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**ECOLE DOCTORALE SCIENCES ET METIERS DE L'INGENIEUR**  
**Laboratoire d'Ingénierie des systèmes physiques et numériques (LISPEN)**  
**Campus de Lille**

# **THÈSE**

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**Ecole Nationale Supérieure d'Arts et Métiers**  
*spécialité :* **automatique**

**Non-asymptotic fractional order state differentiators  
based on improved modulating functions methods**

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# Résumé

Motivée par la complexité et les inefficacités des méthodes de fonctions modulatrices actuelles d'estimation d'état basées sur des équations différentielles d'entrée-sortie, cette thèse vise à introduire des méthodes améliorées d'estimation d'état basées sur des fonctions modulatrices afin de simplifier le processus d'estimation et d'améliorer son efficacité. Elles peuvent être utilisées pour estimer de manière non asymptotique et robuste non seulement les états pour les systèmes linéaires d'ordre entier et non entier dans des environnements bruités, mais aussi leurs dérivées et intégrales fractionnaires. Tout d'abord, un estimateur d'état amélioré est développé pour les systèmes d'ordre entier. Ensuite, cette méthode est généralisée aux systèmes d'ordre fractionnaire pour estimer l'intégrale fractionnaire de l'état. Une analyse d'erreurs dues aux bruits est aussi donnée. Contrairement aux travaux précédents, le schéma présenté s'applique à la forme canonique observable, obtenant directement des formules intégrales algébriques, sans avoir besoin d'estimer les dérivées de la sortie et un ensemble de valeurs initiales fractionnaires. Dans la méthode améliorée proposée, les dérivées fractionnaires de l'état peuvent être obtenues via le schéma de Grünwald-Letnikov. Pour obtenir directement les formules intégrales algébriques des dérivées et intégrales fractionnaires de l'état, nous transformons astucieusement la forme canonique observable en un nouveau système d'ordre fractionnaire. Ainsi, les dérivées et intégrales fractionnaires de l'état sont directement estimées pour les systèmes d'ordre entier et non entier en concevant des fonctions modulatrices, ce qui constitue la deuxième méthode améliorée proposée. De plus, nous examinons les applications pour les moteurs à courant continu. L'efficacité et la robustesse des estimateurs améliorés proposés sont vérifiées par des simulations numériques dans cette thèse. Enfin, un résumé de ces travaux et un aperçu des travaux futurs sont dressés.

Mots-clés: Systèmes d'ordre entier, Systèmes d'ordre fractionnaire, Estimation non asymptotique et robuste, Méthodes basées sur les fonctions modulatrices, Estimation de l'état, Estimations de la dérivée fractionnaire de l'état.

## RESUME

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# Abstract

Motivated by the complexity and inefficiencies in current modulating function state estimation methods based on input-output differential equations, this thesis aims to introduce improved modulating functions-based state estimation methods to simplify the estimation process and improve the estimation efficiency. They can be used to non-asymptotically and robustly estimate not only the states for integer and non-integer order linear systems in noisy environments, but also their fractional derivatives and integrals. First, an improved state estimator is advanced for integer order systems. Then this approach is generalized to fractional order systems for estimating the fractional integral of the state. Noise error analysis is also provided. In contrast to the previous work, the presented scheme applies to the observable canonical form, obtaining algebraic integral formulas directly, without the need for estimating the derivatives of output and a set of fractional initial values. In the proposed improved method, the fractional derivatives of the state can be obtained via the Grünwald-Letnikov scheme. To obtain directly the algebraic integral formulas of the fractional derivatives and integrals of the states, we first cleverly transform the observable canonical form into a new fractional order system. Thus, the fractional derivatives and integrals of the states are directly estimated for integer and non-integer order systems by designing modulating functions, which is the second improved method proposed. Furthermore, we consider applications for DC motors. The efficiency and robustness of the proposed improved estimators are verified by numerical simulations in this thesis. Finally, a summary of this work and an insight into future work are drawn.

Keywords : Integer order systems, Fractional order systems, Non-asymptotic and robust estimation, Modulating functions-based methods, State estimation, Fractional derivative estimations of state.



## ABSTRACT

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# List of publications

## Journals

- [1] **Lei Wang**, Da-Yan Liu and Olivier GIBARU, "A new modulating functions-based non-asymptotic state estimation method for fractional-order systems with MIMO", *Nonlinear Dynamics*, vol. 111, pp. 5533-5546, 2023.
- [2] **Lei Wang**, Da-Yan Liu, Liang Huang and Olivier GIBARU, "A novel modulating function-based non-asymptotic fractional-order state estimator for DC motor drive system", *Communications in Nonlinear Science and Numerical Simulation*, revise.

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- [1] **Lei Wang**, Da-Yan Liu and Olivier GIBARU, "A new modulating functions method for state estimation of integer order system", in *41th Chinese Control Conference (CCC)*, 2022.
- [2] **Lei Wang**, Da-Yan Liu and Olivier GIBARU, "Estimators for fractional derivatives and integral of the state of linear systems based on modulating functions method", in *42th Chinese Control Conference (CCC)*, 2023.

ABSTRACT

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# List of Symbols

$\mathbb{N}$	the set of natural numbers
$\mathbb{Z}$	the set of integers
$\mathbb{R}$	the set of real numbers
$(\cdot)^*$	the set of non-zero numbers
$(\cdot)_+$	the set of positive numbers
$(\cdot)_-$	the set of negative numbers
$\lceil \cdot \rceil$	the smallest integer greater than or equal to $\alpha$
$\circ$	the Hadamard product or element-wise product
$\otimes$	the Kronecker product
$\Gamma(\cdot)$	the Gamma function
$\delta_{i,n}(\cdot)$	the Kronecker delta function
$\mathcal{L}$	Laplace transform
$\mathbf{L}^p(\cdot)$	the function space defined using a natural generalization of the $p$ -norm
$\mathbf{C}^k(\cdot)$	the space of functions $k$ times continuously differentiable on an interval of $\mathbb{R}$
$D_t^{-\alpha}\{\cdot\}$	the Riemann-Liouville fractional integral operator start from 0
$D_t^\alpha\{\cdot\}$	the Riemann-Liouville fractional derivative operator start from 0
$\mathcal{D}_t^{k\alpha}\{\cdot\}$	the Riemann-Liouville fractional sequential derivative operator start from 0
${}^C\mathcal{D}_t^{k\alpha}\{\cdot\}$	the Caputo fractional sequential derivative operator start from 0
${}^C D_t^\alpha\{\cdot\}$	the Caputo fractional derivative operator start from 0
${}^C D_{t,b}^\alpha\{\cdot\}$	the right-sided Caputo fractional derivative operator
${}^G D_t^\alpha\{\cdot\}$	the Grünwald-Letnikov fractional derivative operator start from 0
$\mathcal{AC}(\cdot)$	the space of functions which are absolutely continuous on an interval $\mathbb{R}$
$\mathcal{AC}^n([a, b])$	$= \{f : [a, b] \rightarrow \mathbb{R} \mid f^{(n-1)} \in \mathcal{AC}([a, b])\}$





## Part I

# Introduction



# Chapter 1

## General Introduction

### 1.1 Background of fractional calculus and control

Fractional calculus is an extension of order applied to traditional integer order calculus theory, and its origin can be traced back to 1695, when Leibniz wrote to L'Hôpital a letter in which the concept of fractional derivatives was first time mentioned. Some interesting questions and answers can be found in their letters [1].

- Leibniz: *"Can the meaning of derivatives with integer order be generalized to derivatives with non-integer orders?"*
- L'Hôpital: *"What if the order will be  $1/2$ ?"*
- Leibniz: *"It will lead to a paradox, from which one day useful consequences will be drawn."*

Although the two scholars did not come up with any useful results after contemplation and discussions at that time, fractional calculus was still in the stage of pure mathematical conjecture, but this phase can be regarded as the embryonic phase of fractional calculus, which played a significant role in promoting its subsequent research and development. Subsequently, some notable mathematicians, including L. Euler, J.L. Lagrange, P.S. Laplace, and J. Fourier, made significant contributions to the exploration of fractional calculus, establishing its theoretical groundwork. Mathematician N. Abel made a significant breakthrough in 1823 by introducing fractional operations to address the tautochrone problem [2]. Afterward, various scholars proposed distinct definitions for fractional derivative in their research, such as Grünwald-Letnikov, Riemann Liouville, Weyl, Caputo, Erdelyi Kober, Riesz, and Marchaud Hadamard types [3]. Notably, "fractional" is a relatively less strict expression that has passed down from the history of calculus. Specifically, fractional calculus extends the order of classical calculus from integers to irrational and complex numbers, suggesting its more accurate designation as "non-integer order" calculus. Over the past three hundred years, research in the field of fractional

calculus has developed slowly, with a focus on pure mathematical theory. The first monograph on the theory of fractional calculus was formally published by Oldham and Spanier in 1971 [4]. Until the end of the last century, the concept of fractional calculus was gradually appeared in the vision of researchers with the advancement of computer technology and the demand of researchers. Its theory and applications have flourished in various fields covering electrical engineering [5], [6], viscoelastic [7], [8], signal processing [9], [10], image processing [11], [12], control [13], [14], mechanics [15], environmental science [16], [17], biology [18] and economics [19]. It has been shown that fractional calculus can be particularly valuable in modeling nonlocal and nonlinear behaviors, providing new perspectives on phenomena that are difficult to capture with traditional calculus. As a result, it has provided a new and powerful theoretical tool for researchers and engineers to understand and change the world. Meanwhile, there are many scientific works on fractional calculus, including high-quality papers [20], [21], monographs [1], [7], [22]–[24], software toolkits (e.g., CRONE [25], Ninteger [26], FOTF [27], FOMCON [28], FLOreS [29], etc.), practical applications [30], etc.

In the field of automatic control, the introduction of fractional calculus has brought new exploration ideas for researchers. Fractional calculus has a great advantage over classical calculus in portraying practical problems, and it can describe complex industrial processes and physical phenomena more accurately, especially for systems with memory and genetic properties, such as fractional order electromechanical gyroscope system [31], fractional order gas dynamics equation [32], fractional order dissipation model [33], and fractional order impedance model for battery [34]. Furthermore, within the fractional order framework, numerous classical control schemes can be more comprehensively improved and perfected to enhance the performance and robustness of control design, such as fractional order proportional-integral-differential (PID) controller [15], [35], fractional order adaptive control [36], [37], and fractional order active disturbance rejection control [38], [39], etc. It is worth mentioning that fractional order PID controllers increase the stability margin of the controlled system compared to classical PID controllers.

There are many famous researchers and research groups around the world devoted to the study of fractional order systems and controllers. For instance, Japanese scholar Prof. Manabe extended the study of non-integer order to the application of control systems, introducing the concept of non-integer order control systems [40]. The CRONE team led by the French scholar Prof. Oustalo has made pioneering contributions to fractional order control, not only proposed the concepts and techniques of fractional order robust systems and successfully applied in automotive suspension systems [15], but their contributions also include the dynamics and stability of fractional order systems [41]–[43], controllers [44], along with the development of applications like the Matlab toolbox [25]. Denis Matignon first presented stability results for fractional order systems in [45], [46]. Catherine Bonnet et al [47]–[50] made valuable contributions to stability analysis, stabilization and control of fractional order systems with time-delay. Slovakian scholar Prof. Igor Podlubny and his group were the pioneers in

introducing the structure and applications of fractional order PID controllers [35], and his book [51] published in 1999, cited extensively, particularly stands out as one of the most influential publications in the field of fractional calculus. In addition, Prof. Yangquan Chen's group has been a pioneering force in advancing the research on fractional order systems and controllers for an extended period [1], [52]–[54]. Scientific contributions to fractional order observers cannot be ignored, such as the works of Ibrahima N'Doye, Taous-Meriem Laleg Kirati, Holger Voos, Mohamed Darouach, and so on [55]–[59].

### 1.2 Motivation and objective

In order to construct a fractional order system model and implement a fractional order controller, it is necessary to provide useful parameters and variables of the system, such as the output or state information of the system. Nevertheless, due to economic and technical constraints, this information may not always be measurable. Although some variables can be measured, the measurements are frequently affected by corrupting noises. Thus, it is important to design robust estimation methods against noise. Moreover, for on-line applications, non-asymptotic estimations are usually required. While the introduction of "fractional order control" can effectively enhance system performance and robustness, the non-commutative and non-additive nature of fractional derivatives adds complexity to the study, increasing the difficulty of observation and estimation. Furthermore, the information that the fractional order controller relies on is no longer limited to the state or output of the system, but may also contain the fractional derivatives and integrals of these signals, which puts new requirements on the estimation methods.

Within systems theory, state observers are equipped to fulfill the above estimation tasks. Currently, numerous classical observers have been extended to fractional order case, with approaches often derived from direct extensions of integer order systems, such as the Luenberger-like observers [60], [61], Kalman filters [62], [63], sliding-mode observers [58], adaptive observers [57], [59], high-gain observers [64], reduced-order state observers [65], extended state observers [66], etc. In terms of observer design, accuracy, rapidity, and robustness are the key performance indexes, where high accuracy is the basic requirement of the observer, rapidity is an important guarantee to perform real-time tasks, and robustness is an effective tool against noise interference. However, most of the above fractional order observers usually cannot guarantee the non-asymptotic and robust properties at the same time, and they are greatly affected by the initial value. It should be mentioned that the above work cannot address the requirement of estimating fractional derivatives of states in fractional order controllers. Consequently, the theory of observer design for fractional order systems remains imperfect.

With regard to non-asymptotic and robust estimation, there are two strategies for designing estimators based on algebraic integral formula: the modulation function method in the time domain [67], [68] and the algebraic parametric method in the frequency domain [69], [70]. All these two methods

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were originated from linear identification and have been extended to state estimation [71], [72], [74]. They provide the advantages of high accuracy, fast convergence and robustness. The main reflection is that the resulting estimates are expressed in the form of integral formulas, where the noisy output measurement involves only the integral term. Thus, in the continuous noise-free case, the corresponding estimator does not involve any error sources. In noisy environments, the estimators are robust to corrupting noises, since the involved integrals behave like low-pass filters to attenuate the effects of noises. Recently, the modulating functions method were also applied for fractional order systems [75]–[80], where the estimation issue on the fractional derivatives and integrals of the state was solved in [78]–[80].

Nevertheless, the existing work exhibits certain constraints. On the one hand, with the existing modulating functions method, the considered system needs to be transformed into a differential equation form, which increases the computational complexity. Meanwhile, both the derivatives of the output and a set of fractional order derivative initial values were usually required to estimate states [76], [78], [79], and the obtained integral formulas are very complicated. On the other hand, due to the computational complexity of the fractional order calculus, the modulating functions method is limited in its scope of application. For instance, the modulating functions method in [76], [78] can only be applied for rational-order systems. Consequently, the trajectory of future endeavors needs to shift towards comprehensive research and exploration. This involves not only the consideration of more general fractional-order system models that are not limited by a specific order, but also simplifying the estimation process and enhancing the efficiency of the modulating functions method. This process of exploration has led us in the direction of greater breadth and refinement, opening up new areas for the development of this valuable methodology.

This thesis endeavors to provide improved modulating functions-based state estimation methods, both for integer order and non-integer order linear systems, where the order of the system can be in  $(0, 2)$ . They can be used to non-asymptotically and robustly estimate not only the states of the considered systems but also their fractional derivatives and integrals. In particular, in contrast to previous studies [76], [78], [79], the improved modulating functions-based estimation methods are directly applied to the observable canonical form of the state-space model, they greatly simplify the estimation process and achieve fast, accurate and robust estimation goals. In addition, we consider systems with a practical background and present a novel modulating functions-based speed, current, and their fractional order derivatives and integrals estimators for the design of sensorless fractional order  $PI^\lambda D^\mu$  controllers for DC motors.

### 1.3 Outline

This thesis summarizes my Ph.D. work from 2021 to 2024, which is organized as follows:

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Chapter 2 presents some preliminaries on fractional calculus, including some basic functions, definitions, properties, and fundamental knowledge of fractional calculus. Afterward, the basic concepts and conclusions of fractional order linear systems in input-output differential models and state-space representation models are introduced.

Chapter 3 introduces the history of the development of the modulating functions method, definitions of various modulating functions, and the main thoughts underlying the method. On this basis, the main results of using the modulating functions method in various applications are recalled, such as parameter identification, differentiators, and state estimation.

Based on these two chapters, the main results of this thesis are unfolded in Part III–V.

Part III covers Chapter 4 and 5, where an improved modulating functions-based state estimation method is developed for integer order and non-integer order systems.

Chapter 4 develops an improved modulating functions-based state estimation method for integer order system to simplify the estimation process and enhance the efficiency of the existing modulating functions method, which is non-asymptotic and robust.

Chapter 5 generalizes the non-asymptotic and robust estimator given in Chapter 4 to a class of fractional linear systems with multiple inputs and multiple outputs for estimating the fractional integral of the state. In the considered system model, the differentiation order is a value in  $(0, 2)$  instead of only a rational value. Moreover, in Chapter 4 and 5, the fractional derivatives of the state can be computed by applying the Grünwald-Letnikov scheme.

Part IV covers Chapter 6 and 7, where the second improved modulating functions-based state estimation method is introduced for integer order and non-integer order systems. In order to focus and simplify the analysis, in this part we consider single input single output systems.

Chapter 6 develops the second improved modulating functions method to estimate the fractional integral and derivative of the state directly for integer order system without using the Grünwald-Letnikov scheme. Particularly, the estimation process involves two distinct approaches to initial value terms—one involves their removal through modulating function design, while the other entails a two-step process: first estimating initial value terms and subsequently estimating the fractional derivatives and integrals of the state. Furthermore, we consider applications for DC motors.

Chapter 7 generalizes the non-asymptotic and robust estimators given in Chapter 6 to a class of fractional linear systems for estimating the fractional integral and derivative of the state directly without using the Grünwald-Letnikov scheme.

All the algebraic estimators given in Chapter 4–7 are verified in discrete noisy environment to show the accuracy and robustness.

Finally, conclusions are drawn in Chapter 8 with some perspectives.



### 1.3. OUTLINE

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## Part II

# Preliminaries



## Chapter 2

# Fractional calculus and fractional order control

The basics of fractional calculus and fractional order linear systems are covered in this chapter. It begins by introducing some special functions, including the Gamma function and the Mittag-Leffler function, which are essential elements of fractional calculus. Then, the definitions and properties of the Riemann-Liouville fractional integral are reviewed. Building on this foundation, the Riemann-Liouville, Caputo, and Grünwald-Letnikov fractional derivatives are presented. Notably, fractional integration by parts formulas are summarized, which provide necessary mathematical tools for the following text. Moving on, both the fractional differential equation models and state-space representation models of fractional order linear systems are presented, as well as their properties and behaviors. Finally, the fractional order  $PI^\lambda D^\mu$  controller is recalled.

### 2.1 Fractional calculus

In this section, some basic knowledge of fractional calculus are provided, including some useful functions, the definitions and properties of fractional order integrals and fractional order derivatives.

#### 2.1.1 Basic functions and operators

- **Gamma function**

The Gamma function, as the generalization of the factorial function for non-negative integers, plays a key role in defining the fractional order operator, which is defined as follows.

**Definition 2.1** [81] For  $\forall z \in \mathbb{C}$  with  $\Re(z) > 0$ , the Gamma function is defined by the following infinite integral:

$$\Gamma(z) =: \int_0^{\infty} e^{-t} t^{z-1} dt. \quad (2.1)$$

## 2.1. FRACTIONAL CALCULUS

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According to the following reflection formula of the Gamma function [81]:

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}, \quad 0 < \Re(z) < 1, \quad (2.2)$$

we can extend the integral function in (2.1) analytically to complex numbers with non-positive real parts as follows [82]:

$$\Gamma(z) = \frac{\pi}{\Gamma(1-z) \sin \pi z}, \quad \Re(z) < 1, \quad (2.3)$$

with the non-positive integers  $z = 0, -1, -2, \dots$  as the simple poles of the Gamma function  $\Gamma(z)$ . Hence, the Gamma function is a meromorphic function, which is holomorphic properties across the entire complex plane except for non-positive integers. Fig. 2.1 displays the absolute value of the Gamma function on the complex plane. Furthermore, Fig. 2.2 shows the Gamma function restricted to real numbers.

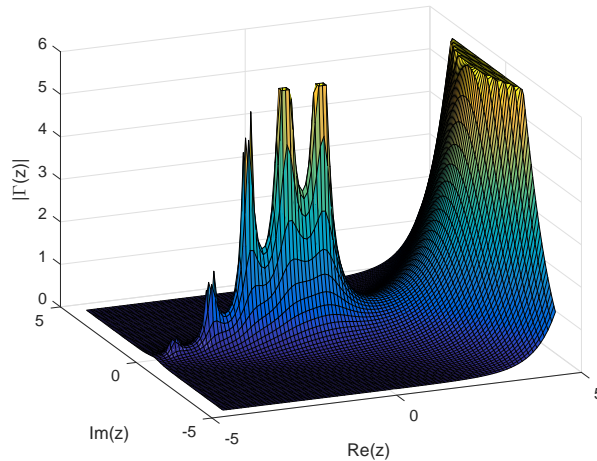


Figure 2.1: The absolute value of Gamma function on complex plane (cited from [83]).

One of the basic properties of the Gamma function is as follows:

$$\Gamma(z+1) = z\Gamma(z). \quad (2.4)$$

In particular, if  $z \in \mathbb{N}_+$ , we have:

$$\Gamma(z) = (z-1)!. \quad (2.5)$$

As an application of the Gamma function, the Beta function is defined as follows:

$$B(z, m) = \int_0^1 t^{z-1}(1-t)^{m-1} dt, \quad \Re(m) > 0, \Re(z) > 0, \quad (2.6)$$

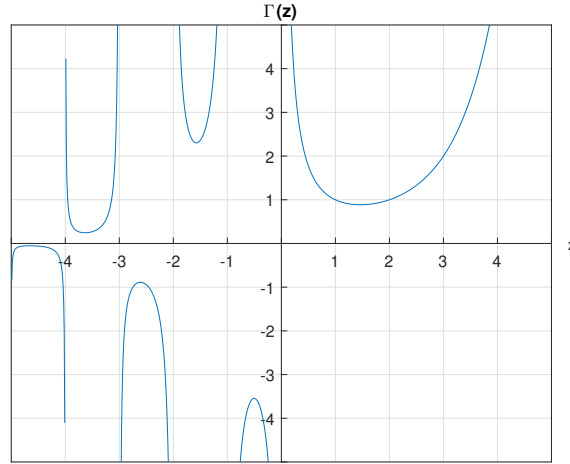


Figure 2.2: The Gamma function  $\Gamma(z)$  (cited from [83]).

which is connected with the Gamma function by the relation [22]:

$$B(z, m) = \frac{\Gamma(z)\Gamma(m)}{\Gamma(z+m)}, \quad z, w \notin \mathbb{N}_-, \quad (2.7)$$

from which it follows that:  $B(z, w) = B(w, z)$ .

• **Mittag-Leffler function**

As a generalization of the classic exponential function, the Mittag-Leffler function serves an important role in the solutions of fractional differential equations. Recall that the Taylor series expansion of the exponential function  $e^z$  with  $z \in \mathbb{C}$  can be expressed by the following formula:

$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!} = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+1)}. \quad (2.8)$$

Then, if a parameter  $\alpha \in \mathbb{C}$  is introduced into (2.8), the one-parameter Mittag-Leffler function can be obtained, defined as follows.

**Definition 2.2** [27] *Let  $\alpha \in \mathbb{C}$  with  $\Re(\alpha) > 0$ , then the one-parameter Mittag-Leffler function is defined as follows:  $\forall z \in \mathbb{C}$ ,*

$$E_\alpha(z) = \sum_0^\infty \frac{z^k}{\Gamma(\alpha k + 1)}. \quad (2.9)$$

Thus, we can see that  $e^z$  is a special case of (2.9) as  $\alpha = 1$ . The comparison between  $E_\alpha(z)$  ( $E_\alpha(-z)$ ) and  $e^z$  ( $e^{-z}$ ) is given in Fig. 2.3.

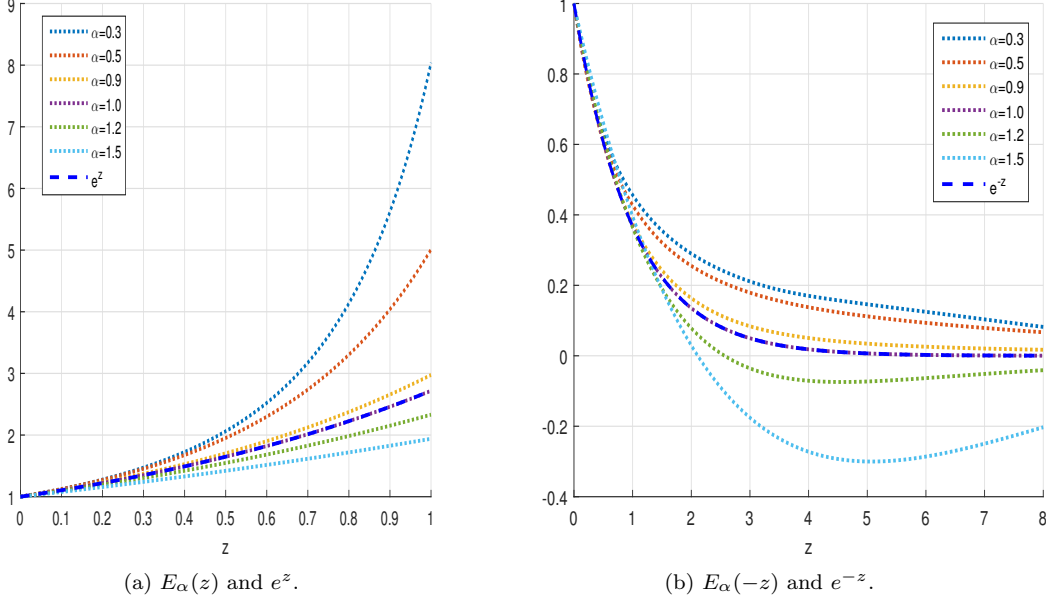


Figure 2.3: Illustration of  $E_\alpha(z)$  ( $E_\alpha(-z)$ ) and  $e^z$  ( $e^{-z}$ ) (cited from [83]).

In addition, if another parameter  $\beta \in \mathbb{C}$  is introduced into (2.9), the two-parameter Mittag-Leffler function can be obtained, defined as follows.

**Definition 2.3** [27] Let  $\alpha, \beta \in \mathbb{C}$  with  $\Re(\alpha), \Re(\beta) > 0$ , then the two-parameter Mittag-Leffler function is defined as follows:  $\forall z \in \mathbb{C}$ ,

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}. \quad (2.10)$$

Therefore,  $e^z$  is a special case of (2.10) as  $\alpha = 1$  and  $\beta = 1$ . Moreover, the two-parameter Mittag-Leffler function is shown in Fig. 2.4.

Some times, a more generalized Mittag-Leffler function with three parameters is required, which is defined as follows.

**Definition 2.4** [84] Let  $\alpha, \beta, \gamma \in \mathbb{C}$  with  $\Re(\alpha) > 0$ ,  $\Re(\beta) > 0$  and  $\Re(\gamma) > 0$ , then the three-parameter Mittag-Leffler function is defined as follows:  $\forall z \in \mathbb{C}$ ,

$$E_{\alpha,\beta}^\gamma(z) = \sum_{k=0}^{\infty} \frac{\Gamma(k + \gamma)}{\Gamma(\gamma)k!} \frac{z^k}{\Gamma(\alpha k + \beta)}. \quad (2.11)$$

In particular, by taking  $z = 0$ , we get:

$$E_{\alpha,\beta}(0) = E_{\alpha,\beta}^\gamma(0) = \frac{1}{\Gamma(\beta)}. \quad (2.12)$$

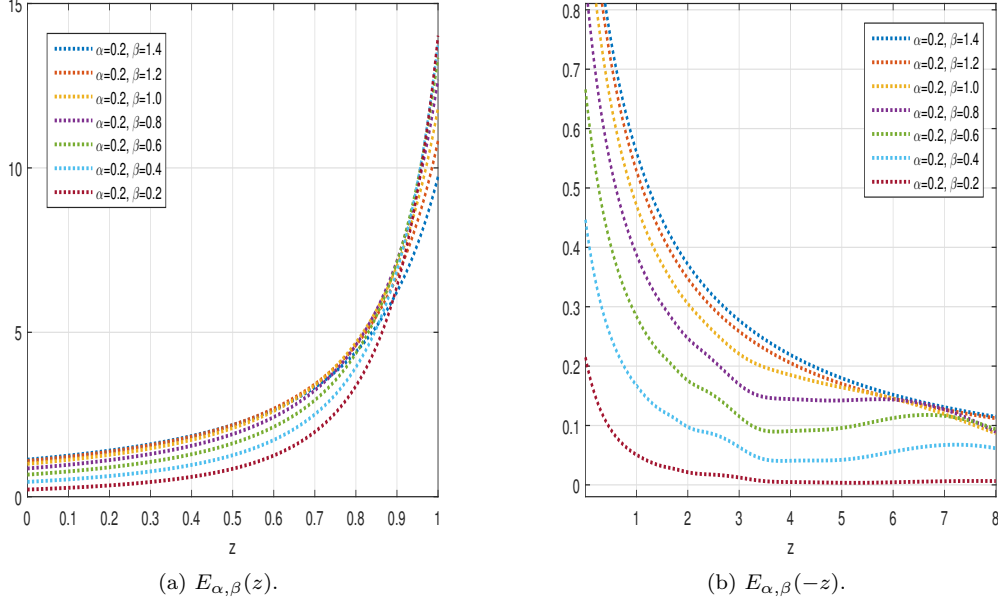


Figure 2.4: Illustration of  $E_{\alpha,\beta}(z)$  and  $E_{\alpha,\beta}(-z)$  (cited from [83]).

Furthermore, by taking some special values to the parameters  $\alpha$  and  $\beta$ , we derive the following well-known classical functions [51]:

- $E_{1,1}(z) = e^z$ ,
- $E_{2,1}(z) = \frac{e^z - 1}{z}$ ,
- $E_{2,1}(z^2) = \cosh(z) = \frac{e^z + e^{-z}}{2}$ ,
- $E_{2,2}(z^2) = \frac{\sinh(z)}{z} = \frac{e^z - e^{-z}}{2z}$ ,
- $E_{\alpha,1}(z) = E_\alpha(z)$ .

### 2.1.2 Fractional integral

In this subsection, the definition of the Riemann-Liouville fractional integral is presented, which generalizes the famous Cauchy iterated integral formula defined below.

**Definition 2.5** [1] Let  $n \in \mathbb{N}^*$ , the Cauchy integral defined on  $I = [0, h] \subset \mathbb{R}_+$  is given by:  $\forall t \in I^*$ ,

$$D_t^{-n} f(t) := \int_0^t \int_0^{x_n} \cdots \int_0^{x_2} f(x_1) dx_1 \cdots dx_n = \frac{1}{(n-1)!} \int_0^t (t-\tau)^{n-1} f(\tau) d\tau. \quad (2.13)$$

For any  $\alpha \in \mathbb{R}_+^*$ , replacing  $n$  with  $\alpha$  in (2.13), then the following Riemann-Liouville fractional integral can be obtained, where the factorial function  $(\cdot)!$  is generalized to the Gamma function  $\Gamma(\cdot)$ .



• **Riemann-Liouville fractional integral**

**Definition 2.6** [51] *Let  $\alpha \in \mathbb{R}_+$ , the Riemann-Liouville fractional integral of a function  $f$  is defined by:  $\forall t \in \mathbb{R}_+$ ,*

$$\begin{cases} D_t^0 f(t) & := f(t), \\ D_t^{-\alpha} f(t) & := \frac{1}{\Gamma(\alpha)} \int_a^t (t - \tau)^{\alpha-1} f(\tau) d\tau. \end{cases} \quad (2.14)$$

Note that if  $f$  is continuous for  $t \geq 0$ , then we have  $\lim_{\alpha \rightarrow 0} D_t^{-\alpha} f(t) = D_t^0 f(t) = f(t)$  [51].

The Riemann-Liouville fractional integral of  $f$  defined by (2.14) can also be described in a view of convolution product [51] in the following way:

$$D_t^{-\alpha} \hat{f}(t) := w(t) * \hat{f}(t), \quad (2.15)$$

where  $w$  is a causal function defined as:

$$w(\tau) =: \begin{cases} \frac{1}{\Gamma(\alpha)} \tau^{\alpha-1}, & \text{for } \tau \in ]0, +\infty[, \\ 0, & \text{else,} \end{cases} \quad (2.16)$$

and the causal function  $\hat{f}$  is given by:

$$\hat{f}(\tau) =: \begin{cases} f(\tau), & \text{for } \tau \in [0, +\infty[, \\ 0, & \text{else.} \end{cases} \quad (2.17)$$

Indeed, if  $t - \tau < 0$ ,  $w(t - \tau) = 0$ . Furthermore, the Laplace transform of the Riemann-Liouville fractional integral is given by:

$$\mathcal{L} \left\{ D_t^{-\alpha} f(t) \right\} (s) := s^{-\alpha} F(s), \quad (2.18)$$

in which  $s$  is the variable in the frequency domain.

**2.1.3 Fractional derivatives**

In this subsection, three well-known types of fractional derivatives, including Riemann-Liouville, Caputo and Grünwald-Letnikov types, are introduced as well as the associated properties [51]. Among them, the Riemann-Liouville fractional derivative serves as the basis for the main contributions in this thesis.

• **Riemann-Liouville fractional derivative**

The Riemann-Liouville fractional derivative can be defined by applying the integer order derivative operator, which is given as follows.

## 2.1. FRACTIONAL CALCULUS

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**Definition 2.7** Let  $\alpha \in \mathbb{R}_+$ , the Riemann-Liouville fractional derivative a function  $f$  is given by:  
 $\forall t \in \mathbb{R}_+$ ,

$$D_t^\alpha f(t) := \frac{d^l}{dt^l} \left\{ D_t^{\alpha-l} f(t) \right\} = \frac{1}{\Gamma(l-\alpha)} \frac{d^l}{dt^l} \int_0^t (t-\tau)^{l-\alpha-1} f(\tau) d\tau, \quad (2.19)$$

where  $l = \lceil \alpha \rceil$ .

**Remark 2.1** If  $\alpha \in \mathbb{N}$ ,  $D_t^\alpha f$  refers to integer order derivative, else it refers to non-integer derivative. Based on (2.14), if  $\alpha \in (0, 1)$ , the Riemann-Liouville fractional integral contains a kernel with an integrable singularity at  $\tau = t$ , thus is defined by an improper integral. Hence, the non-integer derivative is defined by an improper integral in (2.19).

**Remark 2.2** For the convenience of the text narrative, we use  $D_t^{-\alpha}$  with  $\alpha \in \mathbb{R}_+$  to refer to  $I_t^\alpha$ , which means when the order in the Riemann-Liouville fractional derivative operator is negative, it refers to the Riemann-Liouville fractional integral operator.

**Example 2.1** [22] Let  $\alpha \in \mathbb{R}_+^*$ , then the  $\alpha$  order Riemann-Liouville fractional derivative of a power function  $t^\beta$  with  $\beta \in (-1, \infty) \subset \mathbb{R}$  is given by:  $D_t^\alpha \{t^\beta\} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} t^{\beta-\alpha}$  for  $t \in \mathbb{R}_+$ .

It is worth noting that in Example 2.1, if  $\beta$  is equal to 0, the Riemann-Liouville fractional derivative of a constant function is not necessarily zero.

**Example 2.2** [85] Let  $\omega \in \mathbb{R}$ ,  $\alpha \in \mathbb{R}_+^*$ , and  $I = [0, h]$  with  $h \in \mathbb{R}_+^*$ , then the Riemann-Liouville fractional derivatives of the sine and cosine functions are given by:  $\forall t \in I^*$ ,

$$D_t^\alpha \sin(\omega t) = \frac{\omega t^{1-\alpha}}{\Gamma(2-\alpha)} {}_1F_2 \left( 1; \frac{1}{2}(2-\alpha), \frac{1}{2}(3-\alpha); -\frac{1}{4}\omega^2 t^2 \right), \quad (2.20)$$

$$D_t^\alpha \cos(\omega t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)} {}_1F_2 \left( 1; \frac{1}{2}(1-\alpha), \frac{1}{2}(2-\alpha); -\frac{1}{4}\omega^2 t^2 \right), \quad (2.21)$$

in which  ${}_1F_2$  is the generalized hypergeometric function.

From Definition 2.7, it follows that the Riemann-Liouville fractional differentiation is a linear operator, i.e. for  $\alpha \in \mathbb{R}_+^*$ , and  $f, g \in \mathbf{L}^1(I)$ ,

$$D_t^\alpha \left( \lambda f(t) + \mu g(t) \right) = \lambda D_t^\alpha f(t) + \mu D_t^\alpha g(t), \quad \forall \lambda, \mu \in \mathbb{R}, \forall t \in I. \quad (2.22)$$

Besides, the Laplace transform of the Riemann-Liouville fractional derivative of  $f$  is given as follows:

$$\mathcal{L} \{ D_t^\alpha f(t) \} = s^\alpha F(s) - \sum_{k=0}^{\lceil \alpha \rceil - 1} s^k \left[ D_t^{\alpha-k-1} f(t) \right]_{t=0}. \quad (2.23)$$

## 2.1. FRACTIONAL CALCULUS

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It is worth noting that, different from the Laplace transform of the Riemann-Liouville fractional integral (2.18), (2.23) also yields some fractional initial values of  $f$ .

Unlike integer order derivatives, fractional order derivatives are defined by combining a fractional order integral and an integer order derivative, which are not commutative. Therefore, the non-integer order derivative operators exhibit non-commutativity. Thus, the additive index law for Riemann-Liouville fractional integral and derivative is formulated.

**Lemma 2.1** [51] *Let  $\alpha, \beta \in \mathbb{R}$ , then the additive index law of the Riemann-Liouville fractional derivatives of a function  $f$  is given as follows:  $\forall t \in \mathbb{R}_+ \cup \{0\}$ ,*

$$D_t^\beta \{D_t^\alpha f(t)\} = D_t^{\beta+\alpha} f(t) - \psi_{\beta,\alpha} \{f(t)\}, \quad (2.24)$$

where  $\psi_{\beta,\alpha} \{f(t)\}$  is a decreasing function of  $t$  and defined as follows:

$$\psi_{\beta,\alpha} \{f(t)\} := \sum_{i=1}^{[\alpha]} c_{\beta,i} t^{-\beta-i} \left[ D_t^{\alpha-i} f(t) \right]_{t=0} \quad (2.25)$$

with

$$c_{\beta,i} = \begin{cases} 0, & \text{if } \beta \in \mathbb{N} \text{ or } \alpha \in \mathbb{R}_-, \\ \frac{1}{\Gamma(1-\beta-i)}, & \text{else.} \end{cases} \quad (2.26)$$

In particular, we can obtain:

- for  $\beta \in \mathbb{R}$ ,  $\alpha \in \mathbb{R}_+ \cup \{0\}$ ,

$$D_t^\beta \{D_t^{-\alpha} f(t)\} = D_t^{\beta-\alpha} f(t), \quad (2.27)$$

- for  $\beta \in \mathbb{N}$ ,  $\alpha \in \mathbb{R}_+$ ,

$$D_t^\beta \{D_t^\alpha f(t)\} = D_t^{\beta+\alpha} f(t). \quad (2.28)$$

According to the additive index law given in Lemma 2.1 for the Riemann-Liouville fractional derivative, we can see the main difference between fractional calculus and integer calculus, so it is necessary to introduce the following definition of the fractional sequential derivative to distinguish it.

**Definition 2.8** [51] *Let  $k \in \mathbb{N}$ , the Riemann-Liouville fractional sequential derivative of a function  $f$  is defined as follows:*

$$\mathcal{D}_t^{k\alpha} f(t) := \begin{cases} f(t), & \text{for } k = 0, \\ D_t^\alpha \{ \mathcal{D}_t^{(k-1)\alpha} f(t) \}, & \text{for } k \geq 1. \end{cases} \quad (2.29)$$

## 2.1. FRACTIONAL CALCULUS

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Building on this foundation, the following additive index law is derived from Lemma 2.1.

**Lemma 2.2** [51] *The Riemann-Liouville fractional integral and derivative of a fractional sequential derivative can be given as follows:  $\forall \beta \in \mathbb{R}, \forall k \in \mathbb{N}^*$ ,*

$$D_t^\beta \left\{ \mathcal{D}_t^{k\alpha} f(t) \right\} = D_t^{\beta+k\alpha} f(t) - \phi_{\beta,k,\alpha} \{f(t)\}, \quad (2.30)$$

where

$$\phi_{\beta,k,\alpha} \{f(t)\} := \sum_{j=1}^k \psi_{\beta+(j-1)\alpha,\alpha} \left\{ \mathcal{D}_t^{(k-j)\alpha} f(t) \right\}. \quad (2.31)$$

$\psi_{\beta,\alpha} \{f(t)\}$  is a decreasing function of  $t$  defined by:

$$\psi_{\beta,\alpha} \{f(t)\} := \sum_{i=1}^{[\alpha]} c_{\beta,i} t^{-\beta-i} \left[ D_t^{\alpha-i} f(t) \right]_{t=0} \quad (2.32)$$

with

$$c_{\beta,i} = \begin{cases} 0, & \text{if } \beta \in \mathbb{N} \text{ or } \alpha \in \mathbb{R}_-, \\ \frac{1}{\Gamma(1-\beta-i)}, & \text{else.} \end{cases} \quad (2.33)$$

Furthermore, if  $k \geq 2$ , then it hold:

$$\mathcal{D}_t^{k\alpha} f(t) = D_t^{k\alpha} f(t) - \phi_{\alpha,(k-1),\alpha} \{f(t)\}. \quad (2.34)$$

**Remark 2.3** *By the additive index law given in Lemma 2.2, if  $\alpha \in \mathbb{R}_-$ , then it holds:  $\mathcal{D}_t^{k\alpha} f(t) = D_t^{k\alpha} f(t)$ .*

Next, let us recall some results on the Riemann-Liouville fractional derivative at zero, which has been discussed by Igor Podlubny [51].

Let  $f$  be a analytic function on  $[0, \epsilon]$  with  $\epsilon \in \mathbb{R}_+^*$ , then the Taylor series expansion of  $f$  at  $t = 0$  can be given as follows:

$$f(t) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} t^k. \quad (2.35)$$

Let  $\alpha \in \mathbb{R}_+^*$ , using the result in Example 2.1 to (2.35) term-by-term, we get:

$$D_t^\alpha f(t) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{\Gamma(k+1-\alpha)} t^{k-\alpha}, \quad (2.36)$$

hence, we get:

$$D_t^\alpha f(t) \sim \frac{f(0)}{\Gamma(1-\alpha)} t^{-\alpha}, \quad (t \rightarrow 0). \quad (2.37)$$

Moreover,

$$\lim_{t \rightarrow 0} D_t^\alpha f(t) = \begin{cases} 0, & \alpha < 0, \\ f(0), & \alpha = 0, \\ \infty, & \alpha > 0. \end{cases} \quad (2.38)$$

Furthermore, for  $g(t) = t^\beta f(t)$  with  $\beta > -1$  and  $f(0) \neq 0$ , the following results similar to (2.38) can be deduced:

$$\lim_{t \rightarrow 0} D_t^\alpha g(t) = \begin{cases} 0, & \alpha < \beta, \\ \frac{f(0)\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}, & \alpha = \beta, \\ \infty, & \alpha > \beta. \end{cases} \quad (2.39)$$

Additionally, there is still a useful result on the Riemann-Liouville fractional derivative at zero.

**Lemma 2.3** [35] *Let  $I = [0, h] \subset \mathbb{R}_+$  and  $f$  be an  $l$ -times continuously differentiable function on  $I$ , if*

$$f^{(i)}(0) = 0, \quad \text{for } i = 0, \dots, l-1, \quad (2.40)$$

then  $\forall \gamma \in \mathbb{R}$  with  $\gamma \leq l$ ,

$$[D_t^\gamma f(t)]_{t=0} = 0. \quad (2.41)$$

### • Caputo fractional derivative

Different from the Riemann-Liouville fractional derivative, the Caputo one is defined by taking the Riemann-Liouville fractional integral of an integer derivative.

**Definition 2.9** [51] *Let  $\alpha \in \mathbb{R}_+$ , the Caputo fractional derivative of a function  $f$  is given by:  $\forall t \in \mathbb{R}_+^*$ ,*

$${}^C D_t^\alpha f(t) := D_t^{\alpha-l} f^{(l)}(t) = \frac{1}{\Gamma(l-\alpha)} \int_0^t (t-\tau)^{l-\alpha-1} f^{(l)}(\tau) d\tau, \quad (2.42)$$

where  $l = \lceil \alpha \rceil$  with  $l \in \mathbb{N}$ .

**Remark 2.4** *The conditions for the existence of the Riemann-Liouville and Caputo derivatives defined by Definition 2.7 and Definition 2.9 are as follows:*

- i) *If  $f \in \mathcal{AC}^n(I)$ , then  $D_t^\alpha f$  and  ${}^C D_t^\alpha f$  exist almost everywhere on  $I$ .*
- ii) *If  $f$  is  $n$  times continuously differentiable on  $I$ , then  $D_t^\alpha f$  and  ${}^C D_t^\alpha f$  exist on  $I \setminus \{0\}$ .*

## 2.1. FRACTIONAL CALCULUS

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From Definition 2.9, we can conclude that the Caputo fractional differentiation is a linear operator, *i.e.* for  $\alpha \in \mathbb{R}_+^*$ , and  $f, g \in \mathbf{L}^1(I)$ ,

$${}^C D_t^\alpha (\lambda f(t) + \mu g(t)) = \lambda {}^C D_t^\alpha f(t) + \mu {}^C D_t^\alpha g(t), \quad \forall \lambda, \mu \in \mathbb{R}, \forall t \in I. \quad (2.43)$$

Besides, the Laplace transform of the Caputo fractional derivative of  $f$  is given as follows: [51]:

$$\mathcal{L} \left\{ {}^C D_t^\alpha f(t) \right\} = s^\alpha F(s) - \sum_{k=0}^{[\alpha]-1} s^{\alpha-k-1} f^{(k)}(0). \quad (2.44)$$

The Caputo fractional derivative of a polynomial  $t^n$  with  $n \in \mathbb{N}$  is given by [22]:

$${}^C D_t^\alpha \{t^n\} =: \begin{cases} \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)} t^{n-\alpha}, & \text{if } n \geq [\alpha], \\ 0, & \text{else.} \end{cases} \quad (2.45)$$

Particularly, if  $n$  is equal 0, the Caputo fractional derivative of a constant function is zero.

Based on Definition 2.7 and Definition 2.9, the difference between the Riemann-Liouville and Caputo fractional derivatives is the operation order of the involved integer derivative and the fractional integral. Nevertheless, the relationship between the Riemann-Liouville and Caputo fractional derivatives is built as follows.

Let  $\alpha \in \mathbb{R}_+ \setminus \mathbb{N}^*$ , by applying (2.24), the relation of the two fractional derivatives is given as follows:  $\forall t \in \mathbb{R}_+$ ,

$$D_t^\alpha f(t) = {}^C D_t^\alpha f(t) + \sum_{j=0}^{[\alpha]-1} \frac{t^{j-\alpha}}{\Gamma(1+j-\alpha)} f^{(j)}(0). \quad (2.46)$$

Therefore, if the initial values  $f^{(j)}(0)$  with  $j = 0, \dots, [\alpha] - 1$  are known, the two fractional derivatives can be calculated by using each other. In particular, these two types of fractional derivatives are equal if  $f^{(j)}(0) = 0$ .

### • Grünwald-Letnikov fractional derivative

In classic calculus, the first derivative of a continuous function  $f$  is defined by:

$$\frac{d}{dt} f(t) =: f'(t) = \lim_{s \rightarrow 0} \frac{f(t) - f(t-s)}{s}. \quad (2.47)$$

Then, by extending ((2.47)) to the  $k^{\text{th}}$  order derivative of function  $f$ , we have:

$$\frac{d^k}{dt^k} f(t) =: f^{(k)}(t) = \lim_{s \rightarrow 0} \frac{1}{s^k} \sum_{j=0}^k (-1)^j \binom{k}{j} f(t-j s). \quad (2.48)$$

Moreover, by extending the order  $k$  in (2.48) to a real positive number, the following well-known Grünwald-Letnikov fractional derivative can be deduced.

**Definition 2.10** [51] Let  $\alpha \in \mathbb{R}$ , the Grünwald-Letnikov fractional derivative of a function  $f$  is given by:  $\forall t \in \mathbb{R}_+^*$ ,

$${}^G D_t^\alpha f(t) := \lim_{s \rightarrow 0} \frac{1}{s^\alpha} \sum_{j=0}^{\frac{t}{s}} (-1)^j \binom{\alpha}{j} f(t - js). \quad (2.49)$$

Let  $\alpha, s \in \mathbb{R}_+^*$ ,  $l = [\alpha] \in \mathbb{N}^*$ , then for a function  $f \in \mathbf{C}^l([0, s])$ , the Grünwald-Letnikov fractional derivative  ${}^G D_t^\alpha f$  is equivalent to the Riemann-Liouville fractional derivative  $D_t^\alpha f$ . According to Definition 2.10, the numerical scheme of Riemann-Liouville fractional derivative is given by [27]:

$$D_t^\alpha f(t) \approx \frac{1}{s^\alpha} \sum_{j=0}^{[\frac{t}{s}]} w_j^{(\alpha)} f(t - js), \quad (2.50)$$

in which the coefficients  $w_j^\alpha$  are recursively given by:

$$\begin{cases} w_0^{(\alpha)} = 1, & \text{if } j = 0, \\ w_j^{(\alpha)} = \left(1 - \frac{\alpha+1}{j}\right) w_{j-1}^{(\alpha)}, & \text{else.} \end{cases} \quad (2.51)$$

**Remark 2.5** If the initial values  $f^{(i)}(0)$  for  $i = 0, \dots, [\alpha] - 1$  are known, then the Caputo fractional derivative can also be approximated by the Grünwald-Letnikov scheme (2.50) through the relation (2.46).

Furthermore, the Grünwald-Letnikov fractional differentiation is a linear operator, i.e. for  $\alpha \in \mathbb{R}_+^*$ , and functions  $f, g$  on  $I$ ,

$${}^G D_t^\alpha (\lambda f(t) + \mu g(t)) = \lambda {}^G D_t^\alpha f(t) + \mu {}^G D_t^\alpha g(t), \quad \forall \lambda, \mu \in \mathbb{R}, \forall t \in I. \quad (2.52)$$

• **Fractional integration by parts formulae**

The integration by parts formulae play an indispensable role in using modulating function in our work. Next, we will review and summarize the fractional integration by parts formulae. To begin with, we recall the classic (integer order) one.

**Lemma 2.4** [20] Let  $L \in \mathbb{N}^*$ ,  $f, g \in \mathbf{C}^L([a, b])$  with any interval  $[a, b] \subset \mathbb{R}$ , then

$$\int_a^b g(t) f^{(L)}(t) dt = (-1)^L \int_a^b g^{(L)}(t) f(t) dt + \sum_{i=0}^{L-1} (-1)^i \left[ g^{(i)}(t) f^{(L-1-i)}(t) \right]_{t=a}^{t=b}. \quad (2.53)$$

Then, the following fractional integration by parts formulas are presented.

**Lemma 2.5** [52] Let  $\alpha \in (-1, \infty) \subset \mathbb{R}$ ,  $L = \lceil \alpha \rceil$ , and  $f, g \in \mathcal{C}^L([a, b])$  with any interval  $[a, b] \subset \mathbb{R}$ , then

$$\int_a^b g(\tau) D_\tau^\alpha f(\tau) d\tau = \int_a^b {}^C D_{\tau,t}^\alpha g(\tau) f(\tau) d\tau, \text{ for } -1 < \alpha < 0, \quad (2.54)$$

$$\int_a^b g(\tau) D_\tau^\alpha f(\tau) d\tau = \int_a^b {}^C D_{\tau,t}^\alpha g(\tau) f(\tau) d\tau + \sum_{k=0}^{L-1} (-1)^k \left[ g^{(k)}(\tau) D_\tau^{\alpha-1-k} f(\tau) \right]_{\tau=a}^{\tau=b}, \text{ for } \alpha > 0. \quad (2.55)$$

**Lemma 2.6** [80] Let  $\alpha \in \mathbb{R}_+$  and  $t \in \mathbb{R}_+$ , then the following fractional integration by parts formulae hold:

$$\int_0^t g(t-\tau) D_\tau^{-\alpha} f(\tau) d\tau = \int_0^t D_\tau^{-\alpha} g(\tau) f(t-\tau) d\tau, \quad (2.56)$$

$$\int_0^t g(t-\tau) D_\tau^\alpha f(\tau) d\tau = \int_0^t {}^C D_\tau^\alpha g(\tau) f(t-\tau) d\tau + \sum_{k=0}^{\lceil \alpha \rceil - 1} \left[ g^{(k)}(t-\tau) D_\tau^{\alpha-1-k} f(\tau) \right]_{\tau=0}^{\tau=t}. \quad (2.57)$$

## 2.2 Fractional order linear systems

In this section, two models for fractional order linear systems is discussed: the fractional differential equation model and the state-space representation model. In our work, we then focus on the state-space representation of fractional order linear systems.

### 2.2.1 Fractional differential equation model

The fractional order continuous-time linear dynamic system is given as follows: for  $\forall t \in \mathbb{R}$ ,

$$\sum_{i=0}^N a_i D_t^{\alpha_i} y(t) = \sum_{j=0}^M b_j D_t^{\beta_j} u(t), \quad (2.58)$$

in which  $a_i \in \mathbb{R}$ ,  $\alpha_i \in \mathbb{R}_+$ , for  $i = 0, \dots, N$ ,  $b_j \in \mathbb{R}$ ,  $\beta_j \in \mathbb{R}_+$ , for  $j = 0, \dots, M$ ,  $u$  and  $y$  are the input and output respectively,  $N, M \in \mathbb{N}^*$ ,  $M \leq N$ .

If all the fractional derivative orders  $\alpha_i$  and  $\beta_j$  are integer multiples of a base order in (2.58), then the system equation can be simplified by the following commensurate-order one:

$$\sum_{i=0}^N a_i D_t^{i\alpha} y(t) = \sum_{j=0}^M b_j D_t^{j\alpha} u(t), \quad (2.59)$$

in which  $\alpha_i = i\alpha$  for  $i = 0, \dots, N$ , and  $\beta_j = j\alpha$  for  $j = 0, \dots, M$ .

**This thesis centers around the state-space representation model outlined below.**



### 2.2.2 State-space representation model

The state-space representation of fractional order linear systems is given as follows:

$$\begin{cases} D_t^\alpha x &= Ax + Bu, \\ y &= Cx + \eta u, \end{cases} \quad (2.60)$$

where  $A \in \mathbb{R}^{N \times N}$  is the state matrix,  $B \in \mathbb{R}^{N \times 1}$  is the input matrix,  $C \in \mathbb{R}^{1 \times N}$  is the output matrix,  $\eta \in \mathbb{R}$ ,  $D_t^\alpha x = (D_t^\alpha x_1, \dots, D_t^\alpha x_N)^\top$  with  $\alpha \in \mathbb{R}$ ,  $x \in \mathbb{R}^N$  is with unknown initial values,  $y \in \mathbb{R}$  and  $u \in \mathbb{R}$  are the output and the input, respectively. In order to guarantee the stability of the considered fractional order system, it is assumed that  $0 < \alpha < 2$  [86].

Let us denote  $l = [\alpha]$ . By utilizing the Laplace transform in (2.60), one has:

$$s^\alpha X(s) - \sum_{k=0}^{l-1} s^k [D_t^{\alpha-k-1} x(t)]_{t=0} = AX(s) + BU(s), \quad (2.61)$$

it yields,

$$X(s) = (s^\alpha \mathbf{I}_N - A)^{-1} \left( BU(s) + \sum_{k=0}^{l-1} s^k [D_t^{\alpha-k-1} x(t)]_{t=0} \right). \quad (2.62)$$

By applying the inverse Laplace transform in (2.62), the analytical solution of (2.60) can be obtained [24]:

$$x(t) = \sum_{i=1}^{[\alpha]} \Phi_i [D_t^{\alpha-l} x(t)]_{t=0} + \int_0^t \Phi(t-\tau) Bu(\tau) d\tau, \quad (2.63)$$

where  $\Phi(t) = \sum_{k=0}^{\infty} \frac{A^k t^{(k+1)\alpha-1}}{\Gamma((k+1)\alpha)}$  and  $\Phi_i(t) = \sum_{k=0}^{\infty} \frac{A^k t^{(k+1)\alpha-i}}{\Gamma((k+i)\alpha-i+1)}$ .

#### • Observability and controllability

Next, let us recall the controllability and observability of the system given in (2.60).

**Theorem 2.1** [1] *The system defined by (2.60) is observable if the observability matrix given by*

$$\mathcal{O} = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{N-1} \end{pmatrix} \quad (2.64)$$

*has full column rank, i.e.  $\text{rank}(\mathcal{O}) = N$ , which we denote as condition  $(C_1)$ .*

**Theorem 2.2** [1] *The system defined by (2.60) is controllable if the controllability matrix given by*

$$C = \begin{pmatrix} B & AB & \dots & A^{N-1}B \end{pmatrix} \quad (2.65)$$

*has full row rank, i.e.  $\text{rank}(C) = N$ , which we denote as condition  $(C_2)$ .*

Based on the observability and controllability, the state-space representation of the system given by (2.60) can be transformed into the fractional order observable canonical form as presented in the following lemma.

**Lemma 2.7** [79] *By taking the following change of coordinates*

$$z = T x, \quad (2.66)$$

where

$$T = \begin{pmatrix} CA^{N-1} + \sum_{i=1}^{N-1} a_i CA^{i-1} \\ \vdots \\ CA^{N-j} + \sum_{i=j}^{N-1} a_i CA^{i-j} \\ \vdots \\ CA + a_{N-1}C \\ C \end{pmatrix}, \quad (2.67)$$

and  $a_i \in \mathbb{R}$ , for  $i = 0, \dots, N-1$ , are coefficients of the characteristic polynomial of  $A$ , the system defined by (2.60) satisfying  $(C_1) - (C_2)$  can be transformed into the following form:

$$\begin{cases} D_t^\alpha z = \hat{A}z + \hat{B}u, \\ y = \hat{C}z + \eta u, \end{cases} \quad (2.68)$$

where

$$\hat{A} = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 & -a_0 \\ 1 & 0 & \dots & 0 & 0 & -a_1 \\ 0 & 1 & \dots & 0 & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & -a_{N-2} \\ 0 & 0 & \dots & 0 & 1 & -a_{N-1} \end{pmatrix}, \quad (2.69)$$

$$\hat{B} = \left( \sum_{i=1}^N a_i CA^{i-1}B, \dots, \sum_{i=j}^N a_i CA^{i-j}B, \dots, CB \right)^\top \in \mathbb{R}^{N \times 1} \text{ with } a_N = 1, \hat{C} = (0, \dots, 0, 1) \in \mathbb{R}^{1 \times N}.$$

## 2.2. FRACTIONAL ORDER LINEAR SYSTEMS

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Next, the canonical form (2.68) can be expressed by a fractional order linear differential equation in the following lemma.

**Lemma 2.8** [79] *The system given in (2.68) satisfying (C<sub>1</sub>) – (C<sub>2</sub>) can be transformed into the following form:*

$$\sum_{i=0}^N a_i \mathcal{D}_t^{\alpha_i} y(t) = \hat{u}, \quad (2.70)$$

where

$$\hat{u} = \sum_{s=0}^N \lambda_s \mathcal{D}_t^{(N-s)\alpha} u, \quad (2.71)$$

with  $\lambda_0 = \eta$  and

$$\lambda_s = \sum_{i=N-s+1}^N a_i C A^{i-N+s-1} B + \eta a_{N-s}, \quad (2.72)$$

$a_N = 1$ , and  $a_i \in \mathbb{R}$  for  $i = 0, \dots, N-1$ , are the characteristic polynomial coefficients of  $A$ . Moreover, we have:

$$\begin{pmatrix} z_1 \\ \vdots \\ z_{N-1} \\ z_N \end{pmatrix} = M_2 \begin{pmatrix} y \\ \mathcal{D}_t^\alpha y \\ \vdots \\ \mathcal{D}_t^{(N-1)\alpha} y \end{pmatrix} - M_3 \begin{pmatrix} u \\ \mathcal{D}_t^\alpha u \\ \vdots \\ \mathcal{D}_t^{(N-1)\alpha} u \end{pmatrix}, \quad (2.73)$$

where

$$M_2 = \begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_{N-1} & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{N-1} & 1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \quad (2.74)$$

$$M_3 = \begin{pmatrix} \lambda_{N-1} & \lambda_{N-2} & \lambda_{N-3} & \cdots & \lambda_1 & \lambda_0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \lambda_1 & \lambda_0 & 0 & \cdots & 0 & 0 \\ \lambda_0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}. \quad (2.75)$$

Then, by applying Theorem 2.1, the following theorem can be deduced.

**Theorem 2.3** *The system defined by (2.68) satisfying (C<sub>1</sub>) – (C<sub>2</sub>) can be written into the following form on  $I$ :*

$$\sum_{i=0}^N a_i \mathcal{D}_t^{i\alpha} y(t) = \hat{u}(t) + \sum_{i=2}^N a_i \phi_{\alpha, i-1, \alpha} \{y(t)\}, \quad (2.76)$$

## 2.2. FRACTIONAL ORDER LINEAR SYSTEMS

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where

$$\hat{u} = \sum_{s=0}^N \lambda_s \mathcal{D}_t^{(N-s)\alpha} u + \sum_{s=0}^{N-2} \lambda_s \phi_{\alpha, N-s-1, \alpha} \{u(t)\}, \quad (2.77)$$

where  $\phi_{\alpha, i-1, \alpha} \{y(t)\}$  and  $\lambda_s \phi_{\alpha, N-s-1, \alpha} \{u(t)\}$  can be given by (2.31).

Theorem 2.3 establishes the relationship between the fractional order linear differential equation model and the state-space representation model. In particular, if  $\phi_{\alpha, i-1, \alpha} \{y(t)\} \neq 0$  and  $\lambda_s \phi_{\alpha, N-s-1, \alpha} \{u(t)\} \neq 0$ , these two models are not equivalent.

Finally, if condition  $(C_1)$  is fulfilled, the matrix  $M_1$  defined in (2.67) is invertible. Consequently, by applying Lemma 2.7 and Lemma 2.8, the following corollary can be obtained.

**Corollary 2.1** [79] *The state of the system defined by (2.68) satisfying  $(C_1) - (C_2)$  can be given as follows:*

$$x = M_1^{-1} M_2 \begin{pmatrix} y \\ \mathcal{D}_t^\alpha y \\ \vdots \\ \mathcal{D}_t^{(N-1)\alpha} y \end{pmatrix} - M_1^{-1} M_3 \begin{pmatrix} u \\ \mathcal{D}_t^\alpha u \\ \vdots \\ \mathcal{D}_t^{(N-1)\alpha} u \end{pmatrix}, \quad (2.78)$$

where  $M_1$ ,  $M_2$  and  $M_3$  are given in Lemma 2.7 and Lemma 2.8.

### • Stability analysis

System stability analysis is vital for predicting and ensuring the reliability of dynamic systems by assessing their ability to maintain equilibrium over time.

Let us consider the following fractional order system:

$$\mathcal{D}_t^\alpha x = Ax. \quad (2.79)$$

Then the following definitions of various stabilities of the system (2.79) are reviewed.

**Definition 2.11** [87] *Consider the system given in (2.79) with the initial condition  $[\mathcal{D}_t^{\alpha-1} x(t)]_{t=0} = x_0$ , the system (2.79) is said to be*

- *stable if for any  $x_0$ , there exists  $\epsilon > 0$  such that  $\|x(t)\| \leq \epsilon$  for  $t \geq 0$ ;*
- *asymptotically stable if  $\lim_{t \rightarrow +\infty} \|x(t)\| = 0$ .*

**Definition 2.12** [53] *Consider the system given in (2.79) with  $\alpha \in (0, 1]$  and the initial condition  $x_0 = x(0)$ , the solution of (2.79) is said to be Mittag-Leffler stable if*

$$\|x(t)\| \leq \left( m(x_0) E_\alpha(-\lambda t^\alpha) \right)^b, \quad (2.80)$$

where  $\lambda \geq 0$ ,  $b > 0$ ,  $m$  is a locally Lipschitz function such that  $m(x) \geq 0$  and  $m(0) = 0$ .

### 2.3. FRACTIONAL ORDER PI<sup>λ</sup>D<sup>μ</sup> CONTROLLERS

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Next, the following stability analysis results are provided.

**Theorem 2.4** [87], [88] Consider the fractional order linear system (2.60) with  $\alpha \in (0, 2)$  and the initial condition  $[D_t^{\alpha-1}x(t)]_{t=0} = x_0$ ,

- If  $|\arg(\text{spec}(A))| > \frac{\alpha\pi}{2}$ , then the zero solution of system (2.60) is asymptotically stable, where the convergence rate is analogous to  $t^{-1-\alpha}$ .
- If  $|\arg(\text{spec}(A))| \geq \frac{\alpha\pi}{2}$  and the critical eigenvalues

$$\left\{ \lambda_c \in \text{spec}(A) \mid |\arg(\lambda_c)| = \frac{\alpha\pi}{2} \right\}$$

have the same algebraic and geometric multiplicities, then the zero solution of system (2.79) is stable but not asymptotically stable.

where  $\text{spec}(A)$  is the spectrum of the matrix  $A$ , i.e. the set of eigenvalues of  $A$ .

In addition, more results on the stability analysis for fractional order systems with the Riemann-Liouville fractional derivative can be found in [45]–[47], [53].

### 2.3 Fractional order PI<sup>λ</sup>D<sup>μ</sup> controllers

Ever since Oustaloup's [15] groundbreaking introduction of the fractional order controller, a burgeoning wave of innovation has given rise to a plethora of sophisticated fractional order controllers [21], [89]. In particular, the fractional order PI<sup>λ</sup>D<sup>μ</sup> controllers, as the extension of the classical PID controllers, are widely used both for integer order and fractional order systems [51], [90].

In this work, the following fractional order transfer function is considered:

$$G(s) = \frac{1}{a_n s^{\beta_n} + a_{n-1} s^{\beta_{n-1}} + \dots + a_0 s^{\beta_0}}, \quad (2.81)$$

moreover,  $\beta_n > \beta_{n-1} > \dots > \beta_0 > 0$  is assumed. The corresponding fractional order differential equation model can be given as:

$$a_n D^{\beta_n} y(t) + a_{n-1} D^{\beta_{n-1}} y(t) + \dots + a_1 D^{\beta_1} y(t) + a_0 D^{\beta_0} y(t) = u(t). \quad (2.82)$$

Then the fractional order PI<sup>λ</sup>D<sup>μ</sup> controller is presented [35] in the time domain:  $\forall t > 0$ ,

$$u(t) = K_P e(t) + K_I D^{-\lambda} e(t) + K_D D^\mu e(t). \quad (2.83)$$

The transfer function of such a controller (2.83) has the form:

$$G_u(s) = K_P + K_I s^{-\lambda} + K_D s^\mu. \quad (2.84)$$

## 2.4. CONCLUSIONS

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By substituting the fractional order  $PI^\lambda D^\mu$  controller  $u(t)$  in (2.83) to (2.82), the closed-loop the transfer function can be obtained:

$$G_{closed}(s) = \frac{K_P s^\lambda + K_I + K_D s^{\lambda+\mu}}{K_P s^\lambda + K_I + K_D s^{\lambda+\mu} + a_n s^{\beta_n+\lambda} + a_{n-1} s^{\beta_{n-1}+\lambda} + \dots + a_0 s^{\beta_0+\lambda}}. \quad (2.85)$$

Since the fractional order  $PI^\lambda D^\mu$  controller have two more design parameters  $\lambda$  and  $\mu$  than the classical PID controllers, it provides more flexibility and more possibilities for the design. In particular, taking  $\lambda = 1$  and  $\mu = 1$  in (2.83), the classical PID controller is obtained. Moreover, the parameter tuning of these controllers can be found in [91]–[93].

## 2.4 Conclusions

In this chapter, some basics of fractional calculus are introduced. Notably, fractional integration by parts formulas are summarized, which provide necessary mathematical tools for the following text. On this basis, fractional order linear systems with Riemann-Liouville derivatives are introduced. Specifically, based on fractional order linear time systems, two representations of the system model, namely the input-output differential model and the state space representation, were presented. Then, system performances were illustrated, such as stability analysis, controllability, and observability criteria. Moreover, fractional order  $PI^\lambda D^\mu$  controllers are briefly reviewed.

## 2.4. CONCLUSIONS

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## Chapter 3

# Modulating functions method

### 3.1 Introduction

In many practical engineering situations, there always exist some non-measurable variables and parameters for cost and technological reasons. Meanwhile, the measured variables and parameters are usually corrupted with noises. In order to overcome these problems, it is necessary to develop robust estimation methods [68], [94]–[98]. Moreover, sometimes non-asymptotic estimation with fast convergence in finite-time is required in on-line applications [99]–[103].

The modulating functions method is a type of time domain method. This method was originally introduced by Shinbort in 1954 [67] for linear identification and has been widely used for linear and nonlinear identification of continuous-time systems [104]–[107]. The fundamental steps of modulating functions method can be outlined as follows:

- consider (or construct) a linear differential equation;
- multiply designed modulating functions and integrate over a time interval;
- apply integration by parts to remove the derivatives to the modulating functions;
- eliminate undesired terms and keep desired terms;
- solve a linear system to obtained algebraic integral formulas;
- take the noisy signal  $y^\omega = y + \omega$ .

The benefits of the modulating functions method are listed as follows:

- exact algebraic formulation using integral formulas of the observation signal;
- no need of knowing initial conditions;



- without calculating the derivatives of noisy signals;
- non-asymptotic and fast;
- good robustness with respect to corrupting noises without the need of knowing in prior their statistical properties [108].

In this chapter, some of the existing works of the modulating functions for linear identification will be recalled. Drawing inspiration from these established works, the main results of this thesis will be presented.

## 3.2 Definition of modulating functions

In this section, the existing results on modulating functions based parameter identification for integer order and fractional order are recalled. At first, the definition of the classical modulating functions is introduced.

**Definition 3.1** [104] *Let  $[a, b] \subset \mathbb{R}$ ,  $L \in \mathbb{N}^*$ ,  $n \in \mathbb{N}$  with  $n \leq L - 1$ ,  $g$  and  $g_n$  be two functions defined on  $[a, b]$  between which  $g_n$  depends on  $n$ .*

- *If  $g$  satisfies the properties:  $\forall k \in \{0, 1, \dots, L - 1\}$ ,*

$$(P_1) : g \in \mathcal{C}^L([a, b]),$$

$$(P_2) : g^{(k)}(a) = 0,$$

$$(P_3) : g^{(k)}(b) = 0.$$

*Then,  $g$  is called a  $L^{\text{th}}$  order (classical) modulating function on  $[a, b]$ .*

- *If  $g$  only satisfies  $(P_1) - (P_2)$  (resp.  $(P_1), (P_3)$ ), then  $g$  is called left-sided (resp. right-sided)  $L^{\text{th}}$  order (generalized) modulating function on  $[a, b]$ .*
- *If  $g_n$  only satisfies  $(P_1) - (P_2)$  and the following property:  $\forall k \in \{0, 1, \dots, L - 1\}$ ,*

$$(P_4) : g_n^{(k)}(b) = 1 \text{ if } k = n, g_n^{(k)}(b) = 0 \text{ else.}$$

*Then,  $g_n$  is called left-sided  $(L, n)^{\text{th}}$  order (generalized) modulating function on  $[a, b]$ .*

- *If  $g_n$  only satisfies  $(P_1), (P_3)$  and the following property:  $\forall k \in \{0, 1, \dots, L - 1\}$ ,*

$$(P_5) : g_n^{(k)}(a) = 1 \text{ if } k = n, g_n^{(k)}(a) = 0 \text{ else.}$$

*Then,  $g_n$  is called right-sided  $(n, L)^{\text{th}}$  order (generalized) modulating function on  $[a, b]$ .*

### 3.3 Parameter identification

In this section, the parameter identification based on modulating functions method for integer order and fractional order systems are recalled.

#### 3.3.1 Parameter identification for sinusoidal signals

Let us consider the simplest sinusoidal signal as follows:  $\forall t \in I = [0, T]$  with  $T \in \mathbb{R}_+$ ,

$$y = \sin(\omega t), \quad (3.1)$$

with  $\omega \in \mathbb{R}_+$ .  $y$  is the solution of the following harmonic oscillator equation:

$$\ddot{y}(t) + \omega^2 y(t) = 0. \quad (3.2)$$

Then, the identification process of parameter  $\omega$  is provided in following steps.

**Step 1.** By multiplying a  $2^{th}$  order modulating function  $g$  defined on  $[0, t]$  to (3.2) and integrating on  $[0, t]$ , we obtain:

$$\int_0^t g(\tau) \ddot{y}(\tau) d\tau + \omega^2 \int_0^t g(\tau) y(\tau) d\tau = 0. \quad (3.3)$$

**Step 2.** By applying the integration by parts formula given in Lemma 2.4, we have:

$$[g(\tau) \dot{y}(\tau)]_{\tau=0}^{\tau=t} - [\dot{g}(\tau) y(\tau)]_{\tau=0}^{\tau=t} + \int_0^t \ddot{g}(\tau) y(\tau) d\tau + \omega^2 \int_0^t g(\tau) y(\tau) d\tau = 0. \quad (3.4)$$

Using the properties  $(P_2)$  and  $(P_3)$  of  $g$ , the boundary values can be eliminated. Then, we get:

$$\int_0^t \ddot{g}(\tau) y(\tau) d\tau + \omega^2 \int_0^t g(\tau) y(\tau) d\tau = 0. \quad (3.5)$$

**Step 3.** By solving (3.5),  $\omega$  can be given by the following algebraic integral formula:

$$\omega = \left( \frac{\int_0^t \ddot{g}(\tau) y(\tau) d\tau}{\int_0^t g(\tau) y(\tau) d\tau} \right)^{\frac{1}{2}}, \quad (3.6)$$

where  $\ddot{g}$  can be computed from the known function  $g$ .

Next, let us consider a more general sinusoidal signal as follows [107]:  $\forall t \in I = [0, T]$  with  $T \in \mathbb{R}_+$ ,

$$y(t) = (A_0 + A_1 t) \sin(\omega t + \phi), \quad (3.7)$$

where  $A_0, A_1 \in \mathbb{R}^*$ ,  $\omega \in \mathbb{R}_+$  and  $\phi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ .  $y$  is the solution of the following harmonic oscillator equation:

$$y^{(4)}(t) + 2\omega^2 \ddot{y}(t) + \omega^4 y(t) = 0. \quad (3.8)$$

### 3.3. PARAMETER IDENTIFICATION

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Then, the identification process of parameter  $\omega$  is provided in following steps.

**Step 1.** By multiplying a 4<sup>th</sup> order modulating function  $\hat{g}$  on  $[0, t]$  to (3.8) and integrating, we get:

$$\int_0^t \hat{g}(\tau) y^{(4)}(\tau) d\tau + 2\omega^2 \int_0^t \hat{g}(\tau) \ddot{y}(\tau) d\tau + \int_0^t \hat{g}(\tau) \omega^4 y(\tau) d\tau = 0. \quad (3.9)$$

**Step 2.** By applying the integration by parts formula to (3.9), we obtain the following equation:

$$\int_0^t \hat{g}^{(4)}(\tau) y(\tau) d\tau + 2\omega^2 \int_0^t \ddot{\hat{g}}(\tau) y(\tau) d\tau + \omega^4 \int_0^t \hat{g}(\tau) y(\tau) d\tau = 0. \quad (3.10)$$

**Step 3.** By solving (3.10), the algebraic integral formula of  $\omega$  can be give by:

$$\omega = \left( \frac{-B_y + \sqrt{B_y^2 - 4A_y C_y}}{2A_y} \right)^{\frac{1}{2}}, \quad (3.11)$$

where  $A_y = \int_0^t \hat{g}(\tau) y(\tau) d\tau$ ,  $B_y = \int_0^t \ddot{\hat{g}}(\tau) y(\tau) d\tau$  and  $C_y = \int_0^t \hat{g}^{(4)}(\tau) y(\tau) d\tau$ .

**Remark 3.1** *In the noise case, duo to (3.6) and (3.11) are algebraic, the non-asymptotic estimation of  $\omega$  can be achieved. Moreover, the integrals work as low-pass filters, which can reduce the noise effect of the noisy measurement of  $y$  in noisy environments. Therefore, the robust estimation of  $\omega$  is realized by (3.6) and (3.11). Particularly, in the continuous-time noise-free case, the algebraic integral formula (3.6) and (3.11) are exact with no error included.*

#### 3.3.2 Parameter identification for fractional order linear systems

In this subsection, parameter identification for fractional order linear systems is discussed, which extends the result for integer-order linear systems in Subsection 3.3.1. Firstly, the following fractional order integration by parts formula is recalled.

**Lemma 3.1** [75] *Let  $t, \alpha \in \mathbb{R}_+$ ,  $g$  be a  $l^{\text{th}}$  order modulating function on  $[0, t]$  with  $l - 1 \leq \alpha \leq l$ . Then, the following formula holds:*

$$\int_0^t g(t - \tau) D_\tau^\alpha y(\tau) d\tau = \int_0^t D_\tau^\alpha g(\tau) y(t - \tau) d\tau, \quad (3.12)$$

where  $t \in \mathbb{R}_+$ .

Furthermore, as an application of Lemma 2.3, the following properties holds for the  $l^{\text{th}}$  order modulation function  $g$ :

(P<sub>6</sub>) :  $\forall 0 \leq \beta < l$ ,  $D_t^\beta g(t)$  exists,

(P<sub>7</sub>) :  $\forall 0 \leq \beta < l$ ,  $[D_t^\beta g(t)]_{t=0} = 0$ .

### 3.3. PARAMETER IDENTIFICATION

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With the above preparation, let us consider the following fractional order linear system model [75]:  
 $\forall t \in I = [0, h] \subset \mathbb{R}_+$ ,

$$D^{2\alpha}y(t) + a_1 D^\alpha y(t) = u(t), \quad (3.13)$$

where  $y$  is the output and  $u$  is the input,  $\alpha \in \mathbb{R}_+$ .

The identification process for the coefficient  $a_1$  from the noisy  $y$  is given by the following steps.

**Step 1.** Let  $g$  is  $l^{th}$  order modulating function on  $[0, t]$ . By multiplying  $g$  to both sides of (3.13), and integrating from 0 to  $t$ , we have:

$$\int_0^t g(t-\tau) D_\tau^{2\alpha} y(\tau) d\tau + a_1 \int_0^t g(t-\tau) D_\tau^\alpha y(\tau) d\tau = \int_0^t g(t-\tau) u(\tau) d\tau. \quad (3.14)$$

**Step 2.** By applying the Lemma 3.1 to (3.14), we get:

$$\int_0^t D_\tau^{2\alpha} g(\tau) y(t-\tau) d\tau + a_1 \int_0^t D_\tau^\alpha g(\tau) y(t-\tau) d\tau = \int_0^t g(\tau) u(t-\tau) d\tau. \quad (3.15)$$

**Step 3.** Based on (3.15), the parameter  $a_1$  can be calculated by the following algebraic integral formula:

$$a_1 = \frac{\int_0^t (g(\tau) u(t-\tau) - D_\tau^{2\alpha} g(\tau) y(t-\tau)) d\tau}{\int_0^t D_\tau^\alpha g(\tau) y(t-\tau) d\tau}. \quad (3.16)$$

Moreover, if  $y$  is replaced by its noisy observation  $y^\varpi = y + \varpi$  in discrete signal case, the estimation of  $a_1$  is non-asymptotic and robust to corrupting noises.

Moving on, let us consider a more general fractional order linear system model as follows:  $\forall t \in I = [0, h] \subset \mathbb{R}_+$ ,

$$\sum_{i=0}^L a_i D^{\alpha_i} y(t) = \sum_{j=0}^M b_j D^{\beta_j} u(t), \quad (3.17)$$

where  $y$  is the output,  $u$  is the input,  $L, M \in \mathbb{N}$ ,  $b_0 = 1$ ,  $a_i, b_j \in \mathbb{R}^*$  for  $i = 0, \dots, L$  and  $j = 1, \dots, M$  are unknown parameters to be identified, and  $\alpha_i, \beta_j \in \mathbb{R}_+$  are assumed  $0 \leq \alpha_0 < \alpha_1 < \dots < \alpha_L$ ,  $0 \leq \beta_0 < \beta_1 < \dots < \beta_M$ .

The identification process for the system (3.17) is given by the following steps.

**Step 1:** Let  $g_n$  for  $n = 1, \dots, N$  be a set of  $l^{th}$  order modulating functions on  $[0, t]$ , where  $L + M + 1 \leq N \in \mathbb{N}^*$ ,  $\alpha = \max\{\alpha_L, \beta_M\}$  and  $l = \lceil \alpha \rceil$ . By multiplying  $g_n$  to (3.17) and integrating from 0 to  $t$ , we get: for  $n = 1, \dots, N$ ,

$$\sum_{i=0}^L a_i \int_0^t g_n(t-\tau) D_\tau^{\alpha_i} y(\tau) d\tau = \sum_{j=0}^M b_j \int_0^t g_n(t-\tau) D_\tau^{\beta_j} u(\tau) d\tau. \quad (3.18)$$

**Step 2.** By applying the Lemma 3.1 to (3.18), we get: for  $n = 1, \dots, N$ ,

$$\sum_{i=0}^L a_i \int_0^t D_\tau^{\alpha_i} g_n(\tau) y(t-\tau) d\tau = \sum_{j=0}^M b_j \int_0^t D_\tau^{\beta_j} g_n(\tau) u(t-\tau) d\tau. \quad (3.19)$$

### 3.3. PARAMETER IDENTIFICATION

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Due to  $b_0 = 1$ , (3.19) is equivalent to: for  $n = 1, \dots, N$ ,

$$\sum_{i=0}^L a_i Y_N(n, i+1) + \sum_{j=1}^M b_j U_N(n, j) = \int_0^t D_\tau^{\beta_0} g_n(\tau) u(t-\tau) d\tau, \quad (3.20)$$

where  $Y_N(n, i+1) = \int_0^t D_\tau^{\alpha_i} g_n(\tau) y(t-\tau) d\tau$  and  $U_N(n, j) = -\int_0^t D_\tau^{\beta_j} g_n(\tau) u(t-\tau) d\tau$ .

**Step 3.** By denoting the coefficients vectors  $A = (a_0, \dots, a_L)$  and  $B = (b_1, \dots, b_M)$ , the unknown parameters can be obtained by solving the following system:

$$(Y_N, U_N) \begin{pmatrix} A \\ B \end{pmatrix} = \mathbf{I}_N, \quad (3.21)$$

where  $\mathbf{I}_N(n) = \int_0^t D_\tau^{\beta_0} g_n(\tau) u(t-\tau) d\tau$  for  $n = 1, \dots, N$ .

In order to illustrate the accuracy and robustness with respect to corrupting noises of the proposed estimators, we present some numerical results in the following example.

**Example 3.1** [75] *Let us consider the following fractional order linear system:  $\forall t \in [0, 8]$ ,*

$$a_0 D_t^{\alpha_0} y(t) + a_1 D_t^{\alpha_1} y(t) + a_2 D_t^{\alpha_2} y(t) = u(t), \quad (3.22)$$

with  $a_0 = 3$ ,  $a_1 = 2$ ,  $a_2 = 1$ ,  $\alpha_0 = 0$ ,  $\alpha_1 = 0.5$  and  $\alpha_2 = 1.5$ . The output is set as  $y(t) = \sin(3t) + 1$ . Hence, the initial conditions of  $y$  are not equal to 0. Moreover, the expression of the input can be obtained using (2.20) and (2.21). In our identification procedure, the following modulating functions are adopted:

$$g_n(\tau) = (t-\tau)^{6+n} \tau^{6+N+1-n}, \quad (3.23)$$

where  $n = 1, 2, \dots, N$  with  $N = 13$ . Moreover, the trapezoidal numerical integration method is employed to approximate integrals. We assume that  $y^\varpi(t_j) = y(t_j) + 0.5 \sin(10^3 t_j)$  with  $T_s = 0.01$ . The discrete noisy signal is given in Fig. 3.1. By taking  $t_j = jT_s$ , the estimations and relative estimation errors on  $[1.8, 8]$  are given in Fig. 3.2 (a) and Fig. 3.2 (b), respectively. Consequently, the efficiency and robustness of the proposed method are verified.

### 3.3. PARAMETER IDENTIFICATION

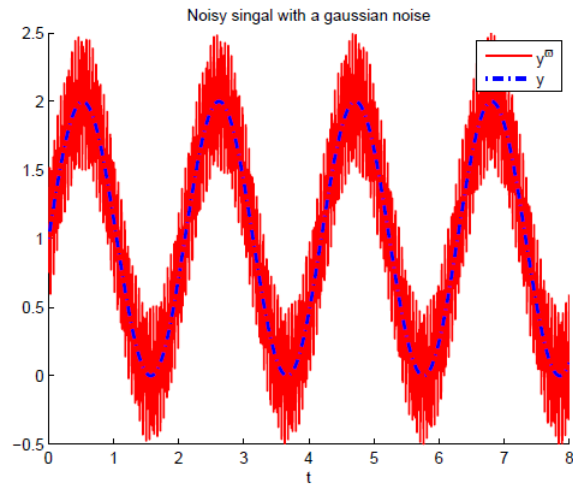


Figure 3.1: Output and its discrete noisy observation.

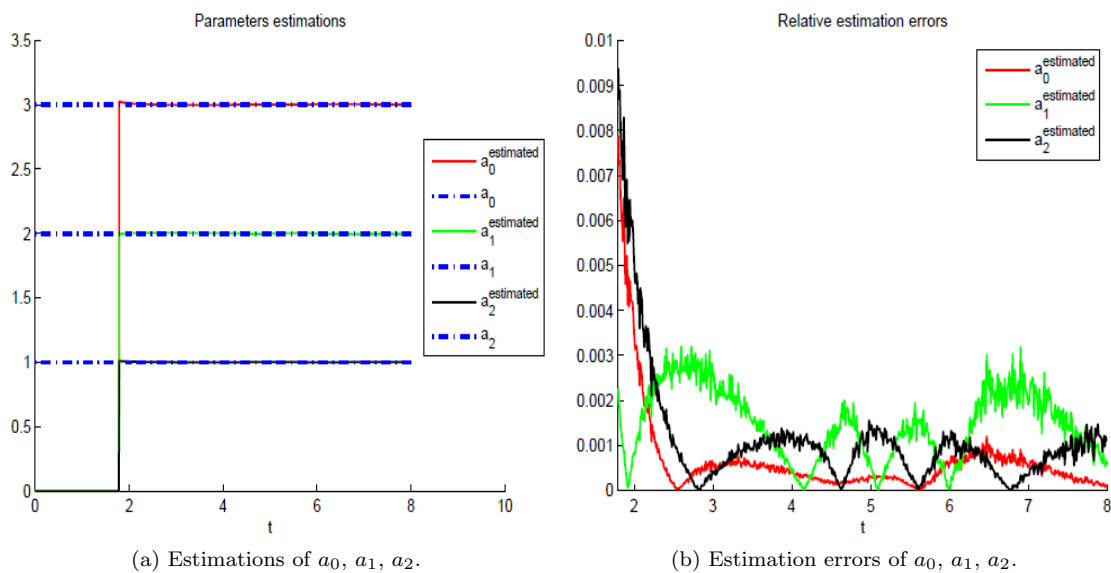


Figure 3.2: Estimation results of  $a_0, a_1, a_2$  (cited from [75]).

### 3.4 Modulating functions based differentiators

In this section, a generalized modulating functions method will be introduced to estimate the output's derivatives of integer order linear systems defined by a differential equation. Then, it will be extended to estimate non-integer derivatives for integer and non-integer systems.

#### 3.4.1 Integer order differentiators for integer order systems

Let us consider the following integer order linear system model on  $\forall t \in I = [0, h] \subset \mathbb{R}_+$ :

$$a_3 y^{(3)}(t) + a_2 \ddot{y}(t) + a_1 \dot{y}(t) + a_0 y(t) = u(t), \quad (3.24)$$

where  $y$  is the output,  $u$  is the input,  $a_3 = 1$  and  $a_i$  with  $i = 0, 1, 2$  are constant parameters.

The estimation process of  $y^{(i)}$  for  $i = 0, 1, 2$  is provided in the following steps.

**Step 1.** Let  $g_n \in \mathcal{C}^3(I)$  for  $n = 0, 1, 2$ , by multiplying  $g_n$  on the both side of (3.24) and integrating from  $t - T$  to  $t$ , we get:

$$\sum_{i=0}^3 a_i \int_{t-T}^t g_n(\tau) y^{(i)}(\tau) d\tau = \int_{t-T}^t g_n(\tau) u(\tau) d\tau. \quad (3.25)$$

**Step 2.** By applying the classic integration by parts formula given in Lemma 2.4 to (3.25), we get:

$$\sum_{i=0}^3 (-1)^i a_i \int_{t-T}^t g_n^{(i)}(\tau) y(\tau) d\tau = -S_n(t) + S_n(t-T) + \int_{t-T}^t g_n(\tau) u(\tau) d\tau, \quad (3.26)$$

where  $S_n(\tau) = \sum_{i=1}^3 \sum_{k=0}^{i-1} (-1)^k a_i g_n^{(k)}(\tau) y^{(i-1-k)}(\tau)$ .

**Step 3.** We assume  $g_n$  satisfies  $(P_2)$ , i.e.  $g_n^{(k)}(\tau)$  for  $k, n = 0, 1, 2$  at the left endpoint  $\tau = t - T$  equal 0, then  $S_n(t - T) = 0$  and (3.26) becomes:

$$\sum_{i=0}^3 (-1)^i a_i \int_{t-T}^t g_n^{(i)}(\tau) y(\tau) d\tau = -S_n(t) + \int_{t-T}^t g_n(\tau) u(\tau) d\tau, \quad (3.27)$$

the boundary values at  $t$  in the right side of (3.27) can be listed in the following matrix: for  $n = 0, 1, 2$ ,

$$\begin{pmatrix} a_1 g_n(t) y(t) & a_2 g_n(t) \dot{y}(t) & a_3 g_n(t) \ddot{y}(t) \\ 0 & -a_2 \dot{g}_n(t) y(t) & -a_3 \dot{g}_n(t) \dot{y}(t) \\ 0 & 0 & a_3 \ddot{g}_n(t) y(t) \end{pmatrix}. \quad (3.28)$$

By taking the following change of index:  $i \rightarrow i + 1 + k$ , we obtain:

$$S_n(t) = \sum_{i=0}^2 \sum_{k=0}^{2-i} (-1)^k a_{i+1+k} g_n^{(k)}(t) y^{(i)}(t). \quad (3.29)$$

Then, the matrix in (3.28) can be rewritten as follows:

$$\begin{pmatrix} a_1 g_n(t) y(t) & -a_2 \dot{g}_n(t) y(t) & a_3 \ddot{g}_n(t) y(t) \\ a_2 g_n(t) \dot{y}(t) & -a_3 \dot{g}_n(t) \dot{y}(t) & 0 \\ a_3 g_n(t) \ddot{y}(t) & 0 & 0 \end{pmatrix}. \quad (3.30)$$

Let us define the matrix  $S(t)$  as follows: for  $n, i = 0, 1, 2$ ,

$$S_{n,i}(t) = \sum_{k=0}^{2-i} (-1)^k a_{i+1+k} g_n^{(k)}(t). \quad (3.31)$$

Hence, we get:  $S(t) = G(t) \times M_1$  where

$$G(t) = \begin{pmatrix} g_0(t) & -\dot{g}_0(t) & \ddot{g}_0(t) \\ g_1(t) & -\dot{g}_1(t) & \ddot{g}_1(t) \\ g_2(t) & -\dot{g}_2(t) & \ddot{g}_2(t) \end{pmatrix}, \quad M_1 = \begin{pmatrix} a_1 & a_2 & 1 \\ a_2 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \quad (3.32)$$

Thus, (3.27) can be presented by the following matrix form:

$$G(t) M_1 \begin{pmatrix} y(t) \\ \dot{y}(t) \\ \ddot{y}(t) \end{pmatrix} = \begin{pmatrix} -\sum_{i=0}^3 (-1)^i a_i \int_{t-T}^t g_0^{(i)}(\tau) y(\tau) d\tau + \int_{t-T}^t g_0(\tau) u(\tau) d\tau \\ -\sum_{i=0}^3 (-1)^i a_i \int_{t-T}^t g_1^{(i)}(\tau) y(\tau) d\tau + \int_{t-T}^t g_1(\tau) u(\tau) d\tau \\ -\sum_{i=0}^3 (-1)^i a_i \int_{t-T}^t g_2^{(i)}(\tau) y(\tau) d\tau + \int_{t-T}^t g_2(\tau) u(\tau) d\tau \end{pmatrix}. \quad (3.33)$$

**Step 4.** Suppose that the matrix  $G(t)$  is invertible, we can obtain the derivative values  $y(t)$ ,  $\dot{y}(t)$  and  $\ddot{y}(t)$  by solving the linear system given in (3.33). Specifically speaking, there exists three cases matrix.

**Case 1.** If  $g_n$  for  $n = 0, 1, 2$  is a set of left-sided 3<sup>th</sup> order modulating functions and the matrix  $G(t)$  is a common invertible matrix, then the derivative values  $y(t)$ ,  $\dot{y}(t)$  and  $\ddot{y}(t)$  can be obtained by:

$$\begin{pmatrix} y(t) \\ \dot{y}(t) \\ \ddot{y}(t) \end{pmatrix} = M_1^{-1} G^{-1}(t) \begin{pmatrix} -\sum_{i=0}^3 (-1)^i a_i \int_{t-T}^t g_0^{(i)}(\tau) y(\tau) d\tau + \int_{t-T}^t g_0(\tau) u(\tau) d\tau \\ -\sum_{i=0}^3 (-1)^i a_i \int_{t-T}^t g_1^{(i)}(\tau) y(\tau) d\tau + \int_{t-T}^t g_1(\tau) u(\tau) d\tau \\ -\sum_{i=0}^3 (-1)^i a_i \int_{t-T}^t g_2^{(i)}(\tau) y(\tau) d\tau + \int_{t-T}^t g_2(\tau) u(\tau) d\tau \end{pmatrix}, \quad (3.34)$$

$$\text{where } M_1^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -a_2 \\ 0 & -a_2 & a_2^2 - a_1 \end{pmatrix}.$$



### 3.4. MODULATING FUNCTIONS BASED DIFFERENTIATORS

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**Case 2.** If  $g_n$  for  $n = 0, 1, 2$  is a set of left-sided  $(3, n)^{th}$  order modulating functions with property

$(P_4)$ : for  $k = 0, 1, 2, \forall t \in [t - T, t], g_n^{(k)}(t) = \delta_{k,n}$ , i.e.  $G(t) = G^{-1}(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ , then using

arecursive computation scheme, the derivative values  $y(t), \dot{y}(t)$  and  $\ddot{y}(t)$  can be obtained by:

$$\begin{pmatrix} y(t) \\ \dot{y}(t) \\ \ddot{y}(t) \end{pmatrix} = \begin{pmatrix} -\sum_{i=0}^3 (-1)^i a_i \int_{t-T}^t g_2^{(i)}(\tau) y(\tau) d\tau + \int_{t-T}^t g_2(\tau) u(\tau) d\tau \\ \sum_{i=0}^3 (-1)^i a_i \int_{t-T}^t g_1^{(i)}(\tau) y(\tau) d\tau - \int_{t-T}^t g_1(\tau) u(\tau) d\tau - a_2 y(t) \\ -\sum_{i=0}^3 (-1)^i a_i \int_{t-T}^t g_0^{(i)}(\tau) y(\tau) d\tau + \int_{t-T}^t g_0(\tau) u(\tau) d\tau - a_1 y(t) - a_2 \dot{y}(t) \end{pmatrix}. \quad (3.35)$$

**Case 3.** If  $g_n$  for  $n = 0, 1, 2$  is a set of left-sided  $3^{th}$  order modulating functions and the matrix  $G(t) = M_1^{-1}$ , i.e.  $M_1^{-1} G^{-1}(t) = I$  then the derivative values  $y(t), \dot{y}(t)$  and  $\ddot{y}(t)$  can be obtained by:

$$\begin{pmatrix} y(t) \\ \dot{y}(t) \\ \ddot{y}(t) \end{pmatrix} = \begin{pmatrix} -\sum_{i=0}^3 (-1)^i a_i \int_{t-T}^t g_0^{(i)}(\tau) y(\tau) d\tau + \int_{t-T}^t g_0(\tau) u(\tau) d\tau \\ -\sum_{i=0}^3 (-1)^i a_i \int_{t-T}^t g_1^{(i)}(\tau) y(\tau) d\tau + \int_{t-T}^t g_1(\tau) u(\tau) d\tau \\ -\sum_{i=0}^3 (-1)^i a_i \int_{t-T}^t g_2^{(i)}(\tau) y(\tau) d\tau + \int_{t-T}^t g_2(\tau) u(\tau) d\tau \end{pmatrix}. \quad (3.36)$$

Next, let us consider a more general system model for (3.24) as follows:  $\forall t \in I = [0, h] \subset \mathbb{R}_+$ ,

$$\sum_{i=0}^M a_i y^{(i)}(t) = u(t), \quad (3.37)$$

where  $a_i$  for  $i = 0, \dots, M \in \mathbb{N}$  are known constants on  $I$  with  $a_M \neq 0$ ,  $y$  is the output which is sufficiently smooth enough, and  $u \in C(I)$  is the known input.

Here, we take the second case shown above as an analysis. Then, the estimation process of  $y^{(i)}$  for  $i = 0, \dots, M - 1$  is provided in the following steps.

**Step 1.** Let  $g_n$  for  $n = 0, \dots, M - 1$  be a set of left-sided  $(M, n)^{th}$  order modulating functions on  $[t - T, t]$ , by multiplying  $g_n$  on both side of (3.37) and integrating from  $t - T$  to  $t$ , we obtain:

$$\sum_{i=0}^M a_i \int_{t-T}^t g_n(\tau) y^{(i)}(\tau) d\tau = \int_{t-T}^t g_n(\tau) u(\tau) d\tau. \quad (3.38)$$

**Step 2.** By applying the classic integration by parts formula given in Lemma 2.4 to the left side of

(3.38), it follows that:

$$\begin{aligned} \sum_{i=0}^M a_i \int_{t-T}^t g_n(\tau) y^{(i)}(\tau) d\tau &= \sum_{i=0}^M (-1)^i a_i \int_{t-T}^t g_n^{(i)}(\tau) y(\tau) d\tau \\ &+ \sum_{i=1}^M \sum_{k=0}^{i-1} (-1)^k a_i \left[ g_n^{(k)}(\tau) y^{(i-1-k)}(\tau) \right]_{\tau=t-T}^{\tau=t}. \end{aligned} \quad (3.39)$$

Then, (3.38) can be rewritten as follows:

$$\begin{aligned} \sum_{i=1}^M \sum_{k=0}^{i-1} (-1)^k a_i \left[ g_n^{(k)}(\tau) y^{(i-1-k)}(\tau) \right]_{\tau=t-T}^{\tau=t} &= - \sum_{i=0}^M (-1)^i a_i \int_{t-T}^t g_n^{(i)}(\tau) y(\tau) d\tau \\ &+ \int_{t-T}^t g_n(\tau) u(\tau) d\tau. \end{aligned} \quad (3.40)$$

**Step 3.** Thanks to the properties ( $P_2$ ) of  $g_n$ , we get:

$$\sum_{i=1}^M \sum_{k=0}^{i-1} (-1)^k a_i g_n^{(k)}(t) y^{(i-1-k)}(t) = - \sum_{i=0}^M (-1)^i a_i \int_{t-T}^t g_n^{(i)}(\tau) y(\tau) d\tau + \int_{t-T}^t g_n(\tau) u(\tau) d\tau. \quad (3.41)$$

The boundary values at  $\tau = t$  of the left side of (3.41) can be given in the following matrix:

$$\begin{pmatrix} a_1 g_n(t) y(t) & \cdots & a_M g_n(t) y^{(M-1)}(t) \\ 0 & \cdots & -a_M g_n^{(1)}(t) y^{(M-2)}(t) \\ \vdots & \ddots & \vdots \\ 0 & \cdots & (-1)^{M-1} a_M g_n^{(M-1)}(t) y(t) \end{pmatrix}. \quad (3.42)$$

Since  $g_n$  for  $n = 0, \dots, M-1$  be a set of left-sided  $(M, n)^{th}$  order modulating functions with the property ( $P_4$ ), the we can obtain one equation (3.41) with  $M$  unknown quantities  $y^{(i)}(t)$  for  $i = 0, \dots, M-1$ . Consequently, the following recursive formulas of the differentiator can be obtained:

- for  $n = M-1$ , we have:

$$y(t) = \frac{(-1)^{M-1}}{a_M} \left( \int_{t-T}^t g_{M-1}(\tau) u(\tau) d\tau - \sum_{i=0}^M (-1)^i a_i \int_{t-T}^t g_{M-1}^{(i)}(\tau) y(\tau) d\tau \right), \quad (3.43)$$

- for  $n = M-2, \dots, 0$ , we have:

$$\begin{aligned} y^{(M-1-n)}(t) &= \frac{(-1)^n}{a_M} \left( \int_{t-T}^t g_n(\tau) u(\tau) d\tau - \sum_{i=0}^M (-1)^i a_i \int_{t-T}^t g_n^{(i)}(\tau) y(\tau) d\tau \right) \\ &- \frac{1}{a_M} \sum_{i=n+1}^{M-1} a_i y^{(i-1-n)}(t). \end{aligned} \quad (3.44)$$

Remark that if  $g_n$  for  $n = 0, \dots, M - 1$  be a set of left-sided  $M^{th}$  order modulating functions on  $[t - T, t]$  satisfying the following property: for  $i = 0, \dots, N - 1$ ,

$$(P_8) : \sum_{k=0}^{M-1-i} (-1)^k a_{i+1+k} g_n^{(k)}(t) = \delta_{i,n}.$$

Then, the derivatives of the output can be independently given as follows: for  $n = 0, \dots, M - 1$ ,

$$y^{(n)}(t) = \int_{t-T}^t g_n(\tau) u(\tau) d\tau - \sum_{i=0}^M (-1)^i a_i \int_{t-T}^t g_n^{(i)}(\tau) y(\tau) d\tau. \quad (3.45)$$

### 3.4.2 Fractional order differentiators for integer order systems

In this subsection, let us consider the following general system model directly, which is the same as the system (3.37):  $\forall t \in I = [0, h] \subset \mathbb{R}_+$ ,

$$\sum_{i=0}^M a_i y^{(i)}(t) = u(t), \quad (3.46)$$

where  $a_M \neq 0$ ,  $y$  is the output,  $u$  is the input. The parameters  $a_i$  and the input  $u$  are known.

The estimation process of  $D_t^\alpha y$  with  $\alpha > -1$  is provided in the following steps.

**Step 1.** By applying the operator  $D^{-\beta}$  with  $\beta = \lceil \alpha \rceil - \alpha$  (thus  $0 \leq \beta < 1$ ) to the both side of (3.46) and utilizing the additive index law given in Lemma 2.1, we can obtain:

$$\sum_{i=0}^M a_i D_t^{i-\beta} y(t) - \sum_{i=1}^M a_i \sum_{k=0}^{i-1} \frac{t^{k-i+\beta}}{\Gamma(k+1-i+\beta)} y^{(k)}(0) = D_t^{-\beta} u(t). \quad (3.47)$$

**Step 2.** Let  $g_n$  for  $n = 0, \dots, M - 1$  be a set of left-sided  $(M, n)^{th}$  order modulating function on  $[0, t]$  and satisfying the following property:

$$(P_9) : \int_0^t \tau^\kappa g(\tau) d\tau = 0, \forall \kappa \in \{\beta - i \mid i = 1, \dots, M\}.$$

Then, by multiplying  $g_n$  on the both side of (3.47) and integrating from 0 to  $t$ , we get: for  $n = 0, \dots, M - 1$ ,

$$\begin{aligned} \sum_{i=0}^M a_i \int_0^t g_n(\tau) D_\tau^{i-\beta} y(\tau) d\tau &= \int_0^t g_n(\tau) D_\tau^{-\beta} u(\tau) d\tau \\ &+ \sum_{i=1}^M a_i \sum_{k=0}^{i-1} \frac{y^{(k)}(0)}{\Gamma(k+1-i+\beta)} \int_0^t g_n(\tau) \tau^{k-i+\beta} d\tau. \end{aligned} \quad (3.48)$$

Due to the property  $(P_9)$  of  $g_n$ , the last term in (3.48) is eliminated. Hence, (3.48) becomes:

$$\sum_{i=0}^M a_i \int_0^t g_n(\tau) D_\tau^{i-\beta} y(\tau) d\tau = \int_0^t g_n(\tau) D_\tau^{-\beta} u(\tau) d\tau. \quad (3.49)$$

**Step 3.** Using the fractional integration by parts formulas given in Lemma 2.5 to (3.49), we get:

$$\sum_{i=1}^M a_i \sum_{k=0}^{i-1} (-1)^k \left[ g_n^{(k)}(\tau) D_{\tau}^{i-\beta-1-k} y(\tau) \right]_{\tau=0}^{\tau=t} = \int_0^t {}^C D_{\tau,t}^{-\beta} g_n(\tau) u(\tau) d\tau - \sum_{i=0}^M a_i \int_0^t {}^C D_{\tau,t}^{i-\beta} g_n(\tau) y(\tau) d\tau. \quad (3.50)$$

Due to the properties  $(P_2)$  and  $(P_4)$  of  $g_n$ , (3.50) becomes: for  $n = 0, \dots, M-1$ ,

$$(-1)^n g_n^{(n)}(t) \sum_{i=n+1}^M a_i D_t^{i-\beta-1-n} y(\tau) = \int_0^t {}^C D_{\tau,t}^{-\beta} g_n(\tau) u(\tau) d\tau - \sum_{i=0}^M a_i \int_0^t {}^C D_{\tau,t}^{i-\beta} g_n(\tau) y(\tau) d\tau. \quad (3.51)$$

Consequently, the following recursive formulas of the differentiators can be obtained:

- for  $n = M-1$ , we have:

$$D_t^{-\beta} y(t) = \frac{(-1)^{M-1}}{a_M} \left( \int_0^t D_{\tau,t}^{-\beta} g_{M-1}(\tau) u(\tau) d\tau - \sum_{i=0}^M (-1)^i a_i \int_0^t {}^C D_{\tau,t}^{i-\beta} g_{M-1}(\tau) y(\tau) d\tau \right), \quad (3.52)$$

- for  $n = M-2, \dots, 0$ , we have:

$$\begin{aligned} D_t^{M-1-n-\beta} y(t) &= \frac{(-1)^n}{a_M} \left( \int_0^t D_{\tau,t}^{-\beta} g_n(\tau) u(\tau) d\tau - \sum_{i=0}^M (-1)^i a_i \int_0^t {}^C D_{\tau,t}^{i-\beta} g_n(\tau) y(\tau) d\tau \right) \\ &\quad - \frac{1}{a_M} \sum_{i=n+1}^{M-1} a_i D_{\tau}^{i-1-n-\beta} y(t). \end{aligned} \quad (3.53)$$

**Example 3.2** [109] In this example, the following sinusoidal signal is considered:  $\forall t \in I \subset \mathbb{R}$ ,

$$y(t) = A \sin(\omega t + \phi), \quad (3.54)$$

with  $A = 3$ ,  $\omega = 5$ , and  $\phi = 0.5$ . Then  $y$  satisfies the following harmonic oscillator equation:  $\forall t \in [0, 10]$ ,

$$y^{(3)}(t) + \omega^2 y^{(1)}(t) = 0. \quad (3.55)$$

The fractional derivatives of  $y$  can be given using the expansion of  $y$  and the linearity of the fractional derivatives [51]:

$$D_t^\alpha y(t) = A \cos \phi D_t^\alpha \sin(\omega t) + A \sin \phi D_t^\alpha \cos(\omega t),$$

where  $D_t^\alpha \sin(\omega t)$  and  $D_t^\alpha \cos(\omega t)$  can be computed by (2.20)-(2.21).

Assumed that  $y^\varpi(t_i) = y(t_i) + \varpi(t_i)$  is a discrete noisy observation and  $T_s = 0.001$ , where the noise  $\{\varpi(t_i)\}$  is simulated from a zero-mean white Gaussian iid sequence, and the variance is adjusted such that  $SNR = 15$  dB. The discrete signal  $y$  and its noisy observation  $y^\varpi$  are plotted in Fig. 3.3. The fractional derivatives  $D_{t_i}^{0.5} y(t_i)$  and  $D_{t_i}^{1.5} y(t_i)$  will be estimated on  $t_i \in [0.5, 10]$ , where the used modulating functions are required to fulfill the property  $(P_9)$ . The trapezoidal numerical integration method is applied to approximate the integrals with the noisy observation  $y^\varpi$ . Finally, the obtained estimators are given in Fig. 3.4(a) and Fig. 3.4(b).

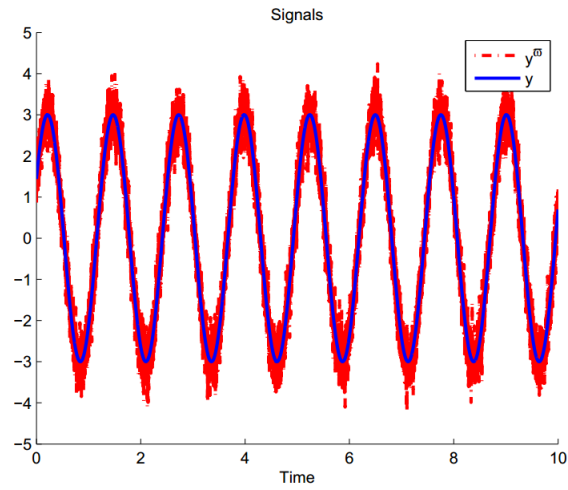
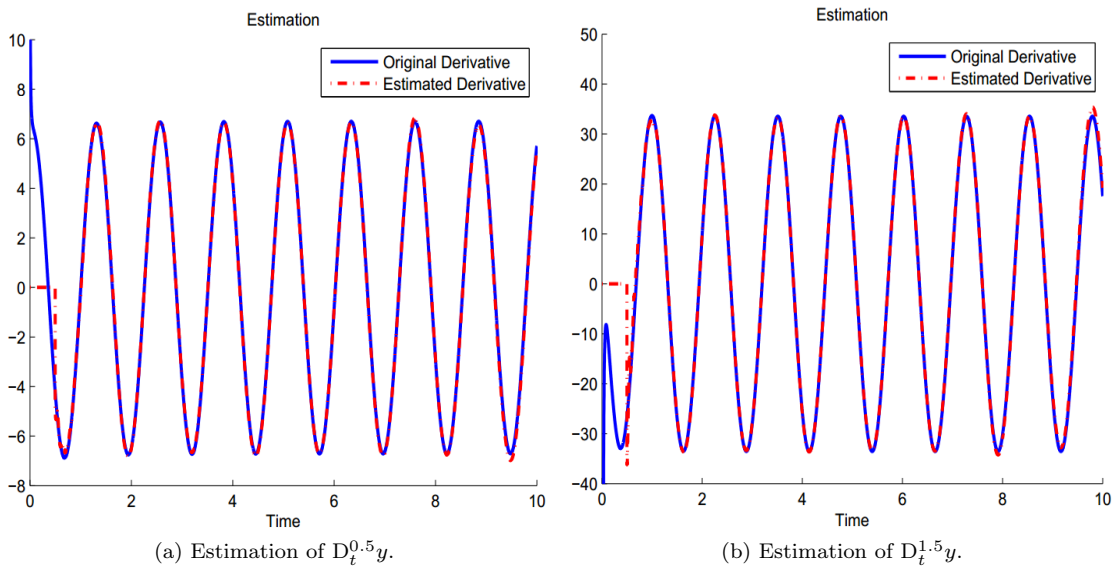


Figure 3.3: Noisy sinusoidal signal.



(a) Estimation of  $D_t^{0.5}y$ .

(b) Estimation of  $D_t^{1.5}y$ .

Figure 3.4: Estimation results (cited from [109]).

### 3.4.3 Fractional order differentiators for fractional order systems

let us consider the following fractional order linear system:  $\forall t \in I = [0, h] \subset \mathbb{R}_+$ ,

$$\sum_{i=0}^M a_i D_t^{\frac{i}{q}} y(t) = u(t), \quad (3.56)$$

where  $q, M \in \mathbb{N}^*$ ,  $a_i$  for  $i = 0, \dots, M-1$  are known constant parameters,  $a_M = 1$ ,  $y$  is the output, and  $u$  is the input.

The estimation process of  $D_t^\beta y$  for  $\beta \in (-1, \frac{M}{q} - 1]$  can be provided in the following step.

**Step 1.** By taking  $D_t^{-\frac{j}{q}-\alpha}$  with  $\alpha \in [0, \frac{1}{q})$  on the both side of (3.56) and applying the additive index law given in Lemma 2.1, we get:

$$\sum_{i=0}^M a_i D_t^{\frac{i-j}{q}-\alpha} y(t) = D_t^{-\frac{j}{q}-\alpha} u(t) + \sum_{i=1}^M a_i \sum_{k=1}^{\lceil \frac{i}{q} \rceil} \frac{t^{\frac{j}{q}+\alpha-k}}{\Gamma(1+\alpha+\frac{j}{q}-k)} \left[ D_t^{\frac{i}{q}-k} y(t) \right]_{t=0}. \quad (3.57)$$

Let  $g_{j,n}$  for  $j = 0, \dots, q-1$  and  $n = 0, \dots, \lceil \frac{M-j}{q} \rceil - 1$  be a set of  $(\lceil \frac{M-j}{q} \rceil - 1, n)^{th}$  order modulating functions on  $[0, t]$  and satisfies the following property:

$$(P_{10}) : \int_0^t \tau^{-\kappa_j} g_{j,n}(\tau) d\tau = 0, \forall \kappa_j \in \{s - \frac{j}{q} - \alpha \mid s = 1, \dots, \lceil \frac{M}{q} \rceil\}.$$

Then, by multiplying  $g_{j,n}$  for  $j = 0, \dots, q-1$  and  $n = 0, \dots, \lceil \frac{M-j}{q} \rceil - 1$  to both side of (3.57) and integrating from 0 to  $t$ , we get:

$$\begin{aligned} & \sum_{i=0}^M a_i \int_0^t g_{j,n}(\tau) D_t^{\frac{i-j}{q}-\alpha} y(t) d\tau - \sum_{i=1}^M a_i \sum_{k=1}^{\lceil \frac{i}{q} \rceil} \frac{\left[ D_t^{\frac{i}{q}-k} y(t) \right]_{t=0}}{\Gamma(1+\alpha+\frac{j}{q}-k)} \int_0^t g_{j,n}(\tau) \tau^{\frac{j}{q}+\alpha-k} d\tau \\ &= \int_0^t g_{j,n}(\tau) D_t^{-\frac{j}{q}-\alpha} u(t) d\tau. \end{aligned} \quad (3.58)$$

Thanks to the property  $(P_{10})$  of  $g_{j,n}$ , (3.58) becomes:

$$\sum_{i=0}^M a_i \int_0^t g_{j,n}(\tau) D_t^{\frac{i-j}{q}-\alpha} y(t) d\tau = \int_0^t g_{j,n}(\tau) D_t^{-\frac{j}{q}-\alpha} u(t) d\tau. \quad (3.59)$$

**Step 2.** By applying the fractional order integration by parts formulas given in Lemma 2.5 to (3.59), we obtain:

$$\begin{aligned} & \sum_{i=0}^M a_i \sum_{k=0}^{\lceil \frac{i-j}{q}-\alpha \rceil - 1} (-1)^k \left[ g_{j,n}^{(k)}(\tau) D_\tau^{\frac{i-j}{q}-\alpha-1-k} y(\tau) \right]_{\tau=0}^{\tau=t} \\ &= \int_0^t {}^C D_{\tau,t}^{-\frac{j}{q}-\alpha} g_{j,n}(\tau) u(\tau) d\tau - \sum_{i=0}^M a_i \int_0^t {}^C D_{\tau,t}^{\frac{i-j}{q}-\alpha} g_{j,n}(\tau) y(\tau) d\tau. \end{aligned} \quad (3.60)$$

### 3.4. MODULATING FUNCTIONS BASED DIFFERENTIATORS

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Due to the property  $(P_2)$  of  $g_{j,n}$  for  $j = 0, \dots, q-1$  and  $n = 0, \dots, \lceil \frac{M-j}{q} \rceil - 1$ , (3.60) becomes:

$$\begin{aligned} & \sum_{i=0}^M a_i \sum_{k=0}^{\lceil \frac{i-j}{q} - \alpha \rceil - 1} (-1)^k g_{j,n}^{(k)}(t) D_t^{\frac{i-j}{q} - \alpha - 1 - k} y(t) \\ &= \int_0^t {}^C D_{\tau,t}^{-\frac{j}{q} - \alpha} g_{j,n}(\tau) u(\tau) d\tau - \sum_{i=0}^M a_i \int_0^t {}^C D_{\tau,t}^{\frac{i-j}{q} - \alpha} g_{j,n}(\tau) y(\tau) d\tau. \end{aligned} \quad (3.61)$$

By regrouping the terms of the left-hand side of (3.61), we have:

$$\begin{aligned} & \sum_{k=0}^{\lceil \frac{M-j}{q} - \alpha \rceil - 1} \sum_{i=j+kq+1}^M a_i (-1)^k g_{j,n}^{(k)}(t) D_t^{\frac{i-j}{q} - \alpha - 1 - k} y(t) \\ &= \int_0^t {}^C D_{\tau,t}^{-\frac{j}{q} - \alpha} g_{j,n}(\tau) u(\tau) d\tau - \sum_{i=0}^M a_i \int_0^t {}^C D_{\tau,t}^{\frac{i-j}{q} - \alpha} g_{j,n}(\tau) y(\tau) d\tau. \end{aligned} \quad (3.62)$$

Since  $g_{j,n}$  for  $j = 0, \dots, q-1$  and  $n = 0, \dots, \lceil \frac{M-j}{q} \rceil - 1$  satisfy the property  $(P_4)$ , then (3.62) becomes:

$$\begin{aligned} (-1)^n \sum_{i=j+nq+1}^M a_i D_t^{\frac{i-j}{q} - \alpha - 1 - n} y(t) &= \int_0^t {}^C D_{\tau,t}^{-\frac{j}{q} - \alpha} g_{j,n}(\tau) u(\tau) d\tau \\ &\quad - \sum_{i=0}^M a_i \int_0^t {}^C D_{\tau,t}^{\frac{i-j}{q} - \alpha} g_{j,n}(\tau) y(\tau) d\tau. \end{aligned} \quad (3.63)$$

As a result, it follows from (3.63) that:

- for  $j + nq + 1 = M$ , we have:

$$D_t^{\frac{1-q}{q} - \alpha} y(t) = \frac{(-1)^n}{a_M} \left( \sum_{i=0}^M \int_0^t {}^C D_{\tau,t}^{-\frac{j}{q} - \alpha} g_{j,n}(\tau) u(\tau) d\tau - \sum_{i=0}^M a_i \int_0^t {}^C D_{\tau,t}^{\frac{i-j}{q} - \alpha} g_{j,n}(\tau) y(\tau) d\tau \right), \quad (3.64)$$

- for  $1 \leq j + nq + 1 \leq M - 1$ , we have:

$$\begin{aligned} D_t^{\frac{M-j-nq-q}{q} - \alpha} y(t) &= \frac{(-1)^n}{a_M} \left( \int_0^t {}^C D_{\tau,t}^{-\frac{j}{q} - \alpha} g_{j,n}(\tau) u(\tau) d\tau - \sum_{i=0}^M a_i \int_0^t {}^C D_{\tau,t}^{\frac{i-j}{q} - \alpha} g_{j,n}(\tau) y(\tau) d\tau \right) \\ &\quad - \sum_{i=j+nq+1}^{M-1} \frac{a_i}{a_M} D_t^{\frac{i-j-nq-q}{q} - \alpha} y(t). \end{aligned} \quad (3.65)$$

Hence, the derivatives  $D_t^{\frac{M-j-nq-q}{q} - \alpha} y$  for  $M - j - nq - q = 1 - q, \dots, M - q$  can be estimated recursively based on (3.64) and (3.65). Noting that  $\alpha = \beta - \frac{d}{q}$  for some  $1 - q \leq d \leq M - q$ , there exist some  $j = 0, \dots, q-1$  and  $n = 0, \dots, \lceil \frac{M-j}{q} \rceil - 1$ , such that  $\frac{M-j-nq-q}{q} - \alpha = \beta$ . Therefore,  $D_t^\beta y$  for  $\beta \in (-1, \frac{M}{q} - 1]$  can be estimated.

### 3.4. MODULATING FUNCTIONS BASED DIFFERENTIATORS

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In [76], the state as well as the fractional derivatives of the output were estimated by using the modulating functions method in noisy environment. In this work, the following fractional order linear system model is considered:

$$\begin{cases} D_t^\alpha x &= Ax + Bu, \\ y &= Cx, \end{cases} \quad (3.66)$$

on  $I \subset \mathbb{R}_+ \cup \{0\}$ , where  $0 < \alpha < 2$ ,  $A \in \mathbb{R}^{N \times N}$ ,  $B \in \mathbb{R}^{N \times 1}$ ,  $C \in \mathbb{R}^{1 \times N}$ ,  $D_t^\alpha x = (D_t^\alpha x_1, \dots, D_t^\alpha x_N)^\top$  with  $\alpha = \frac{p}{q}$ ,  $p, q, N \in \mathbb{N}^*$ ,  $x \in \mathbb{R}^N$  is the state column vectors,  $y \in \mathbb{R}$  and  $u \in \mathbb{R}$  are the output and the input, respectively. Assume that  $u$  is known and the  $y$  is measured by a discrete noisy observation. Meanwhile, the conditions  $(C_1)$ – $(C_2)$  given in Theorem 2.1 and Theorem 2.2 are assumed to be fulfilled. The objective is to estimate the state  $x$ .

In order to apply the idea of the modulating functions method as done for the system (3.56), the system model (3.66) with a form of state-space representation is transformed into a form of fractional differential equation by using Lemma 2.8 as follows:

$$\sum_{i=0}^N a_i \mathcal{D}_t^{\alpha_i} y(t) = \bar{u}(t), \quad (3.67)$$

where  $\alpha_i = i\alpha$ , for  $i = 0, \dots, N$ ,  $a_N = 1$ ,  $a_i$  for  $i = 0, \dots, N - 1$  are coefficients of the characteristic polynomial of  $A$ ,  $\bar{u} = CA^{N-1}Bu$ . Then, the state  $x$  can be expressed by using the fractional sequential derivatives of  $y$  by (2.73). To be brief, a simple case based on (3.67) is considered in the following. Let us assume  $\alpha = \frac{1}{2}$  and  $N = 3$  in (3.66). Then, according to Lemma 2.8, we have:  $\forall t \in I$ ,

$$\sum_{i=0}^3 a_i \mathcal{D}_t^{\alpha_i} y(t) = \hat{u}(t), \quad (3.68)$$

where  $\hat{u} = CA^2Bu$ , and the state  $x$  can be expressed as:

$$x = M_1^{-1} M_2 \begin{pmatrix} y \\ \mathcal{D}_t^\alpha y \\ \mathcal{D}_t^{2\alpha} y \end{pmatrix}. \quad (3.69)$$

By applying the additive index law for fractional sequential derivatives given in Lemma 2.2, we get:  $\forall t \in I$ ,

$$\mathcal{D}_t^{2\alpha} y(t) = \dot{y}(t) - \frac{1}{\Gamma(-\frac{1}{2})} t^{-\frac{3}{2}} \left[ \mathcal{D}_t^{-\frac{1}{2}} y(t) \right]_{t=0}. \quad (3.70)$$

Consequently, the problem of estimating the state  $x$  can be transformed into the problem of estimating the fractional initial value  $\left[ \mathcal{D}_t^{-\frac{1}{2}} y(t) \right]_{t=0}$  and  $\mathcal{D}_t^k y$  for  $k = 0, 1, 2$ , which will be solved in the following text.



- Estimation for fractional initial value  $\left[ D_t^{-\frac{1}{2}} y(t) \right]_{t=0}$

The estimation process of the fractional order initial value is provided in the following step.

**Step 1.** By taking the fractional operator  $D_t^{\frac{i}{2}}$  for  $i = 1, 2$  to the both side of (3.68) and applying Lemma 2.2, we get:

$$\sum_{i=0}^3 a_i D_t^{\frac{i}{2}} y(t) = \hat{u}(t) + \frac{t^{-\frac{3}{2}}}{\Gamma(-\frac{1}{2})} (a_2 H_0 + H_1), \quad (3.71)$$

$$\sum_{i=0}^3 a_i D_t^{\frac{i-1}{2}} y(t) = D_t^{-\frac{1}{2}} \hat{u}(t) + \frac{t^{-\frac{3}{2}}}{\Gamma(-\frac{1}{2})} H_0 + \frac{t^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})} (a_1 H_0 + a_2 H_1 + H_2), \quad (3.72)$$

where  $H_0 = \left[ D_\tau^{-\frac{1}{2}} y(\tau) \right]_{\tau=0}$ ,  $H_1 = \left[ D_\tau^{-\frac{1}{2}} \left\{ D_\tau^{\frac{1}{2}} y(\tau) \right\} \right]_{\tau=0}$ ,  $H_2 = \left[ D_\tau^{-\frac{1}{2}} \left\{ D_\tau^{\frac{1}{2}} \left\{ D_\tau^{\frac{1}{2}} y(\tau) \right\} \right\} \right]_{\tau=0}$ .

**Step 2.** Let  $g_n$  for  $n = 1, 3$  are second order modulating functions on  $[0, t]$  and satisfying the following property: for  $n, j = 1, 3$ ,

$$(P_{11}) : \int_0^t g_n(\tau) \tau^{-\frac{j}{2}} d\tau = \delta_{n,j}.$$

Then, by multiplying  $g_n$  to (3.71)–(3.72) and integrating from 0 to  $t$ , we get:

$$\sum_{i=0}^3 a_i \int_0^t g_3(\tau) D_\tau^{\frac{i}{2}} y(\tau) d\tau = \int_0^t g_3(\tau) \hat{u}(\tau) d\tau + \frac{a_2 H_0 + H_1}{\Gamma(-\frac{1}{2})}, \quad (3.73)$$

$$\sum_{i=0}^3 a_i \int_0^t g_3(\tau) D_\tau^{\frac{i-1}{2}} y(\tau) d\tau = \int_0^t g_3(\tau) D_\tau^{-\frac{1}{2}} \hat{u}(\tau) d\tau + \frac{H_0}{\Gamma(-\frac{1}{2})}, \quad (3.74)$$

$$\sum_{i=0}^3 a_i \int_0^t g_1(\tau) D_\tau^{\frac{i-1}{2}} y(\tau) d\tau = \int_0^t g_1(\tau) D_\tau^{-\frac{1}{2}} \hat{u}(\tau) d\tau + \frac{a_1 H_0 + a_2 H_1 + H_2}{\Gamma(\frac{1}{2})}. \quad (3.75)$$

**Step 3.** By applying the fractional integration by parts given in Lemma 2.5 to (3.73)–(3.75), and duo to the properties  $(P_2)$ – $(P_3)$  of  $g_n$  for  $n = 1, 3$ , we obtain:

$$\sum_{i=0}^3 a_i \int_0^t D_{\tau,t}^{\frac{i}{2}} g_3(\tau) y(\tau) d\tau = \int_0^t g_3(\tau) \hat{u}(\tau) d\tau + \frac{a_2 H_0 + H_1}{\Gamma(-\frac{1}{2})}, \quad (3.76)$$

$$\sum_{i=0}^3 a_i \int_0^t D_{\tau,t}^{\frac{i-1}{2}} g_3(\tau) y(\tau) d\tau = \int_0^t D_{\tau,t}^{-\frac{1}{2}} g_3(\tau) \hat{u}(\tau) d\tau + \frac{H_0}{\Gamma(-\frac{1}{2})}, \quad (3.77)$$

$$\sum_{i=0}^3 a_i \int_0^t D_{\tau,t}^{\frac{i-1}{2}} g_1(\tau) y(\tau) d\tau = \int_0^t D_{\tau,t}^{-\frac{1}{2}} g_1(\tau) \hat{u}(\tau) d\tau + \frac{a_1 H_0 + a_2 H_1 + H_2}{\Gamma(\frac{1}{2})}. \quad (3.78)$$

**Step 4.** Consequently, the fractional initial values  $H_0, H_1, H_2$  can be recursively given by solving the

following linear system:

$$\begin{pmatrix} \frac{1}{\Gamma(\frac{1}{2})} & \frac{a_2}{\Gamma(\frac{1}{2})} & \frac{a_1}{\Gamma(\frac{1}{2})} \\ 0 & \frac{1}{\Gamma(-\frac{1}{2})} & \frac{a_2}{\Gamma(-\frac{1}{2})} \\ 0 & 0 & \frac{1}{\Gamma(-\frac{1}{2})} \end{pmatrix} \begin{pmatrix} H_2 \\ H_1 \\ H_0 \end{pmatrix} = \begin{pmatrix} \sum_{i=0}^3 a_i \int_0^t D_{\tau,t}^{\frac{i-1}{2}} g_1(\tau) y(\tau) d\tau - \int_0^t D_{\tau,t}^{-\frac{1}{2}} g_1(\tau) \hat{u}(\tau) d\tau \\ \sum_{i=0}^3 a_i \int_0^t D_{\tau,t}^{\frac{i}{2}} g_3(\tau) y(\tau) d\tau - \int_0^t g_3(\tau) \hat{u}(\tau) d\tau \\ \sum_{i=0}^3 a_i \int_0^t D_{\tau,t}^{\frac{i-1}{2}} g_3(\tau) y(\tau) d\tau - \int_0^t D_{\tau,t}^{-\frac{1}{2}} g_3(\tau) \hat{u}(\tau) d\tau \end{pmatrix}. \quad (3.79)$$

- Estimations for fractional derivatives  $D_t^{k\alpha} y$  for  $k = 0, 1, 2$

The estimation process of fractional derivatives of  $y$  is provided in the following step.

**Step 1.** Let  $\hat{g}_n$  for  $n = 0, 1$  be  $(2, n)^{th}$  modulating function on  $[0, t]$  and satisfying the following property:  $\forall j \in \{1, 3\}$ ,

$$(P_{12}) : \int_0^t \hat{g}(\tau) \tau^{-\frac{j}{2}} d\tau = 0.$$

Then, by multiplying  $\hat{g}_n$  for  $n = 0, 1$  to (3.71)–(3.72) and integrating from 0 to  $t$ , we get: for  $n = 0, 1$ ,

$$\sum_{i=0}^3 a_i \int_0^t \hat{g}_n(\tau) D_{\tau}^{\frac{i}{2}} y(\tau) d\tau = \int_0^t \hat{g}_n(\tau) \hat{u}(\tau) d\tau, \quad (3.80)$$

$$\sum_{i=0}^3 a_i \int_0^t \hat{g}_n(\tau) D_{\tau}^{\frac{i-1}{2}} y(\tau) d\tau = \int_0^t \hat{g}_n(\tau) D_{\tau}^{-\frac{1}{2}} \hat{u}(\tau) d\tau. \quad (3.81)$$

**Step 2.** By applying fractional integration by parts to (3.80)–(3.81) and thanks to the property  $(P_2)$  of  $\hat{g}_n$  for  $n = 0, 1$ , we get:

$$\sum_{i=0}^3 a_i \int_0^t D_{\tau,t}^{\frac{i}{2}} \hat{g}_n(\tau) y(\tau) d\tau = \int_0^t \hat{g}_n(\tau) \hat{u}(\tau) d\tau - \hat{g}_n(t) \left( D_t^{\frac{1}{2}} y(t) + a_2 y(t) \right) + \dot{\hat{g}}_n(t) D_t^{-\frac{1}{2}} y(t), \quad (3.82)$$

$$\sum_{i=0}^3 a_i \int_0^t D_{\tau,t}^{\frac{i-1}{2}} \hat{g}_n(\tau) y(\tau) d\tau = \int_0^t D_{\tau,t}^{-\frac{1}{2}} \hat{g}_n(\tau) \hat{u}(\tau) d\tau - \hat{g}_n(t) \left( y(t) + a_2 D_t^{-\frac{1}{2}} y(t) \right). \quad (3.83)$$

Furthermore, thanks to  $g_n$  satisfy  $(P_4)$  for  $n = 0, 1$ , then (3.82)–(3.83) become:

$$\sum_{i=0}^3 a_i \int_0^t D_{\tau,t}^{\frac{i}{2}} \hat{g}_0(\tau) y(\tau) d\tau = \int_0^t \hat{g}_0(\tau) \hat{u}(\tau) d\tau - D_t^{\frac{1}{2}} y(t) - a_2 y(t), \quad (3.84)$$

$$\sum_{i=0}^3 a_i \int_0^t D_{\tau,t}^{\frac{i}{2}} \hat{g}_1(\tau) y(\tau) d\tau = \int_0^t \hat{g}_1(\tau) \hat{u}(\tau) d\tau + D_t^{-\frac{1}{2}} y(t), \quad (3.85)$$

$$\sum_{i=0}^3 a_i \int_0^t D_{\tau,t}^{\frac{i-1}{2}} \hat{g}_0(\tau) y(\tau) d\tau = \int_0^t D_{\tau,t}^{-\frac{1}{2}} \hat{g}_0(\tau) \hat{u}(\tau) d\tau - y(t) - a_2 D_t^{-\frac{1}{2}} y(t). \quad (3.86)$$

**Step 3.** Consequently,  $D_t^{\frac{1}{2}}y(t)$ ,  $y(t)$  and  $D_t^{-\frac{1}{2}}y(t)$  can be recursively given by solving the following linear system:

$$\begin{pmatrix} -1 & -a_2 & 0 \\ 0 & 1 & a_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} D_t^{\frac{1}{2}}y(t) \\ y(t) \\ D_t^{-\frac{1}{2}}y(t) \end{pmatrix} = \begin{pmatrix} \sum_{i=0}^3 a_i \int_0^t D_{\tau,t}^{\frac{i}{2}} \hat{g}_0(\tau) y(\tau) d\tau - \int_0^t \hat{g}_0(\tau) \hat{u}(\tau) d\tau \\ \sum_{i=0}^3 a_i \int_0^t D_{\tau,t}^{\frac{i-1}{2}} \hat{g}_0(\tau) y(\tau) d\tau - \int_0^t D_{\tau,t}^{-\frac{1}{2}} \hat{g}_0(\tau) \hat{u}(\tau) d\tau \\ \sum_{i=0}^3 a_i \int_0^t D_{\tau,t}^{\frac{i}{2}} \hat{g}_1(\tau) y(\tau) d\tau - \int_0^t \hat{g}_1(\tau) \hat{u}(\tau) d\tau \end{pmatrix}. \quad (3.87)$$

Moreover,  $\dot{y}$  can also be obtained by (3.72). Finally,  $x$  can be estimated by using (3.69).

**Example 3.3** [76] Let us consider the system model (3.66) with  $\alpha = \frac{1}{2}$ ,  $N = 3$ ,  $B = (0, 1, 0)^\top$ ,  $C = (-1, 0, 1)$  and  $A = \begin{pmatrix} 11 & 1 & -11 \\ 6 & 0 & -6 \\ 18 & 1 & -17 \end{pmatrix}$ . A constant input  $u(t) = 0.5$  on  $[-2, 0]$  and  $u(t) = \sin(5t)$  on  $]0, 5]$  are set. Assume that  $y^\varpi(t_i) = y(t_i) + \sigma\varpi(t_i)$  is the noisy observation of the output  $y$  on  $I = [0, 5]$ , where  $t_i = iT_s$  with  $T_s = 0.0005$ , the noise  $\{\varpi(t_i)\}$  is simulated from a zero-mean white Gaussian iid sequence, and the value of  $\sigma$  is adjusted such that  $SNR = 20$  dB. The output and the associated noisy output are shown in Fig. 3.5. Moreover, the estimation results are depicted in Fig. 3.6 and Fig. 3.7, the efficiency and robustness of the proposed method are verified.

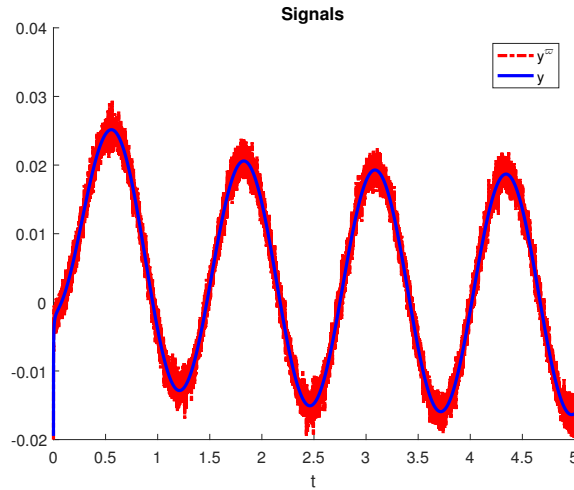


Figure 3.5: The output and its discrete noisy observation (cited from [76]).

### 3.4. MODULATING FUNCTIONS BASED DIFFERENTIATORS

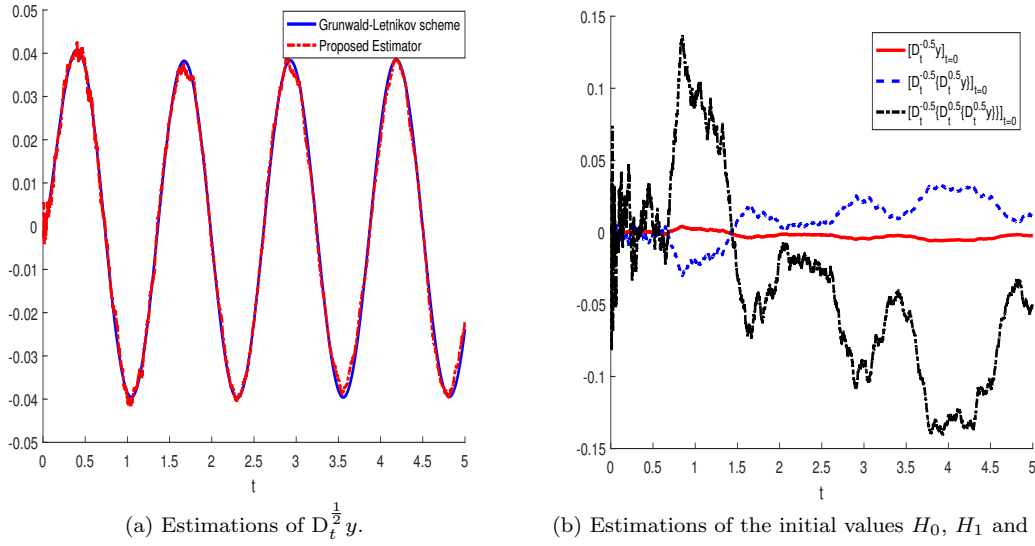


Figure 3.6: Estimation of  $D_t^{\frac{1}{2}} y$  and the initial values  $H_0, H_1$  and  $H_2$  (cited from [76]).

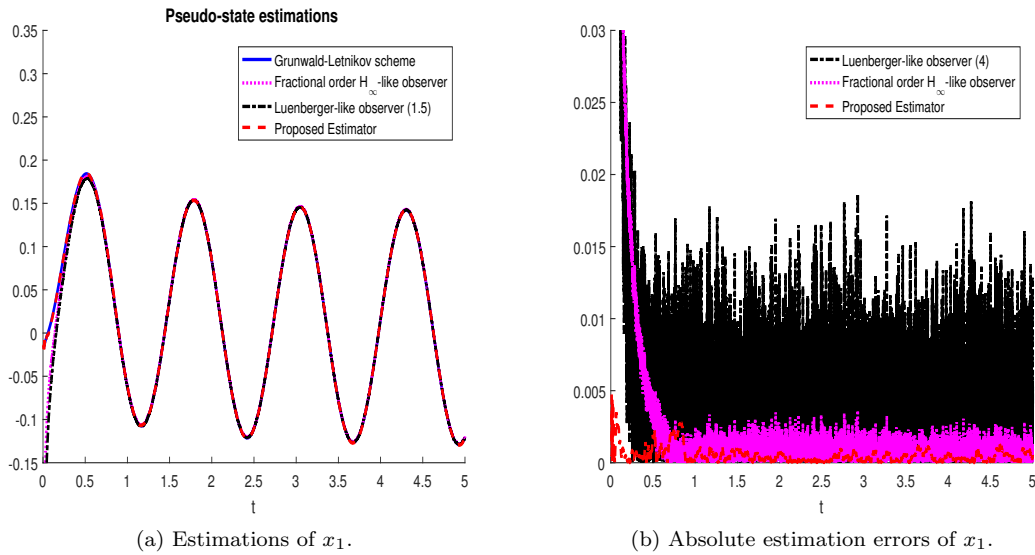


Figure 3.7: Estimations and absolute estimation errors of  $x_1$  (cited from [76]).

### 3.5. CONCLUSIONS

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For more general results on state estimation for fractional order linear systems, see [77], [77]. Furthermore, fractional derivative estimation of the state for rational-order linear systems has been addressed in [78]. As stated in [78], fractional derivative estimation  $D_t^\beta x$  of the state covers the following cases:

- $\beta \leq -1$ : by definition,  $D_t^\beta x$  is a proper integral, which can be estimated via a numerical integration approach.
- $-1 < \beta < 0$ :  $D_t^\beta x$  refers to a improper fractional integral of  $x$ , which was considered in [69].
- $\beta = 0$ :  $D_t^\beta x = x$ , *i.e.* the fractional derivative estimation of the state reduces to the state estimation, which is the topic of the present section.
- $\beta > 0$ :  $D_t^\beta x$  refer to a fractional derivative of  $x$ .

Recently, the fractional derivative estimation of the state for fractional-order linear systems with the differentiation order  $\alpha \in (0, 2)$  as been reported [79].

### 3.5 Conclusions

In this chapter, definitions of modulating functions with various properties were reviewed. On this basis, the main ideas of the applications of the modulating functions method in parameter identification, differentiators, and state estimation were briefly introduced. By converting and eliminating undesired terms and preserving the terms to be estimated, exact algebraic integral formulas can be provided, which enables non-asymptotic and robust estimations. The simulation results illustrated the efficiency and robustness of the modulating functions method. Inspired by these results, the main contributions of this thesis are presented in the following chapters.

## Part III

# Non-asymptotic and robust estimations for state and fractional integral



## Chapter 4

# State estimation for integer order systems

In this chapter, an improved modulating functions method is proposed to non-asymptotically and robustly estimate the state for integer order linear systems in noisy environment. Moreover, the fractional derivatives of the state can be computed by applying the Grünwald-Letnikov scheme. For this purpose, the considered system modeled by the state-space representation is first transformed into the observable canonical form. Based on that, the modulating functions method is applied to obtain algebraic integral formulas for the state. These integral formulas can give robust estimations in discrete noisy cases. Furthermore, it is shown how to design the required modulating functions. To demonstrate the efficiency of the proposed estimator, one numerical example is given by comparing it with the Luenberger observer. The main contributions of this chapter are outlined as follows:

- Different from existing works on the modulating functions method [74], [76]–[79], [97], the improved method proposed in this chapter is directly applied to the observable canonical form, does not require the conversion to the input-output differential equation, nor need to estimate the derivatives of output to obtain algebraic integral formulas for the sought variables, they make the estimation process easier and the obtained formulas simpler.
- The proposed estimator is accurate without any sources of errors in continuous noise-free case and robust against corrupting noises in discrete noisy case. Meanwhile, the proposed estimator is also algebraic (thus non-asymptotic), which can provide estimation results with a finite-time convergence and without knowing initial conditions.



## 4.1 Problem formulation

In this chapter, let us consider the following integer order linear system:

$$\begin{cases} \dot{x} &= Ax + Bu, \\ y &= Cx, \end{cases} \quad (4.1)$$

on  $I = [0, h] \subset \mathbb{R}$ , where  $A \in \mathbb{R}^{N \times N}$ ,  $B \in \mathbb{R}^{N \times 1}$ ,  $C \in \mathbb{R}^{1 \times N}$ ,  $x \in \mathbb{R}^N$  is the state,  $y \in \mathbb{R}$  and  $u \in \mathbb{R}$  are respectively the output and the input. Meanwhile, the conditions  $(C_1)$ – $(C_2)$  given in Theorem 2.1 and Theorem 2.2 are assumed to be fulfilled.

The aim of this chapter is to estimate the state and its fractional derivative in noisy environment. For this purpose, the system defined by (4.1) first needs to be transformed into the observable canonical form, on which the modulating functions method will be applied. By applying the change of coordinate  $z = Tx$  given in (2.66), then system (4.1) can be transformed into the following observable canonical form of state-space representation:

$$\begin{cases} \dot{z} &= \hat{A}z + \hat{B}u, \\ y &= \hat{C}z, \end{cases} \quad (4.2)$$

where

$$\hat{A} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & -a_{N-2} \\ 0 & 0 & \cdots & 0 & 1 & -a_{N-1} \end{pmatrix}, \quad (4.3)$$

and  $\hat{B} = MB = (\hat{B}_0, \hat{B}_1, \dots, \hat{B}_{N-1})^\top \in \mathbb{R}^{N \times 1}$ ,  $\hat{C} = (0, \dots, 0, 1) \in \mathbb{R}^{1 \times N}$ .

## 4.2 Main results

In this section, by applying the improved modulating functions method, the algebraic integral formulas for state of the integer order linear system (4.1) will be given. Moreover, the fractional derivatives of the state are computed by applying the Grünwald-Letnikov scheme. The main results are given in the following theorems.

**Theorem 4.1** *Let us consider the system defined in (4.2) on  $I$ . For  $0 < t \in I$ , suppose that  $\eta$  is a left-sided  $N^{\text{th}}$  order modulating function on  $[0, t]$ . Then, we have:*

$$\sum_{i=1}^N (-1)^{i-1} \eta^{(i-1)}(t) z_i(t) = \sum_{i=0}^{N-1} (-1)^i \hat{B}_i \int_0^t \eta^{(i)}(\tau) u(\tau) d\tau - \sum_{i=0}^N (-1)^i a_i \int_0^t \eta^{(i)}(\tau) y(\tau) d\tau, \quad (4.4)$$

## 4.2. MAIN RESULTS

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where  $a_N = 1$ .

**Proof.** The proof can be given in the following steps.

**Step 1.** By multiplying  $\eta^{(i-1)}$  for  $i = 1, \dots, N$  to (4.2) and integrating on  $[0, t]$ , the following formula holds:

$$\int_0^t \eta^{(i-1)}(\tau) \dot{z}_i(\tau) d\tau = \int_0^t \eta^{(i-1)}(\tau) z_{i-1}(\tau) d\tau + \hat{B}_{i-1} \int_0^t \eta^{(i-1)}(\tau) u(\tau) d\tau - a_{i-1} \int_0^t \eta^{(i-1)}(\tau) y(\tau) d\tau, \quad (4.5)$$

where  $z_0(\tau) = 0$ .

**Step 2.** By applying the integration by parts formula given in Lemma 2.4, then the left side of (4.5) can be written as follows:

$$\int_0^t \eta^{(i-1)}(\tau) \dot{z}_i(\tau) d\tau = \left[ \eta^{(i-1)}(\tau) z_i(\tau) \right]_{\tau=0}^{\tau=t} - \int_0^t \eta^{(i)}(\tau) z_i(\tau) d\tau. \quad (4.6)$$

According to (4.5) and (4.6), we get:

$$\begin{aligned} \left[ \eta^{(i-1)}(\tau) z_i(\tau) \right]_{\tau=0}^{\tau=t} - \int_0^t \eta^{(i)}(\tau) z_i(\tau) d\tau &= \int_0^t \eta^{(i-1)}(\tau) z_{i-1}(\tau) d\tau + \hat{B}_{i-1} \int_0^t \eta^{(i-1)}(\tau) u(\tau) d\tau \\ &\quad - a_{i-1} \int_0^t \eta^{(i-1)}(\tau) y(\tau) d\tau. \end{aligned} \quad (4.7)$$

**Step 3.** By combining the  $N$  equations of (4.7), we obtain:

$$\sum_{i=1}^N (-1)^{i-1} \left[ \eta^{(i-1)}(\tau) z_i(\tau) \right]_{\tau=0}^{\tau=t} = \sum_{i=0}^{N-1} (-1)^i \hat{B}_i \int_0^t \eta^{(i)}(\tau) u(\tau) d\tau - \sum_{i=0}^N (-1)^i a_i \int_0^t \eta^{(i)}(\tau) y(\tau) d\tau. \quad (4.8)$$

Finally, since  $\eta$  satisfy  $(P_2)$  on  $[0, t]$ , this proof is completed. ■

The aim of this chapter is to estimate the state and its fractional derivative by using the new modulating functions method. According to the above theorem, the following results can be deduced.

**Theorem 4.2** *Let us consider the system defined in (4.2). For  $0 < t \in I$ ,  $g_{j'}$  for  $j' = 1, \dots, N$ , are left-sided  $(N, j')$ <sup>th</sup> order modulating functions on  $[0, t]$ , we have:*

$$z_{j'}(t) = \sum_{i=0}^{N-1} (-1)^{i+j'-1} \hat{B}_i \int_0^t g_{j'}^{(i)}(\tau) u(\tau) d\tau - \sum_{i=0}^N (-1)^{i+j'-1} a_i \int_0^t g_{j'}^{(i)}(\tau) y(\tau) d\tau. \quad (4.9)$$

**Proof.** By replacing  $\eta$  by  $g_{j'}$ , for  $j' = 1, \dots, N$ , in (4.4) and using  $(P_4)$  of the modulating functions  $g_{j'}$  in Theorem 4.1, this proof is completed. ■

### 4.3. CONSTRUCTION OF MODULATING FUNCTIONS

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**Corollary 4.1** *Under the same hypotheses of Theorem 4.2,  $g_{j'}$  for  $j' = 1, \dots, N$ , are left-sided  $(N, j')^{th}$  order modulating functions on  $[0, t]$ . Let  $\beta \in \mathbb{R}$ , then  $D_t^\beta x$  can be calculated by the following formulas: for  $j' = 1, \dots, N$ ,*

$$D_t^\beta x_{j'}(t) = D_t^\beta \left( T^{-1} \left( \sum_{i=0}^{N-1} (-1)^{i+j'-1} \hat{B}_i \int_0^t g_{j'}^{(i)}(\tau) u(\tau) d\tau - \sum_{i=0}^N (-1)^{i+j'-1} a_i \int_0^t g_{j'}^{(i)}(\tau) y(\tau) d\tau \right) \right). \quad (4.10)$$

**Proof.** Based on Theorem 4.2, the state  $x$  can be calculated by taking the inverse matrix of  $T$  by using (2.66). Then, the derivatives of state  $D_t^\beta x$  can be easy obtained. This proof is completed. ■

**Remark 4.1** *According to the Corollary 4.1, the state  $x$  can be obtained with  $\beta = 0$ .*

**Remark 4.2** *Different from previous works [76]–[79], the proposed new modulating functions method does not need to estimate the derivatives of output to estimate the state, which makes the estimation process more easier and the obtained formulas simpler.*

### 4.3 Construction of modulating functions

In this section, the construction of the required modulating functions in Theorem 4.2 and Corollary 4.1 is provided. Power functions are adopted to construct the needed modulating functions. The computation schemes are provided in the following proposition. Moreover, we can find a similar method to design modulating functions in [78], [110].

**Proposition 4.1** *Let  $N \in \mathbb{N}^*$  and  $N - 1 < m \in \mathbb{R}_+$ . For  $0 < t \in I$  and  $j' \in \{1, \dots, N\}$ , let  $g_{j'}$  be a function defined as follows:*

$$g_{j'}(\tau) = \sum_{i=1}^N \lambda_i \tau^{m+i} t^{j'-m-i-1}. \quad (4.11)$$

*If the coefficient vector  $\lambda = (\lambda_1, \dots, \lambda_N)^\top$  is the solution of the following system:*

$$B_m \lambda = d_{j'}, \quad (4.12)$$

*where  $B_m \in \mathbb{R}^{N \times N}$ ,  $d_{j'} \in \mathbb{R}^{N \times 1}$  are defined as follows: for  $k, i = 1, \dots, N$ ,*

$$B_m(k, i) = \frac{\Gamma(m+i+1)}{\Gamma(m+i+2-k)}, \quad (4.13)$$

*and  $d_{j'}(k) = 1$  if  $k = j'$  and  $d_{j'}(k) = 0$  else.*

*Then,  $g_{j'}$  is a left-sided  $(N, j')^{th}$  order modulating function on  $[0, t]$ .*

## 4.4 Simulation results

In order to illustrate the accuracy and robustness with respect to corrupting noises of the proposed estimators in Section 4.2, we present some numerical results in this section.

**Example 4.1** Let us take  $N = 3$ ,  $u(t) = \sin(t)$ , and the following matrices in (4.1):

$$A = \begin{pmatrix} 11 & 1 & -11 \\ 6 & 0 & -6 \\ 18 & 1 & -17 \end{pmatrix}, B = (0, 1, 0)^\top, C = (-1, 0, 1),$$

which satisfy the conditions  $(C_1)$ – $(C_2)$ . Assume that the initial value of the state is given by  $x(0) = (-0.1, -0.2, -0.3)^\top$ , and  $y^\varpi(t_i) = y(t_i) + \sigma\varpi(t_i)$  is the noisy observation of the output  $y$  on  $I = [0, 10]$ , where  $t_i = iT_s$  with  $T_s = 0.001$ ,  $\{\varpi(t_i)\}$  is a zero-mean white Gaussian noise, and the value of  $\sigma$  is adjusted such that the signal-to-noise ratio  $SNR = 10 \log_{10} \left( \frac{\sum |y^\varpi(t_i)|^2}{\sum |\sigma\varpi(t_i)|^2} \right)$  is equal to 15dB. The output and the associated noisy output are shown in Fig. 4.1.

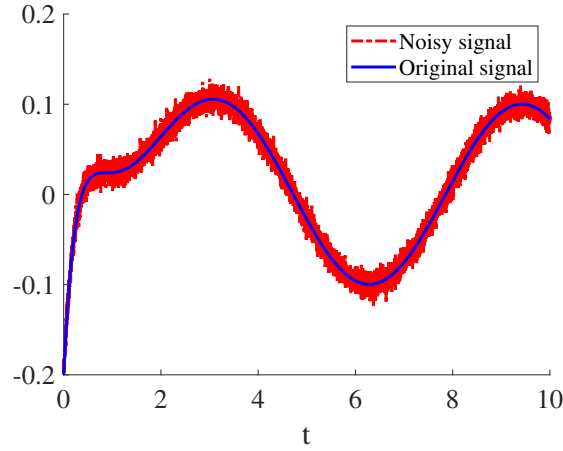


Figure 4.1:  $y$  and its discrete noisy observation in Example 4.1.

The state  $x$  and its fractional derivative  $D_t^\beta x$  can be estimated in this discrete noisy case by applying Corollary 4.1, where the trapezoidal numerical integration method is used to approximate the integral formulas. Then, by means of the constructed modulating functions, the required estimation results can be obtained. Moreover, the Luenberger observer is considered to compare with the proposed method. The first gain is chosen by taking the eigenvalues to be 2 times greater than the ones of the system in order to guaranty the robustness against the noise. The second gain is chosen by taking the eigenvalues to be 7 times greater than the ones of the system in order to improve the convergence speed, which produces larger noise error contribution.

#### 4.4. SIMULATION RESULTS

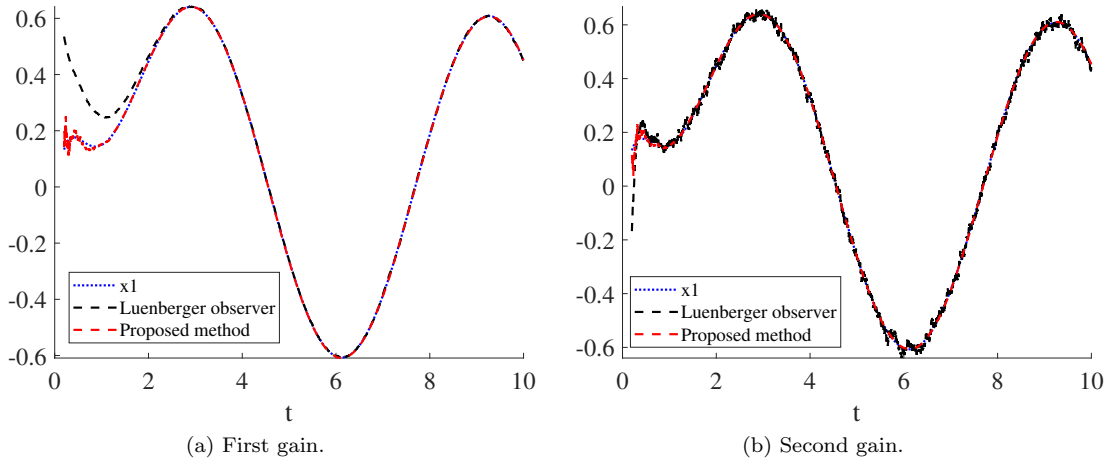


Figure 4.2: Estimations of the state  $x_1$  obtained by the proposed modulating functions method and the Luenberger observer in Example 4.1.

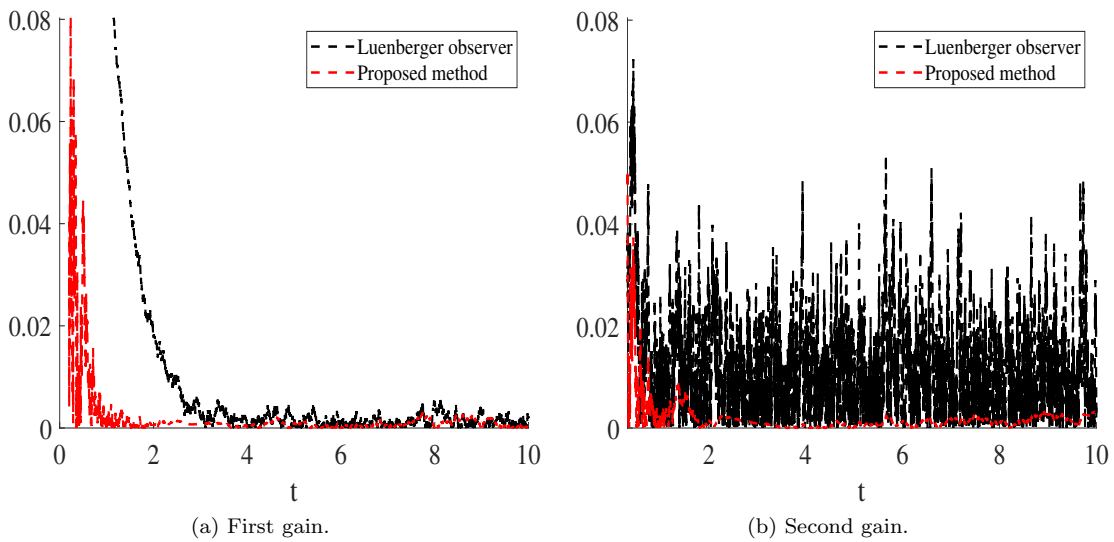


Figure 4.3: Estimation errors of the state  $x_1$  obtained by the proposed modulating functions method and the Luenberger observer in Example 4.1.

#### 4.4. SIMULATION RESULTS

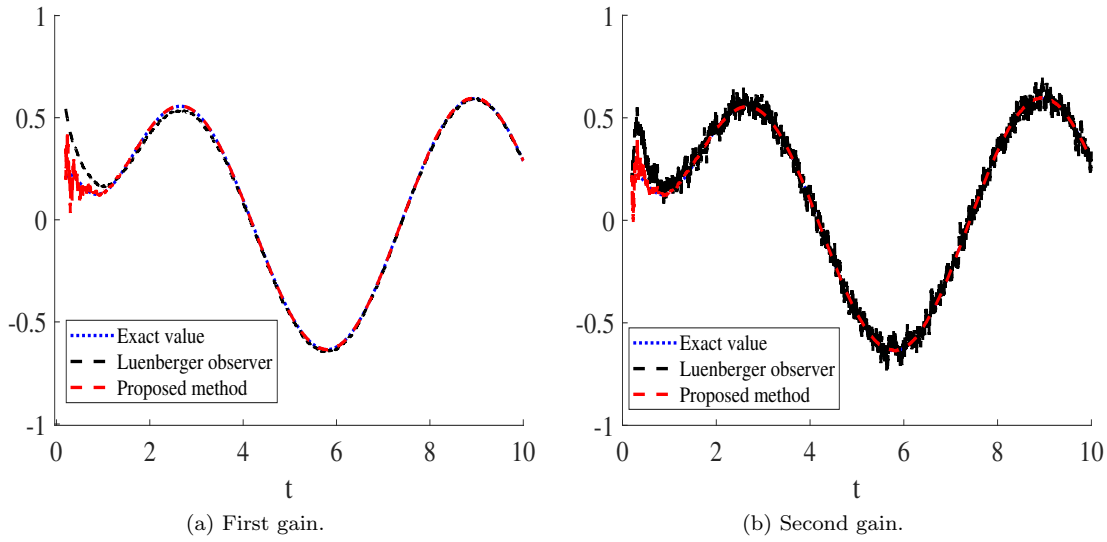


Figure 4.4: Estimations of  $D_t^{0.2}x_1$  obtained by the proposed modulating functions method and the fractional order Luenberger-like observer in Example 4.1.

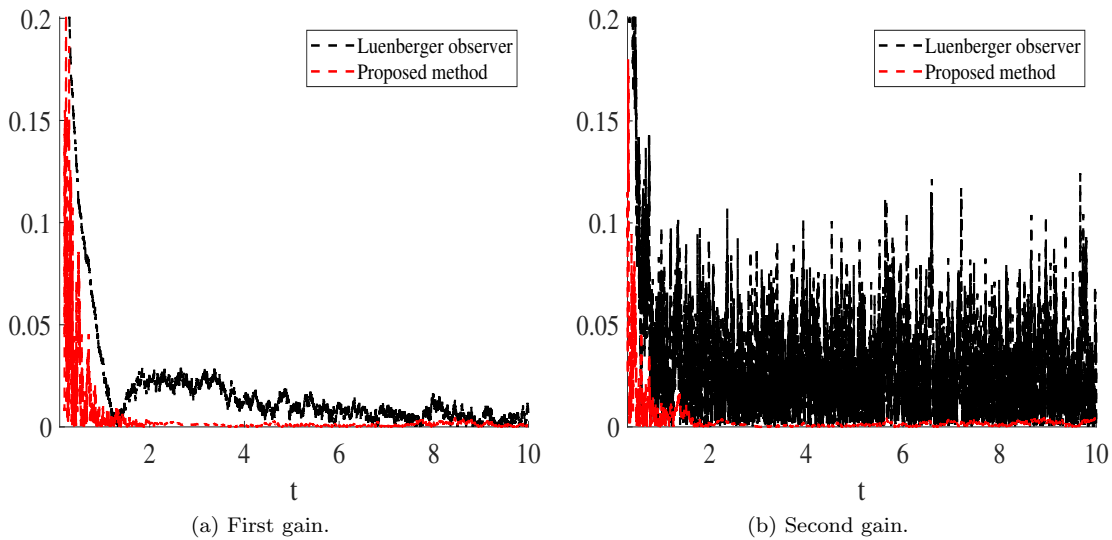


Figure 4.5: Estimation errors of  $D_t^{0.2}x_1$  obtained by the proposed modulating functions method and the fractional order Luenberger-like observer in Example 4.1.

## 4.5. CONCLUSIONS

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By taking  $\beta = 0$ ,  $m = 4$  and the initial value of the Luenberger observer  $\bar{x}(0) = (1, 2, 1)^\top$ , the estimation values and estimation errors of  $x_1$  by using the proposed method and the Luenberger observer are shown in Fig. 4.2 and Fig. 4.3, respectively. Numerical results show the efficiency of the proposed state estimator and the robustness with respect to noises is better than the Luenberger observer.

By taking  $\beta = 0.2$ , the corresponding estimation values and estimation errors of  $D_t^{0.2}x_1$  are shown in Fig. 4.4 and Fig. 4.5, where  $D_t^{0.2}x_1$  is estimated by applying the Grünwald-Letnikov scheme to  $x_1$  obtained using the proposed method and the fractional order Luenberger-like observer.

## 4.5 Conclusions

In this chapter, a non-asymptotic and robust state estimator based on the improved modulating functions method was constructed for integer order systems. Moreover, the fractional derivatives of the state are computed by applying the Grünwald-Letnikov scheme. First, the considered system was transformed from the state-space representation into the observable canonical form. Then, the designed modulating functions are applied to obtain algebraic integral formulas. Finally, numerical example was given to show the efficiency of the proposed estimator. Compared with the Luenberger observer, the proposed modulating functions method is more accuracy. This work has been published in [111].

## Chapter 5

# Fractional integral estimation of state for fractional order systems

In this chapter, the improved modulating functions-based state estimation method introduced in chapter 4 is extended to estimate the fractional integral of the state for fractional order systems in noisy environment, which is non-asymptotic and robust. Moreover, the fractional derivatives of the state can be computed via the Grünwald-Letnikov scheme. The proposed method is directly applied to the fractional order observable canonical form to obtain algebraic integral formulas for the sought variables. For this, a family of the required modulating functions is introduced and built with a design parameter. Furthermore, error analysis in discrete noisy cases is addressed. Finally, the robustness and accuracy of the improved method are verified by numerical simulations. The main contributions of this chapter are outlined as follows:

- The proposed method is more efficient. First, this method is directly applied to the fractional order observable canonical form to obtain algebraic integral formulas for the sought variables, which can make the estimation process simpler, the estimation formulas simpler and the estimation results more accurate. Indeed, when applying the previously mentioned non-asymptotic methods for state estimation, the considered system needed to be transformed into a differential equation [72], [74], [78]. Besides, both the derivatives of the output and a set of fractional order initial values were required to estimate the fractional derivative of the state [76], [78], [79]. Moreover, the required modulating functions were complex to construct.
- The considered system model is more general, thanks to the proposed method. First, this chapter studies fractional systems with multiple inputs and multiple outputs. However, only fractional systems with single input and single output were considered in [72], [74], [76], [78]–[80], [112], [113]. Second, the considered systems are not subject to the fractional differentiation orders. In fact, the methods proposed in [76], [78] can only be applied for rational order systems.



## 5.1 Motivation and problem statement

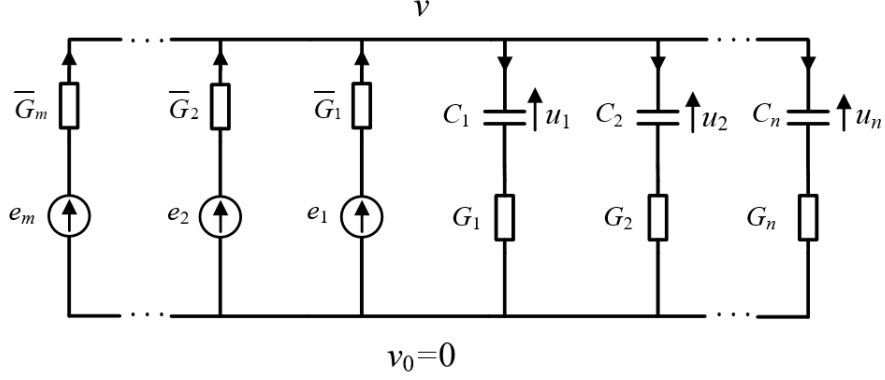


Figure 5.1: Fractional electrical circuit model.

Let us consider an electrical circuit consisting of  $n$  condensators,  $m$  voltage sources and resistors, as showed in Fig. 5.1, where  $G_i$  and  $\bar{G}_j$  for  $i = 0, \dots, n$ ,  $j = 0, \dots, m$  are conductances,  $C_s$  for  $s = 0, \dots, n$  are condensators, and  $e_l$  for  $l = 0, \dots, m$  are the sources. According to [24], by applying the Kirchhoff's laws, we can use the following fractional order model to describe the dynamics of the transient state in this electrical circuit:

$$D_t^\alpha x(t) = Ax(t) + Bu(t), \quad (5.1)$$

where  $\alpha \in (0, 1]$ , the state  $x \in \mathbb{R}^n$  is the vector of the voltages on the condensators,  $u \in \mathbb{R}^m$  is the vector of the source voltages,

$$A = \begin{pmatrix} -\frac{G_1 G - G_1^2}{C_1 G} & \frac{G_1 G_2}{C_1 G} & \dots & \frac{G_1 G_n}{C_1 G} \\ \frac{G_2 G_1}{C_2 G} & -\frac{G_2 G - G_2^2}{C_2 G} & \dots & \frac{G_2 G_n}{C_2 G} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{G_n G_1}{C_n G} & \frac{G_n G_2}{C_n G} & \dots & -\frac{G_n G - G_n^2}{C_n G} \end{pmatrix},$$

and

$$B = \begin{pmatrix} \frac{G_1 \bar{G}_1}{C_1 G} & \frac{G_1 \bar{G}_2}{C_1 G} & \dots & \frac{G_1 \bar{G}_m}{C_1 G} \\ \frac{G_2 \bar{G}_1}{C_2 G} & \frac{G_2 \bar{G}_2}{C_2 G} & \dots & \frac{G_2 \bar{G}_m}{C_2 G} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{G_n \bar{G}_1}{C_n G} & \frac{G_n \bar{G}_2}{C_n G} & \dots & \frac{G_n \bar{G}_m}{C_n G} \end{pmatrix}$$

with  $G = \sum_{i=1}^n G_i + \sum_{j=1}^m \bar{G}_j$ .

The transient state  $x$  is conducive to better analyze and master the change law of the circuit [30], [114]. Moreover, it is required in the state feedback controllers [24]. However, the state is generally unknown and needs to be estimated.

Motivated by the state estimation issue for the previous fractional electrical circuit model, the following fractional order linear systems with multiple inputs and multiple outputs is considered in this work:  $I = [0, h] \subset \mathbb{R}$ ,

$$\begin{cases} D_t^\alpha x(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t), \end{cases} \quad (5.2)$$

where  $A \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{n \times q}$ ,  $C \in \mathbb{R}^{p \times n}$ ,  $D_t^\alpha x = (D_t^\alpha x_1, \dots, D_t^\alpha x_n)^\top$ ,  $x \in \mathbb{R}^n$  is the state,  $u \in \mathbb{R}^q$  and  $y \in \mathbb{R}^p$  with  $1 \leq p, q \leq n$  are respectively the input and the output. It is assumed  $\alpha \in (0, 2)$  to guarantee the stability [23].

The aim of this chapter is to estimate fractional integral and derivative of the state in a noisy environment, which is useful in the fractional order state feedback controller design. To this end, we recall the observability matrix for the system given in (5.2) defined by [115]:

$$Q_k = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{k-1} \end{pmatrix}, \quad (5.3)$$

in which  $k \leq n$  is the smallest integer that satisfies  $\text{rank}(Q_k) = \text{rank}(Q_n)$ . Then, it is assumed that the system defined by (5.2) fulfills the following observability condition:

$$(\widetilde{C}_1) : \text{rank}(Q_k) = n.$$

## 5.2 Main results

An improved modulating functions method is introduced to estimate the fractional integral of the state for fractional order systems in this chapter. Moreover, the fractional derivatives of the state are computed by applying the Grünwald-Letnikov scheme. For this purpose, we first convert the system defined in (5.2) into the fractional order observable canonical form using the following lemma which can be obtained using a similar way as done in [115] for integer order systems.

**Lemma 5.1** *Under condition  $(\widetilde{C}_1)$ , consider the following coordinate transformation:*

$$z = Tx, \quad (5.4)$$

## 5.2. MAIN RESULTS

where  $z = (z_1 \ z_2 \ \cdots \ z_k)^\top$ ,  $z_i \in \mathbb{R}^p$  for  $i = 1, \dots, k$ ,  $z \in \mathbb{R}^{n_z}$  with  $n_z = pk$ , and  $T = (T_1^\top \ T_2^\top \ \cdots \ T_k^\top)^\top$ ,  $T \in \mathbb{R}^{n_z \times n}$  is given by:

$$T = \begin{pmatrix} CA^{k-1} + \sum_{i=1}^{k-1} M_i CA^{i-1} \\ \vdots \\ CA^{k-j} + \sum_{i=j}^{k-1} M_i CA^{i-j} \\ \vdots \\ CA + M_{k-1}C \\ C \end{pmatrix} = \begin{pmatrix} M_1 & \cdots & M_{k-1} & I_p \\ \vdots & \vdots & \ddots & \vdots \\ M_{k-1} & I_p & \cdots & 0 \\ I_p & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{k-1} \end{pmatrix} = \bar{M}Q_k, \quad (5.5)$$

where  $M_i \in \mathbb{R}^{p \times p}$ , for  $i = 0, 1, \dots, k-1$  are determined by the following matrix equation:

$$(M_0 \ M_1 \ \cdots \ M_{k-1}) = -CA^k Q_k^L, \quad (5.6)$$

where  $Q_k$  is given by (5.3),  $Q_k^L \in \mathbb{R}^{n \times n_z}$  is a left-inverse matrix of  $Q_k$  with  $Q_k^L Q_k = I_n$ .

Then, the following fractional order observable canonical form can be obtained:

$$\begin{cases} D_t^\alpha z(t) = \bar{A}z(t) + \bar{B}u(t), \\ y(t) = \bar{C}z(t), \end{cases} \quad (5.7)$$

in which

$$\bar{A} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & -M_0 \\ I_p & 0 & \cdots & 0 & 0 & -M_1 \\ 0 & I_p & \cdots & 0 & 0 & -M_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & I_p & 0 & -M_{k-2} \\ 0 & 0 & \cdots & 0 & I_p & -M_{k-1} \end{pmatrix}, \quad (5.8)$$

and  $\bar{B} = TB = (\bar{B}_0 \ \bar{B}_1 \ \cdots \ \bar{B}_{k-1})^\top$ ,  $\bar{C} = (0 \ \cdots \ 0 \ I_p)$ .

**Remark 5.1** In practical problems, mathematical modeling is established based on the internal structure of the system to obtain state space representation. If  $A$  and  $C$  in the obtained representation satisfy the observable condition  $(\bar{C}_1)$ , the specific coordinate transformation  $T$  can be obtained according to (5.5). Moreover, since  $\bar{M}$  is an inverse matrix, we can get  $\text{rank}(T) = \text{rank}(Q_k) = n$ . Hence, under  $(\bar{C}_1)$ ,  $T$  is a full rank matrix and  $T^L$  can be chosen as  $T^L = Q_k^L \bar{M}^{-1}$  such that  $x = T^L z$  in which  $T^L T = I_n$ .

### 5.2.1 Improved modulating function-based state integral estimation method

The objective of this subsection is to develop algebraic integral formulas for the fractional integral and derivative of the state. For such an estimation task, the existing modulating functions methods were generally applied to a set of constructed fractional order differential equations to estimate some fractional order initial values and the fractional derivatives of the output [76]–[79]. In this work, a new modulating functions method is introduced which is directly applied to estimate the sought derivatives based on the fractional order observable canonical form, thus does not need to construct differential equations. Moreover, no fractional initial values need to be estimated.

First, fractional order modulating functions are introduced in the following definition. For this, let us give a useful set defined as follows: for  $N \in \mathbb{N}^*$  and  $\alpha \in \mathbb{R}_+$ ,

$$\Phi_{N,\alpha} \subset \mathbb{N}^2 := \{(j, r); 0 \leq j \leq N - 1, 0 \leq r \leq \lceil \alpha \rceil - 1\}. \quad (5.9)$$

**Definition 5.1** Let  $[a, b] \subset \mathbb{R}$ ,  $\Phi_{N,\alpha}$  be the set defined in (5.9). Let  $(j', r') \in \Phi_{N,\alpha}$ , if a function  $g_{j',r'}$  satisfies the following properties:  $\forall (j, r) \in \Phi_{N,\alpha}$ ,

$$(P_1^\alpha) : {}_s^C \mathcal{D}_\tau^{j\alpha} g_{j',r'} \text{ exists and is integrable on } [0, b],$$

$$(P_2^\alpha) : \left[ D_\tau^r \{ {}_s^C \mathcal{D}_\tau^{j\alpha} g_{j',r'}(\tau) \} \right]_{\tau=b} = 0,$$

then  $g_{j',r'}$  is called a right-sided  $(N, \alpha)$  fractional order modulating function on  $[0, b]$ . Moreover, if  $g_{j',r'}$  also satisfies the following property:  $\forall (j, r) \in \Phi_{N,\alpha}$ ,

$$(P_3^\alpha) : \left[ D_\tau^r \{ {}_s^C \mathcal{D}_\tau^{j\alpha} g_{j',r'}(\tau) \} \right]_{\tau=0} = 1, \text{ if } (j, r) = (j', r'); \left[ D_\tau^r \{ {}_s^C \mathcal{D}_\tau^{j\alpha} g_{j',r'}(\tau) \} \right]_{\tau=0} = 0, \text{ else,}$$

then,  $g_{j',r'}$  is called a  $(N, \alpha; j', r')$  fractional order modulating function on  $[0, b]$ .

**Remark 5.2** In the case of  $\alpha = 1$  in Definition 5.1, we have:  $r = r' = 0$  and  $D_\tau^r \{ {}_s^C \mathcal{D}_\tau^{j\alpha} g_{j',r'} \} = g_{j',r'}^{(j)}$ . Thus, a  $(N, 1; j', 0)$  fractional order modulating function is a right-sided  $(j', N)^{\text{th}}$  order modulating functions on  $[a, b]$  [80].

**Theorem 5.1** Consider the system given in (5.7). For  $t \in (0, h]$ , let  $\eta$  be a right-sided  $(k, \alpha)$  fractional order modulating function on  $[0, t]$ . Then, the following integral equation holds:

$$\begin{aligned} \sum_{r=0}^{\lceil \alpha \rceil - 1} \sum_{i=1}^k D_t^r \{ {}_s^C \mathcal{D}_t^{(i-1)\alpha} \eta(0) \} D_t^{\alpha-1-r} z_i(t) &= \sum_{i=0}^{k-1} \bar{B}_i \int_0^t {}_s^C \mathcal{D}_\tau^{i\alpha} \eta(\tau) u(t-\tau) d\tau \\ &\quad - \sum_{i=0}^k M_i \int_0^t {}_s^C \mathcal{D}_\tau^{i\alpha} \eta(\tau) y(t-\tau) d\tau, \end{aligned} \quad (5.10)$$

where  $M_k = I_p$ .

**Proof.**

**Step 1. Multiplication and integration.** Let  $\eta_i \in C^{[\alpha]}([0, t])$  for  $i = 1, \dots, k$ . Then, by multiplying  $\eta_i$  to (5.7) and integrating on  $[0, t]$ , we get:

$$\begin{aligned} \int_0^t \eta_i(t-\tau) D_\tau^\alpha z_i(\tau) d\tau &= \int_0^t \eta_i(t-\tau) z_{i-1}(\tau) d\tau + \bar{B}_{i-1} \int_0^t \eta_i(t-\tau) u(\tau) d\tau \\ &\quad - M_{i-1} \int_0^t \eta_i(t-\tau) y(\tau) d\tau, \end{aligned} \quad (5.11)$$

where  $z_0$  is defined by  $z_0(\tau) = 0$  on  $[0, t]$ .

By applying the formula (2.57) given in Lemma 2.6 to the left side of (5.11), we get:

$$\int_0^t \eta_i(t-\tau) D_\tau^\alpha z_i(\tau) d\tau = \int_0^t {}^C D_\tau^\alpha \eta_i(\tau) z_i(t-\tau) d\tau + \sum_{r=0}^{[\alpha]-1} \left[ D_\tau^{\alpha-1-r} z_i(\tau) \eta_i^{(r)}(t-\tau) \right]_{\tau=0}^{\tau=t}. \quad (5.12)$$

By combining (5.11) with (5.12), we obtain: for  $i = 1, \dots, k$ ,

$$\begin{aligned} &\sum_{r=0}^{[\alpha]-1} \left[ D_\tau^{\alpha-1-r} z_i(\tau) \eta_i^{(r)}(t-\tau) \right]_{\tau=0}^{\tau=t} + \int_0^t {}^C D_\tau^\alpha \eta_i(\tau) z_i(t-\tau) d\tau \\ &= \int_0^t \eta_i(\tau) z_{i-1}(t-\tau) d\tau + \bar{B}_{i-1} \int_0^t \eta_i(\tau) u(t-\tau) d\tau - M_{i-1} \int_0^t \eta_i(\tau) y(t-\tau) d\tau. \end{aligned} \quad (5.13)$$

**Step 2. Recursive algorithm.** Let us take  $\eta = \eta_1$ , and  $\eta_i = {}^C D_\tau^{(i-1)\alpha} \eta$  for  $i = 2, \dots, k$ . Then, by combining the  $k$  equations of (5.13), we get:

$$\begin{aligned} &\sum_{r=0}^{[\alpha]-1} \sum_{i=1}^k \left[ D_\tau^r \left\{ {}^C D_\tau^{(i-1)\alpha} \eta(t-\tau) \right\} D_\tau^{\alpha-1-r} z_i(\tau) \right]_{\tau=0}^{\tau=t} \\ &= \sum_{i=0}^{k-1} \bar{B}_i \int_0^t {}^C D_\tau^{i\alpha} \eta(\tau) u(t-\tau) d\tau - \sum_{i=0}^k M_i \int_0^t {}^C D_\tau^{i\alpha} \eta(\tau) y(t-\tau) d\tau. \end{aligned} \quad (5.14)$$

Finally, let  $\eta$  satisfy  $(P_2^\alpha)$  on  $[0, t]$  by replacing  $N$  with  $k$ , the proof can be completed.  $\blacksquare$

According to the above theorem, we can derive the following results.

**Theorem 5.2** Consider the system given in (5.7) with  $\alpha \in (0, 2)$ . For  $t \in (0, h]$ , let  $g_{j', r'}$  for  $j' = 1, \dots, k$ , be  $(k, \alpha; j' - 1, r')$  fractional order modulating functions on  $[0, t]$  with  $r' = [\alpha] - 1$ , then:

$$D_t^{\alpha-[\alpha]} z_{j'}(t) = \sum_{i=0}^{k-1} \bar{B}_i \int_0^t {}^C D_\tau^{i\alpha} g_{j', r'}(\tau) u(t-\tau) d\tau - \sum_{i=0}^k M_i \int_0^t {}^C D_\tau^{i\alpha} g_{j', r'}(\tau) y(t-\tau) d\tau. \quad (5.15)$$

Moreover, in the case of  $\alpha \in (1, 2)$ , let  $g_{j', 0}$  for  $j' = 1, \dots, k$ , be  $(k, \alpha; j' - 1, 0)$  fractional order modulating functions on  $[0, t]$ , then:

$$D_t^{\alpha-1} z_{j'}(t) = \sum_{i=0}^{k-1} \bar{B}_i \int_0^t {}^C D_\tau^{i\alpha} g_{j', 0}(\tau) u(t-\tau) d\tau - \sum_{i=0}^k M_i \int_0^t {}^C D_\tau^{i\alpha} g_{j', 0}(\tau) y(t-\tau) d\tau. \quad (5.16)$$

**Proof.** The left of (5.10) can be written as follows:

$$\text{for } \alpha \in (0, 1], \text{ we have: } \sum_{i=1}^k {}^C \mathcal{D}_t^{(i-1)\alpha} \eta(0) D_t^{\alpha-1} z_i(t);$$

$$\text{for } \alpha \in (1, 2), \text{ we have: } \sum_{i=1}^k {}^C \mathcal{D}_t^{(i-1)\alpha} \eta(0) D_t^{\alpha-1} z_i(t) + \sum_{i=1}^k D_t^1 \left\{ {}^C \mathcal{D}_t^{(i-1)\alpha} \eta(0) \right\} D_t^{\alpha-2} z_i(t).$$

Hence, by replacing  $\eta$  by  $g_{j',r'}$  with  $r' = \lceil \alpha \rceil - 1$ , for  $j' = 1, \dots, k$ , in (5.10), then (5.15) can be obtained. In the case of  $\alpha \in (1, 2)$ , by replacing  $\eta$  by  $g_{j',0}$  for  $j' = 1, \dots, k$  in (5.10), then (5.16) can be obtained, which completes this proof.  $\blacksquare$

**Remark 5.3** *If  $\alpha = 1$  in (5.7) and (5.2), then the considered system is an integer order system and according to (5.15) its state is given by: for  $j' = 1, \dots, k$ ,*

$$z_{j'}(t) = \sum_{i=0}^{k-1} \bar{B}_i \int_0^t g_{j',r'}^{(i)}(\tau) u(t-\tau) d\tau - \sum_{i=0}^k M_i \int_0^t g_{j',r'}^{(i)}(\tau) y(t-\tau) d\tau. \quad (5.17)$$

*Remark that only the integer derivatives of  $g_{j',r'}$  are involved in (5.17). Hence, a sliding integration window can be adopted by replacing the lower limit 0 by  $t-T$  with  $T \in \mathbb{R}_+$  and  $T < t$ .*

Since  $\alpha - \lceil \alpha \rceil < 0$ , based on Lemma 5.1, the following corollary can be obtained using (5.15).

**Corollary 5.1** *Consider the system given in (5.7). For  $t \in (0, h]$ , let  $g_{j',r'}$  for  $j' = 1, \dots, k$ , be  $(k, \alpha; j' - 1, r')$  fractional order modulating functions on  $[0, t]$  with  $r' = \lceil \alpha \rceil - 1$ . Let  $\gamma \in \mathbb{R}$  and  $\beta = \lceil \alpha \rceil - \alpha + \gamma$ , then  $D_t^\gamma z_{j'}(t)$  can be given by:*

$$D_t^\gamma z_{j'}(t) = D_t^\beta \left( \sum_{i=0}^{k-1} \bar{B}_i \int_0^t {}^C \mathcal{D}_\tau^{i\alpha} g_{j',r'}(\tau) u(t-\tau) d\tau - \sum_{i=0}^k M_i \int_0^t {}^C \mathcal{D}_\tau^{i\alpha} g_{j',r'}(\tau) y(t-\tau) d\tau \right). \quad (5.18)$$

Consequently, as done in [76], [78], [79], algebraic formulas are obtained for the fractional derivative of the state, which contains no error in continuous-time noise-free case. In particular, different from observers, the proposed method does not rely on the initial values of the studied system. Moreover, the integral can be used as low-pass filter to reduce the impact of noise [108]. Different from [76], [78], [79],  $D_t^\gamma z_{j'}$  can be directly estimated by applying Corollary 5.1 without estimating the fractional derivatives of the output and fractional initial values. Finally, according to (5.4),  $D_t^\gamma x$  can be obtained by taking a left-inverse matrix of  $T$ . In particular, the state  $x$  can be obtained with  $\gamma = 0$ .

## 5.2.2 Construction of fractional order modulating functions

**Proposition 5.1** *Let  $k \in \mathbb{N}^*$ ,  $\alpha \in (0, 2)$ ,  $r' = \lceil \alpha \rceil - 1$ , and  $k\alpha + r' - 1 < m \in \mathbb{R}_+$ . For  $t \in (0, h]$  and  $j' \in \{1, \dots, k\}$ , let  $g_{j',r'}$  be a power function on  $[0, t]$  as follows:*

$$g_{j',r'}(\tau) = \frac{\tau^{(j'-1)\alpha+r'}}{\Gamma((j'-1)\alpha+r'+1)} + \sum_{i=1}^{(r'+1)k} \frac{\lambda_i \tau^{m+i}}{t^{m+i-(j'-1)\alpha-r'}}, \quad (5.19)$$

## 5.2. MAIN RESULTS

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where  $\lambda = (\lambda_1, \dots, \lambda_{(r'+1)k})^\top$  is the solution of the following linear system:

- In the case of  $0 < \alpha \leq 1$ , we have:  $r' = 0$ , and

$$B_{1,m}\lambda = d_{1,j'}, \quad (5.20)$$

in which  $B_{1,m} \in \mathbb{R}^{k \times k}$ ,  $d_{1,j'} \in \mathbb{R}^{k \times 1}$  are defined as : for  $s, i = 1, \dots, k$ ,

$$B_{1,m}(s, i) = \frac{\Gamma(m + i + 1)}{\Gamma(m + i + 1 - (s - 1)\alpha)}, \quad (5.21)$$

and  $d_{1,j'}(s) = -\frac{1}{\Gamma((j'-s)\alpha+1)}$  if  $s < j'$ ,  $d_{1,j'}(s) = -1$  if  $s = j'$  and  $d_{1,j'}(s) = 0$  else.

- In the case of  $1 < \alpha < 2$ , we have:  $r' = 1$ , and

$$B_{2,m}\lambda = d_{2,j'}, \quad (5.22)$$

in which  $B_{2,m} \in \mathbb{R}^{2k \times 2k}$ ,  $d_{2,j'} \in \mathbb{R}^{2k \times 1}$  are defined as : for  $s, i = 1, \dots, 2k$ ,

$$B_{2,m}(s, i) = \begin{cases} \frac{\Gamma(m + i + 1)}{\Gamma(m + i + 1 - (s - 1)\alpha)}, & \text{if } 1 \leq s \leq k, \\ \frac{\Gamma(m + i + 1)}{\Gamma(m + i - (s - k - 1)\alpha)}, & \text{if } k < s \leq 2k, \end{cases} \quad (5.23)$$

and  $d_{2,j'}(s) = -\frac{1}{\Gamma((j'-s)\alpha+2)}$  if  $s \leq j'$ ,  $d_{2,j'}(s) = 0$  if  $j' < s \leq k$ ,  $d_{2,j'}(s) = -\frac{1}{\Gamma((j'-s+k)\alpha+1)}$  if  $s < j'+k$ ,  $d_{2,j'}(s) = -1$  if  $s = j'+k$  and  $d_{2,j'}(s) = 0$  else.

Then,  $g_{j',r'}$  is a  $(k, \alpha; j' - 1, [\alpha] - 1)$  fractional order modulating functions on  $[0, t]$  satisfying  $(P_1^\alpha) - (P_3^\alpha)$ .

**Proof.** According to the definition of  $g_{j',r'}$  with  $k\alpha + r' - 1 < m \in \mathbb{R}_+$ ,  $(P_1^\alpha)$  is fulfilled, and using (2.45) we get: for  $(j, r) \in \Phi_{k,\alpha}$  ( $0 \leq j \leq k - 1$ ,  $0 \leq r \leq [\alpha] - 1$ ),  $\forall \tau \in [0, t]$ ,

$$D_\tau^r \{ {}_s^C \mathcal{D}_\tau^{j\alpha} g_{j',r'}(\tau) \} = f(\tau) + \sum_{i=1}^{(r'+1)k} \frac{\lambda_i \tau^{m+i-j\alpha-r}}{t^{m+i-(j'-1)\alpha-r'}} \frac{\Gamma(m+i+1)}{\Gamma(m+i+1-r-j\alpha)}, \quad (5.24)$$

where

$$f(\tau) = \begin{cases} 0, & \text{if } j > j' - 1, \\ 1, & \text{if } j = j' - 1 \text{ and } r = r', \\ \frac{\tau^{(j'-1-j)\alpha+r'-r}}{\Gamma((j'-1-j)\alpha+r'-r+1)}, & \text{else.} \end{cases} \quad (5.25)$$

Hence, by taking  $\tau = 0$  in (5.24), it yields  $D_\tau^r \{ {}_s^C \mathcal{D}_\tau^{j\alpha} g_{j',r'}(0) \} = 1$  if  $(j, r) = (j' - 1, r')$  and  $D_\tau^r \{ {}_s^C \mathcal{D}_\tau^{j\alpha} g_{j',r'}(0) \} = 0$  else, *i.e.*  $g_{j',r'}$  satisfies  $(P_3^\alpha)$ . Finally, by taking  $\tau = t$  in (5.24) and using (5.25), we get:

$$D_t^r \{ {}_s^C \mathcal{D}_t^{j\alpha} g_{j',r'}(t) \} = f(t) + t^{(j'-1-j)\alpha+r'-r} \sum_{i=1}^{(r'+1)k} \lambda_i \frac{\Gamma(m+i+1)}{\Gamma(m+i+1-r-j\alpha)}. \quad (5.26)$$

### 5.3. ERROR ANALYSIS

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Consequently, according to (5.26) and (5.25), if  $\lambda$  is the solution of the system given in (5.20) – (5.21) where  $j = s - 1$  in the case of  $0 < \alpha \leq 1$  and the one given in (5.22) – (5.23) where  $j = s - 1$  in the case of  $1 < \alpha < 2$ , then  $(P_2^\alpha)$  is fulfilled. Thus, this proof is completed.  $\blacksquare$

Remark that the modulating functions required in Theorem 5.2 need to be updated with respect to  $t$ . Hence, the modulating functions constructed in Proposition 5.1 depend on  $t$ . However, thanks to the terms of  $\frac{1}{t^{m+i-(j'-1)\alpha-r'}}$  in (5.19), the systems in (5.20) and (5.22) are independent of  $t$ , so  $\lambda_i$  are also. Similar idea was adopted in [78], [79].

**Remark 5.4** *According to Remark 5.2, in the case of  $\alpha = 1$ , the constructed modulating functions is a right-sided  $(j', k)^{th}$  order modulating functions on  $[0, t]$ . It can also be constructed by the method proposed in [80].*

Note that  $m$  is a parameter to built modulating functions, whose role will be shown in the sequel.

### 5.3 Error analysis

Denote  $y^\varpi$  as the discrete noisy observation of  $y$  on  $I = [0, h]$ :

$$y^\varpi(t_{i'}) = y(t_{i'}) + \varpi(t_{i'}), \quad (5.27)$$

where  $t_{i'} = i' T_s$  with  $T_s = \frac{h}{S}$ , for  $i' = 0, 1, \dots, S$ , and  $\varpi = (\varpi_1, \dots, \varpi_p)^\top$  is the corrupting noise vector. According to Theorem 5.2,  $D_t^{\alpha - [\alpha]} z$  is expressed by integrals related to  $y$  and  $u$ . The form involving  $y$  in (5.15) can be simplified as:  $\forall t \in (0, h]$ ,

$$I_G^y(t) := \int_0^t \sum_{i=0}^k M_i G_i(t - \tau) y(\tau) d\tau, \quad (5.28)$$

where  $G_i(\tau) = -{}_s^C D_\tau^{i\alpha} g_{j', r'}(\tau)$ .

Then, for  $i' = 0, 1, \dots, S$ ,  $I_G^y(t_{i'})$  can be approximated by:

$$I_G^{y, i'} := T_s \sum_{j'=0}^{i'} w_{j'} \sum_{i=0}^k M_i G_i(t_{i'} - j') y(t_{j'}), \quad (5.29)$$

where  $w_{j'} \in \mathbb{R}_+ \cup \{0\}$  are the weights for a given numerical integration method. Hence, by taking the discrete noisy observation  $y^\varpi$ , we obtain:

$$I_G^{\varpi, i'} = I_G^{y, i'} + e_G^{\varpi, i'}, \quad (5.30)$$

where

$$e_G^{\varpi, i'} := T_s \sum_{j'=0}^{i'} \sum_{i=0}^k M_i G_i(t_{i'} - j') w_{j'} \varpi(t_{j'}). \quad (5.31)$$



### 5.3. ERROR ANALYSIS

Consequently, this method has both numerical errors and noise errors. On the one hand, it is well-known that the numerical errors converge to zero as  $T_s$  tends to zero. On the other hand, according to [116], the noise errors can be reduced by decreasing  $T_s$ . In this paper, we assume that  $\{\varpi(t), t \in I\}$  is the continuous independent zero-mean stochastic process with a bounded variance function. That is,  $\exists \delta_j \in \mathbb{R}_+, \forall t \in I, \text{Var}[\varpi_j(t)] \leq \delta_j$ , where  $\text{Var}[\cdot]$  is a variance function.

Hence, using (5.31) it is easy to obtain that  $E[e_G^{\varpi, i'}] = 0$ . In order to calculate  $\text{Var}[e_G^{\varpi, i'}]$ ,  $e_G^{\varpi, i'}$  is rewritten as follows:

$$\begin{aligned} e_G^{\varpi, i'} &= T_s \sum_{i=0}^k (M_i \otimes G_i^{i'}) \times ((E_p \otimes w) \circ \varpi^{i'}) \\ &= T_s \left( \sum_{i=0}^k M_i \otimes G_i^{i'} \right) \times ((E_p \otimes w) \circ \varpi^{i'}), \end{aligned} \quad (5.32)$$

where  $\otimes$  refers to the Kronecker product [117],  $\circ$  refers to the Hadamard product (or element-wise product) [117], and

- $G_i^{i'} = (G_i(t_{i'}), \dots, G_i(t_0))$ ,
- $E_p = (1, \dots, 1)^\top \in \mathbb{R}^p$ ,
- $w = (w_0, \dots, w_{i'})^\top \in \mathbb{R}^{i'+1}$ ,
- $\varpi^{i'} = (\varpi_1(t_0), \dots, \varpi_1(t_{i'}), \dots, \varpi_p(t_0), \dots, \varpi_p(t_{i'}))^\top \in \mathbb{R}^{(i'+1)p}$ .

Thus, we get:

$$\begin{aligned} \text{Var}[e_G^{\varpi, i'}] &= T_s^2 \left\{ \left( \sum_{i=0}^k M_i \otimes G_i^{i'} \right) \circ \left( \sum_{i=0}^k M_i \otimes G_i^{i'} \right) \right\} \times \left\{ (E_p \otimes (w \circ w)) \circ \text{Var}[\varpi^{i'}] \right\} \\ &\leq \delta_y T_s^2 \left\{ \left( \sum_{i=0}^k M_i \otimes G_i^{i'} \right) \circ \left( \sum_{i=0}^k M_i \otimes G_i^{i'} \right) \right\} \times \{E_p \otimes (w \circ w)\}, \end{aligned} \quad (5.33)$$

where  $\delta_y = \max_{1 \leq j \leq p} \{\delta_j\}$ .

Then, in similar way as done in [77], the convergence of  $e_G^{\varpi, i'}$  as  $T_s$  tends to zero can be deduced. Moreover, with the help of the Bienaymé-Chebyshev inequality,  $e_G^{\varpi, i'}$  can be bounded as: for  $\kappa \in \mathbb{R}_+$ ,

$$\left| e_G^{\varpi, i'} \right| \stackrel{p_\kappa}{\leq} \kappa \delta_y^{\frac{1}{2}} T_s \times \sqrt{\left\{ \left( \sum_{i=0}^k M_i \otimes G_i^{i'} \right) \circ \left( \sum_{i=0}^k M_i \otimes G_i^{i'} \right) \right\} \times \{E_p \otimes (w \circ w)\}}, \quad (5.34)$$

in which  $f_1 \stackrel{p_\kappa}{<} f_2$  ( $f_1, f_2 \in \mathbb{R}$ ) denotes the probability of  $f_1 < f_2$ , equals to  $p_\kappa$  with  $p_\kappa \geq 1 - \frac{1}{\kappa^2}$ .

Since the parameter  $m$  in the constructed modulating functions affects the noise error bound provided in (5.34), it will be shown in next section that, with the help of the error bound, the influence of noise will be suppressed by selecting suitable values of the designed parameter  $m$ . Moreover, since the error bound is linear with respect to  $\kappa \delta_y^{\frac{1}{2}}$ , the location of its minimum does not depend on the value of  $\kappa \delta_y^{\frac{1}{2}}$ . Consequently, this analysis can be done in an off-line work without knowing the noise levels.

## 5.4 Simulation results

Some examples both on integer order and non-integer order systems are presented to verify the efficiency of improved modulating functions method in this section.

The noisy outputs are considered:  $y^\varpi(t_{i'}) = y(t_{i'}) + \sigma\varpi(t_{i'})$  on  $I = [0, 5]$ , where  $t_{i'} = i'T_s$  with  $T_s = 0.0001$  for  $i' = 0, \dots, 50000$ ,  $\{\varpi(t_{i'})\}$  are zero-mean white Gaussian noises, and  $\sigma$  is set such that signal-to-noise ratio  $\text{SNR} = 10 \log_{10} \left( \frac{\sum |y^\varpi(t_{i'})|^2}{\sum |\sigma\varpi(t_{i'})|^2} \right)$  is equal to  $\rho$  dB. Moreover, the integral of the proposed formula is also approximated using the trapezoidal numerical integration method.

### Example 5.1 Integer order system with multiple inputs and multiple outputs

We take a integer order multiple inputs and multiple outputs system with the following parameters in (5.2):  $\alpha = 1$ ,  $n = 4$ ,  $p = 2$ ,  $q = 2$ , and

$$A = \begin{pmatrix} -2 & 0 & 1 & -1 \\ 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 1 \end{pmatrix}, B = \begin{pmatrix} 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}^\top, C = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

which satisfy the condition  $(\widetilde{C}_1)$  with  $\text{rank}(Q_2) = 4$  and  $k = 2$ . Then, by taking  $u(t) = (\sin(5t), \cos(5t))^\top$ ,  $x(0) = (-0.1351, -0.2990, -0.1546, -0.1633)^\top$ , and  $\rho = 18$ , the noisy output observations are plotted in Fig. 5.2.

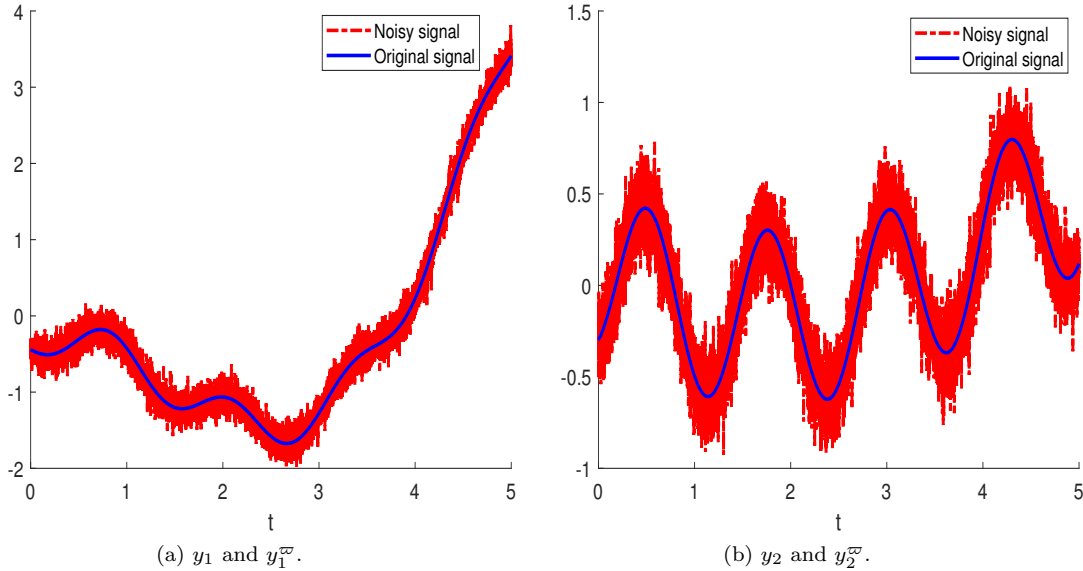


Figure 5.2: Output  $y$  and its discrete noisy observation in Example 5.1.

5.4. SIMULATION RESULTS

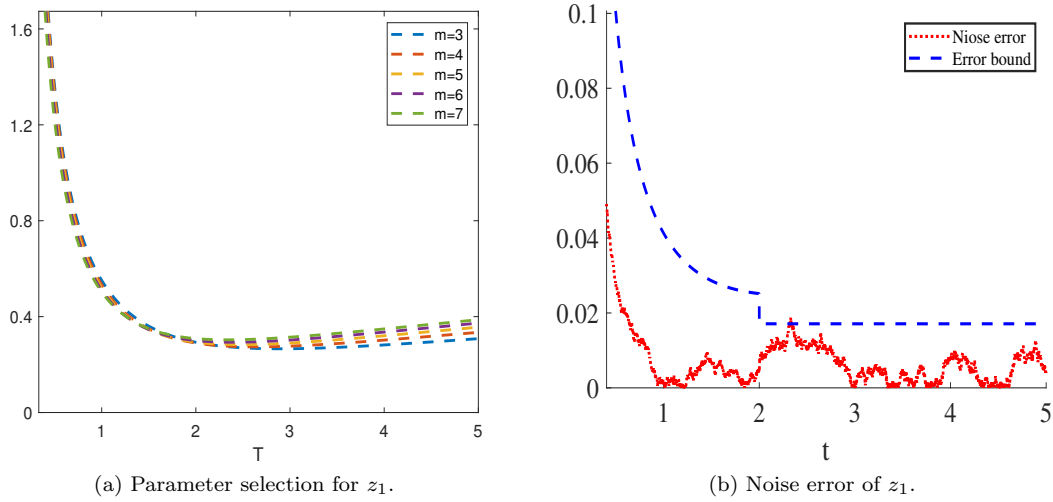


Figure 5.3: Noise error bound and noise error of  $z_1$  in Example 5.1.

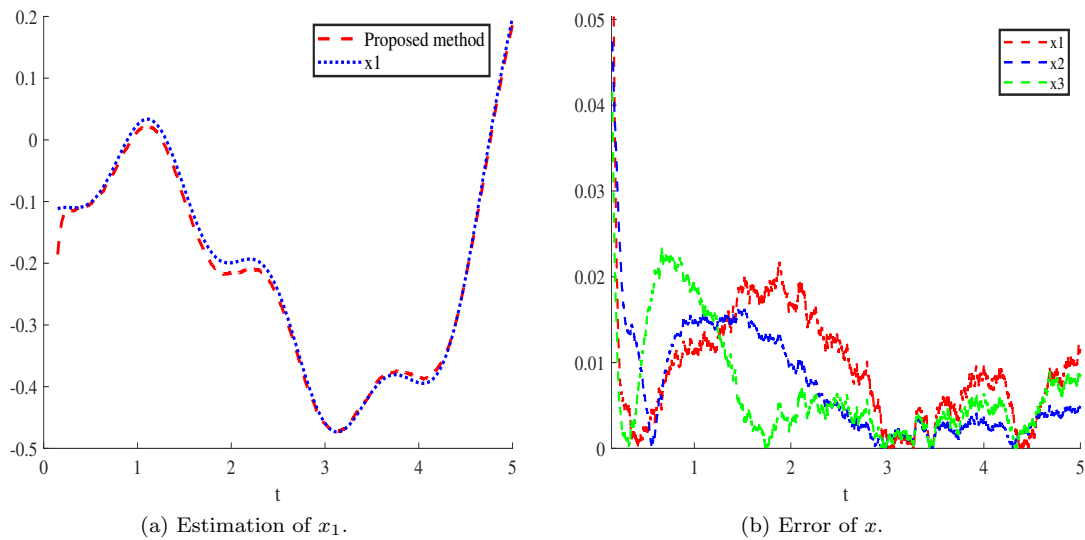


Figure 5.4: Estimation of  $x$  in Example 5.1.

## 5.4. SIMULATION RESULTS

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The objective is to estimate the state  $x$  of the integer order linear system given in (5.2) using  $x = T^L z$ , where  $z$  will be estimated by applying (5.17) with a sliding integration window of length  $T$ , and the required modulating functions are constructed by employing Proposition 5.1. First, it is demonstrated how to choose the design parameters  $T$  and  $m$  using the noise error bound given in (5.34). For this, we take  $\kappa = 3, \delta = 1$  which is unknown in practice, then we discuss the variations of the error bounds with respect to  $m$  and  $T$ , choose the best value that minimize them. In Fig. 5.3 (a), the variation of the error bound for  $z_1$  is shown, where  $m = 3, \dots, 7$ ,  $T = 0.2, 0.2 + T_s, \dots, 5$ . Hence, the following values are chosen to estimate  $z_1$ : when  $t \in [T_s, 2]$ ,  $m = 7$  and  $T = i'T_s$ , then  $m = 3$  and  $T = 2$ . With these parameter values, the estimated  $x_1$  is presented in Fig. 5.4 (a). Similarly,  $x_2, x_3$  and  $x_4$  are estimated, and Fig. 5.4 (b) shows the estimation error of  $x$ . Consequently, it can be seen that the proposed method provides reliable estimation results against the corrupting noises. Moreover, in order to verify the efficiency of the proposed error bound, the associated noise errors and the corresponding error bounds of  $z_1$  obtained by taking  $\delta = 0.12, \kappa = 2$  are shown in Fig. 5.3 (b).

Consequently, it can be seen that the proposed modulating functions method has robust state estimations with fast convergence.

### Example 5.2 Rational order system with single input and single output

To compare with the method in [79] which was designed in the case with single input and single output, we take a fractional order system considered in [79] with the following parameters in (5.2):  $\alpha = \frac{5}{8}$ ,  $n = 3$ ,  $p = 1$ ,  $q = 1$ , and

$$A = \begin{pmatrix} 11 & 1 & -11 \\ 6 & 0 & -6 \\ 18 & 1 & -17 \end{pmatrix}, B = (0, 1, 0)^\top, C = (-1, 0, 1),$$

which satisfy the condition  $(\widetilde{C}_1)$  with  $\text{rank}(Q_3) = 3$  and  $k = 3$ . Moreover, we take  $u(t) = \sin(5t)$ ,  $x(0) = (-0.1297, -0.3234, -0.1470)^\top$ ,  $\rho = 15$ .

It is expected to estimate the state  $x$  from the noisy output observation using the method in [79] and the proposed method by using  $x = T^L z$  and (5.18) with  $\gamma = 0$  and  $\beta = \frac{3}{8}$ . Different from the integer order case in the previous example, no sliding integration window is adopted in fractional cases. Hence, there is only one design parameter  $m$  which can be selected by employing the noise error bound as done in Example 1. Then, the estimated  $x_1$  and the associated estimation errors obtained by the new modulating functions method and the one proposed in [79] are provided in Fig. 5.5 and Fig. 5.6, where  $m = 3$ . The estimation values of  $x_2$  and  $x_3$  can be obtained in a similar way.

## 5.4. SIMULATION RESULTS

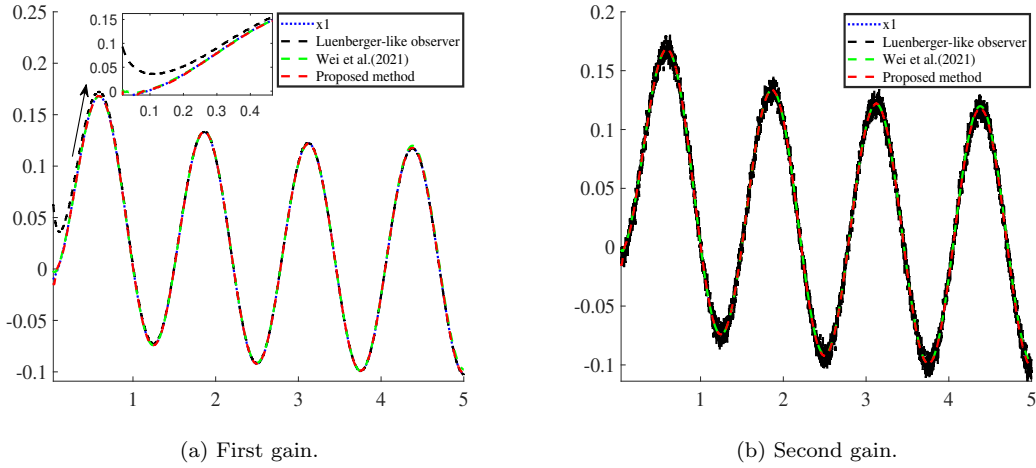


Figure 5.5: Estimations of  $x_1$  obtained by the fractional order Luenberger-like observer (with two gains), the proposed method and the one in [79].

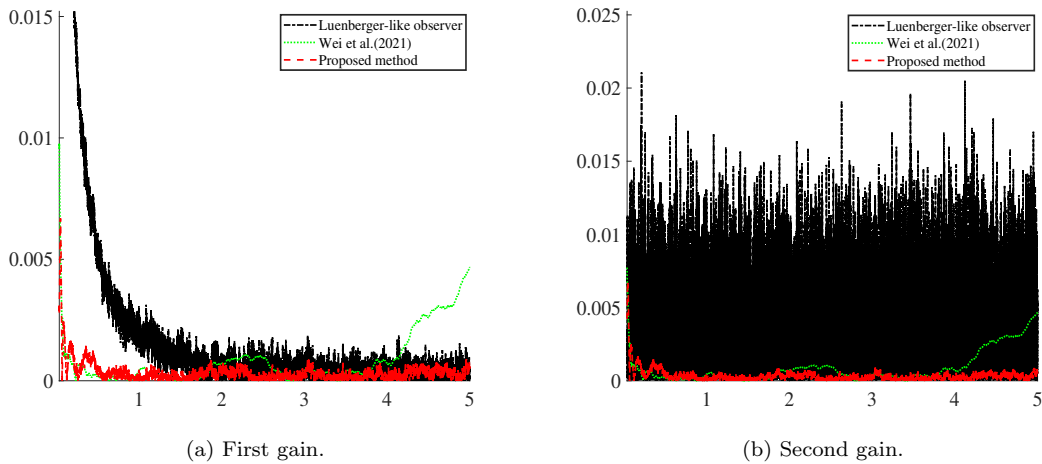


Figure 5.6: Estimation errors of  $x_1$  obtained by the fractional order Luenberger-like observer (with two gains), the proposed method and the one in [79].

## 5.4. SIMULATION RESULTS

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According to the comparison results, the advantages of robustness against noises and fast convergence of these two methods can be observed. However, compared with the method in [79], the proposed new modulating functions method is more accurate. Indeed, the estimation process of the proposed method is easier than the method in [79]. For instance, a set of initial values need to be estimated by the method in [79], which accumulates estimation errors. Consequently, the proposed method is an improvement of the previous work and has potential application prospects.

Moreover, the fractional order Luenberger-like observer is considered to compared with the proposed method. The initial value of the fractional order Luenberger-like observer is taken by  $\bar{x}(0) = (1, 2, 1)^\top$ . We choose two gains. The first gain is chosen by taking the eigenvalues to be 2 times greater than the ones of the system in order to guaranty the robustness against the noise. The second gain is chosen by taking the eigenvalues to be 7 times greater than the ones of the system in order to improve the convergence speed, which produces larger noise error contribution. The estimation results of  $x_1$  are given in Fig. 5.5 and Fig. 5.6, which show the fast estimation ability of the proposed new modulating functions method as well as its robustness against noises. Compared with the proposed new modulating functions method, it is difficult for the fractional order Luenberger-like observer to attend to both robustness against corrupting noises and fast convergence.

### Example 5.3 Fractional order system with multiple inputs and multiple outputs

We take a fractional order multiple inputs and multiple outputs system with the following parameters in (5.2):  $\alpha = \frac{\sqrt{7}}{3}$ ,  $n = 4$ ,  $p = 2$ ,  $q = 2$ , and

$$A = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 1 & -1 & 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}^\top, C = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix},$$

which satisfy the condition  $(\widetilde{C}_1)$  with  $\text{rank}(Q_2) = 4$  and  $k = 2$ . Moreover, we take  $u(t) = (e^{\frac{t}{3}} \sin(10t), e^{\frac{t}{2}} \cos(10t))^\top$ ,  $x(0) = (-0.1, -0.2, -0.3, -0.4)^\top$ , and  $\rho = 25$ . In this example, the fractional integrals and derivatives of the state  $x$  are estimated by utilizing (5.18) for  $\gamma = -0.2, 0, 0.3$ . The obtained estimation results of  $D_t^\gamma x_1$  for  $\gamma = -0.2, 0, 0.3$  are given in Fig. 5.7, where  $m = 3$ . Moreover, the above estimation errors of  $D_t^\gamma x_1$  with different  $\gamma$  are shown in Fig. 5.8.

Moreover, the fractional order Luenberger-like observer is considered to compared with the proposed method, in which the initial value of the fractional order Luenberger-like observer is taken by  $\bar{x}(0) = (-0.5, -0.1, -0.1, -0.2)^\top$ . The gain is chosen by taking the eigenvalues to be twenty times greater than the ones of the system. Since the Luenberger-like observer cannot estimate the fractional integral and derivative of the state, so we use the Grünwald-Letnikov scheme for the calculation, the estimation and estimation error are shown in Fig. 5.7 and Fig. 5.8, respectively. Consequently, according to

## 5.4. SIMULATION RESULTS

the comparison results, it can be seen that the proposed method efficiently provide good noise-tolerant performance.

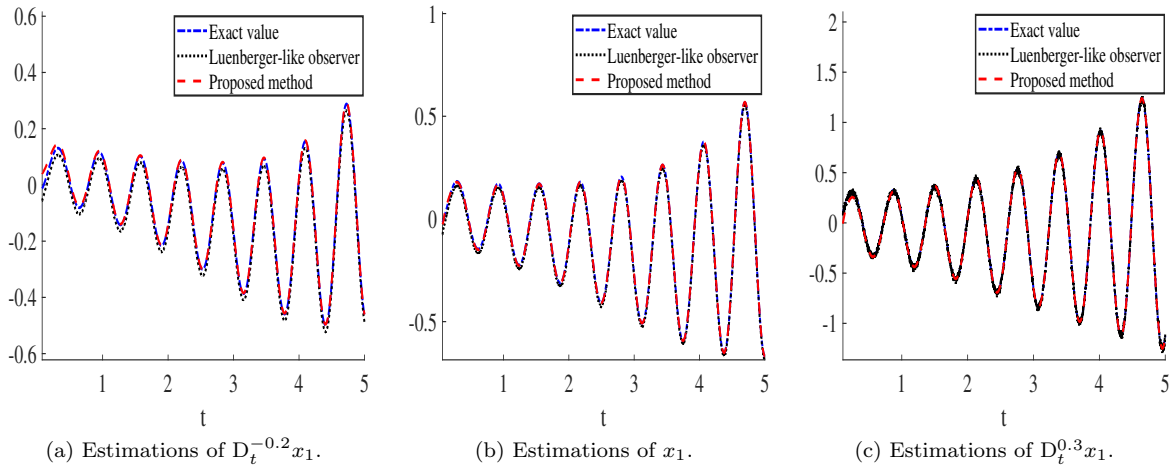


Figure 5.7: Estimations of  $D_t^\gamma x_1$  with different  $\gamma$  in Example 5.3.

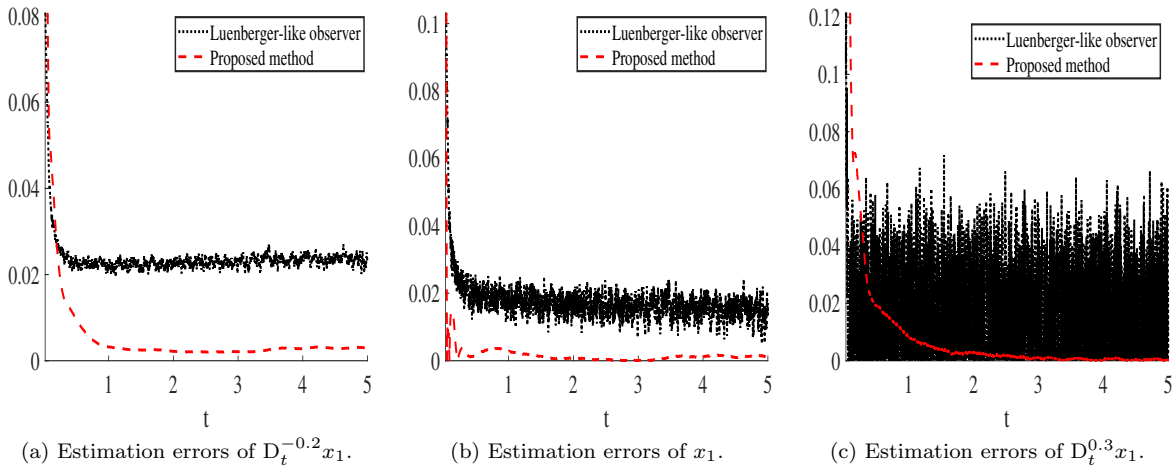


Figure 5.8: Estimation errors of  $D_t^\gamma x_1$  with different  $\gamma$  in Example 5.3.

## 5.5 Conclusions

In the chapter, an improved modulating functions-based state estimation method was addressed, which can be applied to fractional integral estimation of the state for fractional order systems with multiple inputs and multiple outputs in a noisy environment. Moreover, the fractional derivatives of the state are estimated using the Grünwald-Letnikov scheme. Compared with the existing modulating functions methods which were based on fractional differential equation, this method was directly applied to the fractional order observable canonical form. Hence, simpler algebraic integral formulas were obtained by introducing fractional order modulating functions. In particular, different from [76], [78], [79], the fractional order initial values did not need to be estimated. After giving the construction of the required fractional order linear systems, error analysis in discrete noisy cases is addressed. Finally, the effectiveness of the proposed modulating functions method was verified by simulation examples. This work has been published in [118].



## 5.5. CONCLUSIONS

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## Part IV

# Non-asymptotic and robust estimations for fractional integral and derivative of state



## Chapter 6

# Fractional integral and derivative estimations of state for integer order systems

Taking into account the existing works on the modulating functions method [111], [112], [118] as well as the proposed modulating functions method in Chapter 4 where the fractional derivatives of the state are obtained via the Grünwald-Letnikov scheme, in order to directly obtain the algebraic integral formulas of the fractional derivatives and integrals of the states, an improved modulating functions-based fractional derivative and integral of the states estimation method is advanced for integer order systems in this chapter, which is non-asymptotic and robust. For this purpose, the considered system modeled by the state space representation model is firstly transformed into the observable canonical form and then cleverly transformed into a fractional order system. Secondly, based on that, the modulating functions are applied to obtain algebraic integral formulas for fractional derivative and integral of the state directly. These integral formulas can give robust estimations in discrete noisy cases. Particularly, the estimation process involves two distinct approaches to deal with initial value terms—one involves their removal through modulating function design, while the other entails a two-step process: first estimating the initial value terms and subsequently estimating the derivatives and integrals of the state. Besides, it is shown how to design the required modulating functions. Finally, numerical simulations are provided to demonstrate the efficiency of the proposed method in this chapter. Furthermore, we consider applications, *i.e.* we propose a novel modulating functions-based state fractional derivative estimator for DC motor systems. The main contributions of this chapter are outlined as follows:

- Different from [74], [76]–[79], [97], the considered systems do not need to be transformed into the input-output differential equations, which makes the estimation process and the obtained formulas simpler.

- Unlike existing work on the modulating functions method [76]–[79], [113], this chapter can directly obtain algebraic integral formulas for fractional order integral and derivative of the states without estimating the fractional initial values of the output. It is worth mentioning that the fractional order derivative and integral of the state in [111], [112], [118] as well as Part III were estimated by applying the Grünwald-Letnikov scheme to the state  $x$  obtained.
- A novel modulating functions-based estimator is applied for the first time to estimate the current, speed and their fractional derivatives and integrals for DC motor, which is non-asymptotic and robust. Moreover, the efficiency of the presented modulating functions-based estimator is verified by Matlab simulation and comparison with the Lunberger observer. Moreover, we designed a fractional order  $PI^\lambda D^\mu$  controller for the DC motor and obtained good control results.

## 6.1 Problem formulation

In this chapter, we consider the integer order linear system (4.1), which can be transformed into the observable canonical form (4.2) by using the same ideas as Section 4.1. Let us recall the observable canonical form as follows:

$$\begin{cases} \dot{z} &= \hat{A}z + \hat{B}u, \\ y &= \hat{C}z, \end{cases} \quad (6.1)$$

where

$$\hat{A} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & -a_{N-2} \\ 0 & 0 & \cdots & 0 & 1 & -a_{N-1} \end{pmatrix}, \quad (6.2)$$

and  $\hat{B} = TB = (\hat{B}_0, \hat{B}_1, \dots, \hat{B}_{N-1})^\top \in \mathbb{R}^{N \times 1}$ ,  $\hat{C} = (0, \dots, 0, 1) \in \mathbb{R}^{1 \times N}$ .

Before using modulating functions and presenting the algebraic integral formulas of the state and its derivative and integral of the considered system (4.1), the following lemma is provided, which is an important step in this work.

**Lemma 6.1** *Let  $\alpha \in (-1, 1)$ , based on the observable canonical form (6.1) of the system defined in (4.1), the following system can be obtained:  $\forall 0 < t \in I$ ,*

$$D_t^{1+\alpha} z(t) = \hat{A}D_t^\alpha z(t) + \hat{B}D_t^\alpha u(t) + \frac{z(0)}{\Gamma(-\alpha)} t^{-\alpha-1}. \quad (6.3)$$

**Proof.** By taking the  $\alpha$ -order fractional order derivative operator with  $\alpha \in (-1, 1)$  on both sides of (6.1), we get:

$$D_t^\alpha \dot{z}(t) = \hat{A}D_t^\alpha z(t) + \hat{B}D_t^\alpha u(t). \quad (6.4)$$

Then, by applying the additive index law given in Lemma 2.1 to the left side of (6.4), this proof can be completed. ■

**Remark 6.1** *Notably, different from [111], [112], [118], this work estimate directly the fractional derivative and integral of the state without applying the Grünwald-Letnikov scheme, which is more efficient. Indeed, comparing to the existing modulating functions method [73], [74], [76]–[79], the proposed method eliminates the need for conversion into the form of a differential equation, thereby simplifying the estimation process and enhancing the accuracy of the estimation results.*

## 6.2 Main results

In this section, by applying modulating functions, the algebraic integral formulas of fractional derivatives and integrals of the state for integer order systems (4.1) will be provided directly in the following text without using Grünwald-Letnikov scheme. In the estimation process, two approaches are involved to deal with the initial values terms—one involves their removal through modulating function design, while the other entails a two-step process: first estimating initial value terms and subsequently estimating fractional derivatives and integrals of the state.

### 6.2.1 First way—Remove initial value term using modulating function design

**Theorem 6.1** *Let us consider the observable canonical form (6.1) of the system (4.1) on  $I$ . For  $t \in I$ ,  $\alpha \in (-1, 1)$ , the beneficial property of the modulating function  $\eta_j$  for  $j = 1, \dots, N$  is introduced as follows:*

$$(P_4^\alpha) : \forall k \in \{0, \dots, N-1\}, \int_0^t \eta_j^{(k)}(t-\tau) \tau^{-\alpha-1} d\tau = 0.$$

Then,  $D_t^\alpha z$  with  $\alpha \in (-1, 1)$  can be computed as follows:

- In the case of  $\alpha \in (0, 1)$ , let  $\eta$  be a right-sided  $(N, N+1)^{th}$  order modulating function on  $[0, t]$  satisfying  $(P_4^\alpha)$ , then the fractional integral of  $y$  can be given by:

$$D_t^{\alpha-1} y(t) = \sum_{i=1}^N \hat{B}_{i-1} \int_0^t {}^C D_\tau^\alpha \{\eta^{(i-1)}(\tau)\} u(t-\tau) d\tau - \sum_{i=0}^N a_i \int_0^t {}^C D_\tau^\alpha \{\eta^{(i)}(\tau)\} y(t-\tau) d\tau, \quad (6.5)$$

where  $a_N = 1$ .

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Moreover, let  $\eta_j$  be a group of right-sided  $(j - 1, N + 1)^{\text{th}}$  order modulating functions on  $[0, t]$  for  $j = 1, \dots, N$  satisfying  $(P_4^\alpha)$ , the following formula holds:

$$\begin{aligned} D_t^\alpha z_j(t) &= \hat{B}_{j-1} D_t^{\alpha-1} u(t) + \sum_{i=1}^N \hat{B}_{i-1} \int_0^t {}^C D_\tau^\alpha \{\eta_j^{(i-1)}(\tau)\} u(t - \tau) d\tau \\ &\quad - a_{j-1} D_t^{\alpha-1} y(t) - \sum_{i=0}^N a_i \int_0^t {}^C D_\tau^\alpha \{\eta_j^{(i)}(\tau)\} y(t - \tau) d\tau, \end{aligned} \quad (6.6)$$

where  $D_t^{\alpha-1} y(t)$  is given by (6.5).

• In the case of  $\alpha \in (-1, 0)$ , let  $h_j$  be a group of right-sided  $(j - 1, N)^{\text{th}}$  order modulating functions on  $[0, t]$  for  $j = 1, \dots, N$  satisfying  $(P_4^\alpha)$ , the following formula holds:

$$D_t^\alpha z_j(t) = \sum_{i=1}^N \hat{B}_{i-1} \int_0^t D_\tau^\alpha \{h_j^{(i-1)}(\tau)\} u(t - \tau) d\tau - \sum_{i=0}^N a_i \int_0^t D_\tau^\alpha \{h_j^{(i)}(\tau)\} y(t - \tau) d\tau, \quad (6.7)$$

where  $a_N = 1$ .

**Proof.** The proof can be given in the following steps.

**Step 1. Multiplication and integration.**

By multiplying  $\eta^{(i-1)}$  for  $i = 1, \dots, N$  to (6.3) and integrating on  $[0, t]$ , the following formula holds: for  $i = 1, \dots, N$ ,

$$\begin{aligned} \int_0^t \eta^{(i-1)}(t - \tau) D_\tau^{1+\alpha} z_i(\tau) d\tau &= \int_0^t \eta^{(i-1)}(t - \tau) D_\tau^\alpha z_{i-1}(\tau) d\tau + \hat{B}_{i-1} \int_0^t \eta^{(i-1)}(t - \tau) D_\tau^\alpha u(\tau) d\tau \\ &\quad - a_{i-1} \int_0^t \eta^{(i-1)}(t - \tau) D_\tau^\alpha y(\tau) d\tau + \frac{z_i(0)}{\Gamma(-\alpha)} \int_0^t \eta^{(i-1)}(t - \tau) \tau^{-\alpha-1} d\tau, \end{aligned} \quad (6.8)$$

where  $z_0 = 0$ . We know that  $\int_0^t \eta^{(i-1)}(t - \tau) D_\tau^{1+\alpha} z_i(\tau) d\tau = \int_0^t \eta^{(i-1)}(t - \tau) D_\tau^1 \{D_\tau^\alpha z_i(\tau)\} d\tau$ , then by applying the integration by parts formula on the left side of (6.8), we get: for  $i = 1, \dots, N$ ,

$$\begin{aligned} \left[ \eta^{(i-1)}(t - \tau) D_\tau^\alpha z_i(\tau) \right]_{\tau=0}^{\tau=t} + \int_0^t \eta^{(i)}(\tau) D_\tau^\alpha z_i(t - \tau) d\tau &= \int_0^t \eta^{(i-1)}(t - \tau) D_\tau^\alpha z_{i-1}(\tau) d\tau \\ &\quad + \hat{B}_{i-1} \int_0^t \eta^{(i-1)}(t - \tau) D_\tau^\alpha u(\tau) d\tau - a_{i-1} \int_0^t \eta^{(i-1)}(t - \tau) D_\tau^\alpha y(\tau) d\tau \\ &\quad + \frac{z_i(0)}{\Gamma(-\alpha)} \int_0^t \eta^{(i-1)}(t - \tau) \tau^{-\alpha-1} d\tau, \end{aligned} \quad (6.9)$$

where  $\int_0^t \eta^{(i)}(\tau) D_\tau^\alpha z_i(t - \tau) d\tau = \int_0^t \eta^{(i)}(t - \tau) D_\tau^\alpha z_i(\tau) d\tau$ .

Then, by combining  $N$  equations of (6.9), we can obtain:

$$\begin{aligned} \sum_{i=1}^N \left[ \eta^{(i-1)}(t-\tau) D_{\tau}^{\alpha} z_i(\tau) \right]_{\tau=0}^{\tau=t} &= \sum_{i=1}^N \hat{B}_{i-1} \int_0^t \eta^{(i-1)}(t-\tau) D_{\tau}^{\alpha} u(\tau) d\tau \\ &\quad - \sum_{i=0}^N a_i \int_0^t \eta^{(i)}(t-\tau) D_{\tau}^{\alpha} y(\tau) d\tau \\ &\quad + \sum_{i=1}^N \frac{z_i(0)}{\Gamma(-\alpha)} \int_0^t \eta^{(i-1)}(t-\tau) \tau^{-\alpha-1} d\tau. \end{aligned} \quad (6.10)$$

**Step 2. Eliminate the derivatives or integrals of state, input and output.**

• First case  $\alpha \in (0, 1)$ : In this case, using  $(P_4^{\alpha})$  and by applying the fractional order integration by parts formula to the right side of (6.10), then (6.10) becomes:

$$\begin{aligned} &\sum_{i=1}^N \left[ \eta^{(i-1)}(t-\tau) D_{\tau}^{\alpha} z_i(\tau) \right]_{\tau=0}^{\tau=t} \\ &= \sum_{i=1}^N \hat{B}_{i-1} \left[ \eta^{(i-1)}(t-\tau) D_{\tau}^{\alpha-1} u(\tau) \right]_{\tau=0}^{\tau=t} + \sum_{i=1}^N \hat{B}_{i-1} \int_0^t {}^C D_{\tau}^{\alpha} \{ \eta^{(i-1)}(\tau) \} u(t-\tau) d\tau \\ &\quad - \sum_{i=0}^N a_i \left[ \eta^{(i)}(t-\tau) D_{\tau}^{\alpha-1} y(\tau) \right]_{\tau=0}^{\tau=t} - \sum_{i=0}^N a_i \int_0^t {}^C D_{\tau}^{\alpha} \{ \eta^{(i)}(\tau) \} y(t-\tau) d\tau. \end{aligned} \quad (6.11)$$

Since  $\eta$  satisfies  $(P_3)$ , we get:

$$\begin{aligned} \sum_{i=1}^N \eta^{(i-1)}(0) D_t^{\alpha} z_i(t) &= \sum_{i=1}^N \hat{B}_{i-1} \eta^{(i-1)}(0) D_t^{\alpha-1} u(t) + \sum_{i=1}^N \hat{B}_{i-1} \int_0^t {}^C D_{\tau}^{\alpha} \{ \eta^{(i-1)}(\tau) \} u(t-\tau) d\tau \\ &\quad - \sum_{i=0}^N a_i \eta^{(i)}(0) D_t^{\alpha-1} y(t) - \sum_{i=0}^N a_i \int_0^t {}^C D_{\tau}^{\alpha} \{ \eta^{(i)}(\tau) \} y(t-\tau) d\tau. \end{aligned} \quad (6.12)$$

Thanks to  $(P_5)$ , the proof of (6.5) can be completed.

According to (6.12), by replacing  $\eta$  by  $\eta_j$  for  $j = 1, \dots, N$  in (6.12), and then due to the properties of right-sided  $(j-1, N+1)^{th}$  order modulating function and  $(P_4^{\alpha})$ , we can easily get (6.6) where  $D_t^{\alpha-1} y(t)$  is given in (6.5).

• Second case  $\alpha \in (-1, 0)$ : In this case, thanks to (6.10), using  $(P_4^{\alpha})$  and by applying the fractional order integration by parts formula given by Lemma 2.6, then by replacing  $\eta$  with  $h_j$  for  $j = 1, \dots, N$ , (6.10) becomes:

$$\sum_{i=1}^N \left[ h_j^{(i-1)}(t-\tau) D_{\tau}^{\alpha} z_j(\tau) \right]_{\tau=0}^{\tau=t} = \sum_{i=1}^N \hat{B}_{i-1} \int_0^t D_{\tau}^{\alpha} \{ h_j^{(i-1)}(\tau) \} u(t-\tau) d\tau - \sum_{i=0}^N a_i \int_0^t D_{\tau}^{\alpha} \{ h_j^{(i)}(\tau) \} y(t-\tau) d\tau. \quad (6.13)$$

Since  $h_j$  is a right-sided  $(j-1, N)^{th}$  order modulating function on  $[0, t]$  and satisfies  $(P_4^{\alpha})$ , (6.7) can be obtained. Thus, this proof is completed.  $\blacksquare$



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According to Theorem 6.1,  $D_t^\alpha x$  with  $\alpha \in (-1, 1)$  can be calculated by taking the inverse matrix of  $T$  and using (2.66).

Furthermore, the construction of the required modulating functions in Theorem 6.1 is provided. Polynomials are adopted to construct the needed modulating functions. The computation schemes are provided in the following proposition.

**Proposition 6.1** *For  $t \in (0, h]$ , the following cases are considered:*

- *In the case of  $\alpha \in (0, 1)$ , let  $N + 1 \leq m \in \mathbb{N}_+$ ,  $j \in 1, \dots, N + 1$ ,  $\eta_j$  be a polynomial defined on  $[0, t]$  as follows:*

$$\eta_j(\tau) = \sum_{i'=1}^{2N+1} \lambda_{i'} t^{j-m-i'-1} (t-\tau)^{m+i'}, \quad (6.14)$$

*the coefficient  $\Theta_\eta = (\lambda_1, \dots, \lambda_{2N+1})^\top$  is the solution of the following linear system:*

$$B_\eta \Theta_\eta = d_j, \quad (6.15)$$

*where  $B_\eta \in \mathbb{R}^{(2N+1) \times (2N+1)}$  and  $d_j \in \mathbb{R}^{(2N+1) \times 1}$  are defined on as follows: for  $i' = 1, \dots, 2N + 1$ ,*

$$B_\eta(s, i') = \begin{cases} (-1)^{s-1} \frac{\Gamma(m+i'+1)}{\Gamma(m+i'-s+2)}, & \text{for } s = 1, \dots, N+1, \\ \frac{\Gamma(m+i'+1)}{(m+i'+N+2-s-\alpha)\Gamma(m+i'+N+3-s)}, & \text{for } s = N+2, \dots, 2N+1, \end{cases} \quad (6.16)$$

*and  $d_j(s) = 1$ , if  $s = j$ ,  $d_j(s) = 0$ , else. Then  $\eta_j$  is a right-sided  $(j-1, N+1)^{\text{th}}$  order modulating function on  $[0, t]$  satisfying  $(P_4^\alpha)$  required in (6.43) and (6.6).*

- *In the case of  $\alpha \in (-1, 0)$ , let  $N \leq m \in \mathbb{N}_+$ ,  $j \in 1, \dots, N$ ,  $h_j$  be a polynomial defined on  $[0, t]$  as follows:*

$$h_j(\tau) = \sum_{i'=1}^{2N} \lambda_{i'} t^{j-m-i'-1} (t-\tau)^{m+i'}, \quad (6.17)$$

*the coefficient  $\Theta_h = (\lambda_1, \dots, \lambda_{2N})^\top$  is the solution of the following linear system:*

$$B_h \Theta_h = d_j, \quad (6.18)$$

*where  $B_h \in \mathbb{R}^{2N \times 2N}$  and  $d_j \in \mathbb{R}^{2N \times 1}$  are defined on as follows: for  $\hat{i} = 1, \dots, 2N$ ,*

$$B_h(s, \hat{i}) = \begin{cases} (-1)^{s-1} \frac{\Gamma(m+\hat{i}+1)}{\Gamma(m+\hat{i}-s+2)}, & \text{for } s = 1, \dots, N, \\ \frac{\Gamma(m+\hat{i}+1)}{(m+\hat{i}+N+1-s-\alpha)\Gamma(m+\hat{i}+N+2-s)}, & \text{for } s = N+1, \dots, 2N, \end{cases} \quad (6.19)$$

*and  $d_j(s) = 1$ , if  $s = j$ ,  $d_j(s) = 0$ , else. Then  $h_j$  is a right-sided  $(j-1, N)^{\text{th}}$  order modulating function on  $[0, t]$  satisfying  $(P_4^\alpha)$  required in (6.7).*

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**Proof.** Since  $\eta_j$  is a polynomial,  $\eta_j \in \mathcal{C}^{N+1}([0, t])$ , *i.e.*  $(P_1)$  is fulfilled. On the one hand, by calculating the  $k$ -th derivative of  $\eta_j$  for  $k = 0, \dots, N$ , we obtain:  $\forall \tau \in [0, t]$ ,

$$\eta_j^{(k)}(\tau) = \sum_{i'=1}^{2N+1} \lambda_{i'} t^{j-m-i'-1} \frac{(-1)^k \Gamma(m+i'+1)}{\Gamma(m+i'-k+1)} (t-\tau)^{m+i'-k}. \quad (6.20)$$

Since  $m \geq N+1$ ,  $\eta_j^{(k)}(t) = 0$ , *i.e.*  $(P_3)$  is satisfied. Then, by taking  $\tau = 0$  in (6.20), it yields: for  $k = 0, \dots, N$ ,

$$\eta_j^{(k)}(0) = t^{j-k-1} \sum_{i'=1}^{2N+1} \lambda_{i'} (-1)^k \frac{\Gamma(m+i'+1)}{\Gamma(m+i'-k+1)}. \quad (6.21)$$

On the other hand, we have: for  $k = 0, \dots, N+1$ ,

$$\begin{aligned} \int_0^t \eta_j^{(k)}(t-\tau) \tau^{-\alpha-1} d\tau &= \sum_{i'=1}^{2N+1} \lambda_{i'} t^{j-m-i'-1} \frac{(-1)^k \Gamma(m+i'+1)}{\Gamma(m+i'-k+1)} \int_0^t \tau^{m+i'-k-\alpha-1} d\tau \\ &= t^{j-k-\alpha-1} \sum_{i'=1}^{2N+1} \frac{(-1)^k \lambda_{i'}}{m+i'-k-\alpha} \frac{\Gamma(m+i'+1)}{\Gamma(m+i'-k+1)}. \end{aligned} \quad (6.22)$$

Based on (6.21) and (6.22), by solving the system given in (6.15), the construction of  $\eta_j$  is completed. Similarly, the construction of  $h_j$  can be easily completed. Thus, this proof is completed.  $\blacksquare$

**Corollary 6.1** *Under the same hypotheses of Theorem 6.1,  $D_t^\alpha z$  can be obtained by the way:*

- For  $\alpha \in (0, 1)$ , consider  $\eta$  as a right-sided  $(N, N+1)^{th}$  order modulating function satisfying  $(P_4^\alpha)$  on  $[0, t]$ , the fractional integral of  $y$  can be presented by:

$$\begin{aligned} D_t^{\alpha-1} y(t) &= \sum_{i=1}^N \sum_{i'=1}^{2N+1} \sum_{s=1}^{m+i'-i+1} \hat{B}_{i-1} \lambda_{i'} t^{-i-s} (-1)^{i+s-1} \frac{\Gamma(m+i'+1)}{\Gamma(m+i'-i-s+2)} D_t^{\alpha-s-1} u(t) \\ &\quad - \sum_{i=0}^N \sum_{i'=1}^{2N+1} \sum_{s=1}^{m+i'-i} a_i \lambda_{i'} t^{-i-s} \frac{(-1)^{i+s} \Gamma(m+i'+1)}{\Gamma(m+i'-i-s+1)} D_t^{\alpha-s-1} y(t). \end{aligned} \quad (6.23)$$

Moreover, consider  $\eta_j$  as a right-sided  $(j-1, N+1)^{th}$  order modulating functions for  $j = 1, \dots, N$  fulfilling  $(P_4^\alpha)$ , the fractional derivatives of states are calculated as:

$$\begin{aligned} D_t^\alpha z_j(t) &= \hat{B}_{j-1} D_t^{\alpha-1} u(t) + \sum_{i=1}^N \sum_{i'=1}^{2N+1} \sum_{s=1}^{m+i'-i+1} \hat{B}_{i-1} \lambda_{i'} t^{j-i-s} \frac{(-1)^{i+s-1} \Gamma(m+i'+1)}{\Gamma(m+i'-i-s+2)} D_t^{\alpha-s-1} u(t) \\ &\quad - a_{j-1} D_t^{\alpha-1} y(t) - \sum_{i=0}^N \sum_{i'=1}^{2N+1} \sum_{s=1}^{m+i'-i} a_i \lambda_{i'} t^{j-i-s} \frac{(-1)^{i+s} \Gamma(m+i'+1)}{\Gamma(m+i'-i-s+1)} D_t^{\alpha-s-1} y(t). \end{aligned} \quad (6.24)$$

- For  $\alpha \in (-1, 0)$ , consider  $h_j$  as a group of right-sided  $(j-1, N)^{th}$  order modulating functions on

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$[0, t]$  for  $j = 1, \dots, N$  fulfilling  $(P_4^\alpha)$ , the fractional integrals of states are calculated as:

$$\begin{aligned} D_t^\alpha z_j(t) &= \sum_{i=1}^N \sum_{\hat{i}=1}^{2N} \sum_{s=1}^{m+\hat{i}-i+1} \hat{B}_{i-1} \lambda_{\hat{i}} t^{j-i-s} \frac{(-1)^{i+s-1} \Gamma(m + \hat{i} + 1)}{\Gamma(m + \hat{i} + 2 - i - s)} D_t^{\alpha-s-1} u(t) \\ &\quad - \sum_{i=0}^N \sum_{\hat{i}=1}^{2N} \sum_{s=1}^{m+\hat{i}-i} a_i \lambda_{\hat{i}} t^{j-i-s} \frac{(-1)^{i+s} \Gamma(m + \hat{i} + 1)}{\Gamma(m + \hat{i} + 1 - i - s)} D_t^{\alpha-s-1} y(t). \end{aligned} \quad (6.25)$$

**Proof.**

• First case  $\alpha \in (0, 1)$ : According to the modulating function (6.14) given in Proposition 6.1 and the binomial theorem and linearity, the  $\alpha$  order Caputo fractional derivative of  $\eta_j^{(k)}(\tau)$  with  $\forall \tau \in [0, t]$  can be computed by: for  $j = 1, \dots, N$ ,

$${}^C D_\tau^\alpha \eta_j^{(k)}(\tau) = \sum_{i'=1}^{2N+1} \lambda_{i'} \frac{(-1)^{k+s} \Gamma(m + i' + 1)}{\Gamma(m + i' - k + 1)} t^{j-m-i'-1} \sum_{s=0}^{m+i'-k} \binom{m+i'-k}{s} t^{m+i'-k-s} {}^C D_\tau^\alpha \tau^s. \quad (6.26)$$

As a result,  ${}^C D_\tau^\alpha \tau^s = 0$  for  $s = 0, \dots, [\alpha]$  according to (2.45). Thus, it follows from (6.26) that:

$${}^C D_\tau^\alpha \eta_j^{(k)}(\tau) = \sum_{i'=1}^{2N+1} \lambda_{i'} \frac{(-1)^{k+s} \Gamma(m + i' + 1)}{\Gamma(m + i' - k + 1)} t^{j-m-i'-1} \sum_{s=[\alpha]}^{m+i'-k} \binom{m+i'-k}{s} t^{m+i'-k-s} {}^C D_\tau^\alpha \tau^s, \quad (6.27)$$

which, by applying (2.45) to (6.27), yields:

$${}^C D_\tau^\alpha \eta_j^{(k)}(\tau) = \sum_{i'=1}^{2N+1} \sum_{s=[\alpha]}^{m+i'-k} \lambda_{i'} t^{j-k-s-1} \frac{(-1)^{k+s} \Gamma(m + i' + 1)}{\Gamma(m + i' + 1 - k - s) \Gamma(s + 1 - \alpha)} \tau^{s-\alpha}. \quad (6.28)$$

Then, the second term on the right-side of (6.6) read:

$$\begin{aligned} &\sum_{i=1}^N \hat{B}_{i-1} \int_0^t {}^C D_\tau^\alpha \{\eta_j^{(i-1)}(\tau)\} u(t - \tau) d\tau \\ &= \sum_{i=1}^N \sum_{i'=1}^{2N+1} \sum_{s=[\alpha]}^{m+i'-i+1} \hat{B}_{i-1} \lambda_{i'} t^{j-i-s} \frac{(-1)^{i+s-1} \Gamma(m + i' + 1)}{\Gamma(m + i' + 2 - i - s) \Gamma(s + 1 - \alpha)} \int_0^t \tau^{s-\alpha} u(t - \tau) d\tau \\ &= \sum_{i=1}^N \sum_{i'=1}^{2N+1} \sum_{s=[\alpha]}^{m+i'-i+1} \hat{B}_{i-1} \lambda_{i'} t^{j-i-s} \frac{(-1)^{i+s-1} \Gamma(m + i' + 1)}{\Gamma(m + i' + 2 - i - s) \Gamma(s + 1 - \alpha)} \int_0^t (t - \tau)^{s-\alpha} u(\tau) d\tau \\ &= \sum_{i=1}^N \sum_{i'=1}^{2N+1} \sum_{s=[\alpha]}^{m+i'-i+1} \hat{B}_{i-1} \lambda_{i'} t^{j-i-s} \frac{(-1)^{i+s-1} \Gamma(m + i' + 1)}{\Gamma(m + i' + 2 - i - s)} D_t^{\alpha-s-1} u(t). \end{aligned} \quad (6.29)$$

Similarly, the last term on the right-side of (6.6) read:

$$\sum_{i=0}^N a_i \int_0^t {}^C D_\tau^\alpha \{\eta_j^{(i)}(\tau)\} y(t - \tau) d\tau = \sum_{i=0}^N \sum_{i'=1}^{2N+1} \sum_{s=[\alpha]}^{m+i'-i} a_i \lambda_{i'} t^{j-i-s-1} \frac{(-1)^{i+s} \Gamma(m + i' + 1)}{\Gamma(m + i' + 1 - i - s)} D_t^{\alpha-s-1} y(t). \quad (6.30)$$

Then, by substituting (6.29) and (6.30) into (6.6), (6.24) can be obtained.

• **Second case  $\alpha \in (-1, 0)$ :** According to the modulating function (6.17) given in Proposition 6.1 and the fractional Leibniz formula [85], the first term on the right-side of (6.7) read:

$$\sum_{i=1}^N \hat{B}_{i-1} \int_0^t D_{\tau}^{\alpha} \{h_j^{(i-1)}(\tau)\} u(t-\tau) d\tau = \sum_{i=1}^N \sum_{\hat{i}=1}^{2N} \sum_{s=0}^{m+\hat{i}-i+1} \hat{B}_{i-1} \lambda_{\hat{i}} t^{j-i-s} \frac{(-1)^{i+s-1} \Gamma(m+\hat{i}+1)}{\Gamma(m+\hat{i}+2-i-s)} D_t^{\alpha-s-1} u(t). \quad (6.31)$$

Similarly, the second term on the right-side of (6.7) read:

$$\sum_{i=0}^N a_i \int_0^t D_{\tau}^{\alpha} \{h_j^{(i)}(\tau)\} y(t-\tau) d\tau = \sum_{i=0}^N \sum_{\hat{i}=1}^{2N} \sum_{s=0}^{m+\hat{i}-i} a_i \lambda_{\hat{i}} t^{j-i-s-1} \frac{(-1)^{i+s} \Gamma(m+\hat{i}+1)}{\Gamma(m+\hat{i}+1-i-s)} D_t^{\alpha-s-1} y(t). \quad (6.32)$$

Ultimately, by substituting (6.31) and (6.32) into (6.7), (6.25) can be obtained, thereby concluding this proof. ■

According to Theorem 6.1 and Corollary 6.1,  $D_t^{\alpha} x$  with  $\alpha \in (-1, 1)$  can be calculated by taking the inverse matrix of  $T$  and using (2.66).

To illustrate the accuracy and robustness with respect to corrupting noises of the proposed estimators given in Theorem 6.1 and Corollary 6.1, we present numerical results in the following example.

**Example 6.1** *Let us take  $N = 3$  and the following matrices in (4.1):  $A = \begin{pmatrix} 1 & 1 & -11 \\ 6 & 0 & -6 \\ 18 & 1 & -17 \end{pmatrix}$ ,  $B = (0, 1, 0)^{\top}$ ,  $C = (-1, 0, 1)$ , which satisfy the condition  $(C_1)$ . Assume that  $u(t) = \sin(2t)$ , the initial value of the state is given by  $x(0) = (-0.2, -0.4, -0.6)^{\top}$ , and  $y^{\varpi}(t_i) = y(t_i) + \varpi(t_i)$  is a noisy observation of the output  $y$  on  $I = [0, 5]$ , where  $t_i = iT_s$  with  $T_s = 0.001$ . Moreover,  $\varpi(t) = 0.02 \times \sin(2000t)$  is the corrupting noise. Indeed, the high-frequency sinusoidal noises are encountered in many mechanical systems. The output and the associated noisy output are shown in Fig. 6.1 (a).*

*Using Theorem 6.1, the fractional derivative and integral of the state  $D_t^{\alpha} x$  with  $\alpha \in (-1, 1)$  can be estimated directly in this discrete noisy case, where the trapezoidal numerical integration method is used to approximate the integral formulas. Then, by means of the constructed modulating functions, the required estimation results can be obtained. Moreover, the modulating functions method in Chapter 4 [111] is considered to compare with the proposed method. It is worth mentioning that  $D_t^{\alpha} x$  in Chapter 4 was estimated by applying the Grünwald-Letnikov scheme to the obtained state  $x$ . By taking  $m = 4$ , the estimation values and estimation errors of  $D_t^{\alpha} x_3$  with  $\alpha = 0.3, -0.3$  by using the proposed method and the one in Chapter 4 are shown in Fig. 6.1 and Fig. 6.2, respectively. Numerical results show both the proposed estimator and the one in Chapter 4 give satisfactory results with fast convergence and good robustness against noises. Moreover, the proposed estimator is more accurate than the one in Chapter 4.*

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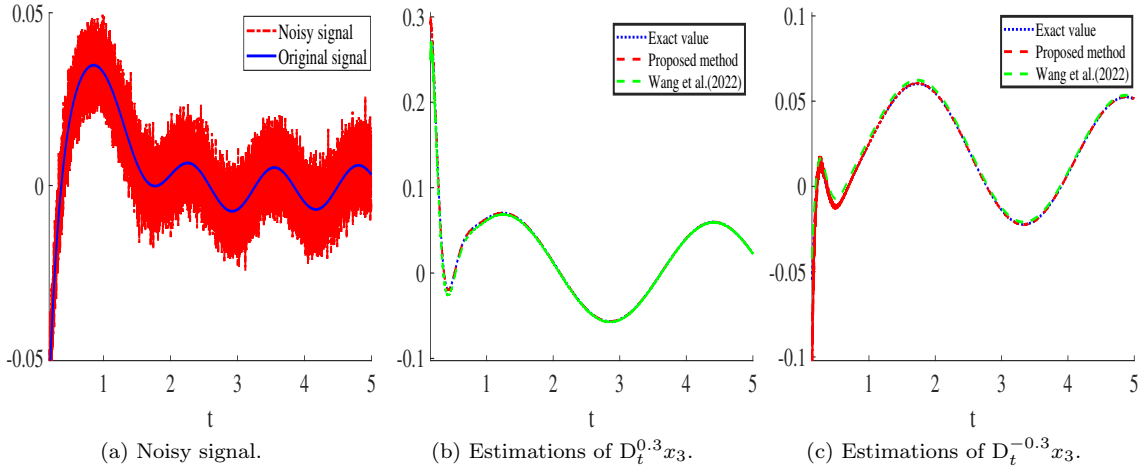


Figure 6.1: Noisy signal and Estimations of  $D_t^\alpha x_3$  with  $\alpha = 0.3, -0.3$  obtained by the proposed method and the one in Chapter 4 [111].

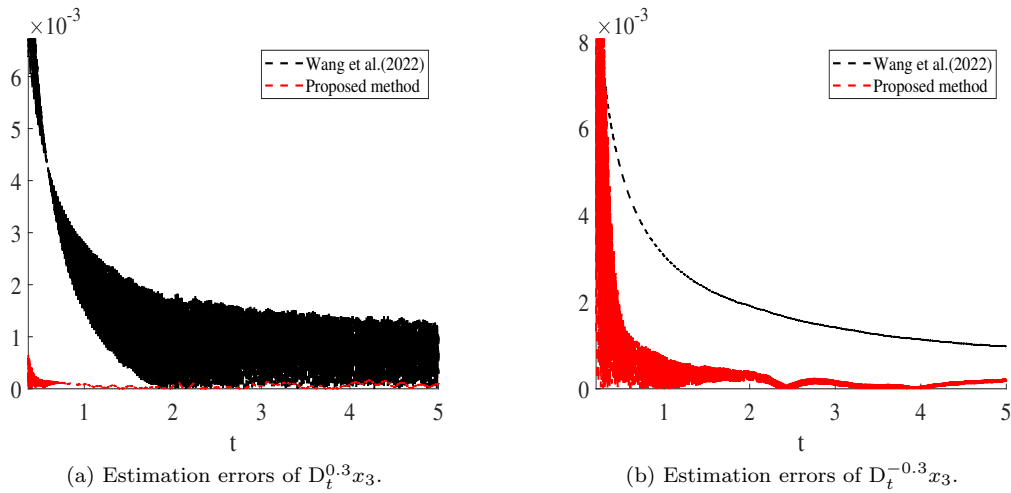


Figure 6.2: Estimation errors of  $D_t^\alpha x_3$  with  $\alpha = 0.3, -0.3$  obtained by the proposed method and the one in Chapter 4 [111].

### 6.2.2 Second way—Estimate initial value term first

Inspired by the Theorem 4.1 and Theorem 4.2 in Chapter 4, by designing modulating functions, we can estimate the initial value of the system (6.1), as shown in the following theorem.

**Theorem 6.2** *Let us consider the observable canonical form (4.2) of the system (4.1) on  $I$ . For  $t \in I$ , let  $g_j$  for  $j = 1, \dots, N$  be a group of right-sided  $(j - 1, N)^{th}$  order modulating function on  $[0, t]$ , the following formula holds:*

$$z_j(0) = \sum_{i=1}^N (-1)^{i+j} \hat{B}_{i-1} \int_0^t g_j^{(i-1)}(\tau) u(\tau) d\tau - \sum_{i=0}^N (-1)^{i+j} a_i \int_0^t g_j^{(i)}(\tau) y(\tau) d\tau. \quad (6.33)$$

where  $a_N = 1$ .

**Proof.** The proof can be given in the following steps.

**Step 1.** By multiplying  $\eta^{(i-1)}$  for  $i = 1, \dots, N$  to (6.1) and integrating on  $[0, t]$ , the following formula holds:

$$\int_0^t \eta^{(i-1)}(\tau) \dot{z}_i(\tau) d\tau = \int_0^t \eta^{(i-1)}(\tau) z_{i-1}(\tau) d\tau + \hat{B}_{i-1} \int_0^t \eta^{(i-1)}(\tau) u(\tau) d\tau - a_{i-1} \int_0^t \eta^{(i-1)}(\tau) y(\tau) d\tau, \quad (6.34)$$

where  $z_0(\tau) = 0$ . Then, by applying the integration by parts formula given in Lemma 2.4 to the left side of (6.34), then the left side of (6.34) can be written as follows:

$$\int_0^t \eta^{(i-1)}(\tau) \dot{z}_i(\tau) d\tau = \left[ \eta^{(i-1)}(\tau) z_i(\tau) \right]_{\tau=0}^{\tau=t} - \int_0^t \eta^{(i)}(\tau) z_i(\tau) d\tau. \quad (6.35)$$

According to (6.34) and (6.35), we get:

$$\begin{aligned} \left[ \eta^{(i-1)}(\tau) z_i(\tau) \right]_{\tau=0}^{\tau=t} - \int_0^t \eta^{(i)}(\tau) z_i(\tau) d\tau &= \int_0^t \eta^{(i-1)}(\tau) z_{i-1}(\tau) d\tau + \hat{B}_{i-1} \int_0^t \eta^{(i-1)}(\tau) u(\tau) d\tau \\ &\quad - a_{i-1} \int_0^t \eta^{(i-1)}(\tau) y(\tau) d\tau. \end{aligned} \quad (6.36)$$

**Step 2.** By combining the  $N$  equations of (6.36), we obtain:

$$\sum_{i=1}^N (-1)^{i-1} \left[ \eta^{(i-1)}(\tau) z_i(\tau) \right]_{\tau=0}^{\tau=t} = \sum_{i=1}^N (-1)^{i+1} \hat{B}_{i-1} \int_0^t \eta^{(i-1)}(\tau) u(\tau) d\tau - \sum_{i=0}^N (-1)^i a_i \int_0^t \eta^{(i)}(\tau) y(\tau) d\tau. \quad (6.37)$$

By replacing  $\eta$  by  $g_j$ , for  $j = 1, \dots, N$ , in (6.37) and using the properties  $(P_3)$  and  $(P_5)$  of the modulating functions  $g_j$ , this proof is completed. ■

The construction of the required modulating functions in Theorem 6.2 is provided as follows.

## 6.2. MAIN RESULTS

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**Proposition 6.2** For  $t \in (0, h]$ , let  $N \leq m \in \mathbb{N}_+$ ,  $j \in 1, \dots, N$ ,  $g_j$  be a polynomial defined on  $[0, t]$  as follows:

$$g_j(\tau) = \sum_{i'=1}^N \lambda_{i'} t^{j-m-i'-1} (t-\tau)^{m+i'}, \quad (6.38)$$

the coefficient  $\Theta_g = (\lambda_1, \dots, \lambda_N)^\top$  is the solution of the following linear system:

$$B_g \Theta_g = d_j, \quad (6.39)$$

where  $B_g \in \mathbb{R}^{N \times N}$  and  $d_j \in \mathbb{R}^{N \times 1}$  are defined on as follows: for  $i', s = 1, \dots, N$ ,

$$B_g(s, i') = (-1)^{s-1} \frac{\Gamma(m+i'+1)}{\Gamma(m+i'-s+2)}, \quad (6.40)$$

and  $d_j(s) = 1$ , if  $s = j$ ,  $d_j(s) = 0$ , else. Then,  $g_j$  is a right-sided  $(j-1, N)^{th}$  order modulating function on  $[0, t]$ .

**Proof.** Since  $g_j$  is a polynomial,  $g_j \in \mathcal{C}^N([0, t])$ , i.e.  $(P_1)$  is fulfilled. By calculating the  $k$ -th derivative of  $g_j$  for  $k = 0, \dots, N-1$ , we obtain:  $\forall \tau \in [0, t]$ ,

$$g_j^{(k)}(\tau) = \sum_{i'=1}^N \lambda_{i'} t^{j-m-i'-1} \frac{(-1)^k \Gamma(m+i'+1)}{\Gamma(m+i'-k+1)} (t-\tau)^{m+i'-k}. \quad (6.41)$$

Since  $m \geq N$ ,  $g_j^{(k)}(t) = 0$ , i.e.  $(P_3)$  is satisfied. Then, by taking  $\tau = 0$  in (6.41), it yields: for  $k = 0, \dots, N-1$ ,

$$g_j^{(k)}(0) = t^{j-k-1} \sum_{i'=1}^N \lambda_{i'} (-1)^k \frac{\Gamma(m+i'+1)}{\Gamma(m+i'-k+1)}. \quad (6.42)$$

Based on (6.42), by solving the system given in (6.39), the construction of  $g_j$  is completed.  $\blacksquare$

Next, the algebraic integration formulas of fractional derivative and integral estimators of the state for the integer order system will be provided by the second way.

**Theorem 6.3** Let us consider the observable canonical form (6.1) of the system (4.1) on  $I$ . Based on Theorem 6.2, for  $t \in I$ ,  $D_t^\alpha z$  with  $\alpha \in (-1, 1)$  can be computed as follows:

• In the case of  $\alpha \in (0, 1)$ , let  $g$  be a right-sided  $(N, N+1)^{th}$  order modulating function on  $[0, t]$ , then the fractional integral of  $y$  can be given by:

$$\begin{aligned} D_t^{\alpha-1} y(t) &= \sum_{i=1}^N \hat{B}_{i-1} \int_0^t {}^C D_\tau^\alpha \{g^{(i-1)}(\tau)\} u(t-\tau) d\tau - \sum_{i=0}^N a_i \int_0^t {}^C D_\tau^\alpha \{g^{(i)}(\tau)\} y(t-\tau) d\tau \\ &\quad + \sum_{i=1}^N \frac{z(0)}{\Gamma(-\alpha)} \int_0^t g^{(i-1)}(t-\tau) \tau^{-\alpha-1} d\tau, \end{aligned} \quad (6.43)$$

where  $a_N = 1$ ,  $z(0)$  is given by (6.33) in Theorem 6.2.

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Moreover, let  $g_j$  be a group of right-sided  $(j-1, N+1)^{th}$  order modulating functions on  $[0, t]$  for  $j = 1, \dots, N$ , the following formula holds:

$$\begin{aligned} D_t^\alpha z_j(t) &= \hat{B}_{j-1} D_t^{\alpha-1} u(t) - a_{j-1} D_t^{\alpha-1} y(t) + \sum_{i=1}^N \hat{B}_{i-1} \int_0^t {}^C D_\tau^\alpha \{g_j^{(i-1)}(\tau)\} u(t-\tau) d\tau \\ &\quad - \sum_{i=0}^N a_i \int_0^t {}^C D_\tau^\alpha \{g_j^{(i)}(\tau)\} y(t-\tau) d\tau + \sum_{i=1}^N \frac{z_j(0)}{\Gamma(-\alpha)} \int_0^t g_j^{(i-1)}(t-\tau) \tau^{-\alpha-1} d\tau, \end{aligned} \quad (6.44)$$

where  $D_t^{\alpha-1} y(t)$  is given by (6.43).

• In the case of  $\alpha \in (-1, 0)$ , let  $h_j$  be a group of right-sided  $(j-1, N+1)^{th}$  order modulating functions on  $[0, t]$  for  $j = 1, \dots, N$ , the following formula holds:

$$\begin{aligned} D_t^\alpha z_j(t) &= \sum_{i=1}^N \hat{B}_{i-1} \int_0^t D_\tau^\alpha \{h_j^{(i-1)}(\tau)\} u(t-\tau) d\tau - \sum_{i=0}^N a_i \int_0^t D_\tau^\alpha \{h_j^{(i)}(\tau)\} y(t-\tau) d\tau \\ &\quad + \sum_{i=1}^N \frac{z_j(0)}{\Gamma(-\alpha)} \int_0^t h_j^{(i-1)}(t-\tau) \tau^{-\alpha-1} d\tau, \end{aligned} \quad (6.45)$$

where  $a_N = 1$ ,  $z(0)$  is given by (6.33) in Theorem 6.2.

### Proof.

• First case  $\alpha \in (0, 1)$ : In this case, by replacing  $\eta$  by  $g$  in (6.10) and by applying the fractional order integration by parts formula to the right side of (6.10), we have:

$$\begin{aligned} &\sum_{i=1}^N \left[ g^{(i-1)}(t-\tau) D_\tau^\alpha z_i(\tau) \right]_{\tau=0}^{\tau=t} \\ &= \sum_{i=1}^N \hat{B}_{i-1} \left[ g^{(i-1)}(t-\tau) D_\tau^{\alpha-1} u(\tau) \right]_{\tau=0}^{\tau=t} + \sum_{i=1}^N \hat{B}_{i-1} \int_0^t {}^C D_\tau^\alpha \{g^{(i-1)}(\tau)\} u(t-\tau) d\tau \\ &\quad - \sum_{i=0}^N a_i \left[ g^{(i)}(t-\tau) D_\tau^{\alpha-1} y(\tau) \right]_{\tau=0}^{\tau=t} - \sum_{i=0}^N a_i \int_0^t {}^C D_\tau^\alpha \{g^{(i)}(\tau)\} y(t-\tau) d\tau \\ &\quad + \sum_{i=1}^N \frac{z(0)}{\Gamma(-\alpha)} \int_0^t g^{(i-1)}(t-\tau) \tau^{-\alpha-1} d\tau. \end{aligned} \quad (6.46)$$

Since  $g$  is a right-sided  $(N, N+1)^{th}$  order modulating function on  $[0, t]$ , the proof of (6.43) can be completed, where  $z(0)$  is given by (6.33) in Theorem 6.2.

By replacing  $g$  by  $g_j$  for  $j = 1, \dots, N$  in (6.46), and since the properties of right-sided  $(j-1, N+1)^{th}$  order modulating function of  $g_j$  for  $j = 1, \dots, N$ , we can easily get (6.44) where  $D_t^{\alpha-1} y(t)$  is given by (6.43).

• Second case  $\alpha \in (-1, 0)$ : In this case, thanks to (6.10) and by replacing  $\eta$  by  $h_j$  for  $j = 1, \dots, N$  in



(6.10), we get:

$$\begin{aligned}
 \sum_{i=1}^N \left[ h_j^{(i-1)}(t-\tau) D_\tau^\alpha z_j(\tau) \right]_{\tau=0}^{\tau=t} &= \sum_{i=1}^N \hat{B}_{i-1} \int_0^t D_\tau^\alpha \{h_j^{(i-1)}(\tau)\} u(t-\tau) d\tau \\
 &\quad - \sum_{i=0}^N a_i \int_0^t D_\tau^\alpha \{h_j^{(i)}(\tau)\} y(t-\tau) d\tau \\
 &\quad + \sum_{i=1}^N \frac{z_j(0)}{\Gamma(-\alpha)} \int_0^t h_j^{(i-1)}(t-\tau) \tau^{-\alpha-1} d\tau.
 \end{aligned} \tag{6.47}$$

Since  $h_j$  is a right-sided  $(j-1, N+1)^{th}$  order modulating function on  $[0, t]$ , (6.45) can be obtained, where  $z(0)$  is given by (6.33) in Theorem 6.2. Thus, this proof is completed.  $\blacksquare$

According to Theorem 6.3,  $D_t^\alpha x$  with  $\alpha \in (-1, 1)$  can be calculated by taking the inverse matrix of  $T$  and using (2.66).

**Remark 6.2** Both the first way and the second way given in Subsection 6.2.1 and Subsection 6.2.2, the algebraic integral formulas of fractional derivatives and integrals of the state for integer order systems (4.1) are provided directly without using Grünwald-Letnikov scheme.

**Remark 6.3** The modulating functions required in Theorem 6.3 are similar to that given in Proposition 6.2, which will not be reiterated at this point.

In order to illustrate the accuracy and robustness with respect to corrupting noises of the proposed estimators in Subsection 6.2.2, the [111] and [119], which introduce in Chapter 4 and Subsection 6.2.1 are considered to be compared with the proposed estimators, with some numerical results given as follows.

**Example 6.2 Estimation  $z(0)$  and  $D_t^\alpha z(t)$**

Let us take  $N = 3$ ,  $u(t) = \sin(5t)$ , and the following matrices in (4.1):

$$A = \begin{pmatrix} 11 & 1 & -11 \\ 6 & 0 & -6 \\ 18 & 1 & -17 \end{pmatrix}, B = (0, 1, 0)^\top, C = (-1, 0, 1),$$

which satisfy the conditions  $(C_1)$ – $(C_2)$ . Assume that the initial value of the state is given by  $x(0) = (-0.1, -0.2, -0.3)^\top$ , and  $y^\varpi(t_i) = y(t_i) + \sigma\varpi(t_i)$  is the noisy observation of the output  $y$  on  $I = [0, 10]$ , where  $t_i = iT_s$  with  $T_s = 0.0001$ ,  $\{\varpi(t_i)\}$  is a zero-mean white Gaussian noise, and the value of  $\sigma$  is adjusted such that the signal-to-noise ratio  $SNR = 10 \log_{10} \left( \frac{\sum |y^\varpi(t_i)|^2}{\sum |\sigma\varpi(t_i)|^2} \right)$  is equal to 15dB. The output and the associated noisy output are shown in Fig. 6.3.

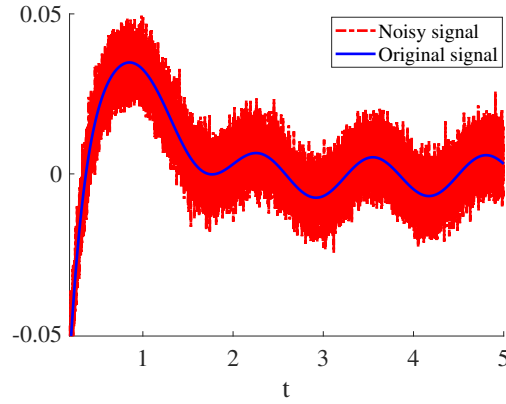


Figure 6.3:  $y$  and its discrete noisy observation in Example 6.2.

According to the estimator 6.33 in Subsection 6.2.2, the state initial value needs to be estimated first before estimating the fractional integral and derivative of the state. The estimation results of  $x_1(0)$ ,  $x_2(0)$  and  $x_3(0)$  are given in Fig. 6.4. Meanwhile, based on these estimated values, the fractional integral and derivative of the state  $x$  are estimated by utilizing Theorem 6.3 for  $\alpha = -0.3, 0.2$ . The estimation results and associated estimation errors of  $D_t^\alpha x_1$  for  $\alpha = -0.3, 0.2$  are presented in Fig. 6.5 and Fig. 6.6, where  $m = 4$ . Hence, it can be seen the proposed estimator in Subsection 6.2.2 is more accurate. Moreover, the estimation values of  $D_t^\alpha x_2$  and  $D_t^\alpha x_3$  can be obtained in a similar way.

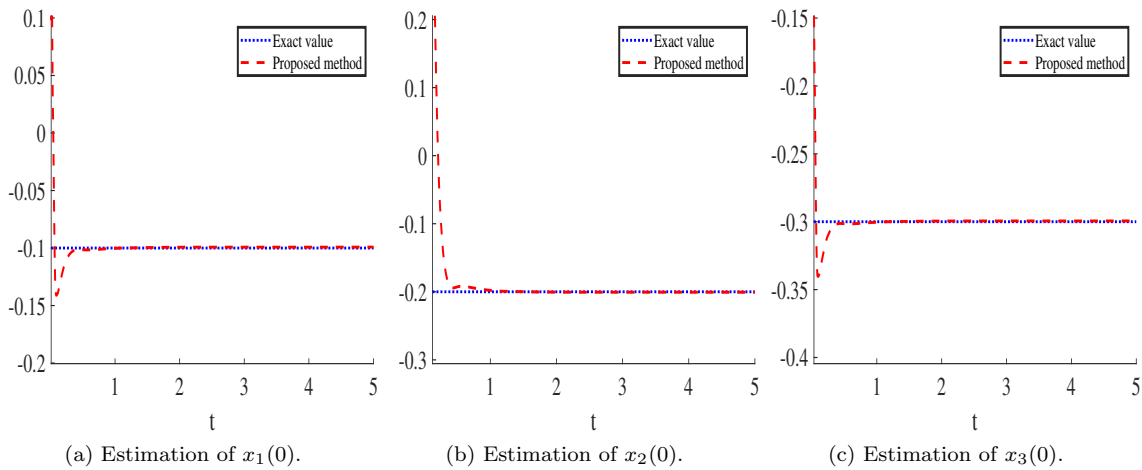


Figure 6.4: Estimations of  $x(0)$  obtained by the proposed method in Example 6.2.

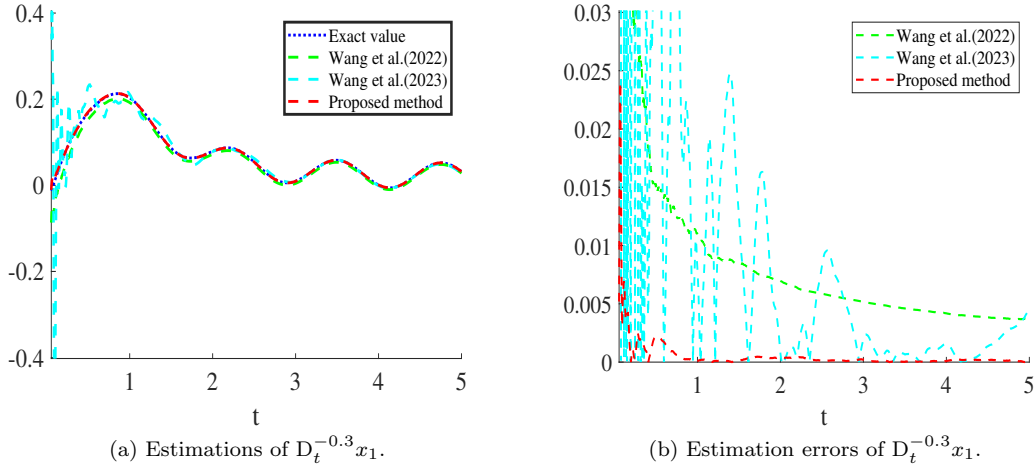


Figure 6.5: Estimations and estimation errors of  $D_t^{-0.3} x_1$  obtained by the proposed method and the one in Chapter 4 [111] and Subsection 6.2.1 [119].

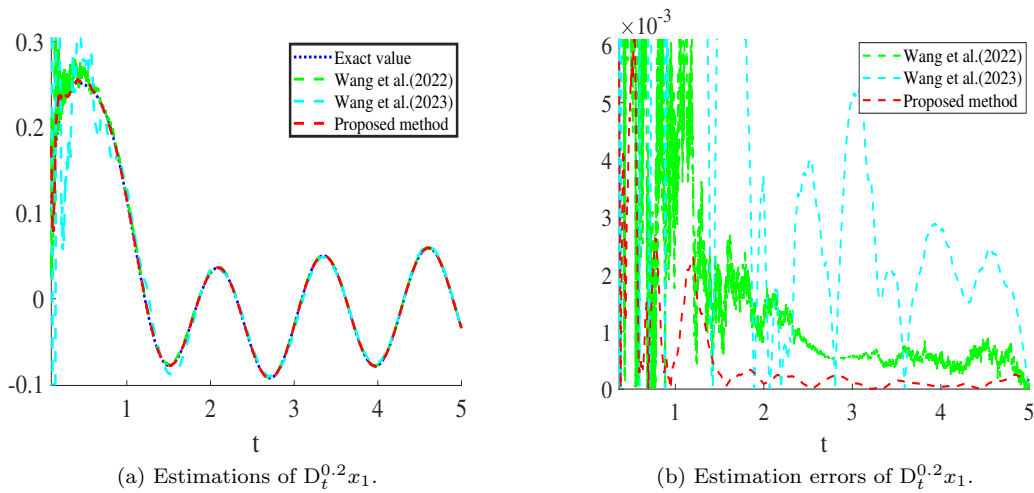


Figure 6.6: Estimations and estimation errors of  $D_t^{0.2} x_1$  obtained by the proposed method and the one in Chapter 4 [111] and Subsection 6.2.1 [119].

### 6.3 Application

The DC motor holds significant importance as one of the key electrical machines in the industry. The distinctive construction advantages of the DC motor, such as compactness, robustness, reliability, and high efficiency have developed the interest in their use in various fields, including extensive applications in robotics, electric vehicles, and manufacturing processes [120], [121]. Accurate knowledge of the shaft speed or position is imperative for the implementation of a DC motor. Contemporary speed control methods for DC motors often rely on position sensors such as tachogenerators or digital shaft encoders to acquire speed data. Besides, given that certain state variables within the motor are not measurable, the observer or estimator are generally used instead of a sensor in order to optimize cost-effectiveness while simultaneously enhancing longevity and reliability [122]–[127]. Nevertheless, these encoders come with a high cost, introduce computational noise and might potentially reduce the lifespan of the drive system. Consequently, a sensorless speed estimation technique is preferred with an outstanding controller that overcomes the aforementioned problems.

Multiple research studies have demonstrated that fractional order calculus, due to its distinctive infinite dimension and non-locality, can provide more precise descriptions of complex industrial processes and physical phenomena compared to integer order calculus. Consequently, system modeling, identification, control prediction, and fault diagnosis, based on fractional order systems, have emerged as focal points of investigation for numerous researchers [128]–[131]. In particular, fractional order controllers are significant in automatic control owing to their performance-enhancing and robustness properties in control design. Compared with the traditional PID controller, the fractional order  $PI^\lambda D^\mu$  controller [35] offers several advantages, these include reduced overflow and oscillation, exhibited resilience to disruptive outcomes, stable adjustments to regulated device parameters and faster response times [132]–[135].

The design of a fractional order  $PI^\lambda D^\mu$  controller typically necessitates access to information pertaining to the state and fractional derivatives and integrals of systems [21], [136], [137]. Therefore, it is valuable to explore methods for acquiring such information. However, the fractional order derivatives and integrals of the state for DC motors often cannot be measured by sensors in practical engineering applications because of cost and technological constraints. Consequently, these values need to be estimated from available measurements, which are commonly subject to noise corrupts. Moreover, the guarantee of only asymptotic convergence provided by most observers falls short of meeting the requirement for rapid convergence within a finite time in online applications. So, it is significant to devise robust estimation techniques with fast convergence to overcome these challenges [67], [68], [70].

With these ideas in consideration, we aim to propose a novel modulating functions-based speed, current and their fractional order derivatives and integrals estimators for designing sensorless fractional order  $PI^\lambda D^\mu$  controllers of DC motor. It is hoped that the novel estimator proposed in this section will

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create valuable information for fractional order controllers in diverse design. The major advantages of this work encompass:

- A novel modulating functions-based estimator is applied for the first time to estimate current, speed and their fractional order derivatives and integrals for DC motor, which is non-asymptotic and robust.
- Unlike existing work [76]–[79] on the modulating functions estimation method, this section directly derives algebraic expressions for state and fractional order integrals and derivatives without converting the given system into input-output differential equations or estimating the fractional order initial values of the output. It is important to note that in [111], [112], [118], the state fractional order integrals and derivatives were obtained using Grünwald-Letnikov scheme. In addition, the novel proposed modulating functions method is more accurate than [111], [119], as we conclude in Subsection 6.2.2.
- The efficiency of the presented modulating functions-based estimator is verified by Matlab simulation and comparison with the Lunberger observer. Moreover, we designed a fractional order  $PI^\lambda D^\mu$  controller for the DC motor and obtained good control results.

#### 6.3.1 DC motor system

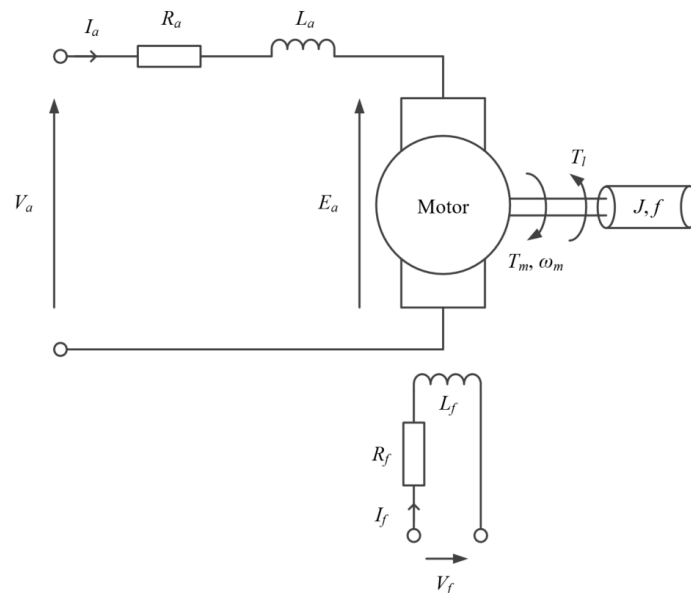


Figure 6.7: Schematic diagram of separately excited DC motor.

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Symbol	Explanation	Symbol	Explanation
$V_a$	Armature voltage	$L_a$	Armature Inductance
$J$	Moment of inertia	$K_b$	Back Emf constant
$R_a$	Armature resistance	$K_t$	Torque constant
$f$	Viscous friction coefficient	$I_a$	Armature current
$\omega_r$	Speed	$I_f$	Field current

Table 6.1: Nomenclature

Fig. 6.7 shows the studied system of this work. The separately excited DC motor can be described schematically as comprising an armature winding and a field winding. Excitation of the DC motor occurs through a field current  $I_f$  and an armature voltage  $V_a$ , resulting in the flow of an armature current  $I_a$  within the armature winding. In response, the motor generates a back electromotive force and a torque to counterbalance the load torque [138]. The state-space equation of this DC motor can be represented as:

$$\begin{cases} \dot{x} = Ax + Bu, \\ y = Cx, \end{cases} \quad (6.48)$$

where  $x = \begin{bmatrix} \omega_r \\ I_a \end{bmatrix}$ ,  $A = \begin{bmatrix} -\frac{f}{J} & \frac{k_t}{J} \\ -\frac{k_b}{L_a} & -\frac{R_a}{L_a} \end{bmatrix}$ ,  $B = \begin{bmatrix} 0 \\ \frac{1}{L_a} \end{bmatrix}$ ,  $C = \begin{bmatrix} 1 & 0 \end{bmatrix}$ ,  $u = V_a$  denotes the control input,  $y$  represents the measurable output. Additionally, the following condition is assumed to hold:

$$(C_1) : \text{rank} \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{f}{J} & \frac{k_t}{J} \end{bmatrix} = 2.$$

It is to be noted that the intention here is to provide a novel modulating functions-based estimator to obtain the current, speed and their fractional order derivatives and integrals for designing sensorless fractional order controllers of motor. Here, we choose DC motor system as an example for study. However, the novel modulating functions-based estimator is not limited to the scope of DC motor system, it may also be applicable to other applications, such as induction motor systems, which will be our future study topics.

#### 6.3.2 Novel estimator based on modulating functions

The algebraic integral formulas of the novel estimator of the speed, current and their fractional order derivatives and integrals of the DC motor system will be obtained directly based on modulating

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functions method. For this purpose, in the following estimator design steps, we first convert the DC motor system given in (6.48) into the observable canonical form.

Using the subsequent change of coordinates [1]:

$$\Psi = Mx \quad (6.49)$$

with

$$M = \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} = \begin{bmatrix} CA + a_1C \\ C \end{bmatrix} = \begin{bmatrix} -\frac{f}{J} + a_1 & \frac{k_t}{J} \\ 1 & 0 \end{bmatrix}, \quad (6.50)$$

where  $a_0, a_1$  are factors of the eigenpolynomial of  $A$ , the DC motor equation (6.48), which satisfies conditions  $(C_1)$ – $(C_2)$ , can be converted into observable canonical form given as:

$$\begin{cases} \dot{\Psi} = \tilde{A}\Psi + \tilde{B}u, \\ y = \tilde{C}\Psi, \end{cases} \quad (6.51)$$

where

$$\tilde{A} = \begin{bmatrix} 0 & -a_0 \\ 1 & -a_1 \end{bmatrix}, \quad (6.52)$$

and  $\tilde{B} = MB = [\tilde{B}_0, \tilde{B}_1]^\top = [\frac{1}{L_a}(-\frac{f}{J} + a_1), 0]^\top$ ,  $\tilde{C} = [0, 1]$ ,  $\Psi = [\Psi_1, \Psi_2]^\top = [\tilde{\omega}_r, \tilde{I}_a]^\top$ ,  $u = V_a$ .

According to the idea of the Subsection 6.2.2, the first step of the novel modulating functions method of estimating speed, current and their derivatives and integrals of a DC motor system is to transform the observable canonical form (6.51) into a fractional order system as follows:

$$D_t^{1+\alpha}\Psi(t) = \tilde{A}D_t^\alpha\Psi(t) + \tilde{B}D_t^\alpha u + \frac{\Psi(0)}{\Gamma(-\alpha)}t^{-\alpha-1}. \quad (6.53)$$

**Theorem 6.4** Consider the DC motor system (6.48), let  $\Psi = [\Psi_1, \Psi_2]^\top = [\tilde{\omega}_r, \tilde{I}_a]^\top$ . For  $t \in (0, h]$ ,  $\alpha \in (-1, 1)$ ,  $D_t^\alpha\Psi$  can be obtained by the way:

• For  $\alpha \in (0, 1)$ , consider  $\xi$  as a right-sided  $(2, 3)^{th}$  order generalized modulating function on  $[0, t]$ , the fractional integrals of  $y$  can be presented by:

$$\begin{aligned} D_t^{\alpha-1}y(t) &= \sum_{i=1}^2 \tilde{B}_{i-1} \int_0^t {}^C D_\tau^\alpha \{\xi^{(i-1)}(\tau)\} u(t-\tau) d\tau - \sum_{i=0}^2 a_i \int_0^t {}^C D_\tau^\alpha \{\xi^{(i)}(\tau)\} y(t-\tau) d\tau \\ &\quad + \sum_{i=1}^2 \frac{\Psi(0)}{\Gamma(-\alpha)} \int_0^t \xi^{(i-1)}(t-\tau) \tau^{-\alpha-1} d\tau, \end{aligned} \quad (6.54)$$

where  $a_2 = 1$ ,  $\Psi(0)$  can be given by (6.33) in Theorem 6.2.

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Moreover, consider  $\xi_j$  as a set of right-sided  $(j-1, 3)^{th}$  order generalized modulating functions on  $[0, t]$  for  $j = 1, 2$ , the fractional derivatives of  $\Psi_j(t)$  are calculated as:

$$\begin{aligned} D_t^\alpha \Psi_j(t) = & -a_{j-1} D_t^{\alpha-1} y(t) + \tilde{B}_{j-1} D_t^{\alpha-1} u(t) + \sum_{i=1}^2 \tilde{B}_{i-1} \int_0^t {}^C D_\tau^\alpha \{\xi_j^{(i-1)}(\tau)\} u(t-\tau) d\tau \\ & - \sum_{i=0}^2 a_i \int_0^t {}^C D_\tau^\alpha \{\xi_j^{(i)}(\tau)\} y(t-\tau) d\tau + \sum_{i=1}^2 \frac{\Psi_j(0)}{\Gamma(-\alpha)} \int_0^t \xi_j^{(i-1)}(t-\tau) \tau^{-\alpha-1} d\tau, \end{aligned} \quad (6.55)$$

where  $D_t^{\alpha-1} y(t)$  is given by (6.54).

• For  $\alpha \in (-1, 0)$ , consider  $\zeta_j$  as a set of right-sided  $(j-1, 2)^{th}$  order generalized modulating functions on  $[0, t]$  for  $j = 1, 2$ , the fractional integral of  $\Psi_j(t)$  are calculated as:

$$\begin{aligned} D_t^\alpha \Psi_j(t) = & \sum_{i=1}^2 \tilde{B}_{i-1} \int_0^t D_\tau^\alpha \{\zeta_j^{(i-1)}(\tau)\} u(t-\tau) d\tau - \sum_{i=0}^2 a_i \int_0^t D_\tau^\alpha \{\zeta_j^{(i)}(\tau)\} y(t-\tau) d\tau \\ & + \sum_{i=1}^2 \frac{\Psi_j(0)}{\Gamma(-\alpha)} \int_0^t \zeta_j^{(i-1)}(t-\tau) \tau^{-\alpha-1} d\tau, \end{aligned} \quad (6.56)$$

where  $a_2 = 1$ ,  $\Psi(0)$  can be given by (6.33) in Theorem 6.2.

#### Proof.

Since Theorem 6.3, this proof can be easily completed. ■

According to Theorem 6.4, the speed, current and their fractional order derivatives and integrals  $D_t^\alpha \Psi$  with  $\alpha \in (-1, 1)$  can be computed by multiplying the inverse of  $M$  and applying (6.49).

Two examples are given below to validate the efficacy of the novel modulating functions-based estimators for the DC motor.

#### Example 6.3 Estimator of a DC motor system

This example will provide the simulation results of speed, current and their fractional order derivatives and integrals by proposed estimator of the DC motor. The parameter of the DC motor is shown in Table 6.2. Assumed that  $y^\varpi(t_i) = y(t_i) + \varpi(t_i)$  is a noisy observation of the output  $y$  on  $I = [0, 3]$ , where  $t_i = iT_s$  with  $T_s = 0.0001$ . Moreover,  $\varpi(t) = 9 \times \sin(100000t)$  is the corrupting noise. Indeed, the high-frequency sinusoidal noises are encountered in many mechanical systems. The output and the associated noisy output are shown in Fig. 6.8.

Moreover, Luenberger observer is considered to be compared with the proposed novel modulating functions-based estimator. Similar to the Example 4.1, we choose two gains to better compare. The first gain of Luenberger observer we chose is based on its robustness, which makes it similar to the robustness of the proposed estimator to noise. The second gain of Luenberger observer we chose is based on improving



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Rated power	3.677 kW	Rated Speed	1750 rpm
Rated armature current ( $I_a$ )	15.3 A	Rated armature voltage ( $V_a$ )	240 V
Moment of inertia ( $J$ )	0.02215 kg · m <sup>2</sup>	Armature Inductance ( $L_a$ )	28 mH
Armature resistance ( $R_a$ )	2.581 Ω	Viscous friction coefficient ( $f$ )	0.002953 N · m · s

Table 6.2: Specifications of the DC motor.

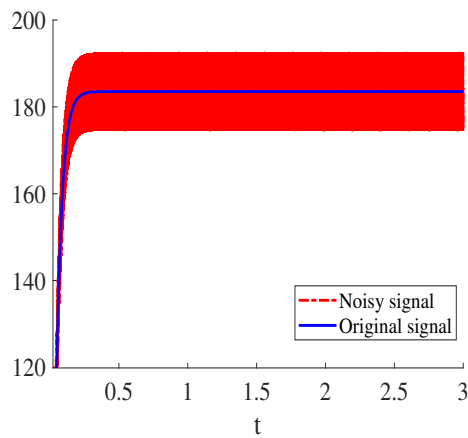


Figure 6.8:  $y$  and its discrete noisy observation in Example 6.3.

convergence speed, which makes it similar to the convergence speed of the proposed estimator. Let  $m = 3$ ,  $\alpha = 0$ , simulations are conducted with Matlab/Simulink software, and the resulting outcomes are analyzed. Fig. 6.9 and Fig. 6.10 show simulation results and errors of speed and current estimated by proposed novel estimator and Luenberger observer, respectively. From these figures, it is seen that the efficiency of proposed novel modulating functions-based estimator and the robustness with respect to noises is better than the Luenberger observer.

Fig. 6.11 and Fig. 6.12 show simulation results of fractional order integrals and derivatives of speed estimated with  $\alpha = 0.2$ ,  $-0.3$  by proposed novel estimator and fractional order Luenberger-like observer respectively. Here we also choose two gain of the observer to better compare with the proposed estimators. Hence, it is confirmed that the speed, current and their fractional order integrals and derivatives estimation of the DC motor are achieved in an effective way by using the proposed novel estimator.

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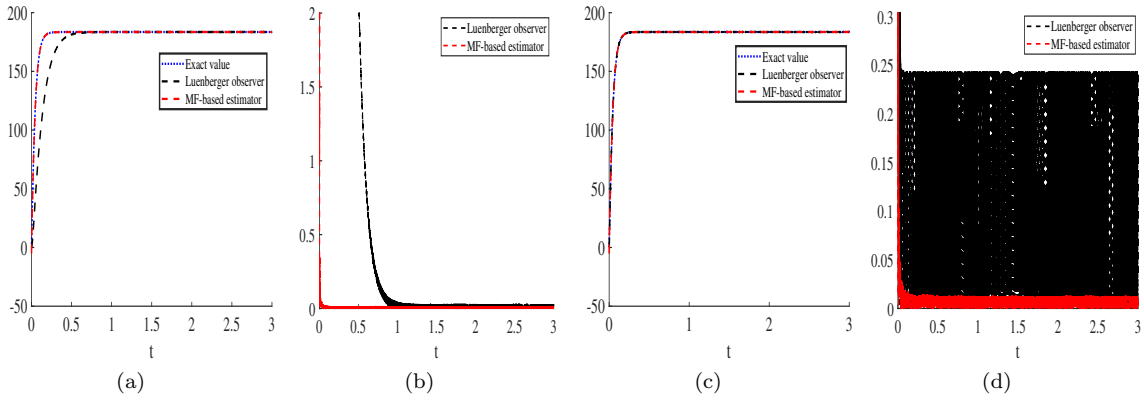


Figure 6.9: Estimations and estimation errors of speed: (a) Estimations with first gain; (b) Estimation errors with first gain; (c) Estimations with second gain; (d) Estimation errors with second gain.

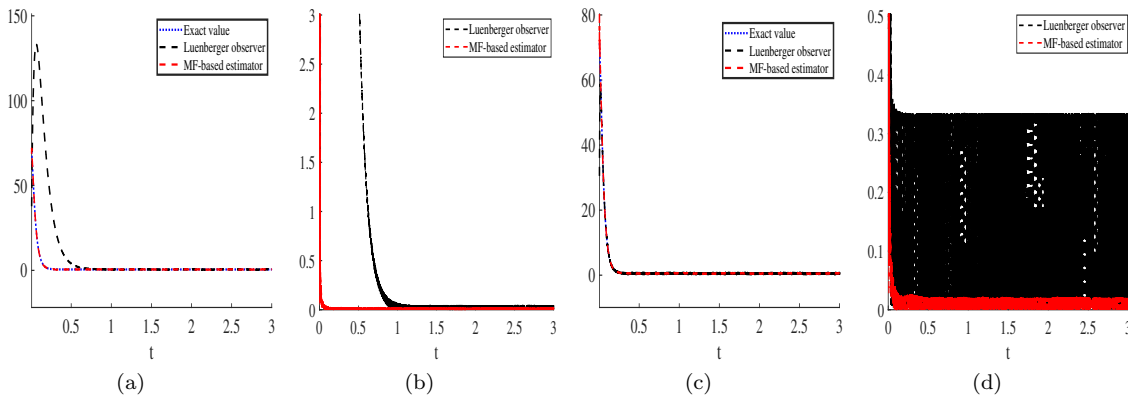


Figure 6.10: Estimations and estimation errors of current: (a) Estimations with first gain; (b) Estimation errors with first gain; (c) Estimations with second gain; (d) Estimation errors with second gain.

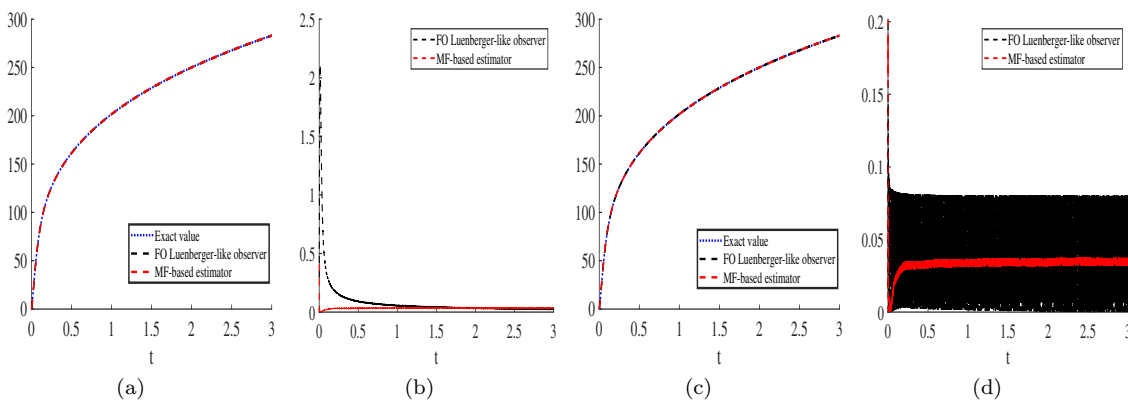


Figure 6.11: Estimations and estimation errors of fractional integral of speed: (a) Estimations with first gain; (b) Estimation errors with first gain; (c) Estimations with second gain; (d) Estimation errors with second gain.

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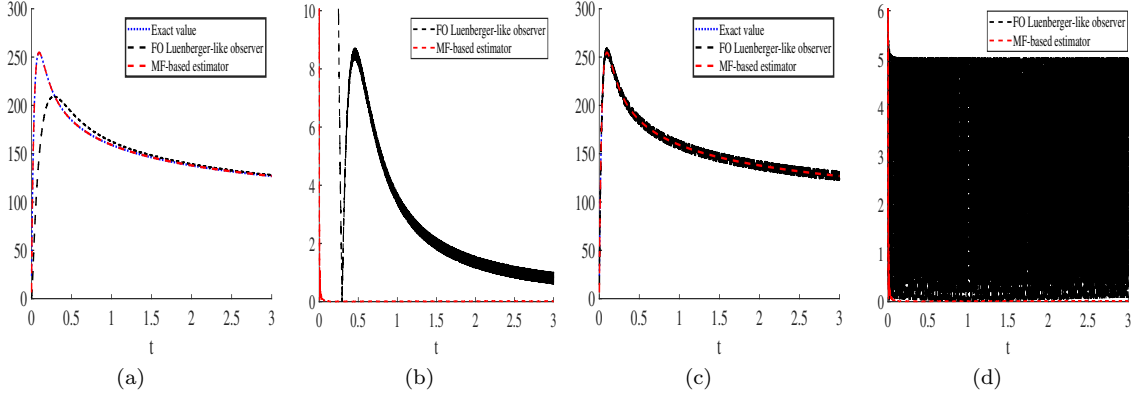


Figure 6.12: Estimations and estimation errors of fractional derivative of speed: (a) Estimations with first gain; (b) Estimation errors with first gain; (c) Estimations with second gain; (d) Estimation errors with second gain.

#### Example 6.4 Fractional order $PI^\lambda D^\mu$ controller for DC motor

In this example, a fractional order  $PI^\lambda D^\mu$  controller is formulated for the purpose of speed control in a DC motor, with the control scheme illustrated in Fig. 6.13. The description of the fractional order  $PI^\lambda D^\mu$  controller [35] is as follows:

$$u(t) = K_p e(t) + K_i D_t^{-\mu} e(t) + K_d D_t^\gamma e(t). \quad (6.57)$$

The fractional order  $PI^\lambda D^\mu$  controller parameters are presented in Table 6.3. The set point for the reference speed is given as 100, rad/s. To better show the estimation results, we do both the closed-loop control with a speed sensor and the novel modulating functions-based state estimator. Fig. 6.14 and Fig. 6.15 illustrate the speed responses of the DC motor, which are captured on a computer using Simulink. When employing fractional order  $PI^\lambda D^\mu$  controllers, we can see that the transient response of the DC motor is free of overshoot, the speed settles smoothly. Moreover, it can be seen that the dynamic effect obtained by the estimator is very similar to that of the speed sensor, which verifies the efficiency of the proposed estimation strategy.

Controllers	$K_p$	$K_i$	$K_d$	$\mu$	$\gamma$
FO-PID	5	40	0.005	0.1	0.2

Table 6.3: Parameters of fractional order  $PI^\lambda D^\mu$  controller in Example 6.4.

### 6.3. APPLICATION

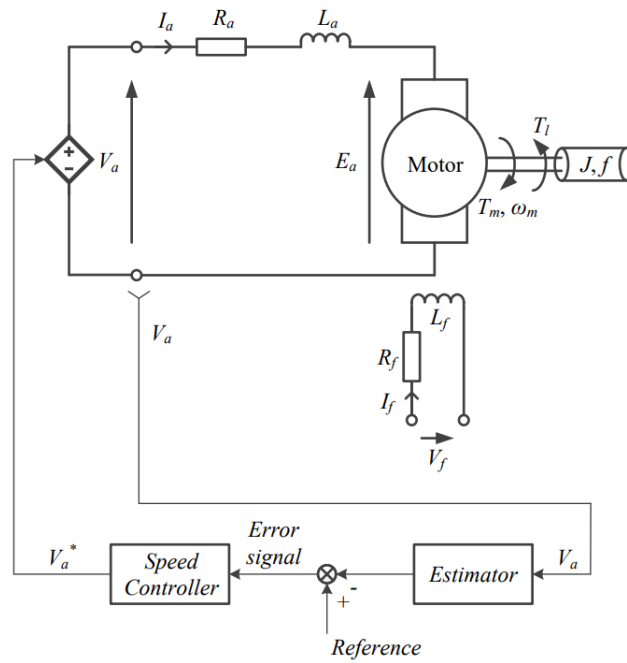


Figure 6.13: Speed control scheme based fractional order  $PI^\lambda D^\mu$  of DC motor in Example 6.4.

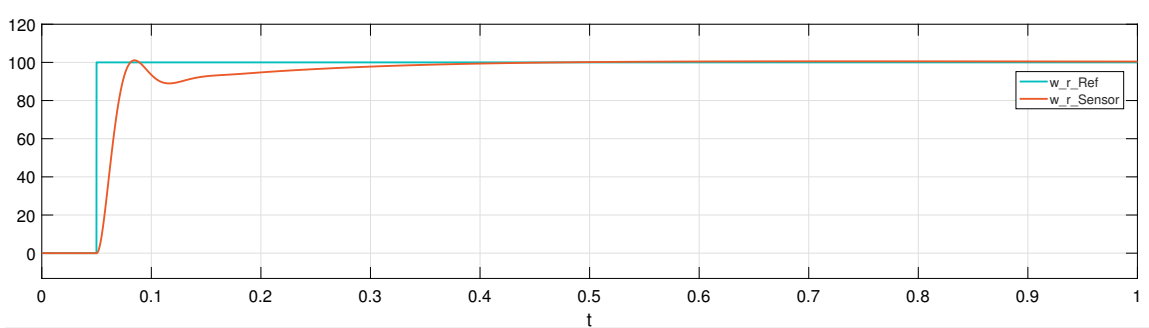


Figure 6.14: Speed response at reference using sensor in Example 6.4.

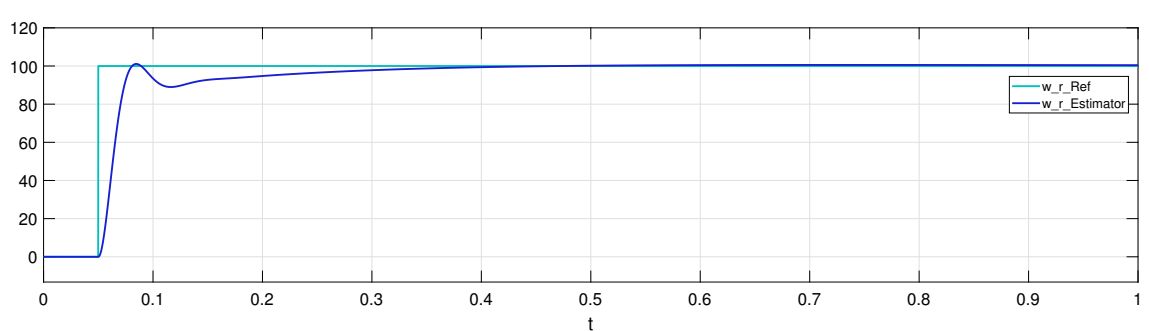


Figure 6.15: Speed response at reference using modulating functions-based estimator in Example 6.4.

## 6.4 Conclusions

In this chapter, non-asymptotic and robust estimators for the state and their fractional derivatives and integrals for integer order linear systems were constructed. First, the considered system was transformed from the state space representation into the observable canonical form, and then cleverly transformed into a fractional order system. Then, modulating functions were applied to obtain algebraic integral formulas of the fractional derivatives and integrals of the state directly. Compared with the existing modulating functions method [74], [76]–[79], [97], the proposed approach eliminates the need for converting to a differential equation form, simplifying the estimation process and enhancing the accuracy of the results. Besides, unlike [111], [112], [118], the proposed novel estimator directly obtained the fractional derivatives and integrals of state without utilizing the Grünwald-Letnikov scheme, resulting in improved efficiency. Finally, numerical examples were given to show the efficiency of the proposed estimators. Besides, we consider applications to DC motors. A part of this work has been published in [119].

## Chapter 7

# Fractional integral and derivative estimations of state for fractional order systems

In this chapter, an improved modulating functions-based fractional derivative and integral estimation method proposed in Chapter 6 is extended to fractional order systems in noisy environment, which is non-asymptotic and robust. Similarly, the considered system modeled by the state-space representation model is firstly transformed into the fractional order observable canonical form and then into a new fractional order linear system. Based on that, modulating functions are applied to obtain algebraic integral formulas for fractional derivative and integral of the state directly. These integral formulas can give robust estimations in discrete noisy cases. Besides, it is shown how to design the required modulating functions. Finally, numerical simulations are provided to demonstrate the efficiency of the proposed method. The main contributions of this chapter are outlined as follows:

- Unlike existing work on the modulating functions method [76]–[79], [113], this chapter can directly obtain algebraic integral formulas for fractional order integral and derivative of the states. It is worth mentioning that the fractional order derivative and integral of the state in [111], [112], [118] as well as Part III were estimated via the Grünwald-Letnikov scheme to the state  $x$  obtained.

### 7.1 Problem formulation

In this chapter, we consider the fractional order linear system (2.60) with  $\eta = 0$  and  $\alpha \in (0, 1)$ , which can be transform into the fractional order observable canonical form by using Lemma 2.7. Let

us recall the fractional order observable canonical form as follows:

$$\begin{cases} D_t^\alpha z &= \hat{A}z + \hat{B}u, \\ y &= \hat{C}z, \end{cases} \quad (7.1)$$

where

$$\hat{A} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & -a_{N-2} \\ 0 & 0 & \cdots & 0 & 1 & -a_{N-1} \end{pmatrix}, \quad (7.2)$$

and  $\hat{B} = TB = (\hat{B}_0, \hat{B}_1, \dots, \hat{B}_{N-1})^\top \in \mathbb{R}^{N \times 1}$ ,  $\hat{C} = (0, \dots, 0, 1) \in \mathbb{R}^{1 \times N}$ ,  $\alpha \in (0, 1)$ .

## 7.2 Main results

Before stating the main results, the following definition is introduced first.

**Definition 7.1** Let  $[0, b] \subset \mathbb{R}$ ,  $j \in \{0, \dots, N-1\}$  and  $g_j$  be a function satisfies the following properties: for  $k = 0, \dots, N-1$ ,

$(\hat{P}_1^\alpha)$ :  ${}_s^C \mathcal{D}_\tau^{k\alpha} g_j$  exists and is integrable on  $[0, b]$ .

- If the following properties are fulfilled: for  $k = 0, \dots, N-1$ ,

$({}^l \hat{P}_2^\alpha)$ :  $\left[ {}_s^C \mathcal{D}_\tau^{k\alpha} g_j(\tau) \right]_{\tau=0} = 0$ ,

$({}^l \hat{P}_3^\alpha)$ :  $\left[ {}_s^C \mathcal{D}_\tau^{k\alpha} g_j(\tau) \right]_{\tau=b} = 1$ , if  $k = j$ ;  $\left[ {}_s^C \mathcal{D}_\tau^{k\alpha} g_j(\tau) \right]_{\tau=b} = 0$ , else,

then  $g_j$  is called a left-sided  $(N, j)^{th}$  fractional modulating function on  $[0, b]$ .

- If the following properties are fulfilled: for  $k = 0, \dots, N-1$ ,

$({}^r \hat{P}_2^\alpha)$ :  $\left[ {}_s^C \mathcal{D}_\tau^{k\alpha} g_j(\tau) \right]_{\tau=0} = 1$ , if  $k = j$ ;  $\left[ {}_s^C \mathcal{D}_\tau^{k\alpha} g_j(\tau) \right]_{\tau=0} = 0$ , else,

$({}^r \hat{P}_3^\alpha)$ :  $\left[ {}_s^C \mathcal{D}_\tau^{k\alpha} g_j(\tau) \right]_{\tau=b} = 0$ ,

then  $g_j$  is called a right-sided  $(N, j)^{th}$  fractional modulating function on  $[0, b]$ .

### 7.2.1 Estimate initial value integral $D_t^{\alpha-1}z(0)$

Inspired by the Theorem 5.1 and Theorem 5.2 in Chapter 4, by designing modulating functions, we can estimate the initial value of the system (7.1), as shown in the following theorem.

**Theorem 7.1** *Consider the system given in (7.1). For  $t \in (0, h]$ , let  $g_j$  be a left-sided  $(N, j - 1)^{th}$  fractional modulating function on  $[0, t]$ . Then, the following integral equation holds:*

$$D_\tau^{\alpha-1}z_j(0) = \sum_{i=1}^N \hat{B}_{i-1} \int_0^t {}^C D_\tau^{(i-1)\alpha} g_j(\tau) u(t-\tau) d\tau - \sum_{i=0}^N a_i \int_0^t {}^C D_\tau^{i\alpha} g_j(\tau) y(t-\tau) d\tau, \quad (7.3)$$

where  $a_N = 1$ .

**Proof.** This proof can be given in the following steps.

**Step 1.** Let  $\eta_i \in C^{[\alpha]}([0, t])$  for  $i = 1, \dots, N$ . Then, by multiplying  $\eta_i$  to (7.1) and integrating on  $[0, t]$ , we get:

$$\int_0^t \eta_i(t-\tau) D_\tau^\alpha z_i(\tau) d\tau = \int_0^t \eta_i(t-\tau) z_{i-1}(\tau) d\tau + \hat{B}_{i-1} \int_0^t \eta_i(t-\tau) u(\tau) d\tau - a_{i-1} \int_0^t \eta_i(t-\tau) y(\tau) d\tau, \quad (7.4)$$

where  $z_0$  is defined by  $z_0(\tau) = 0$  on  $[0, t]$ .

**Step 2.** By applying the fractional order integration by parts formula given in Lemma 2.6 to the left side of (7.4), we get: for  $i = 1, \dots, N$ ,

$$\begin{aligned} & \left[ D_\tau^{\alpha-1} z_i(\tau) \eta_i(t-\tau) \right]_{\tau=0}^{\tau=t} + \int_0^t {}^C D_\tau^\alpha \eta_i(\tau) z_i(t-\tau) d\tau \\ &= \int_0^t \eta_i(\tau) z_{i-1}(t-\tau) d\tau + \hat{B}_{i-1} \int_0^t \eta_i(\tau) u(t-\tau) d\tau - a_{i-1} \int_0^t \eta_i(\tau) y(t-\tau) d\tau. \end{aligned} \quad (7.5)$$

**Step 3.** Let us take  $\eta = \eta_1$ , and  $\eta_i = {}^C D_\tau^{(i-1)\alpha} \eta$  for  $i = 2, \dots, N$ . Then, by combining the  $N$  equations for  $i = 1, 2, \dots, N$  of (7.5), we get:

$$\begin{aligned} \sum_{i=1}^N \left[ {}^C D_\tau^{(i-1)\alpha} \eta(t-\tau) D_\tau^{\alpha-1} z_i(\tau) \right]_{\tau=0}^{\tau=t} &= \sum_{i=1}^N \hat{B}_{i-1} \int_0^t {}^C D_\tau^{(i-1)\alpha} \eta(\tau) u(t-\tau) d\tau \\ &\quad - \sum_{i=0}^N a_i \int_0^t {}^C D_\tau^{i\alpha} \eta(\tau) y(t-\tau) d\tau, \end{aligned} \quad (7.6)$$

by replacing  $\eta$  by  $g_j$  for  $j = 1, \dots, N$  in (7.6), we get:

$$\begin{aligned} \sum_{i=1}^N \left[ {}^C D_\tau^{(i-1)\alpha} g_j(t-\tau) D_\tau^{\alpha-1} z_j(\tau) \right]_{\tau=0}^{\tau=t} &= \sum_{i=1}^N \hat{B}_{i-1} \int_0^t {}^C D_\tau^{(i-1)\alpha} g_j(\tau) u(t-\tau) d\tau \\ &\quad - \sum_{i=0}^N a_i \int_0^t {}^C D_\tau^{i\alpha} g_j(\tau) y(t-\tau) d\tau. \end{aligned} \quad (7.7)$$

Finally, since  $g_j$  satisfy  $({}^l \hat{P}_2^\alpha)$  and  $({}^l \hat{P}_3^\alpha)$  on  $[0, t]$ , this proof can be completed. ■

The construction of the required modulating function in Theorem 7.1 is provided as follows.



**Proposition 7.1** *Let  $N \in \mathbb{N}^*$ ,  $\alpha \in (0, 1)$ ,  $\beta < 0$  and  $N \leq m \in \mathbb{N}_+$ . For  $t \in (0, h]$  and  $j \in \{1, \dots, N\}$ , let  $g_j$  be a polynomial function defined on  $[0, t]$  as follows:*

$$g_j(\tau) = \sum_{i'=1}^N \lambda_{i'} t^{(j-1)\alpha-m-i'} \tau^{m+i'}, \quad (7.8)$$

the coefficient  $\Theta_g = (\lambda_1, \dots, \lambda_N)^\top$  is the solution of the following linear system:

$$B_g \Theta_g = d_j, \quad (7.9)$$

where  $B_g \in \mathbb{R}^{N \times N}$  and  $d_j \in \mathbb{R}^{N \times 1}$  are defined on as follows: for  $s, i' = 1, \dots, N$ ,

$$B_g(s, i') = \frac{\Gamma(m + i' + 1)}{\Gamma(m + i' - (s - 1)\alpha + 1)}, \quad (7.10)$$

and  $d_j(s) = 1$ , if  $s = j$ ,  $d_j(s) = 0$ , else. Then  $g_j$  is a left-sided  $(N, j - 1)^{\text{th}}$  fractional modulating function on  $[0, t]$  required in Theorem (7.1).

**Proof.** Since  $g_j$  is a polynomial,  $g_j \in \mathcal{C}^N([0, t])$ , i.e.  $(\hat{P}_1^\alpha)$  is fulfilled. On the one hand, by calculating the  $k$ -th derivative of  $g_j$  for  $k = 0, \dots, N - 1$ , we obtain:  $\forall \tau \in [0, t]$ ,

$${}_s^C \mathcal{D}_\tau^{k\alpha} g_j(\tau) = \sum_{i'=1}^N \lambda_{i'} t^{(j-1)\alpha-m-i'} \frac{\Gamma(m + i' + 1)}{\Gamma(m + i' - k\alpha + 1)} \tau^{m+i'-k\alpha}. \quad (7.11)$$

Since  $m \geq N$ ,  ${}_s^C \mathcal{D}_\tau^{k\alpha} g_j(0) = 0$ , i.e.  $({}^l \hat{P}_2^\alpha)$  is satisfied. Then, by taking  $\tau = t$  in (7.11), it yields: for  $k = 0, \dots, N - 1$ ,

$${}_s^C \mathcal{D}_\tau^{k\alpha} g_j(t) = \sum_{i'=1}^N \lambda_{i'} \frac{\Gamma(m + i' + 1)}{\Gamma(m + i' - k\alpha + 1)} t^{(j-1-k)\alpha}. \quad (7.12)$$

Based on (7.12), by solving the system given in (7.9), the construction of  $g_j$  is completed. ■

**Lemma 7.1** *Assume that there exists a small  $\varepsilon \in \mathbb{R}_+$  such that  $f$  is essentially bounded on  $[0, \varepsilon]$ , except on a set of measure zero. Then, we have:*

$$\left[ D_t^{-\alpha} f(t) \right]_{t=0} = \lim_{t \rightarrow 0} D_t^{-\alpha} f(t) = 0. \quad (7.13)$$

The proof of the previous lemma can be given using a similar way as done in [139] where  $f$  was assumed to be continuous on  $[0, \varepsilon]$ .

Next, we will given the estimation process of fractional integrals and derivatives of the states.

• **System transformation**

**Lemma 7.2** *Let  $\beta \in \mathbb{R}$ , based on the observable canonical form (7.1) of the system defined in (2.60), the following system can be obtained:  $\forall 0 < t \in I$ ,*

$$D_t^{\alpha+\beta} z(t) = \hat{A}D_t^\beta z(t) + \hat{B}D_t^\beta u(t) + \frac{D_t^{\alpha-1} z(0)}{\Gamma(-\beta)} t^{-\beta-1}. \quad (7.14)$$

**Proof.** By taking the  $\beta$ -order fractional order derivative operator with  $\beta \in \mathbb{R}$  on both sides of (7.1), we get:

$$D_t^\beta \{D_t^\alpha z(t)\} = \hat{A}D_t^\beta z(t) + \hat{B}D_t^\beta u(t). \quad (7.15)$$

By applying the additive index law given by (2.1) to the left side of (7.15), we have:

$$\begin{aligned} D_t^\beta \{D_t^\alpha z(t)\} &= D_t^{\alpha+\beta} z(t) - \frac{D_t^{\alpha-1} z(0)}{\Gamma(-\beta)} t^{-\beta-1} \\ &= \hat{A}D_t^\beta z(t) + \hat{B}D_t^\beta u(t), \end{aligned} \quad (7.16)$$

then, we can get (7.14). Thus, this proof can be completed. ■

### 7.2.2 Fractional integral estimation of state

**Theorem 7.2** *Consider the system given by (7.14) with  $\beta < 0$  on  $I$ . Let  $\eta_j$  for  $j = 1, \dots, N$  be a group of right-sided  $(N, j-1)^{th}$  fractional modulating function on  $[0, t]$ . Then, the fractional integral of the state  $D_t^{\alpha+\beta-1} z$  can be calculated by the following formula: for  $j = 1, \dots, N$ ,*

$$\begin{aligned} D_t^{\alpha+\beta-1} z_j(t) &= \sum_{i=1}^N \hat{B}_{i-1} \int_0^t D_\tau^\beta \{ {}^C \mathcal{D}_\tau^{(i-1)\alpha} \eta_j(\tau) \} u(t-\tau) d\tau - \sum_{i=0}^N a_i \int_0^t D_\tau^\beta \{ {}^C \mathcal{D}_\tau^{i\alpha} \eta_j(\tau) \} y(t-\tau) d\tau \\ &\quad + \sum_{i=1}^N \frac{D_t^{\alpha-1} z_j(0)}{\Gamma(-\beta)} \int_0^t {}^C \mathcal{D}_\tau^{(i-1)\alpha} \eta_j(t-\tau) \tau^{-\beta-1} d\tau, \end{aligned} \quad (7.17)$$

where  $a_N = 1$ ,  $D_\tau^{\alpha-1} z(0)$  is given by (7.3).

**Proof.** This proof can be given in the following steps.

**Step 1.** By multiplying  $g_1, \dots, g_N$  to (7.14) and integrating on  $[0, t]$ , the following formula holds: for  $i = 1, \dots, N$ ,

$$\begin{aligned} \int_0^t g_i(t-\tau) D_\tau^{\alpha+\beta} z_i(\tau) d\tau &= \int_0^t g_i(t-\tau) D_\tau^\beta z_{i-1}(\tau) d\tau + \hat{B}_{i-1} \int_0^t g_i(t-\tau) D_\tau^\beta u(\tau) d\tau \\ &\quad - a_{i-1} \int_0^t g_i(t-\tau) D_\tau^\beta y(\tau) + \frac{D_t^{\alpha-1} z_i(0)}{\Gamma(-\beta)} \int_0^t g_i(t-\tau) \tau^{-\beta-1} d\tau. \end{aligned} \quad (7.18)$$

## 7.2. MAIN RESULTS

By applying the additive index law given by (2.1), we have  $D_\tau^\alpha \{D_\tau^\beta z(\tau)\} = D_\tau^{\alpha+\beta} z(\tau)$ , then (7.18) becomes:

$$\begin{aligned} \int_0^t g_i(t-\tau) D_\tau^\alpha \{D_\tau^\beta z_i(\tau)\} d\tau &= \int_0^t g_i(t-\tau) D_\tau^\beta z_{i-1}(\tau) d\tau + \hat{B}_{i-1} \int_0^t g_i(t-\tau) D_\tau^\beta u(\tau) d\tau \\ &\quad - a_{i-1} \int_0^t g_i(t-\tau) D_\tau^\beta y(\tau) d\tau + \frac{D_t^{\alpha-1} z_i(0)}{\Gamma(-\beta)} \int_0^t g_i(t-\tau) \tau^{-\beta-1} d\tau. \end{aligned} \quad (7.19)$$

**Step 2.** Then, using the fractional order integration by parts formula given in Lemma 2.6 to (7.19), we get:

$$\begin{aligned} &\left[ g_i(t-\tau) D_\tau^{\alpha+\beta-1} z_i(\tau) \right]_{\tau=0}^{\tau=t} + \int_0^t {}^C D_\tau^\alpha g_i(\tau) D_\tau^\beta z_i(t-\tau) d\tau \\ &= \int_0^t g_i(t-\tau) D_\tau^\beta z_{i-1}(\tau) d\tau + \hat{B}_{i-1} \int_0^t D_\tau^\beta g_i(\tau) u(t-\tau) d\tau \\ &\quad - a_{i-1} \int_0^t D_\tau^\beta g_i(\tau) y(t-\tau) d\tau + \frac{D_t^{\alpha-1} z_i(0)}{\Gamma(-\beta)} \int_0^t g_i(t-\tau) \tau^{-\beta-1} d\tau. \end{aligned} \quad (7.20)$$

**Step 2.** Note that if take  $\tau = t - \tau$ , we can easily get  $\int_0^t {}^C D_\tau^\alpha g(\tau) D_\tau^\beta z(t-\tau) d\tau = \int_0^t {}^C D_\tau^\alpha g(t-\tau) D_\tau^\beta z(\tau) d\tau$ . Let us take  $g_1 = g$ ,  $g_i = {}^C \mathcal{D}_\tau^{(i-1)\alpha} g$  for  $i = 2, \dots, N$ . Then, by combining  $N$  equations of (7.20), we get:

$$\begin{aligned} \sum_{i=1}^N \left[ {}^C \mathcal{D}_\tau^{(i-1)\alpha} g(t-\tau) D_\tau^{\alpha+\beta-1} z_i(\tau) \right]_{\tau=0}^{\tau=t} &= \sum_{i=1}^N \hat{B}_{i-1} \int_0^t D_\tau^\beta \{ {}^C \mathcal{D}_\tau^{(i-1)\alpha} g(\tau) \} u(t-\tau) d\tau \\ &\quad - \sum_{i=0}^N a_i \int_0^t D_\tau^\beta \{ {}^C \mathcal{D}_\tau^{i\alpha} g(\tau) \} y(t-\tau) d\tau \\ &\quad + \sum_{i=1}^N \frac{D_t^{\alpha-1} z_i(0)}{\Gamma(-\beta)} \int_0^t {}^C \mathcal{D}_\tau^{(i-1)\alpha} g(t-\tau) \tau^{-\beta-1} d\tau. \end{aligned} \quad (7.21)$$

By replacing  $g$  by  $\eta_j$  for  $j = 1, \dots, N$  in (7.21), we have:

$$\begin{aligned} \sum_{i=1}^N \left[ {}^C \mathcal{D}_\tau^{(i-1)\alpha} \eta_j(t-\tau) D_\tau^{\alpha+\beta-1} z_j(\tau) \right]_{\tau=0}^{\tau=t} &= \sum_{i=1}^N \hat{B}_{i-1} \int_0^t D_\tau^\beta \{ {}^C \mathcal{D}_\tau^{(i-1)\alpha} \eta_j(\tau) \} u(t-\tau) d\tau \\ &\quad - \sum_{i=0}^N a_i \int_0^t D_\tau^\beta \{ {}^C \mathcal{D}_\tau^{i\alpha} \eta_j(\tau) \} y(t-\tau) d\tau \\ &\quad + \sum_{i=1}^N \frac{D_t^{\alpha-1} z_j(0)}{\Gamma(-\beta)} \int_0^t {}^C \mathcal{D}_\tau^{i\alpha} \eta_j(t-\tau) \tau^{-\beta-1} d\tau. \end{aligned} \quad (7.22)$$

Finally, since  $\eta_j$  satisfy  $({}^r \hat{P}_2^\alpha)$  and  $({}^r \hat{P}_3^\alpha)$  on  $[0, t]$ , this proof is completed. ■

### 7.2.3 Fractional derivative estimation of state

**Theorem 7.3** Consider the system given by (7.14) with  $\beta \in (0, 1)$  on  $I$ . Let  $g_j$  be a group of right-sided  $(N + 1, j - 1)^{th}$  fractional modulating functions on  $[0, t]$  for  $j = 1, \dots, N$ , then the fractional derivative of the state  $D_t^{\alpha+\beta-1}z$  can be calculated by the following formula: for  $j = 1, \dots, N$ ,

$$\begin{aligned} D_t^{\alpha+\beta-1}z_j(t) &= \sum_{i=1}^N \hat{B}_{i-1} \int_0^t {}^C D_\tau^{(i-1)\alpha} g_j(t-\tau) D_\tau^\beta u(\tau) d\tau - \sum_{i=0}^N a_i \int_0^t {}^C D_\tau^\beta \{ {}^C D_\tau^{i\alpha} g_j(t-\tau) \} y(\tau) d\tau \\ &\quad - a_{j-1} D_t^{\beta-1} y(t) + \sum_{i=1}^N \frac{D_t^{\alpha-1} z_j(0)}{\Gamma(-\beta)} \int_0^t {}^C D_\tau^{(i-1)\alpha} g_j(t-\tau) \tau^{-\beta-1} d\tau, \end{aligned} \quad (7.23)$$

where  $a_N = 1$ ,  $D_t^{\alpha-1}z(0)$  is given by (7.3).

**Proof.** This proof will be given in the following steps.

**Step 1.** By multiplying  $g_i$  for  $i = 1, \dots, N$  to (7.14) and integrating on  $[0, t]$ , the following formula holds:

$$\begin{aligned} \int_0^t g_i(t-\tau) D_\tau^{\alpha+\beta} z_i(\tau) d\tau &= \int_0^t g_i(t-\tau) D_\tau^\beta z_{i-1}(\tau) d\tau + \hat{B}_{i-1} \int_0^t g_i(t-\tau) D_\tau^\beta u(\tau) d\tau \\ &\quad - a_{i-1} \int_0^t g_i(t-\tau) D_\tau^\beta y(\tau) d\tau + \frac{D_t^{\alpha-1} z_i(0)}{\Gamma(-\beta)} \int_0^t g_i(t-\tau) \tau^{-\beta-1} d\tau. \end{aligned} \quad (7.24)$$

By applying the additive index law given by (2.1), we get  $D_\tau^\alpha \{ D_\tau^\beta z(\tau) \} = D_\tau^{\alpha+\beta} z(\tau) - D_t^{\beta-1} z(0) \frac{\tau^{-\alpha-1}}{\Gamma(-\alpha)}$ , then (7.24) becomes: for  $i = 1, \dots, N$ ,

$$\begin{aligned} \int_0^t g_i(t-\tau) D_\tau^\alpha \{ D_\tau^\beta z_i(\tau) \} d\tau &= \int_0^t g_i(t-\tau) D_\tau^\beta z_{i-1}(\tau) d\tau + \hat{B}_{i-1} \int_0^t g_i(t-\tau) D_\tau^\beta u(\tau) d\tau \\ &\quad - a_{i-1} \int_0^t g_i(t-\tau) D_\tau^\beta y(\tau) d\tau - D_\tau^{\beta-1} z_i(0) \int_0^t g_i(t-\tau) \frac{\tau^{-\alpha-1}}{\Gamma(-\alpha)} d\tau \\ &\quad + \frac{D_t^{\alpha-1} z_i(0)}{\Gamma(-\beta)} \int_0^t g_i(t-\tau) \tau^{-\beta-1} d\tau. \end{aligned} \quad (7.25)$$

**Step 2.** Using the fractional order integration by parts formula given in Lemma 2.6 to the (7.25), we get: for  $i = 1, \dots, N$ ,

$$\begin{aligned} &\left[ g_i(t-\tau) D_\tau^{\alpha-1} \{ D_\tau^\beta z_i(\tau) \} \right]_{\tau=0}^{\tau=t} + \int_0^t {}^C D_\tau^\alpha g_i(\tau) D_\tau^\beta z_i(t-\tau) d\tau \\ &= \int_0^t g_i(t-\tau) D_\tau^\beta z_{i-1}(\tau) d\tau + \hat{B}_{i-1} \int_0^t g_i(t-\tau) D_\tau^\beta u(\tau) d\tau - a_{i-1} \int_0^t g_i(t-\tau) D_\tau^\beta y(\tau) d\tau \\ &\quad - D_t^{\beta-1} z_i(0) \int_0^t \dot{g}_i(t-\tau) \frac{\tau^{-\alpha}}{\Gamma(1-\alpha)} d\tau - D_t^{\beta-1} z_i(0) \left[ g_i(t-\tau) \frac{\tau^{-\alpha}}{\Gamma(1-\alpha)} \right]_{\tau=0}^{\tau=t} \\ &\quad + \frac{D_t^{\alpha-1} z_i(0)}{\Gamma(-\beta)} \int_0^t g_i(t-\tau) \tau^{-\beta-1} d\tau, \end{aligned} \quad (7.26)$$

where  $D_t^{\beta-1} z_i(0) \int_0^t \dot{g}_i(t-\tau) \frac{\tau^{-\alpha}}{\Gamma(1-\alpha)} d\tau = D_t^{\beta-1} z_i(0) {}^C D_t^\alpha g_i(t)$ .

**Step 3.** Let us take  $g_1 = g$ ,  $g_i = {}^C D_\tau^{(i-1)\alpha} g$  for  $i = 2, \dots, N$ . Then, by combining the  $N$  equations of (7.26) for  $i = 1, \dots, N$ , we can obtain:

$$\begin{aligned}
 & \sum_{i=1}^N \left[ {}^C D_\tau^{(i-1)\alpha} g(t-\tau) D_\tau^{\alpha-1} \{ D_\tau^\beta z_i(\tau) \} \right]_{\tau=0}^{\tau=t} \\
 &= \sum_{i=1}^N \hat{B}_{i-1} \int_0^t {}^C D_\tau^{(i-1)\alpha} g(t-\tau) D_\tau^\beta u(\tau) d\tau - \sum_{i=0}^N a_i \int_0^t {}^C D_\tau^{i\alpha} g(t-\tau) D_\tau^\beta y(\tau) d\tau \\
 & \quad - \sum_{i=1}^N D_t^{\beta-1} z_i(0) {}^C D_t^{i\alpha} g(t) - \sum_{i=1}^N D_t^{\beta-1} z_i(0) \left[ {}^C D_\tau^{(i-1)\alpha} g(t-\tau) \frac{\tau^{-\alpha}}{\Gamma(1-\alpha)} \right]_{\tau=0}^{\tau=t} \\
 & \quad + \sum_{i=1}^N \frac{D_t^{\alpha-1} z_i(0)}{\Gamma(-\beta)} \int_0^t {}^C D_\tau^{(i-1)\alpha} g(t-\tau) \tau^{-\beta-1} d\tau.
 \end{aligned} \tag{7.27}$$

By applying the fractional order integration by parts formula given in Lemma 2.6, (7.27) becomes:

$$\begin{aligned}
 & \sum_{i=1}^N \left[ {}^C D_\tau^{(i-1)\alpha} g(t-\tau) D_\tau^{\alpha+\beta-1} z_i(\tau) \right]_{\tau=0}^{\tau=t} - \sum_{i=1}^N \left[ {}^C D_\tau^{(i-1)\alpha} g(t-\tau) \frac{D_t^{\beta-1} z_i(0)}{\Gamma(1-\alpha)} \tau^{-\alpha} \right]_{\tau=0}^{\tau=t} \\
 &= \sum_{i=1}^N \hat{B}_{i-1} \int_0^t {}^C D_\tau^{(i-1)\alpha} g(t-\tau) D_\tau^\beta u(\tau) d\tau \\
 & \quad - \sum_{i=0}^N a_i \int_0^t {}^C D_\tau^\beta \{ {}^C D_\tau^{i\alpha} g(\tau) \} y(t-\tau) d\tau - \sum_{i=0}^N a_i \left[ {}^C D_\tau^{i\alpha} g(t-\tau) D_\tau^{\beta-1} y(\tau) \right]_{\tau=0}^{\tau=t} \\
 & \quad - \sum_{i=1}^N D_t^{\beta-1} z_i(0) {}^C D_t^{i\alpha} g(t) - \sum_{i=1}^N D_t^{\beta-1} z_i(0) \left[ {}^C D_\tau^{(i-1)\alpha} g(t-\tau) \frac{\tau^{-\alpha}}{\Gamma(1-\alpha)} \right]_{\tau=0}^{\tau=t} \\
 & \quad + \sum_{i=1}^N \frac{D_t^{\alpha-1} z_i(0)}{\Gamma(-\beta)} \int_0^t {}^C D_\tau^{(i-1)\alpha} g(t-\tau) \tau^{-\beta-1} d\tau,
 \end{aligned} \tag{7.28}$$

then, by sorting out (7.28), we get:

$$\begin{aligned}
 & \sum_{i=1}^N \left[ {}^C D_\tau^{(i-1)\alpha} g(t-\tau) D_\tau^{\alpha+\beta-1} z_i(\tau) \right]_{\tau=0}^{\tau=t} \\
 &= \sum_{i=1}^N \hat{B}_{i-1} \int_0^t {}^C D_\tau^{(i-1)\alpha} g(t-\tau) D_\tau^\beta u(\tau) d\tau \\
 & \quad - \sum_{i=0}^N a_{i-1} \int_0^t {}^C D_\tau^\beta \{ {}^C D_\tau^{i\alpha} g(\tau) \} y(t-\tau) d\tau - \sum_{i=0}^N a_{i-1} \left[ {}^C D_\tau^{i\alpha} g(t-\tau) D_\tau^{\beta-1} y(\tau) \right]_{\tau=0}^{\tau=t} \\
 & \quad - \sum_{i=1}^N D_t^{\beta-1} z_i(0) {}^C D_t^{i\alpha} g(t) + \sum_{i=1}^N \frac{D_t^{\alpha-1} z_i(0)}{\Gamma(-\beta)} \int_0^t {}^C D_\tau^{(i-1)\alpha} g(t-\tau) \tau^{-\beta-1} d\tau.
 \end{aligned} \tag{7.29}$$

By replacing  $g$  by  $g_j$  for  $j = 1, \dots, N$  in (7.29), since  $g_j$  for  $j = 1, \dots, N$  satisfy  $({}^r \hat{P}_2^\alpha)$  and  $({}^r \hat{P}_3^\alpha)$  on  $[0, t]$ , we can obtain (7.23), which completing this proof.  $\blacksquare$

**Remark 7.1** *The fractional modulating functions required in the Theorem 7.2 and Theorem 7.3 are the same as those given by Proposition 5.1 in the case of  $\alpha \in (0, 1)$ , which will not be repeated here.*

According to Theorem 7.2 and Theorem 7.3,  $D_t^{\alpha+\beta-1}x$  with  $\alpha \in (-1, 1)$  can be calculated by taking the inverse matrix of  $T$  and using (2.66).

### 7.3 Simulation results

In order to illustrate the accuracy and robustness with respect to corrupting noises of the proposed estimators in Subsection 7.2.2 and Subsection 7.2.3, the Chapter 5 [118] and [79] are considered to be compared with the proposed estimators, with some numerical results given as follows.

**Example 7.1** *Let us take  $N = 3$ ,  $u(t) = 5\sin(5t)$ ,  $\alpha = 0.5$ , and the following matrices in (7.1):*

$$A = \begin{pmatrix} 11 & 1 & -11 \\ 6 & 0 & -6 \\ 18 & 1 & -17 \end{pmatrix}, B = (0, 1, 0)^\top, C = (-1, 0, 1),$$

*which satisfy the conditions  $(C_1)$ – $(C_2)$ . Assume that the initial value of the state is given by  $x(0) = (0.1, 0.1, 0.1)^\top$ , and  $y^\varpi(t_i) = y(t_i) + \sigma\varpi(t_i)$  is the noisy observation of the output  $y$  on  $I = [0, 5]$ , where  $t_i = iT_s$  with  $T_s = 0.001$ ,  $\{\varpi(t_i)\}$  is a zero-mean white Gaussian noise, and the value of  $\sigma$  is adjusted such that the signal-to-noise ratio  $SNR = 10 \log_{10} \left( \frac{\sum |y^\varpi(t_i)|^2}{\sum |\sigma\varpi(t_i)|^2} \right)$  is equal to 20dB. The output and the associated noisy output are shown in Fig. 7.1.*

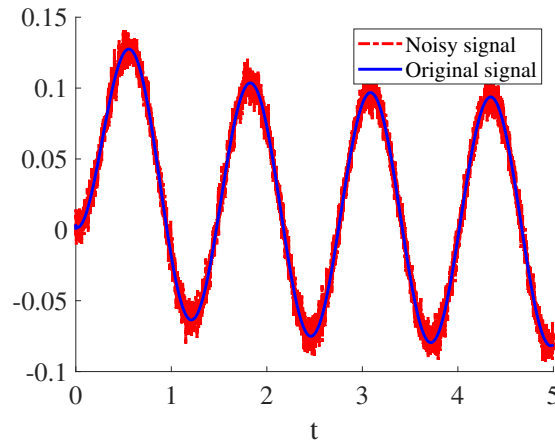


Figure 7.1:  $y$  and its discrete noisy observation in Example 7.1.

*By utilizing Theorem 7.1, the fractional integral initial value  $D_t^{-0.5}x(0)$  can be estimated in this discrete noisy case, as shown in Fig. 7.2. The validity of the estimator is verified according to Lemma*

### 7.3. SIMULATION RESULTS

7.1. Meanwhile, based on these estimated values, the fractional integral and derivative of the state can be estimated by utilizing Theorem 7.2 and Theorem 7.3, where the trapezoidal numerical integration method is used to approximate the integral formulas. By taking  $\beta = -0.3, 0.65$ ,  $m = 4$ , the estimation values and estimation errors of  $D_t^{-0.8}x_1$  and  $D_t^{0.15}x_1$  by using the proposed method and the one in Chapter 5 [118] are shown in Fig. 7.3 and Fig. 7.4, respectively. In addition, [79] is considered to estimate the fractional derivative of the state for comparison with the proposed method in this chapter. Fig. 7.3 and Fig. 7.4 show both the proposed estimator give satisfactory results with fast convergence and good robustness against noises.

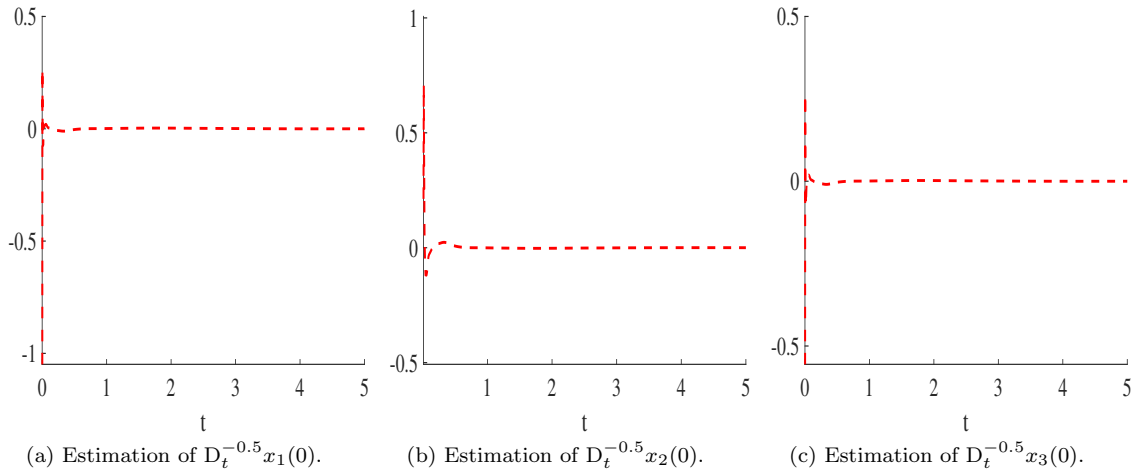


Figure 7.2: Estimations of  $D_t^{-0.5}x(0)$  obtained by the proposed method in Example 7.1.

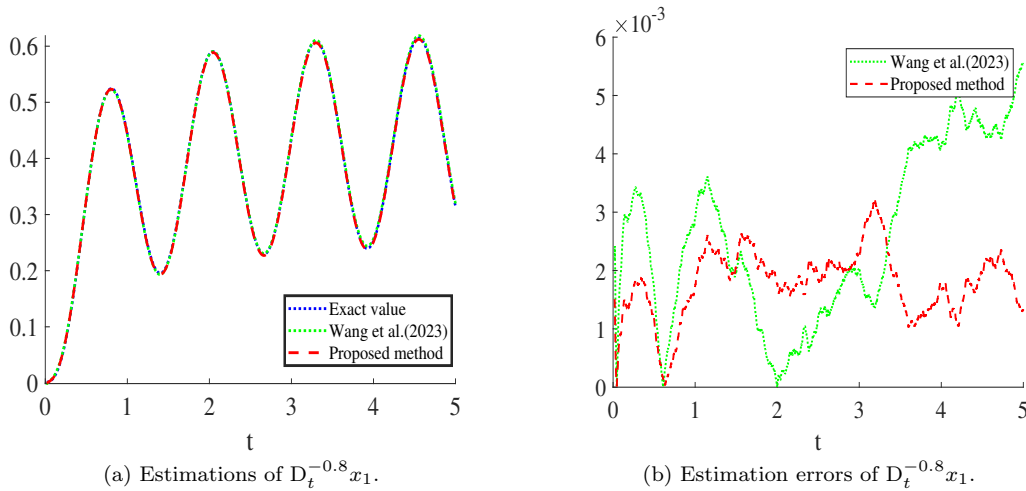


Figure 7.3: Estimations and estimation errors of  $D_t^{-0.8}x_1$  obtained by the proposed method and the one in Chapter 5 [118].

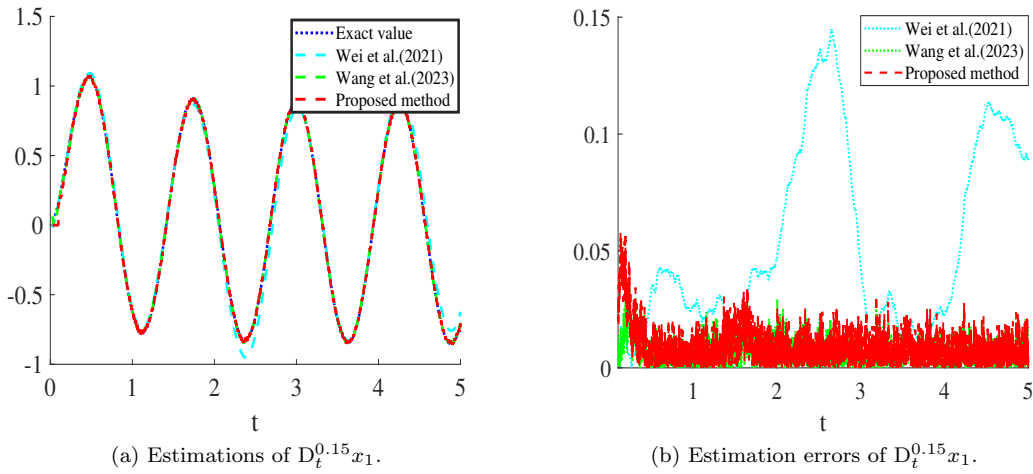


Figure 7.4: Estimations and estimation errors of  $D_t^{0.15} x_1$  obtained by the proposed method and the one in [79] and Chapter 5 [118].

## 7.4 Conclusions

In this chapter, non-asymptotic and robust estimators for the state and its fractional derivative and integral for fractional order linear systems were constructed. First, the considered system was transformed from the state space representation into the fractional order observable canonical form, and then cleverly transformed into a fractional order system. Then, modulating functions were applied to obtain algebraic integral formulas of the fractional derivative and integral of the state directly. Unlike [111], [112], [118], the proposed novel estimator directly obtained the fractional derivative and integral of state without utilizing the Grünwald-Letnikov scheme. Finally, the efficiency of the proposed estimators was verified by numerical simulations.



#### 7.4. CONCLUSIONS

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## Part V

# Conclusions and perspectives



## Chapter 8

# Conclusions and perspectives

### 8.1 Conclusions

This PhD thesis was devoted to developing non-asymptotic and robust state and its fractional derivative and integral estimation methods for integer order systems and non-integer order systems to simplify the estimation process and enhance efficiency. By applying the proposed improved modulating functions-based method, the proposed results were given by algebraic integral formulas. Similar to the existing works, the results of this thesis still maintain the advantages of the modulating functions method. On the one hand, the algebraic formulas can provide analytical computation in continuous-time and non-asymptotically provide estimation results in discrete-time (thus finite-time convergent). On the other hand, since the integrals can work as low-pass filters, the deduced formulas are robust to corrupting noises in discrete-time.

Chapter 2 recalled the foundations of fractional calculus and introduced the basic concepts and conclusions of fractional order linear systems. On this basis, Chapter 3 recalled definitions of modulating functions and reviewed recent developments for fractional order systems in the modulating functions method. In addition, applications of the modulating functions method to parameter identification, design of differentiators, and state estimation were presented through detailed derivations and examples.

On the basis, motivated by the complexity and inefficiencies in existing modulating functions state estimation methods based on input-output differential equations, an improved modulating functions-based state estimation method is developed for integer order and non-integer order systems in Chapter 4 and 5 to simplify the estimation process and improve the estimation efficiency. The main benefits of this work are summarized as follows:

- Different from existing works on the modulating functions method [74], [76]–[79], [97], the improved method proposed in Part III is directly applied to the observable canonical form, does not

require the conversion to the form of differential equation, nor need to estimate the derivatives of output and a set of fractional order initial values to obtain algebraic integral formulas for the sought variables, they make the estimation process easier and the obtained formulas simpler. In particular, the proposed method is more efficient.

- The considered fractional order system model is more general, which is not subject to the fractional differentiation orders. In fact, the methods proposed in [76], [78] can only be applied for rational order systems.

Considering both in existing works [111], [112], [118] and the improved method proposed in Chapter 4 and 5, the fractional derivatives of the state are obtained via Grünwald-Letnikov scheme. To obtain directly the algebraic integral formulas of the fractional derivatives and integrals of the states, the second improved modulating functions-based state estimation method is introduced for integer order and non-integer order systems in Chapter 6 and 7. The main benefits of this work are summarized as follows:

- Different from [74], [76]–[79], [97], the considered systems do not need to be transformed into the input-output differential equations, which makes the estimation process and the obtained formulas simpler.
- Unlike existing work on the modulating functions method [76]–[79], [113], this chapter can directly obtain algebraic integral formulas for fractional order integral and derivative of the states without estimating the fractional initial values of the output. It is worth mentioning that the fractional order derivative and integral of the state in [111], [112], [118] as well as Part III were estimated by applying the Grünwald-Letnikov scheme to the state  $x$  obtained.
- A novel modulating functions-based estimator is applied for the first time to estimate the current, speed and their fractional order derivatives and integrals for DC motors. Moreover, we designed a fractional  $PI^\lambda D^\mu$  controller for the DC motor and obtained good control results.

The efficiency of the proposed estimators in this thesis was verified by numerical simulations. Moreover, the simulation results showed that the designed estimators feature non-asymptotic convergence and robustness against noises.

Finally, let us outline in the following table the recent works on modulating functions-based estimators and differentiators for integer order and non-integer order linear systems.

System Type		Estimation Object	Reference
Integer order systems $\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$		Output derivatives $y^{(i)} \rightarrow$ state $x$	[72]
		State $x \rightarrow$ state derivatives $D_t^\gamma x$	Chapter 4, [111]
		State derivatives $D_t^\gamma x$	Chapter 6, [120]
		Initial values $\rightarrow$ state derivatives $D_t^\gamma x$	Chapter 6
Fractional order systems $\begin{cases} D_t^\alpha x = Ax + Bu \\ y = Cx \end{cases}$	$\alpha = \frac{1}{q}$	Output derivatives $D_t^\beta y$ and a set of fractional initial values $\rightarrow$ state $x$	[77]
	$\alpha = \frac{p}{q}$		[76]
	$\alpha \in (0,2)$	Output derivatives $D_t^\beta y$ and a set of fractional initial values $\rightarrow$ state derivatives $D_t^\gamma x$	[78]
	$\alpha \in (0,2)$	Output derivatives $D_t^\beta y$ and a set of fractional initial values $\rightarrow$ state derivatives $D_t^\gamma x$	[79]
	$\alpha \in (0,1]$	State integral $D_t^{\alpha-[\alpha]} x \rightarrow$ state derivatives $D_t^\gamma x$	Chapter 5, [118]
$\alpha \in (0,1]$	Fractional initial values $\rightarrow$ state derivatives $D_t^\gamma x$	Chapter 7	

## 8.2 Perspectives

The merits of the main contributions of this thesis are summarized in Section 8.1. While in this section, further developments based on the conclusions of this thesis are discussed.

- Generalize the proposed methods to more complex systems, such as fractional multi-agent systems, fractional singular systems, fractional nonlinear systems, fractional order partial differential equation systems, etc. Since the proposed estimators and differentiators have already efficiently and robustly performed on fractional order linear systems, it is natural to consider the extension of these methods to more complex problems. In these cases, there are some challenges.
- Consider applying the designed modulating functions-based estimators and differentiators to fractional order controllers with practical backgrounds, such as electric motors, electric vehicles, viscoelastic materials, etc. With the help of the proposed estimators and differentiators, it is promising to design more stable and robust fractional order controllers.

## 8.2. PERSPECTIVES

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# Chapter 9

## Résumé étendu en Français

### 9.1 Introduction

#### 9.1.1 Motivation et objectif

Pour construire un modèle de système d'ordre fractionnaire et mettre en œuvre un contrôleur d'ordre fractionnaire, il est nécessaire de fournir des paramètres et des variables utiles du système, tels que la sortie ou les informations d'état du système. Néanmoins, en raison de contraintes économiques et techniques, ces informations ne sont pas toujours mesurables. Bien que certaines variables puissent être mesurées, les mesures sont souvent affectées par des bruits de corruption. Ainsi, il est important de concevoir des méthodes d'estimation robustes contre le bruit. De plus, pour les applications en ligne, des estimations non asymptotiques sont généralement requises. Bien que l'introduction du "contrôle d'ordre fractionnaire" puisse améliorer efficacement les performances et la robustesse du système, la nature non commutative et non additive des dérivées fractionnaires ajoute de la complexité à l'étude, augmentant la difficulté d'observation et d'estimation. De plus, les informations sur lesquelles le contrôleur d'ordre fractionnaire repose ne se limitent plus à l'état ou à la sortie du système, mais peuvent également inclure les dérivées et intégrales fractionnaires de ces signaux, ce qui impose de nouvelles exigences aux méthodes d'estimation.

Dans la théorie des systèmes, les observateurs d'état sont équipés pour remplir les tâches d'estimation susmentionnées. Actuellement, de nombreux observateurs classiques ont été étendus au cas de l'ordre fractionnaire, avec des approches souvent dérivées d'extensions directes des systèmes d'ordre entier, tels que les observateurs de type Luenberger [60], [61], les filtres de Kalman [62], [63], les observateurs à mode glissant [58], les observateurs adaptatifs [57], [59], les observateurs à grand gain [64], les observateurs d'état d'ordre réduit [65], les observateurs d'état étendus [66], etc. En termes de conception d'observateur, la précision, la rapidité et la robustesse sont les principaux indices de performance, où une haute précision est l'exigence de base de l'observateur, la rapidité est une garantie importante pour

effectuer des tâches en temps réel, et la robustesse est un outil efficace contre les interférences de bruit. Cependant, la plupart des observateurs d'ordre fractionnaire susmentionnés ne peuvent généralement pas garantir simultanément les propriétés non asymptotiques et robustes, et ils sont fortement affectés par la valeur initiale. Il convient de mentionner que le travail susmentionné ne peut pas répondre à l'exigence d'estimation des dérivées fractionnaires des états dans les contrôleurs d'ordre fractionnaire. Par conséquent, la théorie de la conception d'observateurs pour les systèmes d'ordre fractionnaire reste imparfaite.

En ce qui concerne l'estimation non asymptotique et robuste, il existe deux stratégies pour concevoir des estimateurs basés sur la formule intégrale algébrique : la méthode de fonctions modulatrices dans le domaine temporel [67], [68] et la méthode paramétrique algébrique dans le domaine fréquentiel [69], [70]. Toutes ces deux méthodes sont issues de l'identification linéaire et ont été étendues à l'estimation d'état [71], [72], [74]. Elles offrent les avantages d'une grande précision, d'une convergence rapide et d'une robustesse. La principale caractéristique est que les estimations résultantes sont exprimées sous forme de formules intégrales, où la mesure de sortie bruitée ne concerne que le terme intégral. Ainsi, dans le cas continu sans bruit, l'estimateur correspondant ne comporte aucune source d'erreur. Dans des environnements bruités, les estimateurs sont robustes aux bruits de corruption, car les intégrales impliquées agissent comme des filtres passe-bas pour atténuer les effets des bruits. Récemment, la méthode des fonctions modulatrices a également été appliquée aux systèmes d'ordre fractionnaire [75]–[80], où le problème d'estimation des dérivées fractionnaires et des intégrales de l'état a été résolu dans [78]–[80].

Néanmoins, le travail existant présente certaines contraintes. D'une part, avec les méthodes de fonctions modulatrices existantes, le système considéré doit être transformé en une forme d'équation différentielle, ce qui augmente la complexité de calcul. Par ailleurs, les dérivées de la sortie et un ensemble de valeurs initiales des dérivées d'ordre fractionnaire sont généralement nécessaires pour estimer les états [76], [78], [79], et les formules intégrales obtenues sont très compliquées. D'autre part, en raison de la complexité de calcul fractionnaire, le champ d'application de la méthode des fonctions modulatrices est limité. Par exemple, la méthode des fonctions modulatrices dans [76], [78] ne peut être appliquée que pour les systèmes d'ordre rationnel. Par conséquent, la trajectoire des efforts futurs doit s'orienter vers la recherche globale et l'exploration. Cela implique non seulement la prise en compte de modèles de systèmes d'ordre fractionnaire plus généraux qui ne sont pas limités par un ordre spécifique, mais aussi la simplification du processus d'estimation et l'amélioration de l'efficacité de la méthode de la fonctions modulatrices.

Cette thèse s'efforce de fournir des méthodes améliorées d'estimation d'état basées sur les fonctions modulatrices, à la fois pour les systèmes linéaires d'ordre entier et d'ordre non entier, où l'ordre du système peut être en  $(0, 2)$ . Elles peuvent être utilisées pour estimer de manière robuste et non asymptotique non seulement les états des systèmes considérés, mais aussi leurs dérivées fractionnaires

et leurs intégrales. En particulier, contrairement aux études précédentes [76], [78], [79], les méthodes d'estimation améliorées basées sur les fonctions modulatrices sont directement appliquées à la forme canonique observable du modèle, elles simplifient grandement le processus d'estimation et atteignent des objectifs d'estimation rapides, précis et robustes. En outre, nous considérons les systèmes dans un contexte pratique et présentons une nouvelle estimation de la vitesse, du courant et de leurs dérivées et intégrales d'ordre fractionnaire basée sur les fonctions modulatrices pour la conception de contrôleurs d'ordre fractionnaire  $PI^{\lambda}D^{\mu}$  pour les moteurs à courant continu.

### 9.1.2 Plan

Cette thèse résume mon travail de doctorat de 2021 à 2024, qui est organisé comme suit :

La section 9.2 présente quelques préliminaires sur le calcul fractionnaire, y compris certaines fonctions de base, définitions, propriétés, et connaissances fondamentales du calcul fractionnaire. Ensuite, les concepts de base et les conclusions des systèmes linéaires d'ordre fractionnaire dans les modèles différentiels d'entrée-sortie et les modèles de représentation d'état sont introduits.

La section 9.3 présente l'histoire du développement de la méthode des fonctions modulatrices, les définitions des différentes fonctions modulatrices, ainsi que les principales idées sous-jacentes à la méthode. Sur cette base, les principaux résultats de l'utilisation de la méthode des fonctions modulatrices dans diverses applications sont rappelés, comme l'identification des paramètres, les différentiateurs et l'estimation d'état.

La section 9.4 développe une méthode améliorée d'estimation d'état basée sur les fonctions modulatrices pour les systèmes d'ordre entier afin de simplifier le processus d'estimation et d'améliorer l'efficacité de la méthode existante basée sur les fonctions modulatrices, qui est non asymptotique et robuste.

La section 9.5 généralise l'estimateur non asymptotique et robuste donné dans la section 9.4 à une classe de systèmes linéaires d'ordre fractionnaire avec multiples entrées et multiples sorties pour estimer l'intégrale fractionnaire de l'état. Dans le modèle de système considéré, l'ordre de différenciation est une valeur dans  $(0, 2)$  au lieu d'être seulement une valeur rationnelle. De plus, dans les sections 9.4 et 9.5, les dérivées fractionnaires de l'état peuvent être calculées en appliquant le schéma de Grünwald-Letnikov.

La section 9.6 développe la deuxième méthode améliorée des fonctions modulatrices pour estimer l'intégrale et la dérivée fractionnaires de l'état directement pour les systèmes d'ordre entier sans utiliser le schéma de Grünwald-Letnikov. En particulier, le processus d'estimation implique deux approches distinctes pour les termes de valeur initiale : l'une implique leur élimination grâce à la conception de fonctions modulatrices, tandis que l'autre implique un processus en deux étapes: d'abord estimer les termes de valeur initiale, puis estimer les dérivées et intégrales fractionnaires de l'état. De plus, nous

examinons les applications pour les moteurs à courant continu.

La section 9.7 généralise les estimateurs non asymptotiques et robustes donnés dans la section 9.6 à une classe de systèmes linéaires d'ordre fractionnaire pour estimer directement l'intégrale et la dérivée fractionnaires de l'état sans utiliser le schéma de Grünwald-Letnikov.

Tous les estimateurs algébriques donnés dans les sections 9.4–9.7 sont vérifiés dans un environnement bruité discret pour montrer la précision et la robustesse.

Enfin, des conclusions sont tirées dans la section 9.8 avec quelques perspectives.

## 9.2 Calcul fractionnaire et contrôle d'ordre fractionnaire

Les bases du calcul fractionnaire et des systèmes linéaires d'ordre fractionnaire sont abordées dans cette section. Il commence par introduire certaines fonctions spéciales, notamment la fonction Gamma et la fonction Mittag-Leffler, qui sont des éléments essentiels du calcul fractionnaire. Ensuite, les définitions et les propriétés de l'intégrale fractionnaire de Riemann-Liouville sont passées en revue. Sur cette base, les dérivées fractionnaires de Riemann-Liouville, de Caputo et de Grünwald-Letnikov sont présentées. Notamment, des formules de l'intégration par parties d'ordre fractionnaire sont résumées, fournissant des outils mathématiques nécessaires pour le texte suivant. Ensuite, à la fois les modèles d'équations différentielles fractionnaires et les modèles de représentation d'état des systèmes linéaires d'ordre fractionnaire sont présentés, ainsi que leurs propriétés et comportements. Enfin, le régulateur d'ordre fractionnaire  $PI^{\lambda}D^{\mu}$  est rappelé.

## 9.3 Méthode des fonctions modulatrices

Dans de nombreuses situations pratiques en ingénierie, il existe toujours des variables et des paramètres non mesurables pour des raisons de coût et technologiques. Pendant ce temps, les variables et paramètres mesurés sont généralement corrompus par le bruit. Afin de surmonter ces problèmes, il est nécessaire de développer des méthodes d'estimation robustes [68], [94]–[98]. De plus, parfois, une estimation non asymptotique avec une convergence rapide en temps fini est nécessaire dans des applications en ligne [99]–[103].

La méthode des fonctions modulatrices est un type de méthode de domaine temporel. Cette méthode a été initialement introduite par Shinbort en 1954 [67] pour l'identification linéaire et a été largement utilisée pour l'identification linéaire et non linéaire de systèmes en temps continu [104]–[107]. Les étapes fondamentales de la méthode des fonctions modulatrices peuvent être décrites comme suit:

- considérer (ou construire) une équation différentielle linéaire;

- multiplier les fonctions modulatrices conçues et intégrer sur un intervalle de temps;
- appliquer l'intégration par parties pour éliminer les dérivées des fonctions modulatrices;
- éliminer les termes indésirables et conserver les termes souhaités;
- résoudre un système linéaire pour obtenir des formules intégrales algébriques;
- prendre le signal bruité  $y^\omega = y + \omega$ .

Les avantages de la méthode des fonctions modulatrices sont énumérés comme suit:

- formulation algébrique exacte utilisant des formules intégrales du signal d'observation;
- pas besoin de connaître les conditions initiales;
- sans calcul des dérivées des signaux bruités;
- non asymptotique et rapide;
- bonne robustesse par rapport aux bruits sans nécessité de connaître à l'avance leurs propriétés statistiques [108].

Dans cette section, certains des travaux existants sur les méthode des fonctions modulatrices pour l'identification linéaire seront rappelés. En s'inspirant de ces travaux établis, les principaux résultats de cette thèse seront présentés.

### 9.4 Estimation d'état pour les systèmes d'ordre entier

Dans cette section, une méthode améliorée des fonctions modulatrices est proposée pour estimer de manière non asymptotique et robuste l'état des systèmes linéaires d'ordre entier dans un environnement bruité. De plus, les dérivées fractionnaires de l'état peuvent être calculées en appliquant le schéma de Grünwald-Letnikov. À cette fin, le système considéré, modélisé par la représentation d'état, est d'abord transformé en forme canonique observable. Sur cette base, la méthode des fonctions modulatrices est appliquée pour obtenir des formules intégrales algébriques pour l'état. Ces formules intégrales peuvent fournir des estimations robustes dans des cas discrets bruités. De plus, il est expliqué comment concevoir les fonctions modulatrices requises. Pour démontrer l'efficacité de l'estimateur proposé, un exemple numérique est présenté en le comparant à l'observateur de Luenberger. Les principales contributions de cette section sont énumérées comme suit:



- Contrairement aux travaux existants sur la méthode des fonctions modulatrices [74], [76]–[79], [97], la méthode améliorée proposée dans cette section est directement appliquée à la forme canonique observable, ne nécessite pas la conversion en une équation différentielle d'entrée-sortie, ni besoin d'estimer les dérivées de la sortie pour obtenir des formules intégrales algébriques pour les variables recherchées, ce qui rend le processus d'estimation plus facile et les formules obtenues plus simples.
- L'estimateur proposé est précis sans sources d'erreurs dans le cas continu sans bruit, et robuste contre les bruits dans le cas discret bruité. Par ailleurs, l'estimateur proposé est également algébrique (donc non asymptotique), ce qui permet d'obtenir des résultats d'estimation avec une convergence en temps fini et sans connaître les conditions initiales.

### 9.4.1 Formulation du problème

Dans cette section, considérons le système linéaire d'ordre entier suivant :

$$\begin{cases} \dot{x} &= Ax + Bu, \\ y &= Cx, \end{cases} \quad (9.1)$$

sur  $I = [0, h] \subset \mathbb{R}$ , où  $A \in \mathbb{R}^{N \times N}$ ,  $B \in \mathbb{R}^{N \times 1}$ ,  $C \in \mathbb{R}^{1 \times N}$ ,  $x \in \mathbb{R}^N$  est l'état,  $y \in \mathbb{R}$  et  $u \in \mathbb{R}$  sont respectivement la sortie et l'entrée. Pendant ce temps, les conditions  $(C_1)$ – $(C_2)$  données dans le Théorème 2.1 et le Théorème 2.2 sont supposées être remplies.

L'objectif de cette section est d'estimer l'état et sa dérivée fractionnaire dans un environnement bruité. À cette fin, le système défini par (9.1) doit d'abord être transformé en forme canonique observable, sur laquelle la méthode des fonctions modulatrices sera appliquée. En appliquant le changement de coordonnées  $z = Tx$  donné dans (2.66), alors le système (9.1) peut être transformé en la forme canonique observable suivante de la représentation d'état:

$$\begin{cases} \dot{z} &= \hat{A}z + \hat{B}u, \\ y &= \hat{C}z, \end{cases} \quad (9.2)$$

où

$$\hat{A} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & -a_{N-2} \\ 0 & 0 & \cdots & 0 & 1 & -a_{N-1} \end{pmatrix}, \quad (9.3)$$

et  $\hat{B} = MB = (\hat{B}_0, \hat{B}_1, \dots, \hat{B}_{N-1})^\top \in \mathbb{R}^{N \times 1}$ ,  $\hat{C} = (0, \dots, 0, 1) \in \mathbb{R}^{1 \times N}$ .

### 9.4.2 Résultats principaux

Dans cette sous-section, en appliquant la méthode améliorée des fonctions modulatrices, les formules intégrales algébriques pour l'état du système linéaire d'ordre entier (9.1) seront données. De plus, les dérivées fractionnaires de l'état sont calculées en appliquant le schéma de Grünwald-Letnikov. Les principaux résultats sont donnés dans les théorèmes suivants.

**Theorem 9.1** *Considérons le système défini dans (9.2) sur  $I$ . Pour  $0 < t \in I$ , supposons que  $\eta$  est une fonction modulatrice d'ordre  $N$  du côté gauche sur  $[0, t]$ . Alors, nous avons:*

$$\sum_{i=1}^N (-1)^{i-1} \eta^{(i-1)}(t) z_i(t) = \sum_{i=0}^{N-1} (-1)^i \hat{B}_i \int_0^t \eta^{(i)}(\tau) u(\tau) d\tau - \sum_{i=0}^N (-1)^i a_i \int_0^t \eta^{(i)}(\tau) y(\tau) d\tau, \quad (9.4)$$

où  $a_N = 1$ .

Le but de cette section est d'estimer l'état et sa dérivée fractionnaire en utilisant la nouvelle méthode des fonctions modulatrices. Selon le théorème ci-dessus, les résultats suivants peuvent être déduits.

**Theorem 9.2** *Considérons le système défini dans (9.2). Pour  $0 < t \in I$ ,  $g_{j'}$  pour  $j' = 1, \dots, N$ , sont des fonctions modulatrices d'ordre  $(N, j')$  du côté gauche sur  $[0, t]$ , nous avons:*

$$z_{j'}(t) = \sum_{i=0}^{N-1} (-1)^{i+j'-1} \hat{B}_i \int_0^t g_{j'}^{(i)}(\tau) u(\tau) d\tau - \sum_{i=0}^N (-1)^{i+j'-1} a_i \int_0^t g_{j'}^{(i)}(\tau) y(\tau) d\tau. \quad (9.5)$$

**Corollary 9.1** *Sous les mêmes hypothèses du Théorème 9.2,  $g_{j'}$  pour  $j' = 1, \dots, N$ , sont des fonctions modulatrices d'ordre  $(N, j')$  du côté gauche sur  $[0, t]$ . Soit  $\beta \in \mathbb{R}$ , alors  $D_t^\beta x$  peut être calculée par les formules suivantes: pour  $j' = 1, \dots, N$ ,*

$$D_t^\beta x_{j'}(t) = D_t^\beta \left( T^{-1} \left( \sum_{i=0}^{N-1} (-1)^{i+j'-1} \hat{B}_i \int_0^t g_{j'}^{(i)}(\tau) u(\tau) d\tau - \sum_{i=0}^N (-1)^{i+j'-1} a_i \int_0^t g_{j'}^{(i)}(\tau) y(\tau) d\tau \right) \right). \quad (9.6)$$

### 9.4.3 Résultats de la simulation

Afin d'illustrer la précision et la robustesse vis-à-vis des bruits des estimateurs proposés, nous présentons quelques résultats numériques.

**Example 9.1** *Prenons  $N = 3$ ,  $u(t) = \sin(t)$ , et les matrices suivantes dans (9.1) :*

$$A = \begin{pmatrix} 11 & 1 & -11 \\ 6 & 0 & -6 \\ 18 & 1 & -17 \end{pmatrix}, B = (0, 1, 0)^\top, C = (-1, 0, 1),$$

qui satisfont aux conditions  $(C_1)$ – $(C_2)$ . Supposons que la valeur initiale de l'état soit donnée par  $x(0) = (-0.1, -0.2, -0.3)^\top$  et  $y^\varpi(t_i) = y(t_i) + \sigma\varpi(t_i)$  est l'observation bruitée de la sortie  $y$  sur  $I = [0, 10]$ , où  $t_i = iT_s$  avec  $T_s = 0,001$ ,  $\{\varpi(t_i)\}$  est un bruit gaussien de moyenne nulle, et la valeur de  $\sigma$  est ajustée de telle sorte que le rapport signal sur bruit  $SNR = 10 \log_{10} \left( \frac{\sum |y^\varpi(t_i)|^2}{\sum |\sigma\varpi(t_i)|^2} \right)$  est égal à 15 dB. La sortie et la sortie bruitée associée sont présentées dans la Fig. 9.1.

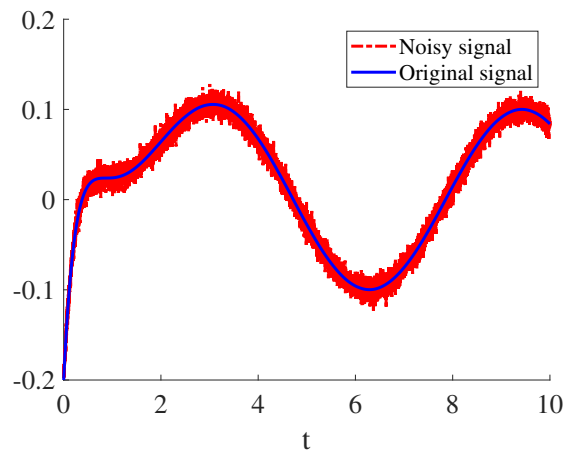


Figure 9.1:  $y$  et son observation discrète bruitée dans l'exemple 9.1.

L'état  $x$  et sa dérivée fractionnaire  $D_t^\beta x$  peuvent être estimés dans ce cas discret bruité en appliquant le corollaire 9.1, où la méthode d'intégration numérique trapézoïdale est utilisée pour approximer les formules intégrales. Ensuite, au moyen des fonctions modulatrices construites, les résultats d'estimation requis peuvent être obtenus. De plus, l'observateur de Luenberger est considéré comme comparable à la méthode proposée. Le premier gain est choisi en prenant les valeurs propres 2 fois supérieures à celles du système afin de garantir la robustesse au bruit. Le deuxième gain est choisi en prenant les valeurs propres 7 fois supérieures à celles du système afin d'améliorer la vitesse de convergence, ce qui produit une plus grande contribution aux erreurs de bruit.

En prenant  $\beta = 0$ ,  $m = 4$  et la valeur initiale de l'observateur de Luenberger  $\bar{x}(0) = (1, 2, 1)^\top$ , les valeurs d'estimation et les erreurs d'estimation de  $x_1$  en utilisant la méthode proposée et l'observateur de Luenberger sont présentées dans la Fig. 9.2 et la Fig. 9.3, respectivement. Les résultats numériques montrent l'efficacité de l'estimateur d'état proposé et la robustesse vis-à-vis des bruits est meilleure que celle de l'observateur de Luenberger.

#### 9.4.4 Conclusions

Dans cette section, un estimateur d'état non asymptotique et robuste basé sur la méthode améliorée des fonctions modulatrices a été construit pour les systèmes d'ordres entiers. De plus, les dérivées

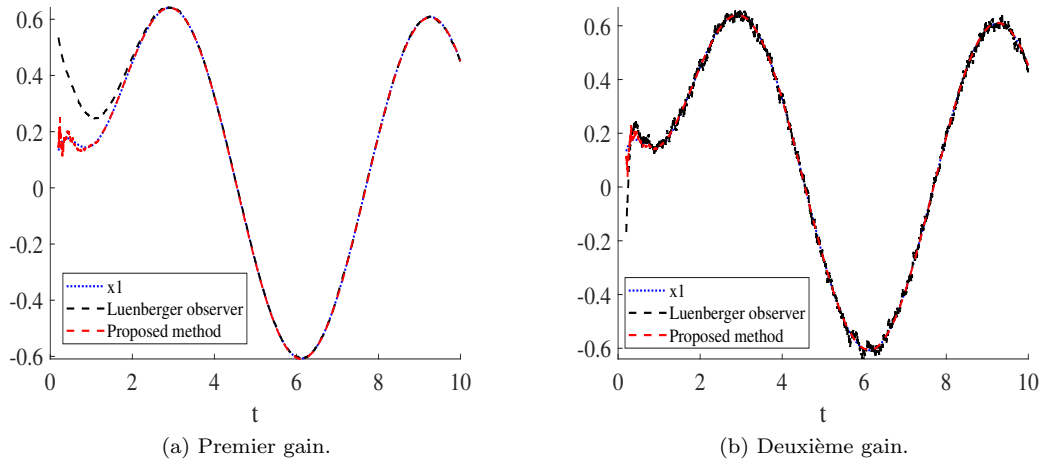


Figure 9.2: Estimations de l'état  $x_1$  obtenues par la méthode des fonctions modulatrices proposée et l'observateur de Luenberger dans l'exemple 9.1.

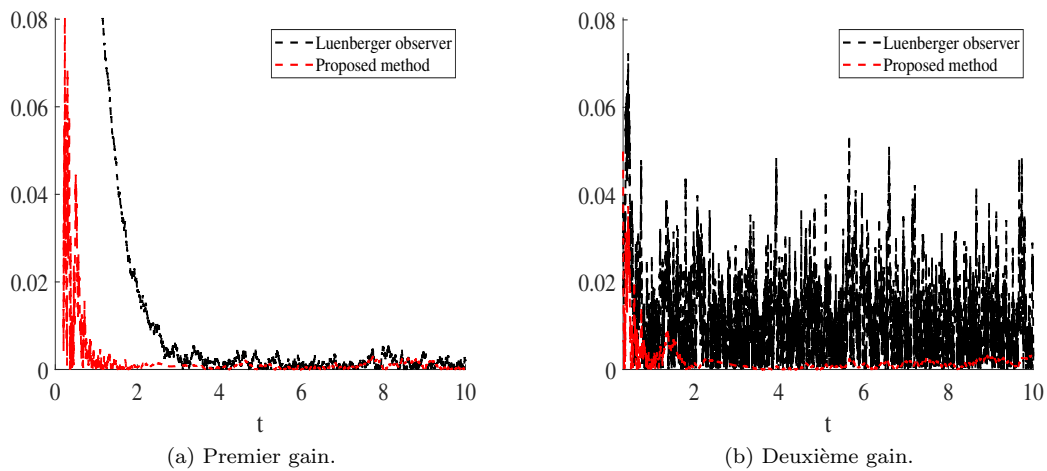


Figure 9.3: Erreurs d'estimation de l'état  $x_1$  obtenues par la méthode des fonctions modulatrices proposée et l'observateur de Luenberger dans l'exemple 9.1.

fractionnaires de l'état sont calculées en appliquant le schéma de Grünwald-Letnikov. Premièrement, le système considéré est passé de la représentation d'état à la forme canonique observable. Ensuite, les fonctions modulatrices conçues sont appliquées pour obtenir des formules algébriques intégrales. Enfin, un exemple numérique a été donné pour montrer l'efficacité de l'estimateur proposé. Par rapport à l'observateur de Luenberger, la méthode des fonctions modulatrices proposée est plus précise. Ce travail a été publié dans [111].

## 9.5 Estimation de l'intégrale fractionnaire de l'état pour les systèmes d'ordre fractionnaire

Dans cette section, la méthode améliorée d'estimation d'état basée sur les fonctions modulatrices introduite dans la section 9.4 est étendue pour estimer l'intégrale fractionnaire de l'état pour les systèmes d'ordre fractionnaire dans un environnement bruité, ce qui est non asymptotique et robuste. De plus, les dérivées fractionnaires de l'état peuvent être calculées via le schéma de Grünwald-Letnikov. La méthode proposée est directement appliquée à la forme canonique observable d'ordre fractionnaire pour obtenir des formules intégrales algébriques pour les variables recherchées. Pour cela, une famille des fonctions modulatrices requises est introduite et construite avec un paramètre de conception. De plus, une analyse d'erreur dans les cas discrets bruités est abordée. Enfin, la robustesse et la précision de la méthode améliorée sont vérifiées par des simulations numériques. Les principales contributions de cette section sont résumées comme suit:

- La méthode proposée est plus efficace. Tout d'abord, cette méthode est directement appliquée à la forme canonique observable d'ordre fractionnaire pour obtenir des formules intégrales algébriques pour les variables recherchées, ce qui peut rendre le processus d'estimation plus simple, les formules d'estimation plus simples et les résultats d'estimation plus précis. En effet, lors de l'application des méthodes non asymptotiques précédemment mentionnées pour l'estimation d'état, le système considéré devait être transformé en une équation différentielle [72], [74], [78]. De plus, à la fois les dérivées de la sortie et un ensemble de valeurs initiales d'ordre fractionnaire étaient nécessaires pour estimer la dérivée fractionnaire de l'état [76], [78], [79]. De plus, les fonctions modulatrices requises étaient complexes à construire.
- Le modèle de système considéré est plus général, grâce à la méthode proposée. Tout d'abord, cette section étudie les systèmes d'ordre fractionnaire avec multiples entrées et multiples sorties. Cependant, seuls les systèmes d'ordre fractionnaire avec une seule entrée et une seule sortie ont été considérés dans [72], [74], [76], [78]–[80], [112], [113]. Deuxièmement, les systèmes considérés ne sont pas soumis aux ordres de différenciation. En fait, les méthodes proposées dans [76], [78] ne peuvent être appliquées que pour des systèmes d'ordre rationnel.

### 9.5.1 Motivation et énoncé du problème

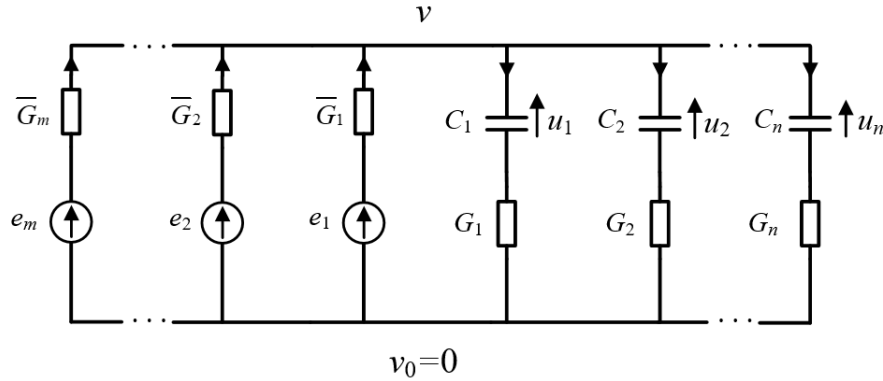


Figure 9.4: Modèle de circuit électrique d'ordre fractionnaire.

Considérons un circuit électrique composé de  $n$  condensateurs,  $m$  sources de tension et résistances, comme le montre la Fig. 9.4, où  $G_i$  et  $\bar{G}_j$  pour  $i = 0, \dots, n$ ,  $j = 0, \dots, m$  sont les conductances,  $C_s$  pour  $s = 0, \dots, n$  sont les condensateurs, et  $e_l$  pour  $l = 0, \dots, m$  sont les sources. Selon [24], en appliquant les lois de Kirchhoff, nous pouvons utiliser le modèle d'ordre fractionnaire suivant pour décrire la dynamique de l'état transitoire dans ce circuit électrique:

$$D_t^\alpha x(t) = Ax(t) + Bu(t), \quad (9.7)$$

où  $\alpha \in (0, 1]$ , l'état  $x \in \mathbb{R}^n$  est le vecteur des tensions sur les condensateurs,  $u \in \mathbb{R}^m$  est le vecteur des tensions de la source,

$$A = \begin{pmatrix} -\frac{G_1 G - G_1^2}{C_1 G} & \frac{G_1 G_2}{C_1 G} & \dots & \frac{G_1 G_n}{C_1 G} \\ \frac{G_2 G_1}{C_2 G} & -\frac{G_2 G - G_2^2}{C_2 G} & \dots & \frac{G_2 G_n}{C_2 G} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{G_n G_1}{C_n G} & \frac{G_n G_2}{C_n G} & \dots & -\frac{G_n G - G_n^2}{C_n G} \end{pmatrix}, B = \begin{pmatrix} \frac{G_1 \bar{G}_1}{C_1 G} & \frac{G_1 \bar{G}_2}{C_1 G} & \dots & \frac{G_1 \bar{G}_m}{C_1 G} \\ \frac{G_2 \bar{G}_1}{C_2 G} & \frac{G_2 \bar{G}_2}{C_2 G} & \dots & \frac{G_2 \bar{G}_m}{C_2 G} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{G_n \bar{G}_1}{C_n G} & \frac{G_n \bar{G}_2}{C_n G} & \dots & \frac{G_n \bar{G}_m}{C_n G} \end{pmatrix}$$

avec  $G = \sum_{i=1}^n G_i + \sum_{j=1}^m \bar{G}_j$ .

L'état transitoire  $x$  permet de mieux analyser et maîtriser la loi de changement du circuit [30], [114]. En outre, il est nécessaire dans les contrôleurs à rétroaction d'état [24]. Cependant, l'état est généralement inconnu et doit être estimé.

Motivé par la question de l'estimation de l'état pour le modèle de circuit électrique d'ordre fractionnaire précédent, le système linéaire d'ordre fractionnaire suivant avec des entrées et des sorties

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multiples est considéré dans ce travail :  $I = [0, h] \subset \mathbb{R}$ ,

$$\begin{cases} D_t^\alpha x(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t), \end{cases} \quad (9.8)$$

où  $A \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{n \times q}$ ,  $C \in \mathbb{R}^{p \times n}$ ,  $D_t^\alpha x = (D_t^\alpha x_1, \dots, D_t^\alpha x_n)^\top$ ,  $x \in \mathbb{R}^n$  est l'état,  $u \in \mathbb{R}^q$  et  $y \in \mathbb{R}^p$  avec  $1 \leq p, q \leq n$  sont respectivement l'entrée et la sortie. On suppose  $\alpha \in (0, 2)$  pour garantir la stabilité [23].

Le but de cette section est d'estimer l'intégrale et la dérivée fractionnaires de l'état dans un environnement bruité, ce qui est utile dans la conception de contrôleurs à rétroaction d'état d'ordre fractionnaire. À cette fin, nous rappelons la matrice d'observabilité pour le système donné dans (9.8) défini par [115]:

$$Q_k = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{k-1} \end{pmatrix}, \quad (9.9)$$

où  $k \leq n$  est le plus petit entier qui satisfait  $\text{rank}(Q_k) = \text{rank}(Q_n)$ . On suppose ensuite que le système défini par (9.8) remplit la condition d'observabilité suivante :

$$(\widetilde{C}_1) : \text{rank}(Q_k) = n.$$

### 9.5.2 Principaux résultats

Dans cette section, une méthode améliorée de fonctions modulatrices est introduite pour estimer l'intégrale fractionnaire de l'état pour les systèmes d'ordre fractionnaire. En outre, les dérivées fractionnaires de l'état sont calculées en appliquant le schéma de Grünwald-Letnikov. À cette fin, nous convertissons d'abord le système défini dans (9.8) dans la forme canonique observable d'ordre fractionnaire à l'aide du lemme suivant, qui peut être obtenu d'une manière similaire à celle utilisée dans [115] pour les systèmes d'ordre entier.

**Lemma 9.1** *Sous la condition  $(\widetilde{C}_1)$ , considérons la transformation de coordonnées suivante:*

$$z = Tx, \quad (9.10)$$

où  $z = (z_1 \ z_2 \ \dots \ z_k)^\top$ ,  $z_i \in \mathbb{R}^p$  pour  $i = 1, \dots, k$ ,  $z \in \mathbb{R}^{n_z}$  avec  $n_z = pk$ , et  $T = (T_1^\top \ T_2^\top \ \dots \ T_k^\top)^\top$ ,

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$T \in \mathbb{R}^{n_z \times n}$  est donnée par:

$$T = \begin{pmatrix} CA^{k-1} + \sum_{i=1}^{k-1} M_i CA^{i-1} \\ \vdots \\ CA^{k-j} + \sum_{i=j}^{k-1} M_i CA^{i-j} \\ \vdots \\ CA + M_{k-1}C \\ C \end{pmatrix} = \begin{pmatrix} M_1 & \cdots & M_{k-1} & I_p \\ \vdots & \vdots & \ddots & \vdots \\ M_{k-1} & I_p & \cdots & 0 \\ I_p & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{k-1} \end{pmatrix} = \bar{M}Q_k, \quad (9.11)$$

où  $M_i \in \mathbb{R}^{p \times p}$ , pour  $i = 0, 1, \dots, k-1$  sont déterminées par l'équation matricielle suivante:

$$(M_0 \ M_1 \ \cdots \ M_{k-1}) = -CA^k Q_k^L, \quad (9.12)$$

où  $Q_k$  est donnée par (9.9),  $Q_k^L \in \mathbb{R}^{n \times n_z}$  est une matrice à inverse gauche de  $Q_k$  avec  $Q_k^L Q_k = I_n$ .

On obtient alors la forme canonique observable d'ordre fractionnaire suivante:

$$\begin{cases} D_t^\alpha z(t) = \bar{A}z(t) + \bar{B}u(t), \\ y(t) = \bar{C}z(t), \end{cases} \quad (9.13)$$

où

$$\bar{A} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & -M_0 \\ I_p & 0 & \cdots & 0 & 0 & -M_1 \\ 0 & I_p & \cdots & 0 & 0 & -M_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & I_p & 0 & -M_{k-2} \\ 0 & 0 & \cdots & 0 & I_p & -M_{k-1} \end{pmatrix}, \quad (9.14)$$

et  $\bar{B} = TB = (\bar{B}_0 \ \bar{B}_1 \ \cdots \ \bar{B}_{k-1})^\top$ ,  $\bar{C} = (0 \ \cdots \ 0 \ I_p)$ .

**Theorem 9.3** *Considérons le système donné dans (9.13). Pour  $t \in (0, h]$ , laissons  $\eta$  être une fonction modulatrice d'ordre  $(k, \alpha)$  du côté droite sur  $[0, t]$ . Dans ce cas, l'équation intégrale suivante est valable:*

$$\begin{aligned} \sum_{r=0}^{[\alpha]-1} \sum_{i=1}^k D_t^r \left\{ {}^C \mathcal{D}_t^{(i-1)\alpha} \eta(0) \right\} D_t^{\alpha-1-r} z_i(t) &= \sum_{i=0}^{k-1} \bar{B}_i \int_0^t {}^C \mathcal{D}_\tau^{i\alpha} \eta(\tau) u(t-\tau) d\tau \\ &\quad - \sum_{i=0}^k M_i \int_0^t {}^C \mathcal{D}_\tau^{i\alpha} \eta(\tau) y(t-\tau) d\tau, \end{aligned} \quad (9.15)$$

où  $M_k = I_p$ .



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**Theorem 9.4** *Considérons le système donné dans (9.13) avec  $\alpha \in (0, 2)$ . Pour  $t \in (0, h]$ , on note  $g_{j', r'}$  pour  $j' = 1, \dots, k$ , soient fonctions modultrices d'ordre  $(k, \alpha; j' - 1, r')$  sur  $[0, t]$  avec  $r' = \lceil \alpha \rceil - 1$ ,*

$$D_t^{\alpha - \lceil \alpha \rceil} z_{j'}(t) = \sum_{i=0}^{k-1} \bar{B}_i \int_0^t {}^C \mathcal{D}_\tau^{i\alpha} g_{j', r'}(\tau) u(t - \tau) d\tau - \sum_{i=0}^k M_i \int_0^t {}^C \mathcal{D}_\tau^{i\alpha} g_{j', r'}(\tau) y(t - \tau) d\tau. \quad (9.16)$$

*De plus, dans le cas de  $\alpha \in (1, 2)$ , prenons  $g_{j', 0}$  pour  $j' = 1, \dots, k$ , soient des fonctions modultrices d'ordre  $(k, \alpha; j' - 1, 0)$  sur  $[0, t]$ , alors:*

$$D_t^{\alpha - 1} z_{j'}(t) = \sum_{i=0}^{k-1} \bar{B}_i \int_0^t {}^C \mathcal{D}_\tau^{i\alpha} g_{j', 0}(\tau) u(t - \tau) d\tau - \sum_{i=0}^k M_i \int_0^t {}^C \mathcal{D}_\tau^{i\alpha} g_{j', 0}(\tau) y(t - \tau) d\tau. \quad (9.17)$$

**Corollary 9.2** *Considérons le système donné dans (9.13). Pour  $t \in (0, h]$ , laissons  $g_{j', r'}$  pour  $j' = 1, \dots, k$ , soient des fonctions modultrices d'ordre  $(k, \alpha; j' - 1, r')$  sur  $[0, t]$  avec  $r' = \lceil \alpha \rceil - 1$ . Let  $\gamma \in \mathbb{R}$  et  $\beta = \lceil \alpha \rceil - \alpha + \gamma$ , alors  $D_t^\gamma z_{j'}(t)$  peut être donnée par:*

$$D_t^\gamma z_{j'}(t) = D_t^\beta \left( \sum_{i=0}^{k-1} \bar{B}_i \int_0^t {}^C \mathcal{D}_\tau^{i\alpha} g_{j', r'}(\tau) u(t - \tau) d\tau - \sum_{i=0}^k M_i \int_0^t {}^C \mathcal{D}_\tau^{i\alpha} g_{j', r'}(\tau) y(t - \tau) d\tau \right). \quad (9.18)$$

Par conséquent, comme cela a été fait dans [76], [78], [79], des formules algébriques sont obtenues pour la dérivée fractionnaire de l'état, qui ne contient pas d'erreur dans le cas d'un temps continu sans bruit. En particulier, contrairement aux observateurs, la méthode proposée ne dépend pas de valeurs initiales du système étudié. De plus, l'intégrale peut être utilisée comme filtre passe-bas pour réduire l'impact du bruit [108]. Différent de [76], [78], [79],  $D_t^\gamma z_{j'}$  peut être directement estimée en appliquant le corollaire 9.2 sans estimer les dérivées fractionnaires de la sortie et les valeurs initiales fractionnaires. Enfin, selon (9.10),  $D_t^\gamma x$  peut être obtenue en prenant une matrice inverse à gauche de  $T$ . En particulier, l'état  $x$  peut être obtenu avec  $\gamma = 0$ .

### 9.5.3 Résultats de la simulation

**Example 9.2** *Pour comparer avec la méthode de [79] qui a été conçue dans le cas d'une entrée et d'une sortie uniques, nous prenons un système d'ordre fractionnaire considéré dans [79] avec les paramètres suivants dans (9.8):  $\alpha = \frac{5}{8}$ ,  $n = 3$ ,  $p = 1$ ,  $q = 1$ , et*

$$A = \begin{pmatrix} 11 & 1 & -11 \\ 6 & 0 & -6 \\ 18 & 1 & -17 \end{pmatrix}, B = (0, 1, 0)^\top, C = (-1, 0, 1),$$

*qui satisfait la condition  $(\widetilde{C}_1)$  avec  $\text{rank}(Q_3) = 3$  et  $k = 3$ . De plus, nous prenons  $u(t) = \sin(5t)$ ,  $x(0) = (-0.1297, -0.3234, -0.1470)^\top$ ,  $\rho = 15$ .*

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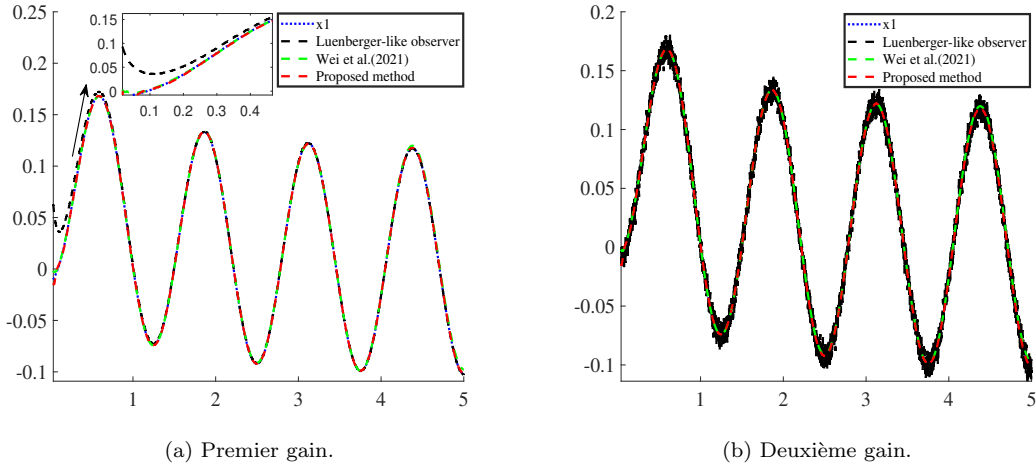


Figure 9.5: Estimations de  $x_1$  obtenues par l'observateur d'ordre fractionnaire de type Luenberger (avec deux gains), la méthode proposée et celle de [79].

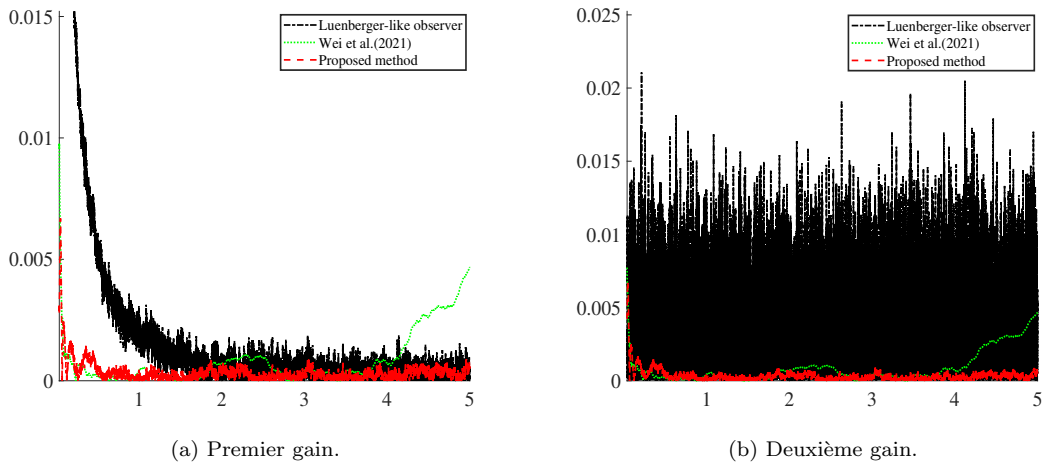


Figure 9.6: Erreurs d'estimation de  $x_1$  obtenues par l'observateur d'ordre fractionnaire de type Luenberger (avec deux gains), la méthode proposée et celle de [79].

## 9.5. ESTIMATION DE L'INTÉGRALE FRACTIONNAIRE DE L'ÉTAT POUR LES SYSTÈMES D'ORDRE FRACTIONNAIRE

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On s'attend à estimer l'état  $x$  à partir de l'observation bruitée de la sortie en utilisant la méthode dans [79] et la méthode proposée en utilisant  $x = T^L z$  et (9.18) avec  $\gamma = 0$  et  $\beta = \frac{3}{8}$ . Ensuite, les valeurs estimées de  $x_1$  et les erreurs d'estimation associées obtenues par la nouvelle méthode des fonctions modulatrices et celle proposée dans [79] sont fournies dans la Fig. 9.5 et la Fig. 9.6, où  $m = 3$ . Selon les résultats de comparaison, les avantages de robustesse contre les bruits et de convergence rapide de ces deux méthodes peuvent être observés. Cependant, comparée à la méthode dans [79], la nouvelle méthode proposée des fonctions modulatrices est plus précise. En effet, le processus d'estimation de la méthode proposée est plus simple que celui de la méthode dans [79]. Par exemple, un ensemble de valeurs initiales doit être estimé par la méthode dans [79], ce qui accumule les erreurs d'estimation. Par conséquent, la méthode proposée est une amélioration du travail précédent et présente des perspectives d'application potentielles.

De plus, un observateur d'ordre fractionnaire de type Luenberger d'ordre fractionnaire est considéré pour être comparé à la méthode proposée. La valeur initiale de l'observateur est prise par  $\bar{x}(0) = (1, 2, 1)^\top$ . Nous choisissons deux gains. Le premier gain est choisi en prenant les valeurs propres 2 fois plus grandes que celles du système afin de garantir la robustesse contre le bruit. Le deuxième gain est choisi en prenant les valeurs propres 7 fois plus grandes que celles du système afin d'améliorer la vitesse de convergence, ce qui produit une contribution plus importante de l'erreur de bruit. Les résultats d'estimation de  $x_1$  sont donnés dans la Fig. 9.5 et la Fig. 9.6, qui montrent la capacité d'estimation rapide de la nouvelle méthode proposée des fonctions modulatrices ainsi que sa robustesse contre les bruits. Comparée à la nouvelle méthode proposée des fonctions modulatrices, il est difficile pour l'observateur d'ordre fractionnaire de concilier à la fois la robustesse contre les bruits de corruption et la convergence rapide.

### 9.5.4 Conclusions

Dans cette section, une méthode améliorée d'estimation de l'état basée sur les fonctions modulatrices a été abordée, qui peut être appliquée à l'estimation de l'intégrale fractionnaire de l'état pour les systèmes d'ordre fractionnaire avec des entrées et des sorties multiples dans un environnement bruité. En outre, les dérivées fractionnaires de l'état sont estimées à l'aide du schéma de Grünwald-Letnikov. Par rapport aux méthodes de fonctions modulatrices existantes qui étaient basées sur l'équation différentielle fractionnaire, cette méthode a été directement appliquée à la forme canonique observable d'ordre fractionnaire. Par conséquent, des formules intégrales algébriques plus simples ont été obtenues en introduisant des fonctions modulatrices d'ordre fractionnaire. En particulier, contrairement à [76], [78], [79], les valeurs initiales d'ordre fractionnaire n'ont pas besoin d'être estimées. Après avoir donné la construction des fonctions modulatrices d'ordre fractionnaire requis, l'analyse d'erreur dans les cas discrets et bruités a été abordée. Enfin, l'efficacité de la méthode des fonctions modulatrices proposée a été vérifiée par des exemples de simulation. Ce travail a été publié dans [118].

## 9.6 Estimations de l'intégrale et de la dérivée fractionnaires de l'état pour les systèmes d'ordre entier

En tenant compte des travaux existants sur la méthode des fonctions modulatrices [111], [112], [118], ainsi que de la méthode des fonctions modulatrices proposée dans la section 9.4 où les dérivées fractionnaires de l'état sont obtenues via le schéma de Grünwald-Letnikov, afin d'obtenir directement les formules intégrales algébriques des dérivées et intégrales fractionnaires des états, une méthode améliorée d'estimation des dérivées et intégrales fractionnaires des états basée sur les fonctions modulatrices est proposée pour les systèmes d'ordre entier dans cette section, qui est non asymptotique et robuste. Dans ce but, le système considéré, modélisé par le modèle de représentation d'état, est d'abord transformé en forme canonique observable, puis astucieusement transformé en un système d'ordre fractionnaire. Deuxièmement, sur cette base, les fonctions modulatrices sont appliquées pour obtenir directement des formules intégrales algébriques pour les dérivées et intégrales fractionnaires de l'état. Ces formules intégrales peuvent fournir des estimations robustes dans des cas discrets bruités. En particulier, le processus d'estimation implique deux approches distinctes pour traiter les termes de valeur initiale—l'une implique leur élimination par la conception de fonctions modulatrices, tandis que l'autre nécessite un processus en deux étapes: d'abord estimer les termes de valeur initiale, puis estimer les dérivées et intégrales de l'état. De plus, il est expliqué comment concevoir les fonctions modulatrices requises. Enfin, des simulations numériques sont fournies pour démontrer l'efficacité de la méthode proposée dans cette section. De plus, nous considérons des applications, *i.e.* nous proposons un nouvel estimateur d'état basé sur les fonctions modulatrices pour les systèmes de moteurs à courant continu. Les principales contributions de cette section sont résumées comme suit:

- Contrairement à [74], [76]–[79], [97], les systèmes considérés n'ont pas besoin d'être transformés en équations différentielles d'entrée-sortie, ce qui rend le processus d'estimation et les formules obtenues plus simples.
- Contrairement aux travaux existants sur la méthode des fonctions modulatrices [76]–[79], [113], cette section peut directement obtenir des formules intégrales algébriques pour les intégrales et dérivées d'ordre fractionnaire des états sans estimer les valeurs initiales fractionnaires de la sortie. Il convient de mentionner que la dérivée et l'intégrale d'ordre fractionnaire de l'état dans [111], [112], [118] ainsi que dans la section 9.4 ont été estimées en appliquant le schéma de Grünwald-Letnikov à l'état  $x$  obtenu.
- Un nouvel estimateur basé sur les fonctions modulatrices est appliqué pour la première fois pour estimer le courant, la vitesse et leurs dérivées et intégrales fractionnaires pour les systèmes de moteurs à courant continu, qui est non asymptotique et robuste. De plus, l'efficacité de l'estimateur basé sur les fonctions modulatrices présenté est vérifiée par une simulation de Matlab

et une comparaison avec l'observateur de Luenberger. De plus, nous avons conçu un régulateur  $PI^\lambda D^\mu$  d'ordre fractionnaire pour le système de moteur à courant continu et obtenu de bons résultats de commande.

### 9.6.1 Formulation du problème

Dans cette section, nous considérons le système linéaire d'ordre entier (9.1), qui peut être transformé en forme canonique observable (9.2). Rappelons la forme canonique observable comme suit:

$$\begin{cases} \dot{z} &= \hat{A}z + \hat{B}u, \\ y &= \hat{C}z, \end{cases} \quad (9.19)$$

où

$$\hat{A} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & -a_{N-2} \\ 0 & 0 & \cdots & 0 & 1 & -a_{N-1} \end{pmatrix}, \quad (9.20)$$

et  $\hat{B} = TB = (\hat{B}_0, \hat{B}_1, \dots, \hat{B}_{N-1})^\top \in \mathbb{R}^{N \times 1}$ ,  $\hat{C} = (0, \dots, 0, 1) \in \mathbb{R}^{1 \times N}$ .

Avant d'utiliser les fonctions modulatrices et de présenter les formules intégrales algébriques de l'état, de sa dérivée et de l'intégrale du système considéré (9.1), le lemme suivant est fourni, ce qui constitue une étape importante dans ce travail.

**Lemma 9.2** *Soit  $\alpha \in (-1, 1)$ , sur la base de la forme canonique observable (9.19) du système défini dans (9.1), le système suivant peut être obtenu:  $\forall 0 < t \in I$ ,*

$$D_t^{1+\alpha} z(t) = \hat{A}D_t^\alpha z(t) + \hat{B}D_t^\alpha u(t) + \frac{z(0)}{\Gamma(-\alpha)} t^{-\alpha-1}. \quad (9.21)$$

**Preuve.** En prenant l'opérateur de dérivée fractionnaire d'ordre  $\alpha$  avec  $\alpha \in (-1, 1)$  à des deux côtés de (9.19), nous obtenons:

$$D_t^\alpha \dot{z}(t) = \hat{A}D_t^\alpha z(t) + \hat{B}D_t^\alpha u(t). \quad (9.22)$$

Ensuite, en appliquant la loi de l'indice additif donnée dans le lemme 2.1 au côté gauche de (9.22), cette preuve peut être complétée. ■

### 9.6.2 Principaux résultats

Dans cette section, en appliquant des fonctions modulatrices, les formules intégrales algébriques des dérivées fractionnaires et des intégrales de l'état pour les systèmes d'ordre entier (9.1) seront fournies directement dans le texte suivant sans utiliser le schéma de Grünwald-Letnikov. Dans le processus d'estimation, deux approches sont utilisées pour traiter les termes de valeurs initiales—l'une implique leur élimination par la conception de fonction modulatrice, tandis que l'autre implique un processus en deux étapes: d'abord l'estimation des termes de valeurs initiales et ensuite l'estimation des dérivées fractionnaires et des intégrales de l'état.

#### 9.6.2.1 Première méthode—Supprimer le terme de la valeur initiale en utilisant la conception de fonction modulatrice

**Theorem 9.5** *Considérons la forme canonique observable (9.19) du système (9.1) sur  $I$ . Pour  $t \in I$ ,  $\alpha \in (-1, 1)$ , la propriété bénéfique de des fonctions modulatrices  $\eta_j$  pour  $j = 1, \dots, N$  est introduite comme suit:*

$$(P_4^\alpha) : \forall k \in \{0, \dots, N-1\}, \int_0^t \eta_j^{(k)}(t-\tau) \tau^{-\alpha-1} d\tau = 0.$$

Dans ce cas,  $D_t^\alpha z$  avec  $\alpha \in (-1, 1)$  peut être calculée comme suit:

Dans le cas de  $\alpha \in (0, 1)$ , soit  $\eta$  une fonction modulatrice d'ordre  $(N, N+1)$  du côté droite sur  $[0, t]$  satisfaisant  $(P_4^\alpha)$ , alors l'intégrale fractionnaire de  $y$  peut être donnée par:

$$D_t^{\alpha-1} y(t) = \sum_{i=1}^N \hat{B}_{i-1} \int_0^t {}^C D_\tau^\alpha \{\eta^{(i-1)}(\tau)\} u(t-\tau) d\tau - \sum_{i=0}^N a_i \int_0^t {}^C D_\tau^\alpha \{\eta^{(i)}(\tau)\} y(t-\tau) d\tau, \quad (9.23)$$

où  $a_N = 1$ .

• De plus, soient  $\eta_j$  un groupe de fonctions modulatrices d'ordre  $(j-1, N+1)$  sur  $[0, t]$  pour  $j = 1, \dots, N$  satisfaisant  $(P_4^\alpha)$ , la formule suivante s'applique:

$$\begin{aligned} D_t^\alpha z_j(t) = & \hat{B}_{j-1} D_t^{\alpha-1} u(t) + \sum_{i=1}^N \hat{B}_{i-1} \int_0^t {}^C D_\tau^\alpha \{\eta_j^{(i-1)}(\tau)\} u(t-\tau) d\tau \\ & - a_{j-1} D_t^{\alpha-1} y(t) - \sum_{i=0}^N a_i \int_0^t {}^C D_\tau^\alpha \{\eta_j^{(i)}(\tau)\} y(t-\tau) d\tau, \end{aligned} \quad (9.24)$$

où  $D_t^{\alpha-1} y(t)$  est donnée par (9.23).

• Dans le cas où  $\alpha \in (-1, 0)$ , soient  $h_j$  un groupe de fonctions modulatrices d'ordre  $(j-1, N)$  sur  $[0, t]$  pour  $j = 1, \dots, N$  satisfaisant  $(P_4^\alpha)$ , la formule suivante s'applique:

$$D_t^\alpha z_j(t) = \sum_{i=1}^N \hat{B}_{i-1} \int_0^t D_\tau^\alpha \{h_j^{(i-1)}(\tau)\} u(t-\tau) d\tau - \sum_{i=0}^N a_i \int_0^t D_\tau^\alpha \{h_j^{(i)}(\tau)\} y(t-\tau) d\tau, \quad (9.25)$$

où  $a_N = 1$ .

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9.6.2.2 Deuxième méthode—Estimer d'abord le terme de la valeur initiale

En s'inspirant du théorème 9.1 et du théorème 9.2 de la section 9.4, en concevant des fonctions modulatrices, nous pouvons estimer la valeur initiale du système (9.19), comme le montre le théorème suivant.

**Theorem 9.6** *Considérons la forme canonique observable (4.2) du système (9.1) sur  $I$ . Pour  $t \in I$ , soient  $g_j$  pour  $j = 1, \dots, N$  un groupe de fonctions modulatrices d'ordre  $(j-1, N)$  du côté droite sur  $[0, t]$ , la formule suivante s'applique:*

$$z_j(0) = \sum_{i=1}^N (-1)^{i+j} \hat{B}_{i-1} \int_0^t g_j^{(i-1)}(\tau) u(\tau) d\tau - \sum_{i=0}^N (-1)^{i+j} a_i \int_0^t g_j^{(i)}(\tau) y(\tau) d\tau. \quad (9.26)$$

où  $a_N = 1$ .

Ensuite, les formules algébriques pour le système d'ordre entier seront fournies par la deuxième méthode.

**Theorem 9.7** *Considérons la forme canonique observable (9.19) du système (9.1) sur  $I$ . D'après le théorème 9.6, pour  $t \in I$ ,  $D_t^\alpha z$  avec  $\alpha \in (-1, 1)$  peut être calculée comme suit:*

• *Dans le cas de  $\alpha \in (0, 1)$ , supposons que  $g$  soit une fonction modulatrice d'ordre  $(N, N+1)$  du côté droite sur  $[0, t]$ , alors l'intégrale fractionnaire de  $y$  peut être donnée par:*

$$\begin{aligned} D_t^{\alpha-1} y(t) = & \sum_{i=1}^N \hat{B}_{i-1} \int_0^t {}^C D_\tau^\alpha \{g^{(i-1)}(\tau)\} u(t-\tau) d\tau - \sum_{i=0}^N a_i \int_0^t {}^C D_\tau^\alpha \{g^{(i)}(\tau)\} y(t-\tau) d\tau \\ & + \sum_{i=1}^N \frac{z(0)}{\Gamma(-\alpha)} \int_0^t g^{(i-1)}(t-\tau) \tau^{-\alpha-1} d\tau, \end{aligned} \quad (9.27)$$

où  $a_N = 1$ ,  $z(0)$  est donnée par (9.26) dans le théorème 9.6.

De plus, si  $g_j$  est un groupe de fonctions modulatrices d'ordre  $(j-1, N+1)$  sur  $[0, t]$  pour  $j = 1, \dots, N$ , la formule suivante s'applique:

$$\begin{aligned} D_t^\alpha z_j(t) = & \hat{B}_{j-1} D_t^{\alpha-1} u(t) - a_{j-1} D_t^{\alpha-1} y(t) + \sum_{i=1}^N \hat{B}_{i-1} \int_0^t {}^C D_\tau^\alpha \{g_j^{(i-1)}(\tau)\} u(t-\tau) d\tau \\ & - \sum_{i=0}^N a_i \int_0^t {}^C D_\tau^\alpha \{g_j^{(i)}(\tau)\} y(t-\tau) d\tau + \sum_{i=1}^N \frac{z_j(0)}{\Gamma(-\alpha)} \int_0^t g_j^{(i-1)}(t-\tau) \tau^{-\alpha-1} d\tau, \end{aligned} \quad (9.28)$$

où  $D_t^{\alpha-1} y(t)$  est donnée par (9.27).

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• Dans le cas de  $\alpha \in (-1, 0)$ , supposons que  $h_j$  soient un groupe de fonctions modulatrices d'ordre  $(j - 1, N + 1)$  du côté droite sur  $[0, t]$  pour  $j = 1, \dots, N$ , la formule suivante s'applique:

$$\begin{aligned} D_t^\alpha z_j(t) = & \sum_{i=1}^N \hat{B}_{i-1} \int_0^t D_\tau^\alpha \{h_j^{(i-1)}(\tau)\} u(t - \tau) d\tau - \sum_{i=0}^N a_i \int_0^t D_\tau^\alpha \{h_j^{(i)}(\tau)\} y(t - \tau) d\tau \\ & + \sum_{i=1}^N \frac{z_j(0)}{\Gamma(-\alpha)} \int_0^t h_j^{(i-1)}(t - \tau) \tau^{-\alpha-1} d\tau, \end{aligned} \quad (9.29)$$

où  $a_N = 1$ ,  $z(0)$  est donnée par (9.26) dans le théorème 9.6.

Afin d'illustrer la précision et la robustesse des estimateurs proposés dans la sous-section 9.6.2.2 par rapport aux bruits de corruption, les estimateurs [111] et [119], présentés la section 9.4 et à la sous-section 9.6.2.1, sont comparés aux estimateurs proposés, et certains résultats numériques sont présentés ci-dessous.

**Exemple 9.3** Prenons  $N = 3$ ,  $u(t) = \sin(5t)$ , et les matrices suivantes dans (9.1):

$$A = \begin{pmatrix} 11 & 1 & -11 \\ 6 & 0 & -6 \\ 18 & 1 & -17 \end{pmatrix}, B = (0, 1, 0)^\top, C = (-1, 0, 1),$$

qui satisfont les conditions  $(C_1)$ – $(C_2)$ . Supposons que la valeur initiale de l'état soit donnée par  $x(0) = (-0.1, -0.2, -0.3)^\top$ , et  $y^\varpi(t_i) = y(t_i) + \sigma\varpi(t_i)$  est l'observation bruitée de la sortie  $y$  sur  $I = [0, 10]$ , où  $t_i = iT_s$  avec  $T_s = 0.0001$ ,  $\{\varpi(t_i)\}$  est un bruit gaussien à moyenne nulle, et la valeur de  $\sigma$  est ajustée de telle sorte que le rapport signal-bruit  $SNR = 10 \log_{10} \left( \frac{\sum |y^\varpi(t_i)|^2}{\sum |\sigma\varpi(t_i)|^2} \right)$  soit égal à 15dB. La sortie et la sortie bruitée associée sont illustrées à la Fig. 9.7. Selon l'estimateur 9.26 de

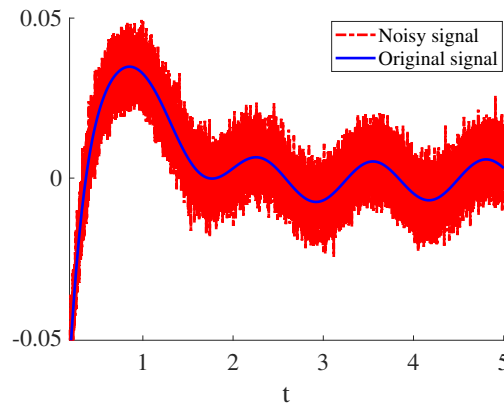


Figure 9.7:  $y$  et son observation discrète et bruitée.

la sous-section 9.6.2.2, il faut d'abord estimer la valeur initiale de l'état avant d'estimer l'intégrale



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fractionnaire et la dérivée de l'état. Entre-temps, sur la base de ces valeurs estimées, l'intégrale et la dérivée fractionnaires de l'état sont estimées, en utilisant le théorème 9.7 pour  $\alpha = -0.3, 0.2$ . Les résultats de l'estimation et les erreurs d'estimation associées de  $D_t^\alpha x_1$  pour  $\alpha = -0.3, 0.2$  sont présentées dans les Fig. 9.8 et 9.9, où  $m = 4$ . On peut donc constater que l'estimateur proposé dans la sous-section 9.6.2.2 est plus précis. En outre, les valeurs d'estimation de  $D_t^\alpha x_2$  et  $D_t^\alpha x_3$  peuvent être obtenues de la même manière.

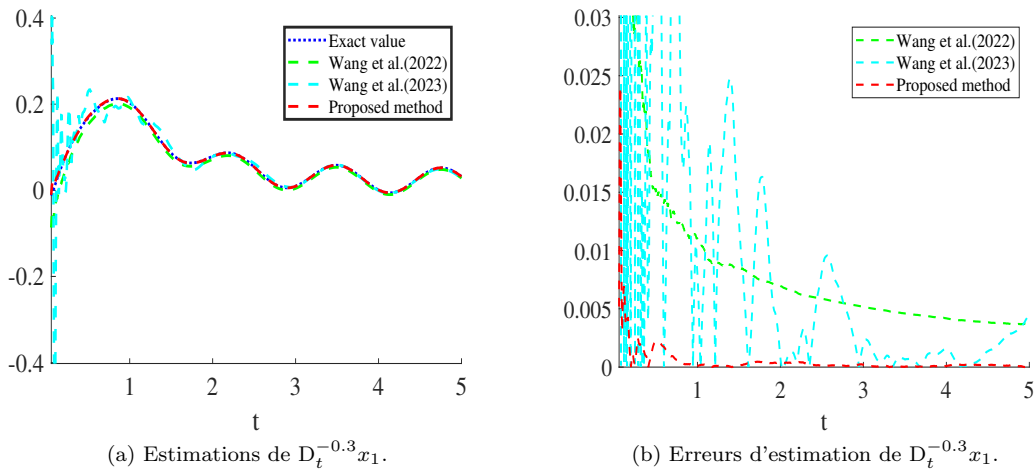


Figure 9.8: Estimations et erreurs d'estimation de  $D_t^{-0.3} x_1$  obtenues par la méthode proposée et celle de la section 9.4 [111] et de la sous-section 9.6.2.1 [119].

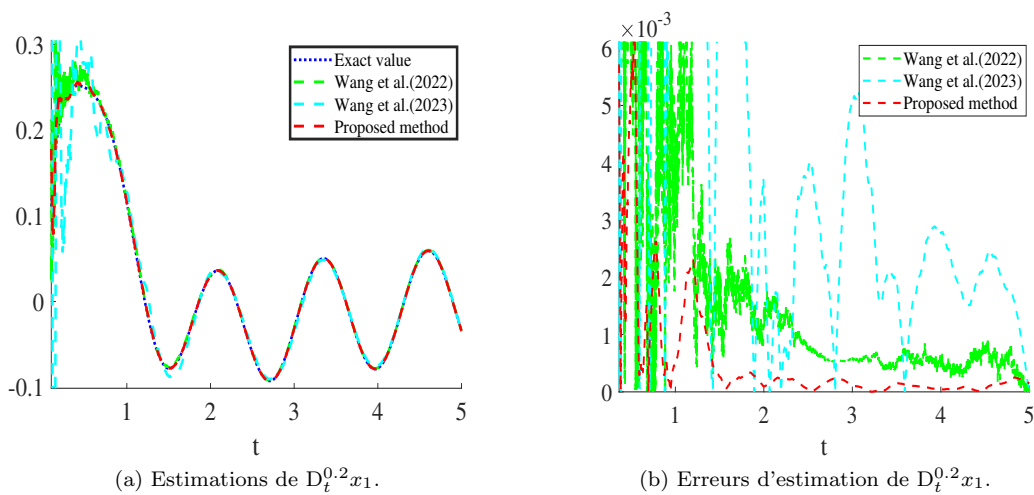


Figure 9.9: Estimations et erreurs d'estimation de  $D_t^{0.2} x_1$  obtenues par la méthode proposée et celle de la section 9.4 [111] et de la sous-section 9.6.2.1 [119].

### 9.6.3 Applications

Le moteur à courant continu est l'une des principales machines électriques de l'industrie. Les avantages distinctifs de la construction du moteur à courant continu, tels que la compacité, la robustesse, la fiabilité et le rendement élevé, ont développé l'intérêt pour leur utilisation dans divers domaines, y compris des applications étendues dans la robotique, les véhicules électriques et les processus de fabrication [120], [121]. Une connaissance précise de la vitesse ou de la position de l'arbre est impérative pour la mise en œuvre d'un moteur à courant continu. Les méthodes contemporaines de contrôle de la vitesse pour les moteurs à courant continu reposent souvent sur des capteurs de position tels que des génératrices tachymétriques ou des encodeurs numériques d'arbre pour acquérir des données sur la vitesse. En outre, étant donné que certaines variables d'état au sein du moteur ne sont pas mesurables, l'observateur ou l'estimateur sont généralement utilisés à la place d'un capteur afin d'optimiser la rentabilité tout en améliorant la longévité et la fiabilité [122]–[127]. Néanmoins, ces codeurs ont un coût élevé, introduisent un bruit de calcul et peuvent potentiellement réduire la durée de vie du système d'entraînement. Par conséquent, une technique d'estimation de la vitesse sans capteur est préférable avec un contrôleur exceptionnel qui surmonte les problèmes susmentionnés.

De nombreuses études de recherche ont démontré que le calcul fractionnaire, en raison de sa dimension infinie et de sa non-localité, peut fournir des descriptions plus précises des processus industriels complexes et des phénomènes physiques que le calcul d'ordre entier. Par conséquent, la modélisation des systèmes, l'identification, la prédiction des commandes et le diagnostic des défaillances, basés sur des systèmes d'ordre fractionnaire, sont devenus des centres d'intérêt pour de nombreux chercheurs [128]–[131]. En particulier, les contrôleurs d'ordre fractionnaire sont importants pour le contrôle automatique en raison de leurs propriétés d'amélioration des performances et de robustesse dans la conception des contrôles. Comparé au contrôleur PID traditionnel, le contrôleur d'ordre fractionnaire  $PI^\lambda D^\mu$  offre multiples avantages, notamment une réduction des débordements et des oscillations, une résilience accrue aux résultats perturbateurs, des ajustements stables aux paramètres régulés du dispositif et des temps de réponse plus rapides [132]–[135].

La conception d'un contrôleur d'ordre fractionnaire  $PI^\lambda D^\mu$  nécessite généralement l'accès à des informations relatives à l'état et à la dérivée et l'intégrale fractionnaires de l'état [21], [136], [137]. Il est donc utile d'explorer les méthodes d'acquisition de ces informations. Cependant, la dérivée d'ordre et l'intégrale fractionnaires de l'état des moteurs à courant continu ne peuvent souvent pas être mesurées par des capteurs dans les applications d'ingénierie pratiques en raison de contraintes de coût et de technologie. Par conséquent, ces valeurs doivent être estimées à partir des mesures disponibles, qui sont généralement sujettes à des bruits. En outre, la garantie de convergence asymptotique fournie par la plupart des observateurs ne permet pas de répondre à l'exigence de convergence rapide en un temps fini dans les applications en ligne. Il est donc important de concevoir des techniques d'estimation

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robustes avec une convergence rapide pour surmonter ces défis [67], [68], [70].

En tenant compte de ces idées, nous visons à proposer un nouvel estimateur de vitesse, de courant et de leur dérivée et intégrale d'ordre fractionnaire basé sur les fonctions modulatrices pour concevoir des contrôleurs d'ordre fractionnaire  $PI^\lambda D^\mu$  pour les moteurs à courant continu. Nous espérons que le nouvel estimateur proposé dans cette section fournira des informations précieuses pour les contrôleurs d'ordre fractionnaire dans diverses conceptions. Les principaux avantages de ce travail sont les suivants:

- Un nouvel estimateur basé sur les fonctions modulatrices est appliqué pour la première fois à l'estimation du courant, de la vitesse et de leurs dérivées et intégrales d'ordre fractionnaire pour un moteur à courant continu, qui est non asymptotique et robuste.
- Contrairement aux travaux existants [76]–[79] sur la méthode d'estimation des fonctions modulatrices, cette section dérive directement des expressions algébriques pour l'état et l'intégrale et la dérivée d'ordre fractionnaire sans convertir le système donné en équations différentielles d'entrée-sortie ou estimer les valeurs initiales d'ordre fractionnaire de la sortie. Il est important de noter que dans [111], [112], [118], l'intégrale et la dérivée d'ordre fractionnaire d'état ont été obtenues en utilisant le schéma de Grünwald-Letnikov. En outre, la nouvelle méthode des fonctions modulatrices proposée est plus précise que [111], [119], comme nous le concluons dans la sous-section 9.6.2.2.
- L'efficacité de l'estimateur basé sur les fonctions modulatrices présenté est vérifiée par une simulation de Matlab et par une comparaison avec l'observateur de Lunberger. En outre, nous avons conçu un contrôleur d'ordre fractionnaire  $PI^\lambda D^\mu$  pour un moteur à courant continu et obtenu de bons résultats de contrôle.

### 9.6.4 Conclusions

Dans cette section, des estimateurs non asymptotiques et robustes pour l'état, sa dérivée et son intégrale fractionnaires pour les systèmes linéaires d'ordre entier ont été construits. Tout d'abord, le système considéré a été transformé à la forme canonique observable, puis astucieusement transformé en un système d'ordre fractionnaire. Ensuite, des fonctions modulatrices ont été appliquées pour obtenir directement des formules intégrales algébriques de la dérivée et de l'intégrale fractionnaires de l'état. Comparée à la méthode des fonctions modulatrices existante [74], [76]–[79], [97], l'approche proposée élimine la nécessité de convertir en une forme d'équation différentielle, ce qui simplifie le processus d'estimation et améliore la précision des résultats. En outre, contrairement à [111], [112], [118], le nouvel estimateur proposé obtient directement la dérivée et l'intégrale fractionnaires de l'état sans utiliser le schéma de Grünwald-Letnikov, ce qui améliore l'efficacité. Enfin, des exemples numériques ont été donnés pour montrer l'efficacité des estimateurs proposés. En outre, nous envisageons des

applications aux moteurs à courant continu. Une partie de ce travail a été publiée dans [119].

## 9.7 Estimations de l'intégrale et de la dérivée fractionnaires de l'état pour les systèmes d'ordre fractionnaire

Dans cette section, une méthode améliorée d'estimation des dérivées et intégrales fractionnaires basée sur les fonctions modulatrices, proposée dans la section 9.6, est étendue aux systèmes d'ordre fractionnaire dans un environnement bruité, ce qui est non asymptotique et robuste. De même, le système considéré, modélisé par le modèle de représentation d'état, est d'abord transformé en une forme canonique observable d'ordre fractionnaire, puis en un nouveau système linéaire d'ordre fractionnaire. Sur cette base, les fonctions modulatrices sont appliquées pour obtenir directement des formules intégrales algébriques pour la dérivée et l'intégrale fractionnaires de l'état. Ces formules intégrales peuvent fournir des estimations robustes dans des cas discrets bruités. De plus, il est expliqué comment concevoir les fonctions modulatrices requises. Enfin, des simulations numériques sont fournies pour démontrer l'efficacité de la méthode proposée. Les principales contributions de cette section sont résumées comme suit:

- Contrairement aux travaux existants sur la méthode des fonctions modulatrices [76]–[79], [113], cette section peut directement obtenir des formules intégrales algébriques pour l'intégrale et la dérivée d'ordre fractionnaire des états. Il convient de mentionner que la dérivée et l'intégrale d'ordre fractionnaire de l'état dans [111], [112], [118] ont été estimées via le schéma de Grünwald-Letnikov appliqué à l'état  $x$  obtenu.

### 9.7.1 Formulation du problème

Dans cette section, nous considérons le système linéaire d'ordre fractionnaire (2.60) avec  $\eta = 0$  et  $\alpha \in (0, 1)$ , qui peut être transformé en forme canonique observable d'ordre fractionnaire en utilisant le lemme 2.7. Rappelons la forme canonique observable d'ordre fractionnaire comme suit:

$$\begin{cases} D_t^\alpha z &= \hat{A}z + \hat{B}u, \\ y &= \hat{C}z, \end{cases} \quad (9.30)$$

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où

$$\hat{A} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & -a_{N-2} \\ 0 & 0 & \cdots & 0 & 1 & -a_{N-1} \end{pmatrix}, \quad (9.31)$$

et  $\hat{B} = TB = (\hat{B}_0, \hat{B}_1, \dots, \hat{B}_{N-1})^\top \in \mathbb{R}^{N \times 1}$ ,  $\hat{C} = (0, \dots, 0, 1) \in \mathbb{R}^{1 \times N}$ ,  $\alpha \in (0, 1)$ .

### 9.7.2 Principaux résultats

#### 9.7.2.1 Estimation de la valeur initiale $D_t^{\alpha-1}z(0)$

En s'inspirant du théorème 9.3 et du théorème 9.4 de la section 9.4, en concevant des fonctions modulatrices, nous pouvons estimer la valeur initiale d'ordre fractionnaire du système (9.30), comme le montre le théorème suivant.

**Theorem 9.8** *Considérons le système donné dans (9.30). Pour  $t \in (0, h]$ , laissons  $g_j$  être une fonction modulatrice d'ordre  $(N, j - 1)$  du côté sur  $[0, t]$ . Dans ce cas, l'équation intégrale suivante est valable:*

$$D_\tau^{\alpha-1}z_j(0) = \sum_{i=1}^N \hat{B}_{i-1} \int_0^t {}^C \mathcal{D}_\tau^{(i-1)\alpha} g_j(\tau) u(t - \tau) d\tau - \sum_{i=0}^N a_i \int_0^t {}^C \mathcal{D}_\tau^{i\alpha} g_j(\tau) y(t - \tau) d\tau, \quad (9.32)$$

où  $a_N = 1$ .

**Lemma 9.3** *Supposons qu'il existe un petit  $\varepsilon \in \mathbb{R}_+$  tel que  $f$  est essentiellement bornée sur  $[0, \varepsilon]$ , sauf sur un ensemble de mesure zéro. On a alors:*

$$\left[ D_t^{-\alpha} f(t) \right]_{t=0} = \lim_{t \rightarrow 0} D_t^{-\alpha} f(t) = 0. \quad (9.33)$$

La preuve du lemme précédent peut être donnée d'une manière similaire que dans [139] où  $f$  était supposée continue sur  $[0, \varepsilon]$ .

**Ensuite, nous donnerons le processus d'estimation des intégrales et des dérivées fractionnaires des états.**

• Transformation du système

**Lemma 9.4** Soit  $\beta \in \mathbb{R}$ , sur la base de la forme canonique observable (9.30) du système défini dans (2.60), le système suivant peut être obtenu:  $\forall 0 < t \in I$ ,

$$D_t^{\alpha+\beta} z(t) = \hat{A}D_t^\beta z(t) + \hat{B}D_t^\beta u(t) + \frac{D_t^{\alpha-1} z(0)}{\Gamma(-\beta)} t^{-\beta-1}. \quad (9.34)$$

**Proof.** En prenant l'opérateur de dérivée fractionnaire d'ordre  $\beta$  avec  $\beta \in \mathbb{R}$  à des deux côtés de (9.30), nous obtenons:

$$D_t^\beta \{D_t^\alpha z(t)\} = \hat{A}D_t^\beta z(t) + \hat{B}D_t^\beta u(t). \quad (9.35)$$

En appliquant la loi de l'indice additif donnée par (2.1) au côté gauche de (9.35), nous avons:

$$\begin{aligned} D_t^\beta \{D_t^\alpha z(t)\} &= D_t^{\alpha+\beta} z(t) - \frac{D_t^{\alpha-1} z(0)}{\Gamma(-\beta)} t^{-\beta-1} \\ &= \hat{A}D_t^\beta z(t) + \hat{B}D_t^\beta u(t), \end{aligned} \quad (9.36)$$

on obtient alors (9.34). Cette preuve peut donc être complétée. ■

9.7.2.2 Estimation de l'intégrale fractionnaire de l'état

**Theorem 9.9** Considérons le système donné par (9.34) avec  $\beta < 0$  sur  $I$ . Soient  $\eta_j$  pour  $j = 1, \dots, N$  un groupe de fonctions modulatrices d'ordre  $(N, j - 1)$  du côté droite sur  $[0, t]$ . Alors, l'intégrale fractionnaire de l'état  $D_t^{\alpha+\beta-1} z$  peut être calculée par la formule suivante: pour  $j = 1, \dots, N$ ,

$$\begin{aligned} D_t^{\alpha+\beta-1} z_j(t) &= \sum_{i=1}^N \hat{B}_{i-1} \int_0^t D_\tau^\beta \{ {}^C \mathcal{D}_\tau^{(i-1)\alpha} \eta_j(\tau) \} u(t - \tau) d\tau - \sum_{i=0}^N a_i \int_0^t D_\tau^\beta \{ {}^C \mathcal{D}_\tau^{i\alpha} \eta_j(\tau) \} y(t - \tau) d\tau \\ &\quad + \sum_{i=1}^N \frac{D_t^{\alpha-1} z_j(0)}{\Gamma(-\beta)} \int_0^t {}^C \mathcal{D}_\tau^{(i-1)\alpha} \eta_j(t - \tau) \tau^{-\beta-1} d\tau, \end{aligned} \quad (9.37)$$

où  $a_N = 1$ ,  $D_\tau^{\alpha-1} z(0)$  est donnée par (9.32).

9.7.2.3 Estimation de dérivée fractionnaire de l'état

**Theorem 9.10** Considérons le système donné par (9.34) avec  $\beta \in (0, 1)$  sur  $I$ . Soient  $g_j$  un groupe de fonctions modulatrices d'ordre  $(N + 1, j - 1)$  du côté droite sur  $[0, t]$  pour  $j = 1, \dots, N$ , alors la dérivée fractionnaire de l'état  $D_\tau^{\alpha+\beta-1} z$  peut être calculée par la formule suivante: pour  $j = 1, \dots, N$ ,

$$\begin{aligned} D_t^{\alpha+\beta-1} z_j(t) &= \sum_{i=1}^N \hat{B}_{i-1} \int_0^t {}^C \mathcal{D}_\tau^{(i-1)\alpha} g_j(t - \tau) D_\tau^\beta u(\tau) d\tau - \sum_{i=0}^N a_i \int_0^t {}^C \mathcal{D}_\tau^\beta \{ {}^C \mathcal{D}_\tau^{i\alpha} g_j(t - \tau) \} y(\tau) d\tau \\ &\quad - a_{j-1} D_t^{\beta-1} y(t) + \sum_{i=1}^N \frac{D_t^{\alpha-1} z_j(0)}{\Gamma(-\beta)} \int_0^t {}^C \mathcal{D}_\tau^{(i-1)\alpha} g_j(t - \tau) \tau^{-\beta-1} d\tau, \end{aligned} \quad (9.38)$$

où  $a_N = 1$ ,  $D_t^{\alpha-1} z(0)$  est donné par (9.32).

### 9.7.3 Résultats de la simulation

Afin d'illustrer la précision et la robustesse des estimateurs proposés dans les sous-sections 9.7.2.2 et 9.7.2.3 par rapport aux bruits de corruption, les sections 9.5 [118] et [79] sont comparés aux estimateurs proposés, avec les résultats numériques suivants.

**Exemple 9.4** Prenons  $N = 3$ ,  $u(t) = 5\sin(5t)$ ,  $\alpha = 0.5$ , et les matrices suivantes dans (9.30):

$$A = \begin{pmatrix} 11 & 1 & -11 \\ 6 & 0 & -6 \\ 18 & 1 & -17 \end{pmatrix}, B = (0, 1, 0)^\top, C = (-1, 0, 1),$$

qui satisfont les conditions  $(C_1)$ – $(C_2)$ . Supposons que la valeur initiale de l'état soit donnée par  $x(0) = (0.1, 0.1, 0.1)^\top$ , et  $y^\varpi(t_i) = y(t_i) + \sigma\varpi(t_i)$  est l'observation bruitée de la sortie  $y$  sur  $I = [0, 5]$ , où  $t_i = iT_s$  avec  $T_s = 0.001$ ,  $\{\varpi(t_i)\}$  est un bruit gaussien à moyenne nulle, et la valeur de  $\sigma$  est ajustée de telle sorte que le rapport signal-bruit  $SNR = 10 \log_{10} \left( \frac{\sum |y^\varpi(t_i)|^2}{\sum |\sigma\varpi(t_i)|^2} \right)$  soit égal à 20dB. La sortie et la sortie bruitée associée sont représentées dans la Fig. 9.10.

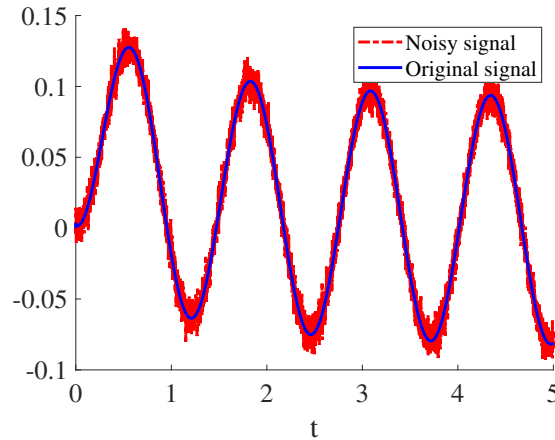


Figure 9.10:  $y$  et son observation discrète et bruitée.

En utilisant le théorème 9.8, la valeur initiale  $D_t^{-0.5}x(0)$  peut être estimée dans ce cas discret et bruité, comme le montre la Fig. 9.11. La validité de l'estimateur est vérifiée conformément au lemme 9.3. Sur la base de ces valeurs estimées, l'intégrale et la dérivée fractionnaires de l'état peuvent être estimées à l'aide du théorème 9.9 et du théorème 9.10, où la méthode d'intégration numérique trapézoïdale est utilisée pour approximer les formules intégrales. En prenant  $\beta = -0.3, 0.65$ ,  $m = 4$ , les valeurs d'estimation et les erreurs d'estimation de  $D_t^{-0.8}x_1$  et  $D_t^{0.15}x_1$  en utilisant la méthode proposée et celle de la section 9.5 [118] sont présentées respectivement dans les figures 9.12 et 9.13.

## 9.7. ESTIMATIONS DE L'INTÉGRALE ET DE LA DÉRIVÉE FRACTIONNAIRES DE L'ÉTAT POUR LES SYSTÈMES D'ORDRE FRACTIONNAIRE

En outre, [79] est considérée pour estimer la dérivée fractionnaire de l'état à des fins de comparaison avec la méthode proposée dans cette section. Les Fig. 9.12 et 9.13 montrent que l'estimateur proposé donne des résultats satisfaisants avec une convergence rapide et une bonne résistance aux bruits.

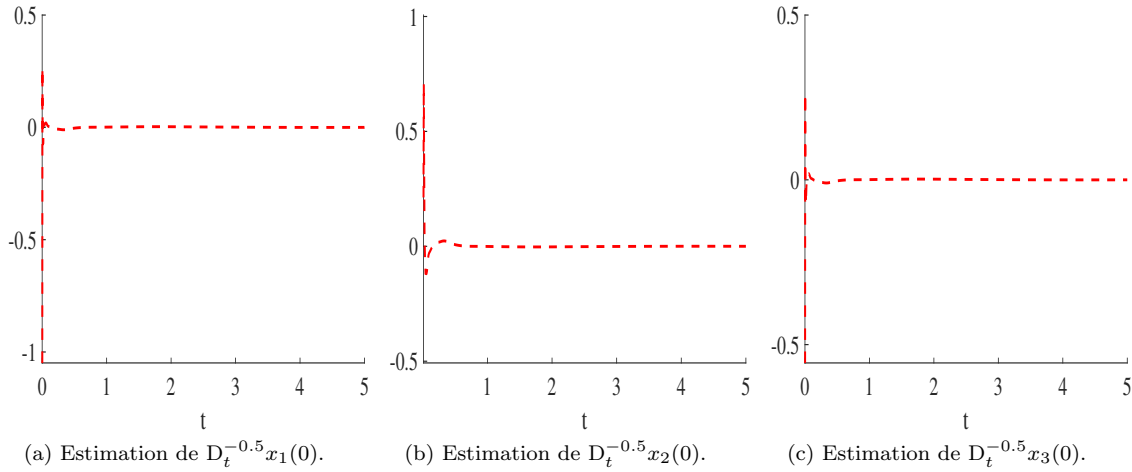


Figure 9.11: Estimations de  $D_t^{-0.5}x(0)$  obtenues par la méthode proposée dans l'exemple 9.4.

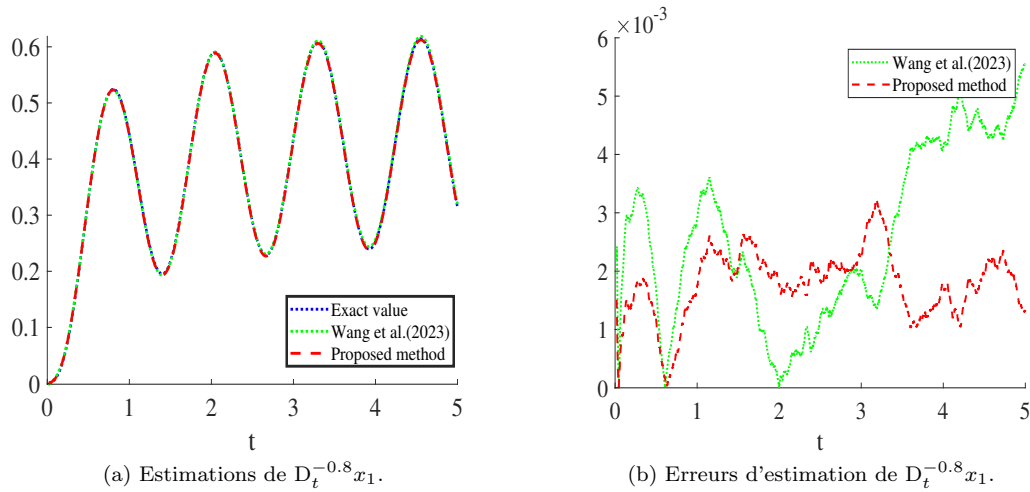


Figure 9.12: Estimations et erreurs d'estimation de  $D_t^{-0.8}x_1$  obtenues par la méthode proposée et celle de la section 9.5 [118].

### 9.7.4 Conclusions

Dans cette section, des estimateurs non asymptotiques et robustes pour l'état, sa dérivée et son intégrale fractionnaires pour les systèmes linéaires d'ordre fractionnaire ont été construits. Tout d'abord, le système considéré a été transformé à la forme canonique observable d'ordre fractionnaire, puis astucieusement transformé en un nouveau système d'ordre fractionnaire. Ensuite, des fonctions modula-



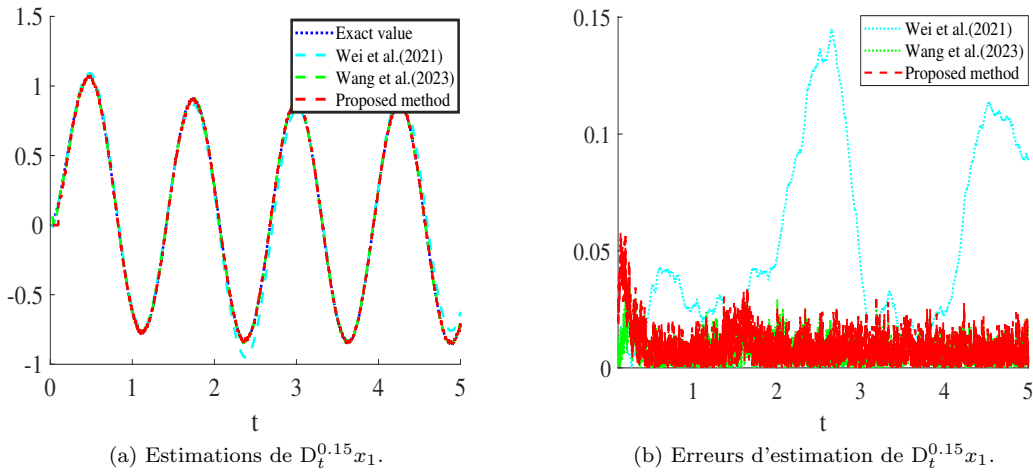


Figure 9.13: Estimations et erreurs d'estimation de  $D_t^{0.15} x_1$  obtenues par la méthode proposée et celle de [79] et de la section 9.5 [118].

trices ont été appliquées pour obtenir des formules intégrales algébriques de la dérivée et de l'intégrale fractionnaires de l'état directement. Contrairement à [111], [112], [118], le nouvel estimateur proposé a obtenu directement la dérivée et l'intégrale fractionnaires de l'état sans utiliser le schéma de Grünwald-Letnikov. Enfin, l'efficacité des estimateurs proposés a été vérifiée par des simulations numériques.

## 9.8 Conclusions et Perspectives

### 9.8.1 Conclusions

Cette thèse de doctorat a été consacrée au développement de méthodes d'estimation non asymptotiques et robustes de l'état et de ses dérivées et intégrales fractionnaires pour les systèmes d'ordre entier et les systèmes d'ordre non entier afin de simplifier le processus d'estimation et d'améliorer l'efficacité. En appliquant les méthodes proposées améliorées basées sur des fonctions modulatrices, les résultats proposés ont été donnés par des formules intégrales algébriques. Similairement aux travaux existants, les résultats de cette thèse conservent toujours les avantages de la méthode des fonctions modulatrices. D'une part, les formules algébriques peuvent fournir un calcul analytique en temps continu et fournir des résultats d'estimation non asymptotiques en temps discret (donc convergents en temps fini). D'autre part, puisque les intégrales peuvent fonctionner comme des filtres passe-bas, les formules déduites sont robustes aux bruits en temps discret.

La section 9.2 a rappelé les fondements du calcul fractionnaire et a introduit les concepts de base ainsi que les conclusions des systèmes linéaires d'ordre fractionnaire. Sur cette base, la section 9.3 a rappelé les définitions des fonctions modulatrices et a passé en revue les développements récents

pour les systèmes d'ordre fractionnaire dans la méthode des fonctions modulatrices. De plus, les applications de la méthode des fonctions modulatrices à l'identification des paramètres, à la conception de différenciateurs et à l'estimation d'état ont été présentées à travers des dérivations détaillées et des exemples.

Sur cette base, motivée par la complexité et les inefficacités des méthodes d'estimation d'état existantes basées sur les équations différentielles d'entrée-sortie en utilisant des fonctions modulatrices, une méthode améliorée d'estimation d'état basée sur les fonctions modulatrices est développée pour les systèmes d'ordre entier et d'ordre non entier dans les sections 9.4 et 9.5 afin de simplifier le processus d'estimation et d'améliorer l'efficacité de l'estimation. Les principaux avantages de ce travail sont résumés comme suit:

- Contrairement aux travaux existants sur la méthode des fonctions modulatrices [74], [76]–[79], [97], la méthode améliorée proposée dans les sections 9.4 et 9.5 est directement appliquée à la forme canonique observable, ne nécessite pas de conversion en une forme d'équation différentielle, ni besoin d'estimer les dérivées de la sortie et un ensemble de valeurs initiales d'ordre fractionnaire pour obtenir des formules intégrales algébriques pour les variables recherchées, ce qui rend le processus d'estimation plus facile et les formules obtenues plus simples. En particulier, la méthode proposée est plus efficace.
- Le modèle de système d'ordre fractionnaire considéré est plus général et n'est pas soumis aux ordres de différenciation fractionnaires. En fait, les méthodes proposées dans [76], [78] ne peuvent être appliquées que pour des systèmes d'ordre rationnel.

En tenant compte à la fois des travaux existants [111], [112], [118] et de la méthode améliorée proposée dans les sections 9.4 et 9.5, les dérivées fractionnaires de l'état sont obtenues via le schéma de Grünwald-Letnikov. Pour obtenir directement les formules intégrales algébriques des dérivées et intégrales fractionnaires des états, la deuxième méthode améliorée d'estimation d'état basée sur les fonctions modulatrices est introduite pour les systèmes d'ordre entier et d'ordre non entier dans les sections 9.6 et 9.7. Les principaux avantages de ce travail sont résumés comme suit:

- Contrairement à [74], [76]–[79], [97], les systèmes considérés n'ont pas besoin d'être transformés en équations différentielles d'entrée-sortie, ce qui rend le processus d'estimation et les formules obtenues plus simples.
- Contrairement aux travaux existants sur la méthode des fonctions modulatrices [76]–[79], [113], ces sections peuvent directement obtenir des formules intégrales algébriques pour les intégrales et dérivées d'ordre fractionnaire des états sans estimer les valeurs initiales fractionnaires de la sortie. Il convient de mentionner que la dérivée et l'intégrale d'ordre fractionnaire de l'état

dans [111], [112], [118] ainsi que dans les sections 9.4 et 9.5 ont été estimées en appliquant le schéma de Grünwald-Letnikov à l'état  $x$  obtenu.

- Un nouvel estimateur basé sur les fonctions modulatrices est appliqué pour la première fois pour estimer le courant, la vitesse et leurs dérivées et intégrales d'ordre fractionnaire pour un moteur à courant continu. De plus, nous avons conçu un régulateur d'ordre fractionnaire pour le moteur à courant continu et obtenu de bons résultats de commande.

L'efficacité des estimateurs proposés dans cette thèse a été vérifiée par des simulations numériques. De plus, les résultats des simulations ont montré que les estimateurs conçus présentent une convergence non asymptotique et une robustesse contre les bruits.

### 9.8.2 Perspectives

Les mérites des principales contributions de cette thèse sont résumés dans la section 9.8.1. Alors que dans cette section, des développements supplémentaires basés sur les conclusions de cette thèse sont discutés.

- Généraliser les méthodes proposées à des systèmes plus complexes, tels que les systèmes singuliers fractionnaires, les systèmes non linéaires fractionnaires, les systèmes d'équations aux dérivées partielles d'ordre fractionnaire, etc. Puisque les estimateurs et différenciateurs proposés ont déjà efficacement et robustement fonctionné sur des systèmes linéaires d'ordre fractionnaire, il est naturel de considérer l'extension de ces méthodes à des problèmes plus complexes. Dans ces cas, il y a quelques défis.
- Envisager d'appliquer les estimateurs et différenciateurs conçus basés sur des fonctions modulatrices à des régulateurs d'ordre fractionnaire avec des contextes pratiques, tels que les moteurs électriques, les véhicules électriques, les matériaux viscoélastiques, etc. Avec l'aide des estimateurs et différenciateurs proposés, il est prometteur de concevoir des régulateurs d'ordre fractionnaire plus stables et robustes.



**Résumé :** Motivée par la complexité et les inefficacités des méthodes de fonctions modulatrices actuelles d'estimation d'état basées sur des équations différentielles d'entrée-sortie, cette thèse vise à introduire des méthodes améliorées d'estimation d'état basées sur des fonctions modulatrices afin de simplifier le processus d'estimation et d'améliorer son efficacité. Elles peuvent être utilisées pour estimer de manière non asymptotique et robuste non seulement les états pour les systèmes linéaires d'ordre entier et non entier dans des environnements bruités, mais aussi leurs dérivées et intégrales fractionnaires. Tout d'abord, un estimateur d'état amélioré est développé pour les systèmes d'ordre entier. Ensuite, cette méthode est généralisée aux systèmes d'ordre fractionnaire pour estimer l'intégrale fractionnaire de l'état. Une analyse d'erreurs dues aux bruits est aussi donnée. Contrairement aux travaux précédents, le schéma présenté s'applique à la forme canonique observable, obtenant directement des formules intégrales algébriques, sans avoir besoin d'estimer les dérivées de la sortie et un ensemble de valeurs initiales fractionnaires. Dans la méthode améliorée proposée, les dérivées fractionnaires de l'état peuvent être obtenues via le schéma de Grünwald-Letnikov. Pour obtenir directement les formules intégrales algébriques des dérivées et intégrales fractionnaires de l'état, nous transformons astucieusement la forme canonique observable en un nouveau système d'ordre fractionnaire. Ainsi, les dérivées et intégrales fractionnaires de l'état sont directement estimées pour les systèmes d'ordre entier et non entier en concevant des fonctions modulatrices, ce qui constitue la deuxième méthode améliorée proposée. De plus, nous examinons les applications pour les moteurs à courant continu.

**Mots clés :** Systèmes d'ordre entier, Systèmes d'ordre fractionnaire, Estimation non asymptotique et robuste, Méthodes basées sur les fonctions modulatrices, Estimation de l'état, Estimations de la dérivée fractionnaire de l'état.

**Abstract :** Motivated by the complexity and inefficiencies in current modulating function state estimation methods based on input-output differential equations, this thesis aims to introduce improved modulating functions-based state estimation methods to simplify the estimation process and improve the estimation efficiency. They can be used to non-asymptotically and robustly estimate not only the states for integer and non-integer order linear systems in noisy environments, but also their fractional derivatives and integrals. First, an improved state estimator is advanced for integer order systems. Then this approach is generalized to fractional order systems for estimating the fractional integral of the state. Noise error analysis is also provided. In contrast to the previous work, the presented scheme applies to the observable canonical form, obtaining algebraic integral formulas directly, without the need for estimating the derivatives of output and a set of fractional initial values. In the proposed improved method, the fractional derivatives of the state can be obtained via the Grünwald-Letnikov scheme. To obtain directly the algebraic integral formulas of the fractional derivatives and integrals of the states, we first cleverly transform the observable canonical form into a new fractional order system. Thus, the fractional derivatives and integrals of the states are directly estimated for integer and non-integer order systems by designing modulating functions, which is the second improved method proposed. Furthermore, we consider applications for DC motors.

**Keywords :** Integer order systems, Fractional order systems, Non-asymptotic and robust estimation, Modulating functions-based methods, State estimation, Fractional derivative estimations of state.