation confirms what we observed with the previous visual based method.

2. For the three frequencies aforementioned we can display the energy ratio of each coefficient resulting from the previous projection, we plot here:

$$\frac{|\alpha_i(f)|^2}{\sum_{n=1}^{10} |\alpha_n(f)|^2}$$

where $f=28, 48$ and 73 Hz.

As shown on Fig. (3.12 d-f) most of the energy (95%, 85% and 69% respectively) is related to a specific mode demonstrating that we are able to excite a particular mode within the cavity.

The three experimental frequencies obtained by decomposition over the basis B and the theoretical frequencies, calculated with Eq. (3.10), for the modes $(m,n)= (2,0)$, $(3,0)$ and $(4,0)$, considering $a' = 1,6$ cm, are summarized in the following table:

m	n	f_{th} [Hz]	f_{ex} [Hz]
2	0	29,9	28
3	0	55,0	48
4	0	84,6	73

Table 3.2: Experimental and theoretical frequencies obtained for the the three resonants modes studied.

We compare them with the previous theoretical values and we confirm a good agreement excepting for the highest frequency. For $f=73$ Hz we can see that the energy ratio of the mode $(m,n)=(4,0)$ is 69% which reflects the gap between the theoretical and experimental frequency. This gap has an explanation: the assumption about the Dirichlet boundary conditions taking $z = 0$ at the edge does not apply well for high frequencies where the typical wavelength inside the membrane is between 1 and 2 cm (for example at $f=73$ Hz is 1,9 cm) hence our cavity behaves like a bigger one because of the boundary's displacement generated by the incoming wave and considering Eq. (3.10) for a bigger cavity the corresponding oscillation frequency is

lower like in our case. For the first two modes the wavelength is bigger than the cavity so the border's deformation is less important.

Our visual and projection method confirm that our system acts like a resonator. It is possible then to adopt the Q factor formalism to fully characterize its behavior.

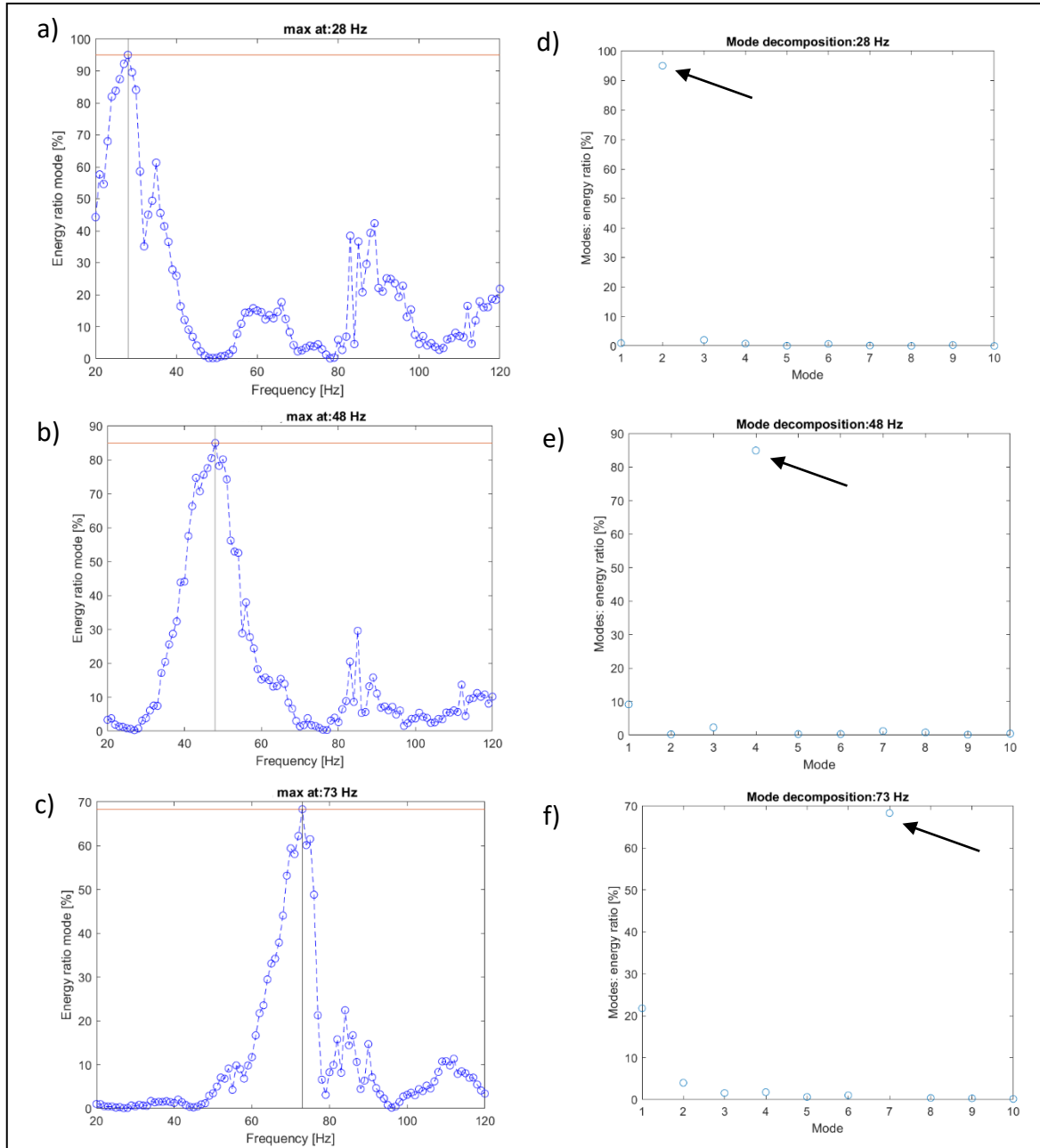


Figure 3.12: (a)-(c) Energy ratio of a single mode $(m,n)=(2,0)$ (a), $(m,n)=(3,0)$ (b), $(m,n)=(4,0)$ (c) in the frequency range from 20 to 90 Hz. (d)-(f) Energy ratio for the modes $(m,n)=(2,0)$ (d), $(m,n)=(3,0)$ (e), $(m,n)=(4,0)$ (f) over the coefficients obtained by projection over the basis B composed by the first 10 theoretical modes of a triangular cavity; on the x-axis the modes are indicated in frequency increasing order, 1 corresponds to the first mode $(m,n)=(1,0)$, 2 to the second mode $(m,n)=(2,0)$, 3 to the third mode $(m,n)=(2,1)$ and so on.

The quality factor has two definitions according to the considered field, we employ here the Q factor defined as [94]:

$$Q \stackrel{\text{def}}{=} \frac{f_c}{\Delta f}$$

where f_c is the resonant frequency and Δf the resonance width.

We can extract this information from Fig. (3.12 a-c) by using the resonant frequencies found and the resonance width calculated with the full width at half maximum (FWHM):

f_c [Hz]	Δf [Hz]	Q
28	13	2,2
48	14	3,4
73	12	6,1

Table 3.3: Q-factor calculated for the the three resonant modes studied.

As shown by the table, all three modes have a very similar resonance width so they are characterized by the same frequency selectivity.

The quality factor is useful to express the response of a point resonator (which is an approximation of our case because the triangular cavity studied has a finite size) based on the Lorentzian function [95], [96]:

$$A(\omega) = \frac{2j \frac{\omega}{c}}{1 + 2jQ \left(1 - \frac{\omega}{\omega_0}\right)}$$

where c is the phase speed (Eq. 3.9), Q the quality factor and $f_0 = \omega_0/2\pi$ the eigenfrequency. This function, for $f = f_0$, presents a peak in the amplitude and a π phase jump coherently with the behavior of a resonator.

3.5 Conclusion

In this chapter we started with the theoretical dispersion relation of HEW focusing on the bending term, in particular on the Young's modulus E that we measured for a sample of the silicone membrane provided by Silex. We reconstructed the dispersion relation via Fourier transform and compared to the theoretical one finding a perfect agreement considering the dynamical value of E . Then, we measured the attenuation

length of a HEW in order to demonstrate that this a suitable platform to study the interactions between a HEW and a triangular cavity (or any pattern or shape on the membrane). The last part of this chapter has been devoted to the characterisation of a triangular cavity that acts like a resonator showing its modes validated via modal decomposition. In the next chapter we will see that the interaction between a HEW and multiple triangular perforations spatially arranged, when the input frequency reaches the resonant frequency of the resonators, will imply a band gap (hybridization gap) in the band diagram. Precisely, the energy of the incoming wave is absorbed by the resonators, at the resonance, resulting in a strong decreasing of the amplitude of the transmitted wave which is represented by a gap in the band diagram.

Chapter 4: Artificial crystals for Hydro-Elastic Waves: Valley transport

4.1 Introduction

In this chapter we extend the study about the triangular resonators, previously seen, creating a 2D pattern of regular perforations on the silicone membrane. So, we will pass from a single cavity to multiple cavities spatially arranged, in particular following the structure of a triangular lattice. These are the so-called artificial crystals for HEWs. An important tool to understand phenomena linked to the spatial arrangement of a crystal is the band structure, so we will start providing the main concepts to extract important information from it.

The first type of crystal that we will study is a triangular lattice made by triangular resonators with same rotation angle (monotype artificial crystal). In other words, we will study first, the easiest 2D pattern where each resonator composing the aforementioned crystal is oriented in the same direction. We will see, in such situation, how the dispersion relation is modified compared to the case where there are no perforations on the membrane. This is the first step to understand how HEWs propagate through periodic media. We verify its band structure in order to observe the expected band gaps due to the Bragg scattering and the local resonances (hybridization gap), presented in Chapter 2. Its band structure is an important tool that indicates for each working frequency when the propagation of a HEW through the crystal is allowed.

Then, we will continue our study with a more complex crystal made by two different patterns where topology plays an important role. The goal, in this case, is to observe a mode that propagates even when the frequency considered belongs to the band gap (the frequency range where there is no propagation). Moreover, it is localized on a precise path like for a waveguide. The explanation of such apparent contradiction comes from Topology or equivalently from the geometry of our crystal. The difference between the monotype crystal, which does not support a topological mode and the two patterns put together consists in the rotation angle of the resonators different for each pattern. Precisely, the two patterns, one next to the other one, form an interface originated by the different spatial arrangement. This interface can support a

topological edge mode according to the theory because the change of geometry on the border between the two patterns implies a topological phase transition. When a topological phase transition occurs, as seen in Chapter 2, we can have interesting transport phenomena or in other terms a topological waveguide. We will present the theory that explains the existence of such a propagating mode (called Valley transport) and the experimental results.

4.2 Band structure of a periodic media: analysis

In Chapter 3, we characterized the behavior of a triangular perforation on the membrane confirming that it acts like a resonator. Now, we want to study a more complex situation for HEW: a 2D triangular pattern perforations. As seen in Chapter 2, with the Bragg reflector and the example of the 1D bi-atomic structure, the dispersion relation exhibits two types of gaps. It is worth noticing that most of the experiments about topological states involves simple scatterers (examples [97], [98], [99]) instead of resonators; a scatterer is an object that interacts like an obstacle for the wave propagation, so there are no resonance modes associated. Using a scatterer or a resonator has an impact on the band structure, in fact more gaps appear when resonators are employed, in particular we have the so-called hybridization gaps originated by the local resonances.

Generally, the band structure associated to a physical problem involving topological modes can be very complicated and usually it is obtained via simulations. Moreover, instead of showing a graph with wavevector on the x-axis and frequency on the y-axis like we usually expect for a dispersion relation, we have on the x-axis the three critical points (or the three high-symmetry points) marked as Γ , M and K . These, as presented in Chapter 2, are the points in the first Brillouin Zone (BZ) positioned at $(0,0)$, $(\pi/L)(1, 1/\sqrt{3})$, and $L(4\pi/3\sqrt{3}, 0)$ respectively [56]. This is the typical way to present a band structure in this field and the information that we can extract is how the dispersion relation varies in those directions which are related to the geometry of the problem considered. This is almost a 3D representation, almost because it is still a 2D representation but with the evolution of the curve following the path $\Gamma \rightarrow K \rightarrow M$. We provide a complex example, Fig. 4.1, from literature about a 2D honeycomb lattice (like graphene, a very famous structure in this field) where on the x-axis we have the

three aforementioned critical points and on the y-axis the frequency. Let us start by the description of Fig. (4.1 a), where at first look, the dispersion relation can appear very confusing. In term of gaps, we can see just one when the normalized frequency $f < 0,65$, so the first information that we can extract is: no waves can propagate in such a crystal in the frequency range $0,45 < f < 0,65$. Now, we can observe that the author highlighted a particular area with a red square corresponding to the critical point Γ . The state associated to this point is four times degenerate and it is usually

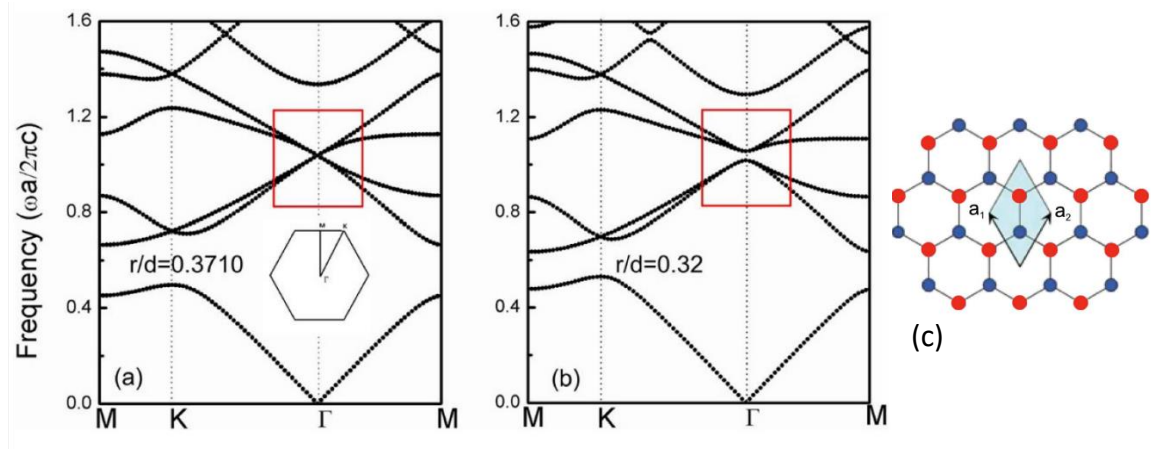


Figure 4.1: a) Band structure for a 2D honeycomb lattice where each unit cell is composed by an iron cylinder in a water wave system; the red square points the double Dirac cone b) Band structure modified as a result of a change of the cylindrical radius of the unit cell; in this case the double Dirac cone is lifted. Adapted from [100]. c) Schematic of the 2D honey-comb lattice with the unit cell defined by the vectors a_1 and a_2 . Adapted from [101].

denoted as quadrupole-degenerate state or 4-fold degeneracy. In fact, we can see that it is the intersection of 4 lines in this “3D representation”. Each curve, in the dispersion relation with axis wavevector (in one direction) and frequency, is linear in the directions $\Gamma \rightarrow M$ or $\Gamma \rightarrow K$. This quadrupole-degenerate state is a direct consequence of the crystal geometry, precisely it corresponds to the two pairs of dipolar modes and quadrupolar modes seen in Chapter 2 (paragraph 2.7). As we know the interesting topological phenomena occur around this state (for the honey-comb lattice the degeneracy occurs at Γ , for the triangular lattice at K). The first who predicted it was Sakoda [102] providing the demonstration of its existence via the group theory. As we are interested in the experimental observation of topological modes, for which its existence has been already proved, we will focus on the elements that we need to create our experiment.

The band structure, Fig. (4.1 a) provided us then two important information:

1 a band gap (a frequency range for which there is no wave propagation)

2 a quadrupole-degenerate state, direct consequence of the geometry of the problem (true for any kind of crystal with this structure)

Let us focus now on Fig. (4.1 b) which is very similar to the previous band structure but not totally equal. The difference in this case concerns the red area, in fact the 4-fold degeneracy is now lifted, in other words we passed from a 4-fold degeneracy to a double degeneracy. Having a deeper look, we can state that this lifting process opens a gap around $f = 1$. So, in this case we have two gaps instead of one.

The reader could be interested in understanding the origin of the lifting process. As previously seen, the 4-fold degeneracy has a strict link with the geometry of the problem considered. In this case, the author changed the radius of its cylindrical scatterers modifying the unit cell of its crystal. This is indicated in both figures with the ratio r/d where r is the radius of the cylinder and d the distance between two scatterers. As we saw in Chapter 2 (paragraph 2.7), when the dipolar modes and quadrupolar modes exchange their position in term of frequency a gap opens. This mechanism is the so-called band inversion. In order to better visualize this 4-fold degeneracy emerged from the band structure, we propose in Fig. (4.2) a real 3D dispersion relation centered at Γ , real because we effectively have 3 axes: k_x, k_y and f .

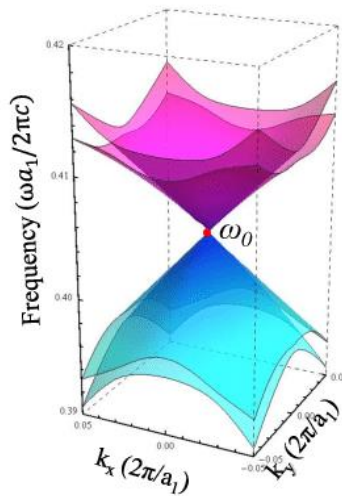


Figure 4.2: 3D Dispersion relation centered at Γ showing the double Dirac cone (4-fold degeneracy). Adapted from [103].

We can see that the 4-fold degeneracy is actually a double cone. In the literature, this cone is called Dirac cone and the intersection point is known as Dirac point (red point in Fig. 4.2).

4.3 Artificial monotype crystal for Hydro-Elastic Waves: the triangular lattice

So far, we saw that the geometry of the unit cell has a direct impact on the band structure. Another way to lift the 4-fold degeneracy is to break the mirror symmetry. In order to explain what it means let us consider a triangular lattice where each unit cell is composed by a triangular perforation. If we want to modify the geometry of the unit cell, which causes the lifting process, we have two options:

- 1 we can change its size, as done with the previous examples involving cylinders
- 2 we can rotate the triangle within the unit cell (this would not have any effect with a cylinder that possesses a rotational symmetry)

Breaking the mirror symmetry is a formal way to say that we apply a rotation (excepting for π and 2π) to the triangular perforation, so it corresponds to the 2nd option. Let us refer to Fig. (4.3) to visualize it, on the left side we have a triangular pattern of triangular resonators and considering a mirror symmetry with respect to the vertical axis the resulting structure is unchanged. On the contrary, if we apply, for example, a rotation angle $\alpha = 10^\circ$ to each resonator the mirror symmetry is broken.

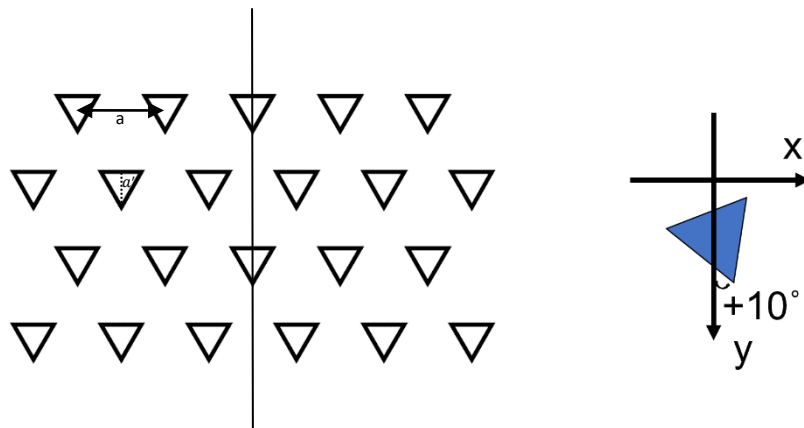


Figure 4.3: a) Schematic of a triangular lattice where the mirror symmetry is preserved. Applying a mirror transformation with respect to the vertical axis would not change the image. b) By applying an angle rotation of 10° to all resonators the mirror symmetry is broken.

This is the strategy that we will employ to lift the 4-fold degeneracy. This choice is justified by the fact that adopting the first option the frequency of the resonant modes

would change (as seen in Chapter 3) because it depends on the size of the cavity. This would modify again the band structure because the hybridization gaps would appear in another range. To avoid a more complex band structure it is easier to break the mirror symmetry by rotating all resonators $\alpha = 10^\circ$ (any angle excepting for π and 2π breaks this symmetry). This particular rotation angle comes from the study of Jiuyang Lu *et al* [63] who demonstrated interesting transport phenomena that occur when $\alpha = 10^\circ$ (and $\alpha = -10^\circ$) with a particular structure (a sonic crystal) composed by two patterns (one with $\alpha = 10^\circ$ and the second one with $\alpha = -10^\circ$). It will be clear in the paragraph 4.5 where we will provide all details, for the moment we focus on the monotype crystal.

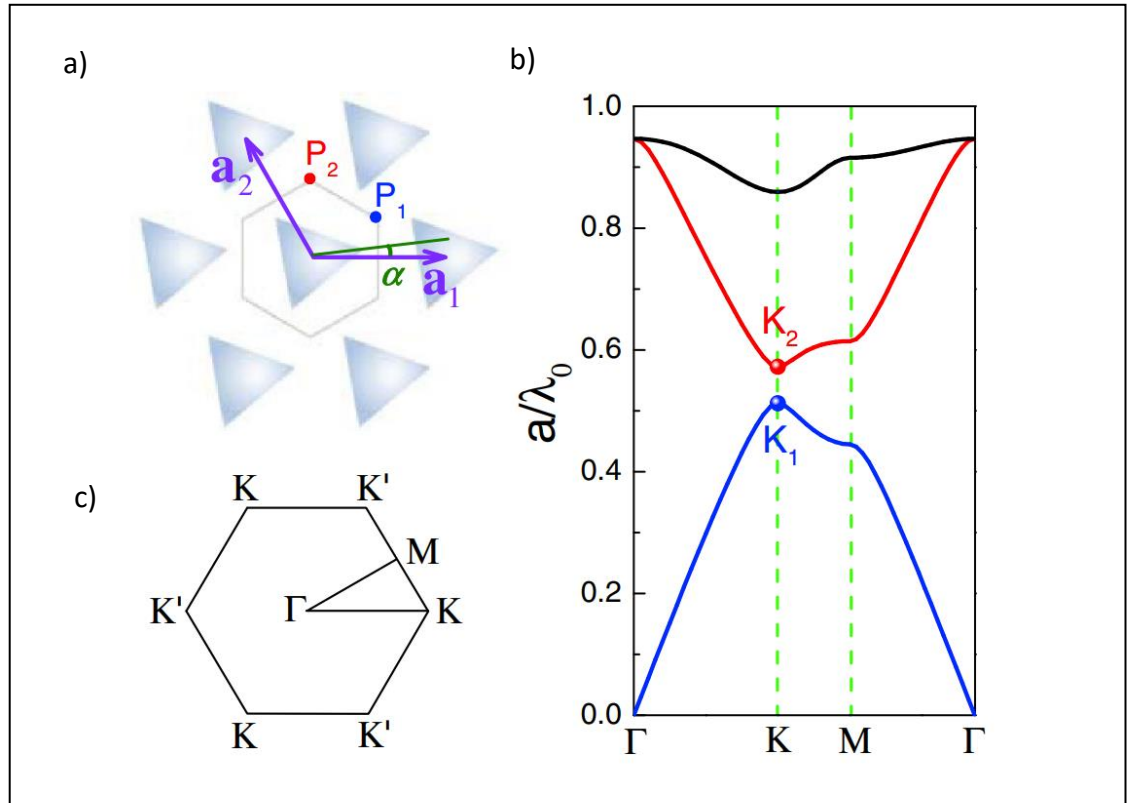


Figure 4.4: a) Schematics of a triangular pattern of triangular resonators with rotation angle $\alpha=10^\circ$, where P_1 , P_2 indicate the positions with C_3 symmetry and a_1 , a_2 are the primitive vectors. b) Dispersion relation along the typical path Γ KM Γ where K_1 and K_2 indicate the degenerate states lifted because of the broken mirror symmetry (resulting in a band gap); on the y-axis a/λ_0 is the dimensionless spatial frequency with λ_0 being the wavelength. c) Typical critical point directions. Adapted from [104].

In Fig. (4.4) we report the band structure where the mirror symmetry is broken for a triangular lattice of triangular resonator. In particular, referring to Fig. (4.4 a), P_1 and

P_2 indicate the positions with C_3 symmetry and the vectors a_1, a_2 are the primitive vectors.

The lattice constant is $|a_1| = |a_2| = a$. Referring to Fig. (4.4 b), on the γ -axis we have the dimensionless spatial frequency a/λ_0 with λ_0 being the wavelength; K_1 and K_2 indicate the degenerate states lifted because of the broken mirror symmetry (resulting in a band gap). Fig. (4.4 c) shows the direction of the usual critical points $\Gamma \rightarrow K \rightarrow M$. We take advantage of this result already demonstrated to create an artificial crystal for HEW. It is worth mentioning that a crystal for gravity-capillary waves has been tested showing topological effects in a water wave system [105].

Let us see, within the context of the HEW, if we can predict the band gaps:

Bragg gap

We know that the first Brillouin zone belongs to the range $k_{FBZ} \in [-\pi/a, \pi/a]$, reminding that for a HEW (Eq. 3.3):

$$\omega = 2\pi f = \sqrt{\frac{Dk_{FBZ}^5}{\rho}}$$

Considering a lattice constant $a = 1 \text{ cm}$ and the parameters of our elastic membrane, we expect the band gap type Bragg centered at $f = 42 \text{ Hz}$.

Hybridization gap

For the hybridization gap, we expect the gap to occur at the resonant frequencies of the triangular resonator analyzed in the paragraph 3.4, equation Eq. (3.10):

$$f_{m,n} = \sqrt{\frac{2\pi\sigma}{\rho a'^3}} (m^2 + n^2 + mn)^{\frac{3}{4}}$$

Considering the height of the triangular resonator $a' = 4 \text{ mm}$ and the fundamental mode $(m,n)=(1,0)$, we expect the band gap type hybridization centered in $f = 85 \text{ Hz}$. These values for a, a' allow us to study such a crystal with our experimental setup.

4.4 Experimental results for a monotype crystal for HEW

Now that the properties of a triangular lattice have been presented, we can pass from theory to the experiment with HEWs to characterize our system.

We start by filling up with water the glass tank up to 17 cm high; we put a membrane of the tank size on the water, with regular perforations forming a monotype crystal (size of the crystal 15 cm x 32 cm) as illustrated in Fig. (4.5 a and d); the distance between resonators is 1 cm, their height is 4 mm and their angle rotation is $\alpha = 10^\circ$ with respect to the crystal area, Fig. (4.5 b). The source generator is a straight paddle, 24 cm long, that generates plane waves in the frequency range 20-120 Hz. A photo of the experimental setup is in Fig. (4.5 c-d). For each frequency we record 60 frames (CCD $fps = 129,8$) and we reconstruct the corresponding height field via the free-surface synthetic Schlieren (1.4.2). This could apparently violate the Nyquist's condition but we do know the input frequency so we are able to reconstruct the signal with the spectral folding technique presented in Chapter 1 (1.4.3). An example of height field reconstruction is proposed in Fig. (4.6). We selected four particular frames to show when the crystal is transparent Fig. (4.6 a-b) and when it acts like a mirror (no propagation allowed, Fig. 4.6 c-d). It means that there exist frequency ranges where the waves propagation has a different behavior. The two selected frequencies, which are the most representative to show the transparency phase, are $f = 30\text{ Hz}$ and $f = 65\text{ Hz}$. On the opposite, for $f = 47\text{ Hz}$ and $f = 87\text{ Hz}$ there is no propagation. In this height field reconstruction, the wave generated by the source comes from the upper part of the image and propagates to the lowest part of the frame.

The yellow-blue color scale indicates the surface elevation and the red dotted lines point the area of the membrane the crystal is located. Each frame contains then 3 parts, the upper one to identify the incident wave, a middle one for the crystal area and the lower one to detect the transmitted wave. It is worth having a deeper look at the frames corresponding to the two gaps (Fig. 4.6 c-d).

The frame in Fig. (4.6 c) shows a very flat height field within the monotype crystal (red dotted lines) confirming that it acts like a mirror (except for the first lines of resonators on the top where we have an evanescent wave).

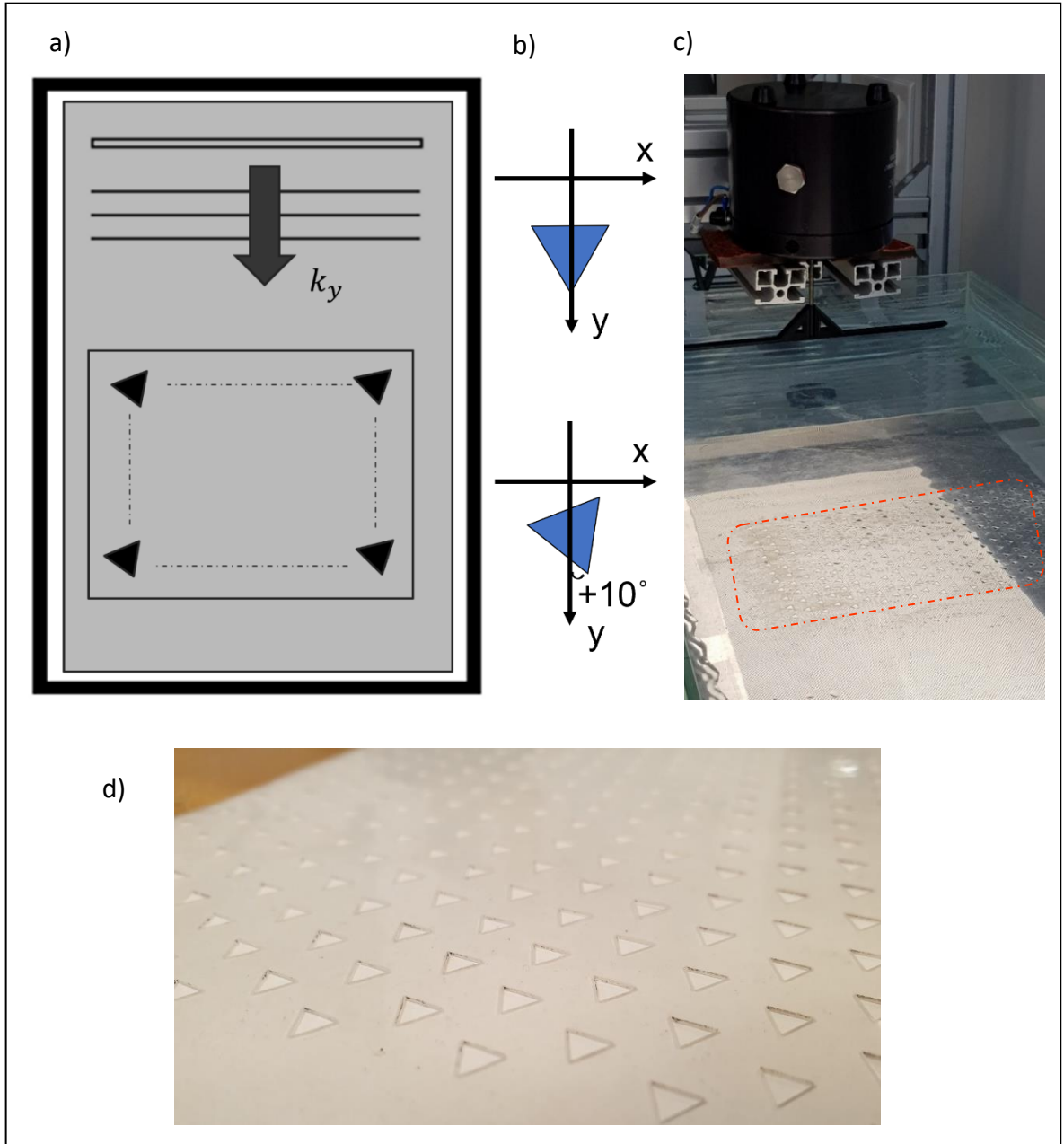


Figure 4.5: a) Experimental setup to study the artificial monotype crystal (32 cm x 15 cm). b) Angle rotation of each resonator composing the artificial crystal. c) Photo of the experimental setup. d) Regular pattern of triangular perforations on a silicone membrane.

Just a small quantity of the incoming wave can penetrate the crystal resulting mostly in an evanescent wave, so in this case the internal resonators (after the first lines) do not interact with the incoming wave. On the other hand, the frame in Fig. (4.6 d), which still corresponds to a gap, does not present a flat height field meaning that the incoming wave interacts with the triangular perforations. The explanation comes from the fact that the latter resonate, in fact, we are in the hybridization gap.

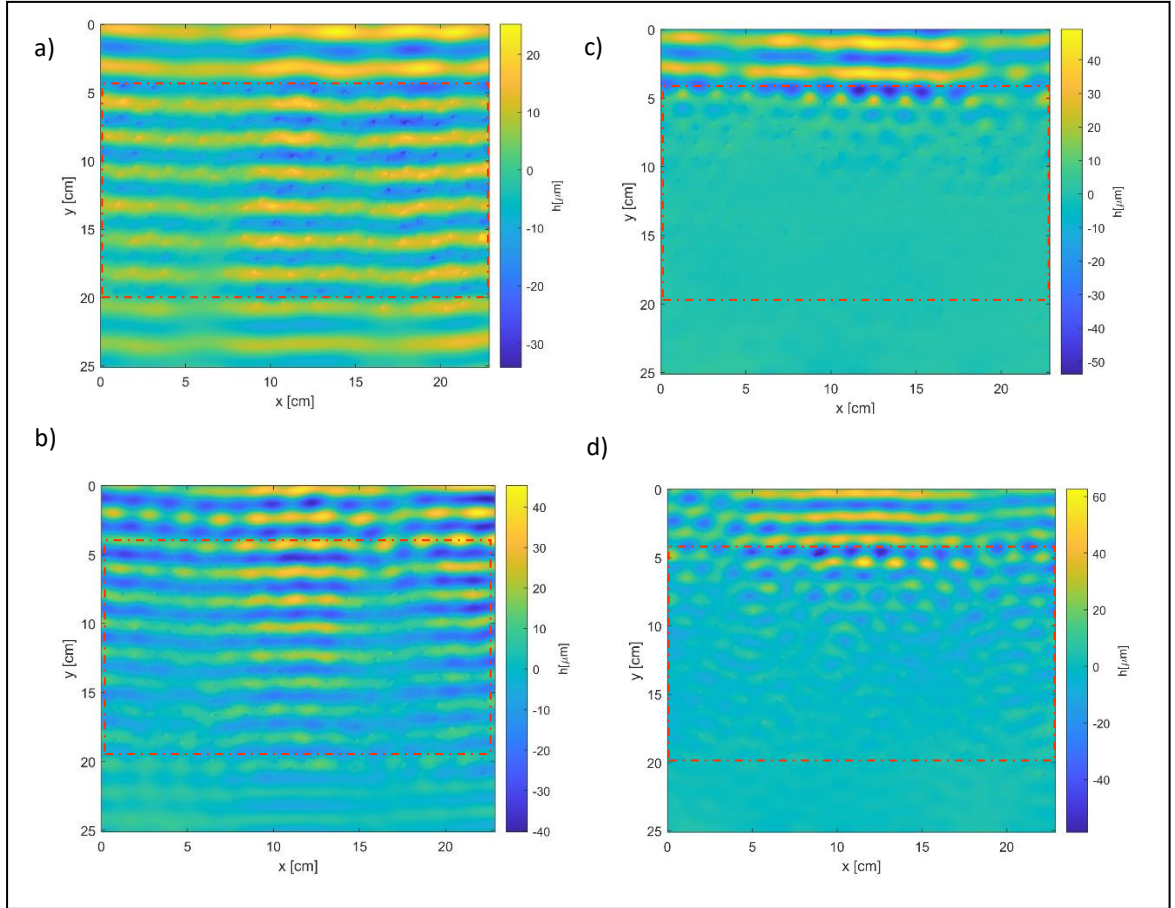


Figure 4.6: Height field reconstruction of the artificial monotype crystal for $f=30$ Hz (a), $f=65$ Hz (b), $f=47$ Hz (c) and $f=87$ Hz (d). Referring to (a-b) the crystal is “transparent” while for (c-d) the propagation is not allowed. The red lines indicate the boundaries of artificial crystal.

Now, we use a temporal Fourier transform, over the recorded frames, to select only the frequency component equal to the frequency used by the source wave generator, in other words we filter the signal at the working frequency in order to avoid unwanted noise. Our system is characterized by the frequency established by us, so we do not need to reconstruct an unknown signal for which a more elevated number of frames would be a necessary condition. We apply then a 2D spatial Fourier transform on the filtered result, within the crystal area, and we obtain in k-space a point accordingly to the wave propagation direction.

We remind that a plane wave in the k-space is represented by a point and in our system and we only employ plane waves. Selecting a line of the 2D spatial Fourier transform where the point corresponding to the plane wave is and piling all lines for each input frequency, we obtain the dispersion relation frequency-wave vector

(precisely $k_y/\pi/a$) in Fig. (4.7). In other words, it is the direction $\Gamma \rightarrow K$ shown in Fig. (4.4 b). The blue line is the theoretical dispersion relation considering a membrane without perforations like seen in Chapter 3.

We can observe that the experimental dispersion relation differs from the theoretical curve associated to the membrane with no perforations. In particular, it exhibits 2 gaps, in the working frequency range, the first one is the Bragg gap expected for $f = 42 \text{ Hz}$ and the second one is the hybridization gap expected for $f = 85 \text{ Hz}$. This experimentally confirms that the spatial arrangement plays a role in the dispersion diagram.

Precisely, we can extract the following information from the experimental dispersion relation:

- The first band gap is located in the frequency range from 40 to 52 Hz while the frequency predicted for the Bragg gap, determined by the spacing between triangles is 42 Hz; in this frequency range there is no propagation and the resonators do not play any role, the relevant element is the triangular structure.
- The second gap is located in the frequency range from 82 to 98 Hz in agreement to our prediction of 85 Hz, corresponding to the resonant frequency of mode $(m,n)=(1,0)$; the surface deformation of the fundamental mode of the first lines of resonators is also visible without height reconstruction for which we provide a timeline sequence of the oscillating mode in Fig. (4.8).
- The two gaps have different frequency width, in fact, the band gap due to the Bragg scattering is smaller compared to the gap originated by the local resonances (12 Hz for the Bragg one and 18 Hz for hybridization one)
- The dispersion relation changes concavity around the 1st gap which is a characteristic of band gap type Bragg while for the 2nd gap, it is the same, confirming that it is a hybridization gap.
- The diagram is symmetric with respect to the vertical axis because we can consider a wave propagating from the upper side to the lower side and vice versa.

Now, the monotype crystal is fully characterized and we are able to predict where band gaps are located. The last observation about Fig. (4.7) concerns the typical border effects. In fact, we can observe a visual artifact, the repeated lines in grey as “echoes”, due to the lack of periodicity of the borders of the window where the spatial Fourier transform is performed. Even though, the band structure is visible presenting 2 gaps as expected.

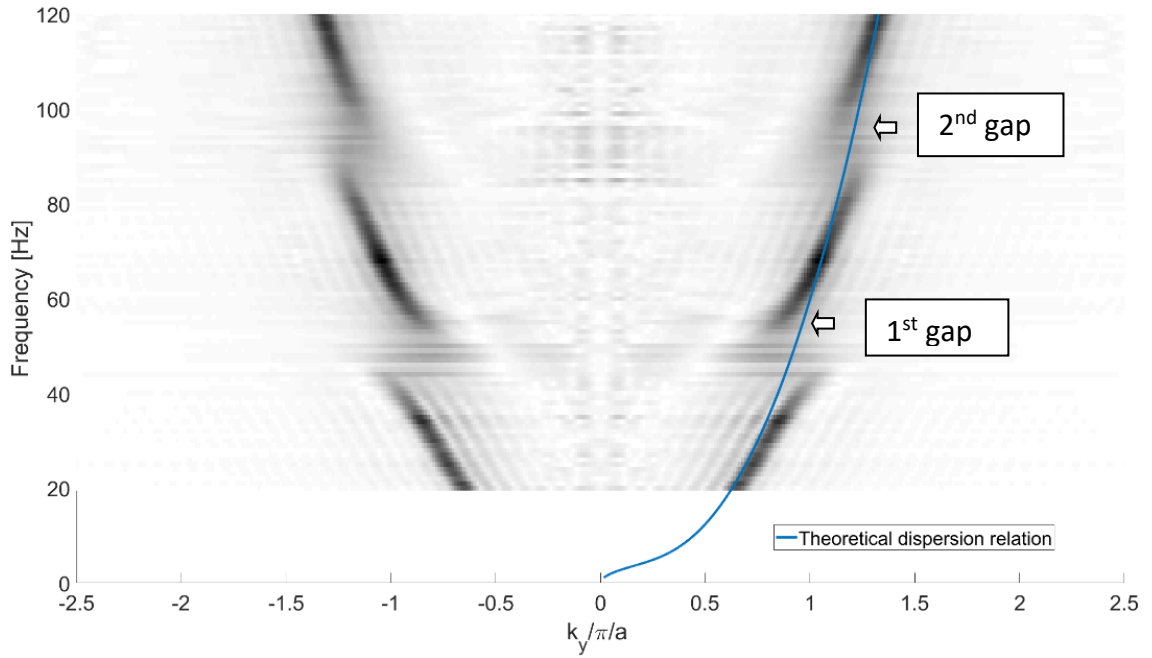


Figure 4.7: Dispersion relation ($f, k/\pi/a$) obtained via Fourier transform compared to the theoretical one when there are no perforations on the membrane (blue line). Two gaps open: type Bragg (1st gap) and hybridization (2nd gap).



Figure 4.8: Time evolution of the fundamental resonant mode within a triangle perforation, $f=87$ Hz and $\Delta t = 1/129$ s.

4.5 Valley transport: main concepts

From the experimental view, we possess now an artificial monotype crystal for HEW that presents two band gaps. For the moment, such a crystal does not support any particular topological mode (because we need a gap closing process as we saw in Chapter 2). It simply allows or not the wave propagation as a “filter” for HEWs. What we would like to observe is a mode propagating through a particular path when we are in the frequency range of the band gap. If this mode exists it would be represented like a line in the band structure (called edge state), meaning that a “conducting” state exists for a particular crystal. The structure that we will present is based on the theory called Valley transport or equivalently on the Valley Hall transition; we present here the main steps following [63].

We start by a pattern of triangular resonators with rotation angle $\alpha = 10^\circ$, then we put together a second similar pattern with rotation angle $\alpha = -10^\circ$ this time. As previously seen in Chapter 2: paragraph 2.7 the geometry tuning of the unit cell (the rotation angle in this case) when the two patterns are put together, implies a band inversion at their interface or in other words we should obtain the two helical edge states at the interface of the two patterns. This, because the two pairs of modes populating the higher and lower bands flip (also called valley states). This flipping process is the so-called band inversion and when it occurs it means that we have a topological phase transition. The extrema on the two sides of the gap are called valleys, for their visual similarity to real valleys, Fig. (4.9) and that is why it is called valley transport.

A topological phase transition is observable as an edge mode that exists in the frequency range of a band gap.

In order to verify if we can observe a topological edge mode, we just need to create an interface putting together a monotype crystal, as previously seen, with the same monotype crystal flipped upside-down. Then, we will analyse the band gaps previously found to see if this conducting state or edge mode really exists in a classical system based on HEWs.

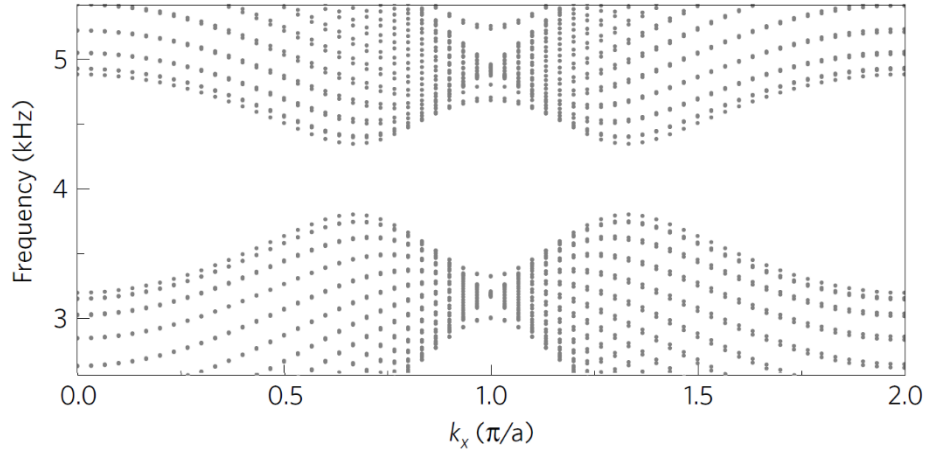


Figure 4.9: Dispersion relation of a structure composed by two patterns of triangular scatterers obtained via simulations. Focusing on its shape the reader can visualize it as “real” valleys from which the theory behind the existence of the topological edge mode takes its name. Adapted from [63].

4.6 Experimental results for the Valley Hall effect for HEW

Now, we want to verify if we are able to demonstrate the Valley transport for HEW or in other words we want to verify if we can detect a topological edge mode that propagates on the interface made by two patterns in the frequency range belonging to a gap. As seen in the previous paragraph, we need to interface two crystals, inversion images of each other. In order to do so, we create a crystal composed by two domains with rotation angle $\alpha = 10^\circ$ and $\alpha = -10^\circ$ like illustrated in Fig. (4.10). In other words, we need two monotype crystals like the one characterized in the paragraph (4.4). We start by filling up with water the glass tank up to 17 cm high; we put a membrane of the tank size on the water; the crystal size is 15 cm x 32 cm, the distance between each resonator is 1 cm and its height is 4 mm. The source generator is a straight paddle, 24 cm long, that generates plane waves in the frequency range 30-120 Hz. For each frequency we record 100 frames (CCD $fps = 129,8$) and we reconstruct the corresponding height field via the free-surface synthetic Schlieren (1.4.2). Now we use a temporal Fourier transform, over the frames recorded, as done for the monotype crystal, to select only the frequency component equal to the frequency used by the source wave generator, in other words we filter the signal at the working frequency. Keeping in mind that we want to compare different areas of the crystal to demonstrate that the expected topological edge mode exists and that it is localized on

the interface between the two patterns, we can choose two different kinds of strategies to analyze our data:

1 we can apply a temporal Fourier transform on a line along the x-axis at the exit of the crystal (see the red line in Fig. 4.10 a); this provides the transmission profile along the x-axis and if a mode really propagates through the interface, we should see a peak corresponding to the interface position (also called channel) in the amplitude-x position graph.

2 we can apply a spatial Fourier transform on the filtered result over the transmitted area just below the crystal divided in three zones called “left” (exit of the domain II, red), “channel” (in front of the interface, green) and “right” (exit of the domain I, blue) as illustrated in Fig. (4.10 a); we repeat this for each frequency (30-120 Hz).

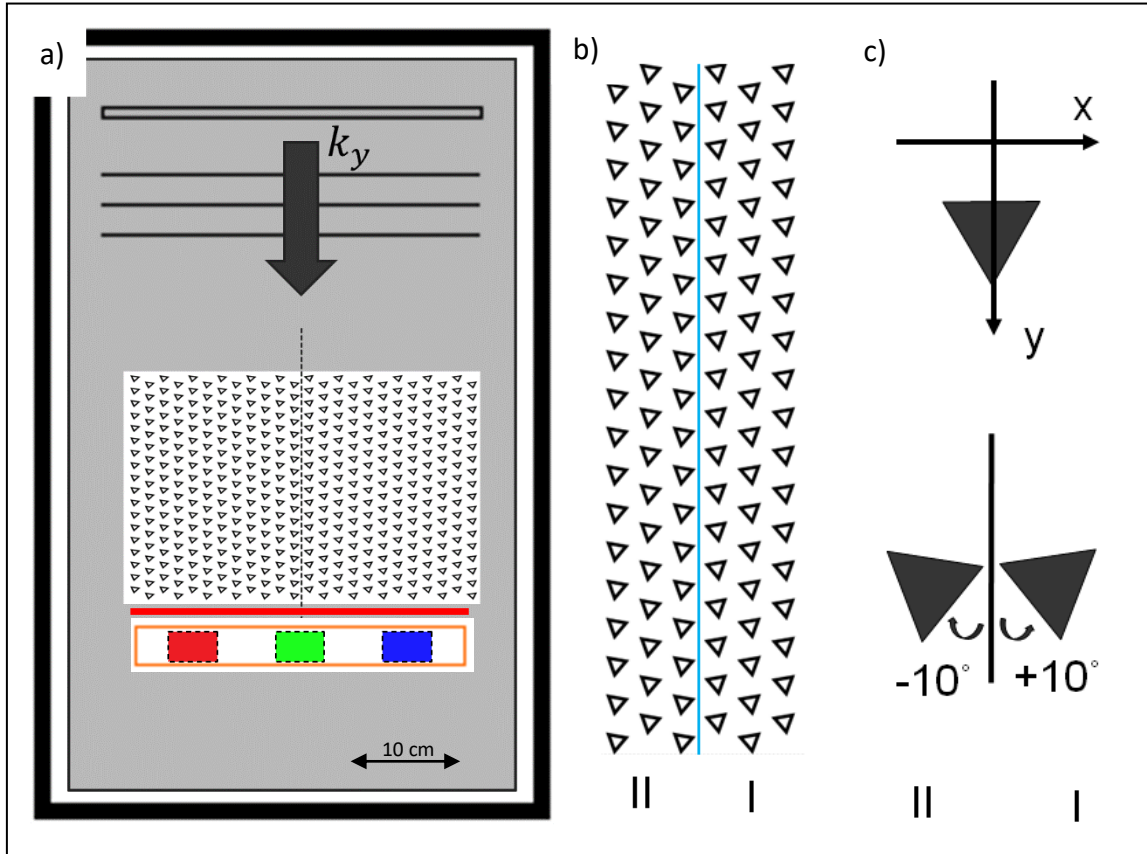


Figure 4.10: a) Experimental setup to investigate the existence of a topological edge mode; crystal size 15 cm x 32 cm. The three areas to evaluate the transmission profile are defined as left (red zone) at the exit of the domain II, channel right in front of the interface (green) and right at the exit of the domain I (blue). b) Zoom over the interface made by two crystals inversion images of each other. c) Angle rotation of the two domains.

The three areas have identical surface area in order to be able to compare them correctly. In Fig. (4.11) we show precisely where the windows for the spatial Fourier

transform are evaluated. If the topological mode exists, we should see a point in the k -space just for the window associated to the “channel” part (green window) for a specific frequency within the gap range. The other two windows are in front of a simple monotype crystal so they should not support any topological mode and in the k -space we should not see any particular point. This method of dividing in areas is the technique usually employed in the experiments involving topological modes in acoustic. Precisely, they place a microphone in each area that they want to compare in order to demonstrate that the transmission profile has a peak just for one of them. Our windows to evaluate the spatial Fourier transform play the role of the microphone.

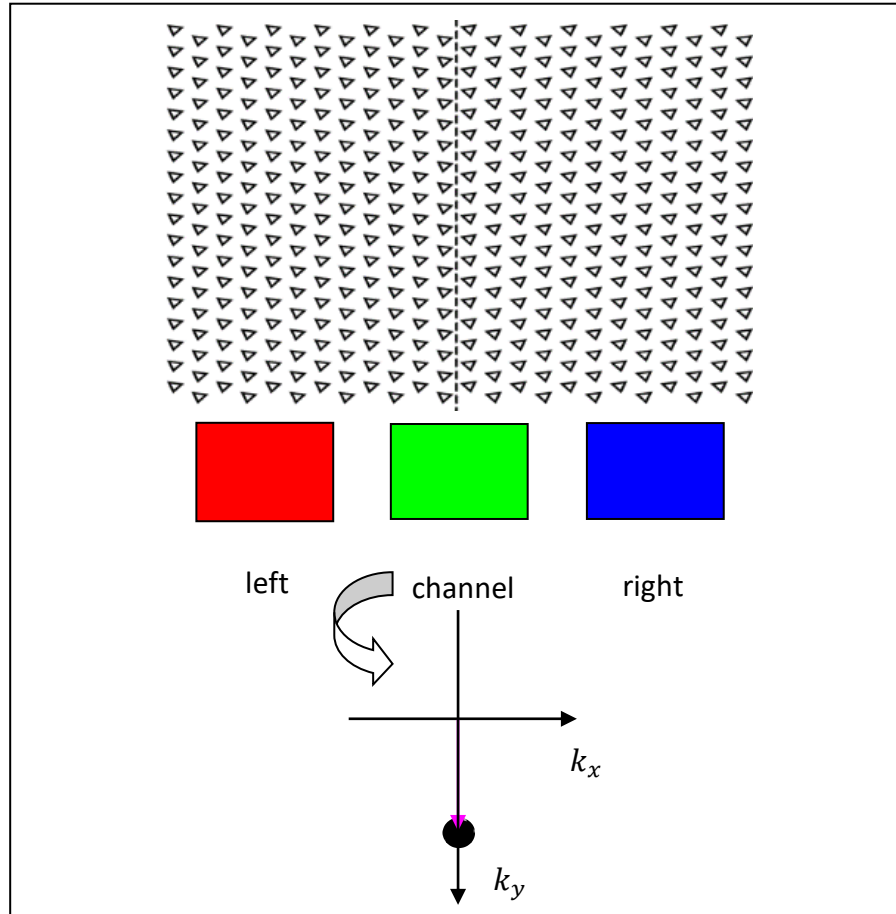


Figure 4.11: a) Schematic of the crystal and the three windows to evaluate the spatial Fourier transform. In the k -space, if a topological mode exists, we should see a point just for the green windows which corresponds to interface. The three areas to evaluate the transmission profile are defined as left (red zone) at the exit of the domain II, channel right in front of the interface (green) and right at the exit of the domain I (blue).

Let us focus on the first method for which there is an experimental difficulty. The first one is that the incoming wave can pass from the sides so, when we evaluate the temporal Fourier transform on the extremities of the red line, we collect also the direct

wave causing an increased amplitude. Moreover, the corners on the lower side cause a diffraction of the incoming wave. The reader could think about extending the crystal along the x-axis in order to block the incoming wave. That would be theoretically a good idea but from the experimental view if the membrane has perforations next to its borders its floatability is compromised resulting in water over the membrane when the source generator runs. The good compromise is to have at least 2 cm of membrane not perforated on the sides, that gives enough stability for this kind of measure. However, the crystal is large enough with respect to the x-axis to compare the three areas: "left", "channel" and "right". The line to perform the temporal Fourier transform has to be chosen right next the exit of the crystal in order to reduce the noise coming from the source.

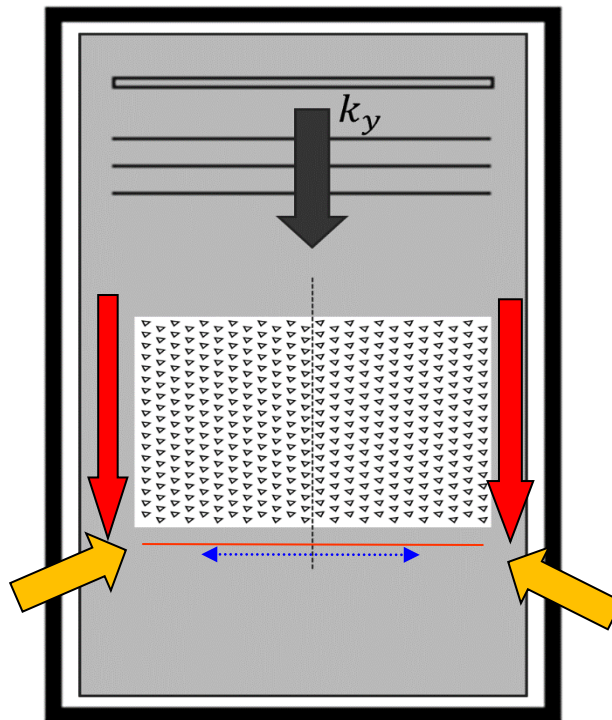


Figure 4.12: Schematic of the experimental setup where the red arrows indicate the propagation of the incoming wave which causes an increased amplitude applying the temporal Fourier transform over the red line. The orange arrows indicate the corners where the incoming wave is diffracted. The blue arrow indicates the line where the Fourier transform is evaluated to avoid the incoming wave.

Both techniques can be used together, we start from the spatial Fourier transform and we obtain the transmission profile shown in Fig. (4.13 a). From this profile we can extract several information:

- We can easily recognize the Bragg and hybridization gap, the first one, due to the spatial arrangement of the resonators, in the frequency range 40 – 52 Hz and the second one 82 – 98 Hz (wider compared to the Bragg one); in fact, like they are band gaps the amplitude goes to zero meaning that no propagation is allowed.
- The crystal max “transparency” is for $f = 38 \text{ Hz}$ and $f = 68 \text{ Hz}$; in other words, those are the local maxima in the transmission profile.
- Not only the frequency width of the two gaps is different, having a deeper look we can observe that the profile reaches its local minimum with a different slope, faster for the Bragg one. The explanation comes from the modes of the triangular cavities which are a linear combination of the basis modes meaning that their possible interaction is larger in term of frequencies.
- We can observe that the blue and red transmission profiles, corresponding to the monotype crystals have almost an identical behavior confirming that we are able to detect changes in the k-space in coherent way.

Let us see what happens for $f = 87 \text{ Hz}$, for the two domains (left and right) the transmission profile (red and blue lines) reaches its local minimum while the channel transmission (green) line does not. This frequency belongs to the band gap so for the monotype crystals there is no propagation. In Fig. (4.13), we can highlight this different behavior between the monotype crystals and the interface, calculating the ratio transmission profile channel/transmission left (and/or right). For $f = 87 \text{ Hz}$ it exists a mode only at the interface where the transmission is up to 6 times higher compared to the monotype crystal (left or right).

This is the first proof of the existence of a topological edge mode.

Now, we can use the other technique based on the temporal Fourier transform in order to visualize how the amplitude varies along the x-axis in a “continuous way” instead of using the windows strategy. We perform the temporal Fourier transform

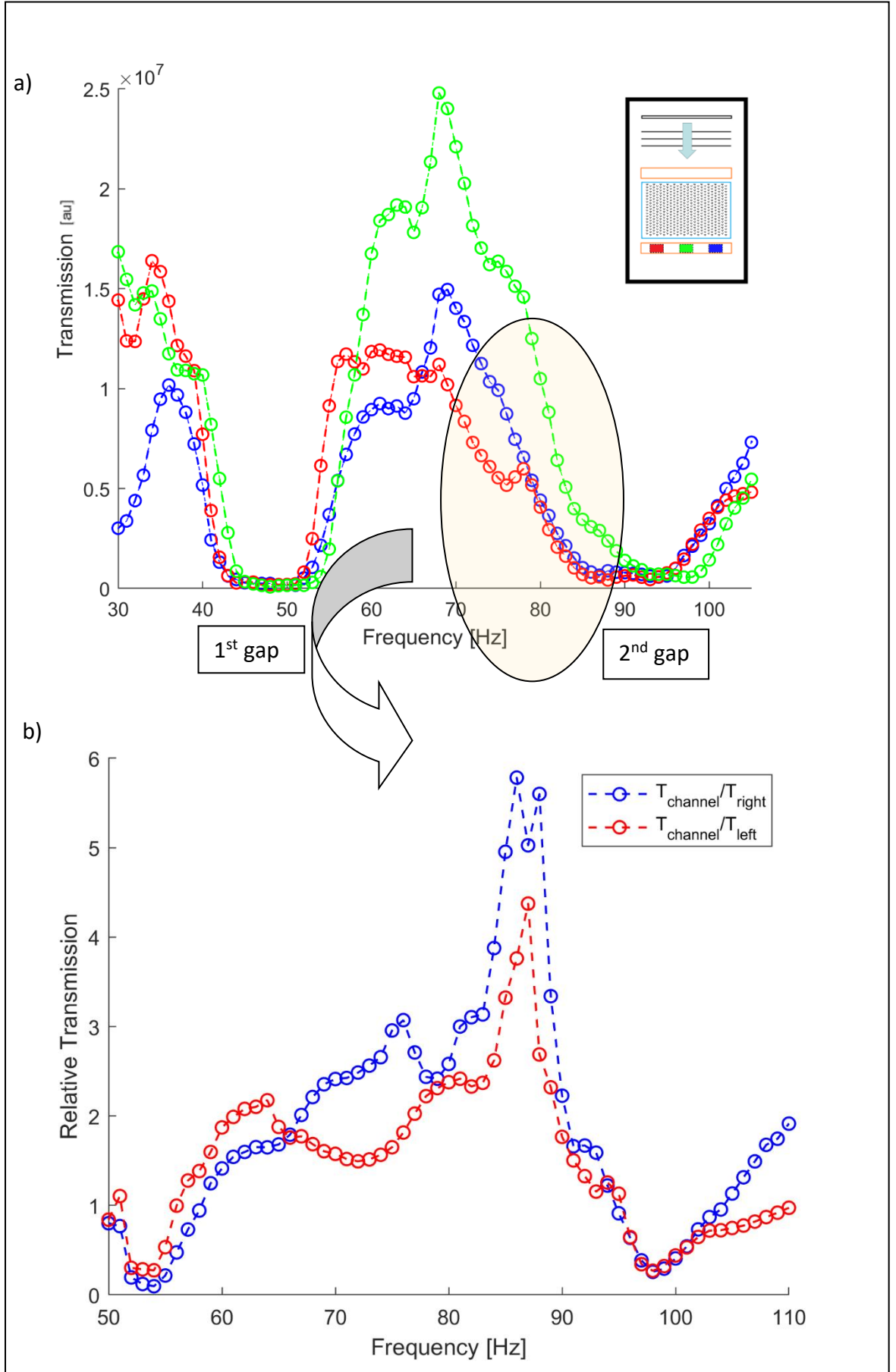


Figure 4.13: a) Transmission profile evaluated in three different positions denoted left (red), right (blue) and channel (green) at the crystal exit. b) Ratio between the transmission profile on the left (red line)/right (blue line) and in the center (corresponding to the channel); for $f=87$ Hz the energy coming out from the interface is almost 6 times compared to the sides.

over a line 16 cm long, where the channel is located at about 8,2 cm (blue arrow in Fig. 4.12). This line is centered with the interface and is located in front of the three areas “left”, “channel” and “right” avoiding the diffracted incoming wave from the sides as previously explained in Fig. (4.12). The result for $f = 87 \text{ Hz}$ which is the frequency found with the spatial technique, is shown in Fig. (4.14) in a graph amplitude-x position. We can easily see that we have a peak corresponding to the position 8,2 cm where the channel is. Far from the interface the amplitude decreases on both sides.

This technique confirms the presence of the topological edge mode and shows also the presence of a modulation along the x-axis. This is due to the crystal structure, in fact, the distance between all peaks is 2 cm which is 2 times the lattice constant. We can also observe that the peak is strongly localized at the interface of our crystal.

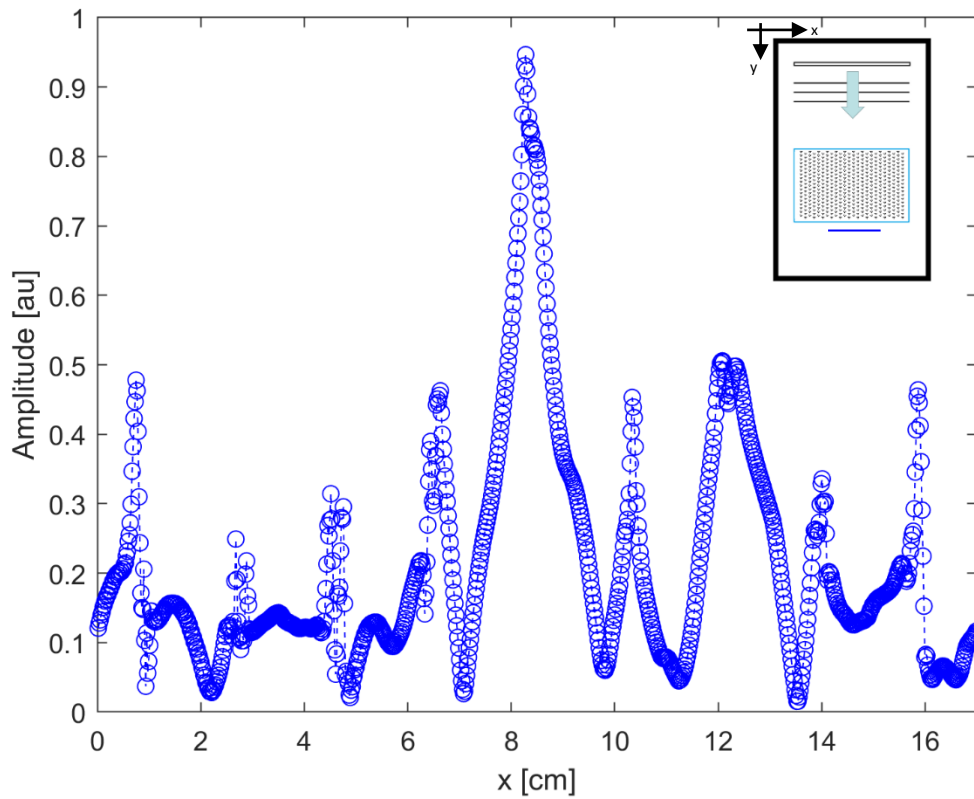


Figure 4.14: Temporal Fourier transform performed over a line right next the exit of the crystal, blue line in the sketch on the upper right corner. The channel is located at 8,2 cm where the amplitude reaches its maximum.

Both techniques confirm the presence of a topological edge mode at $f = 87 \text{ Hz}$ which is a frequency belonging to the band gap of our crystal. We demonstrated that a HEW system can be a new platform to observe interesting transport phenomena explained via Topology.

4.7 Conclusion

In this chapter we saw how a 2D pattern of regular perforations on the silicone membrane, the so-called artificial monotype crystal for HEWs, modifies the dispersion relation showing band gaps due to the Bragg scattering and the local resonances (hybridization gap). This is the first step to understand how a more complex crystal characterized by interesting transport phenomena can be realized. Then, using the concepts of Dirac cone and 4-fold degeneracy illustrated via the dispersion relation of a 2D honeycomb lattice and a triangular lattice, we focused on a structure that supports a topological edge mode that is experimentally observed in a similar experiment performed in Acoustics employing a phononic (or sonic) crystal composed by triangular scatterers [63]. This structure is based on the valley transport theory and does not require the time-reversal breaking like for a Quantum Hall insulator. We presented the experimental results concerning a similar crystal (adapted for HEWs), made by two monotype patterns of triangular resonators with different rotation angles. The interface between them sustains a topological edge mode and we experimentally show its presence within the frequency range of the hybridization gap. In other words, we proved that it is possible to obtain a classical equivalent of the QSH effect for HEWs extending the list of classical physical experiments where such effect has been observed.

Chapter 5: Topological insulator: hexagonal crystal

5.1 Introduction

In this chapter, we present a different triangular lattice compared to the previous one, still composed by triangular resonators and same lattice constant but with a hexagonal shape and different kind of interfaces that should support a topological edge mode. Or, in other words we propose a new structure, characterized by different rotation angles for the resonators, where a topological phase transition should occur.

So far, we observed an edge mode only for a straight interface between two patterns with rotation angle $\alpha = 10^\circ$ and $\alpha = -10^\circ$. The goal is to demonstrate that it is possible for a HEW to follow a particular path different from a straight line. This path is the result of two interfaces, forming a V shape within the hexagonal crystal. We remind that the triangular lattice supports a fourfold degeneracy at the double Dirac cone which is suitable to create the classical equivalent of a Quantum Spin Hall insulator. In fact, when two triangular lattices are put together, at the interface, the four-fold degeneracy can hybridize forming two helical modes (called also symmetric and antisymmetric modes) to mimic the QSH effect.

As we know, the geometry of the problem plays a crucial role in term of topological phase transitions, consequently we will discuss the geometry of our new crystal constructed by rotating the resonators with a particular rotation angle ($\alpha = 30^\circ$ and $\alpha = -30^\circ$). The latter has two important implications that we will examine in detail: it corresponds to the angle for which the band gap reaches its maximum (like experimentally demonstrated in [105]), and makes the interface bidirectional meaning that a mode can be channelled into the interface from both sides of the crystal. Then, we will show the band diagram in order to observe the Bragg and hybridization gaps like previously done, and we will see how an incoming wave can be channelled into a particular path showing the transmission profile, based on the Fourier analysis expressly adapted for this problem, when the topological transition occurs.

5.2 Hexagonal crystal for HEW

Let us consider the usual triangular pattern of triangular resonators and discuss the characteristics of such a crystal when a particular rotation angle α is applied to its resonators. This degree of freedom has an important impact on the symmetries of the problem and allow us to study a different crystal with two interfaces, one supporting an edge mode and the other one blocking it.

In Chapter 4, we saw the dispersion relation related to a sonic crystal, Fig. 4.4, where the gap opened because of the breaking of the mirror symmetry (in that case $\alpha = 10^\circ$ (-10°)). In other words, the rotation angle implies the preservation or not of this symmetry, that is why it plays an important role but it is not the only reason. In terms of dispersion relation, it causes the so-called lifting process of the double Dirac cone (true for any α excepting zero) resulting in a band gap. We remind that the double Dirac cone corresponds to the intersection in the band gap that we can observe in a 2D dispersion relation which is supported by the triangular lattice; it can also be view as a 4-fold degeneracy and, as previously seen, it is a necessary condition to mimic the QSH effect. As we saw, the lifting process is caused by any α , but we could wonder if there is one value for which our crystal has a different behavior compared to the crystal presented in Chapter 4. In order to reply to this question, let us refer to the results obtained by Laforge *et al.* [105] in a water wave system, Fig. (5.1). In particular, in Fig. (5.1 a) we can see the dispersion relation represented with the usual critical points ΓKM on the x-axis. This result is totally similar to the previous example in acoustics (Fig. 4.4). Focusing on the gap, there exists a particular rotation angle for which the band gap reaches its maximum width: for $\alpha = |30^\circ|$ (demonstrated in [105]) and it is what we can see in Fig. (5.1 a). Starting by the rotation angle $\alpha = 0^\circ$, Fig. (5.1 b) for which we have the gap closing process (4-fold degeneracy or double Dirac cone), we can see how the gap opens tuning α (Fig. 5.1 c); the latter is represented on the x-axis with the direction ΓK which is another way to indicate the usual k_y .

Using $\alpha = |30^\circ|$ implies three important consequences:

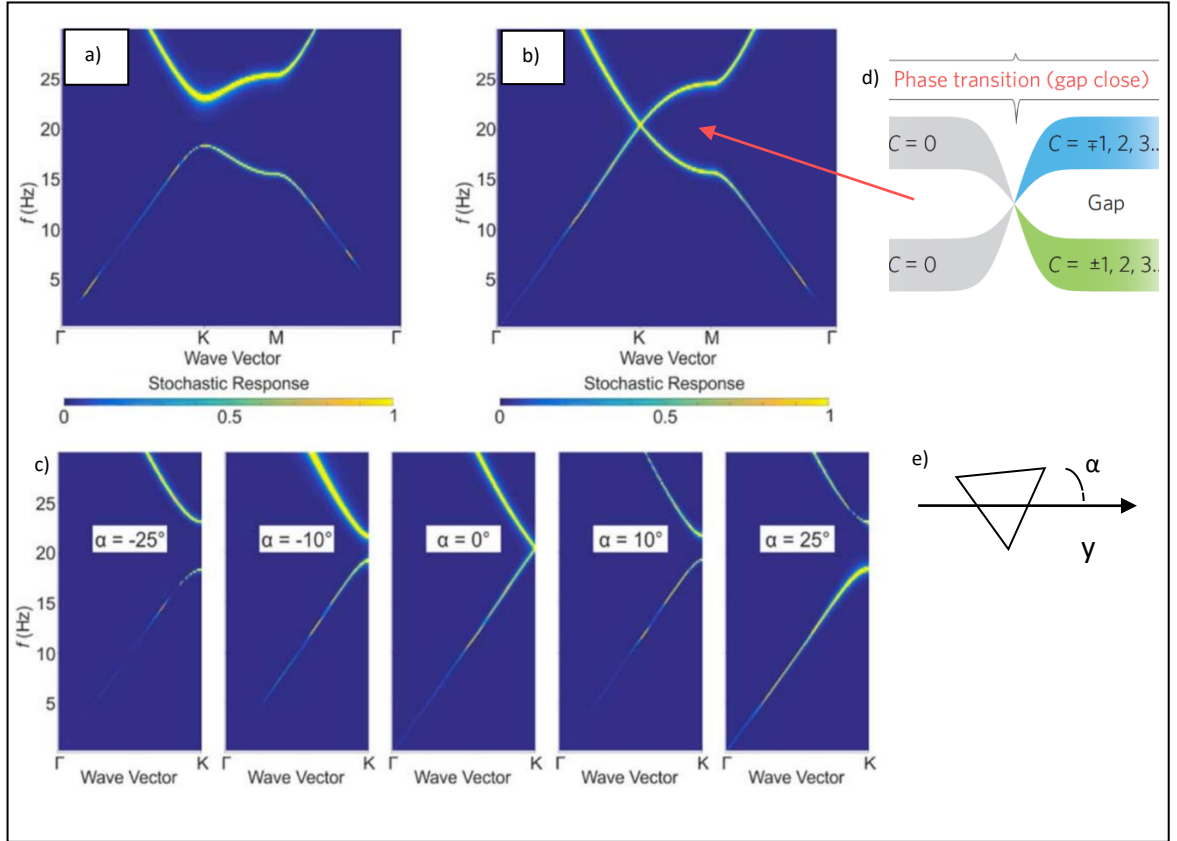


Figure 5.1: a) Dispersion relation for a triangular lattice of triangular scatters (in a water wave system), represented using the typical critical points Γ KM, for $\alpha = 30^\circ$. For this rotation angle, the band gap reaches its maximum width. b) Gap closing process obtained for $\alpha = 0^\circ$; the intersection corresponds to the so-called Dirac point and the four curves to Dirac cones. c) Dispersion relation in the direction Γ K (which corresponds to k_y) for different rotation angles α ; the gap opens reaching its maximum width for $\alpha = 30^\circ$. d) Sketch of a topological phase transition which is characterized by a gap closing process involving a 4-fold degeneracy within the band gap. e) Rotation angle α for the triangular scatterers. a-c) Adapted from [105], d) from [62].

- 1) the gap width reaches its maximum and this allows us to have a well-defined area where we can investigate the existence of an edge mode.
- 2) it allows us to create a crystal where we can test two kinds of interfaces within the same area (more details in Fig. 5.2).
- 3) the interface supporting the edge mode is bi-directional meaning that the edge mode can propagate both ways.

For practical reason, to identify a crystal, we will give the label A to the crystal with the resonators oriented like in Fig. (5.2 c) and the label B to the same crystal with rotation angle $\alpha = -30^\circ$. Then, we define a domain wall (DW) constructed putting together a crystal A and a crystal B. Consequently, we have two possible DW:

- DW A-B, when crystal A and B are placed as shown in Fig. (5.2 a)

- DW B-A, when crystal A and B are placed as shown in Fig. (5.2 b)

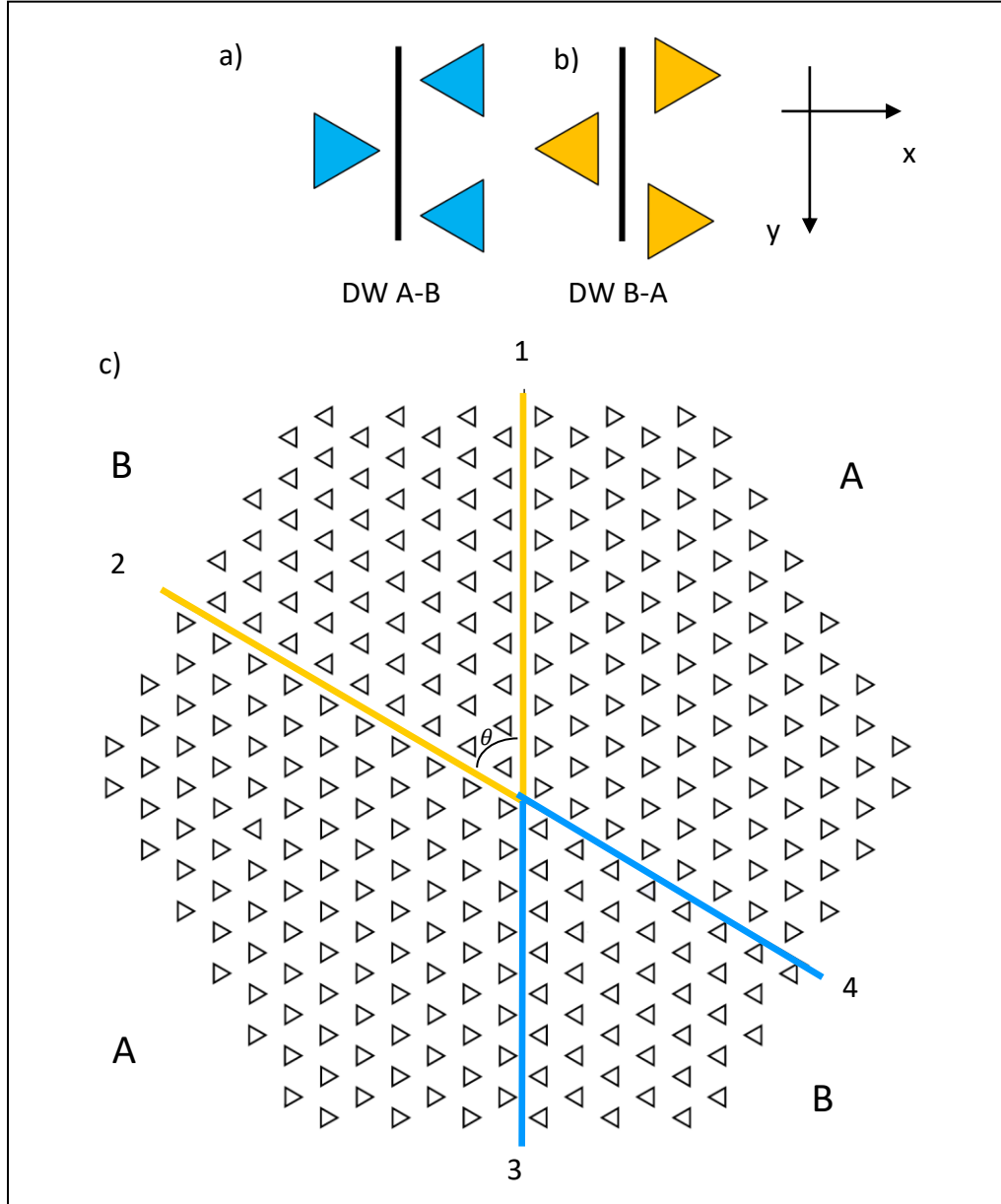


Figure 5.2: a) Domain wall A-B. b) Domain wall B-A. c) Schematic illustration of the hexagonal crystal with 4 interfaces, the path 1-2 allows the propagation of a topological edge mode (orange line); the latter forms an angle $\theta = 60^\circ$. The interfaces 3 and 4 block any propagation (blue line). Hexagon side $l=10$ cm. Crystal adapted from [106].

The band inversion occurs when DW B-A is considered as demonstrated in previous works as [106], meaning that it is possible to observe a topological edge mode in such a domain. We take advantage of this important result to demonstrate that we can obtain interesting transport properties for HEW as a mode following a V-path. We can construct a crystal with both DW in order to show that an edge mode can propagate just through DW B-A and not through DW A-B. This allows to confirm within the same measurement, which interface supports a topological edge mode. In order to do so, we

start from a lattice with a hexagonal shape divided in four areas: two patterns type A and two patterns type B. In this case the referring axis for the crystals A and B is the vertical axis (Fig. 5.2 a-b). The four patterns are arranged as shown in Fig. (5.2 c) and they form four interfaces, as it was proposed by Xia *et al.* [106] for a phononic crystal. Two of them allow the propagation of a topological edge mode because they are a DW B-A (orange line), on the opposite the propagation is not allowed for the other two (blue line).

5.3 Experimental results with a hexagonal crystal

We start by filling up with water the glass tank up to 17 cm high; we put a membrane of the tank size on the water as illustrated in Fig. (5.3 a); the distance between resonators is 1 cm (side of the hexagon $l = 10$ cm) and their height is 4 mm as for the previous crystal analyzed in Chapter 4. The source generator is a straight paddle, 24 cm long, that generates plane waves in the frequency range 38-120 Hz. For each frequency we record 100 frames (CCD $fps = 136,0$) and we reconstruct the corresponding height field taking advantage of the spectral folding which allows us to reconstruct the signal even when the Nyquist condition is not respected (like in our case). The fps is higher than we saw in Chapter 4 because the hexagonal crystal is smaller compared to the previous one and so the frame recorded; this implies an increased fps . Now, we use a temporal Fourier transform over the sequence recorded to select only the frequency component equal to the frequency used by the source wave generator, in other words we filter the signal at the working (or input) frequency. We remind that our system and the effects (output) that we want to observe do not cause a modification of the input frequency so, if a topological mode exists it will have the same frequency imposed by the source generator. That is why we apply this filtering process, because the information that we want to extract just concerns the frequency component equal to the input frequency. We apply then a spatial Fourier transform on the filtered result over the incident area above the hexagonal crystal and over the transmitted area just below the crystal in order to evaluate the ratio transmitted wave/incident wave (dashed lines Fig. 5.3 a). The two areas have identical surface area in order to correctly evaluate the aforementioned ratio. The goal in this case is to see how the amplitude transmitted (output wave after the crystal) varies in the working frequency range. As we know, our system has a frequency-dependent

response and that is what we want to investigate showing the so-called transmission profile. Precisely, in order to normalize this profile, we evaluate the ratio transmitted wave/incident wave.

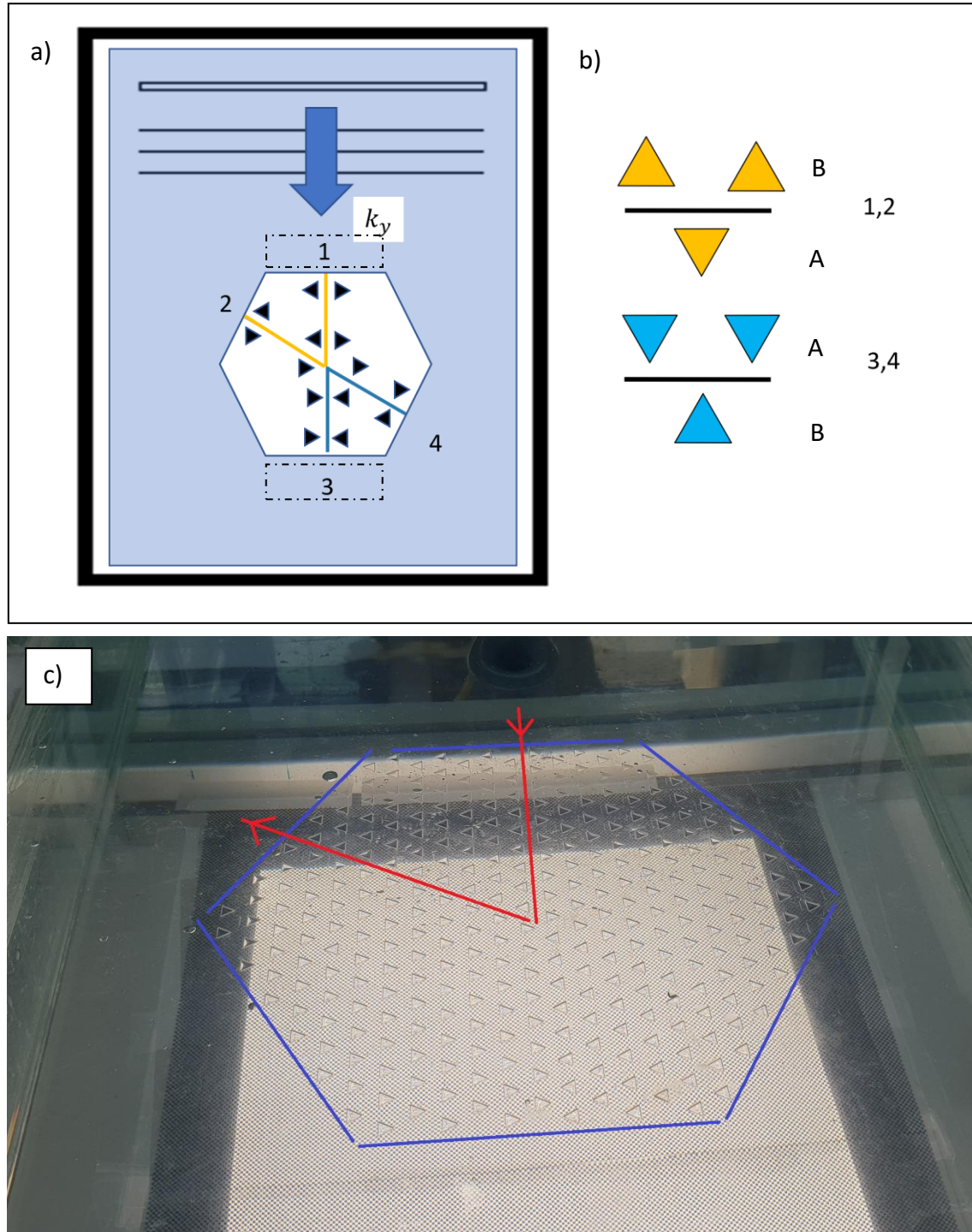


Figure 5.3: a) Sketch of the experimental setup, a plane wave interact with the hexagonal crystal characterized by 4 interfaces; the orange lines indicate the domain wall that allows the propagation of a topological edge mode, on the contrary the blue lines indicate the interfaces where no propagation is allowed. b) Domain walls B-A (orange) and A-B (bleu). c) Photo of the hexagonal crystal for HEW used to verify that an incoming wave (red line) can be channelled

through the DW B-A; the purple lines indicate the hexagonal crystal obtained as a pattern of regular perforations on the silicone membrane. The side of the hexagon is 10 cm long.

Repeating the whole procedure involving the Fourier transforms, for each frequency and calculating the previous ratio we obtain the transmission profile expressed as percentage, see Fig. (5.4). From this profile we can extract several information:

- We can easily recognize the Bragg and hybridization gap, in fact, the transmission profile reaches its local minimum in two distinguished frequency ranges (as seen for the previous crystal in Chapter 4). The first gap, due to the spatial arrangement of the resonators, is located in the frequency range 43 – 58 Hz while the second one in the frequency range 85 – 110 Hz. We can observe that these ranges are similar but not equal compared to the previous crystal. That is due to the different rotation angle of the resonators, in fact, this degree of freedom causes a different lift in the band diagram as seen in Fig. (5.1 c) resulting in a different frequency range for the band gap.
- We can observe that the hybridization gap is wider compared to the Bragg one but not only the frequency width of the two gaps is different, having a closer look we can observe that the profile reaches its local minimum with a different slope, faster for the Bragg one. The explanation comes from the modes of the triangular cavities which are a linear combination of the basis modes meaning that their possible interaction is larger in term of frequencies. In our model we take into account just the ground mode which, as we saw in Chapter 3, is the main term of the linear combination of the basis.
- The crystal is “transparent” for $f < 41 \text{ Hz}$, $70 < f < 80 \text{ Hz}$ and $f > 120 \text{ Hz}$, meaning that the propagation of the HEW is allowed. However, even in the transparent window, the transmission profile never reaches 100% and actually it never exceeds 40%. That is caused by the attenuation losses passing through the crystal where the perforations can be seen as scatterers (the hexagon is 18 cm long in the direction of the propagation axis y).
- We can observe that the percentage of transmission profile for the three local maxima, corresponding to the transparent windows, decreases from 40% to 12% as the frequency increases; this is caused by the attenuation, accordingly to our previous measurements, it plays a dominant role for the high

frequencies and that explains why we chose these particular geometrical properties (lattice constant and height of the triangular resonators) for our crystals. Another choice of the lattice constant or of the height of the resonators would move the band gaps in frequency regions where the analysis would be compromised or difficult with our experimental setup.

- The transmission profile never goes to zero; a possible explication is the fact that we are not dealing with a real crystal but with an artificial one, where the pseudoatoms or scatterers are not strongly localized as in real crystals, allowing then, a small percentage (2-3%) to pass through the artificial crystal even in the band gaps.

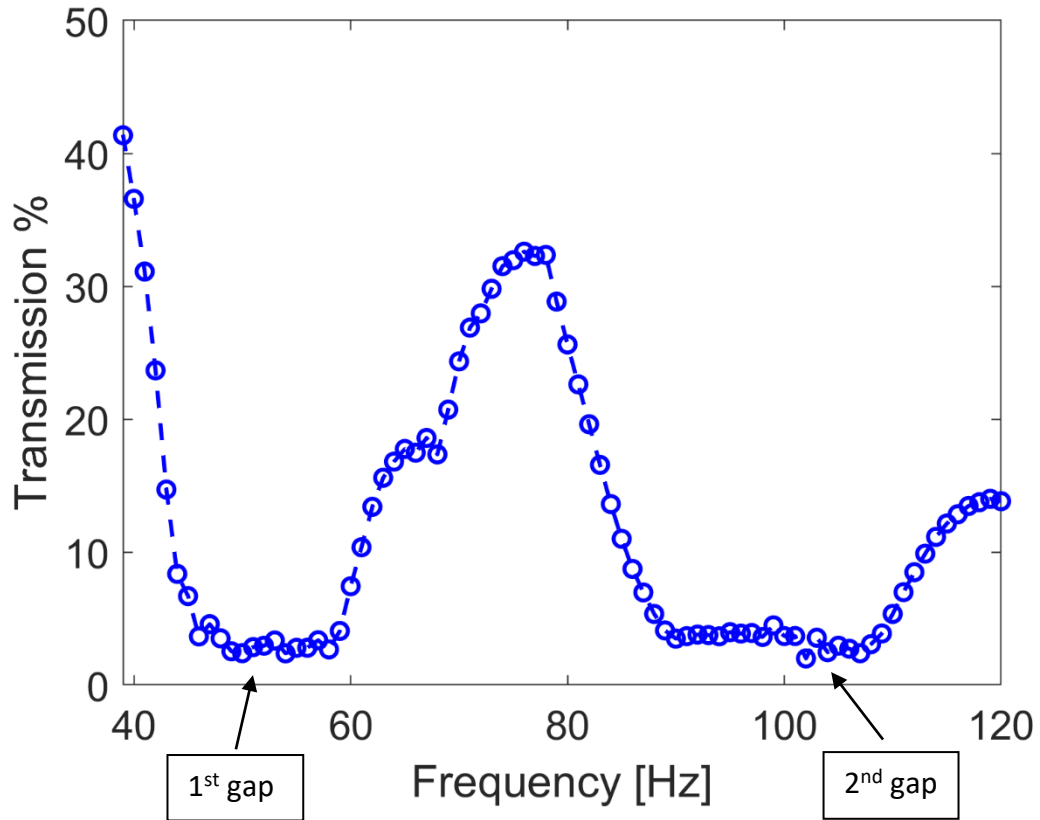


Figure 5.4: Transmission profile for the hexagonal crystal for HEW where we can observe the two gaps: type Bragg and hybridization. The 1st gap is located in the frequency range 43-58 Hz while the second one in the frequency range 85-110 Hz. The crystal is “transparent” for $f < 41$ Hz, $70 < f < 80$ Hz and $f > 120$ Hz.

Now, that the transmission profile is fully described we can pass to some examples of height field reconstruction, shown in Fig. (5.5). We selected four frames to show when

the crystal is transparent Fig. (5.5 a-b) and when it acts like a mirror (no propagation allowed, Fig. 5.5 c-d).

It is worth having a closer look at the frames corresponding to the two gaps (Fig. 5.5 c-d). The frame in Fig. (5.5 c), at $f = 47 \text{ Hz}$ corresponding to the Bragg type, shows a very flat height field (strongly green according to the color bar so almost zero) within the hexagonal crystal confirming that it acts like a mirror (excepting for the first lines of resonators on the top where we have an evanescent wave).

The frame in Fig. (5.5 d), at $f = 87 \text{ Hz}$ corresponding to the hybridization type, which still corresponds to a gap, does not present a flat height field (it has a mix of yellow and blue colors indicating a relevant surface elevation). The explanation comes from the fact that the triangular perforations resonate and interact with the incoming wave producing a relevant surface elevation within the crystal area. Globally, out of the crystal area because of those interactions the transmitted amplitude reaches its minimum. In other words, the resonators absorb the energy of the incoming wave.

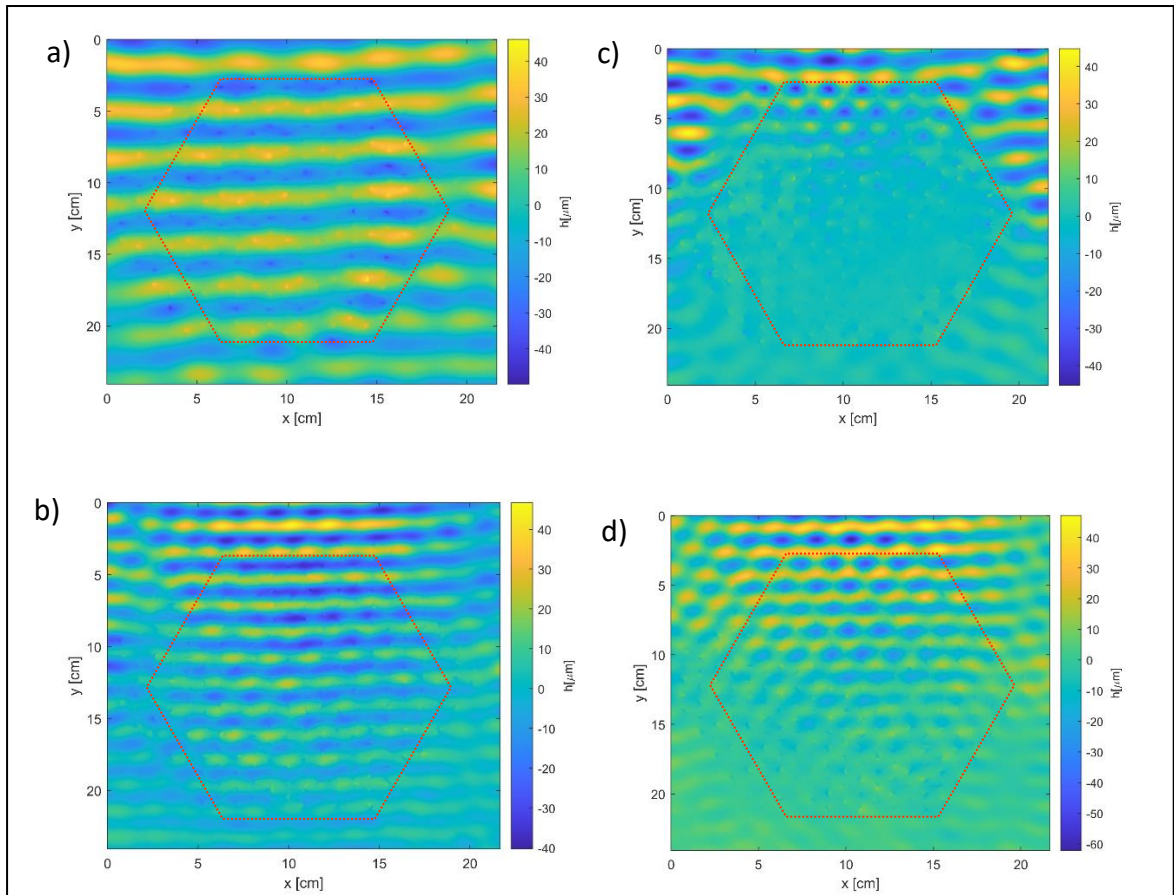


Figure 5.5: Height field reconstruction of the hexagonal crystal for $f=20 \text{ Hz}$ (a), $f=75 \text{ Hz}$ (b), $f=47 \text{ Hz}$ (c) and $f=87 \text{ Hz}$ (d). Referring to (a-b) the crystal is “transparent” while for (c-d) the propagation is not allowed. The red dotted lines indicate the hexagonal crystal.

5.4 Observation of a topological edge mode via Fourier analysis

Now, that the gaps are defined and experimentally observed, we want to focus on the path 1-2 in order to see if we are able to detect a mode propagating through this interface which is the goal of our experiment. As we know, if it exists its frequency belongs to the frequency range of a gap where the topological phase transition occurs. Referring to the experiments concerning topological modes in acoustics (for example [106]) we can observe two important experimental differences with our system:

- 1) the sound wave generator is placed right next to the interface, in other words, the excitation is localized at the interface while for us the wave generator excites the whole membrane.
- 2) the transmission is measured with a microphone at the end of the path 1-2 and compared to the transmission through another part of the hexagon where there is no interface to support a topological mode.

To better visualize it please refer to Fig. (5.6) where there is an example of an artificial sonic crystal formed by triangular scatterers with a wave source on the left next to the interface (indicated by the number 1) and a movable microphone on the right to measure the transmission profile at the output [106]. Both differences concern the position where the input is generated and where the output is recorded. This is necessary to have a clear signal to analyze. Imaging the source or microphone far from the interface, the resulting signals would be more difficult to recognize, in fact, focusing on the microphone, it would record something coming from all directions so it would be hard to separate what comes from the interface and what from somewhere else.

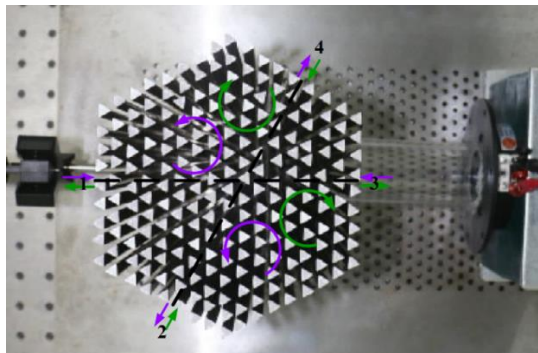


Figure 5.6: Sonic artificial crystal to study edge modes, with a source next to the input (1) and a microphone right next the interface (3) to record the transmitted wave. Adapted from [106].

There is also a second reason: the reconstruction of the height field is based on images. If the source is localized at the interface the images recorded would contain an area covered by the source so the reconstruction would be partially compromised. Focusing on the output, we do not dispose of a simple microphone for HEWs. In order to mimic the microphone's role, our strategy to measure a clear signal with HEWs is to use temporal and spatial Fourier transforms over the sequences recorded. Considering that a plane wave is simply a point in the k -space along the k_y axis we are able to easily separate the incoming wave generated by the straight paddle from a plane wave propagating in a different direction. As we made an interface with a V shape forming an angle of $\theta_{output} = 60^\circ$ with respect to the vertical axis y , if a topological edge mode propagates through that interface, we should just see a point, in k -space, oriented with such an angle. Moreover, as we use a temporal Fourier transform to filter the signal at the working frequency, we have also the information about the propagation direction (performing just a spatial Fourier transform over a frame, for a plane wave we would obtain two symmetrical points in the k -space).

This easily readable k -space employing plane waves is why we did not use a circular wave generator at the interface, which would have the advantage of being a localized source wave (as for the acoustic experiments) but would create multiple reflections in the tank originating a noisy signal to analyze. Precisely, in the k -space, considering waves in all directions as the result of a circular wave source, we would see a circle, in other words a collection of point for $\theta \in [0, 2\pi]$ (or part of it depending on the precise window adopted) and it would not possible to separate the source wave from the topological edge mode because it would be hidden by the incident wave or reflections of the incident wave. We provide an example of height field reconstruction employing a circular wave source right next the interface in Fig. (5.7). In particular, Fig. (5.7 a) is at $f = 30 \text{ Hz}$ and Fig. (5.7 b) at $f = 75 \text{ Hz}$, for both the red dashed lines indicate the hexagonal crystal. In Fig. (5.7 c-d) we can see the result in the k -space for both cases. As previously discussed, we have points in almost all directions compromising the research of a particular mode propagating in a precise direction. As we need to perform a spatial Fourier transform it is important to see how we should choice the correct window to evaluate it. Les us consider the frame in Fig. (5.5 a) which shows that the crystal is transparent. We can see that the incoming wave can pass through

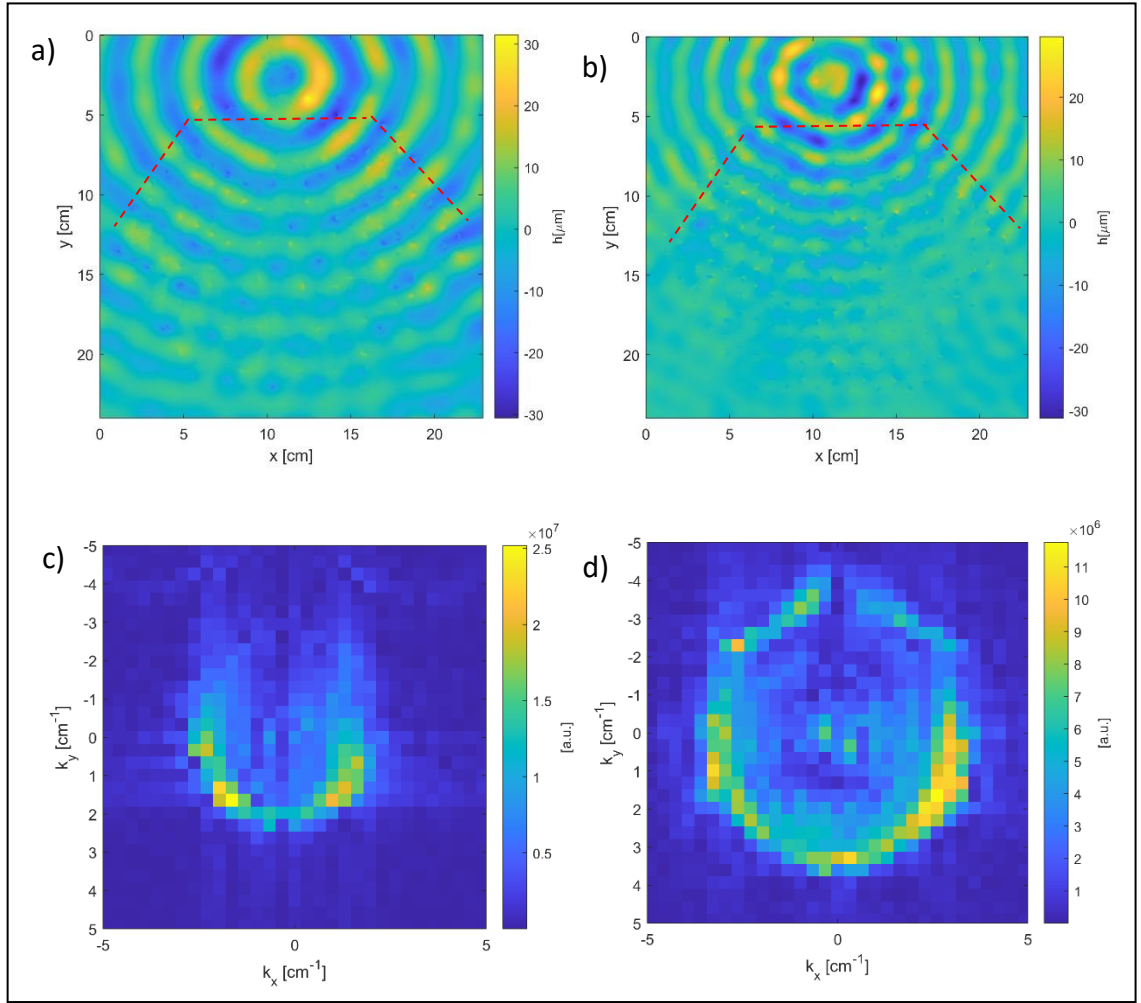


Figure 5.7: a) Height field reconstruction at $f=30$ Hz using a circular wave source placed right next to the upper side of the hexagonal crystal. b) Same reconstruction at $f=75$ Hz. The red dashed lines indicate the upper part of the hexagonal crystal. c) Spatial Fourier transform over the whole frame (a) and frame (b). The result in the k -space does not allow us to make the difference between the source wave and a mode propagating in the direction $\theta = 60^\circ$.

the crystal and it is globally a plane wave. If the window is the whole frame or a part of it, the result in the k -space is always a bright point on the k_y axis. So, in this case, the windows selected has not a major impact in the k -space (except for the resolution), see Fig (5.8 a) where both images show a bright point on the k_y . We cannot say the same if we consider the frame in Fig. (5.5 c). In this case, if we select a window in the upper part ($x \in (0,22), y \in (0,5)$) or within the crystal area ($x \in (5,15), y \in (12,20)$), the result in the k -space is different. In fact, just for the upper part we have a bright point on the k_y axis in the k -space because there is plane wave while in the crystal there is no propagation for that particular frequency (band gap) resulting in a bright area in the middle (0,0) of the k -space, see Fig. (5.8 b).

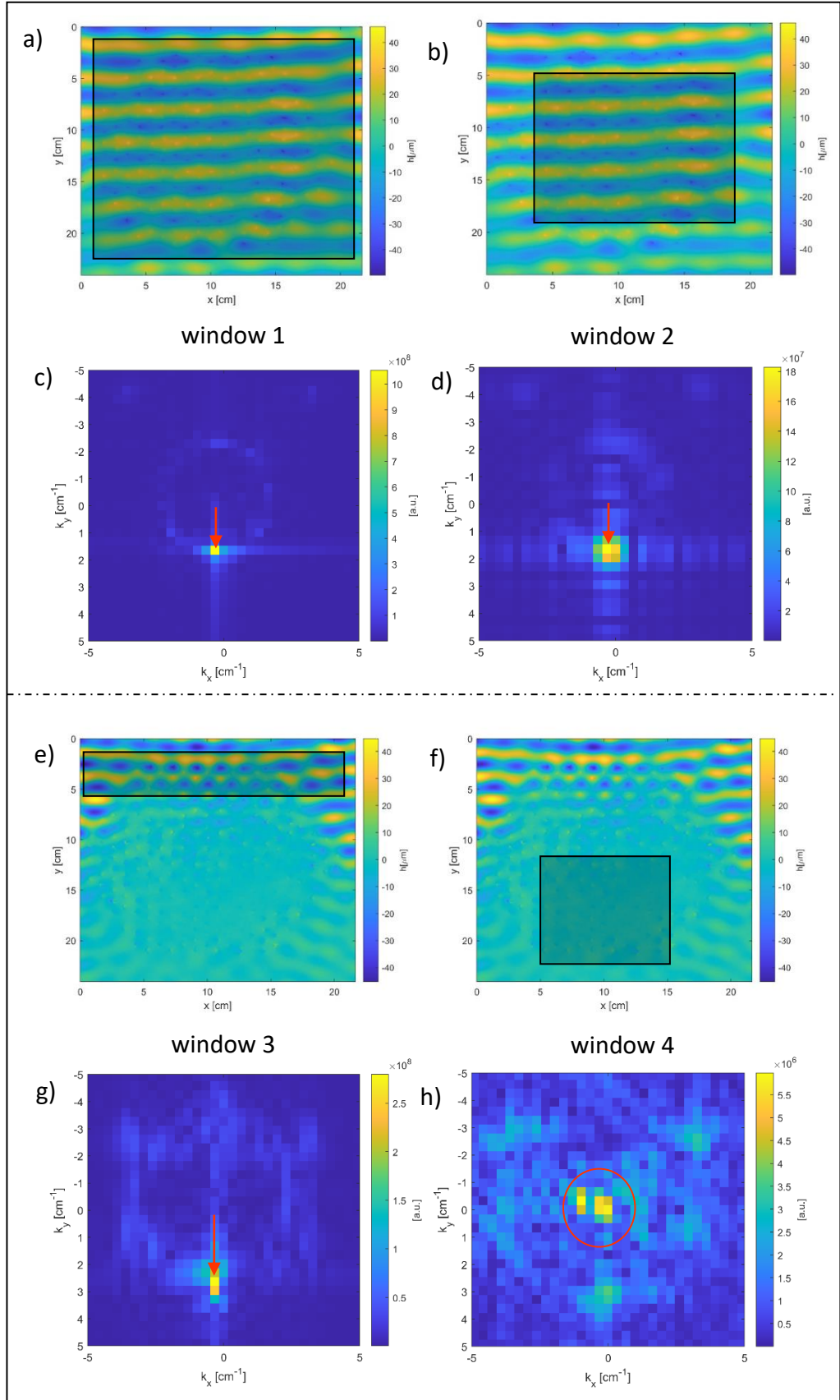


Figure 5.8: Height field reconstruction for $f=20$ Hz showing the window 1 (a) covering almost all the frame (shadow area) and window 2 covering part of it (b). Spatial Fourier transform using window 1 (c) and window 2 (d); both provide a very similar result. Height field reconstruction for $f=47$ Hz showing the window 3 (e) in the upper part of the frame and the window 4 in the middle area (f). Spatial Fourier transform using window 3 (g) and window 4 (h); the result is strongly window-dependent.

In order to detect a topological edge mode, we select two windows of identical size (necessary condition to correctly compare them), one containing the interface that should support such a mode and one without interface which does not support any edge mode as seen for the monotype crystals in Chapter 4 (playing the role of two “microphones”). We apply then a spatial Fourier transform, temporally filtered as previously seen, to the upper left corner first (the window selected is shown in Fig. (5.9 a) $x \in (0,10), y \in (0,12)$) and similarly to the upper right corner ($x \in (10,20), y \in (0,12)$). In order to verify the existence of a topological edge mode on the path 1-2, we need to observe, in the k-space, a point as represented by the blue arrow in Fig. (5.9 b), meaning that from that side of the hexagonal crystal, an incoming wave previously channeled in the path 1-2, comes out.

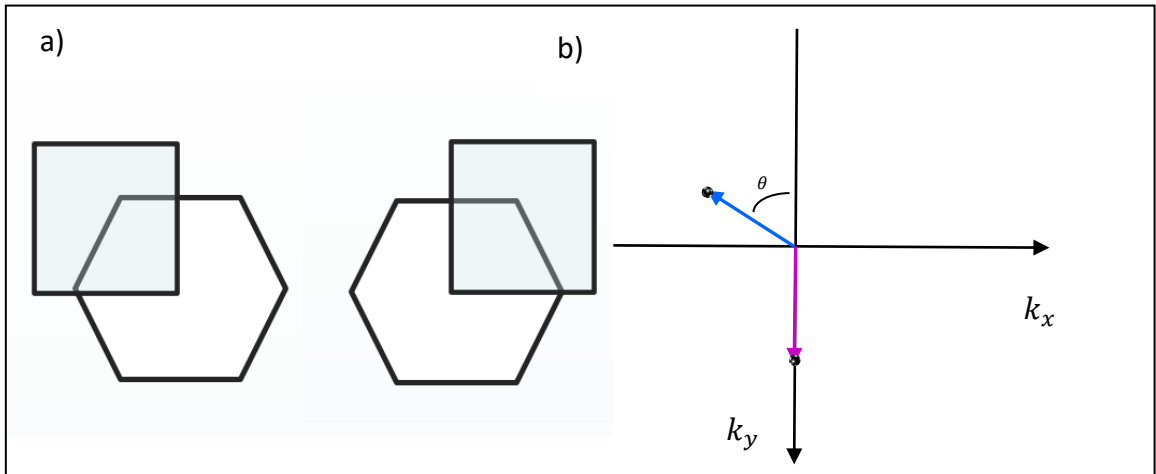


Figure 5.9: a) Windows used for the spatial Fourier transform, upper left and upper right corner of the hexagonal crystal (shadow area). b) Expected result from the spatial Fourier transform applied on the left window, the purple arrow indicates the incoming wave while the blue line indicates the topological edge mode via the path 1-2 ($\theta = 60^\circ$).

We apply the spatial Fourier transform and we attentively study each frame in the frequency range. Scanning all frequencies, we found its maximum, for the area pointed by the blue arrow (Fig. 5.9 b), for $f = 87 \text{ Hz}$, where making the ratio between the energy of the transmitted wave and the incoming wave we obtain 80%. In other words, there exists such a mode (Fig. 5.10 c) and has limited attenuation losses. It is inevitable for such a system because we remind that we are dealing with pseudo-spin

states (so backscattering is not totally suppressed) and not real Quantum physics systems where we have spin-locked transport meaning that there is no backscattering.

Let us discuss all frames: in Fig. (5.10 a-d) we report four spatial Fourier transforms, previously filtered taking into account the temporal information that we possess with the recorded sequence of frames, to show the appearance of the topological edge mode in the second gap. Let us start with Fig. (5.10 a), we can easily see that there is a bright point on the y-axis meaning that for $f = 70 \text{ Hz}$, in the window selected, we just have the incident wave, in fact the point belongs to the vertical axis. For $f = 77 \text{ Hz}$ we have Fig. (5.10 b) which looks similar to the previous result but two more points come out meaning that two waves are reflected from the upper left and right side of the crystal. The explanation comes from the fact that in the gap the crystal acts like a mirror so part of the incident wave is reflected with an angle $\theta = 60^\circ$ (and -60°) with respect to the vertical axis. Again, we have mostly the effect of the incident wave. Fig. (5.10 c) is the main result, this time we have two bright points and one on the right part caused by the reflection of the right side of the crystal (like seen before for $f = 77 \text{ Hz}$). The two bright points are located as expected (Fig. 5.9 b to compare to Fig. 5.10 c) or to rephrase it, we found a mode propagating with an angle $\theta = 60^\circ$ with respect to the vertical axis. This mode does not exist on the right side where we can just see the mirror effect of the crystal. The topological mode exists then where the DW B-A is located.

It is possible to artificially reconstruct the wave propagation of this mode calculating the inverse Fourier transform over the bright point associated to the edge mode obtaining the result in Fig. (5.11). From the latter it is possible to extract another information, the transmitted wave, in terms of height field, is strongly localized in the center (about 2 cm long), where the interface is (red dashed lines). This reminds what we obtained for the previous crystal with a straight interface where the peak was strongly localized around the interface's position ($x = 8,2 \text{ cm}$), see Fig. (4.14). In the end, for the left side, Fig. (5.10 d) simply shows that after the frequency corresponding to pseudo-spin state we come back to the original situation where only the incoming wave is detected (as in Fig. 5.10 a), a point on the vertical axis with the two mirror points due to the reflections caused by the crystal structure.

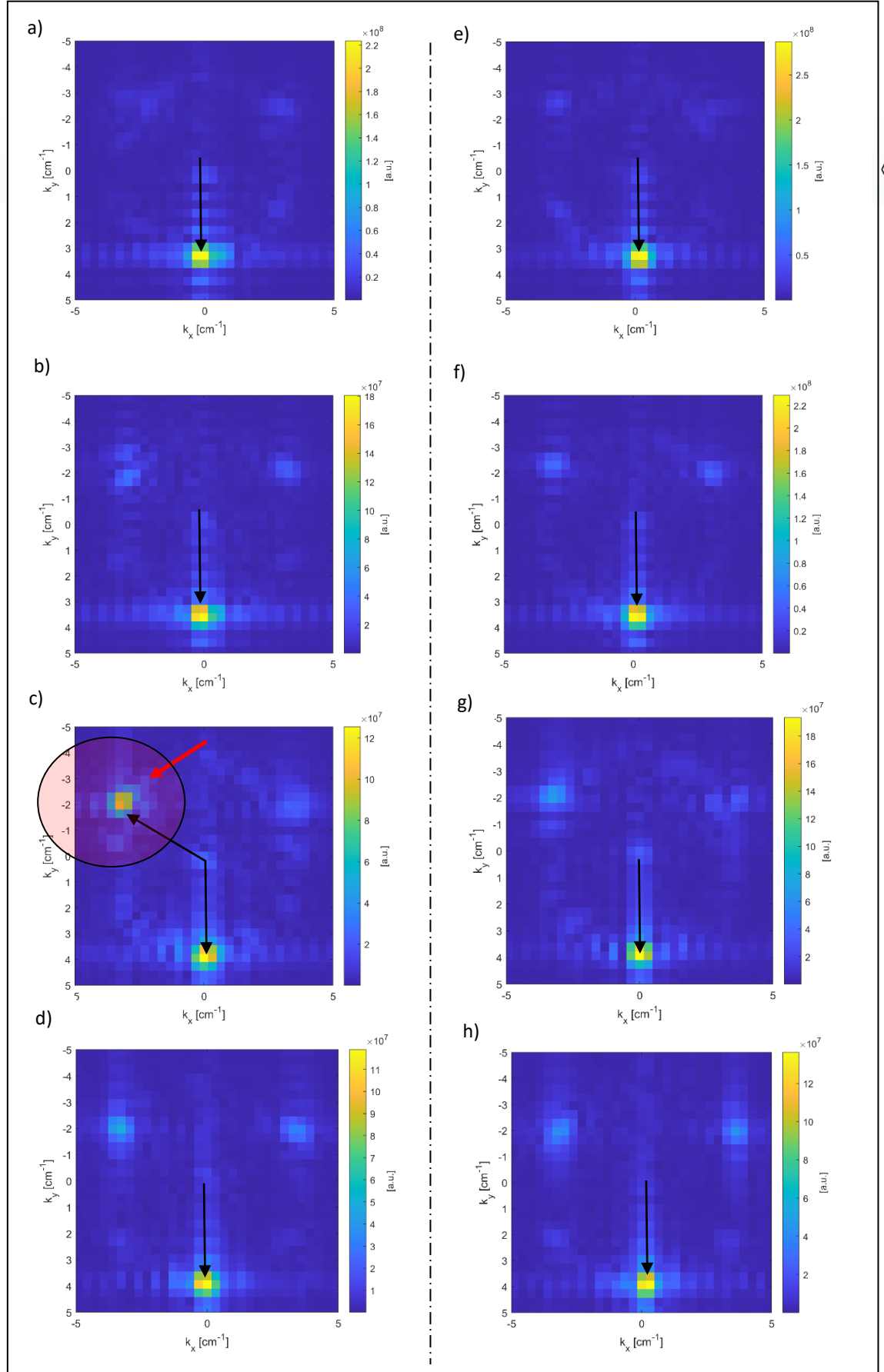


Figure 5.10: Spatial Fourier transform over the upper left corner of the crystal for $f=70$ Hz (a), $f=77$ Hz (b), $f=87$ Hz (c), $f=110$ Hz (d). e-h) Same frequencies for the upper right corner. For $f=87$ Hz the topological mode exists on the left while there is no propagation on the right side.

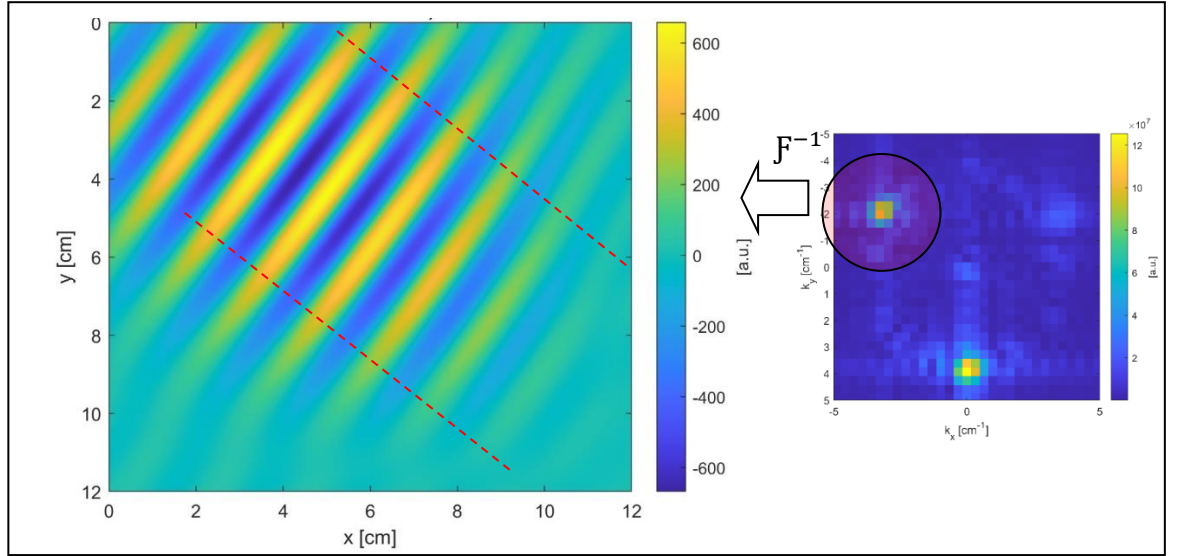


Figure 5.11: a) Artificial reconstruction, via inverse Fourier transform ($\mathcal{F}^{-1}$) taking into account only the bright spot at $\theta = 60^\circ$ with respect to the vertical axis in the k -space (b), of the topological edge mode through the DW B-A in the upper left side corner of the hexagonal crystal for $f=87$ Hz. The red dashed lines indicate where the edge mode is mainly localized.

In Fig. (5.10 e-h) we did the same spatial Fourier transform but over a window placed on the upper right corner just to verify that, where is no interface, there is no propagation. This is like a verification that what we observe on a side is unique. The window has the same size of the left side and totally symmetric with respect to the vertical axis. Indeed, we have mainly just a bright point, along the vertical axis, corresponding to the incoming wave, so the topological mode exists just on the left side of the crystal; the other two spots are due to the symmetry of the triangular pattern meaning that the spatial arrangement implies two weak reflections to the upper left and right corner (the crystal acts like a mirror). To display the difference from what propagate to the left (direction $\theta = 60^\circ$) and right (direction $\theta = -60^\circ$) of the upper part of the crystal, we plot the transmission profile in the frequency range of the hybridization gap and we observe that for $f = 87$ Hz, the transmission related to the left side is 3 times compared to the right side, see Fig. (5.12) where the blue line indicates the left side, the red line indicates the right side. In the upper right corner, the sketch indicates the window where the spatial Fourier transform is performed (the whole upper part of the hexagon). This window choice comes from the fact that now we want to compare all the transmitted wave, in the upper part, to the left (direction $\theta = 60^\circ$) and to the right (direction $\theta = -60^\circ$). As visually seen with the spatial

Fourier transform the left part has a higher transmission profile and the wave that can propagate to that direction can come out only from the interface that we constructed which is aligned with $\theta = 60^\circ$. The right profile concerns just the mirror effect of the crystal. Another important information that we can extract from Fig. (5.12) is that the curves have a very similar trend from $f = 75 \text{ Hz}$ to 80 Hz and from $f = 95 \text{ Hz}$ to 100 Hz . The change concerns the interval $80 < f < 95$ where for $f = 87 \text{ Hz}$ we have the maximum difference in term of transmission of the left and right side.

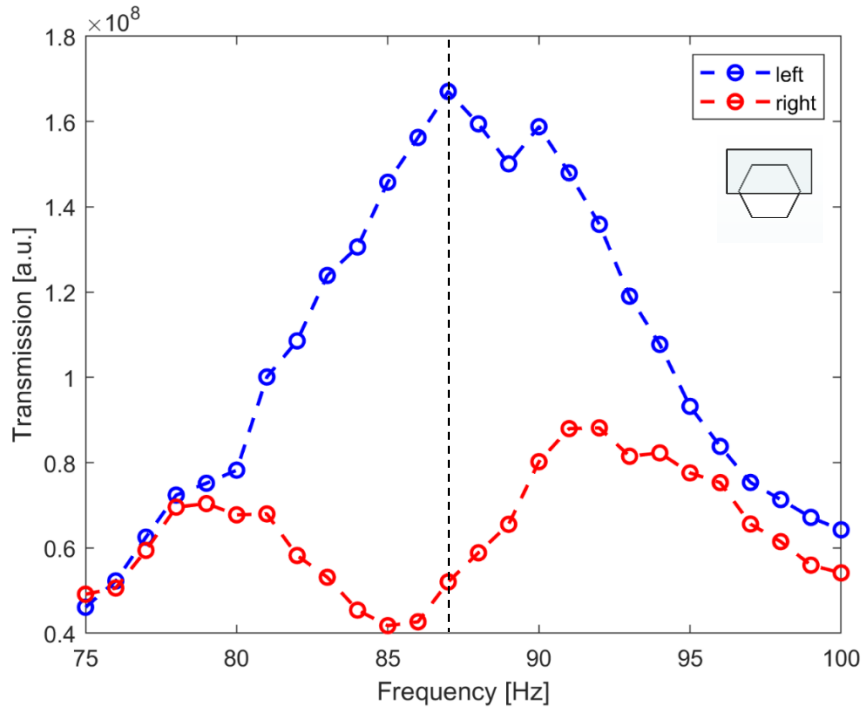


Figure 5.12: Transmission profile evaluated in the upper part of the hexagonal crystal, to compare the waves propagating to the directions $\theta = 60^\circ$ and -60° . The left side, aligned with the interface that supports the topological edge mode, presents a peak for $f=87 \text{ Hz}$ indicating its existence only for $\theta = 60^\circ$.

The whole procedure can be repeated for the same crystal upside-down, where the interfaces not supporting the edge mode are on the same side of the source generator, see Fig. (5.13 a). The black dashed lines indicate a monotype crystal, so as we know it does not support any edge mode (left side). We report here the transmission profile evaluated on the upper side as done previously (green window) where no propagation is allowed and we obtain Fig. (5.13 b). As we can see, this time within the gap ($f = 85 - 100 \text{ Hz}$) there is almost no difference between the left and right side meaning that the same structure inverted has not at all the same behavior.

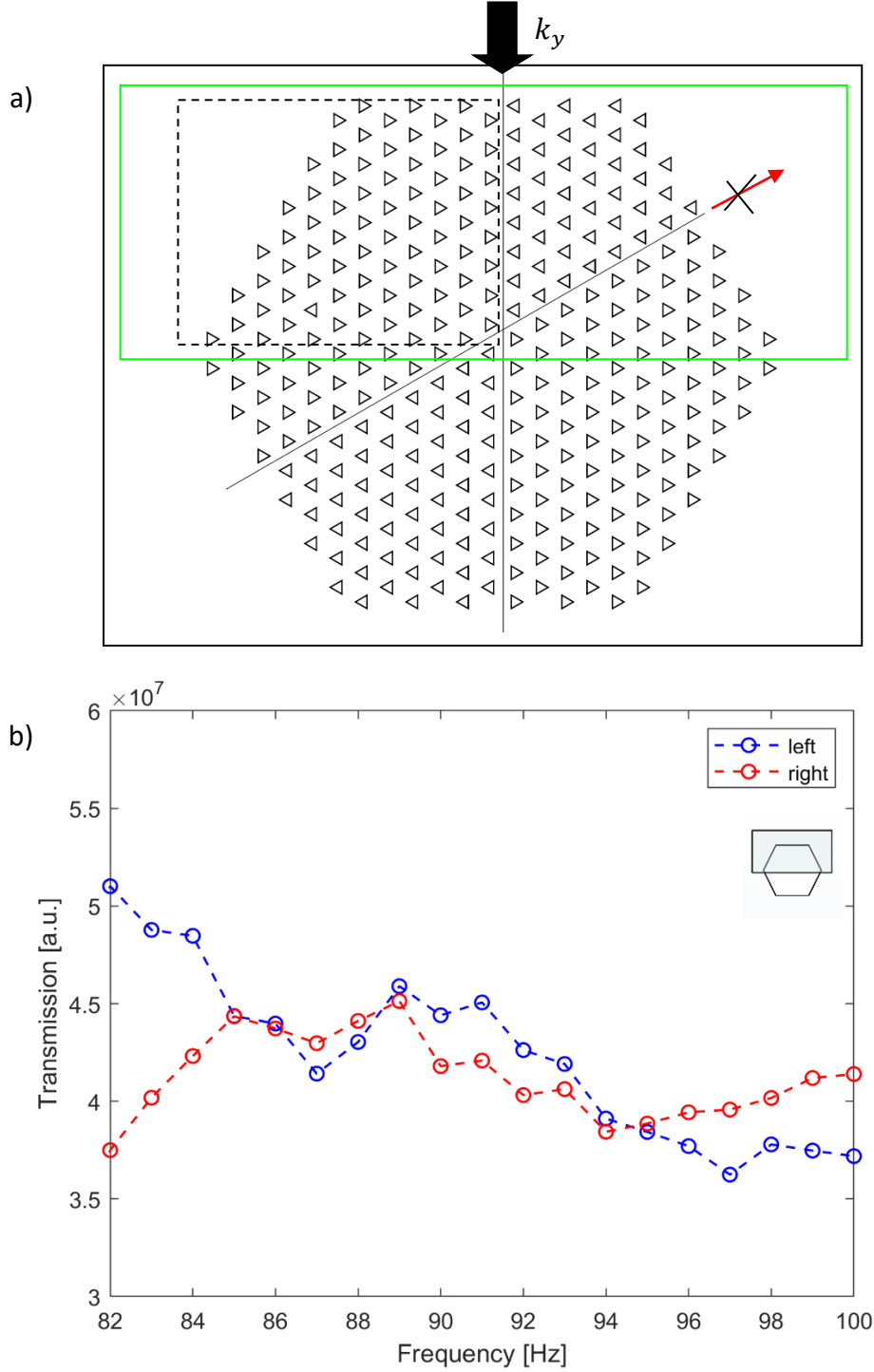


Figure 5.13: a) Sketch of the hexagonal structure flipped up-down; in this case the incident wave interacts with the side where no propagation is allowed. The green area indicates the window to perform the spatial Fourier transform; the dashed black lines highlights that the left part is a monotype crystal so it does not support a topological mode. b) Transmission profile evaluated in the upper part of the hexagonal crystal, to compare the waves propagating to the directions $\theta = 60^\circ$ and -60° ; left and right profile do not show any particular change, in the transmission profile, related to a topological mode within the gap frequency range.

5.5 Conclusion

In this chapter we presented a hexagonal pattern composed by triangular resonators with a particular rotation angle that allow us to study two kinds of interfaces within the same crystal. One interface supports a topological mode, (edge mode in the band diagram) while the other one does not allow any propagation (band gap). In other words, the folded double Dirac cone is lifted by rotating the triangular resonators breaking the mirrored symmetry like done with phononic crystals [106] and also in water wave systems [105]. The hybridized eigenmodes can mimic the pseudo-up-spin and pseudo-down-spin leading to a topological transition resulting in a pattern that supports a topological edge mode through an interface different from a straight line, precisely a V shape. The band gaps, due to Bragg scattering and local resonances, are experimentally observed. We presented the method based on temporal and spatial Fourier transforms to evaluate all the transmission profiles and the strategy to individuate the edge mode. For a particular frequency, the transmission profile comparison shows a clear difference between the left side where the interface supporting the edge mode is and the right side where there is no interface. We experimentally showed that the blocking interfaces do not present a topological edge mode within the same frequency range.

Referring to the phononic crystal studied by Xia *et al.* [106], Fig. (5.6), in order to compare our result with theirs, we can state that our system is more selective, in fact, our frequency range (85-100 *Hz*) is way shorter than 16-20 *KHz* employed for the phononic crystal. However, their transmission profile shows a remarkably difference, between the channel 1-2 that supports the edge mode, and the area that does not support it, precisely 10 *dB*; that means 10 times the sound intensity while for us the difference is 3 times in the transmission profile (Fig. 5.12).

In conclusion, we experimentally demonstrated, within the context of the HEWs, that we can observe a topological edge mode following a path different from a straight line leading to a new transport way for HEWs.

Chapter 6: Conclusions and future perspectives

6.1 Conclusions

Inspired by one of the most mathematically studied example of hydroelasticity, the propagation of Hydro-Elastic Waves (HEW) into and within sea ice fields, we presented the theory to study HEW. We used the model for waves in floating elastic ice plates adapted to a simple laboratory model based on a tank filled with water covered by an elastic sheet. We derived the fundamental equations to obtain the dispersion relation of a HEW. This is the starting point to understand how this kind of waves propagates and under which conditions they are studied (deep water condition and thin elastic plate). We saw that the dispersion relation is composed by three terms: gravity, tension and bending term. The latter, is a peculiarity of HEW that allow us to easily modify the dispersion relation by changing the geometrical properties of the elastic sheet (*i.e* its thickness).

We characterized the elastic sheet by measuring the Young's modulus E (which is contained in the bending term of the dispersion relation). We reconstructed the dispersion relation via Fourier transform and compared to the theoretical one finding a perfect agreement considering the dynamical value of the Young's modulus E instead of the static one. Then, we measured the attenuation length of a HEW in order to demonstrate that this a suitable platform to study the interactions between a HEW and a triangular cavity (or any pattern or shape on the membrane). In simple water waves system that would be harder because the attenuation is too important to measure any possible interaction.

We demonstrated that a triangular perforation in the elastic membrane acts as a resonator showing its modes with a 2D and 3D reconstruction of the height field for the corresponding resonant frequency, validated via modal decomposition (a projection over a basis composed by the first modes of a triangular cavity). We extended this, making artificial crystals of triangular resonators for HEW following a triangular lattice configuration in order to modify the waves propagation in a controlled way.

We described how waves propagate through a structured media and how the dispersion relation can be modified by the spatial arrangement of the lattice

considered and by the local resonances. Consequently, two types of gaps can be observed. We started with the easiest example of 1D crystal, the Bragg mirror, providing the explanation based on the Bloch-Floquet's formalism. We illustrated the relevant properties of the 2D triangular lattice in direct and reciprocal space. Then, we saw how Topology can be applied to Physics leading to new interesting transport properties. In particular, the Quantum Spin Hall effect can have a classical equivalent based on a triangular lattice which naturally sustains a four-fold degeneracy at the Brillouin zone center, necessary condition for the QSH effect. We proposed to study topological phases using HEW interacting with artificial crystals based on the triangular lattice instead of a honey-comb configuration (which is the classical lattice employed when looking for topological phases).

We observed how a HEW interacts with a triangular lattice composed by triangular resonators all oriented in the same way (a monotype artificial crystal). The dispersion relation, obtained via Fourier transform, in this case shows band gaps due to Bragg scattering and local resonances, first and second gap respectively.

We introduced the Valley transport and experimentally verify the propagation of an edge mode through a crystal made by two monotype patterns of triangular resonators in a triangular lattice with different rotations (making an interface), in other words, we verified the classical equivalent of the QSH effect with a valley degree of freedom for HEW. We showed the transmission profile for three distinct areas of the crystal and experimentally observed that a channel exists only at the interface, in the frequency range of the second gap, confirming that we are able to control the propagation of an incoming wave through a specific path.

Then, we made a triangular lattice, composed by triangular resonators, with a hexagonal shape in order to demonstrate that it is possible for a HEW to follow a particular path different from a straight line. The double Dirac cone can be lifted by rotating the triangular resonators with two different orientations, in that way it is possible to create two specific interfaces, within the hexagonal shape, that allow or not the propagation of an edge mode. We showed, via Fourier transform, that the incoming wave follows the channel made by the aforementioned interfaces for a particular frequency within the range corresponding to the second gap.

6.2 Future perspectives

Other configurations of structured media have been proposed to control the wave propagation and either tested experimentally or simulated numerically. These configurations could be tested as well in the case of HEWs.

6.2.1 Pattern to obtain a local excitation of the hexagonal structure

Many physical systems are employed to demonstrate the existence of topological edge mode. Having a look at the literature, for example phononic or photonic crystals are usually excited directly at the interface that supports the topological mode.

Is there a way for HEWs to excite an artificial crystal in a more localized way?

We could replace the straight paddle of the source generator with a circular wave source, *i.e.* a metallic bar to place over the area to excite. The problem with this solution, as previously seen, is that we will have multiple reflected waves in the tank resulting in noisy data. A more sophisticated solution comes from the idea of using artificial crystals to create a waveguide. We saw that our crystals have a band structure presenting band gaps. Changing the geometrical properties of these crystals like the distance between two resonators and their height, we can create a filter that does not allow any propagation for a specific frequency range. This is what we did in Fig. (5.14): two monotype crystals (red lines) next to the hexagonal crystal (purple lines).

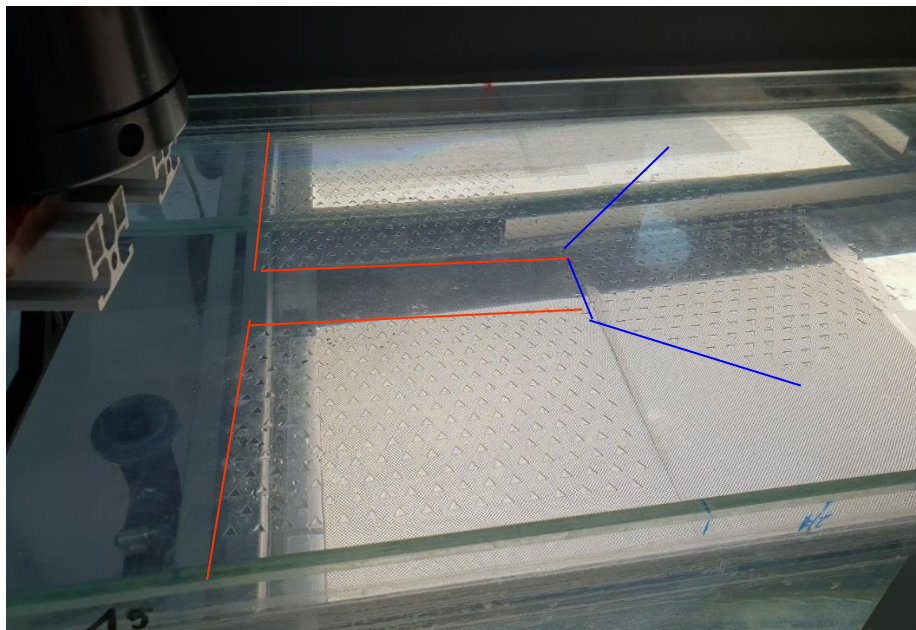


Figure 6.1: Photo of an Hexagonal crystal (purple lines) for HEWs with a waveguide made by two artificial crystals (red lines) to excite just the upper part of the hexagon.

We show in Fig. (6.1) the height field reconstruction for the 1st gap $f = 47 \text{ Hz}$. We can easily see that the plane wave is blocked by the crystals above the hexagon resulting in a plane wave localized just to the upper part of the hexagonal crystal (we realised a waveguide). This kind of configuration can be seen like a future perspective in order to better separate the topological effects from noise caused by the source. However, for this particular filter we faced other problems coming from the corners between the upper part of the hexagonal crystal and the “blocking” crystals. Precisely, we had diffraction effects causing circular waves in the areas that we want to investigate. This could be fixed by employing larger crystals (hexagon and blocking crystals).

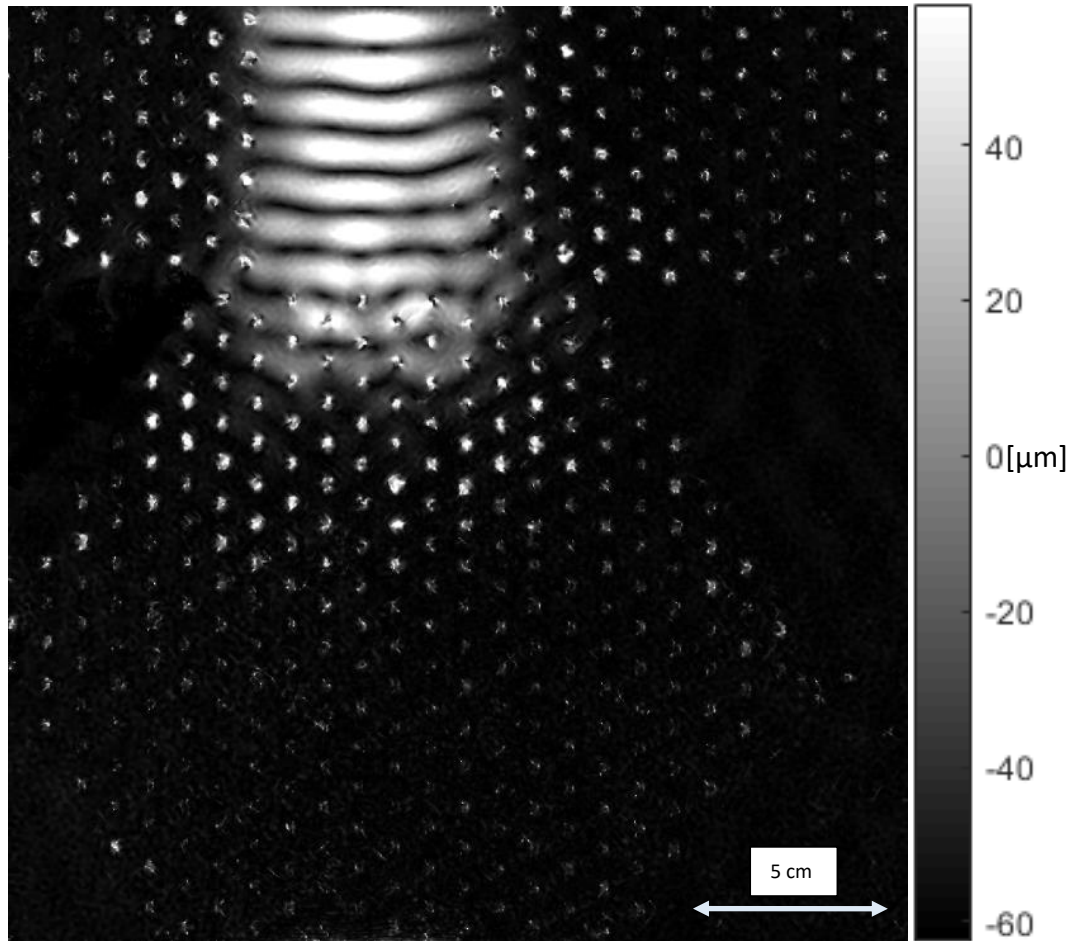


Figure 6.2: Height field reconstruction ($f=47 \text{ Hz}$ 1st gap no propagation allowed) of the hexagonal crystal with two extra blocking crystals to create a waveguide (rotated 90° with respect to Fig. 6.1. In this way we are able to excite just a particular side of the hexagon.

6.2.2 Hexagonal crystal made by circular resonators for HEW

It is possible to study topological phases adopting different shapes for the resonators that compose a specific lattice configuration. So far, we studied and focused on triangular patterns of triangular resonators because it easily allows us to make interfaces between two different patterns just rotating the triangular resonators. Many other options exist to obtain a band inversion, an interesting one is to study the hexagonal shape proposed in Chapter 5, made this time by circular resonators in a honeycomb configuration. Because of its symmetry a rotation would not have any impact, the strategy in this case is to change the resonators' radius. The hexagon is still divided in 4 areas, with different radii, generating two interfaces that allow or not the propagation of an edge mode like did in acoustics [68], [107] (see Fig. 6.3 a-c).

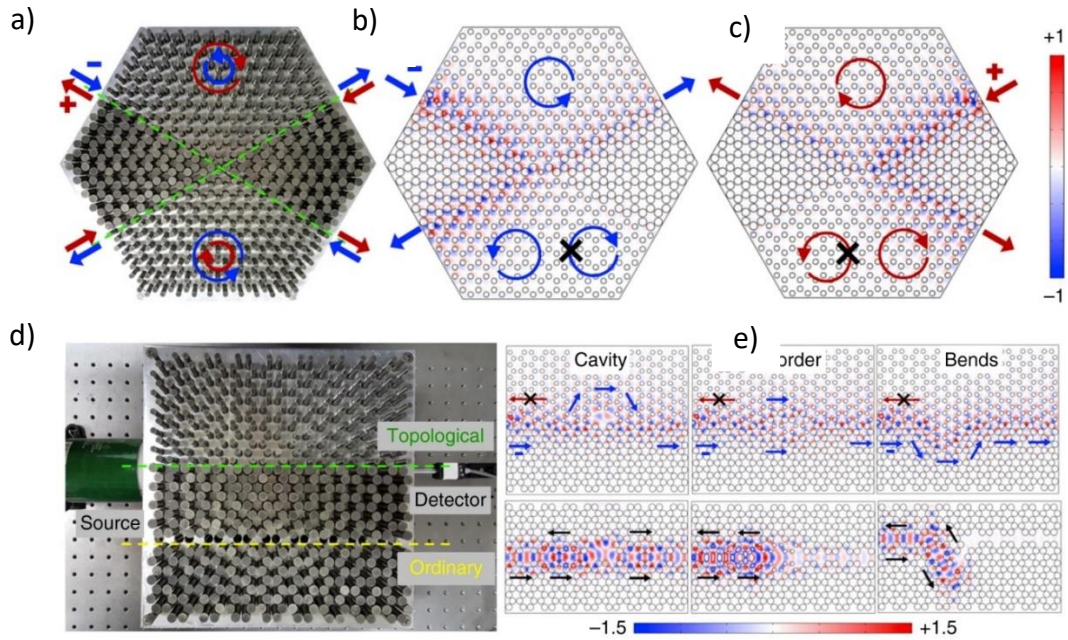


Figure 6.3: a) Spin-locked edge state propagation through a hexagonal crystal made by circular scatterers (metallic cylinders arranged in a trivial and non-trivial lattice). Four interfaces are realized and a band inversion is obtained varying the size of the circular scatterers. b) Pseudo-spin down propagation (anticlockwise). c) Pseudo-spin up propagation (clockwise). d) Experimental setup to study the robustness of the edge state propagation against various defects (green interface) compared to a regular waveguide (yellow line). e) Simulation of the pressure field typical defects confirming the robustness of the upper interface. Adapted from [107].

In Fig. (6.3 a) we can see the experimental hexagonal crystal made by circular scatterers (metallic cylinders arranged in a trivial and non-trivial lattice) where the 4 interfaces are indicated by the green lines (a band inversion is obtained varying the size of the circular scatterers). In Fig. (6.3 b-c) the pseudo-spin down propagation

(anticlockwise) and pseudo-spin up propagation (clockwise) are shown through the interfaces as a simulation of the pressure field. This can be seen as a splitter as well where from an incoming wave we get two outputs.

Using the same strategy, it is possible to study how this kind of waves are good in when exposed to cavities, disorder and bends like experimentally did in Fig. (6.3 d). The edge state propagation against various defects (green interface) is compared to a regular waveguide (yellow line). In Fig. (6.3 e) we have the simulation of the pressure field for three kinds of defects: cavity, disorder and bends for the two interfaces. It confirms that the topological interface possesses interesting transport properties expressed by its robustness to cavity, disorder and bends not achievable via the ordinary waveguide.

6.2.3 Kekulé lattice for HEW

The Kekulé lattice introduces an alternative type of pseudo-spin that emerges from band folding. It supports double Dirac cones. It is still based on circular resonators but this time we have a honeycomb lattice with the so-called Kekulé distortion (experimental representation in Fig. 6.4 a). The peculiarity of such configuration is that we can have an extra type of interface, so in total three interfaces. The first and second one, are simply the two domains one next to the other one (Fig. 6.4 b-c) so for instance domain I on the upper side and domain II on the downer one and vice versa. The third interface is called armchair interface (AC) and it is shown in Fig. (6.4 d).

Having a look of the experimental acoustic setup, Fig. (6.4 a), proposed by [108] we can recognize the two domains I and II and the armchair interface (dashed green lines). The honeycomb lattice, as previously seen, supports double Dirac cones and a complete band gap appears due to the reduced inversion symmetry caused by the Kekulé distortion.

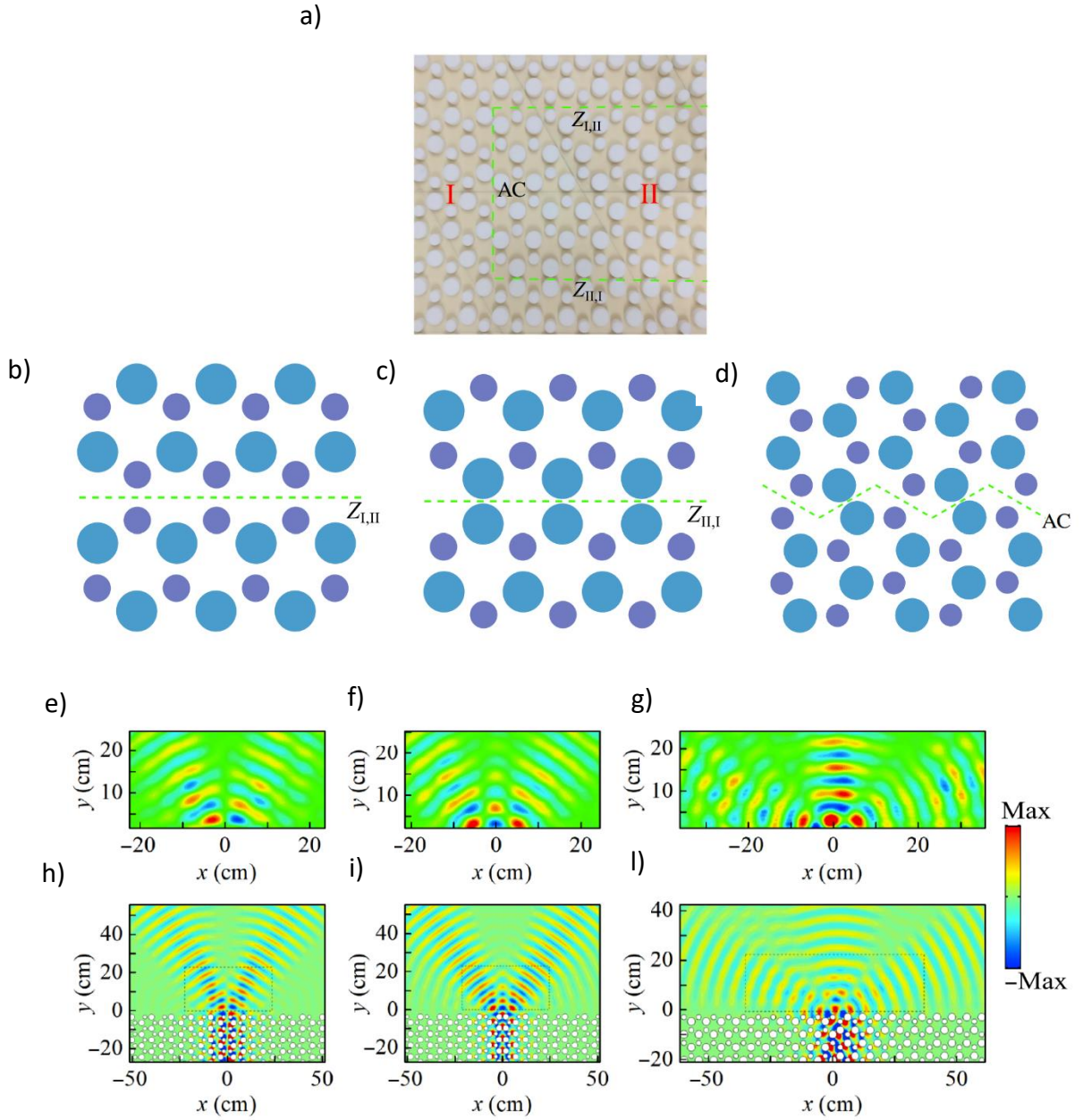


Figure 6.4: a) Experimental setup with two domains I and II forming three interfaces $Z_{I,II}$, $Z_{II,I}$ and the armchair one (AC). b) Schematics illustration of the interface $Z_{I,II}$. c) Schematic illustration of the interface $Z_{II,I}$. d) Schematic illustration of the interface AC. e) Measurement of the pressure field for the interface $Z_{I,II}$. f) Measurement of the pressure field for the interface $Z_{II,I}$. g) Measurement of the pressure field for the AC interface. h) Simulated pressure field for the interface $Z_{I,II}$. i) Simulated pressure field for the interface $Z_{II,I}$. l) Simulated pressure field for the AC interface. Adapted from [108].

These interfaces are robust against defects including cavity, disorder and bends and are have different characteristics in term of transmission of the edge state. Referring to Fig. (6.4 e-l) we can observe how the topologically protected refraction (output wave coming from the interface) differs for the armchair termination in the

measurements and simulations. This could lead to a new way to control the transmission through the interface and a control of the topologically protected refraction of pseudo-spin states.

It would be interesting, as a future perspective, to define a parameter to compare different configurations and establish which one can respond in a better way depending on the problem studied. Another parameter could help making a link between different fields and provide an independent approach to evaluate the efficiency of a crystal (like topology with respect to condensed-matter physics).

It is worth mentioning that other options have been analyzed, a relevant configuration is represented by the Kagome lattice for which we recommend these papers for further details [109], [110].

6.2.4 Zig-zag paths for HEW

The triangular resonators can be employed to create interfaces different from a straight line, like seen in Chapter 5. Starting from this fact, it is possible to make a crystal where the edge mode follows a zig-zag path. So far, such a path has been mostly simulated with multiple configurations, generally in Acoustics. The main limit is the attenuation through a crystal presenting many corners (we remind that the backscattering is never fully suppressed and it inevitably causes losses along the zig-zag path).

In Fig. (6.5 a) we have a zig-zag path made using 2 domains with the simulated pressure field that shows how the edge mode is localized withing the interface [63]. In Fig. (6.5 b) the measurement of the pressure field is presented (red circles). The black circles represent the transmitted pressure for a straight channel. In particular, we can observe that the corners along the zig-zag path have a weak impact on the topological mode demonstrating that it is possible, with a sonic crystal, to create new devices with a good potential in term of transport.

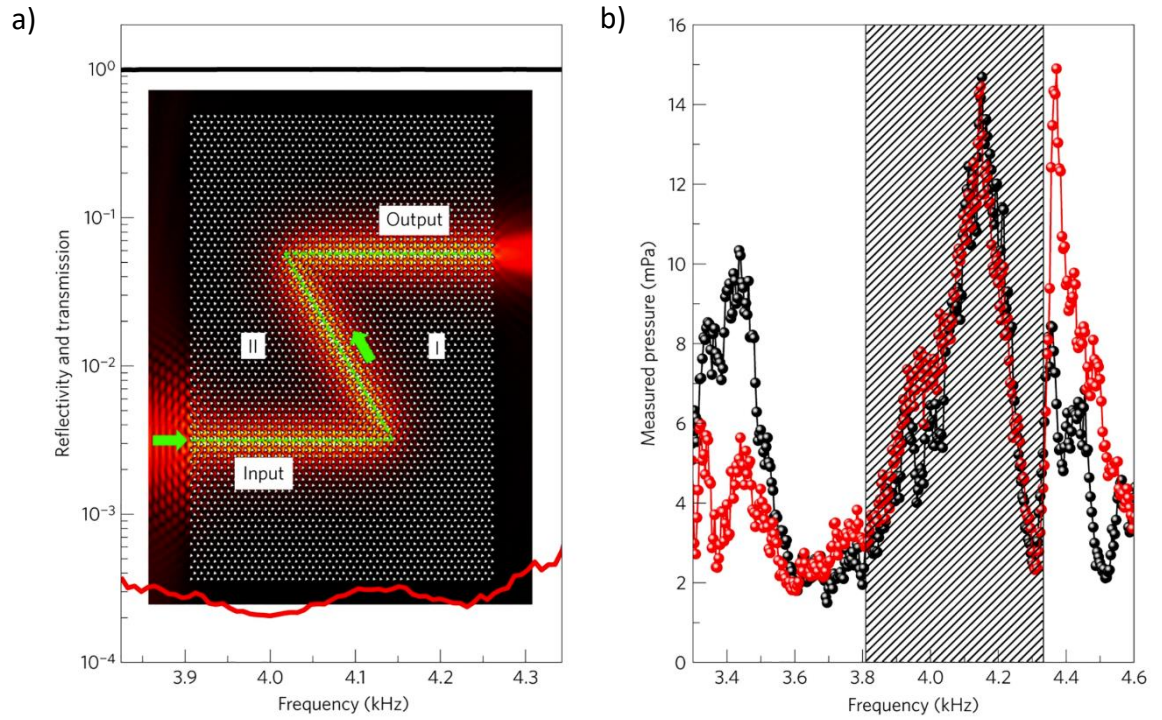


Figure 6.5: a) Reflectivity (red) and power transmission (black) in the frequency range 3,8-4,3 Hz with a pressure field simulated at $f=4,06$ Hz for a zig-zag path. b) Pressure measured for two types of interfaces: a zig-zag path (red) and a straight channel (black). Within the bulk gap (shadow area), they show that the corners of the zig-zag path do not cause a loss of pressure confirming an interesting transport way. Adapted from [63].

6.2.5 On a larger scale: IFREMER, Brest

So far, all experiments have been realized in the PMMH in a small tank filled with water. Would it be possible to transfer all ideas tested in the laboratory in a 50 meters long and 12,5 meters width tank?

That is the goal of the project called “Elastonics” where a plate (16,5 m x 5 m) has been characterized to verify the dispersion relation at large scale using the testing facilities of IFREMER [111]. The experiment has been first simulated via Hydrostar, a hydrodynamic software developed by Bureau Veritas that provides a complete solution for the 1st order wave diffraction and radiation problem [112]. The experimental setup (Fig. 6.6) is composed by a floating plate anchored to all corners to avoid drift movements of the plate, video cameras and black spots on the plate for the real-time visualization of the height field and a wave generator in the background (the incoming waves have a wavelength ranging from 1 to 6 meters). The dispersion relation obtained is in good agreement with the laboratory tests firstly conducted by Lucie Domino [75]. I had the honor to participate to one of the experiments with Antonin Eddi. During the measurements via video camera operated by the IFREMER’s team, we conducted a similar measurement reconstructing the height field using the deformations of black lines on the plate recorded via CCD (Fig. 6.7). Possible uses of the device: control of waves generated by ships in sensitive environments, protection of some harbor structure.

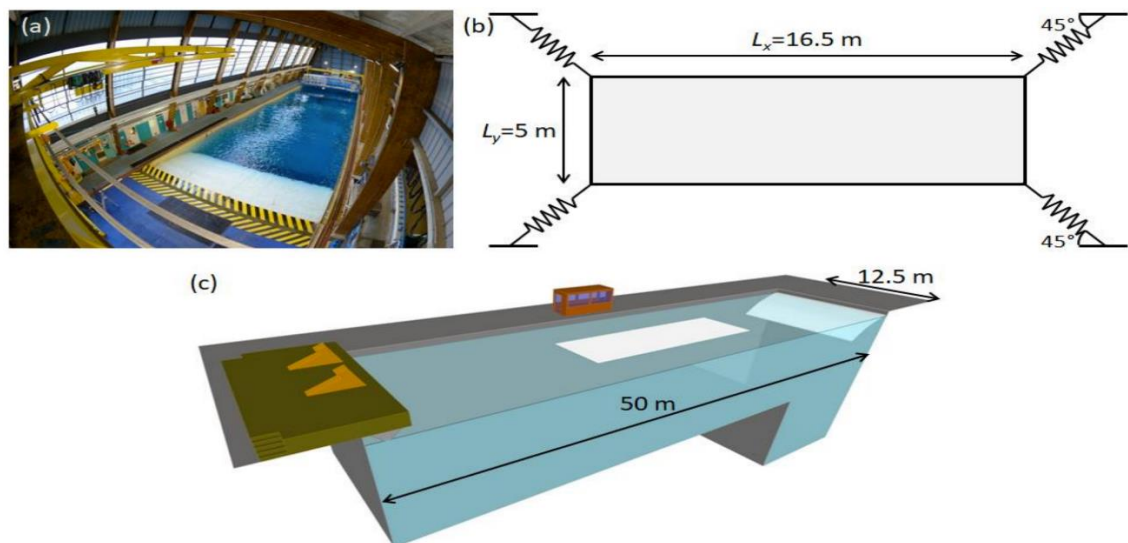


Figure 6.6: a) Top view of the wave tank of IFREMER. b) Anchor’s configuration to keep the floating structure in the same position. c) Side view of the wave tank [111].

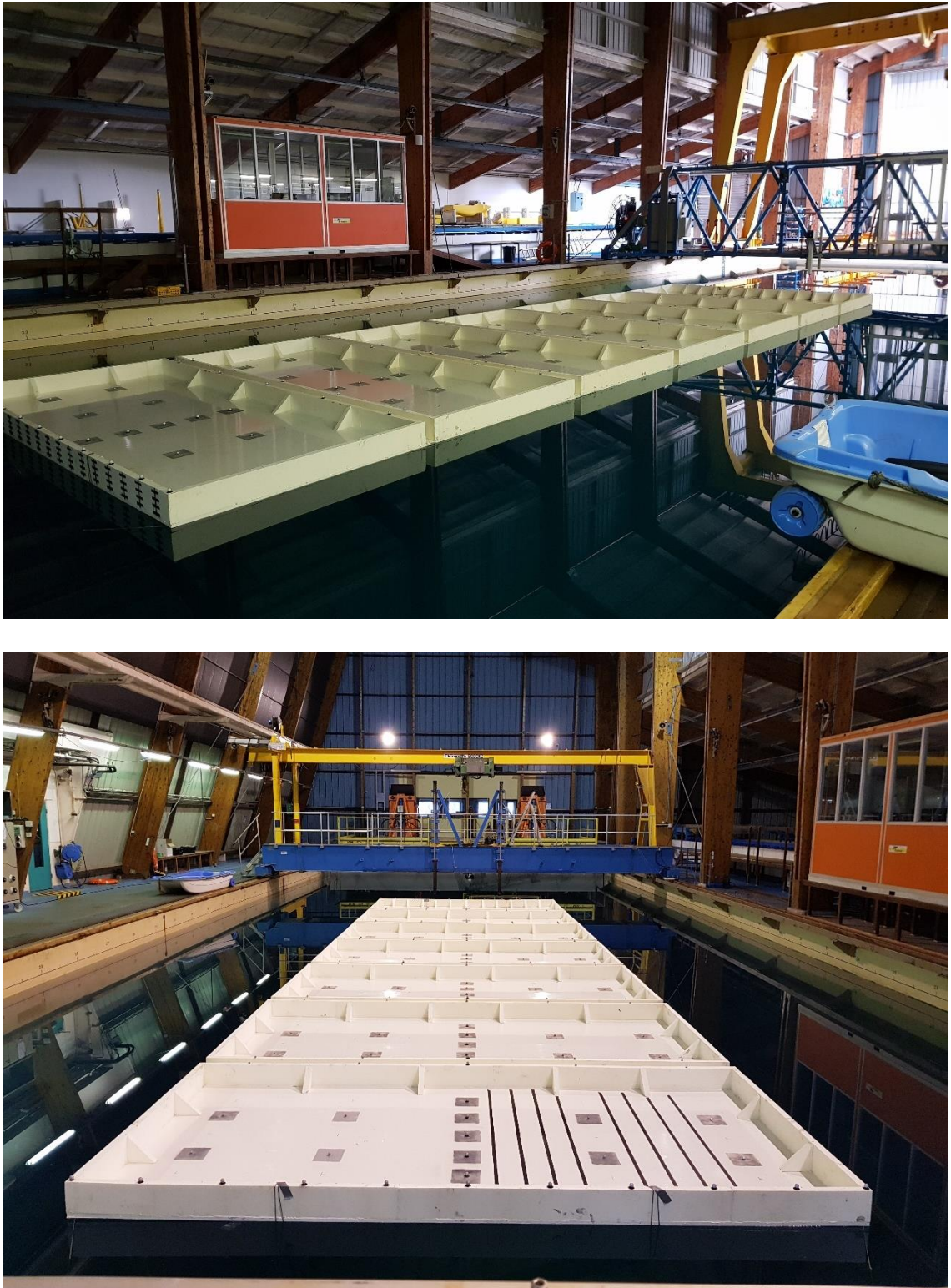


Figure 6.7: Floating structure used in the “Elastonics project” to study Hydro-Elastic Waves in the IFREMER’s tank located in Brest. On the top, a view of the elastic plates floating in the middle of the 50 meters tank. On the bottom, it is possible to see black lines on the plate that have been used to reconstruct the height field using a CCD to record the deformation of its profile caused by the incoming wave. The black spots are part of the testing facilities of IFREMER to reconstruct the height field via video camera which allows a real-time visualization.

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Cette thèse porte sur le comportement des Ondes Hydro-Élastiques (HEW), des ondes à la surface de l'eau qui interagissent avec une membrane élastique, en utilisant les idées et méthodologies qui dérivent de la topologie et de la physique de la matière condensée comme le *Quantum Hall Effect* (QHE). Les phases topologiques sont étudiées dans plusieurs domaines physiques et dans la majorité des cas, afin d'observer les propriétés liées aux isolants topologiques, il est nécessaire de construire un système où la symétrie par renversement du temps (*T symmetry*) est brisée. En se concentrant sur les systèmes avec des ondes à la surface de l'eau, ce n'est pas trivial de briser la *T symmetry*, dans ce cas ils sont nécessaires des flux d'eau rotationnelles qui comportent une forte injection d'énergie dans le système. Une autre approche pour faire face à ce défi expérimental, est de mimer le *Quantum Spin Hall Effect* (QSHE), dans ce cas la *T symmetry* est préservée. Nous proposons d'étudier les phases topologiques en utilisant une nouvelle approche basée sur les HEW ou l'interaction entre une membrane de silicone et des ondes à la surface d'un liquide peut conduire à des guides onde robustes et immunes au *back-scattering*. Cette étude est basée sur des expériences en laboratoire qui permettent de reconstruire le champ de hauteur précisément. D'abord nous caractérisons la membrane de silicone en mesurant son module de Young, la relation de dispersion et l'atténuation, en montrant que nous sommes capables de modifier sa propagation en changeant les propriétés géométriques de la membrane. Nous étudions le comportement d'une perforation triangulaire sur la membrane, qui peut être vue comme une cavité 2D, en montrant ses modes de résonance. Nous utilisons un pattern triangulaire de perforations triangulaires, ce que l'on appelle un cristal artificiel, pour mimer la structure de bande d'un vrai cristal: les gaps de type Bragg et d'hybridation sont présentés expérimentalement. Nous démontrons qu'une inversion de bande se manifeste quand deux patterns avec des orientations différentes sont mis ensemble en créant une interface qui support un mode de bord topologique.

MOTS CLÉS

mécanique des fluides, ondes, hydroélasticité, cristaux artificiels, topologie, isolants topologiques

ABSTRACT

This thesis concerns the study of Hydro-Elastic Waves (HEW): waves at the surface of a liquid interacting with an elastic floating membrane, merging concepts and methodologies from topology typically and recently applied to condensed-matter physics. At this point, the study of topological phases involves many classical physical systems where artificial crystals (*i.e.* sonic or photonic crystals) are employed to observe new transport properties mainly characterized by robustness against defects and suppression of the back-scattering (spin-locked transport). We propose to mimic the Quantum Spin Hall effect and Quantum Valley Hall effect describing the topological phases using a new approach based on HEW where the interaction between a silicone membrane spatially structured and water waves can lead to unidirectional robust waveguides immune to back-scattering. The study is based on a laboratory experiment which makes it possible to measure the wave field precisely via the synthetic Schlieren method. We first characterize the silicone membrane measuring its Young modulus, the dispersion relation and the attenuation of a HEW, showing that we are able to modify its propagation by changing the geometrical properties of the membrane. Then, before analysing the collective response of a spatially arranged structure, we describe the behaviour of a single triangular perforation on the membrane demonstrating that it acts like a 2D cavity; its resonant modes are obtained via modal decomposition and play a crucial role. We manufacture multiple triangular patterns of triangular perforations, the so-called artificial crystal for HEW, to mimic the band structure of a solid-state one: Bragg and hybridization gaps are experimentally shown. We demonstrate that a band inversion takes place when two different patterns with a particular rotational direction are put together (also called domain wall). The latter, in other words, is a straight interface that supports a topological edge mode that leads to a new robust pseudospin-dependent transport. At last, we experimentally prove the existence of a topological edge mode using an interface different from a straight line, a V-shape, where the attenuation losses due to the corner along the path are largely suppressed.

KEYWORDS

fluid mechanics, waves, hydroelasticity, artificial crystals, topology, topological insulators